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Dynamic assignment without money:
Optimality of spot mechanisms

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Abstract

We study a large market model of dynamic matching with no monetary transfers and a continuum of agents. Time is discrete and horizon finite. Agents are in the market from the first date and, at each date, have to be assigned items (or bundles of items). When the social planner can only elicit ordinal preferences of agents over the sequences of items, we prove that, under a mild regularity assumption, incentive compatible and ordinally efficient allocation rules coincide with spot mechanisms. A spot mechanism specifies "virtual prices" for items at each date and, at the beginning of time, for each agent, randomly selects a budget of virtual money according to a (potentially non-uniform) distribution over $[0, 1]$. Then, at each date, the agent is allocated the item of his choice among the affordable ones. Spot mechanisms impose a linear structure on prices and, perhaps surprisingly, our result shows that this linear structure is what is needed when one requires incentive compatibility and ordinal efficiency. When the social planner can elicit cardinal preferences, we prove that, under a similar regularity assumption, incentive compatible and Pareto efficient mechanisms coincide with a class of mechanisms we call *Spot Menu of Random Budgets* mechanisms. These mechanisms are similar to spot mechanisms except that, at the beginning of the time, each agent must pick a distribution in a menu. This distribution is used to initially draw the agent's budget of virtual money.

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1 Introduction

In many contexts, agents have to be assigned streams of items when no monetary transfers are allowed. Examples include the assignment of civil servants—such as teachers—to positions along their career trajectories, the allocation of courses to students from semester-to-semester, spaces in college dorms during university years, organs to hospitals waiting for transplants for their sick patients, etc. However, the literature does not provide much guideline on how to design allocation rules in these dynamic contexts.[1] While the class of possible allocation rules can potentially be quite large, in this paper, we show how efficiency and incentive compatibility requirements narrow down the class of mechanisms to fairly simple assignment rules which conform well with prevalent practices.

Typically in the aforementioned situatuions a real-money market is not allowed, so a fake-money market is a natural option.[2] In practice, agents are often given a budget of token money and at regular intervals of time they can spend their tokens on items with high price or buy cheap items and save tokens for future use. Hence, the assignment proceeds simply by having a sequence of *spot markets*.

One example is provided by Columbia Business School (CBS) for course allocation. In CBS, lifetime tokens are given up-front and carry over from semester-to-semester.[3] A student can spend his budget of token money equally in each semester, spend most of it on courses in the first semester, or save most of them for future use.[4] Prices on courses are set to clear supply and demand for each course. Eventually, the price for a stream of courses simply corresponds to the sum of prices of each course in the stream. Another example is the assignment of teachers to public schools as done in France.[5] Teachers are initially endowed a budget of tokens which depends on their characteristics and is used all along their career.[6] Each year, each teacher can decide to use his budget to transfer to another school, i.e., to "buy" a position in another school. They can use their budget to buy a position in overdemanded schools if they can afford it. For some specific underdemanded schools, mainly disadvantaged schools, prices are actually negative, i.e., these teachers would get a bonus of tokens if they go in these schools (for several years). They could then accumulate more tokens to get assigned schools

[1]As discussed in the related literature section below, there has been a number of attempts to define optimal mechanisms in these dynamic contexts. Most of these papers rely on repeated games structures where preferences are drawn i.i.d. over time and are separable. This rules out many of the applications we have in mind in this paper.

[2]In static single-unit-or multi-unit demand matching environments, see, for instance, Hylland and Zeckhauser (1979), Budish (2011), Budish, Cachon, Kessler, and Othman (2017) or He, Miralles, Pycia, and Yan (2018).

[3]The Wharton School of Business uses a bidding system for courses as well. However, the mechanism used is different: unused budgets from one semester do not carry over to subsequent semesters (see Budish, Cachon, Kessler, and Othman, 2017).

[4]A full description is given in the "Guide to Bidding" of CBS.

[5]See Combe, Tercieux, and Terrier (2016) for institutional details on the french teacher assignment scheme.

[6]For instance, the initial budget of a teacher depends on characteristics such as his number of kids, social situation or medical condition.

that they desire in the future. Here again, the price of a stream of schools along the career trajectory of a teacher is simply the sum of the prices of each school.[7] Thus, spot markets, by construction, have a special linear pricing structure.

One can imagine many other allocation rules. For instance, upon arriving, one could ask an agent his preferences over streams of items and, given the reported preferences, allocate the agent a sequence of items from then on. Indeed, in the context of course allocation, based on students (reported) preferences, a university could decide every year to use an allocation rule to assign students to sequences of courses over the full year spanning several semesters. Similarly, teachers who just graduated could be proposed sequences of schools over the following years. In a context with token money, one could price directly these streams of items. Since this approach does not impose any linear structure on prices, it may be more permissive than using spot markets, i.e., the allocation rules obtained in that way may not be obtained through spot markets.

In this paper, we use a large matching market setting with a continuum of agents introduced by Ashlagi and Shi (2016). However, we study a dynamic market where agents are assigned items sequentially while Ashlagi and Shi (2016) consider static environments. In our framework, agents are present from date 1 through T (the finite horizon) and, at each of these dates, they have to be assigned items which perish at the end of the current period. We first consider the case where the mechanism designer can only elicit ordinal preferences over the sequences of items. We show that, under a mild regularity assumption, the class of incentive compatible and ordinally efficient allocation rules coincides with the class of spot mechanisms.[8] A spot mechanism works as follows. It specifies "virtual prices" for items at each date. At the beginning of time, for each agent, it randomly selects a budget of virtual money according to a distribution over $[0, 1]$. Then, at each date, an item is affordable for this agent if her remaining budget is above the virtual price for this item. At this date, the agent is allocated the item of his choice among affordable ones. The agent pays the price of the assigned item and the budget is adjusted accordingly by the end of the period. Together with our prior observation that spot mechanisms impose a linear structure on prices, our result shows, perhaps surprisingly, that this linear structure is what is needed when one requires incentive compatibility and ordinal efficiency.

We then consider the case where the mechanism designer can elicit cardinal preferences. Under a similar regularity assumption, we show a corresponding result: the class of incentive compatible and Pareto efficient mechanisms coincides with a class of mechanisms that we call *Spot Menu of Random Budget (MRB) mechanisms*. A spot MRB mechanism is similar to a

[7] Dynamic assignment schemes with point systems can also be found in other applications. For instance, to incentivize voluntary participation by hospitals in kidney exchange platforms, point systems rewarding hospitals based on their marginal contribution to the platform have been recently adopted by the National Kidney Registry kidney exchange platform (see Agarwal, Ashlagi, Azevedo, Featherstone, and Karaduman, 2019). Also, the elite french school *Ecole Normale Supérieure* has been using a point system for the assignment of students to dorms over the years of study.

[8] In particular, it implies that the characterization result obtained by Ashlagi and Shi (2016) does not extend to our dynamic setup.

spot mechanism: it sets prices for each object at each date and will initially draw a budget for each agent. The main difference is that, at the beginning of time, each agent is offered a menu of distributions over $[0, 1]$. The distribution chosen in the menu will be used to randomly select an initial budget of virtual money. Then, similarly to spot mechanisms, each agent uses his budget to buy objects at each date.

These theoretical results provide insights into the types of mechanisms used in practice. As we already underlined, spot mechanisms are used in real-world markets. Of course, since under spot mechanisms, at a given date, agents do not have to express their preferences on what items they are willing to consume at further dates, these mechanisms may be seen as offering simplicity in agents' decision making or accommodating shocks in preferences that may occur in the future. However, given the special structure of pricing underlying these mechanisms, one may wonder about the losses induced by this special structure. Our main theoretical result shows that the loss may be small in practice, in particular, in markets with a fairly large number of agents. Further, while the optimality of spot mechanisms accords well with their use in practice, it is interesting to note that, in some contexts, dynamic allocation of items is implemented by market mechanisms differing from spot mechanisms. For instance, as we already mentioned, the Wharton School of Business uses a bidding system for courses where unused budgets from one semester do not carry over to subsequent semesters. We show by means of examples that such mechanisms precluding transfers of budget from one period to the other are inefficient (and, hence, cannot be replicated by spot mechanisms).[9] More generally, our results shed light on the lack of efficiency of the alternative assignment schemes.

These results also provide a path toward setting up the prices and the budgets in applications where spot markets are in use and where a social planner has a clear objective to optimize. For instance, for the assignment of teachers to public schools in France, one of the main objective of the administration / social planner is to ensure that enough experienced teachers are assigned to disadvantaged schools. Maximizing the number of experienced teachers in disadvantaged schools subject to incentive (and efficiency) constraints can then be solved by optimizing over spot mechanisms only. The question then boils down to choices of spot prices for schools and (distribution of) budgets for teachers.

Related literature. Many papers have considered market-like mechanisms with token money. The seminal paper is Hylland and Zeckhauser (1979) which defines competitive equilibrium with equal income in an environment with fake money. In this context, agents buy probability shares of items and prices clear the market. Budish (2011) defines a related concept in combinatorial assignment problems such as course allocation. The closest paper to ours is Ashlagi and Shi (2016) which characterizes incentive compatible and efficient alloca-

[9] While this is a source of inefficiencies, Budish, Cachon, Kessler, and Othman (2017) argue that allowing transfer of budget increases decision complexity since students have to think about how much of their budget they want to reserve for future use.

tion rules with a continuum of agents.[10] When the designer can only elicit ordinal preferences (under a same regularity assumption), they show that the class of incentive compatible and ordinally efficient mechanisms coincides with a class of mechanisms they call *Lottery plus cutoffs mechanisms*. Lottery plus cutoffs correspond to mechanisms fixing a price for each object and drawing the budget of each agent following a uniform distribution.[11] However, all these papers study static settings, whereas we consider a dynamic environment. In particular, we show that the characterization by Ashlagi and Shi (2016) does not extend to our dynamic setup.[12]

There is an extensive literature on dynamic mechanism design problems. Most of the literature focuses on settings in which monetary transfers are allowed (see Bergemann and Said (2011) for a survey). There is a small body of literature on dynamic mechanism without transfers. Jackson and Sonnenschein (2007) study a general framework for resource allocation in a finite horizon model without discounting in which agents learn all private information at time 0.[13] They assume that agents' preferences are additively separable and independently distributed across time and agents. The designer's goal is to achieve ex-ante Pareto-efficient outcomes. In order to achieve this goal, they build a budget-based mechanism in which each agent announces his preferences and announcements of agents are "budgeted" so that the distribution of preferences announced over the different dates must mirror the underlying distribution of preferences. Hence, the mechanism links the different periods to enforce incentives. Related ideas have been developed and applied to infinite horizon models with discounting where a designer has to repeatedly allocate a single resource to one of multiple agents, whose values are private and i.i.d. across agents and periods (e.g., Guo, Conitzer, and Reeves, 2009 and Balseiro, Gurkan, and Sun, 2019).[14] The proposed mechanisms share some similarities with our spot mechanisms, in particular, they are based on artificial currencies. For instance, in Jackson and Sonnenschein (2007), each preference ordering is associated with a budget of token money and announcing a preference ordering has a price which is taken from the associated preference-specific budget.[15] Beyond this type of similarities, our envi-

[10] Miralles and Pycia (2020) establish a Second Welfare Theorem in assignment problems without transfers.

[11] Lottery-plus-cutoffs mechanisms can be implemented using the standard Deferred-Acceptance mechanism with random priorities. Shi (2021) defines a large class of mechanisms which includes Lottery-plus-cutoffs mechanisms. He provides conditions under which one can implement these mechanisms using either Deferred-Acceptance, Top Trading Cycle or Serial Dictatorship.

[12] Instead, to prove our characterization, we introduce a generalization of their class of Lottery plus cutoffs mechanisms that we call *Generalized Lottery plus Cuttoffs* (GLC) mechanisms. GLC mechanisms also define prices over sequences of items in our case but draw the budgets according to a general (possibly non-uniform) distribution. Spot mechanisms can be seen as GLC mechanisms where the prices of sequences has a linear structure. We detail the exact connection in Section 3.2.

[13] Jackson and Sonnenschein (2007) is actually more general, they consider a decision problem that is linked with a large number of independent copies of itself. One possible interpretation is that the same problem is repeated a large number of times.

[14] These papers combine techniques from repeated games (Abreu, Pearce, and Stacchetti, 1990, Fudenberg, Levine, and Maskin, 1994) with some of the ideas in Jackson and Sonnenschein (2007) to show how one can approach efficient outcomes when the discount rate is high enough.

[15] In some related works, the budget may not be preference-specific and endow agents with just a single

ronments differ in important dimensions. The environments in these papers correspond to a large repetition of independent problems (which is reflected in the assumption that preferences are drawn i.i.d. over time and are separable). This is the cornerstone to ensure that one can link the problems to incentivize agents to report truthfully their preferences when implementing an ex-ante efficient allocation. In contrast, our result do not rely at all on any separability or i.i.d. assumptions and we cannot rely on Jackson and Sonnenschein (2007)'s "linkage principle". Dropping the separability and i.i.d. assumptions considerably enlarges the set of applications.[16] Such an advantage comes at the cost of assuming a large number of agents.

Last, this paper also relates to the growing literature on dynamic matching. Bloch and Houy (2012) and Kurino (2014) analyze a dynamic version of the housing market with overlapping generations. In their models, the housing side is fixed at the beginning of time and infinitely durable. In dynamic matching infinite horizon stochastic models Akbarpour, Li, and Shayan (2017), Baccara, Lee, and Yariv (2019), Anderson, Ashlagi, Gamarnik, and Kanoria (2017) and Ashlagi, Burq, Jaillet, and Manshadi (2019) study the trade-off between matching agents right away or match them later in order to benefit from market thickening.[17]

Outline. We begin by introducing a benchmark dynamic allocation problem where each agent is assigned a single object in every period. Although this simple model does not capture a variety of environments described above, it allows for a clear exposition of main ideas. In

artificial currency budget. For instance, in Guo, Conitzer, and Reeves (2009), agents have a budget of token money. In case they have a high valuation for the item today, they can pay the other agent a certain amount of token money to increase their likelihood of getting the item today. In turn, the other agent can use the additional tokens later on to increase his likelihood of getting the item whenever he will get a high valuation for the item. In a finite horizon model, at the cost of satisfying incentive constraints approximately, Gorokh, Banerjee, and Iyer (2021) offer mechanisms endowing agents with a budget of artificial currency, and organizing a static monetary mechanism in each period with payments in the artificial currency.

