A Complete Characterization of Downside Risk Preference

Donald C. Keenan
University of Cergy-Pontoise
dkeenan@uga.edu

Arthur Snow
University of Georgia
snow@uga.edu

Abstract

We characterize third-order risk preference in expected utility theory by utility transformations and by rankings of risk-preference measures. At the second order, a risk-averse transformation is exactly opposite to a risk-loving transformation, and is replicated by the ranking of Arrow-Pratt measures *r*. However, at the third order, transformations that introduce aversion correspond to rankings by utility measures that are not opposites, as *u* being more averse than risk-neutral utility *i* is equivalent to Kimball's prudence measure *p* being positive, but *i* being less averse than *u* requires that *p* exceed three times *r*. We resolve this paradox and shed light on previously reported comparative statics predictions based these extremes.

1 Introduction

In expected utility theory, risk preferences are dictated by the derivatives of the von Neumann-Morgenstern utility function *u* defined on income $y > 0$. As any utility function *u* is a transformation φ of risk-neutral utility $i(y) = y$, these derivatives are exactly those of the transformation generating *u*. Assuming that marginal utilities are always positive, reflecting non-satiation, *direction* of *n*th-order risk preference is indicated by the sign of the *n*th utility derivative divided by the first. Thus, at the second order, aversion to bearing risk for $u = \varphi(i)$ is identified with a positive Arrow-Pratt index $r_u = -u''/u'$, because decision makers with concave utility $(u'' < 0)$ always dislike any increase in risk [Rothschild & Stiglitz (1970)].[1] At the third order, direction is indicated by the sign of the measure $d_u = u'''/u'$, introduced by Crainich & Eeckhoudt (2008), because these decision makers always dislike any increase in downside risk [Menezes et al. (1980)].

As emphasized by Eeckhoudt (2012), risk preference refers to both direction and *intensity*. At the second order, a transformation of utility $\varphi(v)$ increases the intensity of risk aversion if and only if φ is itself risk averse, that is, $r_\varphi > 0$ [Pratt (1964)], and successive risk-averse transformations produce a strict partial *ordering* of utilities by greater risk aversion. At the third order, however, successive downside risk-averse

[1] To accommodate the definitions of increased risk and increased downside risk, we assume that income *y* belongs to a nonempty, compact positive interval. Throughout, inequalities are assumed to hold for all incomes in the interval, and we use primes to denote derivatives.

transformations, satisfying $d_\varphi > 0$, do not necessarily yield strict partial orderings.[2] This deficiency is remedied if the transformations are required to be risk averse as well as downside risk averse. With $r_\varphi > 0$ and $d_\varphi > 0$, successive transformations yield an ordering of utilities by greater risk-averse and downside risk-averse preference [Keenan & Snow (2016)].

When the reference for comparison v is risk neutral, we find that utility $u = \varphi(i)$ is risk averse and downside risk averse if and only if $r_u = r_\varphi > 0$ and $d_u = d_\varphi > 0$. These conditions are equivalent to $r_u > 0$ and $p_u > 0$, where $p_u = -u'''/u''$ is the index of prudence introduced by Kimball (1990), since $d_u = p_u r_u$. However, we also find that risk-neutral utility is less risk averse and less downside risk averse than $u = \varphi(i)$ if and only if $r_u > 0$ and $p_u - 3r_u > 0$. Thus, the measure conditions required for less aversion are stronger than those required for greater aversion.

