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Fiscal Policy under Secular Stagnation: An Optimal Pump Priming Strategy

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Fiscal Policy under Secular Stagnation: An Optimal Pump Priming Strategy*

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Abstract

This paper provides a new perspective on fiscal policy. A permanent depression in aggregate demand results in multiple equilibria: a secular stagnation equilibrium characterized by a binding zero lower bound, low inflation, and underemployment; and a neoclassical equilibrium where inflation is sufficiently high for the zero lower bound to be non-binding at the (very low) natural real interest rate, resulting in full employment. The optimal fiscal policy under secular stagnation consists in moving the economy to the neoclassical equilibrium. This requires a temporary, but massive, amount of government spending to overheat the economy such as to raise the inflation anchor. The lack of fiscal space *cannot* prevent the government from pump priming the economy through fiscal policy. It may in fact help spur inflation. To keep a tight control over the price level, the government can finance the stimulus through a sufficiently long maturity structure of government debt.

Keywords: Fiscal policy, Liquidity trap, Ponzi scheme, Secular stagnation

JEL Classification: E12, E62, E63, H63

1 Introduction

For the past quarter of a century, the Japanese economy has been liquidity trapped. Inflation, long-term interest rates, and economic growth have remained subdued, despite highly expansionary monetary and fiscal policy, resulting in a 250% debt-to-GDP ratio, 40% of which has been bought by the central bank. For over a decade, the Eurozone has been trapped into a similar situation, with no end in sight. The spectre of Japanification now looms throughout the industrialized world.

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These manifestations of secular stagnation can be accounted for by a persistent lack of demand, resulting in a very depressed natural real interest rate (Michau 2018, Eggertsson, Mehrotra, and Robbins 2019). This leads to a binding zero lower bound on the nominal interest rate, while inflation is determined by some nominal frictions, such as downward nominal wage rigidities. The resulting real interest rate is above the natural real interest rate that clears the market for capital at full employment.[1]

However, in addition to the *secular stagnation equilibrium* characterized by a zero nominal interest rate, low inflation, and underemployment, there must exist a *neoclassical equilibrium* with sufficiently high inflation for the zero lower bound to be non-binding at the natural real interest rate, resulting in full employment.[2] This begs the question: How can we move the economy from the secular stagnation to the neoclassical equilibrium? In this paper, I investigate how this can be achieved through fiscal policy.

My analysis relies on a parsimonious representative-household model of secular stagnation. Following Michau (2018), I start from the Ramsey model, to which I add money (to have the zero lower bound), a preference for wealth (to depress aggregate demand and to have a finite elasticity of steady state consumption with respect to the steady state real interest rate), and a downward nominal wage rigidity (to put a break on the deflationary spiral under stagnation). In addition, to have an upward slopping Phillips curve and non-trivial inflation dynamics, I assume sluggish wage adjustments, which are partly determined by a backward looking inflation anchor. Importantly, this framework allows for the possibility of a Ponzi debt scheme, which is essential for a careful analysis of debt sustainability. One contribution of this paper is therefore to offer an analytically simple, yet rich, model of the macroeconomy (which is fully summarized by equations (23) to (29) below).

Initially, the economy is assumed to be trapped into the secular stagnation steady state, with inflation expectations anchored at a very low level. To permanently move the economy to the neoclassical equilibrium, the inflation anchor needs to increase sufficiently to make the zero lower bound non-binding. But, to raise the inflation anchor, the economy needs to overheat for some time, with labor demand rising above desired labor supply.

In the ideal case of full commitment and state-contingent government spending, this can be achieved through a fiscal policy of inflation targeting. The threat of massive public spending whenever inflation falls below target induces households to rationally expect

[1]Importantly, the nominal rigidity is not the fundamental cause of secular stagnation. As wages become more flexible, inflation is even lower, the real interest rate even higher, which further depresses aggregate demand. This is the paradox of flexibility (Michau 2018, Eggertsson, Mehrotra, and Robbins 2019).