[16]For instance, coming back to our leading examples, students have different sets of choices of courses across semesters and teachers' preferences on the schools they want to attend today may depend on the school they have been assigned to yesterday (for example, because they decided to move near their current school). More generally, preferences over courses or schools in these applications are likely to be persistent across time. Hence, these applications typically violate the assumptions in Jackson and Sonnenschein (2007).

[17]More tangentially related to our work, the literature on online resource allocation and online fair division studies the problem of allocating indivisible items arriving over time over a fixed time horizon to a set of agents. The agents' valuations for the item arriving at a given date is known only after the item arrives, and are unknown until then. One main question is how the offline setting where items are all available upfront compares with the online setting where items arrive one at a time (e.g., Karp, Vazirani, and Vazirani, 1990). Other papers deal with how much envy can be generated in the online context and how it conflicts with efficiency (e.g., Benade, Kazachkov, Procaccia, and Psomas (2018), Zeng and Psomas (2020) and Bogomolnaia, Moulin, and Sandomirskiy (Forthcoming)). A difficulty in this literature is to deal with uncertain future. One common view is that an adversary selects a distribution of values from which each agent's values are drawn. Results vary depending on the class of distributions that the adversary can select from. In our model, we assume that distribution of agents' preferences is known to the designer and our continuum model rules out uncertainty.

Section 3, we then proceed to formally define ordinal mechanisms (i.e., mechanisms where agents only report their ordinal preferences) and state our main result in the context of the benchmark model. We also provide the intuition and sketch the proof of the main result. In Section 4, we extend the analysis to cardinal mechanisms. Section 5 introduces the general framework which encompasses our benchmark model and can be applied to many other settings including, for instance, the allocation of bundles of objects. In particular, it subsumes the dynamic course allocation application discussed in the introduction. We provide formal proofs in the Appendix.

2 The dynamic allocation problem

We consider a dynamic version of the allocation problem introduced by Ashlagi and Shi (2016). There is a continuum of agents, a sequence of T dates, and at each date t, a finite set of object types O_t. Every date each agent must be allocated exactly one object, and the set of pure allocations is given by $\mathbf{O} = O_1 \times \cdots \times O_T$. We allow individuals to receive **random allocations** which are elements of the probability simplex,

$$\Delta = \left\{ \mathbf{q} \in \mathbb{R}^{|\mathbf{O}|} : \mathbf{q} \geq 0, \sum_{\mathbf{o} \in \mathbf{O}} q_{\mathbf{o}} = 1 \right\},$$

where $q_{\mathbf{o}} \geq 0$ is the probability of pure allocation $\mathbf{o} \in \mathbf{O}$.

The problem of the social planner is to design a mechanism that allocates objects taking into account the preferences of agents. We separately study the two types of mechanisms corresponding to the elicited preferences being either ordinal or cardinal. We begin with ordinal mechanisms because all the applications mentioned in the introduction involve ordinal preferences and the main argument for the proof in the cardinal case heavily relies on the construction of the ordinal one. We extend our results to cardinal preferences in Section 4.

Remark 1 (Birkhoff-Von Neumann)**.** *We allow individuals to be assigned distributions over pure allocations. However, in order to actually implement a random allocation, one must find a lottery over pure allocations that resolves the randomness. This is doable in the static one-to-one environment, since the Birkhoff-von Neumann Theorem states that any random allocation can be expressed as a convex combination of pure allocations. When agents are allocated distributions over bundles of items (or as in our model over sequences of items), the theorem does not hold anymore (e.g., Nguyen, Peivandi, and Vohra (2016)). However, one can show that the theorem is restored in a model such as ours with a continuum of agents.*

3 Ordinal mechanisms

In this section we assume that the social planner can elicit only **ordinal preferences** over $\mathbf{O}$. We assume that the preferences are strict and let π denote such an ordinal preference, i.e., a permutation of $\mathbf{O}$, and Π denote the set of all such preferences. Hence, we allow for arbitrary complementarities in preferences between objects consumed by an agent on different dates. For $h = 1, \ldots, |\mathbf{O}|$, we let $\pi(h)$ be the element of $\mathbf{O}$ on the h-th place in an agent's ranking according to the preferences $\pi \in \Pi$. Let $F(\pi)$ be a commonly known probability distribution over ordinal preferences of agents. We say that F has **full support** if for any preference π, we have $F(\pi) > 0$.

A social planner allocates objects available at each date among agents taking into account their reported ordinal preferences. A **mechanism** (or allocation rule) $\mathbf{x}$ is a mapping from the set of strict ordinal preferences to a set of random allocations, $\mathbf{x} : \Pi \to \Delta$. Given mechanism $\mathbf{x}$, we denote a corresponding random allocation of an agent with preference profile π by $\mathbf{x}(\pi) \in \Delta$.[18,19] We say that a mechanism is **incentive compatible** (IC) if for any π, π' and each $m = 1, \ldots, |\mathbf{O}|$, we have

$$\sum_{k=1}^{m} x_{\pi(k)}(\pi) \geq \sum_{k=1}^{m} x_{\pi(k)}(\pi').$$

In other words, a mechanism is incentive compatible if the random allocation obtained by reporting each agent's true preferences first-order stochastically dominates for this agent each random allocation that can be obtained by reporting some other preferences.[20] Another requirement that we impose is that it must be impossible for agents to improve their random allocations in the sense of the first order stochastic dominance by trading their allocation probabilities. Given date t and object $i \in O_t$, let S_{it} be a set of pure allocations with object i at date t, i.e., $S_{it} = \{\mathbf{o} \in \mathbf{O} : o_t = i\}$. We say that a mechanism $\mathbf{x}$ is **ordinally efficient** (OE) if there is no other mechanism $\mathbf{x}'$ such that:

1. For each date t and object type $i \in O_t$ we have

$$\int \sum_{\mathbf{o} \in S_{it}} x'_{\mathbf{o}}(\pi) dF(\pi) = \int \sum_{\mathbf{o} \in S_{it}} x_{\mathbf{o}}(\pi) dF(\pi).$$

2. For each $m = 1, \ldots, |\mathbf{O}|$ and for each π we have: $\sum_{h=1}^{m} x'_{\pi(h)}(\pi) \geq \sum_{h=1}^{m} x_{\pi(h)}(\pi)$, with a strict inequality for some m and π.

[18] Our definition of a mechanism assumes that agents are treated symmetrically, i.e., agents with the same reported ordinal preferences will get the same random allocation. In particular, the social planner cannot discriminate based on observed characteristics of agents. However, it is easy to enrich our environment allowing for observed characteristics of agents. As in Ashlagi and Shi (2016), we would index mechanisms by these observed "types" and focus on mechanisms that treat agents of the same type symmetrically and is ordinally efficient within types. It is straightforward to extend our results to this richer environment.

[19] Note that with a continuum of agents and a full support distribution, there is formally no difference between a mechanism and an assignment of random allocations to agents.

[20] Since the model is ordinal, we use a definition purely based on ordinal preferences. As is well known, this is equivalent to requiring that each agent maximizes his expected utility by reporting his true preferences π for all cardinal representations of the π.

The first condition requires that at every date the mass of allocated objects of every type is the same in $\mathbf{x}$ and $\mathbf{x}'$. The second condition requires that for each agent a random allocation in $\mathbf{x}'$ first order stochastically dominates for this agent a random allocation in $\mathbf{x}$. We denote by $\mathcal{M}^e_{IC}$ the set of all IC and OE mechanisms.

Remark 2 (Capacities)**.** *We do not introduce object capacities in our model. Instead, capacities appear implicitly in Condition 1 of the definition of ordinal efficiency. Indeed each allocation induces some capacity utilization, and is OE if the utilized capacities cannot be reassigned in a way that makes agents better-off. A more standard definition would include capacities as a primitive of the model and say that an allocation* $\mathbf{x}$ *is ordinally efficient if there exists no other allocation* $\mathbf{x}'$ *that agents prefer (formally condition 2) and, at every date, the mass of allocated objects of every type at* $\mathbf{x}'$ *is smaller than the capacity for this object type. Clearly, if an allocation is OE with capacities, then it is OE in our model, but not vice versa. As shown in Ashlagi and Shi (2016), the framework without explicit capacities is more suitable in environments where a designer optimizes some objective which may take into account fairness and welfare considerations and where objects' capacities are part of the designer's choice variables. In this sense, our model provides a greater flexibility compared to models with exogenous capacities.*

The main goal in this paper is to characterize the set of mechanisms that are incentive compatible and ordinally efficient. Incidentally, we will show that these mechanisms share similarities with assignment schemes that are used in practice (e.g., for course allocation at universities, for the assignment of teachers to schools in France, etc. See the introduction for further details).

3.1 Spot mechanisms and main characterization

The mechanisms used in practice and described in the introduction share a common feature: they give a budget of artificial currency to each agent and allocate the objects "on the spot", i.e., they let agents manage their budget over time to buy some available objects at each date. To capture this feature, we introduce the following definition. Fix a distribution of budget G over $[0,1]$ and, for each date $t = 1, ..., T$, prices $p^t = (p^t_i)_{i\in O_t}$ for the objects available at these dates. Consider the following procedure:

- **Date 1.** Each agent independently draws a budget according to distribution G. Let b^1_a be the realized budget of agent a. Each agent picks an object among the affordable ones, i.e., in $\{i \in O_1 : p^1_i \leq b^1_a\}$. If agent a chooses object $i \in O_1$, the budget is adjusted to $b^2_a := b^1_a - p^1_i$;

- **Date $t \geq 2$.** Each agent picks an object among the affordable ones, i.e., in $\{i \in O_t : p^t_i \leq b^t_a\}$. If agent a chooses object $i \in O_t$, the budget is adjusted to $b^{t+1}_a := b^t_a - p^t_i$.

We let object prices and budget distribution be such that for each budget realization there is an affordable pure allocation, i.e., $\min_{\mathbf{o}\in\mathbf{O}} \sum_{t=1,\ldots,T} p^t_{o_t} \leq \inf\{z : G(z) > 0\}$.

For each agent a and a realization of budget b_a, we assume that a must choose an object at each date and, moreover, the sequence of choices of a is optimal, i.e., it corresponds to agent a's most preferred vector $\mathbf{o} = (o_t)_{t=1,\ldots,T}$ in $\mathbf{O}$ such that $\sum_{t=1,\ldots,T} p^t_{o_t} \leq b^1_a$. Using a simple backward induction argument and given our assumption of strict preferences, for each agent a, there is a unique such $\mathbf{o}$. Hence, the procedure induces a deterministic allocation of objects. Using G, we can integrate over all possible realizations of random draws of the budgets to define a corresponding allocation rule $\mathbf{x}$. Our assumption that for each budget realization there is an affordable pure allocation, ensures that spot mechanisms induce allocation rules.

A mechanism $\mathbf{x}$ is a **spot mechanism** if its allocation rule can be obtained by the above procedure. We let $\mathcal{G}_{sm}$ denote the set of allocation rules which are spot mechanisms. Note that the definition captures, in particular, the course allocation procedure used at CBS modulo the fact that we have not allowed situations where *bundles* of objects are allocated. Section 5 presents a straightforward extension of our model which captures this aspect as well. It also resembles the procedure to assign teachers to school in France that we described in the introduction.

To illustrate the functioning of the spot mechanisms and the restrictions imposed by them on the set of allocations we discuss several examples in this section. In every example there are two dates and two objects to be allocated at every date, $O_1 = O_2 = \{1, 2\}$. A set of pure allocations is $\mathbf{O} = \{(11), (21), (12), (22)\}$, where the first digit stands for an object consumed in the first date and the second stands for an object consumed in the second date. We begin with an example illustrating the functioning of the spot mechanisms.

Example 1. *Fix a spot mechanism with a uniform budget distribution and prices* $p^1_1 = 0$, $p^1_2 = 1/3$, $p^2_1 = 0$, $p^2_2 = 2/3$. *Consider an allocation of an agent with ordinal preferences* $(22) \succ (12) \succ (21) \succ (11)$. *For example, if the realized budget is in* $[1/3, 2/3)$, *then the agent will opt for object 2 in the first date, and then spend her budget on object 1 in the second date, thus obtaining a pure allocation (21). The probability of such realization is 1/3, and hence the probability of pure allocation (21) is 1/3. With a similar logic, we obtain the following probabilities for each pure allocation.*

Pure allocation	Probability
(22)	0
(12)	1/3
(21)	1/3
(11)	1/3

The main result of this section is that spot mechanisms characterize the entire set of incentive compatible and ordinally efficient allocation rules in dynamic environments.

Theorem 1. *Suppose that the distribution F over Π has full support. A mechanism $\mathbf{x}$ is incentive compatible and ordinally efficient if and only if it is a spot mechanism. Formally, $\mathcal{M}^e_{IC} = \mathcal{G}_{sm}$.*

Some comments are in order. As we already underlined, spot mechanisms are used in real-world markets. However, one can imagine other mechanisms and, indeed, other types of mechanisms are used in practice. Our result shows that, with a continuum of agents, the restriction to spot mechanisms is without loss as long as one wants to achieve ordinally efficient allocations. Methodologically, this brings some simplification to a designer's problem having a social objective to optimize. Indeed, if the objective is ordinally efficient, then one has to optimize over spot mechanisms and the question then boils down to choices of spot prices for items and distribution of budgets for agents. In addition, our results shed some light on the lack of efficiency of alternative assignment schemes some of which are used in practice. Indeed, the following examples illustrate two natural modifications of the spot mechanisms which, however, turn out to be inefficient.

