

## WORKING PAPER SERIES

## EXTERNAL INSTRUMENT SVAR ANALYSIS FOR NONINVERTIBLE SHOCKS

**Mario Forni, Luca Gambetti and Giovanni Ricco**

# External Instrument SVAR Analysis for Noninvertible Shocks

Mario Forni<sup>1</sup>, Luca Gambetti<sup>2</sup>, and Giovanni Ricco<sup>3</sup>

<sup>1</sup>*Università di Modena e Reggio Emilia, CEPR and RECent*

<sup>2</sup>*Universitat Autònoma de Barcelona, BSE, Università di Torino, CCA*

<sup>3</sup>*École Polytechnique, University of Warwick, OFCE-SciencesPo, and CEPR*

29th January 2023

## Abstract

We propose a novel External-Instrument SVAR procedure, the Generalised External-Instrument SVAR, to identify and estimate the impulse response functions, regardless of the shock being invertible or recoverable. When the shock is recoverable, we also show how to estimate the unit variance shock and the ‘absolute’ response functions. When the shock is invertible, the method collapses to the standard External-Instrument SVAR procedure. We show how to test for recoverability and invertibility. We apply our techniques to a monetary policy VAR. It turns out that, using standard specifications, the monetary policy shock is not invertible, but is recoverable. When using our procedure, results are plausible even in a parsimonious specification, not including financial variables. Contrary to previous findings, monetary policy has significant and sizeable effects on prices.

**JEL classification:** C32, E32.

**Keywords:** Proxy-SVAR, SVAR-IV, Impulse response functions, Variance Decomposition, Historical Decomposition, Monetary Policy Shock.

We thank Mikkel Plagborg-Møller and Christian Wolf for reading a previous version of the present paper and providing useful comments and suggestions. Luca Gambetti acknowledges the financial support from the Spanish Ministry of Science and Innovation, through the Severo Ochoa Programme for Centres of Excellence in R&D (CEX2019-000915-S), the financial support from the Spanish Ministry of Science, Innovation and Universities and FEDER through grant PGC2018-094364-B-I00 and the BSE Research Network.

# 1 Introduction

Since the seminal contributions of [Stock \(2008\)](#), [Mertens and Ravn \(2013\)](#) and [Stock and Watson \(2018\)](#), SVAR-IV (or Proxy-SVAR) methods have become a popular approach to structural macroeconomic analysis. A very partial list of recent noticeable applications include [Stock and Watson \(2012\)](#), [Mertens and Ravn \(2014\)](#), [Gertler and Karadi \(2015\)](#), [Mertens and Montiel-Olea \(2018\)](#), [Paul \(2020\)](#) and [Miranda-Agrippino and Ricco \(2021\)](#). In this paper we refer to the method proposed in [Mertens and Ravn \(2013\)](#) as the ‘standard External-Instrument SVAR’.

A severe limitation of the method is that it requires invertibility ([Stock and Watson, 2018](#), [Miranda-Agrippino and Ricco, forthcoming](#)).<sup>1</sup> A shock is invertible if it is a linear combination of the present and past values of the VAR variables, or, equivalently, a contemporaneous linear combination of the VAR residuals. Invertibility is a demanding property, unlikely to be satisfied in the presence of ‘news’ technology shocks ([Forni et al., 2014](#)), forward guidance ([Ramey, 2016](#)) or fiscal foresight ([Mertens and Ravn, 2010](#), [Ramey, 2011](#), [Leeper et al., 2013](#)) and necessarily failing for the so-called ‘noise’ shocks ([Blanchard et al., 2013](#), [Forni et al., 2017](#)). The problem is that current and past values of macroeconomic variables do not convey enough information to recover the shock. Here we show that indeed invertibility does not hold for a few standard monetary-policy VAR specifications.

[Plagborg-Møller and Wolf \(2021\)](#) have proposed to overcome this limitation by adopting an ‘Internal-Instrument SVAR’ approach (see also [Ramey, 2016](#)). The approach consists of a standard Cholesky identification where the instrument enters the VAR as the first variable, and the IRFs to the first shock are the IRFs of interest. With this procedure, the impulse response functions are consistently estimated even in absence of invertibility. Other approaches which are valid under noninvertibility are the LP-IV (discussed in [Stock and Watson, 2018](#)), based on [Jordà \(2005\)](#)’s Local Projections, and the VARX where the proxy plays the role of the exogenous variable ([Paul, 2020](#)). With these alternative methods, however, an important flexibility of the original External-Instrument approach is lost: the possibility of having different time spans for the model variables and the proxy, a property which is very useful when the time span of the proxy is relatively short, as it is often in practice.

The main methodological contribution of this paper is to generalise the External-Instrument SVAR approach to the case of noninvertibility. We refer to our proposed method as the Generalised External-Instrument SVAR. We show that all types of analysis that can be conducted with the above alternative methods can be also conducted, under similar validity

<sup>1</sup>[Stock and Watson \(2018\)](#) provide validity conditions under global invertibility. [Miranda-Agrippino and Ricco \(forthcoming\)](#) show the method can be generalised to the case of partial invertibility, at the cost of reinforcing the validity conditions for the instrument.

conditions, with our approach. Furthermore, as compared to the Internal-Instrument SVAR approach, our method maintains the full flexibility of the original External-Instrument methodology. First, the important time-span flexibility cited above is retained. Second, the conditional exogeneity of the instrument can be ensured by using an information set different from the VAR information set. Third, by keeping the instrument out of the VAR, the dynamics of the instrument are not constrained by those of the VAR variables.

Our proposed method is the following. Instead of regressing the VAR residuals onto the current proxy only (the standard External-Instrument method), we regress the VAR residuals onto the current proxy and its lags. The impulse response functions are then estimated by combining the coefficients of this regression with the reduced-form impulse response functions obtained from the VAR. Moreover, we show how to implement, within our setting, the upper and lower bounds for the relative IRFs and the variance decomposition proposed by [Plagborg-Møller and Wolf \(2022\)](#).

While the ‘relative’ impulse-response functions can be estimated independently of the shock of interest being invertible or recoverable, the ‘absolute’ response functions and the structural shock itself cannot, unless the shock is recoverable. Recoverability ([Chahrour and Jurado, 2021](#)) is much less demanding than invertibility.<sup>2</sup> A shock is recoverable if it is a linear combination of the present, past and future values of the VAR variables, or, equivalently, it is a linear combination of the present and future values of the VAR residuals.

Here we show how to test for recoverability and estimate the structural shock of interest when it is recoverable. We regress the instrument onto the present and future values of the VAR residuals. If the shock is recoverable, the fitted value is a consistent estimate of the shock and therefore must be serially uncorrelated. Following a valuable suggestion of [Plagborg-Møller and Wolf \(2022\)](#), we then propose to test for recoverability by testing for serial uncorrelation of such projection. If the test does not reject the null, the above fitted value provides an estimate of the structural shock. Moreover, we show how to estimate the corresponding ‘absolute’ impulse-response functions.

Having an estimate of the shock, historical decomposition can be performed as usual. Standard variance decomposition is downward biased at short horizons when the model is not globally invertible. However, we can get an unbiased variance decomposition by looking at the integrals of the spectral densities over specific frequency bands, as suggested in [Forni et al. \(2019\)](#).<sup>3</sup>

The regression of the proxy onto the present and future values of the VAR residuals

<sup>2</sup>Consider that, if the number of shocks is equal to the number of variables, recoverability is always fulfilled for all shocks, whereas invertibility requires special conditions on the impulse-response functions (see e.g. [Lippi and Reichlin, 1993](#), [Lippi and Reichlin, 1994](#) and [Fernandez-Villaverde et al., 2007](#)).

<sup>3</sup>The method is the same used, for different purposes, in [Angeletos et al. \(2020\)](#).

also allows us to test for invertibility. For, under the invertibility assumption, the shock is a linear combination of current VAR residuals only. Hence a simple  $F$ -test for the null of zero coefficients for future residuals tells us whether the shock is invertible or not. Both the recoverability test and the invertibility test provide a valuable guidance for the choice of the VAR variables. If the null of invertibility is not rejected, our proposed procedure collapses to the standard External-Instrument SVAR procedure. [Stock and Watson \(2018\)](#) and [Plagborg-Møller and Wolf \(2022\)](#) also propose invertibility tests based on the proxy. The former requires estimation of LP-IV. The latter consists in testing whether the proxy Granger causes the VAR variables. In our setting, the  $F$ -test on the joint significance of future VAR residuals seems more appropriate than the Granger causality test, being a natural by-product of the estimation procedure.

A few Monte Carlo exercises validate our proposed estimation and testing method in small samples. Two of these simulations show that our procedure has an excellent performance in estimating the structural impulse response functions in comparison to the Internal-Instrument SVAR.

From a theoretical point of view, our contribution is twofold. On the one hand, we have representation results relating the structural impulse response functions and shocks to the reduced form VAR coefficients and residuals. Such results do not involve the instrument. They can be regarded as a generalisation of [Lippi and Reichlin \(1994\)](#) to the important case in which there are more shocks than variables. On the other hand, we present identification results relating the structural impulse response functions (and the shock of interest) to the proxy. For these results, we are closely connected to [Plagborg-Møller and Wolf \(2022\)](#). The basic difference between our formulas and the ones in that paper is that, in order to develop our approach, we write such relations in terms of the reduced form VAR coefficients and residuals, using the representation results cited above.

In the empirical application, we study the effects of US monetary policy using the proxy of [Gertler and Karadi \(2015\)](#). Such instrument is based on surprises in federal fund futures with three-month maturity, so that it is likely to capture both conventional monetary policy shocks and shocks to forward guidance about the path rate at short horizons. This news component might induce noninvertibility, thus providing a strong motivation for our analysis.

Our main findings are the following. First, in standard VAR specifications the monetary policy shock turns out to be noninvertible according to our test. This is true even for specifications including the excess bond premium or other financial variables. Hence, the results obtained so far with the standard External-Instrument SVAR approach should be taken with caution.

Second, when using our External-Instrument method, a contractionary shock reduces

inflation and output consistently across different VAR specifications and independently of the inclusion of financial variables. By contrast, when using the standard approach, results are dramatically different across VAR specifications, with large price and real activity puzzles emerging when the financial variables are not included.

Finally, the monetary policy shock is recoverable, so that we can perform variance decomposition. The variance decomposition shows that the contribution of the monetary policy shock on both output and prices is sizeable and larger than previously reported. This is a noticeable result, in that it suggests that monetary policy can be effective in controlling prices, contrary to what found in most of the existing literature.

The remainder of the paper is organised as follows. Sections 2 present our structural MA model and our representation results. In Section 3 we present our identification results and our proposed estimation and testing procedure, which is summarised in Section 4. Section 5 collects our Monte Carlo exercises. Section 6 presents our empirical application. The last section provides some conclusions. The Online Appendix provides the proofs of all propositions, examples, additional simulations and a few robustness checks for the empirical application.

## 2 Representation theory

In this section we introduce our theoretical framework and study the relation between the structural representation and the VAR representation when the structural shock of interest is not recoverable, recoverable but not invertible, and invertible.

### 2.1 The model

Let us start from our assumptions about the structural macroeconomic model and the VAR representation.

**Assumption 1.** (Structural MA representation) *The observable macroeconomic variables in the  $n$ -dimensional vector  $y_t$  (possibly after suitable transformations) have the representation*

$$y_t = B(L)u_t, \tag{1}$$

where (i)  $B(L) = B_0 + B_1L + B_2L^2 + \dots$  is an  $n \times q$  matrix of rational impulse-response functions in the lag operator  $L$ ; (ii)  $n \leq q$  and  $B(L)$  has maximum rank  $n$ ; (iii)  $u_t$  is a  $q$ -dimensional white noise vector including the structural shocks, whose variance covariance

matrix is  $I_q$ .<sup>4</sup>

The above model is sometimes referred to as the Slutsky-Frisch representation of the macro economy. It can be thought of as resulting from the linearisation of a DSGE model and can easily be derived from its state-space representation. Notice that we do not assume that the number of shocks is equal to the number of variables, so that the matrix  $B(L)$  is not necessarily square. However, we assume that the number of variables cannot be larger than the number of shocks ( $n \leq q$ ). This assumption can be justified by recognising that the variables are observed with error and such errors are allowed to enter the vector  $u_t$  together with the structural shocks. Since the entries of  $B(L)$  are rational functions, the assumption that  $B(L)$  has maximum rank  $n$  implies that  $B(z)$ ,  $z$  being a complex variable, has rank  $n$  almost everywhere in the complex plane. This is tantamount to assuming that the spectral density matrix of  $y_t$ , i.e.  $S^y(\theta) = \frac{1}{2\pi}B(e^{-j\theta})B'(e^{j\theta})$ ,  $j$  being the imaginary unit, is nonsingular a.e. in  $[\pi \pi]$ .<sup>5</sup>

In this paper we are concerned with identification of a single shock of interest, say  $u_{it}$ . In order to highlight the shock of interest and the corresponding response functions, it is convenient to re-write (1) in the form

$$y_t = b_i(L)u_{it} + \tilde{B}(L)\tilde{u}_t, \quad (2)$$

where  $b_i(L) = b_{i0} + b_{i1}L + b_{i2}L^2 + \dots$  is the  $i$ -th column of  $B(L)$ ,  $\tilde{B}(L)$  includes the other columns of  $B(L)$  and  $\tilde{u}_t = (u_{1t} \dots u_{i-1,t} u_{i+1,t} \dots u_{qt})'$ .