[2]While the rate of unemployment is currently very low in Japan, the number of part-time jobs has been steadily increasing over time. This, together with the lack of inflationary pressures, is symptomatic of underemployment (Hashimoto, Ono, and Schlegl 2020). Blanchflower (2019) forcefully argues that "underemployment has replaced unemployment as the main measure of labor market slack".

high inflation and, hence, to spend sufficiently to hit the inflation target. Thus, the secular stagnation equilibrium is eliminated by an off-the-equilibrium threat of massive government spending. A policy of forward guidance, committing to keep the nominal interest rate at zero for longer than strictly necessary, can be used to fine-tune the policy.

However, this scenario relies on a very optimistic view of the flexibility of fiscal policy. I therefore subsequently focus on the more realistic case where the government can only commit to a non-contingent spending plan over a fixed horizon. I assume that, for households to coordinate on the neoclassical equilibrium, the inflation anchor under secular stagnation must exceed a given threshold. Hence, a large fiscal stimulus is necessary to induce the economy to overheat sufficiently to raise the inflation anchor, despite depressed consumption from households who expect the economy to return to the secular stagnation steady state. In other words, the only way to convince households that stagnation is over is to generate high inflation, even if households are so pessimistic as to expect stagnation to persist forever.

I consider two scenarios: under naive expectations, the economy only jumps to the neoclassical equilibrium path once the inflation anchor reaches the threshold; while, under rational expectations, households immediately realize that the path of government spending is sufficient to put an end to stagnation. In this latter case, the fiscal stimulus that raises the inflation anchor in the (off-the-equilibrium) stagnation path causes a consumption boom in the (on-the-equilibrium) neoclassical path, resulting in excessive overheating. Under my calibration, for the inflation anchor to reach the 4.2% threshold under stagnation, it ends up reaching 7.9% in the neoclassical equilibrium. This can however be avoided through state-contingent monetary policy, whereby the government implements a contractionary monetary policy during the fiscal stimulus episode.

Depending on the scenario, the optimal reflation policy consists of a total fiscal stimulus of 22 to 35% of GDP, which is spread over 6 to 18 months. A legitimate concern is that some countries may not have the required fiscal space to finance such a stimulus program. I therefore carefully investigate the consequences of financing the stimulus through public debt, instead of lump-sum taxes. A first possibility is that this triggers an upward jump in the initial price level, such as to reduce the real value of public liabilities, in line with the fiscal theory of the price level. This is equivalent to a lump-sum tax on the representative household, except that the jump in the price level can help stimulate the economy by raising the inflation anchor.

Under stagnation, the natural real interest rate is likely to be so low as to make a Ponzi debt scheme sustainable. Hence, an alternative possibility, is for the fiscal stimulus to generate a Ponzi scheme. This raises household wealth, which helps stimulate aggregate demand.

These results show that the lack of fiscal space *cannot* prevent the government from reflating the economy through expansionary fiscal policy. Fundamentally, a debt sustainability problem can only help generate inflation, which is the goal of the fiscal stimulus.

However, if households do not believe the Ponzi scheme to be sustainable, it must trigger an upward jump in the initial price level. To avoid losing its control over the price level, the government can alternatively finance the fiscal stimulus by extending the maturity structure of its debt before implementing the policy. In this case, the government is simply exploiting the fact, by changing the equilibrium of the economy, it will change asset prices.

Finally, I show that the nature of the optimal reflation policy is robust to the introduction of capital with adjustment costs for investment.

Related Literature. The Great Recession has led to a resurgence of interest for fiscal policy under liquidity trap circumstances. This literature has emphasized the desirability of relying on government spending to prop up aggregate demand such as to break the deflationary spiral (Werning 2012, Schmidt 2013, 2017, Murota and Ono 2015, Nakata 2016, Bilbiie, Monacelli, and Perotti 2019, Michau 2019). This has resulted in a large emphasis on the magnitude of the fiscal multiplier (Christiano, Eichenbaum, and Rebelo 2011, Woodford 2011, Farhi and Werning 2016, Hills and Nakata 2018, Roulleau-Pasdeloup 2018). However, under secular stagnation, even with a large multiplier, it is not desirable to permanently replace a lack of private demand by high public spending. Hence, this paper provides a complementary perspective on fiscal policy by emphasizing its ability to pump prime the economy. Importantly, this only justifies very large stimulus packages, as small ones cannot do the job of permanently lifting the economy out of stagnation.