Example 2. *Alternative mechanisms sometimes used in practice are mechanisms where at the start a separate budget is drawn independently for each date, and these budgets are not transferable across dates.*[21] *In a setting with two dates, it means that there are two budget realizations. Suppose that a budget distribution at each date is uniform and spot prices are $p^1_1 = 0$, $p^1_2 = 1/3$, $p^2_1 = 0$, $p^2_2 = 2/3$. Consider the allocations of two agents, agent 1 with ordinal preferences $(12) \succ (21) \succ (22) \succ (11)$, and agent 2 with ordinal preferences $(21) \succ (12) \succ (22) \succ (11)$, given in the table below.*

Pure allocation	Probability for agent 1	Probability for agent 2
(22)	0	0
(12)	3/9	1/9
(21)	4/9	6/9
(11)	2/9	2/9

Note that Agent 1 gets (21) with positive probability, i.e., when budget at period 1 is in $[1/3, 1]$ and budget at period 2 is in $[0, 2/3)$. Similarly, Agent 2 gets (12) with positive probability, i.e., when budget at period 1 is in $[0, 1/3)$ and budget at period 2 is in $[2/3, 1]$. However, if these two agents trade the probabilities of (21) and (12), then they improve their allocations, and hence the initial allocation is not OE. The only difference of such mechanism from a spot mechanism is that the budget cannot be transferred across dates. This turns out to be the source of inefficiency. Clearly, by Theorem 1 this mechanism cannot be replicated by a spot mechanism.

Example 3. *Spot mechanisms induce prices on pure allocations given by the sums of the corresponding spot prices across dates. In the next subsection we introduce a more general*

[21] The mechanism "Course Match" used for the assignment of courses in the MBA program in Wharton is an example where unused budgets from one semester do not carry over to subsequent semesters.

class of mechanisms where prices of pure allocations cannot be decomposed into spot prices. Specifically, one can assign prices directly to each pure allocation. Consider such mechanism where the budget is drawn uniformly and the prices of pure allocations are $p_{12} = 0$, $p_{21} = 1/3$, $p_{22} = 2/3$, *and* $p_{11} = 1$. *Now, consider an allocation of an agent with ordinal preferences same as the order of prices, specifically* $(11) \succ (22) \succ (21) \succ (12)$. *The random allocation is described below.*

Pure allocation	Probability
(11)	0
(22)	1/3
(21)	1/3
(12)	1/3

Note that this mechanism is not a spot mechanism and each agent is asked to choose an entire allocation at the start. Moreover, the allocation is not OE. Indeed, the distribution in the table below allocates the same mass of each object at each date and stochastically dominates the given allocation for the agent with ordinal preferences $(11) \succ (22) \succ (21) \succ (12)$.

Pure allocation	Probability
(11)	1/3
(22)	2/3
(21)	0
(12)	0

Note that, by Theorem 1, the initial allocation cannot be induced by a spot mechanism. Importantly, one cannot decompose the above prices into per-period prices, i.e., one cannot find a vector of "spot prices": $(p_1^1, p_2^1, p_1^2, p_2^2)$ *such that* $p_{11} = p_1^1 + p_1^2$, $p_{21} = p_2^1 + p_1^2$, $p_{12} = p_1^1 + p_2^2$ *and* $p_{22} = p_2^1 + p_2^2$. *Indeed,* $p_{22} > p_{12}$ *implies that* $p_2^1 > p_1^1$, *whereas* $p_{11} > p_{21}$ *implies that* $p_2^1 < p_1^1$, *a contradiction. This observation is a core element of the proof of Theorem 1. The following section presents a sketch of this proof.*

3.2 Sketch of the proof

In this subsection, we discuss the connection of our model with the one of Ashlagi and Shi (2016) and provide a sketch of the proof of Theorem 1.

Spot mechanisms are a special case of a larger class of mechanisms. This class is a straightforward generalization of the "lottery-plus-cutoff mechanisms" introduced by Ashlagi and Shi (2016). Fix a collection of cutoffs $\boldsymbol{\alpha} := (\alpha_{\mathbf{o}})_{\mathbf{o} \in \mathbf{O}} \in [0,1]^{|\mathbf{O}|}$ and a distribution G over $[0,1]$. An allocation rule $\mathbf{x}$ is a **Generalized Lottery-plus-Cutoff (GLC) mechanism with parameters** $L := (\boldsymbol{\alpha}, G)$ if $x_{\pi(h)}(\pi) = G(\min_{m=1,\dots,h-1} \alpha_{\pi(m)}) - G(\min_{m=1,\dots,h} \alpha_{\pi(m)})$ for every π and

$h = 1, \ldots, |\mathbf{O}|$. Plainly, under a generalized lottery-plus-cutoffs allocation rule, each agent a independently draws a budget b_a from distribution G on unit interval and chooses her favorite pure allocation $\mathbf{o}$ among those with cutoffs below her budget, i.e., in $\{\mathbf{o} \in \mathbf{O} : \alpha_{\mathbf{o}} \leq b_a\}$. For parameters $L = (\boldsymbol{\alpha}, G)$, we denote by $\mathbf{x}^L$ the allocation rule defined by the GLC mechanism with parameters L. We sometimes simply refer to $\mathbf{x}^L$ as a GLC mechanism with parameters L. We denote by $\mathcal{G}$ the set of allocation rules which are GLC mechanisms.

Spot mechanisms are a subclass of GLC mechanisms with a special "linear" structure of cutoffs. Formally, a spot mechanism is a GLC mechanism $\boldsymbol{\alpha}^L$ with parameters $L = (\mathbf{x}, G)$ such that there exists a sequence of profiles of prices $\mathbf{p} = (p^t)_{t=1,\ldots,T}$ where $p^t = (p^t_i)_{i \in O_t}$ for each $t = 1, ..., T$ satisfying

$$\alpha_{\mathbf{o}} = \sum_{t=1}^{T} p^t_{o_t}$$

for each $\mathbf{o} = (o_1, ..., o_T) \in \mathbf{O}$. We will say that cutoffs satisfying the above condition are **linear**. Hence spot mechanisms restrict the GLC mechanisms that one can use. Because generally the cutoffs $\boldsymbol{\alpha}$ in the definition of a GLC mechanism are not linear, GLC mechanisms cannot be reproduced by allocating objects "on the spot" as a spot mechanism does. Therefore, one can implement a larger set of allocation rules using GLC mechanisms. However, as shown in Example 3 of the previous section, GLC mechanisms need not be efficient.

The model of Ashlagi and Shi (2016) can be seen as a static version of ours. In a static environment, i.e., when $T = 1$, they characterized ordinally efficient and incentive compatible allocation rules as *lottery-plus-cutoff mechanisms.* Formally, using our above terminologies, an allocation rule $\mathbf{x}$ is a **lottery-plus-cutoffs** mechanism if it is a GLC mechanism with parameters $L = (\boldsymbol{\alpha}, G)$ where $G = U_{[0,1]}$. Under a lottery-plus-cutoffs allocation rule, each agent a independently draws a budget b_a from the uniform distribution on the unit interval and chooses her favorite pure allocation $\mathbf{o}$ among those in $\{\mathbf{o} \in \mathbf{O} : \alpha_{\mathbf{o}} \leq b_a\}$. Let $\mathcal{G}_{AS}$ be the set of lottery-plus-cutoffs mechanisms.

The following result is Ashlagi and Shi (2016)'s characterization in the static case.

Theorem 2 (Ashlagi and Shi, 2016)**.** *Let $T = 1$. Suppose that the distribution F over Π has full support. An allocation rule $\mathbf{x}$ is ordinally efficient and incentive compatible if and only if it is a lottery-plus-cutoff mechanism.*

As long as we are in a non-trivial dynamic environment, i.e., when $T \geq 2$, the above result fails as we have demonstrated in Example 3. However, when $T \geq 2$, we can still interpret an allocation $\mathbf{o} \in \mathbf{O}$ as "an item" in a static environment and use Ashlagi and Shi (2016)'s "static" notion of ordinal efficiency on these items. Call this notion the *AS ordinal efficiency.* This AS ordinal efficiency in our dynamic setting is not natural. It imposes that there is no alternative allocation rule $\mathbf{x}'$ satisfying for each $\mathbf{o} \in O$

$$\int x'_{\mathbf{o}}(\pi) dF(\pi) = \int x_{\mathbf{o}}(\pi) dF(\pi)$$

(where we recall that $\mathbf{o} = (o_t)_t$ is a pure allocation, one for each date) together with Condition 2 in our definition of ordinal efficiency. Typically, reallocation of objects within a period is not allowed.[22] For instance, in Example 3, $\mathbf{x}'$ violates the above condition while it uses the same mass of each object at each date. Of course, AS ordinal efficiency is stronger than OE as stated in the following lemma.

Lemma 1. *If an allocation rule* $\mathbf{x}$ *is ordinally efficient then it is AS-ordinally efficient.*

PROOF. If $\mathbf{x}$ is not AS-ordinally efficient, then one can find another allocation $\mathbf{x}'$ s.t. Condition 2 of ordinal efficiency is satisfied and for each $\mathbf{o} \in \mathbf{O}$:

$$\int x'_{\mathbf{o}}(\pi) dF(\pi) = \int x_{\mathbf{o}}(\pi) dF(\pi)$$

Fix an object $i \in O_t$. Clearly, summing the above equalities over all $\mathbf{o} \in S_{it}$ gives us condition 1 in the definition of ordinal efficiency. Thus, we conclude that $\mathbf{x}$ is not ordinally efficient. ∎

Equipped with Lemma 1, we obtain that, in our dynamic environment with $T \geq 2$, one direction of Theorem 2 by Ashlagi and Shi (2016) holds.

Proposition 1. *Suppose that the distribution* F *over* Π *has full support. An allocation rule is incentive compatible and ordinally efficient only if it is a lottery-plus-cutoffs mechanism. Formally,* $\mathcal{M}^e_{IC} \subset \mathcal{G}_{AS}$.

As will be explained below, our main result (Theorem 1) can be proved using Proposition 1 together with the following proposition:

Proposition 2. *Suppose that the distribution* F *over* Π *has full support. Fix an ordinally efficient lottery-plus-cutoffs mechanism* $\mathbf{x}^L$ *with* $L = (\boldsymbol{\alpha}, U_{[0,1]})$. *Then, there exists a linear collection of cutoffs* $\bar{\boldsymbol{\alpha}}$ *which has the same strict order as* $\boldsymbol{\alpha}$, *i.e.,* $(\alpha_{\mathbf{o}} < \alpha_{\mathbf{o}'}) \Rightarrow (\bar{\alpha}_{\mathbf{o}} < \bar{\alpha}_{\mathbf{o}'})$.

The cornerstone of the proof of Proposition 2 is the following result from the theory of linear inequalities.

Lemma 2 (Carver, 1921)**.** *For an arbitrary matrix* A, $Ax < 0$ *is feasible, if and only if* $y = 0$ *is the only solution for* $y \geq 0$ *and* $A'y = 0$.

To get a sense of how we use Lemma 2, consider Example 4 below.

Example 4. *There are two periods and two objects. Fix some random assignment* $\mathbf{q}$, *and suppose it is induced by a lottery-plus-cutoffs mechanism consisting of strict cutoffs* $\boldsymbol{\alpha} = (\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$ *such that* $\alpha_{12} < \alpha_{21} < \alpha_{22} < \alpha_{11} < 1$. *It is easily checked that these cutoffs are not linear. Here, we use Lemma 2 to explain how non-linearity of cutoffs implies failure*

[22] Of course, as mentioned, in a static environment where $T = 1$, both notions coincide.

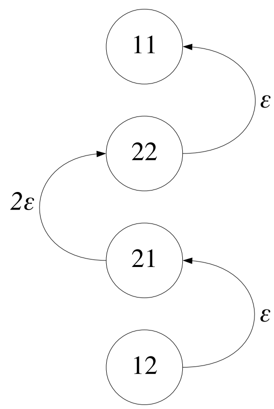


Figure 1: Improving mass transfer for an agent with ordinal preferences $(1,2) \prec (2,1) \prec (2,2) \prec (1,1)$.

of ordinal efficiency of the random assignment. This will give a sense for why Proposition 2 holds true.