On the other hand, being stationary and purely nondeterministic by (1),  $y_t$  necessarily admits the Wold representation

$$y_t = C(L)\varepsilon_t \quad (3)$$

where  $C(L) = C_0 + C_1L + C_2L^2 + \dots$  and  $\varepsilon_t$  is a vector white-noise process with covariance matrix  $\Sigma_\varepsilon$ . Of course, this representation is square.

For VAR estimation, it is convenient to add the following assumption.

**Assumption 2.** (VAR representation) *The matrix of the Wold representation  $C(L)$  has an inverse in the non-negative powers of  $L$ , possibly of infinite order. Hence, letting its inverse be  $A(L)$ ,  $y_t$  has the VAR representation*

$$A(L)y_t = \varepsilon_t. \quad (4)$$

<sup>4</sup>We omit the constant term for notational simplicity. A rational impulse-response function is the ratio of two polynomials in  $L$ .

<sup>5</sup>A treatment of the case  $q < n$ , where the variables have a singular spectral density matrix, can be found in [Forni et al. \(2020\)](#).

We remark that we do not require invertibility of the structural representation, but only invertibility of the Wold representation. Moreover, we do not assume that the VAR has a finite order.

## 2.2 Structural shocks and VAR residuals

What is the relation between the structural shocks  $u_t$  and the VAR residuals  $\varepsilon_t$ ? What is the relation between the structural response functions  $B(L)$  and the VAR coefficients  $A(L)$ ? To begin, we present a very general result, not requiring either invertibility or recoverability. Then we define these important properties of the structural shocks and specialise our results to the recoverability and the invertibility cases.

The results in the present section can be regarded as a generalisation of the representation results in [Lippi and Reichlin \(1994\)](#), which refer to a square system, to the case of possibly “short” structural models, where  $q \geq n$  and recoverability does not hold for all structural shocks.

### 2.2.1 A general result

Clearly, the VAR residuals  $\varepsilon_t$  are linear combinations of the current and lagged structural shocks  $u_t$ , since from Equations (1) and (4) we see that

$$\varepsilon_t = A(L)y_t = A(L)B(L)u_t = Q(L)u_t. \quad (5)$$

But what about the inverse relation, i.e. the one relating  $u_t$  to  $\varepsilon_t$ ? Unfortunately, in the general case we cannot write  $u_t$  as an exact function of the  $\varepsilon$ 's. To see this, consider the case  $q > n$ , i.e. there are more structural shocks than VAR residuals: intuition suggests that in this case we cannot have an exact linear mapping relating all of the shocks in  $u_t$  to those in  $\varepsilon_t$ . We show below that this is in fact the case. The best we can do, within a linear setting, is to approximate  $u_t$  by taking the projection of  $u_t$  onto  $\mathcal{H}$ , the space spanned by the present, past and future of the Wold shocks  $\varepsilon_t$ :  $\mathcal{H} = \overline{\text{span}}(\varepsilon_{j,t-k}, j = 1, \dots, n, k \in \mathbb{Z})$ . Denoting with  $P$  the linear projection operator, we have

$$u_t = P(u_t|\mathcal{H}) + s_t = D'(F)\varepsilon_t + s_t, \quad (6)$$

where  $s_t$  is the residual of the projection,  $F = L^{-1}$  is the forward operator such that  $F\varepsilon_t = \varepsilon_{t+1}$  and  $D'(F)$  is a  $q \times n$  matrix of linear filters. Existence of the above infinite sum representation is guaranteed by the fact that  $\varepsilon_t$  is a vector white noise.

The following result shows that there is a precise relation between (5) and (6).

**Proposition 1.** (Basic representation theorem)

(i)  $D(F)$  is one-sided in the non-negative powers of  $F$ .

(ii)  $Q(L)$  defined in (5) is linked to the projection coefficients in (6) by the relations

$$Q(L) = \Sigma_\varepsilon D(L); \quad D(L) = \Sigma_\varepsilon^{-1} Q(L). \quad (7)$$

(iii) The structural impulse-response functions are linked to the Wold impulse response functions by the relation

$$B(L) = C(L)Q(L) = C(L)\Sigma_\varepsilon D(L).$$

In particular, for the impulse-response functions of interest the relation is

$$b_i(L) = C(L)q_i(L) = C(L)\Sigma_\varepsilon d_i(L). \quad (8)$$

Proposition 1 establishes a mapping between the Wold impulse-response functions  $C(L)$  and the structural impulse-response functions  $B(L)$  which holds true independently of invertibility or recoverability of the structural shocks.

### 2.2.2 Recoverable shocks

An important special case is the one of  $u_{it}$  being recoverable. In this case, we have an exact mapping relating  $u_{it}$  to the present and future of  $\varepsilon_t$ .

**Definition 1.** (Recoverability) Let  $\mathcal{H}^y$  be the closed linear space spanned by present, past and future values of  $y_t$ :  $\mathcal{H}^y = \overline{\text{span}}(y_{j,t-k}, j = 1, \dots, n, k \in \mathbb{Z})$ . We say that the structural shock  $u_{it}$  is recoverable with respect to  $y_t$  if and only if  $u_{it} \in \mathcal{H}^y$ . We say that  $u_t$  is (globally) recoverable with respect to  $y_t$  if and only if all of the structural shocks are recoverable.

**Proposition 2.** (Structural shocks and VAR residuals) If  $u_{it}$  is recoverable with respect to  $y_t$ ,

$$u_{it} = d'_i(F)\varepsilon_t = q'_i(F)\Sigma_\varepsilon^{-1}\varepsilon_t, \quad (9)$$

where  $d_i(F) = d_{i0} + d_{i1}F + d_{i2}F^2 + \dots$  is the  $i$ -th column of  $D(F)$  and  $q_i(F) = q_{i0} + q_{i1}F + q_{i2}F^2 + \dots$  is the  $i$ -th column of  $Q(F)$ . Moreover

$$d'_i(F)\Sigma_\varepsilon d_i(L) = q'_i(F)\Sigma_\varepsilon^{-1}q_i(L) = 1.$$

**Remark 1.** (Global recoverability and square systems) An immediate consequence of the above result is that, if  $u_t$  is globally recoverable, then  $s_t = 0$  and  $u_t = D'(F)\varepsilon_t$ . Hence from

(5) we see that  $D'(F)$  is a left-inverse of  $Q(L)$ . This can be true in our setting, where  $n \leq q$ , only if  $q = n$  (so that  $D'(F)$  is the inverse of  $Q(L)$ ). The converse is also true, i.e. if  $q = n$ , then  $u_t$  is recoverable. For, from (5) we get  $\Sigma_\varepsilon = Q(z)Q'(z^{-1})$ . Hence, if  $\det Q(z)$  vanishes for  $z = z^*$ , then  $\det Q(z^{-1})$  must have a pole in  $z^*$ , which is impossible if  $|z^*| = 1$ , since  $\varepsilon_t$  is stationary. Hence  $Q(L)$  is invertible (possibly toward the future) and  $u_{it} \in \mathcal{H}$  for any  $i$ . In conclusion,  $u_t$  is globally recoverable if and only if  $q = n$ .<sup>6</sup>

**Remark 2.** (Global recoverability and Blaschke matrices) [Lippi and Reichlin \(1994\)](#), in the context of a square model, show that the structural IRFs and shocks are related to the Cholesky IRFs and shocks by a Blaschke matrix, i.e. a matrix  $M(z)$  of rational functions in  $z$  such that  $M(z)M'(1/z) = I$ . This can be easily seen using the above results. From (3) we see that the Cholesky representation of  $y_t$  is  $y_t = [C(L)H]\eta_t$ , where  $H$  is the Cholesky factor of  $\Sigma_\varepsilon$ , such that  $HH' = \Sigma_\varepsilon$ , and  $\eta_t = H^{-1}\varepsilon_t$ . We have seen in Remark 2 that, under global recoverability,  $u_t = D'(F)\varepsilon_t$ . Hence  $u_t = M'(F)\eta_t$ , where  $M'(F) = D'(F)H$ . Moreover,  $M(z)$ ,  $z$  being a complex variable, is a Blaschke matrix. For, we have seen in Remark 2 that, under global recoverability,  $D'(F)Q(L) = I_n$ . Since  $Q(L) = \Sigma_\varepsilon D(L)$  by (7), we have  $M'(F)M(L) = M(L)M'(F) = I_n$ . Hence  $\eta_t = M(L)u_t$  and the structural impulse response functions are related to the Cholesky impulse-response functions by the equation  $B(L) = [C(L)H]M(L)$ .

**Remark 3.** (Recoverability measure) Let us consider the spectral density function of the projection of  $u_{it}$  onto  $\mathcal{H}$ , i.e.  $d'_i(F)\varepsilon_t = q'_i(F)\Sigma_\varepsilon^{-1}\varepsilon_t$ . We have

$$R_r^2(\theta) = d'_i(e^{j\theta})\Sigma_\varepsilon d_i(e^{-j\theta}) = q'_i(e^{j\theta})\Sigma_\varepsilon^{-1}q_i(e^{-j\theta}),$$

where  $j$  denotes the imaginary unit. This quantity represents the fraction of the total variance of  $u_{it}$  explained by the projection, decomposed by frequency, and is equal to 1 at all frequencies if  $u_{it}$  is recoverable. Correspondingly, the variance of the projection, i.e.

$$R_r^2 = \sum_{k=0}^{\infty} d'_{ik}\Sigma_\varepsilon d_{ik} = \sum_{k=0}^{\infty} q'_{ik}\Sigma_\varepsilon^{-1}q_{ik}$$

is the fraction of total variance explained by the  $\varepsilon$ 's and is equal to 1 when  $u_{it}$  is recoverable. Hence  $R_r^2(\theta)$  and  $R_r^2$  can be regarded as measures of recoverability.

<sup>6</sup>See also [Chahrour and Jurado \(2021\)](#).

### 2.2.3 Fundamental/invertible shocks

If  $u_{it}$  is recoverable, it may be the case that it fulfils a more demanding property, that is fundamentalness. In the literature, fundamentalness is often regarded as a synonymous of invertibility. Indeed, fundamentalness is somewhat weaker than invertibility, in that invertibility requires that  $u_{it}$  can be written as a linear combination of the present and past history of  $y_t$ , whereas fundamentalness does not. For instance, if  $y_t = (1 - L)u_t$ , then  $u_t$  is fundamental but does not have a VAR representation since  $(1 - L)$  is not invertible. In our setting however we are assuming that  $y_t$  has a VAR representation, so that fundamentalness and invertibility coincide.

**Definition 2.** (Fundamentalness) *Let  $\mathcal{H}_t^-$  be the closed linear space spanned by present and past values of  $y_t$ :  $\mathcal{H}_t^- = \overline{\text{span}}(y_{j,t-k}, j = 1, \dots, n, k \geq 0)$ . We say that the structural shock  $u_{it}$  is fundamental with respect to  $y_t$  if and only if  $u_{it} \in \mathcal{H}_t^-$ . We say that  $u_t$  is fundamental with respect to  $y_t$  if and only if all structural shocks are fundamental with respect to  $y_t$ .*

From the definition of fundamentalness we see that if  $u_{it}$  is fundamental for  $y_t$ , then  $u_{it}$  is recoverable, whereas the converse is not necessarily true. The following well-known result holds.

**Proposition 3.** (Structural shocks and VAR residuals) *If  $u_{it}$  is fundamental for  $y_t$ , then  $d_i(F) = d_{i0} = d_i$  and  $q_i(F) = q_{i0} = q_i$ , so that*

$$u_{it} = d'_i \varepsilon_t = q'_i \Sigma_\varepsilon^{-1} \varepsilon_t. \quad (10)$$

**Remark 4.** (Global fundamentalness) An immediate consequence of Proposition 3 is that, if  $u_t$  is globally fundamental for  $y_t$ , then it is a contemporaneous linear combination of the  $\varepsilon$ 's, that is  $u_t = D' \varepsilon_t$ . Hence using Proposition 1 we get  $Q(L) = \Sigma_\varepsilon D'$ , so that  $Q(L) = Q = B_0$  (since  $A_0 = I_n$ , see Equation 5). Moreover, we see from Equation (5) that  $D' \varepsilon_t = u_t = D' Q u_t$ , so that  $D' Q = I_n$ . Since we have global fundamentalness,  $B_0, Q$  and  $D$  must be square (see Remark 1), so that  $D'$  is the inverse of the matrix of the impact effects  $B_0$ .