Benhabib, Schmitt-Grohé, and Uribe (2001) were the first to point out that the liquidity trap could result from self-fulfilling deflationary expectations.[3] While Mertens and Ravn (2014) have established that government spending are deflationary within such an expectations-driven liquidity trap, Nakata and Schmidt (2019) have shown that the response of government spending can be so strong as to eliminate this liquidity trap equilibrium. This is very similar to my state-contingent fiscal policy that eliminates the secular stagnation equilibrium. However, the underlying mechanism is diametrically opposed: in my fundamentals-driven liquidity trap the inflationary effect of government spending can be sufficiently strong to eliminate the low inflation secular stagnation equilibrium, whereas in their expectations-driven liquidity trap the deflationary effect of government spending can be so strong as to be inconsistent with the existence of a fixed-point at the zero lower bound.[4]

[3]Benhabib, Schmitt-Grohé, and Uribe (2002) have argued that the implementation of a non-Ricardian policy could eliminate the liquidity trap equilibrium by making it fiscally unsustainable. However, this argument relies on the fiscal theory of the price level, which does not hold when Ponzi schemes can be sustainable (Bassetto and Cui 2018).

[4]Bilbiie (2019) has combined the expectations-driven and the fundamentals-driven liquidity trap

A number of papers have investigated the effects of public debt under liquidity trap circumstances. Eggertsson (2006) was the first to point out that, in the absence of commitment, public debt can make promises of future inflation credible. Similarly, Burgert and Schmidt (2014) have found that a high level of public debt makes discretionary monetary policy more accommodative, while reducing the magnitude of the optimal fiscal stimulus. Bianchi and Melosi (2019) and Bianchi, Faccini, and Melosi (2020) have emphasized that, under monetary and fiscal coordination, high public debt can enhance the effectiveness of a fiscal stimulus by raising the inflation that is tolerated by the central bank. Nakata (2017) has documented that, under full commitment, a high level of public debt makes expansionary fiscal policy even more desirable. Relative to this literature, my paper incorporates the possibility of sustainable Ponzi schemes. It also shows that, when the optimal policy consists in moving to a different equilibrium, the government can raise resources by exploiting the maturity structure of its debt. My analysis concurs with Blanchard's (2019) insight that public debt should not be a big concern in a low interest rate environment.

Interestingly, Bhattarai, Eggertsson, and Gafarov (2019) have shown that, by reducing the duration of government liabilities, quantitative easing strengthens the government's commitment to keeping the nominal interest rate low once aggregate demand has recovered. By contrast, in my secular stagnation framework, even though there is no commitment problem, a long maturity structure of public debt strengthens the benefits to the government of shifting to the neoclassical equilibrium, where the nominal interest rate is typically no longer at the zero lower bound. This is another illustration of the difference between the management of a temporary and of a permanent liquidity trap.[5]

Models of demand-driven secular stagnation can rely on a preference for wealth (Michau 2018) or an OLG structure (Eggertsson, Mehrotra, and Robbins 2019).[6] However, despite different micro-foundations, the properties of the secular stagnation and of the neoclassical equilibrium are identical under these two model structures, both of which allow for the possibility of Ponzi schemes.[7] Hence, the results of this paper do not require the preference for wealth and could alternatively be derived under an OLG model

within a single framework, showing the they advocate for opposite monetary and fiscal policy responses. Cuba-Borda and Singh (2020) have emphasized that the critical difference between the two is that expectations-driven liquidity traps are locally indeterminate, while the fundamentals-driven ones are locally determinate.

[5] Bouakez, Oikonomou, and Priftis (2018) have investigated how the maturity structure of government debt should be used to manage the uncertainty associated with the zero lower bound.

[6] The first micro-founded model of demand-driven secular stagnation was offered by Ono (1994, 2001), who assumed an insatiable preference for liquidity. Michaillat and Saez (2019) have also built a model of the business cycle with matching frictions, where a preference for wealth can generate a permanent liquidity trap. Geerolf (2019) has relied on a superelliptic production function to show that the demand for investment can be insensitive to the real interest rate, leading to secular stagnation.