If cutoffs are linear, then by definition there exists a vector $\mathbf{p} = (p_1^1, p_2^1, p_1^2, p_2^2)^T$ *such that* $\alpha_{ij} = p_i^1 + p_j^2$ *for all* i *and* j*. Therefore, if our strict cutoffs are linear, the following system of weak inequalities must be feasible:*

$$\begin{aligned} p_1^1 + p_1^2 &> p_2^1 + p_2^2, \\ p_2^1 + p_2^2 &> p_2^1 + p_1^2, \\ p_2^1 + p_1^2 &> p_1^1 + p_2^2. \end{aligned}$$

We can rewrite the above system in matrix form as $\mathbf{Ap} < 0$*, where*

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

However, our cutoffs are not linear, and so we know that $\mathbf{Ap} < 0$ *is not feasible. Therefore Lemma 2 guarantees that there exists* $\mathbf{y} \geq 0$*,* $\mathbf{y} \neq 0$ *such that* $\mathbf{A}^T\mathbf{y} = 0$*. In particular, for any* $\varepsilon > 0$*,* $\mathbf{y} = (\varepsilon, 2\varepsilon, \varepsilon)$ *is such a solution of* $\mathbf{A}^T\mathbf{y} = 0$*. It turns out that we can interpret this* $\mathbf{y}$ *as specifying a sequence of bilateral mass transfers between consecutive bundles. In particular, let* $y_1 = \varepsilon$ *be the probability mass to be transferred from* (22) *to* (11)*,* $y_2 = 2\varepsilon$ *be the probability mass to be transferred from* (21) *to* (22)*, and* $y_3 = \varepsilon$ *be the probability mass to be transferred from* (12) *to* (21)*. Then* $\mathbf{A}^T\mathbf{y} = 0$ *implies that after implementing these three transfers, the mass of each object in each period remains the same. For example, consider object 1 in period 1. When we transfer* ε *from* (22) *to* (11)*, the mass of the object increases by* ε*, whereas its mass does not change when we transfer* 2ε *from* (21) *to* (22)*, and its mass decreases by* ε *when we transfer* ε *from* (12) *to* (21)*. In particular, the change in the assigned mass of object* i

in period t is captured by the negative of the dot product of the corresponding row of $\mathbf{A}^T$ and $\mathbf{y}$. Now, to show that random allocation $\mathbf{q}$ is not ordinally efficient, consider the agent whose ordinal preferences are the same as the order of cutoffs, i.e., $(12) \prec (21) \prec (22) \prec (11)$. By the full support assumption, there is a positive mass of such agents in our economy. Because the cutoffs are strict, such agent is assigned a strictly positive probability of each bundle. Hence, for small $\varepsilon > 0$, we can implement the above sequence of bilateral mass transfers. Moreover, each bilateral transfer moves the probability from a lower to a higher ranked bundle according to this agent's preferences (See Figure 1 for an illustration). Hence, after implementing transfers y, she obtains a dominating random assignment while keeping the mass of each object assigned in each period constant. Therefore, the random assignment $\mathbf{q}$ is not ordinally efficient.

The above example illustrates why a linear structure in cutoffs is needed for a lottery-plus-cutoff mechanism to be ordinally efficient as stated in Proposition 2.[23] Then, we can use Proposition 1 to deduce that if $\mathbf{x} \in \mathcal{M}_{IC}^e$ then it is induced by a lottery-plus-cutoffs mechanism, i.e., there exists a collection of cutoffs $\boldsymbol{\alpha}$ such that $\mathbf{x} = \mathbf{x}^L$ with $L = (\boldsymbol{\alpha}, U_{[0,1]})$. From Proposition 2, we can deduce that there exists a collection of prices $\mathbf{p} = (p^t)_{t=1,...,T}$ where $p^t = (p_i^t)_{i \in O_t}$ for each $t = 1, ..., T$ where the collection of linear cutoffs $\bar{\boldsymbol{\alpha}}$ induced by $\mathbf{p}$ has the same strict order as the collection $\boldsymbol{\alpha}$, i.e., $(\alpha_{\mathbf{o}} < \alpha_{\mathbf{o}'}) \Rightarrow (\bar{\alpha}_{\mathbf{o}} < \bar{\alpha}_{\mathbf{o}'})$.[24]

However, the GLC mechanism with parameters $(\bar{\boldsymbol{\alpha}}, U_{[0,1]})$ does not generate the same allocation rule as $\mathbf{x}$. But, using a properly defined distribution G, we can show that the GLC mechanism $L' := (\bar{\boldsymbol{\alpha}}, G)$ is s.t. $\mathbf{x}^{L'} = \mathbf{x}$ so that the "only if part" of Theorem 1 is proved. While we believe this part of the theorem is surprising, the "if part" of Theorem 1 is a bit more expected and its proof, which also uses Lemma 2, is relegated to Section B of the Appendix.

Remark 3 (Linear cutoffs and uniform budget distribution.)**.** *One cannot use a uniform distribution together with linear cutoffs to generate all the incentive compatible and ordinally efficient rules (contrary to the static case studied in Ashlagi and Shi, 2016). To illustrate this, Example 7 in Appendix A provides an ordinally efficient allocation that cannot be implemented by a lottery-plus-cutoffs mechanism (i.e., with a uniform distribution over budgets) with linear cutoffs.*

4 Cardinal mechanisms

We have studied a dynamic allocation problem where a social planner can only elicit ordinal preferences of agents. In this section, we extend the analysis to the case where a planner can

[23]The argument presented in Example 4 only works with a collection $\boldsymbol{\alpha}$ of strict cutoffs where $(\mathbf{o} \neq \mathbf{o}') \Rightarrow (\alpha_{\mathbf{o}} \neq \alpha_{\mathbf{o}'})$. It is easy to construct examples with an ordinally efficient random allocation that can only be implemented by a lottery-plus-cutoffs mechanism with non-strict cutoffs. In that case, one has to properly build the resulting probability masses to be transferred and an important part of the proof is devoted to this construction.

[24]Cutoffs $\bar{\boldsymbol{\alpha}}$ is induced by $\mathbf{p}$ means that $\bar{\boldsymbol{\alpha}}_{\mathbf{o}} = \sum_{t=1}^{T} p_{o_t}^t$ for each $\mathbf{o} = (o_1, ..., o_T) \in \mathbf{O}$.

elicit a complete cardinal preference profile. Our results here are twofold. First, we introduce a new cardinal allocation mechanism tailored to the dynamic environment. Second, we use this mechanism to prove the main result resembling the spot market characterization in the ordinal case.

Consider the dynamic allocation problem from Section 2. In contrast to the previous section, here we let agents have cardinal preferences over a set of pure allocations $\mathbf{O}$ represented by utility vector $\mathbf{u} \in U$, each coordinate denoting utility from consuming a corresponding pure allocation. We let U denote the set of all utility vectors inducing strict ordinal preferences and assume that these utility vectors are distributed according to a continuous probability measure F. For a measurable subset $A \subset U$, we let $F(A)$ denote the mass of agents with utility vectors in A.

We follow Ashlagi and Shi (2016) and impose a full relative support assumption on distribution F. In order to state this condition, let $D := \{\mathbf{u} \in U : \mathbf{u} \cdot \mathbf{1} = 0\}$. One could understand our regularity condition as imposing that, a priori, an agent's relative preference could, with positive probability, take any direction in D. As will become clear, this regularity condition is implied by the stronger but simpler assumption that F assigns positive mass to each open set in U.

In order to formally define our regularity assumption, let us define $\tilde{D} := \{\mathbf{u} \in D : \|\mathbf{u}\| = 1\}$ where $\|\cdot\|$ is the Euclidean norm. Sets U, D and $\tilde{D}$ are all endowed with standard topologies.[25] Let $\mathcal{C}$ be the collection of cones in D.[26] We endow $\mathcal{C}$ with the following topology: $\mathcal{C}' \subset \mathcal{C}$ is open if $\mathcal{C}' \cap \tilde{D}$ is open in $\tilde{D}$. Following Ashlagi and Shi (2016), we say that distribution F has **full relative support** if, for any open cone C in $\mathcal{C}$, $F(\text{Proj}_D^{-1}(C)) > 0$ where $\text{Proj}_D(\cdot)$ stands for the projection of U into D.

An allocation rule $\mathbf{x}$ is a mapping from utility vectors to random allocations, $\mathbf{x} : U \to \Delta$. An allocation rule $\mathbf{x}$ is **incentive compatible** if for each $\mathbf{u} \in U$ reporting the true preferences maximizes the expected utility:

$$\mathbf{u} \in \arg\max_{\mathbf{u}' \in U} \mathbf{u} \cdot \mathbf{x}(\mathbf{u}').$$

An allocation rule $\mathbf{x}$ is **Pareto efficient** if there is no other allocation rule $\mathbf{x}'$ which allocates the same mass of each object at each date (condition 1 in the definition of ordinal efficiency holds), and delivers a weakly higher expected utility to every agent and a strictly higher one for a positive mass of agents, i.e., there does not exist $\mathbf{x}'$ such that $\mathbf{u} \cdot \mathbf{x}'(\mathbf{u}) \geq \mathbf{u} \cdot \mathbf{x}(\mathbf{u})$ for each $\mathbf{u} \in U$ and there is a set $A \subset U$ such that $F(A) > 0$ and the inequality is strict for each $\mathbf{u} \in A$. In what follows, we introduce a new cardinal mechanism which can be decentralized through a sequence of spot markets, and use it to characterize the set of incentive compatible and Pareto efficient allocation rules.

[25] $U = \mathbb{R}^{|\mathbf{O}|}$ is endowed with the topology induced by the Euclidean norm and D is endowed with the relative topology, i.e., a set is open in D if it is the intersection of an open set in $\mathbb{R}^{|\mathbf{O}|}$ with D. We endow $\tilde{D}$ with the relative topology, i.e., a set is open in $\tilde{D}$ if it is the intersection of an open set in D with $\tilde{D}$.

[26] Recall that a cone is a set C such that for all $\lambda > 0 : x \in C \implies \lambda x \in C$.

A GLC mechanism is an ordinal mechanism in the sense that it is not flexible enough to differentiate cardinal preferences: if ordinal preferences of two agents coincide, then they receive the same allocation. Therefore we modify a GLC mechanism in order to obtain a mechanism which is responsive to the cardinal preferences of agents. Whereas a GLC mechanism has a single distribution from which each agent independently draws a budget of artificial currency, we now allow agents to choose from a menu of such distributions. We begin with a collection of cutoffs $\boldsymbol{\alpha} := (\alpha_{\mathbf{o}})_{\mathbf{o}\in\mathbf{O}} \in [0,1]^{|\mathbf{O}|}$ and a collection of distributions $\mathcal{G} := (G_j)_{j\in J}$ over $[0,1]$. Then, a random allocation can be constructed by drawing from an agent's ex ante favorite distribution, and then choosing the agent's most preferred affordable allocation given her budget realization. For an agent with utility vector $\mathbf{u} \in U$, let $\mathbf{x}^G(\mathbf{u})$ be the expected utility-maximizing random allocation induced by budget distribution G, that is

$$x^G_{u(h)}(\mathbf{u}) = G(\min_{m=1,\dots,h-1} \alpha_{u(m)}) - G(\min_{m=1,\dots,h} \alpha_{u(m)})$$

for each $h = 1,\dots,|\mathbf{O}|$, where $u(h) \in \mathbf{O}$ is a an allocation on h-th place in a preference ranking according to utility vector $\mathbf{u}$. An allocation rule $\mathbf{x}$ is a **Menu of Random Budgets (MRB) mechanism with parameters** $L := (\boldsymbol{\alpha}, \mathcal{G})$ if, for every utility vector $\mathbf{u}$, there is distribution $G_{j(\mathbf{u})} \in \mathcal{G}$ such that $\mathbf{x}(\mathbf{u}) = \mathbf{x}^{G_{j(\mathbf{u})}}(\mathbf{u})$, and

$$G_{j(\mathbf{u})} \in \arg\max_{G\in\mathcal{G}} \mathbf{x}^G(\mathbf{u}) \cdot \mathbf{u}.$$

Note that agents with identical ordinal but different cardinal preferences can choose different random budgets and hence receive different random allocations.

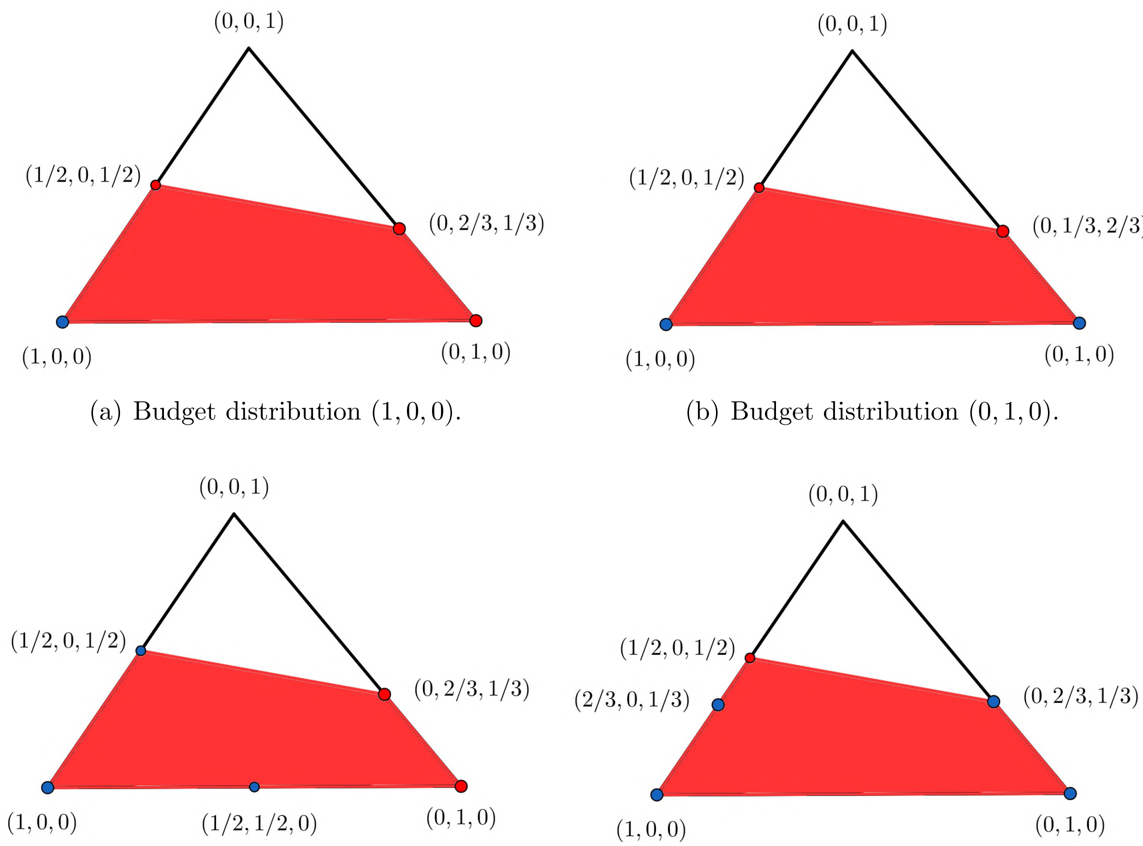
Similarly to the case of a GLC mechanism, we can introduce a spot version of a MRB mechanism. Fix a sequence of profiles of prices $\mathbf{p} = (p^t)_{t=1,\dots,T}$, where $p^t = (p^t_i)_{i\in O_t}$ for each $t = 1,\dots,T$ and a collection of distributions $\mathcal{G} := (G_j)_{j\in J}$ over $[0,1]$. Then, it applies the following procedure:

- **Date 1.** Each agent chooses a distribution from collection $\mathcal{G}$ and independently draws a budget from it. Let b^1_a be the realized budget of each agent a. Each agent picks an object among the feasible ones, i.e., in $\{i \in O_1 : p^1_i \le b^1_a\}$. If a chooses object $i \in O_1$, the budget is adjusted to $b^2_a := b^1_a - p^1_i$;
- **Date $t \ge 2$.** Each agent picks an object among the feasible ones, i.e., in $\{i \in O_t : p^t_i \le b^t_a\}$. If agent a chooses object $i \in O_t$, the budget is adjusted to $b^{t+1}_a := b^t_a - p^t_i$.