**Remark 5.** (Fundamentalness measure) In analogy with Remark 3, we can define a measure of fundamentalness as  $R_f^2 = d'_i \Sigma_\varepsilon d_i$ . This measure corresponds to the fundamentalness measure proposed by Forni et al. (2019).

Regarding the impulse-response functions, we see from Proposition 3 that, if  $u_{it}$  is fundamental, Equation (8) reduces to

$$b_i(L) = C(L)q_i = C(L)\Sigma_\varepsilon d_i, \quad (11)$$

where  $d'_i \Sigma_\varepsilon d_i = q'_i \Sigma_\varepsilon^{-1} q_i = 1$ .

## 2.3 A general representation

From the above result we can derive a general representation of  $y_t$  in the fundamental shocks, the recoverable but nonfundamental shocks and the nonrecoverable shocks.<sup>7</sup>

**Proposition 4.** (General Representation) *Any vector process  $y_t$  satisfying Assumptions 1 and 2 can be represented as*

$$\begin{aligned} y_t &= B^f(L)u_t^f + B^r(L)u_t^r + B^n(L)u_t^n \\ &= C(L)Q^f u_t^f + C(L)Q^r(L)u_t^r + C(L)Q^n(L)u_t^n \\ &= C(L)\Sigma_\varepsilon D^f u_t^f + C(L)\Sigma_\varepsilon D^r(L)u_t^r + C(L)\Sigma_\varepsilon D^n(L)u_t^n. \end{aligned} \quad (12)$$

where  $u_t^f$  is the sub-vector of the fundamental structural shocks,  $u_t^r$  of the recoverable (but nonfundamental) shocks, and  $u_t^n$  of the nonrecoverable ones.  $\Sigma_\varepsilon$  is the covariance matrix of the Wold innovations  $\varepsilon_t$  and  $C(L)$  is the matrix of the Wold representation.  $Q^h(L)u_t^h$ , for  $h = f, r, n$ , is the projection of  $\varepsilon_t$  onto  $u_{t-k}^h$ , with  $k \geq 0$ ; and  $D^h(L)$ , for  $h = f, r, n$ , is such that  $D^h(L)\varepsilon_t$  is the projection of  $u_t^h$  onto  $\varepsilon_{t+k}$ , with  $k \geq 0$ . Moreover, the following properties hold:

- (i)  $D^f$  and  $Q^f$  are such that  $D^{f'}\Sigma_\varepsilon D^f = Q^{f'}\Sigma_\varepsilon^{-1}Q^f = I_{q_f}$ ,  $q_f$  being the number of fundamental shocks;
- (ii)  $D^r(L)$  and  $Q^r(L)$  are such that  $D^{r'}(L)\Sigma_\varepsilon D^r(L) = Q^{r'}(L)\Sigma_\varepsilon^{-1}Q^r(L) = I_{q_r}$ ,  $q_r$  being the number of recoverable but nonfundamental shocks.

## 3 Identification, estimation, and testing

We now discuss how the structural impulse-response functions and the structural shocks can be identified and estimated by using an external proxy in the VAR framework discussed in the previous section.

### 3.1 The instrument

Let us start by introducing the instrument.

<sup>7</sup>Proposition 4 generalises representation results provided in [Lippi and Reichlin \(1994\)](#) and [Miranda-Agrippino and Ricco \(2021\)](#).

**Assumption 3.** (The Instrument) *The researcher can observe the proxy  $\tilde{z}_t$ , following the relation*

$$\tilde{z}_t = \beta(L)\tilde{z}_{t-1} + \mu'(L)x_{t-1} + \alpha u_{it} + w_t, \quad (13)$$

where  $w_t$  is an error orthogonal to  $u_{j,t-k}$ ,  $j = 1, \dots, q$ , for any integer  $k$  and to  $z_{t-k}, x_{t-k}$ ,  $k \geq 0$ , and  $\beta(L), \mu(L)$  are rational functions in the lag operator  $L$  (for simplicity we omit the constant term).

This assumption is similar to the one used in [Plagborg-Møller and Wolf \(2022\)](#), which in turn is essentially equivalent to the weak LP-IV condition of [Stock and Watson \(2018\)](#). A remarkable difference, however, is that the vector  $x_t$  appearing on the right side of (13) is not necessarily equal to the VAR vector  $y_t$ .

[Stock and Watson \(2018\)](#) stresses that Equation (13) is more restrictive than the standard validity conditions for the SVAR-IV, i.e. (i)  $\text{cov}(\tilde{z}_t, u_{it}) \neq 0$  and (ii)  $\text{cov}(\tilde{z}_t, u_{kt}) = 0$  for  $k \neq i$ . Let us observe, however, that (i) and (ii) are sufficient only when assuming *global* invertibility, which of course is more demanding than invertibility of  $u_{it}$  alone. If we require only partial invertibility, we need a stronger validity condition, similar to Equation (13) above (see [Miranda-Agrippino and Ricco, forthcoming](#)).<sup>8</sup>

The above proxy cannot be used directly in our setting unless  $\beta(L) = 0$ . Hence, in place of  $\tilde{z}_t$ , we shall consider the residual of the projection of  $\tilde{z}_t$  onto the past history of  $\tilde{z}_t$  and  $x_t$ , i.e.

$$z_t = \alpha u_{it} + w_t. \quad (14)$$

Indeed the first step of our proposed procedure is to ‘clean’  $\tilde{z}_t$  by estimating (13) and then using the residual  $z_t$  in place of  $\tilde{z}_t$ .

### 3.2 The IRFs and the shock

In this subsection, we present our main identification results. First we consider the case of nonrecoverable shocks. In this case the shock and the impulse response functions corresponding to a unit-variance shock (the absolute IRFs) are not identified, but the relative response functions are identified. Moreover, we can estimate upper and lower bounds for the absolute IRFs. Then we turn to the case of a recoverable shock. In this case both the shock and the absolute IRFs are identified. Finally we consider the case of a fundamental shock. In this case the IRFs and the shock are identified by the standard External-Instrument SVAR formulas.

<sup>8</sup>As an example, consider the proxy  $\tilde{z}_t = u_{1t} + u_{2,t-1} + w_t$ , where  $u_{2t}$  is not fundamental. If the shock of interest is the first one ( $i = 1$ ), this proxy fulfills (i) and (ii). But in general  $u_{2,t-1}$  will be correlated with  $\varepsilon_t$ , thus inducing a bias in the estimated response functions.

Throughout this and the following subsections, we present results for the ‘cleaned’ proxy  $z_t = \alpha u_{it} + w_t$  (see Equation 14). Let us observe, however, that all results hold true also for the original proxy  $\tilde{z}_t$ , provided that  $\beta(L) = 0$  in Equation (13).

The identification results of the present subsection are essentially equivalent to those of Plagborg-Møller and Wolf (2022); the difference is that our identification formula, basing on the representation results of the previous section, relate the structural IRFs to the coefficients of the Wold representation of  $y_t$ , on the one hand, and the projection of the VAR residuals on the current and lagged instrument, on the other hand. As for the structural shock, we relate it to the coefficients of the projection of the instrument on the present and future of the VAR residuals. This is useful to introduce our proposed estimation method, where, unlike Plagborg-Møller and Wolf (2022), the instrument is kept outside the VAR.

### 3.2.1 Nonrecoverable shocks

Let us consider the projection of  $\varepsilon_t$  onto the present and past of the proxy:

$$\varepsilon_t = \psi(L)z_t + e_t. \quad (15)$$

The following result holds.

**Proposition 5.** (Relative IRFs) *The coefficients of the projection (15) are related to  $q_i(L)$  appearing in (8) by the equation*

$$\psi(L)\sigma_z^2 = q_i(L)\alpha. \quad (16)$$

Hence the impulse-response functions fulfil the relation

$$b_i(L)\alpha = C(L)\psi(L)\sigma_z^2. \quad (17)$$

A consequence of Proposition 5 is that a possible strategy to estimate the impulse response functions is to perform the OLS regression of  $\varepsilon_t$  onto the present and past values of  $z_t$  until a maximum lag  $r$  to get an estimate of  $\psi(L) = q_i'(L)\alpha/\sigma_z^2$ , say  $\widehat{\psi}(L)$ , and estimate  $b_i(L)\alpha$  as

$$\widehat{b_i(L)\alpha} = \widehat{C}(L)\widehat{\psi}(L)\widehat{\sigma}_z^2 = \widehat{\gamma}(L)\widehat{\sigma}_z^2. \quad (18)$$

**Remark 6.** (Consistency) The estimator (18), as well as the estimators suggested below, are elementary functions of the estimators appearing on the right-hand side of the equation, so that consistency is ensured by consistency of such estimators. Notice that the population VAR is possibly of infinite order and projection (15) is necessarily of infinite order, so that, when setting  $p$  and  $r$ , we truncate such relations. The approximation works since the coefficients

shrink the further you go into the past; however, consistency requires that  $p$  and  $r$  go to infinity with  $T$  at a suitable rate.

Unfortunately, in the general case  $\alpha$  cannot be estimated consistently so that we cannot estimate the impulse response functions corresponding to a unit-variance shock. However, we can estimate the relative IRFs, by normalising the IRFs obtained according to (18) by dividing by the effect on a pre-specified variable at a given lag, as suggested in [Stock and Watson \(2018\)](#). For instance we can normalise the impulse-response functions by dividing by the impact effect on the first variable:

$$\frac{\widehat{b_i(L)}}{\widehat{b_{i1}(0)}} = \frac{\widehat{\gamma(L)}}{\widehat{\gamma_1(0)}}, \quad (19)$$

where  $\widehat{\gamma_1(L)}$  is the first entry of  $\gamma(L)$ . The IRFs are then the ones corresponding to a shock having impact effect 1 on the first variable.

**Remark 7.** (An alternative estimator for the IRFs) An alternative strategy for the estimation of the relative impulse response functions is to perform the opposite projection, i.e. the OLS projection of  $z_t$  onto the present and future of  $\varepsilon_t$ , until a maximum lead  $r$ , to get an estimate of  $\delta(F) = \alpha d_i(F)$  (see Equation 6) and use the equality  $b_i(L) = C(L)\Sigma_\varepsilon d_i(L)$  (see Formula 8). This strategy however implies estimation of a single equation with  $(r + 1)n$  regressors as against the  $r + 1$  regressors of each one of the  $n$  equations in (15). Our simulations, not reported here, confirm that the procedure proposed above performs better than this alternative.

[Plagborg-Møller and Wolf \(2022\)](#) show that, while it is impossible to estimate the absolute response functions, it is nonetheless possible to compute upper and lower bounds for the parameter  $\alpha$ ; such upper and lower bounds can be derived in our setting and provide lower and upper bounds, respectively, for the absolute response functions  $b_i(L) = \gamma(L)\sigma_z^2/\alpha$ .

**Proposition 6.** (Upper and lower bounds) *Letting  $\bar{\alpha}^2$  and  $\underline{\alpha}^2$  be the upper and the lower bound of  $\alpha^2$ , respectively, we have*

$$\begin{aligned} \alpha^2 &\leq \sigma_z^2 = \bar{\alpha}^2 \\ \alpha^2 &\geq \alpha^2 \sup_{\theta \in (0, \pi]} R_r^2(\theta) = \sigma_z^4 \sup_{\theta \in (0, \pi]} \psi'(e^{j\theta})\Sigma_\varepsilon^{-1}\psi(e^{-j\theta}). \end{aligned} \quad (20)$$

In practice, the upper bound can be easily estimated as  $\widehat{\bar{\alpha}} = \widehat{\sigma}_z$ ; the lower bound can be estimated by taking the maximum of  $\widehat{\psi}'(e^{j\theta})\widehat{\Sigma}_\varepsilon^{-1}\widehat{\psi}(e^{-j\theta})$  over the Fourier frequencies

$\theta_f = f(2\pi/T)$ , for  $f = 1, \dots, T$ ,  $T$  being the number of observations:

$$\hat{\alpha} = \hat{\sigma}^2 \max_{0 < f \leq T} \sqrt{\hat{\psi}'(e^{j\theta_f}) \hat{\Sigma}_\varepsilon^{-1} \hat{\psi}(e^{-j\theta_f})}.$$

Having an estimate of the two bounds, the lower bound for the absolute impulse response functions can be estimated as

$$\underline{b}_i(L) = \hat{\gamma}(L) \hat{\sigma}_z^2 / \hat{\alpha} = \hat{\gamma}(L) \hat{\sigma}_z = \hat{C}(L) \hat{\psi}(L) \hat{\sigma}_z. \quad (21)$$

Similarly, the upper bound for the impulse response functions can be estimated as

$$\bar{b}_i(L) = \frac{\hat{\gamma}(L) \hat{\sigma}_z^2}{\hat{\alpha}} = \frac{\hat{C}(L) \hat{\psi}(L)}{\max_{0 < f \leq T} \sqrt{\hat{\psi}'(e^{jf\pi/T}) \hat{\Sigma}_\varepsilon^{-1} \hat{\psi}(e^{-jf\pi/T})}}. \quad (22)$$

The upper and lower bounds for  $\alpha$  can also be used to get upper and lower bounds for the fraction of variance accounted for by the shock of interest. We explain in detail how to perform variance decomposition in subsection 3.3 below.