In this paper, we resolve this paradox between greater and less aversion, and in so doing develop a complete characterization of downside risk preference that encompasses greater and less aversion. In the next section, we characterize direction and intensity for third-order risk preference, and conditions necessary and sufficient for strict partial orderings by greater intensity in terms of restrictions on the risk preferences of utility transformations. It is rarely possible to determine the transformation that converts the

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[2] A strict partial ordering is asymmetric and transitive. Lacking these properties, a ranking of utilities cannot consistently yield unambiguous comparative statics predictions. A case in which a ranking by $d_\varphi > 0$ is symmetric rather than asymmetric is provided by utility $u = \varphi(i) = -1/y$ and utility $i = \psi(u) = -1/u$, which are downside risk-averse transformations of each other. Liu & Meyer (2012) provide examples illustrating both intransitivity and symmetry.

risk preferences of one utility function into those of another. Hence, a characterization of risk preference defined in terms of utility transformations is lent tractability when it has a parallel representation in terms of utility *measures* as illustrated by the preceding discussion. At the second order, this role is served by the Arrow-Pratt index r_u, which indicates direction and intensity, and yields partial orderings of utility functions by greater risk aversion. Characterizing conditions for downside risk preference defined in terms of utility measure are identified in section 3.

In section 4, we investigate reversibility for third-order risk preference, an intrinsic although seldom recognized property of greater risk aversion. Specifically, reversing the risk preference embodied in a transformation from averse to loving, reverses the resulting utility ranking. For greater downside risk aversion, *reversibility* ensures that predictions for less aversion reverse those for greater aversion. Hence, by ensuring reversibility, we resolve the conflict between the conditions for greater and less aversion outlined above.

Finally, a complete characterization identifies a *comparative statics* thought experiment that identifies greater aversion toward bearing risk. At the second order, this role is served by the risk premium, the decision maker's willingness to pay to avoid risk, which is always greater after a risk-averse transformation of utility. At the third order, it is common to associate downside risk aversion with prudence and greater prudence with greater downside risk aversion, as observed by Crainich & Eeckhoudt (2008). In section 5, we contrast direction and comparative statics predictions for greater prudence and greater downside risk aversion. Conclusions are offered in section 6.

2 Direction, Intensity, and Ordering for Downside Risk Preference

We begin with a transformation of utility *v*, $u = \varphi(v)$, and the relationship between the attitude toward risk-bearing embodied in the transformation φ and the risk aversion measures for *v* and *u*,

$$r_u = r_\varphi v' + r_v \,, \tag{1}$$

obtained by dividing the second derivative of *u* by its first. Unless otherwise specified, the risk preferences assumed for the transformation φ are independent of those for either the final utility *u* or the reference utility *v*. Equation (1) shows that, in contrast, second-order risk preference for *u* is conditional on the second-order preference of both the transformation and the reference utility.

However, whether *v* or *u* is chosen as the reference has no bearing on either the direction of risk preference for *u* or its preference intensity relative to *v*, both as indicated by their Arrow-Pratt measures. Thus, with the inverse transformation denoted by $\psi = \varphi^{-1}$, we have $v = \psi(u)$, and

$$r_v = r_\psi u' + r_u \,. \tag{2}$$

Together, equations (1) and (2) imply

$$r_\varphi = -r_\psi / \psi' \,, \tag{3}$$

indicating that φ is risk averse if and only if its inverse ψ is risk loving. For the special case in which the reference *v* is risk neutral, utility $u = \varphi(i)$ takes on the risk preferences of φ.

Extending the link between risk-averse transformations $(r_\varphi > 0)$ and greater risk aversion from the second to the third order ties downside risk-averse transformations

$(d_{\varphi} > 0)$ to greater downside risk aversion [Keenan & Snow (2002), (2009)]. However, as noted, a ranking of utility functions by successive transformations satisfying $d_{\varphi} > 0$ is not necessarily either asymmetric or transitive, and therefore cannot produce reliable comparative statics predictions. By requiring that the transformations are risk averse as well as downside risk averse, that is,

$$r_{\varphi} > 0 \text{ and } d_{\varphi} > 0, \tag{4}$$

rankings generated by these transformations are asymmetric and transitive, and therefore constitute strict partial orderings [Keenan & Snow (2016)].

