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WORKING PAPER SERIES

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N°7 /May 2023

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On the growth rate of superadditive processes and the stability of functional GARCH models

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May 6, 2023

Abstract

We extend the result of Kesten (Proc. Am. Math. Soc., 49:205-211, 1975) on the growth rate of random walks with stationary increments to superadditive processes. We show that superadditive processes which remain positive after a certain time diverge at least linearly to infinity. Our proof relies on new techniques based on concepts from ergodic theory. Different versions of this result are also given, generalizing Lemma 3.4 of Bougerol and Picard (Ann. Probab., 20:1714-1730, 1992) on the contraction property of products of random matrices. We use our results to provide necessary and sufficient conditions for the stability of a class of Stochastic Recurrent Equations (SRE) with positive coefficients in the space of continuous functions with compact support, including continuous functional GARCH models.

Keywords: Ergodic theorem Contraction property, functional Garch, Lyapunov exponent, Stochastic Recurrence Equation, Strict stationarity, Subadditive sequence.
MSC2020 subject classifications: Primary 60G10, 60F15; Secondary 47A35.

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1 Introduction

Let $(\Omega, \mathcal{B}, \mu, T)$ a measure-preserving dynamical system, i.e $(\Omega, \mathcal{B}, \mu)$ is a probability space and for all $A \in \mathcal{B}$, $\mu(T^{-1}(A)) = \mu(A)$. A $\mathcal{B}$-measurable sequence $\{\boldsymbol{S}_n\}_{n\geq 1}$ with value in $(-\infty, \infty]$ is said to be superadditive if

$$\text{for all } \ n, s \in \mathbb{N}^* \quad \boldsymbol{S}_n + \boldsymbol{S}_s \circ T^n \leq \boldsymbol{S}_{n+s} \quad a.s. \tag{1.1}$$

A subadditive sequence is defined as the opposite of a superadditive process. Since their introduction by Hammersley and Welsh (1965), one of the most significant contributions to the study of subadditive stochastic processes is the Kingman's subadditive ergodic theorem (see Kingman (1973)). Kingman showed that if $\{\boldsymbol{S}_n\}_{n\geq 1}$ is a superadditive process and $\boldsymbol{S}_1^-$ is integrable then $n^{-1}\boldsymbol{S}_n$ converges $a.s.$ to a function $\boldsymbol{S} : \Omega \to \overline{\mathbb{R}}$. Moreover, $\boldsymbol{S}^-$ is integrable and

$$\int \boldsymbol{S} d\mu = \lim_{n\to\infty} \frac{1}{n} \int \boldsymbol{S}_n d\mu = \sup_n \frac{1}{n} \int \boldsymbol{S}_n d\mu \in (-\infty, +\infty].$$

This result is a generalization of the well-known ergodic theorem of Birkhoff for additive processes, such that for all $n, s \in \mathbb{N}^*$ $\boldsymbol{S}_n + \boldsymbol{S}_s \circ T^n = \boldsymbol{S}_{n+s}$ $a.s.$ For these additive processes, even if the integrability condition does not hold, Kesten (1975) (see also Atkinson (1976)) showed that

$$\liminf n^{-1}\boldsymbol{S}_n > 0 \quad a.s \text{ on the set } \{\boldsymbol{S}_n \to \infty, n \to \infty\}. \tag{1.2}$$

This well-known result has found numerous applications in ergodic theory and was a precursor in the study of the recurrence of stationary random walks, see Atkinson (1976), Berbee (1981) and Schmidt (2006)). A similar result under an integrability condition has been obtained by Bougerol and Picard (1992, Lemma 3.4) for the product of random matrices, which characterizes the case where the so-called top-Lyapunov coefficient is negative. As in Bougerol and Picard (1992), this contraction property is often used to establish necessary and sufficient conditions for the existence of stationary solutions for Stochastic Recurrence Equations (SRE) in $\mathbb{R}^n$.

In this paper, we extend Kesten's result to superadditive processes by showing that a superadditive process that stays positive for a certain period grows at least linearly to infinity. As a corollary, we deduce the lemma 3.4 of Bougerol and Picard (1992). Our results provide a characterization of the top-Lyapunov's exponent sign for a class of discrete-time dynamical systems. The top-Lyapunov exponent is used to quantify the stability or instability of a system, and is often associated with stability when it is negative. For instance, we use our result to provide, under mild conditions, a necessary and sufficient condition for the existence of stationary solutions of functional

GARCH models in the space of continuous functions introduced by Aue et al. (2017) and Hörmann et al. (2013).

The rest of the paper is organized as follows. Section 2 is reserved for the main results. The study of the existence of stationary solution of functional GARCH models is the object of Section 3. Section 4 discusses perspectives for future work.

2 The growth rate of superadditive processes

Let us start with some remainders and conventions. A set $I \in \mathcal{B}$ is said to be invariant if $\mu\left(I \Delta T^{-1}(I)\right) = 0$. The invariant σ-algebra $\mathcal{I}_\mu$ is the collection of all such invariant sets I. It is easy to verify that for all $A \in \mathcal{B}, \quad \mu(A) = 1$ implies that $A \in \mathcal{I}_\mu$.

We set $\boldsymbol{S}_0 = 0$ throughout the paper. The convention that $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$ is used, a sum over an empty set will be equal to zero, and $T^0 = id_\Omega$. Pointwise convergence will be denoted by $\xrightarrow{\text{pw}}$. For all measurable functions Y from Ω to a measurable space $(F, \mathcal{F})$, and all $A \in \mathcal{F}$ and $(B, C) \in \mathcal{B}^2$, we say that:

$$\begin{aligned} &Y \in A \quad a.s \quad on \quad B && \text{if} \quad \mu(\{Y \in A\} \cap B) = \mu(B), \\ &C \subset B \quad a.s && \text{if} \quad \mathbb{1}_C \leq \mathbb{1}_B \quad a.s. \end{aligned}$$

Remark that results obtained for superadditive processes can be easily adapted for subadditive processes. Let us state our main result.

Theorem 2.1. *Let $\{\boldsymbol{S}_n\}_{n\geq 1}$ be a superadditive sequence and let $\boldsymbol{\tau}_0 = \sup_{n\in\mathbb{N}}\{n : \quad \boldsymbol{S}_n \leq 0\}$. We have*

$$\liminf n^{-1}\boldsymbol{S}_n > 0 \quad a.s \quad on \ \{\boldsymbol{\tau}_0 < \infty\}.$$

Noting that $\{\boldsymbol{\tau}_0 < \infty\} = \liminf\{\boldsymbol{S}_n > 0\}$, Theorem 2.1 includes Kesten's result for additive sequences. Unlike Kesten's assumption that $\boldsymbol{S}_n$ goes to infinity, we only require that the process is positive for sufficiently large values of n.

We need two technical lemmas before proving the theorem.

Lemma 2.1. *Let $\{\boldsymbol{S}_n\}_{n\geq 1}$ be a real valued superadditive sequence and let $\boldsymbol{\tau}_0 = \sup_{n\in\mathbb{N}}\{n : \quad \boldsymbol{S}_n \leq 0\}$. We have*

$$\liminf \boldsymbol{S}_n > 0 \quad a.s \quad on \ \{\boldsymbol{\tau}_0 < \infty\}. \tag{2.1}$$

Proof. Let $A = \{\liminf \boldsymbol{S}_n = 0, \ \ \boldsymbol{\tau}_0 < \infty\}$, it is clear that (2.1) is equivalent to $\mu(A) = 0$. We argue by contradiction: suppose that $\mu(A) > 0$, let

$$V = A \cap \{\boldsymbol{S}_n + \boldsymbol{S}_s \circ T^n \leq \boldsymbol{S}_{n+s} \quad \text{for all} \ \ n, s \in \mathbb{N}^*\}.$$

By countability of $\mathbb{N}^2$, we have

$$\mu(\{\boldsymbol{S}_n + \boldsymbol{S}_s \circ T^n \leq \boldsymbol{S}_{n+s} \quad \text{for all} \ \ n, s \in \mathbb{N}^*\}) = 1,$$

so we have $\mu(V) = \mu(A) > 0$. By Birkhoff's ergodic theorem we have

$$g_n := n^{-1} \sum_{k=1}^{n} \mathbb{1}_V \circ T^k \longrightarrow g := \mathbb{E}^\mu(\mathbb{1}_V | \mathcal{I}_\mu) \ \ a.s \ \ as \quad n \to \infty.$$

Since $\mathbb{E}^\mu(g) = \mu(V)$, then $\mu(\{g > 0\}) = \mu(\{\liminf g_n > 0\}) > 0$ and so $\{\liminf g_n > 0\} \neq \varnothing$. Letting $\omega \in \{\liminf g_n > 0\}$, this implies that $\{k : \ \ T^k(\omega) \in V\}$ is not finite, and so there is a strictly increasing sequence of integers $\{n_k(\omega)\}_{n\geq 1}$ such that $T^{n_k}(\omega) \in V$. Since $\omega' := T^{n_1}(\omega) \in V$ and $V \subset \{\boldsymbol{\tau}_0 < \infty\}$, then $\boldsymbol{\tau}_0(\omega') < \infty$. Let $p(\omega)$ such that $s := n_p - n_1 \geq \boldsymbol{\tau}_0(\omega') + 1$. Since $s > \boldsymbol{\tau}_0(\omega')$ it follows that

$$\boldsymbol{S}_s(\omega') > 0. \tag{2.2}$$

The fact that $T^{n_p}(\omega) \in V$ implies that

$$\liminf \boldsymbol{S}_n(T^{n_p}(\omega)) = 0. \tag{2.3}$$

By the fact that $\omega' \in V$, for all $n \geq s$,

$$\boldsymbol{S}_s(\omega') + \boldsymbol{S}_{n-s}\left(T^{n_p}(\omega)\right) = \boldsymbol{S}_s(\omega') + \boldsymbol{S}_{n-s} \circ T^s(\omega') \leq \boldsymbol{S}_n(\omega'), \tag{2.4}$$

It follows by (2.2)-(2.4), that $\liminf \boldsymbol{S}_n(\omega') \geq \boldsymbol{S}_s(\omega') > 0$, which contradicts the fact that $\omega' \in V$ and thus $\mu(A) = 0$. ☐

The second lemma gives a property on series with terms in $\{0, 1\}$. Let $p > 0$ and let $\boldsymbol{u} = \{\boldsymbol{u}_n\}_{n\geq 0}$ a sequence of elements of $\{0, 1\}^{\mathbb{N}}$. For all $n > p$, define $\{\boldsymbol{v}_k^n(\boldsymbol{u})\}_{k=0}^{\infty}$ a sequence of elements of $\{n, n-1, \cdots, 0, \ \ -\infty\}^{\mathbb{N}}$ by

$\boldsymbol{v}_0^n(\boldsymbol{u}) = n$ and for all $k > 0$ $\boldsymbol{v}_k^n(\boldsymbol{u}) = \sup\{r \in \mathbb{N} : r \leq \boldsymbol{v}_{k-1}^n(\boldsymbol{u}) - p, \ \boldsymbol{u}_r = 1\}$.
Define also $q^n(\boldsymbol{u}) = \sup\{k : \ \boldsymbol{v}_k^n(\boldsymbol{u}) > -\infty\}$ and $s(\boldsymbol{u}) = \inf\{k : \ \boldsymbol{s}_k \geq p\}$,

where $\boldsymbol{s}_k := \sum_{i=0}^{k} \boldsymbol{u}_i$.

(2.5)

It is clear that the sequence $\{\boldsymbol{v}_k^n(\boldsymbol{u})\}_{k=0}^{\infty}$ is decreasing and becomes $-\infty$ eventually. Hence, the largest index k for which $\boldsymbol{v}_k^n(\boldsymbol{u})$ is finite, denoted as $q^n(\boldsymbol{u})$, is well-defined. Note also that

$$\text{for all } k < q^n(\boldsymbol{u}), \quad \boldsymbol{v}_k^n(\boldsymbol{u}) - \boldsymbol{v}_{k+1}^n(\boldsymbol{u}) \geq p. \tag{2.6}$$

If $\sum_{i=1}^{\infty} \boldsymbol{u}_i \geq p$ then $s(\boldsymbol{u})$ is finite and, since $\{\boldsymbol{u}_n\}_{n\geq 0}$ takes values in $\{0,1\}$, there exists an integer n such that $\sum_{k=0}^{n} \boldsymbol{u}_k = p$, which implies that $\sum_{k=0}^{s(\boldsymbol{u})} \boldsymbol{u}_k = p$.

Lemma 2.2. *If* $\liminf n^{-1}\boldsymbol{s}_n > 0$ *then*

1. *for all* $n > s(\boldsymbol{u})$, $\boldsymbol{v}_{q^n(\boldsymbol{u})}^n(\boldsymbol{u}) \leq s(\boldsymbol{u})$,
2. $\liminf n^{-1} q^n(\boldsymbol{u}) > 0$.