### 3.2.2 Recoverable but nonfundamental shocks

If the shock is recoverable, we can estimate the absolute IRFs and the shock itself. Let us begin with the absolute IRFs. The following results holds.

**Proposition 7.** (Absolute IRFs) *If  $u_{it}$  is recoverable, its (absolute) impulse response functions are given by the equation*

$$b_i(L) = \gamma(L) \sigma_z^2 / \alpha = \frac{C(L) \psi(L)}{\sqrt{\sum_{k=0}^{\infty} \psi'_k \Sigma_\varepsilon^{-1} \psi_k}}.$$

From the above proposition we see that  $b_i(L)$  can be estimated as

$$\hat{b}_i(L) = \hat{\gamma}(L) \hat{\sigma}_z^2 / \hat{\alpha} = \frac{\hat{C}(L) \hat{\psi}(L)}{\sqrt{\sum_{k=0}^r \hat{\psi}'_k \hat{\Sigma}_\varepsilon^{-1} \hat{\psi}_k}}. \quad (23)$$

Coming to the shock, let us consider the projection of  $z_t$  onto the space spanned by the present and the future of the VAR residuals:

$$z_t = \delta'(F) \varepsilon_t + v_t. \quad (24)$$

The following proposition holds.

**Proposition 8.** (The structural shock) *If  $u_{it}$  is recoverable, then*

$$u_{it} = \frac{\delta'(F)\varepsilon_t}{\sqrt{\sum_{k=0}^{\infty} \delta'_k \Sigma_{\varepsilon} \delta_k}}.$$

From the above proposition we see that, if the shock is recoverable, it can be estimated as

$$\hat{u}_{it} = \frac{\hat{\delta}'(F)\hat{\varepsilon}_t}{\sqrt{\sum_{k=0}^r \hat{\delta}'_k \hat{\Sigma}_{\varepsilon} \hat{\delta}_k}}. \quad (25)$$

Having an estimate of the shock, we can perform historical and variance decomposition as explained in Section 3.3 below.

**Remark 8.** (An alternative estimator for the shock) An alternative strategy for the estimation of the shock is to use the equality  $q_i(F) = \psi(F)\sigma_z/\alpha$  (see (17)), which, coupled with Equation (9), gives  $\tilde{u}_{it} = \hat{\psi}(F)\hat{\Sigma}_{\varepsilon}^{-1}\hat{\sigma}_z/\hat{\alpha}$ . We do not follow this alternative route since our simulations, available on request, show that the estimator  $\hat{u}_{it}$  performs better than  $\tilde{u}_{it}$ .

**Remark 9.** (Measuring instrument validity) Having an estimate of the shock, we can measure the Instrument Relevance (IR), and test for it, by using the correlation coefficient of  $\hat{u}_{it}$  and  $z_t$ :

$$\widehat{\text{IR}} = \text{corr}(\hat{u}_{it}, z_t).$$

### 3.2.3 Fundamental shocks

If we have fundamentalness, the following result holds.

**Proposition 9.** (IRFs and shocks under fundamentalness)

(i) *Let us consider the projection equation  $\varepsilon_t = \psi'z_t + e_t$ . If  $u_{it}$  is fundamental, then*

$$b_i(L) = \frac{C(L)\psi}{\sqrt{\psi'\hat{\Sigma}_{\varepsilon}^{-1}\psi}}.$$

(ii) *Let us consider the projection equation  $z_t = \delta'\varepsilon_t + e_t$ . If  $u_{it}$  is fundamental, then*

$$u_{it} = \frac{\delta'\varepsilon_t}{\sqrt{\delta'\Sigma_{\varepsilon}\delta}}.$$

From Proposition 9 (i) we see that, if the shock is fundamental, we can estimate (15) without including the lags of  $z_t$ , i.e. we can estimate by OLS the projection  $\varepsilon_t = \psi'z_t + e_t$  to

get an estimate of  $\psi$ . The impulse-response functions can then be estimated as

$$\hat{b}_i(L) = \frac{\widehat{C}(L)\widehat{\psi}}{\sqrt{\widehat{\psi}'\widehat{\Sigma}_\varepsilon^{-1}\widehat{\psi}}}. \quad (26)$$

Notice that the above procedure is nothing else than the standard estimation procedure, which is usually applied without testing (see below for our proposed fundamentalness test).

Turning to estimation of the shock, by Proposition 9 (ii) we can estimate (24) including only the current  $\varepsilon_t$  among the regressors, in order to estimate  $\delta$ ; having  $\hat{\delta}$ , the unit variance shock can be estimated as

$$\hat{u}_{it} = \frac{\hat{\delta}'\hat{\varepsilon}_t}{\sqrt{\hat{\delta}'\widehat{\Sigma}_\varepsilon\hat{\delta}}}. \quad (27)$$

### 3.3 Historical and variance decomposition

In this subsection we discuss historical decomposition and variance decomposition. First we consider the case of recoverability, then we turn to nonrecoverability.

#### 3.3.1 Recoverable shocks

We have shown that, if the shock of interest is recoverable, it can be estimated. Having an estimate of the shock and the corresponding impulse-response functions, historical decomposition can be performed in the standard way.

Variance decomposition is more problematic. The standard forecast error variance decomposition (FVD) can be computed only for globally invertible models. This is because the forecast error depends on all structural shocks and the corresponding impulse response functions. Having an estimate of the IRFs of  $u_{it}$  we can of course compute the numerator of the ratio, but we cannot estimate the denominator without estimating the whole structural model.

Plagborg-Møller and Wolf (2022) replace this denominator with the forecast error variance based on present and past values of  $y_t$ , which can be estimated, and name this ratio FVR. At long horizons, these denominators are equal, since the forecast error coincides with the variable itself; but in the short run the past values of  $y_t$  are less informative than the structural shocks and have larger forecast errors. As a consequence, the FVR underestimates the variance contribution of the structural shock of interest as given by the unfeasible FVD.<sup>9</sup> In the Online Appendix, Subsection B.3, we provide an example showing that the downward bias can be

<sup>9</sup>See Forni et al. (2019), Section 3.4.

very large, so that a low FVR at short horizons is scarcely informative.<sup>10</sup>

To solve this problem, in place of the FVR we suggest to use the variance decomposition given by the integral of the spectral density over suitable frequency bands. This decomposition has been proposed in [Forni et al. \(2019\)](#) for the case of partial fundamentalness and has become popular with [Angeletos et al. \(2020\)](#), where it is used to identify the so called Main Business-Cycle Shock. It is the one referred to as VD in [Plagborg-Møller and Wolf \(2022\)](#).<sup>11</sup> The relation linking the VD to the FVD is not simple. However, the VD for the whole interval  $[-\pi \ \pi)$  is the total variance of the variable accounted for by the shock of interest, divided by the total variance of the variable, so that it is equivalent to both FVD and FVR at horizon infinity.

Let  $b_{ih}(L)$  be the  $h$ -th element of  $b_i(L)$ . The total variance of the component of  $y_{ht}$  which is attributable to  $u_{it}$  can be computed as  $\int_0^\pi b_{ih}(e^{-j\theta})b_{ih}(e^{j\theta})d\theta/\pi$ , where  $j$  denotes the imaginary unit. But we can also compute the variance on a specific frequency band  $[\theta_1 \ \theta_2)$ . If we are interested for instance in the variance of waves of business cycle periodicity, say between 8 and 32 quarters, the corresponding angular frequencies (with quarterly data) are  $\theta_1 = \pi/16$  and  $\theta_2 = \pi/4$  and the corresponding variance is  $\int_{\pi/16}^{\pi/4} b_{ih}(e^{-j\theta})b_{ih}(e^{j\theta})d\theta/\pi$ . In practice, the integral must be approximated by averaging over a suitable frequency grid within the relevant interval (for instance, the Fourier frequencies  $\theta_f = f(2\pi/T)$ ,  $f$  being a natural number such that  $\theta_1(T/2\pi) \leq f \leq \theta_2(T/2\pi)$ ). This is the numerator of the suggested variance decomposition. As for the denominator, we need the spectral density of  $y_{ht}$ , say  $\frac{1}{2\pi}S_h(\theta)$ , where

$$\widehat{S}_h(\theta) = \widehat{C}_h(e^{-j\theta})\widehat{\Sigma}_\varepsilon\widehat{C}_h(e^{j\theta})',$$

$C_h(L)$  being the  $h$ -th row of the matrix  $C(L)$  appearing in the Wold representation. The total variance of  $y_{ht}$  on the frequency band  $[\theta_1 \ \theta_2)$  is given by  $\int_{\theta_1}^{\theta_2} \widehat{S}_h(\theta)d\theta/\pi$ . Hence the contribution of  $u_{it}$  to the variance of  $y_{ht}$  on the frequency band  $[\theta_1 \ \theta_2)$ , say  $c_{ih}(\theta_1, \theta_2)$  can be estimated as the ratio

$$\hat{c}_h(\theta_1, \theta_2) = \frac{\int_{\theta_1}^{\theta_2} \hat{b}_{ih}(e^{-j\theta})\hat{b}_{ih}(e^{j\theta})d\theta}{\int_{\theta_1}^{\theta_2} \widehat{S}_h(\theta)d\theta}. \quad (28)$$

We can also evaluate the fraction of total variance explained by the shock of interest at each

<sup>10</sup>Were it possible to have an estimate of all structural shocks, the sum of the variance contributions of these shocks, computed with the FVR, is smaller than 100% (in the example of the Online Appendix it is just 20%).

<sup>11</sup>We recommend to use this variance decomposition even if fundamentalness of the shock of interest is not rejected according to the test described in the following subsection, since of course we might have fundamentalness of this shock but not global fundamentalness; and, as stated above, the standard forecast error variance decomposition requires global fundamentalness.

frequency, i.e.

$$\hat{c}_h(\theta) = \frac{\hat{b}_{ih}(e^{-j\theta})\hat{b}_{ih}(e^{j\theta})}{\hat{S}_h(\theta)}. \quad (29)$$

We stress that the above formulas provide a variance decomposition for the variable itself. In this respect, its interpretation is more direct than the one of the standard variance decomposition, which refers to the variance of the forecast errors.

### 3.3.2 Nonrecoverable shocks

In the case of nonrecoverability we can only estimate  $\gamma(L)\sigma_z^2 = \alpha b_i(L)$  according to Equation (18). Hence we can get an estimate of

$$\alpha^2 c_h(\theta_1, \theta_2) = \frac{\sigma_z^4 \int_{\theta_1}^{\theta_2} \gamma_h(e^{-j\theta})\gamma_h(e^{j\theta})d\theta}{\int_{\theta_1}^{\theta_2} S_h(\theta)d\theta}. \quad (30)$$

Since  $\alpha$  cannot be estimated, we cannot perform the variance decomposition. However, the upper and lower bounds for  $\alpha^2$  provided in subsection 3.2.1 provide respectively lower and upper bounds for  $c_h(\theta_1, \theta_2)$ . Precisely, we can estimate the lower bound as

$$\hat{c}_h(\theta_1, \theta_2) = \hat{\sigma}_z^2 \frac{\int_{\theta_1}^{\theta_2} \hat{\gamma}_h(e^{-j\theta})\hat{\gamma}_h(e^{j\theta})d\theta}{\int_{\theta_1}^{\theta_2} \hat{S}_h(\theta)d\theta} \quad (31)$$

and the upper bound as

$$\hat{c}_h(\theta_1, \theta_2) = \hat{\sigma}_z^4 \frac{\int_{\theta_1}^{\theta_2} \hat{\gamma}_h(e^{-j\theta})\hat{\gamma}_h(e^{j\theta})d\theta}{\underline{\hat{\alpha}}^2 \int_{\theta_1}^{\theta_2} \hat{S}_h(\theta)d\theta}, \quad (32)$$

where  $\theta_f = f(2\pi/T)$ ,  $f = 1, \dots, T$ . Finally, the lower and upper bounds for the frequency-by-frequency decomposition of Equation (29) can be computed as

$$\hat{c}_h(\theta) = \hat{\sigma}_z^2 \frac{\hat{\gamma}_h(e^{-j\theta})\hat{\gamma}_h(e^{j\theta})}{\hat{S}_h(\theta)} \quad (33)$$

$$\hat{c}_h(\theta) = \frac{\hat{\sigma}_z^4 \hat{\gamma}_h(e^{-j\theta})\hat{\gamma}_h(e^{j\theta})}{\underline{\hat{\alpha}}^2 \hat{S}_h(\theta)}, \quad (34)$$

where  $\theta_f = f(2\pi/T)$ ,  $f = 1, \dots, T$ .

## 3.4 Testing for recoverability and fundamentalness

In this subsection we propose a test for recoverability and a test for fundamentalness.