When the reference utility is risk neutral, the inequality conditions stated at (4) imply that utility $u = \varphi(i)$ exhibits risk averse and downside risk averse preferences, $r_u = r_{\varphi} > 0$ and $d_u = d_{\varphi} > 0$. Accordingly, we associate conditions (4) with *direction* of downside risk-averse preference. When the reference utility *v* satisfies these direction conditions, we associate transformations satisfying conditions (4) with greater *intensity* of downside risk-averse preference and with *orderings* by greater downside risk aversion.

3 Measures for Downside Risk Preference

To derive measure representations of the transformation conditions $r_{\varphi} > 0$ and $d_{\varphi} > 0$, we obtain

$$d_u = d_{\varphi} v'^2 + 3 r_{\varphi} r_v v' + d_v \tag{5}$$

from the first and third derivatives of $u = \varphi(v)$, then substitute for r_{φ} from equation (1), add and subtract $3r_u^2$, and rearrange terms to arrive at

$$d_\varphi = [(d_u - 3r_u^2) - (d_v - 3r_v^2) + 3r_u(r_u - r_v)]/v'^2. \tag{6}$$

To consolidate notation, we introduce the measure

$$D_v = d_v - 3r_v^2. \tag{7}$$

The following is now an immediate consequence of equations (1) and (6), equating inequality restrictions (4) for greater downside risk aversion with restrictions on utility measures.

Proposition 1 [Keenan & Snow (2022)] Given $u = \varphi(v)$, we have

$$r_\varphi > 0 \text{ and } d_\varphi > 0 \text{ if and only if } r_u > r_v \text{ and } (D_u - D_v) + 3r_u(r_u - r_v) > 0.$$

Thus, a ranking of utility functions defined by transformations that are risk averse and downside risk averse is equally represented by restrictions on the changes in the utility *measures* r and D. The Proposition identifies restrictions sufficient for $r_\varphi > 0$ and $d_\varphi > 0$ as, with $r_v > 0$, if both r_v and D_v increase after $u = \varphi(v)$ replaces v, then $r_\varphi > 0$ and $d_\varphi > 0$, implying that the transformation increases downside risk aversion.

4 Reversibility for Downside Risk Preference

A transformation of utility that increases risk aversion is reversible since the transformation must be risk averse and its inverse, a risk-loving transformation, yields the reverse ranking by less risk aversion. As a complement to Proposition 1, the following is a further consequence of equations (1) and (6) establishing *reversibility* for an ordering by greater downside risk-averse preference.

Proposition 2 [Keenan & Snow (2022)] Given $u = \varphi(v)$, we have

(a) $r_\varphi > 0$ and $d_\varphi > 0$ if and only if $r_u > r_v$ and $D_u - D_v + 3r_u(r_u - r_v) > 0$;

(b) $r_\varphi < 0$ and $d_\varphi < 0$ if and only if $r_u < r_v$ and $D_u - D_v + 3r_u(r_u - r_v) < 0$.

Part (a) restates Proposition 1 characterizing greater downside side risk aversion, while part (b) reverses the inequality conditions (4) and characterizes less downside risk aversion. The two parts provides sufficient conditions in terms of the measures r and D for greater and less aversion, respectively. By exploiting the relation $d_v = p_v r_v$, we can rewrite equation (7) as $D_v = r_v(p_v - 3r_v)$. Then the final inequality in part (a) is satisfied if $r_u(p_u - 3r_u) > r_v(p_v - 3r_v)$, while the reverse inequality is sufficient for the final inequality in part (b).

Corollary 1 Given $u = \varphi(v)$, we have

(a) if $r_u > r_v > 0$ and $p_u - 3r_u > p_v - 3r_v > 0$, then $r_\varphi > 0$ and $d_\varphi > 0$;

(b) if $0 < r_u < r_v$ and $0 < p_u - 3r_u < p_v - 3r_v$, then $r_\varphi < 0$ and $d_\varphi < 0$.

Thus, conditional on $r_v > 0$ and $p_v - 3r_v > 0$, if these measures both increase when $u = \varphi(v)$ replaces v, then φ increases downside risk aversion, and φ reduces downside risk aversion if $r_u > 0$ and $p_u - 3r_u > 0$, and both increase when the reference utility v replaces the final utility u.