Proof. For the first part, we have $s(\boldsymbol{u}) + 1 \geq p$ and then

$$\begin{aligned}\sum_{k=0}^{s(\boldsymbol{u})+1-p} \boldsymbol{u}_k = \sum_{k=0}^{s(\boldsymbol{u})} \boldsymbol{u}_k - \sum_{k=s(\boldsymbol{u})+1-p+1}^{s(\boldsymbol{u})} \boldsymbol{u}_k &= p - \sum_{k=s(\boldsymbol{u})+2-p}^{s(\boldsymbol{u})} \boldsymbol{u}_k \\ &\geq p - (p-1) = 1 > 0.\end{aligned}$$

It follows that there exists $r_0 \leq s(\boldsymbol{u}) + 1 - p$ such that $\boldsymbol{u}_{r_0} = 1$. Therefore, if $\boldsymbol{v}_{q^n(\boldsymbol{u})}^n(\boldsymbol{u}) > s(\boldsymbol{u})$, i.e $s(\boldsymbol{u}) + 1 \leq \boldsymbol{v}_{q^n(\boldsymbol{u})}^n(\boldsymbol{u})$, then $r_0 \leq \boldsymbol{v}_{q^n(\boldsymbol{u})}^n(\boldsymbol{u}) - p$. It follows that $r_0 \in \{r \in \mathbb{N} : r \leq \boldsymbol{v}_{q^n(\boldsymbol{u})}^n(\boldsymbol{u}) - p,\ \boldsymbol{u}_r = 1\}$ and thus $\boldsymbol{v}_{q^n(\boldsymbol{u})+1}^n \neq -\infty$. This contradicts the definition of $q^n(\boldsymbol{u})$. We thus have shown 1. To show 2, noting that $\boldsymbol{v}_0^n(\boldsymbol{u}), \boldsymbol{v}_1^n(\boldsymbol{u}), \cdots, \boldsymbol{v}_{q^n(\boldsymbol{u})}^n(\boldsymbol{u})$ is a strictly decreasing sequence of integers with $\boldsymbol{v}_0^n(\boldsymbol{u}) = n$, one has

$$s_n = \sum_{k=\boldsymbol{v}_1^n(\boldsymbol{u})+1}^{\boldsymbol{v}_0^n(\boldsymbol{u})} \boldsymbol{u}_k + \sum_{k=\boldsymbol{v}_2^n(\boldsymbol{u})+1}^{\boldsymbol{v}_1^n(\boldsymbol{u})} \boldsymbol{u}_k + \cdots + \sum_{k=0}^{\boldsymbol{v}_{q^n(\boldsymbol{u})}^n(\boldsymbol{u})} \boldsymbol{u}_k. \tag{2.7}$$

Since, by definition, $\boldsymbol{v}_{l+1}^n(\boldsymbol{u})$ is the largest index l below $\boldsymbol{v}_l^n(\boldsymbol{u}) - p$ such that $\boldsymbol{u}_l = 1$ then $\boldsymbol{u}_k = 0$ for all $\boldsymbol{v}_{l+1}^n(\boldsymbol{u}) < k \leq \boldsymbol{v}_l^n(\boldsymbol{u}) - p$. We thus have

$$\sum_{k=\boldsymbol{v}_{l+1}^n(\boldsymbol{u})+1}^{\boldsymbol{v}_l^n(\boldsymbol{u})} \boldsymbol{u}_k = \sum_{k=\boldsymbol{v}_l^n(\boldsymbol{u})-p+1}^{\boldsymbol{v}_l^n(\boldsymbol{u})} \boldsymbol{u}_k \leq p, \text{ for all } l < q^n(\boldsymbol{u}). \tag{2.8}$$

By the first part of the lemma we have

$$\sum_{k=0}^{\boldsymbol{v}_{q^n(\boldsymbol{u})}^n(\boldsymbol{u})} \boldsymbol{u}_k \leq \sum_{k=0}^{s(\boldsymbol{u})} \boldsymbol{u}_k = p, . \tag{2.9}$$

It follows by (2.7), (2.8) and (2.9) that $\boldsymbol{s}_n \leq p(q^n(\boldsymbol{u})+1)$. Therefore,

$$\liminf n^{-1} q^n(\boldsymbol{u}) \geq \liminf n^{-1}(p^{-1}\boldsymbol{s}_n - 1) = p^{-1} \liminf n^{-1}\boldsymbol{s}_n > 0,$$

which concludes the proof. □

We are now ready to give the proof of the theorem.

Proof of Theorem 2.1. The proof does not follow that of Kesten and only uses the ergodic theorem as an external result. Since the real valued sequence $\tilde{\boldsymbol{S}} := \{\min(\boldsymbol{S}_n, n)\}_{n\geq 1}$ is superadditive and $\liminf\{\boldsymbol{S}_n > 0\} = \liminf\{\tilde{\boldsymbol{S}}_n > 0\}$, and $\boldsymbol{S} \geq \tilde{\boldsymbol{S}}$ then one can assume without loss of generality that $\boldsymbol{S}$ is a real valued process. By (2.1), it suffices to prove that

$$\liminf n^{-1}\boldsymbol{S}_n > 0 \quad a.s \quad \text{on} \quad \{\liminf \boldsymbol{S}_n > 0\}\,. \tag{2.10}$$

Let $B = \{\liminf n^{-1}\boldsymbol{S}_n = 0,\ \liminf \boldsymbol{S}_n > 0\}$. Since on $\{\liminf \boldsymbol{S}_n > 0\}$, one has $\liminf n^{-1}\boldsymbol{S}_n = 0$ or $\liminf n^{-1}\boldsymbol{S}_n > 0$ $a.s.$ then to show (2.10), it is equivalent to prove that $\mu(B) = 0$. We argue by contradiction: assume that $\mu(B) > 0$. Let $f = \liminf n^{-1}\boldsymbol{S}_n$. Note that for all $\omega \in \Omega$,

$$f(\omega) = \liminf \frac{1}{n}\boldsymbol{S}_{n+1}(\omega) \geq \liminf_{n\to\infty} \frac{\boldsymbol{S}_n \circ T(\omega) + \boldsymbol{S}_1(\omega)}{n} = f\left(T(\omega)\right), \quad a.s$$

hence for all $a \in \overline{\mathbb{R}}$, $\{\omega : \ f \circ T(\omega) > a\} \subset \{\omega : \ f(\omega) > a\}$ $a.s.$ i.e.

$$T^{-1}\left(\{f > a\}\right) \subset \{f > a\} \quad a.s.$$

Because $\mu\left(T^{-1}(\{f > a\})\right) = \mu(\{f > a\})$, we have $\mu\left(\{f > a\}\,\Delta T^{-1}(\{f > a\})\right) = 0$, and therefore

$$\text{for all } a \in \overline{\mathbb{R}},\ \{f > a\} \in \mathcal{I}_\mu. \tag{2.11}$$

Let $N = \{f \leq 0\}$. Since $B \subset N$, then $\mu(N) > 0$. Let ν the probability measure in $(\Omega,\ \mathcal{B})$ given by the conditional probability given N. By Lemma A.1, $(\Omega, \mathcal{B}, \nu, T)$ is a measure-preserving dynamical system and since ν is absolutely continuous with respect to μ then $\{\boldsymbol{S}_n\}_{n\geq 1}$ is a superadditive sequence on $(\Omega, \mathcal{B}, \nu, T)$. Noting that $\{f > 0\} \cap N = \{f > 0\} \cap \{f \leq 0\} = \varnothing$, we have

$$\nu(f > 0) = \mu(N)^{-1}\mu(\{f > 0\} \cap N) = 0. \tag{2.12}$$

Let us now show that under the condition $\mu(B) > 0$, one also has $\nu(f > 0) > 0$, which contradicts (2.12). Since $\{\liminf \boldsymbol{S}_n > 0\} \cap N = B$, one has

$$\nu(\liminf \boldsymbol{S}_n > 0) = \mu(N)^{-1}\mu(B) > 0$$

and thus

$$\text{there exists } \eta > 0 \text{ such that } \nu(\liminf \boldsymbol{S}_n > \eta) > 0. \tag{2.13}$$

Since

$$\nu(\liminf \boldsymbol{S}_n > \eta) = \nu(\cup_n \{\inf_{k\geq n} \boldsymbol{S}_k > \eta\}) = \lim_{n\to\infty} \nu(\inf_{k\geq n} \boldsymbol{S}_k > \eta),$$

it follows by (2.13) that

$$\text{there exists } p > 0 \text{ such that } \nu(\inf_{k\geq p} \boldsymbol{S}_k > \eta) > 0. \tag{2.14}$$

For this p, let

$$W = \{\inf_{k\geq p} \boldsymbol{S}_k > \eta\}.$$

By Birkhoff's ergodic theorem we have

$$h_n := n^{-1} \sum_{k=1}^{n} \mathbb{1}_W \circ T^k \longrightarrow h := \mathbb{E}^\nu(\mathbb{1}_W | \mathcal{I}_\nu) \quad \nu - a.s \quad as \quad n \to \infty.$$

Since $\mathbb{E}^\nu(h) = \nu(W) > 0$, then $\nu(\{\liminf h_n > 0\}) = \nu(\{h > 0\}) > 0$. Let

$$U = \{\liminf h_n > 0\} \cap \{\boldsymbol{S}_n + \boldsymbol{S}_s \circ T^n \leq \boldsymbol{S}_{n+s} \quad \text{for all} \quad n, s \in \mathbb{N}^*\}.$$

By arguments already given, we have

$$\nu(U) = \nu(\{\liminf h_n > 0\}) > 0. \tag{2.15}$$

Let $\boldsymbol{u} = \{\boldsymbol{u}_n\}_{n\geq 0} := \{\mathbb{1}_W \circ T^n\}_{n\geq 0}$, on U, i.e. for all $\omega \in U$, $\boldsymbol{u}^\omega = \{\mathbb{1}_W \circ T^n(\omega)\}_{n\geq 0}$. Define $\{\boldsymbol{v}_k^n(\boldsymbol{u})\}_{k=0}^{\infty}$, $\{q^n(\boldsymbol{u})\}_{n\geq 0}$, $s(\boldsymbol{u})$ and $\{\boldsymbol{S}_n\}_{n\geq 1}$ as in (2.5) with p defined in (2.14). Note that $s(\boldsymbol{u}) < \infty$ and for all $n > s(\boldsymbol{u})$, $q^n(\boldsymbol{u}) \geq 1$. Remark also that $n^{-1}\boldsymbol{s}_n = h_n$ for all n and thus on U

$$\liminf n^{-1}\boldsymbol{s}_n = \liminf h_n > 0 \tag{2.16}$$

Since $\boldsymbol{v}_0^n(\boldsymbol{u}) = n$, then on U

$$\text{for all } n \geq s(\boldsymbol{u}), \quad \boldsymbol{S}_n = \boldsymbol{S}_{\boldsymbol{v}^n_{q^n(\boldsymbol{u})}(\boldsymbol{u})} + \sum_{k=0}^{q^n(\boldsymbol{u})-1} \left(\boldsymbol{S}_{\boldsymbol{v}_k^n(\boldsymbol{u})} - \boldsymbol{S}_{\boldsymbol{v}_{k+1}^n(\boldsymbol{u})}\right). \tag{2.17}$$

By the first point of Lemma 2.2 one has $\boldsymbol{S}_{\boldsymbol{v}^n_{q^n(\boldsymbol{u})}(\boldsymbol{u})} \geq \inf_{i\leq s(\boldsymbol{u})}(\boldsymbol{S}_i)$ and by the definition of U, for all $n \geq s(\boldsymbol{u})$ and $k < q^n(\boldsymbol{u})$,

$$\boldsymbol{S}_{\boldsymbol{v}_k^n(\boldsymbol{u})} - \boldsymbol{S}_{\boldsymbol{v}_{k+1}^n(\boldsymbol{u})} \geq \boldsymbol{S}_{\boldsymbol{v}_k^n(\boldsymbol{u}) - \boldsymbol{v}_{k+1}^n(\boldsymbol{u})} \circ T^{\boldsymbol{v}_{k+1}^n(\boldsymbol{u})} \quad \text{on} \quad U.$$

It follows by (2.17) that on U,

$$\text{for all } n \geq s(\boldsymbol{u}), \quad \boldsymbol{S}_n \geq \inf_{i \leq s(\boldsymbol{u})} (\boldsymbol{S}_i) + \sum_{k=0}^{q^n(\boldsymbol{u})-1} \boldsymbol{S}_{\boldsymbol{v}_k^n(\boldsymbol{u})-\boldsymbol{v}_{k+1}^n(\boldsymbol{u})} \circ T^{\boldsymbol{v}_{k+1}^n(\boldsymbol{u})}. \tag{2.18}$$

Since on U for all $k < q^n(\boldsymbol{u})$, $\boldsymbol{u}_{\boldsymbol{v}_{k+1}^n(\boldsymbol{u})} = 1$, i.e. $T^{\boldsymbol{v}_{k+1}^n(\boldsymbol{u})} \in W$ and by Eq. (2.6) $\boldsymbol{v}_k^n(\boldsymbol{u}) - \boldsymbol{v}_{k+1}^n(\boldsymbol{u}) \geq p$, then the definition of W implies that

$$\text{for all } k < q^n(\boldsymbol{u}), \ \boldsymbol{S}_{\boldsymbol{v}_k^n(\boldsymbol{u})-\boldsymbol{v}_{k+1}^n(\boldsymbol{u})} \circ T^{\boldsymbol{v}_{k+1}^n(\boldsymbol{u})} > \eta \quad \text{on } U.$$

Thus, by (2.18), one has on U

$$\text{for all } n \geq s(\boldsymbol{u}), \quad \boldsymbol{S}_n \geq \inf_{i \leq s(\boldsymbol{u})} (\boldsymbol{S}_i) + \eta q^n(\boldsymbol{u}).$$

It follows by (2.16) and the second point of Lemma 2.2, that

$$f = \liminf n^{-1} \boldsymbol{S}_n \geq \eta \liminf n^{-1} q^n(\boldsymbol{u}) > 0 \ \text{ on } U.$$

Thus $\nu(\{f > 0\}) \geq \nu(U) > 0$, where the last inequality is due to (2.15). This contradicts (2.12) and concludes the proof. □

Remark 2.1. *Following the result of Theorem 2.1, we can wonder if*

$$\limsup n^{-1} \boldsymbol{S}_n < 0 \ \ a.s \quad on \ \liminf\{\boldsymbol{S}_n < 0\}.$$

However, this statement is incorrect. A simple counter-example is the superadditive process that is identically equal to -1. A counter-example of non-a.s. constant process can be constructed.[1].