We let object prices and budget distributions in $\mathcal{G}$ be such that for each budget realization there is an affordable pure allocation, i.e., $\min_{\mathbf{o}\in\mathbf{O}} \sum_{t=1,\dots,T} p^t_{o_t} \le \inf\{z : G(z) > 0\}$ for each $G \in \mathcal{G}$. Under this assumption, spot mechanisms will always induce a random allocation (i.e., no agent can be unassigned).

Hence spot MRB mechanisms constitute MRB mechanisms with linear cutoffs. Formally, $L := (\boldsymbol{\alpha}, \mathcal{G})$ is a **spot MRB mechanism** if there exists a sequence of profiles of prices $\mathbf{p} = (p^t)_{t=1,\dots,T}$, where $p^t = (p^t_i)_{i\in O_t}$ for each t satisfying $\alpha_{\mathbf{o}} = \sum_{t=1}^T p^t_{o_t}$ for each $\mathbf{o} \in \mathbf{O}$.

The possibility of the spot market implementation of a MRB mechanism is in contrast to the more standard CEEI approach adopted in Ashlagi and Shi (2016). Recall that an allocation rule $\mathbf{x}$ is a **Competitive Equilibrium with Equal Income** (CEEI) with prices $\mathbf{p}$ if for any $\mathbf{o} \in \mathbf{O}$ and any $\mathbf{u}$, $\mathbf{x}(\mathbf{u}) \in \arg\max_{\mathbf{q}\in\Delta}\{\mathbf{u}\cdot\mathbf{q} : \mathbf{p}\cdot\mathbf{q} \leq 1\}$. Thus, given a profile of prices, agents use a budget of one unit of artificial currency to buy probability shares of objects. A CEEI approach does not fit our dynamic framework because each agent must choose the entire dynamic allocation at the very first date. Nevertheless, it turns out that there is a connection between the two mechanisms. Each CEEI can be implemented as a MRB mechanism as the following static example illustrates.



(a) Budget distribution $(1, 0, 0)$.

(b) Budget distribution $(0, 1, 0)$.

(c) Budget distribution $(1/2, 0, 1/2)$.

(d) Budget distribution $(0, 2/3, 1/3)$.

Figure 2: Affordable allocations for different random budgets.

Example 5. *Take a static model with $T = 1$ and consider an economy where each agent is endowed with a single unit of artificial currency and there are three objects with prices of probability shares $\hat{p}_1 = 0$, $\hat{p}_2 = 0.5$, and $\hat{p}_3 = 2$. In the CEEI, an agent chooses an allocation in the probability simplex which maximizes her expected utility subject to a budget constraint. We shall construct a MRB mechanism which induces the same allocation rule as the CEEI* *above. First, let a collection of cutoffs for the MRB mechanism be given by the above prices, normalized to lie inside the unit interval by dividing each price by the highest price, i.e.,* $p_1 = 0, p_2 = 0.25, p_3 = 1$*. Second, for each random allocation which is a part of the CEEI we associate a distribution of random budget. In particular, for such an allocation* $\mathbf{x}$*, let the corresponding distribution* $G_{\mathbf{x}}$ *assign probability* x_i *to* p_i*, for* $i = 1, 2, 3$*. For instance, in each figure of Figure 2, the four allocations* $(1/2, 0, 1/2)$*,* $(1, 0, 0)$*,* $(0, 1, 0)$ *and* $(0, 2/3, 1/3)$*, corresponding to the vertices of the budget set, give rise to four budget distributions:* $(1, 0, 0)$*,* $(0, 1, 0)$*,* $(1/2, 0, 1/2)$*, and* $(0, 2/3, 1/3)$*, where the first number in each of the last 3-tuples is the probability that the budget is equal to* p_1*, the second – to* p_2*, and the third – to* p_3*. Consider the random allocations that can be induced by an agent who draws a random budget from each of these distributions and optimally chooses his pure allocation given this realization. We illustrate them by the blue dots in Figure 2. For instance, an agent who chooses the random budget distribution* $(0, 1, 0)$ *gets 0.25 units of artificial currency with probability 1. In that case, he can buy either object 1 or 2 under the resulting MRB. Hence, depending on his preferences, the agent will choose one of these two pure allocations represented by the two blue allocations in Figure 2(b). Similarly, an agent who chooses the budget distribution* $(1/2, 0, 1/2)$ *obtains a null budget with probability 1/2. In that case, he can only buy object 1 under the resulting MRB. With probability 1/2, he gets a budget of 1 and can buy any of the available objects. The random allocation induced by his optimal choices (integrating over all possible realizations of the budget) correspond to one of the three blue dots in Figure 2(c). For instance, if the agent prefers 2 over 1 over 3, his optimal choices will generate random allocation* $(1/2, 1/2, 0)$*. Notice that the random allocations that can be generated by the choice of a random budget distribution all lie inside the CEEI budget set represented by the red region in Figure 2. Hence, if an agent receives an allocation in the CEEI, then this agent weakly prefers the random budget distribution generated by this allocation to any distribution generated by another allocation, and obtains this allocation in the MRB mechanisms with the above menu of budgets and prices. Hence, this MRB mechanism induces the same allocation rule as the CEEI.*

The following result generalizes the observation in the example. Its proof relies on the characterization of incentive compatible and Pareto efficient allocation rules as CEEIs by Ashlagi and Shi (2016).

Proposition 3. *Suppose that the distribution* F *over* U *is continuous and has full relative support. Then any incentive-compatible and Pareto-efficient allocation rule is a MRB mechanism with some parameters* $L = (\boldsymbol{\alpha}, \mathcal{G})$*.*

Proof. Fix an incentive compatible and Pareto efficient allocation rule $\mathbf{x}$. Ashlagi and Shi (2016) show that $\mathbf{x}$ is supported as CEEI for some collection of prices $\hat{\mathbf{p}} = (\hat{p}_{\mathbf{o}})_{\mathbf{o} \in \mathbf{O}}$ and a unit budget for each agent. Let $p_{\mathbf{o}} := \frac{\hat{p}_{\mathbf{o}}}{\max_{\mathbf{o}} \hat{p}_{\mathbf{o}}}$ for each $\mathbf{o} \in \mathbf{O}$ and $\mathbf{p} = (p_{\mathbf{o}})_{\mathbf{o} \in \mathbf{O}}$. For each $\mathbf{u} \in U$ and $\mathbf{x}(\mathbf{u}) \in \Delta$, let $G_{\mathbf{x}(\mathbf{u})}$ be a discrete distribution that assigns probability $x_{\mathbf{o}}(\mathbf{u})$ to $p_{\mathbf{o}}$ for each $\mathbf{o}$, and let $\mathcal{G} = (G_{\mathbf{x}(\mathbf{u})})_{\mathbf{u} \in U}$ be a collection of such distributions. We show that a MRB mechanism with $L = (\mathbf{p}, \mathcal{G})$ induces $\mathbf{x}$.

Fix a discrete distribution of a random budget G. By choosing some affordable allocation at each realization of random budget, we induce some ex ante distribution over allocations. Define a feasible choice rule to be a function that chooses an affordable pure allocation for each realization of a random budget. Formally, a feasible choice rule is a function $\psi : [0,1] \to \mathbf{O}$ such that $p_{\psi(z)} \leq z$. Then, given distribution G, let the set of random allocations which can be induced by some feasible choice rule be:

$$B(G) = \{\mathbf{y} \in \Delta(\mathbf{O}) | \text{there exists feasible } \psi \text{ such that } y_{\mathbf{o}} = \sum_{z:\psi(z)=\mathbf{o}} P_G(z)\},$$

where $P_G(z)$ is the probability of realization z given G.

Now, if an agent with utility $\mathbf{u}$ chooses the most preferred affordable bundle for each realization of the random budget $G_{\mathbf{x}(\mathbf{u})}$, then, by construction of $G_{\mathbf{x}(\mathbf{u})}$, the induced ex ante distribution is $\mathbf{x}(\mathbf{u})$. Hence, $\mathbf{x}(\mathbf{u}) \in B(G_{\mathbf{x}(\mathbf{u})})$. Next, we show that if $\mathbf{y} \in B(G_{\mathbf{x}(\mathbf{u})})$ for some $G_{\mathbf{x}(\mathbf{u})} \in \mathcal{G}$, then distribution $\mathbf{y}$ also belongs to the original budget set in the CEEI mechanism with the collection of prices $\hat{\mathbf{p}}$, i.e., $\sum_{\mathbf{o}} y_{\mathbf{o}} \hat{p}_{\mathbf{o}} \leq 1$. Therefore, when choosing from a collection of random budgets $\mathcal{G}$, it is optimal for an agent with utility $\mathbf{u}$ to choose distribution $G_{\mathbf{x}(\mathbf{u})}$.

Suppose $\mathbf{y} \in B(G_{\mathbf{x}(\mathbf{u})})$, and let ψ be a feasible choice rule that induces $\mathbf{y}$. We have

$$\sum_{\mathbf{o}} \hat{p}_{\mathbf{o}} y_{\mathbf{o}} = \sum_{\mathbf{o}} \hat{p}_{\mathbf{o}} \sum_{\mathbf{o}':\psi(p_{\mathbf{o}'})=\mathbf{o}} x_{\mathbf{o}'}(\mathbf{u}).$$

Note that each $x_{\mathbf{o}}(\mathbf{u})$ enters the sum on the right hand side only once. Specifically, we can rewrite the above as:

$$\begin{aligned} \sum_{\mathbf{o}} \hat{p}_{\mathbf{o}} \sum_{\mathbf{o}':\psi(p_{\mathbf{o}'})=\mathbf{o}} x_{\mathbf{o}'}(\mathbf{u}) &= \sum_{\mathbf{o}} x_{\mathbf{o}}(\mathbf{u}) \hat{p}_{\psi(p_{\mathbf{o}})}, \\ &\leq \sum_{\mathbf{o}} x_{\mathbf{o}}(\mathbf{u}) \hat{p}_{\mathbf{o}}, \\ &\leq 1. \end{aligned}$$

Here, the first and the second lines follow from the relation between choice rule ψ and distribution $\mathbf{x} \in B(G_{\mathbf{x}(\mathbf{u})})$; the final line follows from $\mathbf{x}(\mathbf{u})$ being a part of CEEI. ∎

Hence, if the prices in CEEI are linear, then it can be decentralized using a MRB mechanism and sequence of spot markets. Our main result in this section is a cardinal version of Theorem 1.

Theorem 3. *Suppose that the utility distribution F over U is continuous and has full relative support. An allocation rule $\mathbf{x}$ is Pareto efficient and incentive compatible if and only if it is a spot MRB mechanism.*

We generalize the cardinal model to the environment with bundles in the next section and provide a generalization of the above result in Section C of the Appendix.

5 The general framework

Throughout the paper we have focused on a simple dynamic environment where agents are assigned a single object at every date. Although this model describes applications such as the assignment of teachers to jobs and students to dormitories, it does not address all the situations where bundles of objects are allocated. For instance, in our motivating example of course allocation, students can typically take some number of electives per semester. Moreover, some courses can be pre- or anti-requisites to other courses and students may be required to earn a certain number of credits over the years to graduate. In order to capture this as well as a variety of other settings, we generalize our benchmark model and state the two theorems which generalize Theorem 1 and 3.

The general model. Fix a finite set of generalized object types O. Each agent must be allocated a bundle of objects which contains at most one object of each type. We denote the set of all admissible (nonempty) bundles by $B \subset 2^O$, and write $i \in b$ to denote that bundle $b \in B$ contains object type $i \in O$. A set of random allocations is

$$\Delta = \left\{ \mathbf{q} \in \mathbb{R}^{|B|} : \mathbf{q} \geq 0, \sum_{b \in B} q_b = 1 \right\}.$$

Ordinal preferences. Agents have ordinal strict preferences over B. As before, π denotes such a preference and Π is the set of all preferences, while $\pi(h) \in B$ is the bundle on h-th place in the ranking according to $\pi \in \Pi$. Let $F(\pi)$ be a probability distribution over ordinal preferences of agents with full support. An allocation rule $\mathbf{x}$ is a mapping from a set of ordinal preferences to a set of random allocations, i.e., $\mathbf{x} : \Pi \to \Delta$. Definitions of incentive compatibility and ordinal efficiency are similarly adapted to the bundle framework. An allocation rule $\mathbf{x}$ is **incentive compatible** if for any $\pi, \pi' \in \Pi$ and each $m = 1, \ldots, |B|$, we have

$$\sum_{k=1}^{m} x_{\pi(k)}(\pi) \geq \sum_{k=1}^{m} x_{\pi(k)}(\pi').$$

An allocation rule $\mathbf{x}$ is **ordinally efficient** if there is no other allocation rule $\mathbf{x}'$ such that:

1. For each object type $i \in O$ we have

$$\int \sum_{b: i \in b} {x'}_b(\pi) dF(\pi) = \int \sum_{b: i \in b} x_b(\pi) dF(\pi).$$

2. For each $m = 1, \ldots, |B|$ and each $\pi \in \Pi$ we have: $\sum_{h=1}^{m} x'_{\pi(h)}(\pi) \geq \sum_{h=1}^{m} x_{\pi(h)}(\pi)$, with a strict inequality for some m and π.