Reversibility for transformation φ at the third order, however, does not imply reversibility for its inverse $\psi = \varphi^{-1}$, as it does at the second order where reversibility follows from equation (3). For the inverse transformation, d_ψ is given by

$$d_\psi = [(D_v - D_u) + 3r_v(r_v - r_u)]/u'^2, \tag{8}$$

in parallel with equation (6). The next result characterizes reversibility at the third order for the inverse transformation ψ.

Proposition 3 Given $v = \psi(u)$, we have

(a) $r_\psi < 0$ and $d_\psi < 0$ if and only if $r_u > r_v$ and $D_u - D_v + 3r_v(r_u - r_v) > 0$;

(b) $r_\psi > 0$ and $d_\psi > 0$ if and only if $r_u < r_v$ and $D_u - D_v + 3r_v(r_u - r_v) < 0$.

The inequality restrictions imposed in Propositions 2 and 3 differ precisely because the reference and final utilities differ in their preference intensity with respect to risk aversion. Assuming that v and u are risk averse, the inequalities in part (a) of Proposition 3 imply those in part (a) of Proposition 2, since in both instances we then have $r_u > r_v > 0$ and therefore $r_u(r_u - r_v) > r_v(r_u - r_v) > 0$, while those in part ($b$) of Proposition 2 imply those in part (b) of Proposition 3, as in that case we have $0 > r_u(r_u - r_v) > r_v(r_u - r_v)$. Thus, the inverse ψ being a downside risk-loving transformation implies that φ is a downside risk-averse transformation, but not vice versa, while φ being a downside risk-loving transformation implies that ψ is a downside risk-averse transformation, but not vice versa.

For the special case in which the reference utility *v* is risk neutral, Propositions 2 3 imply the following paradox, described in the introduction, after exploiting the relations $d_v = p_v r_v$ and $d_u = p_u r_u$.

Corollary 2 Given $u = \varphi(i)$ and $i = \psi(u)$, we have

(*a*) $r_\varphi > 0$ and $d_\varphi > 0$ if and only if $r_u > 0$ and $p_u > 0$;

(*b*) $r_\psi < 0$ and $d_\psi < 0$ if and only if $r_u > 0$ and $p_u - 3r_u > 0$.

Part (*a*) states that $u = \varphi(i)$ is more downside risk-averse than *i* if and only if $r_u > 0$ and $p_u > 0$, while part (*b*) states that $i = \psi(u)$ is less downside risk averse than *u*. Clearly, the difference between the characterizing measure conditions for the two parts is traceable to the fact that Propositions 2 and 3 are not equivalent. Moreover part (*b*) is the stronger condition, since the transformation conditions $r_\psi < 0$ and $d_\psi < 0$ imply $r_\varphi > 0$ and $d_\varphi > 0$.

However, part (*b*) can also be viewed as an alternative to part (*a*) as a definition of greater downside risk aversion. Exploring this avenue reveals several logical inconsistencies, among them that compensated increases in downside risk for utility $u = \varphi(i)$ are not necessarily liked by utility *i*, which is neutral to all changes in risk

Keenan & Snow (2023)].[3] Here we take a complementary tack, and observe that the necessary and sufficient conditions for $d_\varphi > 0$ can be written as

$$D_u - D_v + 3r_u(r_u - r_v) = d_u - d_v - 3r_v(r_u - r_v) > 0. \tag{9}$$

Given $r_v \geq 0$ and $r_\varphi > 0$, the inequality condition $d_u - d_v > 0$ is necessary for $d_\varphi > 0$, and is also sufficient for the special case in which $v = i$ leading to the characterization $r_u > 0$ and $p_u > 0$ in part (*a*) of Corollary 2, indicating direction of risk preference for *u* with respect to risk aversion and prudence. Since $i = \psi(u)$ is less downside risk averse than *u* implies that $u = \varphi(i)$ is more downside risk averse than *i*, the conditions $r_u > 0$ and $p_u - 3r_u > 0$ with intensity of preference aversion with respect to downside risk. These observations lead us to examine prudence and downside risk aversion with respect to direction and intensity.