Theorem 2.1 could be stated in a weaker form if the set where $(\boldsymbol{S}_n)_{n \geq 1}$ is not always non-positive and becomes non-negative for sufficiently large values of n is invariant. This variant result is the following:

Theorem 2.2. *Let $\{\boldsymbol{S}_n\}_{n \geq 1}$ be a superadditive sequence, let $\boldsymbol{\tau} = \sup_{n \in \mathbb{N}}\{n : \ \boldsymbol{S}_n < 0\}$ and let E be an invariant subset of $\{\sup_{n \in \mathbb{N}} \boldsymbol{S}_n > 0, \ \boldsymbol{\tau} < \infty\}$. One has*

$$\liminf n^{-1} \boldsymbol{S}_n > 0 \ \ a.s \ \ on \ E. \tag{2.19}$$

[1]Let $\{\boldsymbol{X}_n\}_{n=1}^{\infty}$ a positive strictly stationary and ergodic process with a positive finite moment and let $\alpha \in (0,1)$. Using the inequality $(a+b)^\alpha \leq a^\alpha + b^\alpha$ for all $a, b \geq 0$, we have $\{\boldsymbol{S}_n := -(\sum_{k=1}^n \boldsymbol{X}_k)^\alpha\}_{n=0}^{\infty}$ is superadditive. However, since the ergodic theorem implies that $n^{-1}\sum_{k=1}^n \boldsymbol{X}_k \to \mathbb{E}\boldsymbol{X}_1 \in (0,\infty)$ *a.s.* as $n \to \infty$ then $\boldsymbol{S}_n \to -\infty$ *a.s.* as $n \to \infty$ and $\limsup n^{-1}\boldsymbol{S}_n = \limsup n^{-(1-\alpha)}(n^{-1}\sum_{k=1}^n \boldsymbol{X}_k)^\alpha = 0$ *a.s.*

Proof of Theorem 2.2. Note that (2.19) is equivalent to $\mu(\{\liminf n^{-1}\boldsymbol{S}_n > 0\} \cap E) = \mu(E)$. Letting $\upsilon = \inf_{n\in\mathbb{N}}\{n : \boldsymbol{S}_n > 0\}$, one has $\{\upsilon < \infty\} = \{\sup_{n\in\mathbb{N}} \boldsymbol{S}_n > 0\}$ and then $E \subset \{\upsilon < \infty, \boldsymbol{\tau} < \infty\}$. Let $P = \{\boldsymbol{S}_n + \boldsymbol{S}_s \circ T^n \leq \boldsymbol{S}_{n+s}$ for all $n, s \in \mathbb{N}^*\}$ and let

$$E' = E \cap P \text{ and } C(E') = \bigcap_{n\geq 0} T^{-n}(E').$$

Since $\mu(P) = 1$ then P is invariant and $\mu(E') = \mu(E)$. Hence E' is also invariant and it follows by Lemma A.2, that

$$\mu(C(E')) = \mu(E). \tag{2.20}$$

Since $C(E') \subset E \subset \{\upsilon < \infty, \boldsymbol{\tau} < \infty\}$, it follows by Lemma A.2 that for all $\omega \in C(E')$ one has,

$$\boldsymbol{S}_\upsilon(\omega) > 0,\ \boldsymbol{S}_n(\omega) \geq 0 \text{ for all } n > \boldsymbol{\tau} \text{ and } T^n(\omega) \in C(E') \text{ for all } n \geq 0$$

and then on $C(E')$,

$$\text{for all } n > \boldsymbol{\tau} \circ T^\upsilon + \upsilon, \quad \boldsymbol{S}_n(\omega) \geq \boldsymbol{S}_{n-\upsilon} \circ T^\upsilon + \boldsymbol{S}_\upsilon \geq \boldsymbol{S}_\upsilon > 0. \tag{2.21}$$

The second last inequality comes from the fact that $\boldsymbol{S}_{n-\upsilon} \circ T^\upsilon \geq 0$ because $n - \upsilon > \boldsymbol{\tau} \circ T^\upsilon$ and $T^\upsilon \in C(E')$ on $C(E') \subset \{\boldsymbol{\tau} < \infty\}$. It follows by (2.21) that $C(E') \subset \liminf\{\boldsymbol{S}_n > 0\} \cap E$ and thus, by Theorem 2.1, one has

$$\mu(C(E')) \leq \mu(\liminf\{\boldsymbol{S}_n > 0\} \cap E) = \mu(\{\liminf n^{-1}\boldsymbol{S}_n > 0\} \cap E). \tag{2.22}$$

Hence, in views of (2.20) and (2.22) we have $\mu(\{\liminf n^{-1}\boldsymbol{S}_n > 0\} \cap E) \geq \mu(E)$. Since $\{\liminf n^{-1}\boldsymbol{S}_n > 0\} \cap E \subset E$, it follows that

$$\mu(\{\liminf n^{-1}\boldsymbol{S}_n > 0\} \cap E) = \mu(E).$$

This concludes the proof. □

Remark 2.2. *It is clear that the condition $\sup_{n\in\mathbb{N}} \boldsymbol{S}_n > 0$ in Theorem 2.2 is necessary and cannot be replaced by a weaker condition. Moreover, the invariance assumption cannot be weakened without adding a supplementary condition. To illustrate this, we consider a process from Kesten (1975). Let $\Omega = \mathbb{R}^\mathbb{N}$, T the left shift operator and $\mathbb{P}[\{(-1)^n\}_n] = \mathbb{P}[\{(-1)^{n+1}\}_n] = 1/2$. The sequence $\boldsymbol{S}_n(\{x_i\}_n) := \sum_{i=0}^{n-1} x_i$ is an additive process. However, $(\boldsymbol{S}_n)_n$ is almost surely bounded on $\{\sup_{n\in\mathbb{N}} \boldsymbol{S}_n > 0\} \cap \liminf\{\boldsymbol{S}_n \geq 0\} = \{[(-1)^n]_n\}$. It can be observed that $\{[(-1)^n]_n\}$ is not invariant.*

Remark 2.3. *An interesting consequence of Theorem 2.2 is that*

$$\liminf n^{-1}\boldsymbol{S}_n > 0 \ a.s \quad \text{if} \ \sup_{n\in\mathbb{N}} \boldsymbol{S}_n > 0 \ \ \mu\text{-}a.s \ \text{and} \ \mu(\liminf\{\boldsymbol{S}_n \geq 0\}) = 1, \tag{2.23}$$

(i.e if $\mu(\{\sup_{n\in\mathbb{N}} \boldsymbol{S}_n > 0\} \cap \liminf\{\boldsymbol{S}_n \geq 0\}) = 1$*). Indeed, since for all* $A \in \mathcal{B}$, $\mu(A) = 1$ *implies that* $A \in \mathcal{I}_\mu$*, then* (2.23) *follows from Theorem 2.2.*

We say that T is ergodic if for all $I \in \mathcal{I}_\mu$, $\mu(I) \in \{0,1\}$.

Corollary 2.1. *Let* $\{\boldsymbol{S}_n\}_{n\geq 1}$ *a superadditive sequence. If* T *is ergodic then* $\liminf n^{-1}\boldsymbol{S}_n$ *is almost surely constant in* $\overline{\mathbb{R}}$ *and*

$$\liminf n^{-1}\boldsymbol{S}_n > 0 \ \ a.s \quad \text{if and only if} \quad \mu(\liminf\{\boldsymbol{S}_n > 0\}) > 0. \tag{2.24}$$

Proof. For the first point, using (2.11) and the ergodicity of T, we can deduce that for all $a \in \overline{\mathbb{R}}$, the function $F(a) := \mu(\liminf n^{-1}\boldsymbol{S}_n \leq a)$ takes its values on $\{0,1\}$. Since $F(\infty) = 1$, and $a \mapsto F(a)$ is right continuous and non-decreasing on $\overline{\mathbb{R}}$, we can conclude that there exists an $a_0 \in \overline{\mathbb{R}}$ such that for all $a < a_0$, $F(a) = 0$ and $F(a_0) = 1$. Therefore, we can conclude that $\mu(\liminf n^{-1}\boldsymbol{S}_n = a_0) = 1$, which implies that $\liminf n^{-1}\boldsymbol{S}_n = a_0$ almost surely.

The necessary condition in (2.24) is trivial. To show the sufficient condition, using Theorem 2.1, we deduce that $\mu(\liminf n^{-1}\boldsymbol{S}_n > 0) = \mu(\liminf \boldsymbol{S}_n > 0) > 0$. Therefore, in view of Equation (2.11) and the ergodicity of T, it follows that $\mu(\liminf n^{-1}\boldsymbol{S}_n > 0) = 1$. □

In the next theorem, we state the last main result of this section. Let $A \in \mathcal{B}$ with $\mu(A) > 0$. Let $\tau = \inf\{n \geq 1 : \ T^n \in A\}$. Define L by $L = T^\tau$ if τ is finite, and $L = id_\Omega$ otherwise. Let ν be the probability measure given by the conditional probability given A. By the Poincaré recurrence theorem we know that the set of points ω of A for which $T^n(\omega) \notin A$ for all $n \geq 1$ has zero measure. Therefore, τ is almost surely finite under ν and then $L = T^\tau$ ν-a.s. We can also define the ν-a.s. finite sequence of integers $(\tau_n)_{n\geq 1}$: $\tau_n = \tau \circ L^{n-1}$. For all $n \geq 0$, let $\upsilon_n = \sum_{k=1}^n \tau_k$. It is easy to see that υ_n is the index k where $T^k \in A$ for the n-th time. Therefore (υ_n) is ν-a.s. strictly increasing and grows to infinity. We have the following:

Theorem 2.3. *Let* $\{\boldsymbol{S}_n\}_{n\geq 1}$ *a superadditive sequence with* $\boldsymbol{S}_1^-$ *integrable, one has*

$$\lim n^{-1}\boldsymbol{S}_n > 0 \ \ \nu\text{-}a.s \ \ \text{on} \ \liminf_n\{\boldsymbol{S}_{\upsilon_n} > 0\}.$$

Noting that $(\upsilon_n) = (n: \ T^n \in A)$, Theorem 2.3 states that if $\boldsymbol{S}_1^-$ is integrable, then $\lim n^{-1}\boldsymbol{S}_n > 0$ μ-*a.s* on the intersection of set A and the set where the sequence $(\boldsymbol{S}_n: \ T^n \in A)$ is positive from a certain period. This means that under the set A, the positivity condition only involves the values of $(\boldsymbol{S}_n)$ with indices in $(n: \ T^n \in A)$. Note also that, under the integrability of $\boldsymbol{S}_1^-$, this result is more general than Theorem 2.1, which is obtained by taking $A = \Omega$.

In Remark 2.4 below, we show that the integrability condition in Theorem 2.3 is not superfluous.

We immediately deduce the following result that extends a variant of Kesten's result, established by Eskin and Mirzakhani (2018, Lemma C.8), for additive sequences to superadditive processes.

Corollary 2.2. *Suppose that T is ergodic and let $\{\boldsymbol{S}_n\}_{n\geq 1}$ a superadditive sequence with $\boldsymbol{S}_1^-$ integrable. Let $A \in \mathcal{B}$ with $\mu(A) > 0$. If, almost surely, the sequence $(\boldsymbol{S}_n: \ T^n \in A)$ is positive from a certain period, then*

$$\lim n^{-1}\boldsymbol{S}_n = \lim n^{-1}\mathbb{E}\boldsymbol{S}_n = \sup_n n^{-1}\mathbb{E}\boldsymbol{S}_n > 0 \quad a.s. \tag{2.25}$$

Proof. The first two equalities in (2.25) follows from Kingman's subadditive ergodic theorem. To prove that $\lim n^{-1}\boldsymbol{S}_n > 0$ *a.s.*, observe that if $\mu(\{(\boldsymbol{S}_n: \ T^n \in A) \text{ is positive from a certain period}\}) = 1$, then by Theorem 2.3, $\mu(\lim n^{-1}\boldsymbol{S}_n > 0, \ A) = \mu(A) > 0$. This means that $\mu(\lim n^{-1}\boldsymbol{S}_n > 0) > 0$, which implies the result by already given arguments. □

The proof of Theorem 2.3 is based on Theorem 2.1 and the following additional result.

Lemma 2.3. *We claim that: i) $(\Omega, \mathcal{B}, \nu, L)$ is a measure-preserving dynamical system and, ii)*

$$\mathbb{E}^\nu \tau = \mu(\cup_{k\geq 1}\{T^k \in A\})/\mu(A) < \infty. \tag{2.26}$$

Moreover, iii) $(\boldsymbol{S}_{\upsilon_n})_n$ is a superadditive sequence on $(\Omega, \mathcal{B}, \nu, L)$.