We denote the set of incentive compatible and ordinally efficient allocation rules by $\mathcal{M}^e_{IC}$.

The above model encompasses our benchmark dynamic allocation model with ordinal preferences. Recall that the dynamic model begins with a finite set of object types O_t for each date t. Without loss of generality we can let O_t's be disjoint sets. The set of pure allocations was a product $\mathbf{O} = O_1 \times \cdots \times O_T$. Now, define the corresponding set of generalized object types to be $O = O_1 \cup \cdots \cup O_T$. Moreover, a bundle is feasible if and only if it contains exactly one object from each O_t. Then, the set of pure allocations $\mathbf{O}$ corresponds to the set of admissible bundles.

Example 6. *The generalization allows us to include into our benchmark model the possibility of allocating bundles and arbitrarily restricting feasible allocations. As an example, consider a course allocation problem with two semesters and three courses a, b, and c. Suppose that each course is available in both semesters, but course a is a prerequisite for course c, and the same course cannot be taken twice. Moreover, to graduate, each student is required to take at least two courses. We can model this situation by letting $O = \{a_1, a_2, b_1, b_2, c_1, c_2\}$, where a subscript denotes a semester at which a course is taken. The corresponding set of feasible bundles is $B = \{(a_1, b_1), (a_1, b_2), (a_2, b_2), (b_1, a_2), (a_1, c_2), (a_1, b_1, c_2), (a_1, b_2, c_2)\}$.*

As before, our goal is to characterize all incentive compatible and ordinally efficient allocation rules. To do so, we now introduce the appropriately modified version of a GLC mechanism. Fix a collection of cutoffs $\boldsymbol{\alpha} := (\alpha_b)_{b \in B} \in [0, 1]^{|B|}$ and a distribution G over $[0, 1]$. An allocation rule $\mathbf{x}$ is a **Generalized Lottery-plus-Cutoff (GLC) mechanism with parameters** $L := (\boldsymbol{\alpha}, G)$ if

$$x_{\pi(h)}(\pi) = G(\min_{m=1,\ldots,h-1} \alpha_{\pi(m)}) - G(\min_{m=1,\ldots,h} \alpha_{\pi(m)})$$

for every π and $h = 1, \ldots, |B|$. Note that not every GLC mechanism defines an allocation rule. If a GLC mechanism with parameters $L = (\boldsymbol{\alpha}, G)$ defines an allocation rule, we denote this rule by $\mathbf{x}^L$. Cutoffs $\boldsymbol{\alpha}$ are **linear** if there exist object prices $\mathbf{p} = (p_i)_{i \in O} \in \mathbb{R}^{|O|}$, such that

$$\boldsymbol{\alpha}_b = \sum_{i \in b} p_i,$$

for each $b \in B$. Let $\mathcal{G}_L$ be the set of all allocation rules which are GLC mechanisms with linear cutoffs. Now, we are ready to state our main result.

Theorem 4. *Suppose that the distribution F over Π has full support. An allocation rule is incentive compatible and ordinally efficient if and only if it is a GLC mechanism with linear cutoffs, i.e., $\mathcal{M}^e_{IC} = \mathcal{G}_L$.*

It is easy to see that our dynamic framework is embedded into the current one so that Theorem 1 is a corollary of Theorem 4. The sketch of the proof is similar to the one we presented in Section 3.2. The actual proof is provided in Section B of the Appendix.

Cardinal preferences. The generalization for cardinal preferences is the mirror analogue of the previous section. Agents have cardinal strict preferences over B and we let $\mathbf{u}$ be the utility vector where each coordinate gives the utility for a bundle in B. The distribution F over cardinal utility vectors can also be easily generalized and the full relative support definition does not change from the one given in Section 4. An allocation rule $\mathbf{x}$ now maps the set U of cardinal utility vectors to Δ, the set of random allocations. The definitions of incentive compatibility and Pareto efficiency can easily be adapted from Section 4 to the case with bundles in B. We can similarly define what is a **Menu of Random Budgets (MRB) mechanism with parameters** $L := (\boldsymbol{\alpha}, \mathcal{G})$ in this new framework where now the collection of cutoffs $\boldsymbol{\alpha}$ is defined over the bundles in B. The spot version of a MRB mechanism defines a vector of prices $\mathbf{p} = (p_i^t)_{i \in O}$ and the cutoffs $\boldsymbol{\alpha}$ are linear if $\alpha_b = \sum_{i \in b} p_i$ for each $b \in B$. We restrict our attention to MRB mechanisms inducing an allocation rule. We now state the generalization of Theorem 3 to the general setting with bundles:

Theorem 5. *Suppose that the utility distribution F over U is continuous and has full relative support. An allocation rule $\mathbf{x}$ is Pareto efficient and incentive compatible if and only if it is a MRB mechanism with linear cutoffs.*

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A Example 7

In Example 7 below, we exhibit a random assignment that is OE and cannot be replicated by a lottery-plus-cutoff mechanism with linear cutoffs. This, in particular, implies that we need to allow non-uniform distributions of budgets in the definition of spot mechanisms to achieve all OE and IC allocation rules.

Example 7. *Let $T = 2$, $O_1 = O_2 = \{1, 2\}$ and consider the following spot mechanism where $p_1^1 = 0.4, p_2^1 = 0, p_1^2 = 0.6$ and $p_2^2 = 0$. The cutoffs are summarized below.*

Allocation	Cutoff
(11)	1
(12)	0.6
(21)	0.4
(22)	0

Distribution G over possible budgets in $[0, 1]$ is assumed to satisfy $P(z = 1) = 0.2, P(z = 0.6) = 0.2, P(z = 0.4) = 0.1$ and $P(z = 0) = 0.5$. By Theorem 1, this random allocation is ordinally efficient. Now, we claim that this random allocation cannot be replicated by a lottery-plus-cutoff mechanism with linear cutoffs. First, to replicate this allocation, it is clear that the order of cutoffs must remain the same, i.e., $\alpha_{11} > \alpha_{12} > \alpha_{21} > \alpha_{22}$. Given that, by definition of a lottery-plus-cutoff mechanism, the distribution over budgets in $[0, 1]$ must be uniform, we must have the following cutoffs to replicate the random allocation:

Allocation	Cutoff
(11)	0.8
(12)	0.6
(21)	0.5
(22)	0

However, it is easily checked that these cutoffs are non-linear.[27] *To recap, we need to use spot mechanisms with non-uniform distributions to reproduce the above OE random allocation rule.*

[27]To see that these cutoffs are non-linear, we need to argue that there is no vector $\mathbf{p} = (p_1^1, p_2^1, p_1^2, p_2^2)^T$ such that $\alpha_{ij} = p_i^1 + p_j^2$ for all $i, j = 1, 2$. Note that these equalities for $ij = 11, 12$ imply that $p_1^2 - p_2^2 = 0.2$ while equalities for $ij = 21, 22$ imply that $p_1^2 - p_2^2 = 0.5$.

B Proof of Theorem 4

To prove the theorem, we need some additional notations and preliminary results. We begin by describing bundles by vectors. Each feasible bundle $b \in B$ is assigned a row vector $\mathbf{d}_b$ with $|O|$ columns, one column for each generalized object. For each object $i \in O$, we let $d_i = 1$ if $i \in b$, and $d_i = 0$ otherwise. It turns out that it is useful to describe the differences in a composition between bundles b and b' by another row vector $\mathbf{a}_{b,b'}$ given by

$$\mathbf{a}_{b,b'} = \mathbf{d}_b - \mathbf{d}_{b'}.$$

Hence, each vector $\mathbf{a}_{b,b'}$ is composed only of 1's, -1's, and 0's:

- If $i \in b$ and $i \notin b'$, then the row of $\mathbf{a}_{b,b'}$ corresponding to object i is equal to 1.
- If $i \notin b$ and $i \in b'$, then the row of $\mathbf{a}_{b,b'}$ corresponding to object i is equal to -1.
- If i either belongs or does not belong to both bundles, then the row of $\mathbf{a}_{b,b'}$ corresponding to object i is equal to 0.

For any total order $\leq$ on the set of feasible bundles B, we associate a matrix $\mathbf{A}$ that captures the differences in a composition between each pair of strictly ordered bundles. In particular, let matrix $\mathbf{A}$ contain row $\mathbf{a}_{b,b'}$ if and only if $b < b'$. Each column of $\mathbf{A}$ corresponds to a generalized object. Let $\mathbf{a}^i$ be the column of $\mathbf{A}$ corresponding to object i. The following two properties of matrix $\mathbf{A}$ are instrumental for the proof.

Linear cutoffs. For object prices $\mathbf{p}$ and two bundles b and b', consider the vector of linear cutoffs $\bar{\boldsymbol{\alpha}}$ induced by these prices, i.e., $\bar{\alpha}_b = \mathbf{d}_b \cdot \mathbf{p}$ for each $b \in B$.[28] The difference between cutoffs for any two bundles b and b' is $\bar{\alpha}_b - \bar{\alpha}_{b'} = \mathbf{a}_{b,b'} \cdot \mathbf{p}$. So, in particular, $\mathbf{a}_{b,b'} \cdot \mathbf{p} < 0$ means that bundle b has lower cutoff than b'. Now, take any cutoffs $\boldsymbol{\alpha}$, and let $\mathbf{A}$ be the matrix associated with the total order on bundles induced by these cutoffs, i.e., $\mathbf{A}$ contains row $\mathbf{a}_{b,b'}$ if and only if $\alpha_b < \alpha_{b'}$. If for some price vector $\mathbf{p}$, we have $\mathbf{Ap} < 0$, then the linear cutoffs $\bar{\boldsymbol{\alpha}}$ induced by $\mathbf{p}$ are such that if $\alpha_b < \alpha_{b'}$, then $\bar{\alpha}_b < \bar{\alpha}_{b'}$ for each $b, b' \in B$.

Probability mass transfers. Consider a column vector $\mathbf{y}$, each coordinate of which, $y_{b,b'} \in \mathbb{R}$, corresponds to a row $\mathbf{a}_{b,b'}$ of $\mathbf{A}$. We can view each $y_{b,b'}$ as a probability mass to be transferred from a bundle with a lower cutoff b to a bundle with a higher cutoff b'. So, $\mathbf{y}$ specifies a set of bilateral mass transfers from lower to strictly higher bundles in the order of cutoffs. Now, take a row i of matrix $\mathbf{A}^T$. Each coordinate of this row corresponds to some pair of bundles b and b'. For example, suppose b' has object i, while b does not. Then, the corresponding coordinate of row i is equal to -1. Imagine transferring mass $y_{b,b'}$ from b to b'. Then, the total allocated mass of object i changes by $y_{b,b'}$. Therefore, the negative of the dot product of row i of $\mathbf{A}^T$

[28] We assume that the coordinates of $\mathbf{p}$ and $\mathbf{a}_{b,b'}$ are ordered in the same way, so that the vector operations make sense.

and the vector $\mathbf{y}$, $-(\mathbf{a}^i)^T\mathbf{y}$, gives the total change in the allocated mass of object i resulting from the transfers defined by the vector $\mathbf{y}$. Accordingly, $-\mathbf{A}^T\mathbf{y}$ is a vector that captures the change in the allocated mass of each object. In particular, if $A^T\mathbf{y} = 0$, then transfers $\mathbf{y}$ simply redistribute the masses of objects across bundles.

We summarize the discussion above in the following lemma.

Lemma 3. *For cutoffs $\boldsymbol{\alpha}$, let $\mathbf{A}$ be the matrix associated with the total order on bundles induced by these cutoffs. Then:*

(i) There exist linear cutoffs $\bar{\boldsymbol{\alpha}}$ such that for each $b, b' \in B$, if $\alpha_b < \alpha_{b'}$, then $\bar{\alpha}_b < \bar{\alpha}_{b'}$, if and only if there exists vector $\mathbf{p}$ of prices such that $\mathbf{Ap} < 0$;

(ii) For each $b, b' \in B$ such that $\alpha_b < \alpha_{b'}$, let $y_{b,b'}$ be the mass to be transferred from b to b'. Then $\mathbf{A}^T\mathbf{y} = 0$ if and only if the transfers in $\mathbf{y}$ do not change the allocated mass of each object.