[7] More specifically, Michau, Ono, and Schlegl (2020) have shown that the characterization of rational bubbles or Ponzi schemes under a preference for wealth is exactly the same as under an OLG structure.

of secular stagnation.

Finally, while the fiscal policy of this paper might seem rather extreme, there are few other solutions to bring secular stagnation to an end, none of which is easy to implement. One is to abolish cash, such as to remove the zero lower bound. Another is to stimulate aggregate demand through tax policy. This either requires a rising path of consumption taxes and a falling path of labor income taxes (Eggertsson 2010, Correia, Farhi, Nicolini, and Teles 2013), which requires a high degree of tax flexibility and cannot be sustained forever, or a wealth tax (Michau 2018), which is fraught with wealth measurement problems. Helicopter drops of money, i.e. money-financed transfers to the representative household, can work, but can also induce the government to lose control over the price level (Michau 2020). Finally, in a heterogeneous-agent economy, the government can try raise the natural real interest rate by redistributing resources across households (Rachel and Summers 2019); however these policies are far from optimal if such redistribution is not otherwise desirable. While these policy options are not mutually exclusive, a pump-priming fiscal policy, financed by a fall in the value of long-term debt, offers a serious candidate solution.

This paper begins with a careful exposition of the model structure. Section 3 provides a definition of equilibrium, while the steady state equilibria are characterized in Section 4. The calibration of the model is performed in Section 5. The optimal fiscal policy, under lump-sum taxes, is derived in Section 6. The following section carefully investigates the issue of debt sustainability. Section 8 incorporates capital into the analysis. The paper ends with a conclusion.

2 Economy

This section exposes the setup of the economy, starting with households and firms, before turning to the determination of sluggish wages and, finally, to the derivation of the government budget constraint.

2.1 Households

Time is continuous. There is a mass 1 of infinitely lived households. Population within each household grows at rate n. The total population of the economy is equal to $N_t = e^{nt}$.

The representative household discounts the future at rate ρ, with $\rho > n$. Let c_t denote private consumption per capita and g_t public consumption per capita. At any point in time, the household derives utility $u(c_t)$ and $\chi(g_t)$ from consuming a quantity c_t of private consumption goods and a quantity g_t of public consumption goods, with $u'(\cdot) > 0$, $u''(\cdot) < 0$, $\lim_{c \to 0} u'(c) = \infty$, and $\chi'(\cdot) > 0$, $\chi''(\cdot) < 0$, $\lim_{g \to 0} \chi'(g) = \infty$. For simplicity, I do not allow for complementarity between private and public consumption. The household incurs disutility $v\left(l_t^s\right)$ from supplying l_t^s units of labor per capita, with $v'(\cdot) > 0$, $v''(\cdot) > 0$, $v'(0) = 0$, and $\lim_{l^s \to \bar{l}} v'(l^s) = \infty$ where $\bar{l}$ is the maximum feasible supply of labor, which can be infinite.

The household also derives utility from holding wealth a_t. However, government debt b_t is a liability to the government and, hence, to the tax payer; unless the government intends to run a Ponzi scheme. The representative household therefore perceives its net wealth to be equal to $a_t - b_t + \Delta_t$, where Δ_t denotes the magnitude of the government's Ponzi scheme.[8] The household derives utility $\gamma(a_t - b_t + \Delta_t)$ from holding net wealth $a_t - b_t + \Delta_t$, with $\gamma'(\cdot) > 0$, $\gamma''(\cdot) < 0$, $\gamma'(0) < \infty$, and $\lim_{k\to\infty} \gamma'(k) = 0$. Note that, if the household cared about wealth rather than net wealth, then the government could artificially increase welfare by making a large lump-sum payment that would eventually be offset by a large lump-sum tax. In other words, we assume that households are Ricardian and that they do not suffer from any wealth illusion from government transfers. The household's intertemporal utility function is given by:[9]

$$\int_0^\infty e^{-(\rho-n)t} \left[u\left(c_t\right) + \chi\left(g_t\right) - v\left(l_t^s\right) + \gamma\left(a_t - b_t + \Delta_t\right)\right] dt. \tag{1}$$