Proof. We prove *i)*. *Recall that τ is almost surely finite under ν and $L = T^\tau$ ν-a.s. Thus, we must show that for all $B \in \mathcal{B}$, $\nu(T^\tau \in B)) = \nu(B)$. We* *have*

$$\begin{aligned}\nu(T^\tau \in B)) &= \sum_{k=1}^{\infty} \nu(\tau = k, T^k \in B) \\ &= \mu(A)^{-1} \sum_{k=1}^{\infty} \mu(A, T^1 \notin A, \cdots, T^{k-1} \notin A, T^k \in A, T^k \in B) \\ &= \mu(A)^{-1} \sum_{k=1}^{\infty} \mu(\boldsymbol{X}_0 = 1, \boldsymbol{X}_1 = 0, \cdots, \boldsymbol{X}_{k-1} = 0, \boldsymbol{X}_k = 1, \boldsymbol{Y}_k = 1),\end{aligned}$$

where $(\boldsymbol{X}_n, \boldsymbol{Y}_n) = (\mathbb{1}_A \circ T^n, \mathbb{1}_B \circ T^n)$ *for all* $n \geq 0$*. It is clear that* $(\boldsymbol{X}_n, \boldsymbol{Y}_n)_{n\geq 0}$ *is a stationary sequence on* $(\Omega, \mathcal{B}, \mu)$*. Therefore, it is well-known that we can extend that sequence into the past to obtain a full stationary process* $(\boldsymbol{X}_n, \boldsymbol{Y}_n)_{n\in\mathbb{Z}}$*, see for instance Elton (1990, Lemma 1). Hence*

$$\begin{aligned}\nu(T^\tau \in B)) &= \mu(A)^{-1} \sum_{k=1}^{\infty} \mu(\boldsymbol{X}_{-k} = 1, \boldsymbol{X}_{-k+1} = 0, \cdots, \boldsymbol{X}_{-1} = 0, \boldsymbol{X}_0 = 1, \boldsymbol{Y}_0 = 1) \\ &= \mu(A)^{-1} \sum_{k=1}^{\infty} \mu(\boldsymbol{X}_0 = 1, \boldsymbol{Y}_0 = 1, \cup_{k\geq 1}\{\boldsymbol{X}_{-k} = 1\}) \\ &= \mu(A)^{-1} \mu(\boldsymbol{X}_0 = 1, \boldsymbol{Y}_0 = 1\}) \\ &= \nu(B).\end{aligned}$$

The second equality is derived from the fact that the sets $(\{\boldsymbol{X}_{-k} = 1, \boldsymbol{X}_{-k+1} = 0, \cdots, \boldsymbol{X}_{-1} = 0\})_{k\geq 1}$ *are disjoint and their union constitutes* $\cup_{k\geq 1}\{\boldsymbol{X}_{-k} = 1\}$*. The third equality follows from the Poincaré recurrence theorem, which implies that* $\{\boldsymbol{X}_0 = 1\} \subset \cup_{k\geq 1}\{\boldsymbol{X}_{-k} = 1\}$*. The conclusion follows.*

The proof of ii) (Eq. (2.26)) uses similarly arguments. We have

$$\begin{aligned}\mathbb{E}^\nu \tau = \sum_{k=1}^{\infty} \nu(\tau \geq k) &= \mu(A)^{-1} \sum_{k=1}^{\infty} \mu(\boldsymbol{X}_{-k} = 1, \boldsymbol{X}_{-k+1} = 0, \cdots, \boldsymbol{X}_{-1} = 0) \\ &= \mu(A)^{-1} \mu(\cup_{k\geq 1}\{\boldsymbol{X}_{-k} = 1\}) \\ &= \mu(\cup_{k\geq 1}\{T^k \in A\})/\mu(A),\end{aligned}$$

because $\mu(\cup_{k\geq 1}\{\boldsymbol{X}_{-k} = 1\}) = \lim_n \mu(\cup_{k=1}^n\{\boldsymbol{X}_{-k} = 1\}) = \lim_n \mu(\cup_{k=1}^n\{\boldsymbol{X}_k = 1\}) = \mu(\cup_{k\geq 1}\{\boldsymbol{X}_k = 1\}) = \mu(\cup_{k\geq 1}\{T^k \in A\})$*.*

To show iii), first note that,

$$\text{for all } n, s \in \mathbb{N}^*, \ L^n = T^{\upsilon_n} \text{ and } \upsilon_n + \upsilon_s \circ T^{\upsilon_n} = \upsilon_{n+s} \ \ \nu\text{-a.s.} \tag{2.27}$$

Thus, by superadditivity and the fact that ν is absolutely continuous with respect to μ, for all $n, s \in \mathbb{N}^*$,

$$\begin{aligned}\boldsymbol{S}_{\upsilon_n} + \boldsymbol{S}_{\upsilon_s \circ L^n} \circ L^n &= \boldsymbol{S}_{\upsilon_n} + \boldsymbol{S}_{\upsilon_s \circ T^{\upsilon_n}} \circ T^{\upsilon_n} \\ &\leq \boldsymbol{S}_{\upsilon_n + \upsilon_s \circ T^{\upsilon_n}} \\ &= \boldsymbol{S}_{\upsilon_{n+s}} \quad \nu\text{-}a.s.\end{aligned}$$

This concludes the proof. □

Proof of Theorem 2.3. By superadditivity, for all n, one has ν-almost surely $\boldsymbol{S}_n - \boldsymbol{S}_{\upsilon_k} \geq \boldsymbol{S}_{n-\upsilon_k} \circ T^{\upsilon_k}$ where $\upsilon_k \leq n < \upsilon_{k+1}$. Thus

$$\begin{aligned}\boldsymbol{S}_n &= (\boldsymbol{S}_n - \boldsymbol{S}_{\upsilon_k}) + \boldsymbol{S}_{\upsilon_k} \\ &\geq \boldsymbol{S}_{n-\upsilon_k} \circ T^{\upsilon_k} + \boldsymbol{S}_{\upsilon_k} \\ &\geq \min_{0 \leq i < \upsilon_{k+1} - \upsilon_k} \boldsymbol{S}_i \circ T^{\upsilon_k} + \boldsymbol{S}_{\upsilon_k} \\ &\geq \min_{0 \leq i < \tau \circ L^k} \boldsymbol{S}_i \circ L^k + \boldsymbol{S}_{\upsilon_k}, \quad \text{by Eq. (2.27).} \\ &\geq \min_{0 \leq i < \tau \circ L^k} -\boldsymbol{S}_i^- \circ L^k + \boldsymbol{S}_{\upsilon_k} \ \nu\text{-}a.s.\end{aligned}$$

Since, by superadditivity $\boldsymbol{S}_i \geq \sum_{j=0}^{i-1} \boldsymbol{S}_1 \circ T^j$ ν-a.s. for all $0 \leq i < \tau \circ L^k$, we have $-\boldsymbol{S}_i^- \circ L^k \geq (\sum_{j=0}^{i-1} -\boldsymbol{S}_1^- \circ T^j) \circ L^k \geq -(\sum_{j=0}^{\tau-1} \boldsymbol{S}_1^- \circ T^j) \circ L^k$ ν-a.s. Therefore,

$$\begin{aligned}n^{-1}\boldsymbol{S}_n &\geq -n^{-1}(\sum_{j=0}^{\tau-1} \boldsymbol{S}_1^- \circ T^j) \circ L^k + n^{-1}\boldsymbol{S}_{\upsilon_k} \\ &\geq -k^{-1}(\sum_{j=0}^{\tau-1} \boldsymbol{S}_1^- \circ T^j) \circ L^k + \upsilon_{k+1}^{-1}\boldsymbol{S}_{\upsilon_k} \ \nu\text{-}a.s.\end{aligned}$$

The last inequality is due to the fact that υ_k is ν-a.s. strictly increasing, which implies that $k \leq \upsilon_k \leq n$. Under the integrablity of $\boldsymbol{S}_1^-$, the Kingman ergodic theorem implies that $\lim n^{-1}\boldsymbol{S}_n$ exists μ-a.s. Thus the limit also exists ν-a.s. Since k grows to infinity with n, it follows that

$$\lim_n n^{-1}\boldsymbol{S}_n \geq -\limsup_k k^{-1}(\sum_{j=0}^{\tau-1} \boldsymbol{S}_1^- \circ T^j) \circ L^k + \liminf_k \upsilon_{k+1}^{-1}\boldsymbol{S}_{\upsilon_k} \ \nu\text{-}a.s.$$

To conclude, it suffices to show that on $\liminf_k\{\boldsymbol{S}_{\upsilon_k} > 0\}$, one has ν-*a.s* :

$$i) \ \liminf_k \upsilon_{k+1}^{-1}\boldsymbol{S}_{\upsilon_k} > 0, \ \text{and} \ ii) \ \limsup_k k^{-1}(\sum_{j=0}^{\tau-1} \boldsymbol{S}_1^- \circ T^j) \circ L^k = 0.$$

Let us show i). Since $\upsilon_n = \sum_{k=0}^{n-1} \tau \circ L^k$ ν-a.s., it follows by Lemma 2.3 and the Birkhoff ergodique theorem that

$$\lim_k k^{-1}\upsilon_{k+1} \text{ exists and is finite } \quad \nu\text{-}a.s. \tag{2.28}$$

We also have, by Lemma 2.3 and Theorem 2.1, that

$$\liminf_k k^{-1}\boldsymbol{S}_{\upsilon_k} > 0 \ \ \nu\text{-}a.s \quad \text{on } \liminf_k \{\boldsymbol{S}_{\upsilon_k} > 0\}.$$

Therefore, the result follows:

$$\liminf_k \upsilon_{k+1}^{-1}\boldsymbol{S}_{\upsilon_k} = \liminf_k (k^{-1}\upsilon_{k+1})^{-1}(k^{-1}\boldsymbol{S}_{\upsilon_k}) > 0 \ \ \nu\text{-}a.s \quad \text{on } \liminf_k \{\boldsymbol{S}_{\upsilon_k} > 0\}.$$

Now let us turn to the proof of ii). Since $\boldsymbol{S}_1^-$ is integrable, we have by the Birkhoff ergodique theorem and the "absolutely continuous" argument that

$$\lim_n n^{-1} \sum_{i=1}^{n} \boldsymbol{S}_1^- \circ T^i \text{ exists and is finite } \nu\text{-}a.s. \tag{2.29}$$

Since $\upsilon_k \to \infty$ ν-a.s., it follows that $\lim_k \upsilon_k^{-1} \sum_{i=0}^{\upsilon_k - 1} \boldsymbol{S}_1^- \circ T^i$ exists and is finite ν-a.s. Letting $f = \sum_{j=0}^{\tau-1} \boldsymbol{S}_1^- \circ T^j$, it is no difficult to see that

$$\sum_{i=0}^{\upsilon_k - 1} \boldsymbol{S}_1^- \circ T^i = \sum_{i=0}^{k-1} (\sum_{j=0}^{\tau-1} \boldsymbol{S}_1^- \circ T^j) \circ L^i = \sum_{i=0}^{k-1} f \circ L^i \ \ \nu\text{-}a.s.$$

It follows by (2.28) and (2.29) that

$$\lim_k k^{-1} \sum_{i=0}^{k-1} f \circ L^i = \lim_k (k^{-1}\upsilon_k)(\upsilon_k^{-1} \sum_{i=0}^{\upsilon_k - 1} \boldsymbol{S}_1^- \circ T^i) \text{ exists and is finite } \ \nu\text{-}a.s.$$

This implies that $k^{-1} f \circ L^k$ converges to 0 ν-a.s., which concludes the proof. □

The following result, which we state without proof, follows directly from Corollary 2.1 for Point 1. and Corollary 2.2 for Point 2. through the application of the function $-\log$.

Corollary 2.3. *Suppose that T is ergodic and let $\{\gamma_n\}_{n\geq 1} \in \mathbb{R}^{\mathbb{N}}$ be a positive sub-multiplicative process (i.e for all $n, s \in \mathbb{N}^*$ $0 \leq \gamma_{n+s} \leq \gamma_n \times \gamma_s \circ T^n$ a.s.). Let $A \in \mathcal{B}$ with $\mu(A) > 0$.*

1. *If* $\mu(\liminf_n\{\boldsymbol{\gamma}_n < 1\}) > 0$, *then* $\boldsymbol{\gamma}$ *is almost surely constant in* $\overline{\mathbb{R}}$ *and*

$$\gamma := \limsup_n n^{-1} \log \boldsymbol{\gamma}_n < 0 \quad a.s.$$

2. *If* $\mathbb{E} \log^+ \boldsymbol{\gamma}_1$ *is finite and, almost surely, the sequence* $(\boldsymbol{\gamma}_n : T^n \in A)$ *is strictly less than* 1 *from a certain period, then*

$$\gamma = \lim_n n^{-1} \log \boldsymbol{\gamma}_n = \lim_n n^{-1} \mathbb{E} \log \boldsymbol{\gamma}_n = \inf_n n^{-1} \mathbb{E} \log \boldsymbol{\gamma}_n < 0 \quad a.s.$$

Recall that γ is almost surely constant. The previous corollary is more general than Lemma 3.4 of Bougerol and Picard (1992), see the next corollary. Point 1. of Corollary 2.3 does not require any integrability condition, it applies to all sub-multiplicative ergodic sequences and only needs its values to be negative for n large enough. This result also enables the characterization of the case where the top-lyapunov exponent of a class of cocycles on a measure preserving transformation is negative.