Next, we characterize first order stochastic dominance employed in the definition of ordinal efficiency using bilateral mass transfers. Fix a random allocation $\mathbf{q}$, a preference ordering π, and a pair of bundles b and b'. We say that $\tau_{b,b'}(\pi) \in \mathbb{R}$ is a **bilateral transfer from b to b' for π at $\mathbf{q}$**, or simply a bilateral transfer, if $0 < \tau_{b,b'}(\pi) \leq q_b$ and $q_{b'} + \tau_{b,b'}(\pi) \leq 1$. A bilateral transfer $\tau_{b,b'}(\pi)$ is **improving** if $\pi^{-1}(b') < \pi^{-1}(b)$. In words, an improving bilateral transfer moves mass from lower ranked bundles to higher ranked ones. Now, fix two random allocations $\mathbf{q}'$ and $\mathbf{q}$. We say that $\mathbf{q}'$ can be derived from $\mathbf{q}$ by an **improving bilateral transfer** for π if there are bundles b and b' such that $q_{b''} = q'_{b''}$ for all bundles $b'' \in B \setminus \{b, b'\}$, and $q_b > 0$ and, moreover, $\tau_{b,b'}(\pi) := q_b - q'_b = q'_{b'} - q_{b'} > 0$ is an improving bilateral transfer from b to b' for π at $\mathbf{q}$. The following lemma applies the characterization of first order stochastic dominance in terms of improving bilateral transfers to our framework.[29]

Lemma 4. *Fix a preference ordering π and two random allocations $\mathbf{q}$ and $\mathbf{q}'$. The random allocation $\mathbf{q}' \neq \mathbf{q}$ first order stochastically dominates $\mathbf{q}$ if and only if $\mathbf{q}'$ can be derived from $\mathbf{q}$ by a finite sequence of improving bilateral transfers. Formally, there exists a sequence $(\mathbf{q}_1, \ldots, \mathbf{q}_n)$ of random allocations s.t. $\mathbf{q}_1 = \mathbf{q}$, $\mathbf{q}_n = \mathbf{q}'$ and for $k = 1, \ldots, n-1$, $\mathbf{q}_{k+1}$ can be derived from $\mathbf{q}_k$ by an improving bilateral transfer for π.*

In light of Lemma 4, we can restate the second condition in the definition of ordinal efficiency. Specifically, for each preferences π such that $\mathbf{x}(\pi) \neq \mathbf{x}'(\pi)$, it requires to find a sequence of improving bilateral transfers to go from random allocation $\mathbf{x}(\pi)$ to random allocation $\mathbf{x}'(\pi)$.

Lemma 5. *A random allocation $\mathbf{x}$ is ordinally efficient if and only if there is no other random allocation $\mathbf{x}'$ such that:*

1. *For each object type $i \in O$ we have*

$$\int \sum_{b: i \in b} {x'}_b(\pi) dF(\pi) = \int \sum_{b: i \in b} x_b(\pi) dF(\pi).$$

[29]See for instance Østerdal (2010).

2. *For each $\pi \in \Pi$ such that $\mathbf{x}(\pi) \neq \mathbf{x}'(\pi)$, random allocation $\mathbf{x}'(\boldsymbol{\pi})$ can be derived from $\mathbf{x}(\pi)$ by a sequence of improving bilateral transfers for π.*

Our proof also relies on the following result from the theory of linear inequalities.

Lemma 6. *(Carver, 1921) For an arbitrary matrix $\mathbf{A}$, $\mathbf{Ax} < 0$ is feasible, if and only if $\mathbf{y} = 0$ is the only solution for $\mathbf{y} \geq 0$ and $\mathbf{A^T y} = 0$.*

Finally, we are ready to prove Theorem 4. We split the proof into two parts, with the first part being the result below.

Proposition 4. *Suppose that the distribution F over Π has full support. Let $\mathbf{x}^L$ be an ordinally efficient GLC mechanism with $L = (\boldsymbol{\alpha}, U_{[0,1]})$. There exist linear cutoffs $\bar{\boldsymbol{\alpha}}$ such that for all $b, b' \in B$ if $\alpha_b < \alpha_{b'}$, then $\bar{\alpha}_b < \bar{\alpha}_{b'}$.*

Proof. Let $\mathbf{A}$ be the matrix associated with the total order on bundles induced by cutoffs α. In order to show that these cutoffs are linear, by Lemma 3, it suffices to show that there exists a vector $\mathbf{p}$ of prices such that $\mathbf{Ap} < 0$. For the sake of contradiction, suppose that $\mathbf{Ap} < 0$ is not feasible, so that no such vector exists. Then by Lemma 6, there exists $\mathbf{y}$ such that $\mathbf{y} \geq \mathbf{0}$,$\mathbf{y} \neq \mathbf{0}$ and $\mathbf{A}^T\mathbf{y} = 0$. Next, we show that $\mathbf{y}$ can be used to construct improving bilateral transfers for some preference profiles.

Let $\Pi_{\boldsymbol{\alpha}}$ be the set of preference profiles which ranking over bundles is consistent with the strict rankings induced by cutoffs $\boldsymbol{\alpha}$, i.e., if $\alpha_b < \alpha_{b'}$, then $\pi^{-1}(b') < \pi^{-1}(b)$ for all $\pi \in \Pi_{\boldsymbol{\alpha}}$ and $b, b' \in B$. Below, we define a function f which, for each coordinate $y_{b,b'} > 0$ of $\mathbf{y}$, chooses a preference profile $f(b, b') \in \Pi_\alpha$ such that $x_b(f(b, b')) > 0$. For each $b \in B$, denote the set of bundles with a cutoff equal to α_b by $I(b) = \{b'' \neq b : \alpha_{b''} = \alpha_b\}$. Consider two cases:

- First, suppose $I(b) = \emptyset$. Then, let $f(b, b')$ be any $\pi \in \Pi_\alpha$. Indeed, for all such π we must have $x_b(\pi) > 0$ because a GLC mechanism with $L = (\boldsymbol{\alpha}, U_{[0,1]})$ picks the budget of each agent uniformly from the unit interval. Hence, there is a positive probability for the event $E = \{\alpha_b \leq z < \hat{\alpha}_b\}$, where $\hat{\alpha}_b = \min\{\alpha_{b''} : \alpha_{b''} > \alpha_b\}$ is well-defined since α_b is not the highest cutoff. Indeed, recall that $\mathbf{y}$ contains coordinate $y_{b,b'}$ only when $\mathbf{A}$ contains row $\mathbf{a}_{b,b'}$, which is true if and only if $\alpha_b < \alpha_{b'}$.
- Second, suppose $I(b) \neq \emptyset$. Then, by the full support assumption there exists a preference profile $\pi_b \in \Pi_{\boldsymbol{\alpha}}$ that ranks b ahead of each $b'' \in I(b)$. Hence, for the same reason as before, we must have $x_b(\pi_b) > 0$. So, we define $f(b, b') = \pi_b$.

Now, for each $y_{b,b'} > 0$, pick the preference profile $\pi = f(b, b')$. For $\varepsilon > 0$, let all the agents with such preferences transfer a probability mass of $\frac{\varepsilon}{F(\pi)} y_{b,b'}$ from b to b' at their random allocation $\mathbf{x}(\pi)$. Note that this is well-defined given that, by the full support assumption, $F(\pi) > 0$ for all π. Hence, the total mass transferred from b to b' is $\varepsilon y_{b,b'} \geq 0$. Then, clearly, for a small enough $\varepsilon > 0$ these are improving bilateral transfers. Moreover, because $\mathbf{A}^T\mathbf{y} = 0$, by Lemma 3 these transfers do not change the allocated mass of each object. Therefore $\mathbf{x}$

is not ordinally efficient, which is a contradiction. It follows that there exist linear cutoffs $\bar{\alpha}$ such that for each $b, b' \in B$, if $\alpha_b < \alpha_{b'}$, then $\bar{\alpha}_b < \bar{\alpha}_{b'}$. ∎

To finish the proof, we construct a distribution G' which, together with the linear cutoffs from Proposition 4, yields the required GLC mechanism.

Proof. ($\Rightarrow$) Let $\mathbf{x}$ be an incentive compatible and ordinally efficient allocation rule. Define the new set of generalized objects $\hat{O}$ to be the set of feasible bundles B and a new set of feasible bundles $\hat{B}$ containing all the singleton subsets of $\hat{O}$. This new environment directly corresponds to the case studied by Ashlagi and Shi (2016). Let $\hat{\mathbf{x}}$ be an allocation rule in the new environment that corresponds to $\mathbf{x}$. It is clear that $\hat{\mathbf{x}}$ must also be incentive compatible and ordinally efficient in the new environment. Hence, by Proposition 2 (Ashlagi and Shi, 2016), we know that there exists a GLC mechanism with $\hat{L} = (\hat{\boldsymbol{\alpha}}, U_{[0,1]})$ which defines the same allocation rule as $\hat{\mathbf{x}}$, i.e., $\mathbf{x}^{\hat{L}} = \hat{\mathbf{x}}$. Moreover, the corresponding GLC mechanism $L = (\boldsymbol{\alpha}, U_{[0,1]})$ in our initial environment must also define the same allocation rule as $\mathbf{x}$, i.e., $\mathbf{x}^L = \mathbf{x}$.

Now, by Proposition 4, there exists a collection of linear cutoffs $\bar{\boldsymbol{\alpha}}$ such that, for all $b, b' \in B$, if $\alpha_b < \alpha_{b'}$, then $\bar{\alpha}_b < \bar{\alpha}_{b'}$. Next, we define the distribution G' such that the GLC mechanism $\mathbf{x}^{L'}$ with $L' = (\bar{\boldsymbol{\alpha}}, G')$ induces allocation $\mathbf{x}$. In particular, we construct G' such that for each subset $S \subset B$, the probability to be able to afford each bundle $b \in S$ is the same in mechanisms $(\boldsymbol{\alpha}, G)$ and $(\bar{\boldsymbol{\alpha}}, G')$. Note that the probability to afford a subset of bundles is equal to the probability to afford the bundle with the highest cutoff among those in the subset. Therefore, it is enough to find G' such that the probability to afford each bundle is the same in both mechanisms. Let G' be a discrete distribution such that

$$P(z = \bar{\alpha}_b) = \begin{cases} \hat{\alpha}_b - \alpha_b & \text{if } \bar{\alpha}_b \in \arg\max\{\bar{\alpha}_{b''} : b'' \in I(b) \cup \{b\}\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\hat{\alpha}_b = \min\{\alpha_{b''} : \alpha_{b''} > \alpha_b\}$ if α_b is not the highest cutoff, and $\hat{\alpha}_b = 1$ otherwise. Then, the mechanism $(\bar{\boldsymbol{\alpha}}, G')$ defined above indeed induces allocation rule $\mathbf{x}$.

($\Leftarrow$) Let $\mathbf{x}^L$ be an allocation rule defined by a GLC mechanism with parameters $L = (\boldsymbol{\alpha}, G)$ and linear cutoffs. We show that $\mathbf{x}^L \in \mathcal{M}^e_{IC}$. The incentive compatibility is straightforward, so we focus on proving ordinal efficiency. For the sake of contradiction, suppose $\mathbf{x}^L$ is not ordinally efficient. Then, there exists $\mathbf{x}'$ such that $\mathbf{x}^L$ and $\mathbf{x}'$ allocate the same mass of each object and, for each π, the random allocation $\mathbf{x}'(\pi)$ can be derived from $\mathbf{x}^L(\pi)$ via a sequence of improving bilateral transfers. Given such a sequence for π, let $\tau_{b,b'}(\pi)$ be the total mass transferred from bundle b to bundle b'. We first note that, if $\tau_{b,b'}(\pi) > 0$, then we must have that $\alpha_b < \alpha_{b'}$. Indeed, assume that $\alpha_{b'} \leq \alpha_b$. By definition of improving transfers, we must have that $\pi^{-1}(b') < \pi^{-1}(b)$ and whenever an agent with preferences π has budget $z \geq \alpha_b$, both b and b' can be chosen by this agent so that she always picks b'. Hence, it implies $x^L_b(\pi) = 0$, a contradiction to $\tau_{b,b'}(\pi)$ being the sum of the improving bilateral transfers from b to b'. Now, we aggregate the bilateral transfers across all agents into a column vector $\mathbf{y}$. In particular, for each $b, b' \in B$ such that $\alpha_b < \alpha_{b'}$ we let

$$y_{b,b'} = \int \tau_{b,b'}(\pi) dF(\pi).$$

Hence, $y_{b,b'}$ is the total mass transferred by all agents from b to b'. Let $\mathbf{A}$ be the matrix associated with the total order on bundles induced by cutoffs $\boldsymbol{\alpha}$. Because $\mathbf{x}^L$ and $\mathbf{x}'$ allocate the same mass of each object, by Lemma 3, we have $\mathbf{A}^T\mathbf{y} = 0$. In addition, since $\mathbf{x}^L \neq \mathbf{x}'$, by construction we have $\mathbf{y} \neq 0$. But then, by Lemma 6, $\mathbf{Ap} < 0$ is not feasible, a contradiction to $\boldsymbol{\alpha}$ being linear. Therefore, allocation rule $\mathbf{x}^L$ is ordinally efficient. ∎

C Proof of Theorem 5

Our proof will follow the one of Theorem 4 in Section B. To do so, we need the equivalent of Proposition 4.

Proposition 5. *Suppose that the utility distribution F over U is continuous and has full relative support. If $\mathbf{x}$ is incentive compatible and Pareto efficient, then $\mathbf{x}$ is a CEEI with prices α. Moreover, there exist linear prices $\bar{\boldsymbol{\alpha}}$ such that for all $b, b' \in B$ if $\alpha_b < \alpha_{b'}$, then $\bar{\alpha}_b < \bar{\alpha}_{b'}$.*

Proof. Let $\mathbf{x}$ be an incentive compatible and Pareto efficient allocation rule. Define the new set of generalized objects $\hat{O}$ to be the set of feasible bundles B and the new set of feasible bundles $\hat{B}$ to be singleton subsets of $\hat{O}$. Hence, in this new environment we treat bundles as objects, so that it exactly fits the model studied by Ashlagi and Shi (2016). Let $\hat{\mathbf{x}}$ be an allocation rule in the new environment that corresponds to $\mathbf{x}$. It is clear that $\hat{\mathbf{x}}$ must also be incentive compatible and Pareto efficient in the new environment. Hence, by Theorem 1 of Ashlagi and Shi (2016), we know that the allocation rule $\hat{\mathbf{x}}$ is a CEEI for some prices $(\alpha_b)_{b\in\hat{B}}$. Hence, the allocation rule $\mathbf{x}$ is also a CEEI for the same prices $(\alpha_b)_{b\in B}$ in our original environment where agents are buying the probability shares of bundles.