5 Comparative Statics for Prudence and Downside Risk Aversion

Whereas we have identified risk-averse utility functions with dislike of mean preserving spreads in the distribution for income, Arrow (1965) and Pratt (1964) link risk aversion to a positive risk premium whose magnitude increases with greater risk aversion. Insofar as the premium approach to characterizing either direction or intensity of risk preference relies on the absence of risk as the benchmark, this approach is not applicable beyond the second order. In this section, we contrast prudence and downside risk aversion with respect to the direction of preference imparted by a transformation of

[3] A compensated increase in downside risk is a shift the distribution for income *y* that induces an increase in downside risk for utility $u(y)$, and must compensate for the decline in expected utility experienced by a downside risk-averse utility.

risk-neutral utility and intensity of preference as reflected in characteristic comparative statics predictions.

While part (*a*) of Corollary 2 shows that downside risk aversion implies positive prudence when the reference utility is risk neutral, the implication is not valid when the reference is downside risk averse in the sense that $p_v - 3r_v$ is positive.

Proposition 4 [Keenan & Snow (2010)] Given $u = \varphi(v)$, for all increasing transformations φ, we have

(*a*) $r_\varphi > 0$ and $d_\varphi > 0$ implies $p_u > p_v$ if and only if $p_v - 3r_v \le 0$;

(*b*) $r_\varphi > 0$ and $p_u > p_v$ implies $d_\varphi > 0$ if and only if $p_v - 3r_v \ge 0$.

When the reference is risk neutral, these necessary and sufficient conditions are satisfied, and we have $p_u > 0$ in both cases. Thus, introducing downside risk aversion introduces prudence, although the converse is not true, since risk-loving and downside risk-loving utility displays positive prudence.

When the reference utility is risk averse, one finds that transformations increasing prudence also increase risk aversion. As shown by Kimball, just as an increasing, concave transformation of (an increasing) utility function increases risk aversion, an increasing, convex transformation of a decreasing marginal utility, $u' = \xi(v')$, increases prudence. Given $r_v > 0$, calculation shows $r_u = r_v(\xi' v' / \xi)$ and $\xi' v' / \xi > 1$, implying that risk aversion increases. Thus, greater preference intensity for the preference directions $r_u > 0$ and $p_u > 0$ is identified with greater risk aversion and greater prudence.

The contrast between greater prudence and greater downside risk aversion is illustrated by the simple two-period saving problem with time-separable preferences described by Eeckhoudt et al (2005). The decision maker chooses saving s to maximize expected utility given by $\int v(\bar{y}-s)+v(\bar{y}+s+\varepsilon)\,dF(\varepsilon)$, where utility present and future utility functions are the same, both the interest rate and the subjective rate of time preference are set equal zero, $\bar{y}$ is the sure value of endowed income in both periods, and F is the cumulative distribution function for a zero-mean additive risk to future income, ε. Analysis of this problem by Leland (1968) showed that the introduction of an additive risk to future income increases saving by a "precautionary" amount if the decision maker exhibits risk aversion and downside risk aversion, that is, if $r_u > 0$ and $d_u > 0$, implying positive risk aversion and positive prudence, $r_u > 0$ and $p_u > 0$. Optimal saving in the absence of risk is equal to zero, and therefore in the presence of risk, optimal saving is entirely precautionary.

Kimball (1990) introduced the measure of prudence to characterize the precautionary motive in saving, in direct parallel with the characterization direction and intensity for risk aversion. Optimal precautionary saving for v, denoted by s_v, satisfies the first-order condition, $-v'(\bar{y}-s_v)+\int v'(\bar{y}+s_v+\varepsilon)\,dF=0$, or equivalently

$$v'(\bar{y}-s_v)=v'(\bar{y}+s_v-\theta_v)\,, \tag{10}$$

where θ_v is the prudence premium for v. By analogy with risk aversion and the risk premium, the prudence premium reflects the direction and intensity of the prudence measure p_v. Solving for s_v yields $s_v=\theta_v/2$. Hence, precautionary saving is positive and increases with positive and increasing prudence.