Remark 2.4. *If* $\mathbb{E} \log^+ \boldsymbol{\gamma}_1$ *is not finite, then the conclusion of Point 2. of Corollary 2.3 is no longer valid. Indeed, for all* n*, let* $\boldsymbol{\alpha}_n = e^{-1} \frac{\boldsymbol{u}_n}{\boldsymbol{u}_{n-1}}$ *where* $(\boldsymbol{u}_n)_{n \in \mathbf{z}}$ *is a positive iid sequence such that* $\mathbb{E} \ln^+ \boldsymbol{u}_0 = \infty$*. Consider the measure-preserving dynamical system given by the quadruplet:* $R^{\mathbb{Z}}$ *and its Borel* σ*-algebra, the push-forward probability measure* $\mathbb{P}_{\boldsymbol{u}}$ *of* $(\boldsymbol{u}_n)$ *and the shift operator* T*. It is easy to see that the process* $(\boldsymbol{\gamma}_n)_{n \geq 1}$*, where* $\boldsymbol{\gamma}_n = \prod_{k=1}^n \boldsymbol{\alpha}_n$*, is a sub-multiplicative process. We show, in Appendix A.1, that there exists a measurable set* A *with* $\mathbb{P}_{\boldsymbol{u}}(A) > 0$ *such that the sequence* $(\boldsymbol{\gamma}_n : T^n \in A)$ *converges almost surely to* 0 *and on the other hand that* $\limsup_n n^{-1} \log \boldsymbol{\gamma}_n \geq 0$*. This is because* $\mathbb{E} \ln^+ \boldsymbol{\gamma}_1 = \infty$*.*

Corollary 2.4. *(Bougerol and Picard, 1992, Lemma 3.4) Let* $\{M_n\}_{n \geq 1}$ *be an ergodic strictly stationary sequence in the space of the* $d \times d$ *real matrices. We suppose that* $\mathbb{E}\left(\log^+ \|M_0\|\right)$ *is finite and that, almost surely,* $\lim_{n \to +\infty} \|M_n M_{n-1} \cdots M_1\| = 0$*. Then*

$$\gamma := \inf_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E}\left(\log \|M_n M_{n-1} \cdots M_1\|\right) < 0.$$

Proof. Let $\boldsymbol{\gamma}_n = \log \|M_n M_{n-1} \cdots M_1\|$ for all $n \in \mathbb{N}$. Note that $\mathbb{P}(\liminf_n\{\boldsymbol{\gamma}_n < 1\}) = 1$ if $\lim_{n \to \infty} \boldsymbol{\gamma}_n = 0$ $a.s.$ Since $\{\boldsymbol{\gamma}_n\}_{n \geq 1}$ is a sub-multiplicative sequence, the result follows from Corollary 2.3 and the Kingman's subadditive ergodic theorem (see also Furstenberg and Kesten (1960)). ∎

3 Stationarity of fGARCH models in $\mathcal{C}^0$

In this section, we study the existence of a stationary solution of the functional GARCH models in the space of continuous functions (see Aue et al. (2017)).

A sequence $(\boldsymbol{r}_t : t \in \mathbb{Z})$ of random elements where each random object $\boldsymbol{r}_t$ is a curve $(\boldsymbol{r}_t(u) : u \in [0,1])$ in $\mathcal{C}[0,1]$, the space of continuous functions on $[0,1]$, is called a functional GARCH process of orders $(1,1)$, abbreviated by fGARCH$(1,1)$, if it satisfies the equations

$$\begin{aligned} \boldsymbol{r}_t &= \boldsymbol{\sigma}_t \boldsymbol{\eta}_t, \\ \boldsymbol{\sigma}_t^2 &= \delta + \alpha \boldsymbol{r}_{t-1}^2 + \beta \boldsymbol{\sigma}_{t-1}^2 = \delta + \int \boldsymbol{\gamma}_t(\cdot, s) \boldsymbol{\sigma}_{t-1}^2(s) ds = \delta + \boldsymbol{\gamma}_t \boldsymbol{\sigma}_{t-1}, \end{aligned} \tag{3.1}$$

where $(\boldsymbol{\eta}_t : t \in \mathbb{Z})$ is a sequence of independent and identically distributed (iid) random functions in $\mathcal{C}[0,1]$, δ is a positive function and the integral operators α and β, i.e. $(\alpha x)(u) = \int \alpha(u,s) x(s) ds$ and $(\beta x)(u) = \int \beta(u,s) x(s) ds$ are positive, i.e. they map nonnegative functions to nonnegative function. $\boldsymbol{\gamma}_t(u,s) = \alpha(u,s) \boldsymbol{\eta}_{t-1}^2(s) + \beta(u,s)$ is an element of $\mathcal{C}[0,1]^2$.

By extending our considerations to include strictly stationary and ergodic but non-id innovations, and by replacing the interval $[0,1]$ with an arbitrary compact set K, we can generalize the autoregressive model with non-negative random functional coefficients with Eq. (3.2) below to include a wide range of conditional volatility models. Furthermore, by allowing the coefficients δ, α, and β to be stochastic processes rather than constants, and by dropping the assumption that $\gamma = \alpha + \beta$ must be an integral operator, we can obtain even more flexibility in our modeling approach. The model is as follows:

$$\boldsymbol{h}_t = \delta(\boldsymbol{\eta}_{t-1}) + \gamma(\boldsymbol{\eta}_{t-1}) \boldsymbol{h}_{t-1}, \tag{3.2}$$

where the positive stochastic curve $\boldsymbol{\delta}_t = \delta(\boldsymbol{\eta}_t)$ and linear operator $\boldsymbol{\gamma}_t = \gamma(\boldsymbol{\eta}_t)$ are measurable functions of $\boldsymbol{\eta}_t$.

We can see that Model (3.2) include the functional GARCH considered in (3.1). If $K = \{1\}$, we obtain the univariate class of GARCH$(1,1)$ model of He and Teräsvirta (1999) and if K is finite we get the class of multivariate-constant conditional correlation and univariate asymmetric power GARCH(p,q), see the AR(1) representation of Maïnassara et al. (2022).

Across the different normed vector spaces, we will unambiguously use the classical notation of the norm, $\|\cdot\|$. We recall that $F := \mathcal{C}(K)$ equipped with the uniform norm $\|x\| = \sup\{|x(u)|, \quad u \in K\}$ is a Banach space. The space of the linear endomorphisms in F is equipped with the usual operator norm $\|\alpha\| = \sup\{\|\alpha(x)\|, \quad \|x\| \leq 1\}$. Denoting $\boldsymbol{e} : K \ni u \longmapsto 1$, remark that for all positive operator α, $\|\alpha\| = \|\alpha \boldsymbol{e}\|$. For all $x \in F$, let $\inf x = \inf\{|x(u)|, \quad u \in K\}$.

The stationarity of Model (3.1) has been studied in (Aue et al., 2017, Theorem 2.2) and in (Hörmann et al., 2013, Theorem 2.3) when the model is reduced to a pure functional ARCH. In both papers, they give a sufficient condition for the existence of a stationary solution. The weakest condition is obtained by Aue et al. (2017). They show that if

$$-\infty \leq \mathbb{E} \log \|\boldsymbol{\gamma}_0\| < 0, \tag{3.3}$$

then Model (3.1) have a unique, strictly stationary and nonanticipative solution in $\mathcal{C}[0,1]$.

Contrary to the multivariate setup, to our knowledge, necessary and sufficient conditions for the existence of a stationary solution of Model (3.1) have never been established. As noted by Cerovecki et al. (2019, Remark 1), one of the main challenges in establishing these conditions is to extend the contraction property of random matrices to linear operators. Since we have established this result in Corollary 2.3, we are ready to provide necessary and sufficient conditions for the existence of a stationary solution for the general functional GARCH models considered in (3.2). To establish theses conditions, the following assumptions will be made.

A1 $(\boldsymbol{\eta}_t)$ is iid.

A2 $(\boldsymbol{\eta}_t)$ is strictly stationary and ergodic and $\mathbb{E}(\log^+ \|\boldsymbol{\gamma}_0\|)$ is finite.

For all $t \geq 0$, let

$$\boldsymbol{\gamma}_t^{(0)} = id_F \text{ and } \boldsymbol{\gamma}_t^{(n)} = \boldsymbol{\gamma}_t \circ \cdots \circ \boldsymbol{\gamma}_{t-n+1} \text{ for all } n \geq 1.$$

Consider the following assumption.

$$\mathbb{P}(\inf\{\sum_{k=0}^{+\infty} \boldsymbol{\gamma}_0^{(k)} \boldsymbol{\delta}_{-k}, \quad u \in K\} = 0) < 1. \tag{3.4}$$

Note that if $\mathbb{P}(\inf\{\boldsymbol{\delta}_0(u), \ u \in K\} = 0) < 1$ then we have (3.4). Since we deal with volatility curves, it is not restrictive to assume that (3.4) holds. This condition is satisfied by most commonly used volatility models and ensures that the solutions are positive on the entire curve in a non-negligible set. Indeed, by iterating (3.2), we can see that any non-negative solution $(\boldsymbol{h}_t)$ of (3.2) satisfies: $\boldsymbol{h}_t \geq \sum_{k=0}^{\infty} \boldsymbol{\gamma}_t^{(k)} \boldsymbol{\delta}_{t-k}$. Thus, (3.4) implies that $\mathbb{P}(\inf\{\boldsymbol{h}_0(u), \quad u \in K\} = 0) < 1$.

For all $t \in \mathbb{Z}$, define $\boldsymbol{w}_t := \sum_{k=0}^{+\infty} \boldsymbol{\gamma}_t^{(k)} \boldsymbol{\delta}_{t-k} \in [0, \infty]^K$. Note that the sequence of continuous, positive, and non-decreasing functions $(\sum_{k=0}^{n} \boldsymbol{\gamma}_t^{(k)} \boldsymbol{\delta}_{t-k})_n$ converges pointwise to $\boldsymbol{w}_t$ a.s., even though the limit may not be finite at some points. It is also important to note that the convergence may not be uniform. Therefore, $\boldsymbol{w}_t$ is not necessarily continuous.

Now we state the main result of this section.

Theorem 3.1. *Let* $\boldsymbol{\gamma} = \limsup_n \frac{1}{n} \log(\|\boldsymbol{\gamma}_0^{(n)}\|)$.

1. *Suppose that (3.4) hold. If* ***A1*** *or* ***A2*** *hold and Equation (3.2) has a positive stationary solution in F then*

$$\boldsymbol{\gamma} < 0 \quad a.s.$$

2. *Conversely, if $\mathbb{E}(\log^+ \|\boldsymbol{\delta}_0\|) < \infty$ and $\boldsymbol{\gamma} < 0$ then $(\sum_{k=0}^{n} \boldsymbol{\gamma}_t^{(k)} \boldsymbol{\delta}_{t-k})_n$ converges in F to $\boldsymbol{w}_t$ and $(\boldsymbol{w}_t)$ is the unique (continuous, positives and non-anticipative) stationary solution of (3.2).*

Remark 3.1.

1. *In views of Corollary 2.3, under* ***A1****, $\boldsymbol{\gamma}$ is almost surely constant with value in $[-\infty, \infty[$. Under* ***A2****, the subadditive ergodic theorem implies that*

$$\begin{aligned} \boldsymbol{\gamma} &= \lim_{n\to+\infty} \frac{1}{n} \log(\|\boldsymbol{\gamma}_0^{(n)}\|) \\ &= \lim_{n\to+\infty} \frac{1}{n} \mathbb{E} \log(\|\boldsymbol{\gamma}_0^{(n)}\|) = \lim_{n\to+\infty} \frac{1}{n} \mathbb{E} \log(\|\boldsymbol{\gamma}_n \circ \cdots \circ \boldsymbol{\gamma}_1\|) \\ &= \inf_{n\geq 1} \frac{1}{n} \mathbb{E} \log(\|\boldsymbol{\gamma}_n \circ \cdots \circ \boldsymbol{\gamma}_1\|). \end{aligned} \tag{3.5}$$

2. *If $(\boldsymbol{\eta}_t)$ is strictly stationary and ergodic and $\mathbb{E}(\log^+ \|\boldsymbol{\gamma}_0\|)$ is not finite, the following example, in the scalar case, shows that it is possible to have a stationary solution and at the same time $\boldsymbol{\gamma} \geq 0$. Let us take $\boldsymbol{\eta}_t = \boldsymbol{u}_t$, $\boldsymbol{\delta}_t = 1/\boldsymbol{u}_t$, $\boldsymbol{\gamma}_t = e^{-1}\boldsymbol{u}_{t-1}/\boldsymbol{u}_t$, where $(\boldsymbol{u}_t)$ is defined in Remark 2.4. We have $\sum_{k=0}^{+\infty} \boldsymbol{\gamma}_t^{(k)} \boldsymbol{\delta}_{-k} = (1/\boldsymbol{u}_t) \sum_{k=0}^{+\infty} e^{-k} < \infty$ a.s. It is easy to see that this process is a strictly stationary (and ergodic) solution. However, using the arguments used in Appendix A.1, we can see that $\boldsymbol{\gamma} = \limsup_n \frac{1}{n} \log \boldsymbol{\gamma}_0^{(n)} \geq 0$ and $\mathbb{E} \log^+ \boldsymbol{\gamma}_0 = \infty$.*

For all $\delta \in \mathcal{C}[0,1]$, we can remark that if

$$\text{for all } u \in [0,1],\ \delta(u) > 0,\ i.e. \inf_{u\in[0,1]} \delta(u) > 0, \tag{3.6}$$

then (3.4) holds. Therefore, we have the following immediate corollary.