At time t, the real wage is equal to w_t, the dividends per capita from firm ownership to ξ_t, the lump-sum tax per capita to τ_t, and the real interest rate to r_t. Population growth within the household results in a dilution of wealth. Hence, the expected net return on wealth per capita is equal to $r_t - n$, which implies the following flow of funds constraint:[10]

$$\dot{a}_t = \left(r_t - n\right) a_t + w_t l_t^s + \xi_t - \tau_t - c_t. \tag{2}$$

The household is subject to an intertemporal budget constraint that prevents it from running Ponzi schemes:

$$\lim_{t\to\infty} e^{-\int_0^t (r_s - n) ds} a_t \geq 0. \tag{3}$$

The household maximizes its intertemporal utility (1) subject to its budget constraint

[8] Michau (2019b) provides a careful justification for this specification of net household wealth.

[9] All the results of the paper would hold under a constant rate κ of exogenous technical progress, provided that the household has balanced growth preferences:

$$\int_0^\infty e^{-(\rho-n)t} \left[\ln\left(c_t\right) + \ln\left(g_t\right) - v\left(l_t^s\right) + \gamma\left(\frac{a_t - b_t - m_t + \Delta_t}{y_t}\right)\right] dt,$$

where y_t denotes output per capita (or alternatively, to obtain exactly the same formulae as in the paper, $y_t = e^{\kappa t}$). Under all steady states, including the secular stagnation steady state, the economy would grow at rate $n + \kappa$ instead of n.

[10] A similar wealth accumulation equation is formally derived in a nominal economy without population growth in Michau (2018) and in a real economy with population growth in Michau, Ono, and Schlegl (2020).

(2) and (3) with a_0 given. By the maximum principle, the solution to the household's problem is characterized by a consumption Euler equation:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\varepsilon_u(c_t)} \left[r_t - \rho + \frac{\gamma'(a_t - b_t + \Delta_t)}{u'(c_t)} \right], \tag{4}$$

where $\varepsilon_u(c_t) = -c_t u''(c_t)/u'(c_t)$, a labor supply function:

$$v'(l_t^s) = w_t u'(c_t), \tag{5}$$

and a transversality condition:

$$\lim_{t \to \infty} e^{-(\rho - n)t} u'(c_t)\, a_t = 0. \tag{6}$$

From the consumption Euler equation (4), the preference for wealth makes households more patient. In steady state (i.e. when the square bracket is equal to zero), it also implies a finite elasticity of consumption with respect to the real interest rate.

Importantly, I am assuming a nominal economy. By the Fisher identity, the real interest rate r_t is equal to the nominal interest rate i_t net of inflation π_t. The nominal interest rate cannot be negative:

$$i_t \geq 0. \tag{7}$$

While I just impose this constraint, it can easily be derived by inserting money in the utility function. This would yield a money demand equation, which would imply that the nominal return on bonds cannot be smaller than the zero nominal return on money (Michau 2018, 2020).

2.2 Firms

For simplicity, I assume for now that labor is the only factor of production. Total population is equal to N_t. The representative firm employs L_t^d units of labor per capita. The aggregate production function is $N_t f(L_t^d)$ with $f'(\cdot) > 0$, $f''(\cdot) \leq 0$, and $f(0) = 0$. Thus, the economy displays constant returns to scale with respect to the total population of the economy, but non-increasing returns with respect to the employment level per capita.

Allowing for decreasing returns makes labor demand a decreasing function of the real wage, which facilitates the analysis. However, the special case of constant returns to scale, where labor demand is perfectly elastic, does not cause any theoretical problem.

Aggregate output $N_t f\left(L_t^d\right)$ consists of private consumption goods $c_t N_t$ and of public consumption goods $g_t N_t$. We therefore have:

$$c_t + g_t = f\left(L_t^d\right), \tag{8}$$

The firms chooses labor demand L_t^d such as to maximize profits $N_t f\left(L_t^d\right) - w_t N_t L_t^d$, which implies that the equilibrium real wage must always be equal to marginal product of labor:

$$w_t = f'\left(L_t^d\right). \tag{9}$$

Aggregate profits $\xi_t N_t$ are therefore equal to $N_t f\left(L_t^d\right) - f'\left(L_t^d\right) N_t L_t^d$ or, equivalently:

$$\xi_t = f\left(L_t^d\right) - f'\left(L_t^d\right) L_t^d. \tag{10}$$

Profits are strictly positive whenever the production function is characterized by decreasing returns to scale.