Although the response of saving to the introduction of future-income risk is dictated by the direction and intensity of prudence, the same is not true for increases in an existing risk. Assume that an increase in the shift parameter γ for the distribution function $F(\varepsilon,\gamma)$ induces a mean preserving spread, denoted by $F_\gamma(\varepsilon,\gamma)$. The effect on precautionary saving is given by $ds_v / d\gamma = (\partial\theta_v / \partial\gamma)/2$, where equation (10) yields

$$\begin{aligned} \partial\theta_v / \partial\gamma &= -\int v' \, dF_\gamma / \hat{v}'' \\ &= -\int v''' \int^{\varepsilon} F_\gamma \, d\tau \, d\varepsilon / \hat{v}'', \end{aligned} \tag{11}$$

where the second line follows using integration by parts twice, and $\hat{v}'' = v''(\bar{y} + s_v - \theta_v)$. For the increase in saving to be greater for utility *u* than for *v*, we must have

$$\frac{\partial\theta_u}{\partial\gamma} - \frac{\partial\theta_v}{\partial\gamma} = \int \left(p_u \frac{u''}{\hat{u}''} - p_v \frac{v''}{\hat{v}''} \right) \int^{\varepsilon} F_\gamma \, d\tau \, d\varepsilon > 0, \tag{12}$$

where $\hat{u}'' = u''(\bar{y} + s_u - \theta_u)$. However, greater prudence for *u* than for *v* is not sufficient for this inequality, and therefore does not imply that a greater increase in precautionary saving in response to an increase in future-income risk.

A contrasting thought experiment introduced by Crainich & Eeckhoudt (2008) yields a complementary but distinct comparative statics prediction concerning the change in the interest rate required to maintain optimal saving equal to zero when future-income risk is introduced. Let m_v denote the compensating (gross) interest rate for *v* under which optimal saving is equal to zero when future income risk is present, defined by the first-order condition

$$-v'(\bar{y}) + m_v \int v'(\bar{y} + \varepsilon) \, dF(\varepsilon,\gamma) = 0. \tag{13}$$

In the absence of risk, $m_v = 1$. Assume that $F_\gamma(\varepsilon, \gamma)$ denotes a simple mean preserving spread with single crossing at $\varepsilon = 0$.[4] When initially there is no risk, F_γ represents an introduction of risk, and otherwise F_γ represents an increase in risk with a positive cumulative increase in probability below the mean balanced by a cumulative reduction above the mean. The effect of an increase in γ on the compensating interest rate

$$
\begin{aligned}
dm_v / d\gamma &= \int v'' F_\gamma \, d\varepsilon / \bar{v}' \\
&= -\int v''' \int^{\varepsilon} F_\gamma \, d\tau \, d\varepsilon / \bar{v}'
\end{aligned}
\tag{14}
$$

is obtained from equation (13) using integration by parts twice, where $\bar{v}' = v'(\bar{y})$. Since the partial integrals on the third line are non-negative, the compensating interest rate falls below one when risk is introduced if $d_v > 0$, which is implied by $r_v > 0$ and $p_v - 3r_v > 0$.