Corollary 3.1. *If Eq. (3.6) hold and **A1** or **A2** is verified, Model (3.1) admits a (unique and non-anticipative) positive and strictly stationary (and ergodic) solution in $\mathcal{C}([0,1])$ if and only if*

$$\limsup_n \frac{1}{n} \log(\|\boldsymbol{\gamma}_0 \circ \cdots \circ \boldsymbol{\gamma}_{-n+1}\|) < 0 \quad a.s. \tag{3.7}$$

*Moreover, under **A2**, Eq. (3.7) is equivalent to*

$$\mathbb{E}\left[\log \|\boldsymbol{\gamma}_n \circ \cdots \circ \boldsymbol{\gamma}_1\|\right] < 0 \text{ for some } n. \tag{3.8}$$

Proof. Under **A2**, the equivalence between equations (3.7) and (3.8) comes from Eq. (3.5). Thus, Corollary 3.1 is a direct consequence of Theorem 3.1. □

Since (3.8) is necessary and sufficient under **A2**, which does not require the iid assumption, it is clear that this condition is weaker than the sufficient condition, Eq. (3.3), given by Aue et al. (2017).

In order to prove Theorem 3.1, we will use the following general result. It is used, under **A1**, to address the other challenge mentioned in (Cerovecki et al., 2019, Remark 1), which consists in showing (3.21) from (3.20). Point 2. of Corollary 2.3 is used to handle this step under **A2**.

Lemma 3.1. *Let $(\boldsymbol{x}_n)_{n\geq 0}$ and $(\boldsymbol{y}_n)_{n\geq 0}$ be real value processes. If (i) $(\boldsymbol{x}_n)_{n\geq 0}$ is identically distributed, (ii) $\boldsymbol{x}_{n+1}$ and $\sigma((\boldsymbol{x}_s, \boldsymbol{y}_{s+1}), s \leq n)$ are independent and (iii) $\mathbb{P}(\boldsymbol{x}_0 = 0) < 1$ then*

$$\boldsymbol{y}_n \to 0 \text{ a.s when } n \to \infty \text{ on } G := \{\boldsymbol{x}_n \boldsymbol{y}_n \to 0 \text{ a.s when } n \to \infty\}$$

By replacing the condition $\mathbb{P}(\boldsymbol{x}_0 = 0) < 1$ in the previous lemma by the slightly stronger assumption that $\boldsymbol{x}_0$ is not almost surely constant, we can establish the following more general result:

$$\boldsymbol{y}_n \to 0 \text{ a.s when } n \to \infty \text{ on } \{(\boldsymbol{x}_n \boldsymbol{y}_n) \text{ converges}\},$$

see Proposition A.1 in Appendix A.2.

Proof of Lemma 3.1. It suffices to prove that

$$\text{for all } \varepsilon > 0,\ \mathbb{P}\left(\liminf_n \{|\boldsymbol{y}_n| < \varepsilon\} \cap G\right) = \mathbb{P}(\{\sum_{n\geq 1} \mathbb{1}_{\{|\boldsymbol{y}_n|\geq \varepsilon\}} < \infty\} \cap G) = \mathbb{P}(G).$$

Define $G_\varepsilon = \{\sum_{n\geq 1} \mathbb{1}_{\{|\boldsymbol{x}_n \boldsymbol{y}_n|\geq \varepsilon\}} < \infty\} = \liminf_n \{|\boldsymbol{x}_n \boldsymbol{y}_n| < \varepsilon\}$. Note that (iii) implies that

$$\text{there exists } \delta > 0 \text{ such that } \mathbb{P}(|\boldsymbol{x}_0| \geq \delta) > 0. \tag{3.9}$$

Fix ε and for this δ, let's show first that for all $0 < \varepsilon' \leq \delta\varepsilon$,

$$\mathbb{P}(\{\sum_{n\geq 1} \mathbb{1}_{\{|\boldsymbol{y}_n|\geq \varepsilon\}} < \infty\} \cap G_{\varepsilon'}) = \mathbb{P}(G_{\varepsilon'}). \tag{3.10}$$

Since $\{\sum_{n\geq 1} \mathbb{1}_{\{|\boldsymbol{y}_n|\geq \varepsilon'/\delta\}} < \infty\} \subset \{\sum_{n\geq 1} \mathbb{1}_{\{|\boldsymbol{y}_n|\geq \varepsilon\}} < \infty\}$ for all $\varepsilon' \leq \delta\varepsilon$, to prove (3.10), it suffices to show that for all $\varepsilon' \leq \delta\varepsilon$,

$$\mathbb{P}(\{\sum_{n\geq 1} \mathbb{1}_{\{|\boldsymbol{y}_n|\geq \varepsilon'/\delta\}} < \infty\} \cap G_{\varepsilon'}) = \mathbb{P}(G_{\varepsilon'}). \tag{3.11}$$

To prove this, we will use a conditional version of the Borel-Cantelli lemma. Let $\varepsilon' > 0$. Since

$$|\boldsymbol{x}_n \boldsymbol{y}_n| \geq \delta \mathbb{1}_{\{|\boldsymbol{x}_n|\geq \delta\}} |\boldsymbol{y}_n| \quad \text{and} \quad \mathbb{1}_{\{\delta \mathbb{1}_{\{|\boldsymbol{x}_n|\geq\delta\}} |\boldsymbol{y}_n| \geq \varepsilon'\}} = \mathbb{1}_{\{|\boldsymbol{x}_n|\geq \delta\}} \mathbb{1}_{\{|\boldsymbol{y}_n|\geq \varepsilon'/\delta\}},$$

we have

$$\mathbb{1}_{\{|\boldsymbol{x}_n \boldsymbol{y}_n|\geq \varepsilon'\}} \geq \mathbb{1}_{\{|\boldsymbol{x}_n|\geq \delta\}} \mathbb{1}_{\{|\boldsymbol{y}_n|\geq \varepsilon'/\delta\}}.$$

Define, for all $n \geq 0$, $\boldsymbol{z}_n = \mathbb{1}_{\{|\boldsymbol{x}_n|\geq \delta\}} \mathbb{1}_{\{|\boldsymbol{y}_n|\geq \varepsilon'/\delta\}}$, it follows that

$$\sum_{n\geq 1} \boldsymbol{z}_n < \infty \quad a.s \quad \text{on} \quad G_{\varepsilon'}. \tag{3.12}$$

Let $\mathcal{F}_n = \sigma((\boldsymbol{x}_s, \boldsymbol{y}_{s+1}), s \leq n)$, for all $n \geq 0$. Since $\boldsymbol{y}_n$ is $\mathcal{F}_{n-1}$-measurable and in view of (ii), $\boldsymbol{x}_n$ and $\mathcal{F}_{n-1}$ are independent, then, also by (i), for all $n \geq 1$,

$$\boldsymbol{m}_n := \mathbb{E}(\boldsymbol{z}_n | \mathcal{F}_{n-1}) = \mathbb{P}(|\boldsymbol{x}_0| \geq \delta) \mathbb{1}_{\{|\boldsymbol{y}_n|\geq \varepsilon'/\delta\}},$$

hence, by (3.9), $\{\sum_{n\geq 1} \boldsymbol{m}_n < \infty\} = \{\sum_{n\geq 1} \mathbb{1}_{\{|\boldsymbol{y}_n|\geq \varepsilon'/\delta\}} < \infty\}$ $a.s.$ This result, the fact that $(\mathcal{F}_n)_{n\geq 0}$ is a sequence of nondecreasing σ-algebras, $\boldsymbol{z}_n$ is $\mathcal{F}_n$-measurable, and Theorem 1 of Chen (1978) (see also Freedman (1973, Eq. 5 and 6)) imply that

$$\sum_{n\geq 1} \mathbb{1}_{\{|\boldsymbol{y}_n|\geq \varepsilon'/\delta\}} < \infty \quad a.s \quad \text{on} \quad \{\sum_{n\geq 1} \boldsymbol{z}_n < \infty\}. \tag{3.13}$$

Equation (3.11) is a direct consequence of (3.12) and (3.13). This show (3.10). Since $(G_{1/m})_{m\in\mathbb{N}}$ is a nonincreasing sequence of sets and

$$G = \cap_{m\geq 1} \cup_{N\geq 1} \cap_{n\geq N} \{|\boldsymbol{x}_n \boldsymbol{y}_n| < 1/m\} = \cap_{m\geq 1} G_{1/m},$$

it follows by the monotone convergence theorem and (3.10) that

$$\begin{aligned}\mathbb{P}(G) &= \lim_{m\to\infty} \mathbb{P}(G_{1/m}) \\ &= \lim_{m\to\infty} \mathbb{P}(\{\sum_{n\geq 1} \mathbb{1}_{\{|\boldsymbol{y}_n|\geq\varepsilon\}} < \infty\} \cap G_{1/m}) \\ &= \mathbb{P}(\{\sum_{n\geq 1} \mathbb{1}_{\{|\boldsymbol{y}_n|\geq\varepsilon\}} < \infty\} \cap G),\end{aligned}$$

which completes the proof. □

Proof of Theorem 3.1.

We prove *1*. Let us first consider that **A1** holds. For all $t \in \mathbb{Z}$ and $n \in \mathbb{N}$, let $\boldsymbol{w}_{t,n} = \sum_{k=0}^{n} \boldsymbol{\gamma}_t^{(k)} \boldsymbol{\delta}_{t-k}$. Suppose that (3.2) has a positive stationary solution $(\boldsymbol{h}_t)_{t\in\mathbb{Z}}$ in F. By iterating (3.2), it follows that for all $n \geq 0$ and $t \in \mathbb{Z}$, $\boldsymbol{w}_{t,n} \leq \boldsymbol{h}_t$. This implies that *a.s.* $(\boldsymbol{w}_{t,n})_{n\geq 0}$ is a sequence of nondecreasing functions bounded by $\boldsymbol{h}_t$. Therefore, *a.s.* $(\boldsymbol{w}_{t,n})_n$ converges pointwise (with finite limit) to $\boldsymbol{w}_t$. Noting that for all n and t, $\boldsymbol{w}_0 = \boldsymbol{w}_{t,n-1} + \boldsymbol{\gamma}_0^{(n)} \boldsymbol{w}_{-n}$, one has a.s.

$$\boldsymbol{\gamma}_0^{(n)} \boldsymbol{w}_{-n} \xrightarrow{\text{pw}} 0 \quad when \quad n \to \infty$$

Since

$$(\inf \boldsymbol{w}_{-n}) \boldsymbol{\gamma}_0^{(n)} \boldsymbol{e} \leq \boldsymbol{\gamma}_0^{(n)} \boldsymbol{w}_{-n}, \tag{3.14}$$

we have

$$(\inf \boldsymbol{w}_{-n}) \boldsymbol{\gamma}_0^{(n)} \boldsymbol{e} \xrightarrow{\text{pw}} 0 \quad when \quad n \to \infty \quad a.s. \tag{3.15}$$

Consider that (3.4) holds. The proof relies on the following intermediate results.

a) there exists $n_0 \geq 1$ such that $\mathbb{P}(\inf \boldsymbol{w}_{0,n_0} = 0) < 1$.

b) $\limsup_k (n_0 k)^{-1} \log(\|\boldsymbol{\gamma}_0^{(n_0 k)}\|) < 0 \quad a.s,$

c) $\limsup_k (n_0 k + p)^{-1} \log(\|\boldsymbol{\gamma}_0^{(n_0 k + p)}\|) < 0 \quad a.s$ for all $p = 0, 1, \cdots n_0 - 1$

Let us proceed by contradiction to prove *a)*. Suppose that

$$\text{for all } n \geq 1, \quad \inf \boldsymbol{w}_{0,n} = 0 \; a.s. \tag{3.16}$$

For all $n \geq 1$, let $\boldsymbol{J}_n^{\omega} = \{u \in K: \ \boldsymbol{w}_{0,n}(u) = 0\}$. The sequence $(\boldsymbol{w}_{0,n})_{n\geq 0}$ is continuous, positive, and non-decreasing, therefore, a.s., $(\boldsymbol{J}_n)_{n\geq 0}$ is a sequence of non-empty, non-increasing, random, compact sets. By Cantor's intersection theorem and Eq. (3.16), $\boldsymbol{J} := \bigcap_{n\geq 0} \boldsymbol{J}_n \neq \varnothing$ *a.s.* The sequence $(\boldsymbol{w}_{0,n})_{n\geq 1}$ converges pointwise to $\boldsymbol{w}_0$, and almost surely $\boldsymbol{w}_{0,n} = 0$ on $\boldsymbol{J}$ for all $n \geq 1$. This implies that $\boldsymbol{w}_0 = 0$ on $\boldsymbol{J}$ *a.s.*, which contradicts equation (3.4). This completes the proof of part *a)*.