### 3.4.1 Recoverability test

In the proof of Proposition 8 we have seen that, if  $u_{it}$  is recoverable, then the projection in (24),  $\delta'(F)\varepsilon_t$ , is equal to  $\alpha u_{it}$  and therefore is a white noise process. By contrast, if recoverability does not hold,  $s_{it} \neq 0$  and the projection is not equal to  $\alpha u_{it}$ . Being a Moving Average of present and future VAR residuals, the projection will in general be autocorrelated.

Hence, to check whether the shock is recoverable or not, following a suggestion of [Plagborg-Møller and Wolf \(2022\)](#), we propose to test for zero serial correlation of the projection  $\delta(F)\varepsilon_t$ .<sup>12</sup> Precisely, we propose to perform the OLS regression of  $z_t$  onto the present and future values of  $\hat{\varepsilon}_t$ , until a maximum lead  $r$ , to get an estimate of  $\delta(F)$ . Then apply the Ljung-Box Q-test to the estimated projection  $\hat{\delta}(F)\hat{\varepsilon}_t$ . The null hypothesis is recoverability (serial uncorrelation) and the alternative is nonrecoverability (serial correlation). In the following section we present a Monte Carlo exercise in which the autocorrelation test has a reasonably good power in rejecting recoverability when it is false.

If recoverability is rejected, we have two options: (1) estimate the relative impulse response functions and the upper and lower bounds for variance decomposition; (2) amend the VAR specification by adding variables (or use a FAVAR model in place of the VAR) and perform the test with the novel VAR specification.

It is worth noticing that the above test is valid under the maintained hypothesis that Equation (13) is fulfilled. If it is not, the lags of  $u_{it}$  may appear in Equation (14) and the test may reject serial uncorrelation even if the shock is recoverable. Hence the serial uncorrelation test is indeed a joint test about recoverability and instrument validity.

If recoverability is not rejected, we can estimate the ‘absolute’ response function, the unit variance shock and the variance decomposition as explained above.

### 3.4.2 Fundamentalness test

If  $u_{it}$  is fundamental with respect to  $y_t$  we see from Proposition 9 that in Equation (24)  $\delta_k = 0$  for all positive  $k$  and  $\delta(F)$  reduces to  $\delta_0 = \delta$ . Hence, we can test for the null of fundamentalness against the alternative of nonfundamentalness by estimating (24) by OLS as explained above and perform a standard  $F$ -test for the joint significance of the coefficients of the leads. Notice that the test is valid even if recoverability does not hold; hence, in principle we can test directly for fundamentalness without testing for recoverability.<sup>13</sup>

**Remark 10.** (Estimating the degree of fundamentalness) If fundamentalness does not hold, but recoverability does, we can estimate consistently the degree of fundamentalness  $R_f^2$  (see

<sup>12</sup>In [Plagborg-Møller and Wolf \(2022\)](#), the practical implementation of the test is left for future research.

<sup>13</sup>However, the recoverability test can provide a useful check in that, if recoverability is rejected, fundamentalness is rejected as well.

Remark 5). From Proposition 8 we see that the fraction of the variance of  $u_{it}$  explained by the current VAR residuals is  $R_f^2 = \delta_0' \Sigma_\varepsilon \delta_0 / \sum_{k=0}^{\infty} \delta_k' \Sigma_\varepsilon \delta_k$ . Hence we can estimate  $R_f^2$  as

$$\hat{R}_f^2 = \hat{\delta}_0' \hat{\Sigma}_\varepsilon \hat{\delta}_0 / \sum_{k=0}^r \hat{\delta}_k' \hat{\Sigma}_\varepsilon \hat{\delta}_k.$$

Stock and Watson (2018) and Plagborg-Møller and Wolf (2022) also propose invertibility tests based on the proxy. The latter paper, in particular, proposes to test whether the proxy Granger causes the VAR variables.<sup>14</sup> By using Sims' theorem, it can be shown that their method is asymptotically equivalent to ours. It would be interesting to compare the small sample performances of all these tests; this however is left for further research. In the online appendix we present a Monte Carlo exercise showing that the proposed  $F$ -test has a good performance in small samples.

If fundamentalness is not rejected, we can estimate the 'absolute' response function, the unit variance shock and the variance decomposition as explained in the previous subsections.

### 3.5 Inference

For inference purposes, we suggest the following bootstrap procedure. For simplicity, we assume that the sample size  $T$  is the same for  $z_t$  and  $y_t$ . The generalisation to the case of different time spans is straightforward.

First, draw with reintroduction  $T - (p + r)$  integers  $i(t)$ ,  $t = 1, \dots, T - (p + r)$ , uniformly distributed between 1 and  $T - (p + r)$ , and construct the artificial sequences of shocks  $\varepsilon_t^1 = \hat{\varepsilon}_{i(t)}$  and  $v_t^1 = \hat{v}_{i(t)}$ ,  $t = p + 1, \dots, T - r$ ,  $\hat{v}_t$  being the estimated residual of regression (24). Set the final conditions  $\varepsilon_t^1 = \hat{\varepsilon}_t$  for  $t = T - r + 1, \dots, T$ . Repeat the procedure  $H$  times to get the sequences  $\varepsilon_t^h$ ,  $t = p + 1, \dots, T$  and  $v_t^h$ ,  $t = p + 1, \dots, T - r$ , for  $h = 1, \dots, H$ .

Second, compute  $y_t^h$ ,  $h = 1, \dots, H$ , according to the VAR Equation (4). Precisely, set the initial conditions  $y_t^h = y_t$ ,  $t = 1, \dots, p$ , for all  $h$ . Then compute

$$y_t^h = - \sum_{k=1}^p \hat{A}_k y_{t-k}^h + \varepsilon_t^h$$

for  $t = p + 1, \dots, T$ . As for the proxy, set the initial and final conditions  $z_t^h = z_t$ ,  $t = 1, \dots, p$

<sup>14</sup>Fundamentalness test based on Granger causality have been previously proposed, in the context of standard VAR identification, in Giannone and Reichlin (2006) and Forni and Gambetti (2014).

and  $t = T - r + 1, \dots, T$ , for all  $h$ . Then compute

$$z_t^h = \sum_{k=0}^r \hat{\delta}_k \varepsilon_{t+k}^h + v_t^h$$

for  $t = p + 1, \dots, T - r$ .

Finally, repeat the estimation procedure for any one of the artificial data sets  $y_1^h, \dots, y_T^h$ ,  $h = 1, \dots, H$  to get the sequences of absolute IRFs  $b_i^h(L)$ ,  $h = 1, \dots, H$ , or the corresponding sequence of relative IRFs. Compute the confidence band as usual, by taking appropriate percentiles of the distribution of  $b_{ik}^h$ , for each lag  $k$ .

## 4 The proposed procedure

On the basis of the above considerations and results, we propose the following estimation and testing procedure.

1. As a first step, regress the available proxy  $\tilde{z}_t$  onto the first  $m$  lags (notice that  $m$  is not necessarily equal to  $p$ ) of  $\tilde{z}_t$  itself and a set of regressors  $x_t$  —which can in principle be different from  $y_t$ — to get an estimate of the residual  $z_t$ , say  $\hat{z}_t$ . If the  $F$ -test does not reject the null that the coefficients of past  $\tilde{z}_t$ 's are all zero, i.e.  $H_0 : \beta(L) = 0$ , step 1 is unnecessary and can be skipped.
2. Estimate a VAR( $p$ ) with OLS to obtain  $\hat{A}(L)$ ,  $\hat{C}(L) = \hat{A}(L)^{-1}$ ,  $\hat{\varepsilon}_t$  and  $\hat{\Sigma}_\varepsilon$ .

3. Regress with OLS the proxy  $\hat{z}_t$  on the current value and the first  $r$  leads of the Wold residuals:

$$\hat{z}_t = \sum_{k=0}^r \hat{\delta}'_k \hat{\varepsilon}_{t+k} + \hat{v}_t = \hat{\delta}(F) \hat{\varepsilon}_t + \hat{v}_t.$$

Save the fitted value of the above regression, let us call it  $\hat{\eta}_t$ . Test for invertibility by performing the  $F$ -test for the null  $H_0 : \delta_1 = \delta_2 = \dots = \delta_r = 0$  against the alternative that at least one of the coefficients is non-zero.

4. Case 1: invertibility is not rejected. In this case estimate (24) without the leads of  $\varepsilon_t$  to get an estimate of  $\delta$  and estimate the unit-variance shock according to (27). To estimate the corresponding IRFs, apply the standard procedure, i.e. estimate (15) without the lags of  $z_t$  to get  $\hat{\psi}$  and estimate the IRFs according to (26). Estimate the variance decomposition according to equation (28) or (29).
- 4'. Case 2: invertibility is rejected. In this case perform the recoverability test by testing for the null of serial uncorrelation of the fitted value  $\hat{\eta}_t$  by using the Ljung-Box Q-test.

5. Case 1: recoverability is not rejected. In this case estimate the unit-variance shock according to (25) and the corresponding IRFs according to (23). Estimate the variance decomposition according to equation (28) or (29). Historical decomposition can be performed in the standard way.
- 5'. Case 2: recoverability is rejected. In this case, either amend the VAR specification and repeat steps 2-4, or estimate (15) with a maximum lead  $r$  and the ‘relative’ IRFs according to (19). Estimate lower and upper bounds according to (21) and (22) and the corresponding variance contributions according to (31) and (32) or (33) and (34).

Let us stress that the above External-Instrument SVAR procedure is more flexible than the Internal-Instrument method, recently re-proposed in Plagborg-Møller and Wolf (2021, 2022), and the VARX method, proposed in Paul (2020).<sup>15</sup> This is mainly because the sample span of the proxy can be different from the sample span of the VAR (or the VARX). Moreover, with respect to the internal proxy SVAR method, our procedure is more flexible because the number of lags  $m$  and the regressors  $x_t$  used in step 1 of the above procedure are not necessarily equal to the number of lags  $p$  and the regressors  $y_t$  used in the VAR. In particular, if the regression in step 1 is not significant, or the coefficient in the lags of the proxy are not significant, the proxy can be used without treatment ( $m = 0$ ). When including the proxy into the VAR model, this would require to impose restrictions on the VAR parameters. Finally, notice that the number of lags  $r$  used in the regression of the VAR residuals  $\varepsilon_t$  onto the current and lagged proxy can be different from  $p$  – the number of lags used in the VAR. In Simulation 5 below we show that this flexibility may translate into better small-sample estimation performances.

## 5 A simulated economy with fiscal foresight

In this section, we assess the small sample performance of the proposed estimation and testing procedure in two Monte Carlo exercises. The simulations show that the method works and outperforms the Internal-Instrument method in estimating the relative IRFs, conditional on the data generating process studied here. Additional Monte Carlo exercises are provided in Section C of the Online Appendix.

<sup>15</sup>The internal proxy method consists in including the proxy into the VAR model, ordered first, and identifying the shock as the first one in a Cholesky scheme; the VARX method consists in estimating a VARX with the proxy used as the exogenous variable.

## 5.1 Simulation 1: A recoverable shock

In this simulation exercise we use the fiscal foresight model of [Leeper et al. \(2013\)](#) (LWY henceforth). The model is a simple Real Business Cycle model with log preferences, inelastic labor supply and two shocks:  $u_{a,t}$ , a technology shock, and  $u_{\tau,t}$ , a tax shock. A nonstandard feature of the model is the fact that the tax shocks are announced to the agents before being implemented, thus inducing fiscal foresight. The equilibrium capital accumulation is

$$k_t = \alpha k_{t-1} + a_t - \kappa \sum_{i=0}^{\infty} \theta^i E_t \tau_{t+i+1} \quad (35)$$

where  $0 < \alpha < 1$ ,  $0 < \theta < 1$ ,  $\kappa = (1 - \theta)\tau/(1 - \tau)$ ,  $\tau$  being the steady state tax rate,  $0 \leq \tau < 1$ , and  $a_t$ ,  $k_t$  and  $\tau_t$  are the log deviations from the steady state of technology, capital and the tax rate, respectively. Technology and taxes are given by

$$\begin{aligned} a_t &= u_{a,t} \\ \tau_t &= u_{\tau,t-2}, \end{aligned}$$

where  $u_{\tau,t}$  and  $u_{a,t}$  are i.i.d. shocks. Solving for  $k_t$  we obtain the following equilibrium MA representation for capital and taxes:

$$\begin{pmatrix} \tau_t \\ k_t \end{pmatrix} = \begin{pmatrix} L^2 & 0 \\ \frac{-\kappa(L + \theta)}{1 - \alpha L} & \frac{1}{1 - \alpha L} \end{pmatrix} \begin{pmatrix} u_{\tau,t} \\ u_{a,t} \end{pmatrix} = B(L)u_t. \quad (36)$$

The determinant of the above matrix vanishes for  $L = 0$ ; hence the shocks are not fundamental. However, they are recoverable, since the system is square (see Remark 1). In particular, the tax shock is equal to taxes two periods ahead:  $u_{\tau,t} = \tau_{t+2}$ .