Now, we shall prove that there exist linear prices $\bar{\boldsymbol{\alpha}}$ such that, for all $b, b' \in B$, if $\alpha_b < \alpha_{b'}$, then $\bar{\alpha}_b < \bar{\alpha}_{b'}$. Let b^{max} be the bundle with the highest price (fix any such bundle if there are several ones), i.e., $\alpha_{b^{max}} = \max_{b''} \alpha_{b''}$. First, if all bundles are affordable, i.e., $\alpha_{b^{max}} \leq 1$, then the task is trivial because any collection of linear prices which keeps all bundles affordable induces the same CEEI. So suppose that $\alpha_{b^{max}} > 1$ and, toward a contradiction, such linear prices $\bar{\boldsymbol{\alpha}}$ do not exist. Then, by the same argument as in the proof of proposition 4, there exists $\mathbf{y}$ such that $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{A}^T\mathbf{y} = 0$, where matrix $\mathbf{A}$ is defined in Section B given the prices α. Next, we show that $\mathbf{y}$ can be used to construct improving bilateral transfers for a positive mass of agents so that $\mathbf{x}$ is not Pareto efficient, a contradiction.

Let $U_\alpha \subset U$ be the set of utility vectors which ranking over bundles is consistent with the strict ranking induced by prices α, i.e., if $\alpha_b < \alpha_{b'}$, then $u_b < u_{b'}$ for all $\mathbf{u} \in U_\alpha$ and $b, b' \in B$. We shall define mapping f which, for each coordinate $y_{b,b'} > 0$ of $\mathbf{y}$, chooses an open subset of preferences $f(b, b') \subset U_\alpha$ such that $x_b(\mathbf{u}) > \varepsilon$ for each $\mathbf{u} \in f(b, b')$ and some $\varepsilon > 0$. Now, let $f(b, b')$ consists of all $\mathbf{u} \in U_\alpha$ such that for $b'' \neq b, b^{max}$ we have $u_{b''} = \alpha_{b''} + \varepsilon_{b''}$ where $\varepsilon_{b''} \in (0, \bar{\varepsilon})$, and $u_b = \alpha_b + \delta_b + \varepsilon_b$ where $\varepsilon_b \in (0, \bar{\varepsilon})$, and $u_{b^{max}} = \alpha_{b^{max}} + \delta_{b^{max}} + \varepsilon_{b^{max}}$ where $\varepsilon_{b^{max}} \in (0, \bar{\varepsilon})$. So, utility vectors in $f(b, b')$ assign to each bundle b'' an utility equal to the bundle's price $\alpha_{b''}$ perturbed by some positive constant. For each bundle $b'' \neq b^{max}$, let $\hat{\alpha}_{b''}$ be the next highest price after $\alpha_{b''}$, i.e., $\alpha_{b''} < \hat{\alpha}_{b''}$ and there is no bundle b^* such that $\alpha_{b''} < \alpha_{b^*} < \hat{\alpha}_{b''}$. We shall choose three positive constants δ_b, $\delta_{b^{max}}$, and $\bar{\varepsilon}$, so that they satisfy the following constraints:

(i) For each b'' such that $\alpha_{b''} \neq \alpha_{b^{max}}$ we have $\alpha_{b''} + \delta_b + \bar{\varepsilon} < \hat{\alpha}_{b''}$.

(ii) $\delta_b > \bar{\varepsilon}$

(iii) If $\alpha_b > 0$ and $b \neq b^{max}$, then for each $b'' \neq b, b^{max}$ such that $\alpha_{b''} > 0$ we have

$$\frac{\delta_b}{\alpha_b} > \frac{\delta_{b^{max}} + \bar{\varepsilon}}{\alpha_{b^{max}}} + \left(\frac{1}{\alpha_b} - \frac{1}{\alpha_{b^{max}}}\right)\bar{\varepsilon} > \frac{\delta_{b^{max}}}{\alpha_{b^{max}}} > \frac{\bar{\varepsilon}}{\alpha_{b''}}. \tag{C.1}$$

In words, the first constraint makes sure that the ranking induced by the perturbed utilities is consistent with the strict ranking induced by prices $\boldsymbol{\alpha}$. The second constraint implies that if bundle b is a free bundle, then it is the most attractive among all free bundles. The last constraint implies that bundles b and b^{max} deliver the highest utility per unit of artificial currency among all non-free bundles, and, roughly speaking, bundle b is sufficiently more attractive than bundle b^{max}.

Clearly, set $f(b, b')$ is open in U as a product of open intervals in $\mathbb{R}$. Next, we show that there exists some $\varepsilon > 0$ so that $x_b(\mathbf{u}) > \varepsilon$ for each $\mathbf{u} \in f(b, b')$.

We begin by showing that in the CEEI, there does not exist $b'' \neq b, b^{max}$ and $\mathbf{u} \in f(b, b')$ such that $\alpha_{b''} > 0$ and $x_{b''}(\mathbf{u}) > 0$. For the sake of contradiction, suppose such b'' and $\mathbf{u}$ exist. Consider reducing expenditures of such agents on b'' by $\eta > 0$, and increasing their expenditures on b^{max} by η. So their probability share of b'' decreases by $\frac{\eta}{\alpha_{b''}}$, and their probability share of b^{max} increases by $\frac{\eta}{\alpha_{b^{max}}} \leq \frac{\eta}{\alpha_{b''}}$. To keep the sum of probability shares equal to 1, increase the share of any free bundle by $\frac{\eta}{\alpha_{b''}} - \frac{\eta}{b^{max}}$ (note that there always exists a bundle with price zero). For a sufficiently small $\eta > 0$, such transfer of mass is feasible and increases the utility of agents with $\mathbf{u} \in f(b, b')$ by assumption (C.1), a contradiction to the allocation being a CEEI.

Now we show that, in the CEEI, we must have $x_b(\mathbf{u}) \geq 1 - \frac{1}{\alpha_{b^{max}}} > 0$ for all $\mathbf{u} \in f(b, b')$. First, suppose $\alpha_b = 0$. Then, given the above result (i.e., that there does not exist $b'' \neq b, b^{max}$ and $\mathbf{u} \in f(b, b')$ such that $\alpha_{b''} > 0$ and $x_{b''}(\mathbf{u}) > 0$), each agent with $\mathbf{u} \in f(b, b')$ must spend her entire budget on b^{max} in purchasing a $\frac{1}{\alpha_{b^{max}}} < 1$ probability share of b^{max}, and complete the allocation with the free bundle b in purchasing a $1 - \frac{1}{\alpha_{b^{max}}} > 0$ probability share of b because $\delta_b > \bar{\varepsilon}_{b''}$ for each $b'' \neq b$ such that $\alpha_{b''} = 0$. Second, suppose $\alpha_b > 0$, and, for the sake of contradiction, suppose $x_b(\mathbf{u}) = 0$. Then, given the above result, it must be that agents with utilities in $f(b, b')$ acquire strictly positive shares of only bundle b^{max} and some free bundle b_0. Consider reducing their expenditures on b^{max} by $\eta > 0$ and similarly increasing their expenditures on b. To keep the sum of probability shares equal to 1, we decrease the share of the free bundle b_0. Such transfer of mass is feasible for small a enough η and increases the utility of these agents by assumption (C.1). Hence, we must have $x_b(\mathbf{u}) > 0$, and moreover either $x_{b^{max}}(\mathbf{u}) = 0$ or $x_{b_0}(\mathbf{u}) = 0$. Therefore, the probability share of b is bounded from below by $1 - \frac{1}{\alpha_{b^{max}}} > 0$. We note, for future use, that if $\mathbf{u} \in f(b, b')$, then $x_b(\mathbf{u}') \geq 1 - \frac{1}{\alpha_{b^{max}}} > 0$ for any $\mathbf{u}' = \lambda \mathbf{u} - \xi$ with $\lambda > 0$ and $\xi \in \mathbb{R}$ since the above argument would still apply to such linear transformations of $\mathbf{u}$.

We now show how from our open set $f(b, b')$ in U, one can build an open cone $C(b, b')$ in $\mathcal{C}$ where $x_b(\mathbf{u}) \geq 1 - \frac{1}{\alpha_{b^{max}}}$ for any $\mathbf{u} \in C(b, b')$. In the sequel, we recall that Proj_D stands for the projection from U into D, i.e.,

$$\text{Proj}_D(\mathbf{u}) := (u_b - \frac{\sum_b u_b}{|B|})_b.$$

As we already noted, for any $\mathbf{u} \in f(b, b')$, $\lambda \mathbf{u}$ also has $x_b(\lambda \mathbf{u}) \geq 1 - \frac{1}{\alpha_{b^{max}}}$ for any $\lambda > 0$. Given $\lambda > 0$, we denote $X_\lambda := \{\mathbf{u}' \in U : \mathbf{u}' = \lambda \mathbf{u} \text{ for some } \mathbf{u} \in f(b, b')\}$. Note that for any $\lambda > 0$, X_λ is open in U (since the function $\mathbf{u} \mapsto \lambda \mathbf{u}$ is an homeomorphism). Now, let us consider $\mathcal{Z} := \cup_{\lambda > 0} X_\lambda$. Note that, as a union of open sets, $\mathcal{Z}$ is open in U. Let $C := \text{Proj}_D(\mathcal{Z})$. Here again, for any $\mathbf{u} \in C$, we must have $x_b(\mathbf{u}) \geq 1 - \frac{1}{\alpha_{b^{max}}}$ since such $\mathbf{u}$ are simple linear transformations of utility vectors in $f(b, b')$.

We first claim that C is a cone. Take any $\mathbf{u}' \in C$ and any $\lambda > 0$. We must show that $\lambda \mathbf{u}' \in C$. Indeed, since $\mathbf{u}' \in C$, we must have that for some $\mathbf{u} \in \mathcal{Z}$, $\text{Proj}_D(\mathbf{u}) = \mathbf{u}'$. Hence, $\text{Proj}_D(\lambda \mathbf{u}) = \lambda \text{Proj}_D(\mathbf{u}) = \lambda \mathbf{u}'$ where the first equality uses the linearity of Proj_D. Since, by definition of set $\mathcal{Z}$, it must be that $\lambda \mathbf{u}$ belongs to $\mathcal{Z}$, $\text{Proj}_D(\lambda \mathbf{u}) = \lambda \mathbf{u}'$ implies that $\lambda \mathbf{u}' \in \text{Proj}_D(\mathcal{Z}) = C$, as claimed.

Now, we show that C is open in D in order to eventually show that C is open in $\mathcal{C}$. This comes from the feature that Proj_D is an open map together with the fact that $\mathcal{Z}$ is open in U.[30] Finally, we want to show that our cone C is open in $\mathcal{C}$, i.e., $C \cap \tilde{D}$ is open in $\tilde{D}$. This is true since, as we just claimed, C is open in D and so $C \cap \tilde{D}$ is open in $\tilde{D}$ by definition of the relative topology. Thus, we can set $C(b, b') := C$. The open cone $C(b, b')$ satisfies $x_b(\mathbf{u}) \geq 1 - \frac{1}{\alpha_{b^{max}}}$ for any $\mathbf{u} \in C(b, b')$. Further, since $f(b, b') \subset U_\alpha$, by construction, we must have $C(b, b') \subset U_\alpha$.

Finally, we construct improving bilateral transfers for agents with utilities in $C(b, b')$. For each $y_{b,b'} > 0$ and $\epsilon > 0$, let agents with $\mathbf{u} \in C(b, b')$ transfer a probability mass of $\frac{\epsilon}{F(C(b,b'))} y_{b,b'}$ from b to b' at their random allocation $\mathbf{x}(\mathbf{u})$. Recall that $F(C(b, b')) > 0$ because F has full relative support. Hence, the total mass transferred from b to b' is $\epsilon y_{b,b'} \geq 0$. Then, for some small enough $\epsilon > 0$, these are improving bilateral transfers since $C(b, b') \subset U_\alpha$. Moreover, because $\mathbf{A}^T \mathbf{y} = 0$, by Lemma 3 these transfers do not change the allocated mass of each object. Therefore, $\mathbf{x}$ is not Pareto efficient, which is a contradiction. It follows that there exist linear cutoffs $\bar{\alpha}$ such that for each $b, b' \in B$, if $\alpha_b < \alpha_{b'}$, then $\bar{\alpha}_b < \bar{\alpha}_{b'}$.

Note that in order to ensure that $\bar{\alpha} \in [0, 1]^{|B|}$, we can normalize the cutoff by $\max_b \bar{\alpha}_b$. ∎

[30] Proj_D is a continuous mapping under our topologies and it is surjective and linear. By the open mapping theorem, Proj_D is an open mapping, i.e., for any open set $\mathcal{O}$ in U, $\text{Proj}_D(\mathcal{O})$ is open in D.

To finish the proof of Theorem 5, we construct a MRB mechanism with linear cutoffs that implements allocation rule $\mathbf{x}$.

PROOF. By Proposition 3 and 5 there exists a MRB mechanism $L = (\boldsymbol{\alpha}, \mathcal{G})$ such that there exist linear prices $\bar{\boldsymbol{\alpha}}$ such that for all $b, b' \in B$ if $\alpha_b < \alpha_{b'}$, then $\bar{\alpha}_b < \bar{\alpha}_{b'}$. We now construct a collection of distributions $\mathcal{G}'$ such that the mechanism $L' = (\bar{\boldsymbol{\alpha}}, \mathcal{G}')$ implements the allocation rule $\mathbf{x}$, where $\bar{\boldsymbol{\alpha}}$ are the linear prices obtained from Proposition 5. For each distribution $G_{\mathbf{x}(\mathbf{u})} \in \mathcal{G}$, let the corresponding distribution $G'_{\mathbf{x}(\mathbf{u})}$ assign probability $x_b(\mathbf{u})$ to $\bar{\alpha}_b$ instead of α_b. Note that in $L' = (\bar{\boldsymbol{\alpha}}, \mathcal{G}')$, for each realization of a random budget, the set of affordable bundles is the same as in L for each distribution. Then we have that, for each distribution and for each set of bundles, the probability that this set is affordable is the same in L and L'. Hence, the induced allocation rules must be the same.

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