We now prove *b)*. By iteration, note that for all $n \geq 0$,

$$\boldsymbol{h}_0 = \boldsymbol{w}_{t,n-1} + \boldsymbol{\gamma}_0^{(n)} \boldsymbol{h}_{-n}. \tag{3.17}$$

It follows that $(\boldsymbol{\gamma}_0^{(n)} \boldsymbol{h}_{-n})$ is a sequence of nonincreasing functions, pointwise bounded by $\boldsymbol{h}_t$ and then, almost surely, it converges pointwise. Since, by continuity, $\boldsymbol{h}_0$ is almost surely bounded then $\lim_{K\to\infty} \mathbb{P}(\sup \boldsymbol{h}_0 < K) = 1$. It follows by (3.4) that there exists $K > 0$ and $\epsilon >$ such that $\mathbb{P}(\sup \boldsymbol{h}_0 < K, \ \inf \boldsymbol{w}_0 > \epsilon) > 0$. Noting that

$$\boldsymbol{\gamma}_0^{(n)} \boldsymbol{h}_{-n} \mathbb{1}_{\{\sup \boldsymbol{h}_{-n} \leq K, \ \inf \boldsymbol{w}_{-n} > \epsilon\}} < (K/\epsilon) \inf \boldsymbol{w}_{-n} \boldsymbol{\gamma}_0^{(n)} \boldsymbol{e},$$

it follows by (3.15) that

$$\boldsymbol{\gamma}_0^{(n)} \boldsymbol{h}_{-n} \mathbb{1}_{\{\sup \boldsymbol{h}_{-n} < K, \ \inf \boldsymbol{w}_{-n} > \epsilon\}} \xrightarrow{\text{pw}} 0 \quad as \quad n \to \infty. \tag{3.18}$$

Since the ergodic theorem implies that almost surely, $\mathbb{1}_{\{\sup \boldsymbol{h}_{-n} < K, \ \inf \boldsymbol{w}_{-n} > \epsilon\}} = 1$ for an infinite number of n, it follows that the sub-sequence $(\boldsymbol{\gamma}_0^{(n)} \boldsymbol{h}_{-n} : \ \mathbb{1}_{\{\sup \boldsymbol{h}_{-n} < K, \ \inf \boldsymbol{w}_{-n} > \epsilon\}} = 1)$ converges, almost surely, pointwise to the limit of $(\boldsymbol{\gamma}_0^{(n)} \boldsymbol{h}_{-n})$. It follows by (3.18) that this limit is 0, i.e.

$$\boldsymbol{\gamma}_0^{(n)} \boldsymbol{h}_{-n} \xrightarrow{\text{pw}} 0 \quad when \quad n \to \infty \quad a.s. \tag{3.19}$$

From this and Eq. (3.17) one has $F \ni \boldsymbol{h}_0 = \boldsymbol{w}_0$ *a.s.* It follows by Dini's Theorem that $(\boldsymbol{w}_{0,n})_n$ converges uniformly to $\boldsymbol{w}_0$ *a.s.* Hence,

$$\|\boldsymbol{w}_0 - \boldsymbol{w}_{0,n}\| = \|\boldsymbol{\gamma}_0^{(n)} \boldsymbol{w}_{-n}\| \longrightarrow 0 \quad when \quad n \to \infty \quad a.s. \tag{3.20}$$

For all $k \geq 0$, let $\boldsymbol{x}_k = \inf \boldsymbol{w}_{-n_0 k, n_0}$ and $\boldsymbol{y}_k = \|\boldsymbol{\gamma}_0^{(n_0 k)}\|$. Note that $(\boldsymbol{x}_k, \ \boldsymbol{y}_k(u))$ verifies the conditions of Lemma 3.1 because of *a)* and the fact that $(\boldsymbol{\eta}_t)$ is iid. Since $inf \boldsymbol{w}_{-n_0 k, n_0} \|\boldsymbol{\gamma}_0^{(n_0 k)} \boldsymbol{e}\| = inf \boldsymbol{w}_{-n_0 k, n_0} \|\boldsymbol{\gamma}_0^{(n_0 k)}\| \leq \|\boldsymbol{\gamma}_0^{(n_0 k)} \boldsymbol{w}_{-n_0 k}\|$, it follows by (3.20), and Lemma 3.1 that

$$\|\boldsymbol{\gamma}_0^{(n_0 k)}\| \longrightarrow 0 \quad when \quad k \to \infty \quad a.s. \tag{3.21}$$

Since $(\|\boldsymbol{\gamma}_0^{(n_0k)}\|)_k$ is a sub-multiplicative sequence, by (3.21) and Point 1. of Corollary 2.3, we have

$$\limsup_k (n_0k)^{-1}\log(\|\boldsymbol{\gamma}_0^{(n_0k)}\|) = \limsup_k (\frac{k}{n_0k})[k^{-1}\log(\|\boldsymbol{\gamma}_0^{(n_0k)}\|)] < 0 \quad a.s.$$

This concludes the proof of *b)*. To prove *c)*, remark by stationarity and *b)* that for all $p = 0, 1, \cdots n_0 - 1$, $\limsup_k (n_0k)^{-1}\log(\|\boldsymbol{\gamma}_{-p}^{(n_0k)}\|) < 0$. Therefore

$$\begin{aligned}
\psi :&= \limsup_k (n_0k+p)^{-1}\log(\|\boldsymbol{\gamma}_0^{(n_0k+p)}\|) \\
&\le \limsup_k (n_0k+p)^{-1}\log(\|\boldsymbol{\gamma}_0^{(p)}\|) + \limsup_k (n_0k+p)^{-1}\log(\|\boldsymbol{\gamma}_{-p}^{(n_0k)}\|) \\
&= \limsup_k (\frac{n_0k}{n_0k+p})[(n_0k)^{-1}\log(\|\boldsymbol{\gamma}_{-p}^{(n_0k)}\|)] < 0 \quad a.s.
\end{aligned} \tag{3.22}$$

Noting that $\mathrm{N} = \cup_{0\le p\le n_0-1}\{n_0k+p : \quad k \in \mathbb{N}\}$, it follows that

$$\limsup_n \frac{1}{n}\log\|\boldsymbol{\gamma}_0^{(n)}\| \le \max_{0\le p\le n_0-1}\left(\limsup_k (n_0k+p)^{-1}\log(\|\boldsymbol{\gamma}_0^{(n_0k+p)}\|)\right) < 0.$$

which gives the first point under **A1**.

We now prove the claim under **A2**. First observe that the iid assumption is only used, in Lemma 3.1, to derive (3.21) from (3.20). Therefore, all the results showed before (3.20) hold under **A2**. Hence, (3.20) implies that

$$\mathbb{1}_{\{\inf \boldsymbol{w}_{-n}>\epsilon\}}\|\boldsymbol{\gamma}_0^{(n)}\| \longrightarrow 0 \quad when \quad k \to \infty \quad a.s.$$

Therefore, the sequence $(\|\boldsymbol{\gamma}_0^{(n)}\| : \quad \inf \boldsymbol{w}_{-n} > \epsilon\})$ converges almost surely to 0. The result follows from arguments used in Remark 2.4 (to define the dynamic system), Appendix A.1 (to verifies the condition of the corollary) and Point 2. of Corollary 2.3.

We now prove 2. We have [2]

$$\begin{aligned}
\limsup_n \frac{1}{n}\log(\|\boldsymbol{\gamma}_0^{(n)}\boldsymbol{\delta}_{-n}\|) &\le \limsup_n \frac{1}{n}[\|\boldsymbol{\gamma}_0^{(n)}\| + \log^+(\|\boldsymbol{\delta}_{-n}\|)] \\
&\le \boldsymbol{\gamma} + \limsup_n \frac{1}{n}\log^+(\|\boldsymbol{\delta}_{-n}\|) \\
&= \boldsymbol{\gamma} < 0.
\end{aligned}$$

[2]For all non negative stationary process $(X_n)_{n\ge1}$ such that $\mathbb{E}X_1 < \infty$, one has $\limsup_{n\to\infty} n^{-1}X_n = 0$. Indeed, for all $\epsilon > 0$, noting that the function $f(t) = P\left(t^{-1}X_1 > \epsilon\right)$ is decreasing, we have $\sum_{n=1}^{\infty} P\left(n^{-1}X_n > \epsilon\right) \le \int_0^{\infty} P\left(\epsilon^{-1}X_1 > t\right) dt = \epsilon^{-1}EX_1 < \infty$. The convergence follows from the Borel-Cantelli lemma.

Therefore, by Cauchy's rule, $(\boldsymbol{w}_{0,n})_{n\geq 1}$ converges absolutely almost surely. Thus, $\boldsymbol{w}_0 \in F$ *a.s.* It is easy to verify that the continuous, positive, stationary process $(\boldsymbol{w}_t)_n$ is non-anticipative and satisfies (3.2). The proof of the uniqueness is standard, see for instance (Kandji, 2023, Appendix A). This completes the proof. □

4 Perspective

The main result of this paper extends the result of Kesten (1975) on the growth rate of sums of stationary sequences to superadditive processes. Our result is established under weaker conditions than those of Kesten (1975). Using a result from Tanny (1974), Kesten show in the same paper that if $\{\boldsymbol{S}_n\}_{n=1}^{\infty}$ is an additive sequence, then on $\{\boldsymbol{S}_n \to \infty\}$, $0 < \liminf n^{-1}\boldsymbol{S}_n < \limsup n^{-1}\boldsymbol{S}_n = \infty$ *a.s.* or $\lim n^{-1}\boldsymbol{S}_n$ exists and $\lim n^{-1}\boldsymbol{S}_n > 0$. An interesting question that could be considered for further work is to see if this result also generalizes to superadditive processes. That is, in which cases can the limit superior in Theorem 2.1 be replaced by a limit.

We also provide a necessary and sufficient condition, without moment condition, for the stability of a class of functional GARCH$(1,1)$ models in the space of the continuous functions. Our results can be easily extended, in the same space, to higher-order GARCH(p,q) models using the same argument. However, since norms are not equivalent in the infinite-dimensional setting, it would be interesting to investigate whether these conditions remain true when considering a different space, such as L^p spaces.

Acknowledgements

I am most thankful to Christian Francq and Jean-Michel Zakoïan for their guidance and feedback.

Appendix: Proofs

A Appendix: Complementary Proofs

A.1 Complement to Remark 2.4

Let $\boldsymbol{\delta}_n = \frac{1}{\boldsymbol{u}_{n-1}}$. Since $\boldsymbol{u}_0$ is not almost surely constant then there exists $s > 0$ such that $\mathbb{P}_{\boldsymbol{u}}(\boldsymbol{\delta}_0 > s) > 0$. Let $A = \{\boldsymbol{\delta}_0 > s\}$. It is easy to see that

$\boldsymbol{\gamma}_n \boldsymbol{\delta}_n = \frac{e^{-n}}{\boldsymbol{u}_0} \to 0$ $a.s.$ as $n \to 0$. Thus $\boldsymbol{\gamma}_n \mathbb{1}_A \circ T^n = \boldsymbol{\gamma}_n \mathbb{1}_{\boldsymbol{\delta}_n > s} \leq \boldsymbol{\gamma}_n \boldsymbol{\delta}_n \to 0$ $a.s.$ It follows that $(\boldsymbol{\gamma}_n : T^n \in A)$ converges almost surely to 0.

On the other hand, since $\boldsymbol{\gamma}_n = \frac{\boldsymbol{u}_n}{\boldsymbol{u}_0} e^{-n}$, then $\limsup_n n^{-1} \log \boldsymbol{\gamma}_n < 0$ $a.s.$ implies that $(\boldsymbol{u}_n e^{-n})$ converges to 0 $a.s.$ However, one has

$$\sum_{n=0}^{\infty} \mathbb{P}(\boldsymbol{u}_n e^{-n} > 1) = \sum_{n=0}^{\infty} \mathbb{P}(\ln^+ \boldsymbol{u}_0 > n) \geq \int_0^{\infty} \mathbb{P}(\ln^+ \boldsymbol{u}_0 > t) dt$$
$$= \mathbb{E}(\ln^+ \boldsymbol{u}_0) = \infty.$$

It follows by the second Borel-Cantelli lemma that $\mathbb{P}(\limsup\{\boldsymbol{u}_n e^{-n} > 1\}) = 1$ and then $(\boldsymbol{z}_n e^{-n})$ does not converge to 0.

Now we compute $\mathbb{E} \ln^+ \boldsymbol{\gamma}_1$. Let a real $K > 0$ such that $\mathbb{P}(\ln^+ \boldsymbol{u}_0 \leq K) > 0$. Since $\ln^+ \boldsymbol{\gamma}_1 \geq \ln^+ \boldsymbol{u}_1 - \ln^+ \boldsymbol{u}_0$, it follows that $\ln^+ \boldsymbol{\gamma}_1 \geq \ln^+ \boldsymbol{u}_1 \mathbb{1}_{\ln^+ \boldsymbol{u}_0 \leq K} - \ln^+ \boldsymbol{u}_0 \mathbb{1}_{\ln^+ \boldsymbol{u}_0 \leq K}$. Hence $\mathbb{E} \ln^+ \boldsymbol{\gamma}_1 \geq \mathbb{E} \ln^+ \boldsymbol{u}_1 \mathbb{P}(\ln^+ \boldsymbol{u}_0 \leq K) - \mathbb{E} \ln^+ \boldsymbol{u}_0 \mathbb{1}_{\ln^+ \boldsymbol{u}_0 \leq K} = \infty$, because the second term is finite.

A.2 On the convergence of the product of two independent random elements

The following result, which generalises Lemma 3.1 is of independent interest.