We generate 1000 different dataset with 240 time observations from model (36) using the parameterisation in [Leeper et al. \(2013\)](#):  $\alpha = 0.36$ ,  $\theta = 0.2673$  and  $\tau = 0.25$  and  $u_t \sim N(0, I)$ . The proxy is generated according to the equation

$$\tilde{z}_t = u_{\tau,t} + 0.5z_{t-1} + 0.4k_{t-1} - 0.6\tau_{t-1} + v_t,$$

where  $v_t \sim iid \mathcal{N}(0, 1)$ ; here the parameters are arbitrarily chosen. For each dataset, we test for invertibility and recoverability, and estimate the tax shock along with its response functions as explained in the previous section, by setting  $p = m = 2$ ,  $r = 0$  and  $p = m = 2$ ,  $r = 4$ . Invertibility is correctly rejected in all cases. Recoverability is (wrongly) rejected at the 5% level in 10% of the cases, showing that the test is somewhat oversized in small samples

with this DGP. Simulation 4 and 5 in the Online Appendix provide additional information about the empirical size and power of the invertibility and recoverability tests.

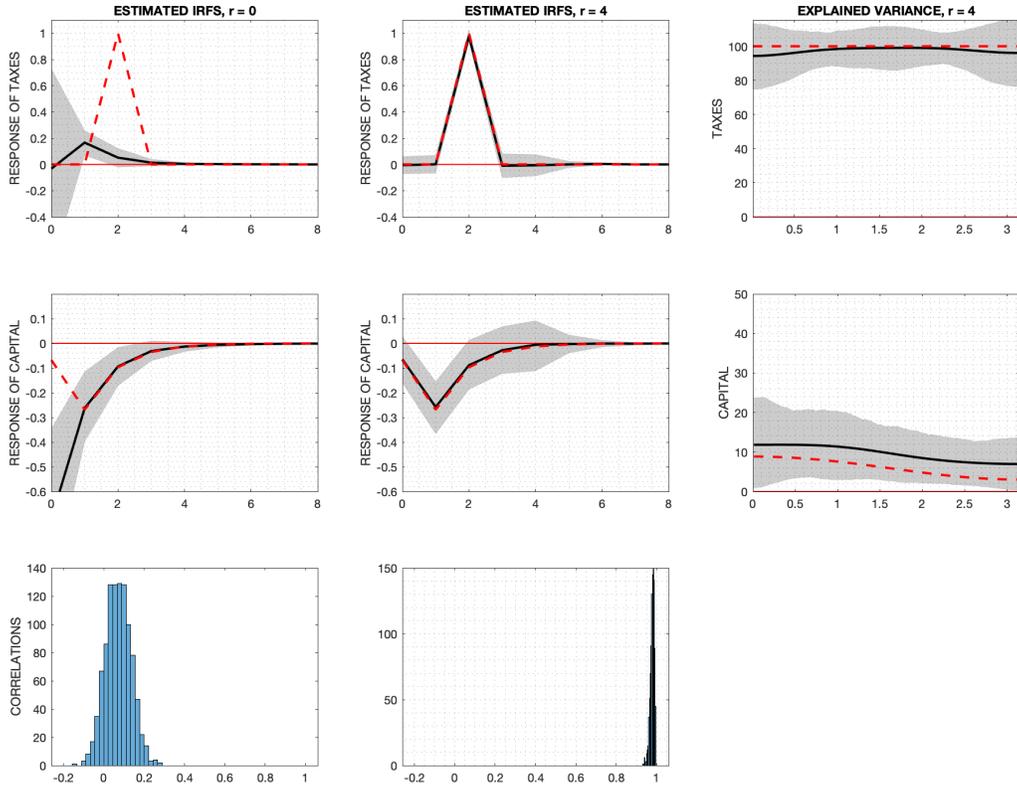


Figure 1: Simulation 1: LWY fiscal foresight model (recoverable shock). Left panels: IRF estimation with  $r = 0$  (standard method). Middle panels: IRF estimation with  $r = 4$ . Right panels: estimation of the variance contribution according to Equation (29), expressed in percentage terms. Upper and middle panels: red dashed line, true response functions and variance decomposition; black solid line, mean of the 1000 estimated response function and variance decompositions; grey area, 16th to 84th percentiles. Lower panels: frequency distribution of the correlation coefficients between the estimated shock and the true shock.

Estimation results are shown in Figure 1. The first column shows the estimates obtained with the standard method ( $r = 0$ ). The upper and middle panels show the estimated response functions: the black solid line is the mean across the 1000 experiments; the grey area shows the estimates between the 16th and the 84th percentiles; the red dashed lines are the true IRFs. The bottom panel shows the frequency distribution of the correlation coefficients between the estimated shock and the true shock. As expected, the estimates are dramatically wrong, since  $r = 0$  requires invertibility, which does not hold in this case. The second column shows the estimates obtained with  $r = 4$ . Here the estimates are very good, both because black and

red lines are almost perfectly overlapping, and because the grey area is not large, particularly for taxes. The correlations coefficients between the estimated shock and the true shock are very close to 1. The third column shows the variance decomposition obtained according to Equation (29), expressed in percentage terms. The estimates are fairly good, albeit somewhat less precise than those of the IRFs.

## 5.2 Simulation 2: A comparison with the Internal-Instrument SVAR

In this second Monte Carlo exercise we compare the small-sample performance of our proposed procedure in estimating the relative IRFs with the one of the Internal-Instrument SVAR approach, using the LWY model of Simulation 1. In particular, we want to see whether the flexibility of the Generalised External-Instrument SVAR translates into smaller estimation errors. The instrument is generated as in the previous exercise. We focus on the relative IRFs of the tax shocks. Since  $b_{i1}(0) = 0$ , the normalisation in Equation (19) cannot be used. Hence we normalise the IRFs by dividing by the effect on the first variable (taxes) at lag 2, i.e. we estimate

$$\mu(L) = \frac{\gamma(L)}{\gamma_{1,2}} = \begin{pmatrix} L^2 \\ -\kappa(L + \theta) \\ 1 - \alpha L \end{pmatrix}.$$

For our procedure, the instrument is preliminarily ‘cleaned’ by setting  $x_t = y_t$  and the number of lags  $m$  according to the BIC. For the Internal-Instrument method, we estimate a VAR for the vector  $(\tilde{z}_t \ y'_t)'$ ; the IRFs are identified as the ones corresponding to the first shock of a Cholesky scheme. As in the previous exercise, we generate 1000 different dataset with 240 time observations. The estimation error of the two competing methods is measured as the sum of the squared errors divided by the sum of the squared coefficients of the true IRFs:

$$100 \times \frac{\sum_{h=1}^n \sum_{k=0}^K (\hat{\mu}_{hk} - \mu_{hk})^2}{\sum_{h=1}^n \sum_{k=0}^K \mu_{hk}^2}. \quad (37)$$

We set  $K = 10$  and  $n = 2$ . This ratio is equal to 100 for the flat estimate  $\hat{\mu}(L) = 0$ .

Results are shown in Table 1. The first column report results for the Internal-Instrument method. The second column reports results obtained with our procedure, with  $r$  selected according to the BIC, applied to Equation (17). Columns 3-7 report results obtained by setting arbitrarily  $r = 3, 4, 5, 6, 7$ . Boldface numbers are the best estimates of each column.

Let us focus first on columns 3-7. Clearly,  $r = 3$  is the best choice for our DGP. Column 2 shows that the BIC criterion is successful in detecting the correct value of  $r$ , since the errors are very close to those of the best column (indeed, using just one decimal figure, we cannot see any difference).

VAR order	Internal IV	External IV					
		$r = BIC$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$
$p = 1$	410.8	<b>4.3</b>	<b>4.3</b>	<b>5.0</b>	<b>6.4</b>	<b>7.9</b>	<b>9.4</b>
$p = 2$	34.8	5.2	5.2	5.9	<b>6.4</b>	<b>7.9</b>	<b>9.4</b>
$p = 3$	<b>7.6</b>	6.0	6.0	6.8	7.3	8.0	9.5
$p = 4$	9.5	7.1	7.1	7.8	8.4	9.1	9.6
$p = 5$	11.2	8.0	8.0	8.8	9.3	10.0	10.6
$p = 6$	12.9	8.9	8.9	9.6	10.2	10.9	11.5
$p = BIC$	<b>7.6</b>	<b>4.3</b>					

Table 1: Results of Simulation 2. Estimation errors for the relative IRFs, measured according to (37), for the internal instrument SVAR (first column) and our proposed procedure, with  $r$  determined by the BIC (second column) and  $r = 3, 4, 5, 6, 7$  (columns 3-7). Boldface numbers are the best results obtained for each column.

The Generalised External-Instrument SVAR performs better than the Internal-Instrument SVAR for all values of  $p$  (small values of  $p$  seem particularly effective with our method). When using the BIC for the VAR order  $p$  (last row), our method performs better than the internal proxy SVAR and the relative improvement is large (about 40%).

Our interpretation is the following. We have three dynamic relations: the one used to clean the proxy  $\tilde{z}_t$ , the VAR for  $y_t$  and the relation linking the VAR residuals and the cleaned proxy  $z_t$ . Such relations may have different optimal orders. With the External-Instrument method we can set different values for such orders ( $m$ ,  $r$  and  $p$ ): a flexibility that the Internal-Instrument method does not have.

A similar Monte Carlo exercise with a different DGP, but similar results, is provided in Section C of the Online Appendix (see Simulation 6). A more extensive comparison between our proposed procedure and existing alternatives would be interesting but is beyond the aim of the present paper.

## 6 The effects of monetary policy shocks

In this section, we provide an empirical application of our method and study the propagation of monetary policy shocks. We first show that the monetary policy shocks identified with standard high-frequency surprises at the short end of the yield curve are likely to be nonfundamental but recoverable, in a few routinely used VAR specifications. We then show that the standard Internal-Instrument SVAR procedure delivers price and output puzzles. Conversely, when using our suggested procedure for noninvertible shocks, we get results in line with the textbook effects of monetary policy.

## 6.1 Data, VAR specification, instruments

Our baseline VAR specification includes three variables at monthly frequency: the 1-year government bond rate (1YB), industrial production (IP) in growth rates and CPI inflation (Specification I). We also present results for two additional specifications, one including [Gilchrist and Zakrajšek \(2012\)](#)'s excess bond premium (EBP) (Specification II); the other including EBP along with the mortgage spread (MS) and the commercial paper spread (CPS) (Specification III). We use 1YB as the policy indicator variable.<sup>16</sup>

Our benchmark sample spans the period 1983:1-2008:12. In two robustness exercises, we consider alternative initial dates, i.e. 1979:7, 1987:8, and 1990:1, as well as two alternative ending dates, i.e. 2012:6 and 2019:6.<sup>17</sup> The trending variables, CPI and IP, are taken in differences since these variables are unlikely to be cointegrated. In a robustness exercise, we consider a VAR specification with all the trending variables in levels.

The instrument for monetary policy shocks consists of the [Gürkaynak et al. \(2005\)](#)'s intra-daily monetary policy surprises triggered by Federal Open Market Committee (FOMC) decisions in the three month ahead monthly Fed Funds futures (FF4), as proposed in [Gertler and Karadi \(2015\)](#) (GK from now on). The use of this instrument provides scope for testing our approach to noninvertibility since, as discussed in [Gertler and Karadi \(2015\)](#) and [Ramey \(2016\)](#), surprises in futures with a three month maturity are likely to capture both conventional monetary policy shocks, and shocks to forward guidance about the path rate at short horizon. We 'clean' the instrument by regressing it onto its own lags and the lags of the three variables of Specification I, using 6 lags.<sup>18</sup> The instrument turns out to be relevant, with a measure of relevance  $\widehat{IR}$  between 0.4 and 0.6 depending on the specification adopted.

## 6.2 Fundamentalness and recoverability

We start our analysis by applying our fundamentalness test (Table 2a) to verify whether our specifications turns out to be fundamental or not when using our instrument GK. The main takeaway is that the results obtained with the standard proxy-SVAR approach in the monetary

<sup>16</sup>We use monthly data taken from the FRED-MD data set of [McCracken and Ng \(2015\)](#). Specifically, we use industrial production (FRED mnemonic INDPRO, IP from now on), taken in log differences, the CPI index (FRED mnemonic CPIAUCSL), taken in log differences, and the 1-year government bond rate (FRED mnemonic GS1, 1YB from now on). In addition, we use the excess bond premium (EBP), the mortgage spread (MS) and the commercial paper spread (CPS) taken from the replication files of [Gertler and Karadi \(2015\)](#).

<sup>17</sup>The samples starting in 1983:1 and 1987:8 are chosen in line with [Sims and Zha \(2006\)](#). Moreover, 1979:7 is the beginning of Volcker's mandate; 1987:8 is the beginning of Greenspan's mandate; 2008:12 is the first month in which the 1-year bond rate falls below 1%, so that cutting our sample to 2008:12 excludes the zero lower bound period.

<sup>18</sup>The regression is significant at the 5% level and the residual is serially uncorrelated according to the Ljung-Box Q-test.