Replacing *v* with $u = \varphi(v)$ yields

$$
\begin{aligned}
dm_u / d\gamma &= \int u'' F_\gamma \, d\varepsilon / \bar{\varphi}' \bar{u}' \\
&= \int (\varphi''' v^3 + 2\varphi'' v'' v') \int^{\varepsilon} F_\gamma \, d\tau \, d\varepsilon / \bar{\varphi}' \bar{v}' + \int \varphi' v'' F_\gamma \, d\varepsilon / \bar{\varphi}' \bar{v}' ,
\end{aligned}
\tag{15}
$$

where the second line is obtained using integration by parts, and $\bar{\varphi}' = \varphi(v(\bar{y}))$. Since the partial integrals are nonnegative, the first integral is positive if *v* is risk averse and the transformation satisfies $r_\varphi > 0$ and $d_\varphi > 0$. Hence, we have $dm_u / d\gamma < dm_v / d\gamma$, and the compensating interest rate falls, if the final integral in equation (15) is at least as great as equation (14), that is, if

$$
\int v'' (1 - \varphi' / \bar{\varphi}') F_\gamma \, d\varepsilon \geq 0 . \tag{16}
$$

[4] A simple, or single mean preserving spread satisfies a single crossing property such that

As a simple mean preserving spread with single crossing at zero, F_γ changes sign from positive to negative at zero as ε increases, while $\varphi'' > 0$ implies that $1-\varphi'/\overline{\varphi}'$ behaves in the opposite manner. Hence, if φ is downside risk averse, then utility $u = \varphi(v)$ requires a greater decline in the interest rate than v for saving to remain constant after the introduction of, or a simple increase in, future-income risk.

Proposition 5 Given $u = \varphi(v)$ and a single-crossing increase in future-income risk induced by $d\gamma > 0$, we have,

(a) $dm_u / d\gamma < dm_v / d\gamma$ if $r_\varphi > 0$ and $d_\varphi > 0$;

(b) $dm_u / d\gamma > dm_v / d\gamma$ if $r_\varphi < 0$ and $d_\varphi < 0$.

Part (a) shows that greater downside risk aversion implies a stronger reaction to to introductions or simple increases in future-income risk as measured by the decline in the interest rate required to maintain saving constant [Keenan & Snow (2016)]. Part (b) follows since φ is reversible, and demonstrates the reverse, that less downside risk aversion implies a weaker reaction to these increases in income risk. Thus, in contrast with greater prudence, greater downside risk aversion holds unambiguous comparatives statics implications for some mean preserving spreads of an existing risk as well as for introduction of risk into initially riskless saving decisions.

$F_\gamma(\varepsilon,\gamma) \geq [=](\leq) 0$ as $\varepsilon < [=](>) 0$ [Rothschild & Stiglitz (1970].

6 Conclusions

We characterize downside risk preference in expected utility theory with respect to direction and with respect to intensity for both greater and less aversion in terms of the risk preferences utility transformations. Downside risk preference is inherited from a transformation of risk-neutral utility, with the direction of third-order preference indicated by the sign of the third derivative. However, to obtain a strict partial ordering of utility functions by intensity of third-order risk preference requires specifying the transformations' second-order risk preferences as well as their third-order preferences. An ordering by greater third-order risk aversion is obtained when transformations are risk averse and downside risk averse, and the ordering is representable in terms of inequality restrictions on the utility measures of risk aversion r_v and prudence p_v. Moreover, reversing the preference directions from averse to loving yields an ordering by less third-order risk aversion. In particular, positive and increasing (decreasing) values for the measures r_v and $p_v - 3r_v$ are sufficient conditions for greater (less) downside risk aversion. Finally, we show that, the decline in the interest rate needed to maintain constant precautionary saving increases with greater downside risk aversion with the introduction of a zero-mean income risk or with a simple increase in risk with single crossing at zero.

Reversibility unlocks the paradox outlined in the introduction. Transformations that increase downside risk aversion are reversible, since the restrictions on r_v and p_v that characterize these transformations are reversed when love replaces aversion. For the same reason, their inverse transformations are reversible, but the reference utility *v* and final utility *u* switch roles thereby altering the characterizing inequality restrictions on the utility measures. As a consequence, we find that a transformation introducing downside risk averse preference necessarily introduces positive prudence, $p_u > 0$, while its inverse eliminates downside risk averse preference only if $p_u - 3r_u > 0$, the difference being an artifact of the switch in direction to and from utility u.

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