Proposition A.1. *Let $(\boldsymbol{x}_n)_{n \geq 0}$ and $(\boldsymbol{y}_n)_{n \geq 0}$ be real value processes. If (i) $(\boldsymbol{x}_n)_{n \geq 0}$ is identically distributed, (ii) $\boldsymbol{x}_{n+1}$ and $\mathcal{F}_n := \sigma((\boldsymbol{x}_s, \boldsymbol{y}_{s+1}), s \leq n)$, are independent and (iii) $\boldsymbol{x}_0$ is not almost surely constant, then*

1. $\boldsymbol{x}_n \boldsymbol{y}_n \to 0$ *a.s when* $n \to \infty$ *on* $\{\boldsymbol{x}_n \boldsymbol{y}_n$ *converges*$\}$
2. $\boldsymbol{y}_n \to 0$ *a.s when* $n \to \infty$ *on* $\{\boldsymbol{x}_n \boldsymbol{y}_n$ *converges*$\}$
3. *If $(\boldsymbol{x}_n \boldsymbol{y}_n)$ converges in probability then the limit is 0.*

Proof. For the first point, it suffices to prove that

$$\mathbb{P}\left(\{|\limsup \boldsymbol{x}_n \boldsymbol{y}_n| > 0,\ \boldsymbol{x}_n \boldsymbol{y}_n \text{ converges}\}\right) = 0. \tag{A.1}$$

For all $\epsilon > 0$ and $t \in \mathbb{R}$, let $B(t, \epsilon) := (t - \epsilon,\ t + \epsilon)$. Let $\boldsymbol{z} = \limsup \boldsymbol{x}_n \boldsymbol{y}_n$ and $G = \{\boldsymbol{x}_n \boldsymbol{y}_n \text{ converges}\}$. Note that on G, $\boldsymbol{z}$ is finite and $(\boldsymbol{x}_n \boldsymbol{y}_n)$ converges to $\boldsymbol{z}$. We argue by contradiction: suppose that $\mathbb{P}\left(\{|\boldsymbol{z}| > 0\} \cap G\right) > 0$. Since this condition implies that $\mathbb{P}(G) > 0$, let $\mathbb{P}^G$ be the conditional probability given G. Noting that $\mathbb{P}^G(|\boldsymbol{z}| > 0) > 0$ we have that the support of $\boldsymbol{z}$ under $\mathbb{P}^G$ contains a non-zero element z_0. Thus, for all $\epsilon > 0$ we have $\mathbb{P}^G\left(\boldsymbol{z} \in B(z_0, \epsilon)\right) > 0$ i.e.

$$\mathbb{P}\left(\{\boldsymbol{z} \in B(z_0, \epsilon)\} \cap G\right) > 0. \tag{A.2}$$

The condition (iii) implies that the support of $\boldsymbol{x}_0$ under $\mathbb{P}$ contains at least two different elements x_1 and x_2. Since $x_1 \neq x_2$, we can assume without loss of generality that $x_1 \neq 0$. Let $y_0 = z_0/x_1$, and $\epsilon_0,\ \epsilon_1,\ \epsilon_2,\ \epsilon_3 > 0$ such that

$$\begin{aligned} &1)\ \{xy:\ (x,y) \in B(x_2,\epsilon_2) \times B(y_0,\epsilon_3)\} \cap B(z_0,\epsilon_0) = \varnothing, \\ &2)\ \text{if } (z,x) \in B(z_0,\epsilon_0) \times B(x_1,\epsilon_1) \text{ then } z/x \in B(y_0,\epsilon_3). \end{aligned} \tag{A.3}$$

The point *1*) in (A.3) comes from the fact that $(x,y) \mapsto xy$ is continuous at (x_2, y_0) and $x_2y_0 = z_0(x_2/x_1) \neq z_0$. The second point is because $(z,x) \mapsto z/x$, defined on $B(z_0,\epsilon) \times B(x_1,\epsilon)$ for ϵ small enough, is continuous at (z_0, x_1). Indeed, for *1*), take $\epsilon_0 > 0$ and $\delta > 0$ such that

$$B(x_2y_0,\delta)\} \cap B(z_0,\epsilon_0) = \varnothing.$$

Choose $\epsilon_2,\ \epsilon_3 > 0$ such that

$$(x,y) \in B(x_2,\epsilon_2) \times B(y_0,\epsilon_3),$$

we have $xy \in B(x_2y_0,\delta)$ and thus

$$\{xy:\ (x,y) \in B(x_2,\epsilon_2) \times B(y_0,\epsilon_3)\} \cap B(z_0,\epsilon_0) = \varnothing.$$

Noting that this statement remains true for smaller ϵ_0, ϵ_2 and ϵ_3, let's fix ϵ_2 and ϵ_3, and choose ϵ_0 smaller than its previous value and take also ϵ_1 such that

$$(z,x) \in B(z_0,\epsilon_0) \times B(x_1,\epsilon_1).$$

We thus have $z/x \in B(y_0,\epsilon_3)$.

For $i \in \{1,\ 2\}$, we have by the strong law of large numbers that

$$n^{-1}\sum_{k=0}^{n} \mathbb{1}_{\{\boldsymbol{x}_k \in B(x_1,\epsilon_1)\}} \to \mathbb{P}(\boldsymbol{x}_0 \in B(x_1,\epsilon_1))\ a.s \text{ when } n \to \infty.$$

Since $\mathbb{P}(\boldsymbol{x}_0 \in B(x_1,\epsilon_1)) > 0$, then $\mathbb{P}(S) = 1$ where $S = \{\sum_{k=0}^{n} \mathbb{1}_{\{\boldsymbol{x}_k \in B(x_1,\epsilon_1)\}} \to \infty\}$. We have by this result and (A.2) that $\mathbb{P}(E) > 0$ where

$$E = S \cap \{\boldsymbol{z} \in B(z_0,\epsilon_0)\} \cap G.$$

Since on E $(\boldsymbol{x}_n\boldsymbol{y}_n)$ converges to $\boldsymbol{z}$, which is in the open set $B(z_0,\epsilon_0)$, then there exists an integer N (random integer) such that if $n \geq N$, then $\boldsymbol{x}_n\boldsymbol{y}_n \in B(z_0,\epsilon_0)$. It follows by *1*) (A.3) that

$$\sum_{k\geq 1} \mathbb{1}_{\{\boldsymbol{x}_k \in B(x_2,\epsilon_2),\ \boldsymbol{y}_k \in B(y_0,\epsilon_3)\}} < \infty \quad a.s \quad on \quad E. \tag{A.4}$$

Since on S, and thus on E, $\boldsymbol{x}_n \in B(x_1, \epsilon_1)$ for infinitely many n, it follows also that

$$\{n: \ \boldsymbol{x}_n\boldsymbol{y}_n \in B(z_0, \epsilon_0), \ \boldsymbol{x}_n \in B(x_1, \epsilon_1)\}$$

is infinite. Therefore, we have by *2)* (A.3) that

$$\sum_{k\geq 1} \mathbb{1}_{\{\boldsymbol{y}_k \in B(y_0,\epsilon_3)\}} = \infty \quad on \quad E. \tag{A.5}$$

To arrive at a contradiction, let's also show that

$$\sum_{k\geq 1} \mathbb{1}_{\{\boldsymbol{y}_k \in B(y_0,\epsilon_3)\}} < \infty \quad a.s \quad on \quad E.$$

In views of (A.4), it is equivalent to show that

$$\sum_{k\geq 1} \mathbb{1}_{\{\boldsymbol{y}_k \in B(y_0,\epsilon_3)\}} < \infty \quad a.s \quad on \quad \{\sum_{k\geq 1} \boldsymbol{z}_k < \infty\} \tag{A.6}$$

where

$$\boldsymbol{z}_k = \mathbb{1}_{\{\boldsymbol{x}_k \in B(x_2,\epsilon_2), \ \boldsymbol{y}_k \in B(y_0,\epsilon_3)\}} = \mathbb{1}_{\{\boldsymbol{x}_k \in B(x_2,\epsilon_2)} \mathbb{1}_{\boldsymbol{y}_k \in B(y_0,\epsilon_3)\}}.$$

To get this result, remark that $\boldsymbol{z}_n$ is $\mathcal{F}_n$-measurable and since $\boldsymbol{y}_n$ is $\mathcal{F}_{n-1}$-measurable and $\boldsymbol{x}_n$ and $\mathcal{F}_{n-1}$ are independent, then for all $n \geq 1$

$$\boldsymbol{m}_n := \mathbb{E}(\boldsymbol{z}_n|\mathcal{F}_{n-1}) = \mathbb{P}(\boldsymbol{x}_0 \in B(x_2, \epsilon_2))\mathbb{1}_{\{\boldsymbol{y}_n \in B(y_0,\epsilon_3)\}}.$$

It follows from the converse part of Theorem 1 of Chen (1978) that

$$\sum_{k\geq 1} \boldsymbol{m}_k < \infty \quad a.s \quad on \quad \{\sum_{k\geq 1} \boldsymbol{z}_k < \infty\}.$$

Noting that

$$\{\sum_{k\geq 1} \boldsymbol{m}_n < \infty\} = \{\sum_{k\geq 1} \mathbb{1}_{\{\boldsymbol{y}_k \in B(y_0,\epsilon_3)\}} < \infty\} \quad a.s,$$

by the fact that $\mathbb{P}(\boldsymbol{x}_0 \in B(x_2, \epsilon_2)) > 0$, (A.6) follows from the previous result. This contradicts (A.5) since $\mathbb{P}(E) > 0$, and thus we have (A.1).

The second point follows from the first point and Lemma 3.1.

For the last point, note that the convergence in probability implies convergence on a sub-sequence $(\boldsymbol{x}_{\phi(n)}\boldsymbol{y}_{\phi(n)})$ almost surely. Since $(\boldsymbol{x}_{\phi(n)}, \boldsymbol{y}_{\phi(n)})$ checks the conditions of Proposition A.1, the result follows from the first point. This concludes the proof. □

A.3 Ergodic Lemmas

This results may not be new. Since we have not been able to find it in the literature, we provide a proof.

Lemma A.1. *Let $I \in \mathcal{I}_\mu$ with $\mu(I) > 0$. Let ν be the probability measure in $(\Omega,\ \mathcal{B})$ given by the conditional probability given I (i.e for all $A \in \mathcal{B}, \quad \nu(A) = \mu(I)^{-1}\mu(A \cap I)$). Then $(\Omega, \mathcal{B}, \nu, T)$ is a measure-preserving dynamical system, i.e.*

$$\textit{for all } A \in \mathcal{B},\ \nu(T^{-1}(A)) = \nu(A),$$

Proof. For all $I \in \mathcal{I}_\mu$ and $A \in \mathcal{B}$, because

$$\begin{aligned} &\mu(I\Delta T^{-1}(I)) = 0, \\ &I \cup I\Delta T^{-1}(I) = T^{-1}(I) \cup I\Delta T^{-1}(I) \quad \text{and} \\ &T^{-1}(A) \cap T^{-1}(I) = T^{-1}(A \cap I), \end{aligned}$$

one has

$$\begin{aligned} \mu(T^{-1}(A) \cap I) &= \mu(T^{-1}(A) \cap (I \cup I\Delta T^{-1}(I)) \\ &= \mu\left(T^{-1}(A) \cap (T^{-1}(I) \cup I\Delta T^{-1}(I))\right) \\ &= \mu(T^{-1}(A) \cap T^{-1}(I)) = \mu(T^{-1}(A \cap I)) \\ &= \mu(A \cap I). \end{aligned} \tag{A.7}$$

The result follows by dividing by $\mu(I)$. □

Lemma A.2. *For all invariant set I, Let $C(I) := \bigcap_{n=0}^{\infty} T^{-n}(I), \quad$ where $T^0 = Id_\Omega$. One has*

1. *$C(I) \in \mathcal{I}_\mu$*
2. *$\mu(C(I)) = \mu(I)$*
3. *for all $\omega \in C(I)$ and $n \geq 0,\ T^n(\omega) \in C(I)$.*

Proof. Let show the first point. If $I \in \mathcal{I}$ then

$$\begin{aligned} \mu\left(T^{-2}(I)\Delta T^{-1}(I)\right) &= \mu(T^{-2}(I) \cup T^{-1}(I)) - \mu(T^{-2}(I) \cap T^{-1}(I)) \\ &= \mu(T^{-1}(I) \cup I) - \mu(T^{-1}(I) \cap I) \\ &= \mu\left(T^{-1}(I)\Delta I\right) = 0, \end{aligned}$$

and then $T^{-1}(I) \in \mathcal{I}_\mu$. Hence, by recurrence, we show that $C(I)$ is the intersection of elements of the σ-algebra $\mathcal{I}_\mu$, and thus $C(I) \in \mathcal{I}_\mu$.

For the second statement, using (A.7) for $A = I$, one has $\mu(T^{-1}(I) \cap I) = \mu(I)$. Therefore, by doing the same operation on $\bigcap_{k=0}^{n} T^{-k}(I)$ for $n = 1, 2, \cdots$, one has by recurrence and the monotone convergence theorem that $\mu(C(I)) = \mu(I)$.

For the last one, note that $\omega \in C(I)$ is equivalent to, for all $n \geq 0$, $T^n(\omega) \in I$. It follows that, $\omega \in C(I)$ implies that for all $p \geq 0$ and for all $n \geq 0$, $T^{n+p}(\omega) = T^n(T^p(\omega)) \in I$, i.e. for all $p \geq 0$, $T^p(\omega) \in C(I)$.

□

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