	Number of leads $r$						Number of leads $r$						
	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$	$r = 9$	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$	$r = 9$	
<i>Specification I</i>							<i>Specification I</i>						
$p = 6$	0.008	0.028	0.002	0.003	0.001	0.001	$p = 6$	0.619	0.662	0.251	0.469	0.037	0.060
$p = 9$	0.016	0.051	0.003	0.003	0.002	0.001	$p = 9$	0.350	0.571	0.114	0.435	0.050	0.042
$p = 12$	0.011	0.045	0.003	0.002	0.001	0.000	$p = 12$	0.880	0.944	0.324	0.820	0.466	0.285
<i>Specification II</i>							<i>Specification II</i>						
$p = 6$	0.080	0.195	0.027	0.001	0.000	0.000	$p = 6$	0.441	0.473	0.308	0.777	0.394	0.357
$p = 9$	0.180	0.351	0.034	0.002	0.000	0.000	$p = 9$	0.119	0.186	0.104	0.517	0.222	0.193
$p = 12$	0.221	0.457	0.059	0.003	0.000	0.000	$p = 12$	0.472	0.558	0.269	0.913	0.701	0.575
<i>Specification III</i>							<i>Specification III</i>						
$p = 6$	0.060	0.184	0.089	0.003	0.001	0.002	$p = 6$	0.034	0.315	0.446	0.608	0.738	0.546
$p = 9$	0.184	0.362	0.220	0.020	0.002	0.003	$p = 9$	0.005	0.064	0.148	0.046	0.391	0.103
$p = 12$	0.215	0.353	0.250	0.060	0.031	0.027	$p = 12$	0.032	0.037	0.065	0.057	0.343	0.022

(a) Fundamentalness test

(b) Recoverability test

Table 2:  $P$ -values for the invertibility (a) and the recoverability tests (b), for different values of  $p$  and  $r$ . Specification I includes the 1-year bond rate (1YB, industrial production growth (IP) and CPI inflation (CPI). Specification II includes 1YB, IP, CPI and the excess bond premium (EBP). Specification III includes 1YB, IP, CPI, EBP, the mortgage spread and the commercial paper spread. The proxy is the one of [Gertler and Karadi \(2015\)](#).

policy literature should be taken with caution, since the VAR specification might be affected by nonfundamentalness. In fact, results in Table 2a show that, for  $r \geq 6$ , fundamentalness is rejected at the 1% level with Specifications I and II and for  $r \geq 7$  is rejected either at the 5% level or the 10% level with Specification III. The degree of fundamentalness  $\widehat{R}_f^2$  is below 0.5 for all specification for  $r \geq 6$ . We conclude that the inclusion of financial variables in the VAR may not be sufficient to solve fundamentalness problems. These results are in line with the findings of [Plagborg-Møller and Wolf \(2022\)](#) and the arguments in [Ramey \(2016\)](#), who cautions against the standard SVAR-IV approach.

Yet, the good news is that the shock is recoverable. The  $p$ -values of the Ljung-Box Q-test for serial correlation of the estimated monetary policy shock, for our three specifications, with different values of  $p$  and  $r$  (maximum lag 24) are reported in Table 2b. The result is that recoverability cannot be rejected at the 5% level for most parameter configurations. We conclude that, at least for our time span, the monetary policy shock is recoverable, even with the three-variable Specification I, and that financial variables are not needed to find the policy shock.

### 6.3 The three-variable VAR

In this subsection we compare the impulse response functions obtained with the standard method with those obtained with our proposed method. We choose Specification I with the

GK instrument as our benchmark. We set the number of lags in the VAR equal to 12 ( $p = 12$ ) and the number of leads of the VAR residuals included in the regression of the proxy equal to 6 ( $r = 6$ ). In a robustness exercise, we try different values of  $p$  and  $r$ .<sup>19</sup>

The basic insight delivered by our exercise is that by incorrectly assuming invertibility without testing one can get dramatically misleading results. On the contrary, when the proposed procedure is applied, the estimation delivers results in line with textbook effects of monetary policy, even with a small VAR, and not including the EBP or other financial variables.

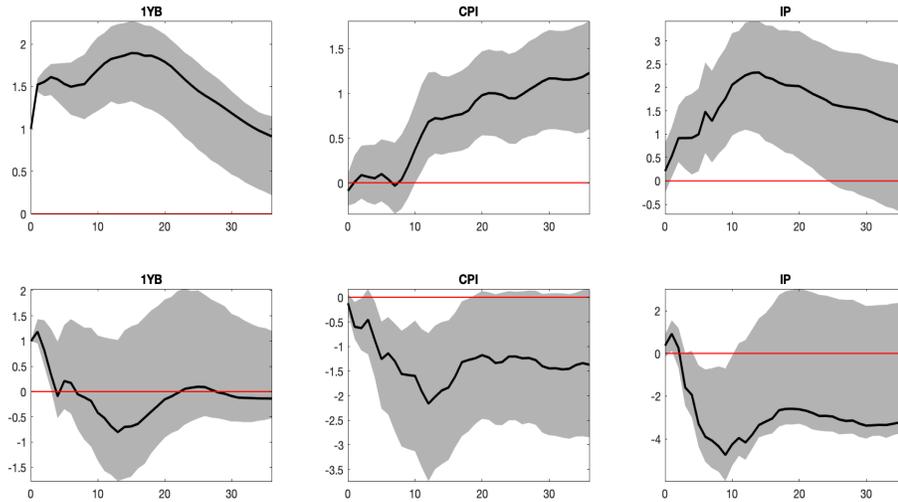


Figure 2: VAR results: Specification I,  $p = 12$ , GK instrument. Top panels: estimated response functions with  $r = 0$  (standard method). Bottom panels: estimated response functions with our proposed method  $r = 6$ . Black line: point estimate. Grey area: 68% confidence bands.

The results from the baseline model are reported in Figure 2. All responses are normalised to have an impact effect of 100 basis points on the 1-year bond rate. The top panels show the estimated impulse response functions obtained with  $r = 0$ , i.e. the standard proxy SVAR procedure. The response of 1YB is hump-shaped and very persistent (the zero line is not reached after 3 years). Both prices and industrial production significantly increase after a tightening shock, so that we have both a large price puzzle and a large real activity puzzle.

The bottom panels show the result obtained with our proposed procedure with  $r = 6$ . Results are completely different and much more plausible. The reaction of the policy variable is much less persistent: after the significant impact effect it reaches the zero line in about 4 months and further on it is no longer significant. Both inflation and output puzzles disappear:

<sup>19</sup>For simplicity, here we do not make any attempt at cleaning the instrument from the information effects recently discussed in the literature (see, for example, [Jarociński and Karadi, 2020](#) and [Miranda-Agrippino and Ricco, 2021](#)). This is left for future research.

prices and industrial production reduce significantly after an impact effect which is very close to zero. The effects on real activity are no longer significant after about one year, showing that the effects of monetary policy on real activity are transitory, in line with the consensus. In Section D of the Online Appendix, we report several robustness checks. The overall conclusion is that the above results are reasonably robust.

## 6.4 Medium-size VAR specifications

Are results sensitive to the VAR information set? To answer this question, in this subsection we examine results for Specification II, that includes EBP, and Specification III, that incorporates the mortgage spread and the commercial paper spread, as well as EBP. We set  $p = 12$ ,  $r = 0$  and  $r = 6$ , as in the previous subsection. The main conclusion of this exercise is that results obtained with our proposed method are reasonably robust, whereas results obtained with the standard method are not.

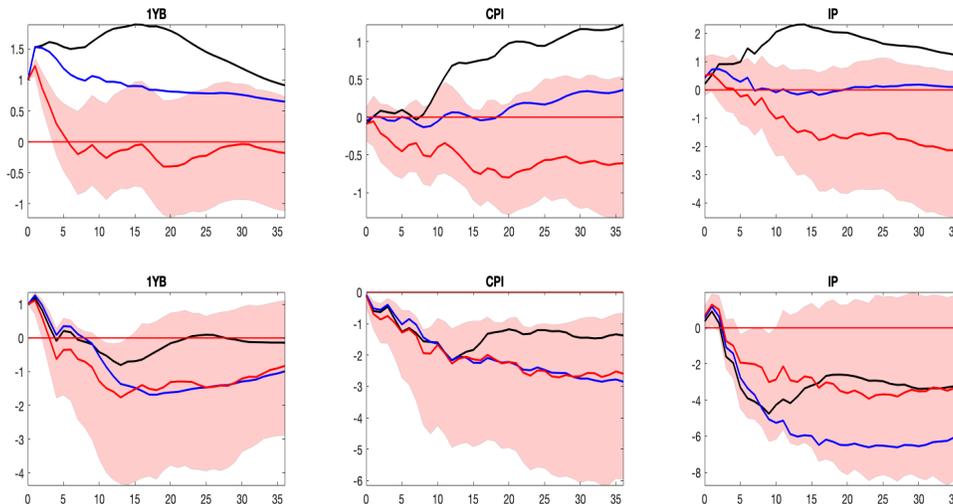


Figure 3: IRFs for Specification III (1YB, CPI inflation, IP growth, EBP, MS CPS). The instrument is GK. Red line: point estimates; blue line: point estimates for Specification II; black line: point estimates for Specification I. Top panels: estimated response functions with  $p = 12$ ,  $r = 0$  (standard method). Bottom panels: estimated response functions with our proposed method,  $p = 12$ ,  $r = 6$ . Pink shaded area: 68% confidence bands for Specification III.

In the top panels, Figure 3 reports the case  $r = 0$  (standard method). The lines are respectively the point estimates for Specification I (black), Specification II (blue), and Specification III (red). We also report the confidence bands of Specification III (pink shaded areas). Results appear to be very sensitive to the set of variables included in the VAR. With Specification II, the effects on prices and real activity are essentially zero. With Specification

III, the sign of the IRFs of prices and industrial production are negative, and the puzzles of Specification I disappear. Still, the effects are quantitatively small, especially for prices, and not significant.

The IRFs obtained with the Generalised External-Instrument approach ( $r = 6$ ) of the first three variables are similar to those obtained with Specifications I and II (bottom panels in Figure 3). The reaction of prices is large and significant. The reaction of IP is barely significant, since the confidence bands are very large, however the point estimates consistently show a sizeable reduction of about 3-5 percentage points of production after one year, depending on the VAR specification.

## 6.5 Variance decomposition

Do monetary policy shocks account for a sizeable share of the variance of prices and output? To answer this question, it is useful to evaluate the variance decomposition VD of CPI inflation and industrial production growth obtained with our VAR specifications ( $p = 12$ ,  $r = 6$ ). We report results for waves of periodicity 2-18 months (short run), 18-96 month (business cycle) and 2+ months (overall variance). The main finding is that the effects of monetary policy on prices are much larger than previously reported, suggesting that it can be used successfully in controlling inflation.

Table 3 reports the point estimates and the 68% confidence bands (in brackets) for the percentage of variance explained by the monetary policy shock. The estimates for the cyclical variance (18-96 months) are not very reliable because of the large confidence bands, so that we focus mainly on the short-run and overall variances. The point estimates of the short-run volatility contributions range from 12.3% to 19.2% for inflation and from 16.1% to 27.7% for industrial production growth. As for the overall variance, the estimates range from 12.5% to 20.8% for inflation and from 13.0% to 28.3% for industrial production.

With Specification III, which provides the smallest estimates, the monetary policy shocks explains 12.5% of the overall variance of inflation and 13% of the overall variance of production, with the 68% confidence bands ranging between a minimum of around 10% and a maximum of around 20% for both variables. We conclude that, contrary to previous findings, the effects of discretionary monetary policy on inflation are far from negligible. These results are at odds with the ones in [Plagborg-Møller and Wolf \(2022\)](#), where, according to FVR estimates, the contribution of policy shocks to inflation fluctuations is negligible at all horizons between 0 and 24 months.

To understand the sources of the difference, we compute the point estimates of the FVR of inflation, reported in Table 4. In the lower part of the table, we also report results for CPI in

	Waves of periodicity		
	2 – 18 months	18 – 96 months	2+ months
<i>Specification I</i>			
CPI inflation	19.2 (13.5—29.1)	27.6 (12.8—64.2)	20.8 (16.2—35.1)
IP growth	27.7 (19.1—36.4)	33.8 (13.1—55.4)	28.3 (20.0—37.6)
<i>Specification II</i>			
CPI inflation	12.3 (10.4—23.1)	12.9 (9.7—45.1)	13.2 (13.4—26.8)
IP growth	20.3 (15.8—28.2)	29.5 (11.4—51.5)	22.5 (16.7—31.3)
<i>Specification III</i>			
CPI inflation	12.5 (10.2—19.5)	10.3 (6.9—34.2)	12.5 (11.2—21.5)
IP growth	16.1 (12.2—22.2)	5.2 (4.2—22.0)	13.0 (11.2—20.7)

Table 3: Percentage of variance accounted for by the monetary policy shock, for waves of periodicity 2-18 months (short run), 18-96 months (business cycle), 2+ months (overall variance). 68% confidence bands in brackets.

	FVR Horizon					VD
	impact	3 months	6 months	12 months	24 months	2+ months
<i>CPI inflation</i>						
Specification I	0.5	7.2	15.3	18.4	20.7	20.8
Specification II	0.2	4.7	9.1	13.3	13.4	13.2
Specification III	0.3	5.6	7.4	12.5	12.4	12.5
<i>CPI index in levels</i>						
Specification I	0.5	4.2	9.9	20.0	21.5	
Specification II	0.2	2.6	5.3	13.7	22.5	
Specification III	0.3	4.4	7.1	13.8	18.5	

Table 4: Percentage of variance of CPI inflation and prices accounted for by the monetary policy shock, according to the FVR measure of [Plagborg-Møller and Wolf \(2022\)](#), on impact and at 3, 6, 12, 24 months horizons.

levels,<sup>20</sup> as is common practice in the literature. At short horizons, the variance contributions are small. As argued in Subsection 3.3, these numbers should be taken with caution because of the downward bias. By contrast, the estimate of the FVR at the 24-month horizon is reliable, since we see in the upper part of the table that the numbers are almost identical

<sup>20</sup>This is obtained simply by taking the cumulated sums of the IRFs.

Time span	VD: waves of periodicity			FVR: horizon
	2 – 18 months	18 – 96 months	2+ months	24 months
1983:1–2008:12	10.4	22.0	16.1	15.5
1990:1–2012:6	6.3	15.5	8.0	8.1
1987:1–2008:12	7.3	15.4	11.3	10.6
1983:1–2012:6	10.0	24.6	12.7	12.8
1979:7–2012:6	17.2	19.3	17.4	17.5
1979:7–2019:6*	15.7	18.2	15.3	15.1

Table 5: Variance decomposition of inflation for different time spans, Specification IV: FFR, CPI inflation, IP growth, EBP. VD: percentage of inflation variance accounted for by the monetary policy shock, for waves of periodicity 2-18 months (short run), 18-96 months (business cycle), 2+ months (overall variance). FVR: percentage of forecast error variance of inflation accounted for by the monetary policy shock at the 2-year horizon. For the sample 1979:7–2019:6 in place of the EBP series we use three financial variables: the 10-year treasury bond rate, the BAA corporate bond yield and the S&P500 stock price index.

to those of the overall VD, reported in the last column for convenience. This means that at horizon 24 all of the IRFs of inflations are already close to zero and the bias has disappeared. We conclude that the use of FVR in place of VD cannot explain the inconsistency of estimates. Coming to the bottom part of the table and focusing on the 2-year horizon, we see that the variance contribution of monetary policy to the forecast error of prices is even larger when considering the price index taken in levels.

Another potential source of differences in the empirical estimates is the policy indicator. Following [Plagborg-Møller and Wolf \(2022\)](#), we consider a model (Specification IV) incorporating the same variables of Specification II but with the federal funds rate (FFR) in place of 1YB. In addition, we set  $p = 6$ , as in [Plagborg-Møller and Wolf \(2022\)](#) (instead of  $p = 12$ ). We retain  $m = 6$  and set  $x_t = y_t$  for the preliminary treatment of the instrument. We compute the point estimates of VD and FVR at horizon 24 for different time spans (Table 5). The first row reports the estimates for the full sample, which are somewhat larger than the ones obtained for Specification II with  $p = 12$ . We conclude that the number of lags used in VAR estimation and the use of FFR in place of 1YB cannot explain the difference.

In the second row, we report the estimates for the time span 1990:1–2012:6, the same used in [Plagborg-Møller and Wolf \(2022\)](#). The explained variances for this sample are sizeably smaller than the ones of our time span: the overall VD is 8.0% as against 16.1%. This points to the fact that the different time spans explain part of the discrepancy. The remaining difference can only be due to the estimation methods.

The span of the GK instrument, 1990:1–2012:6, by excluding the 80’s and including the first years of the zero-lower-bound period, exhibits little variation of both inflation and interest

rates, which could be detrimental to the reliability of the estimates. To verify how different spans affect the estimates, we report in the bottom part of the table the results for four additional time spans: 1987:1–2008:12, 1983:1–2012:6, 1979:7–2012:6 (the same of [Gertler and Karadi, 2015](#)) and 1979:7–2019:6.<sup>21</sup> Notice that, for these time spans (as well as our benchmark 1983:1–2008:12) the Internal-Instrument method cannot be used, at least with the GK instrument. This is a nice illustration of the advantages of our proposed method. Despite results vary considerably across different samples, the overall picture emerging from Table 5 confirms our main finding: discretionary monetary policy has non-negligible effects on prices.

## 7 Concluding remarks

In this paper we propose a new estimation procedure for structural VARs with an external instrument. The procedure includes a test for invertibility and a test for recoverability, a method to estimate the relative impulse response functions when the shock is not recoverable and a method to estimate the absolute response functions and the shock itself when the shock is recoverable but not invertible. The procedure reduces to the standard method when the shock is invertible. Results reported in this paper indicate that all procedures work remarkably well under simulation, when the sample size is comparable with those typical of macroeconomic empirical analyses.

An application to monetary policy shocks, using the instrument of [Gertler and Karadi, 2015](#), indicates that the policy shocks are not invertible in a few popular monetary policy VAR specifications. While the standard method produces puzzling results, our procedure delivers results in line with textbook effects. Finally, we find that the policy shock is recoverable, so that we can estimate its variance contributions. Variance decomposition shows that monetary policy has sizeable effects on both real activity and inflation, suggesting that monetary policy can be effective in controlling prices.

## References

- Angeletos, George-Marios, Fabrice Collard, and Harris Dellas**, “Business-Cycle Anatomy,” *American Economic Review*, October 2020, *110* (10), 3030–70.
- Bernanke, Ben S. and Kenneth N. Kuttner**, “What Explains the Stock Market’s Reaction to Federal Reserve Policy?,” *Journal of Finance*, June 2005, *60* (3), 1221–1257.

<sup>21</sup>For the latter time span in place of the EBP series, which is not available, we use three financial variables: the 10-year treasury bond rate, the BAA corporate bond yield and the S&P500 stock price index.

- Blanchard, Olivier J., Jean-Paul L’Huillier, and Guido Lorenzoni**, “News, Noise, and Fluctuations: An Empirical Exploration,” *American Economic Review*, December 2013, *103* (7), 3045–70.
- Chahrour, Ryan and Kyle Jurado**, “Recoverability and Expectations-Driven Fluctuations,” *The Review of Economic Studies*, 03 2021, *89* (1), 214–239.
- Fernandez-Villaverde, Jesus, Juan F. Rubio-Ramirez, Thomas J. Sargent, and Mark W. Watson**, “ABCs (and Ds) of Understanding VARs,” *American Economic Review*, June 2007, *97* (3), 1021–1026.
- Forni, Mario and Luca Gambetti**, “Sufficient information in structural VARs,” *Journal of Monetary Economics*, 2014, *66* (C), 124–136.
- , – , and **Luca Sala**, “No News in Business Cycles,” *Economic Journal*, December 2014, *124* (581), 1168–1191.
- , – , and – , “Structural VARs and noninvertible macroeconomic models,” *Journal of Applied Econometrics*, 2019, *34* (2), 221–246.
- , – , **Marco Lippi, and Luca Sala**, “Noisy News in Business Cycles,” *American Economic Journal: Macroeconomics*, October 2017, *9* (4), 122–152.
- , – , – , and – , “Common Component Structural VARs,” CEPR Discussion Papers 15529, C.E.P.R. Discussion Papers December 2020.
- Gertler, Mark and Peter Karadi**, “Monetary Policy Surprises, Credit Costs, and Economic Activity,” *American Economic Journal: Macroeconomics*, 2015, *7* (1), 44–76.
- Giannone, Domenico and Lucrezia Reichlin**, “Does information help recovering structural shocks from past observations?,” *Journal of the European Economic Association*, 04-05 2006, *4* (2-3), 455–465.
- Gilchrist, Simon and Egon Zakrajšek**, “Credit Spreads and Business Cycle Fluctuations,” *American Economic Review*, 2012, *102* (4), 1692–1720.
- Gürkaynak, Refet S, Brian Sack, and Eric Swanson**, “Do Actions Speak Louder Than Words? The Response of Asset Prices to Monetary Policy Actions and Statements,” *International Journal of Central Banking*, May 2005, *1* (1).
- Jarociński, Marek and Peter Karadi**, “Deconstructing Monetary Policy Surprises– The Role of Information Shocks,” *American Economic Journal: Macroeconomics*, April 2020, *12* (2), 1–43.
- Jordà, Òscar**, “Estimation and Inference of Impulse Responses by Local Projections,” *American Economic Review*, March 2005, *95* (1), 161–182.
- Leeper, Eric M., Todd B. Walker, and Shu-Chun Susan Yang**, “Fiscal Foresight and Information Flows,” *Econometrica*, May 2013, *81* (3), 1115–1145.

- Lippi, Marco and Lucrezia Reichlin**, “The Dynamic Effects of Aggregate Demand and Supply Disturbances: Comment,” *American Economic Review*, June 1993, 83 (3), 644–652.
- **and** – , “VAR analysis, nonfundamental representations, Blaschke matrices,” *Journal of Econometrics*, July 1994, 63 (1), 307–325.
- McCracken, Michael W. and Serena Ng**, “FRED-MD: A Monthly Database for Macroeconomic Research,” Working Papers 2015-12, Federal Reserve Bank of St. Louis June 2015.
- Mertens, Karel and José Luis Montiel-Olea**, “Marginal Tax Rates and Income: New Time Series Evidence,” *The Quarterly Journal of Economics*, 2018, 133 (4), 1803–1884.
- **and Morten O. Ravn**, “Measuring the Impact of Fiscal Policy in the Face of Anticipation: A Structural VAR Approach\*,” *The Economic Journal*, 2010, 120 (544), 393–413.
- **and** – , “The Dynamic Effects of Personal and Corporate Income Tax Changes in the United States,” *American Economic Review*, June 2013, 103 (4), 1212–47.
- **and** – , “A reconciliation of SVAR and narrative estimates of tax multipliers,” *Journal of Monetary Economics*, 2014, 68, S1–S19. Supplement issue: October 19-20, 2012 Research Conference on “Financial Markets, Financial Policy, and Macroeconomic Activity” Sponsored by the Study Center Gerzensee and the Swiss National Bank.
- Miranda-Agrippino, Silvia and Giovanni Ricco**, “The Transmission of Monetary Policy Shocks,” *American Economic Journal: Macroeconomics*, July 2021, 13 (3), 74–107.
- **and** – , “Identification with External Instruments in Structural VARs,” *Journal of Monetary Economics*, forthcoming.
- Paul, Pascal**, “The Time-Varying Effect of Monetary Policy on Asset Prices,” *The Review of Economics and Statistics*, 2020, 102 (4), 690–704.
- Plagborg-Møller, Mikkel and Christian K. Wolf**, “Local Projections and VARs Estimate the Same Impulse Responses,” *Econometrica*, 2021, 89 (2), 955–980.
- **and** – , “Instrumental Variable Identification of Dynamic Variance Decompositions,” *Journal of Political Economy*, 2022, 130 (8), 2164–2202.
- Ramey, Valerie A.**, “Identifying Government Spending Shocks: It’s all in the Timing,” *The Quarterly Journal of Economics*, 2011, 126 (1), 1–50.
- , “Macroeconomic Shocks and Their Propagation,” in John B. Taylor and Harald Uhlig, eds., *Handbook of Macroeconomics*, Vol. 2 of *Handbook of Macroeconomics*, Elsevier, 2016, chapter 2, pp. 71 – 162.
- Sims, Christopher A. and Tao Zha**, “Were There Regime Switches in U.S. Monetary Policy?,” *American Economic Review*, March 2006, 96 (1), 54–81.

**Stock, James H.**, “What’s New in Econometrics: Time Series, Lecture 7,” NBER Summer Institute, Short course lectures 2008.

– **and Mark W. Watson**, “Disentangling the Channels of the 2007-09 Recession,” *Brookings Papers on Economic Activity*, 2012, 44 (1 (Spring), 81–156.

– **and** – , “Identification and Estimation of Dynamic Causal Effects in Macroeconomics Using External Instruments,” *The Economic Journal*, 2018, 128 (610), 917–948.



CREST  
Center for Research in Economics and Statistics  
UMR 9194

5 Avenue Henry Le Chatelier  
TSA 96642  
91764 Palaiseau Cedex  
FRANCE

Phone: +33 (0)1 70 26 67 00

Email: [info@crest.science](mailto:info@crest.science)

<https://crest.science/>

The Center for Research in Economics and Statistics (CREST) is a leading French scientific institution for advanced research on quantitative methods applied to the social sciences.

CREST is a joint interdisciplinary unit of research and faculty members of CNRS, ENSAE Paris, ENSAI and the Economics Department of Ecole Polytechnique. Its activities are located physically in the ENSAE Paris building on the Palaiseau campus of Institut Polytechnique de Paris and secondarily on the Ker-Lann campus of ENSAI Rennes.