2.3 Wage Sluggishness

Nominal wages adjust sluggishly over time. This generates a discrepancy between the quantity l_t^s of labor that households would like to supply at time t, and the quantity L_t^d that firms demand. I assume that households do supply whatever quantity of labor L_t^d firms demand, while putting an upward pressure on sluggish wages whenever firms' labor demand L_t^d is above households' desired labor supply l_t^s and a downward pressure in the opposite case. In addition, households impose a downward nominal wage rigidity.

The profit maximizing behavior of firms implies, by (9), that the nominal wage W_t is always equal to the marginal product of labor $P_t f'(L_t^d)$. Hence, for a given employment level L_t^d, the nominal wage W_t grows at rate π_t. The wage rigidity, for a given employment level L_t^d,[11] is specified as follows:

$$\left(1 + \pi_t dt\right) W_t = \max\left\{ \left(1 + \pi_t^A dt\right) W_t + \beta dt \left[\frac{P_t v'\left(L_t^d\right)}{u'\left(c_t\right)} - W_t \right], \left(1 + \pi^R dt\right) W_t \right\}, \tag{11}$$

together with:

$$\dot{\pi}_t^A = \theta \left[\pi_t - \pi_t^A\right], \tag{12}$$

where $\beta > 0$ and $\theta > 0$. The first term within the maximization on the right-hand side of (11) corresponds to the nominal wage sluggishness, the second to the downward nominal wage rigidity. Let us now provide an interpretation for each of these two terms.

Wage sluggishness implies that an inflation anchor π_t^A partly determines the growth rate of nominal wages, for a given employment level L_t^d. The deviation from the anchor is proportional to the wedge between the (money-metric) marginal disutility of labor

[11] For simplicity, I assume that, when the employment level L_t^d changes, nominal wages adjust in line with the resulting evolution of the marginal product of labor $f'\left(L_t^d\right)$. This is consistent with the "fair wage" microfoundation of Ono and Ishida (2014). This assumption is not needed under a constant marginal product of labor, i.e. $f\left(L_t^d\right) = L_t^d$.

$P_t v'\left(L_t^d\right)/u'(c_t)$ and the nominal wage rate W_t. Recall that, by the labor supply function (5), $W_t = P_t v'(l_t^s)/u'(c_t)$. Hence, whenever workers supply more labor than they would like to, i.e. whenever $L_t^d > l_t^s$, the growth of nominal wages exceeds the anchor. The anchor itself is slowly adjusting over time. Integrating (12) from $-\infty$ to time t, subject to $\lim_{T\to-\infty} e^{\theta T}\pi_T^A = 0$, yields:

$$\pi_t^A = \int_{-\infty}^{t} \theta e^{-\theta(t-s)}\pi_s ds. \tag{13}$$

The anchor is therefore determined as a weighted average of past inflation realizations. Note that perfectly flexible wages correspond to the limit as either β or θ tends to infinity.

In addition to this wage sluggishness, I impose a downward nominal wage rigidity. For a given employment level L_t^d, workers never accept the growth rate of their nominal wages to fall below a reference rate of inflation π^R. For instance, the celebrated downward nominal wage rigidity, whereby workers do not accept nominal wage cuts, corresponds to $\pi^R = 0$. The reference rate of inflation π^R, unlike the inflation anchor π^A, is a fixed parameter that does not adjust over time. This feature is necessary to obtain a secular stagnation steady state with constant inflation and under-employment, i.e. $L_t^d < l_t^s$.[12]

Using the labor supply function (5), the wage sluggishness equation (11) can be written as:

$$\pi_t = \max\left\{\pi_t^A + \beta\left[\frac{v'\left(L_t^d\right)}{v'\left(l_t^s\right)} - 1\right], \pi^R\right\}. \tag{14}$$

This resembles the expectation-augmented Phillips curve, whereby the updating rule for the anchor (12) prevents the economy from permanently operating above full capacity. In addition, the downward wage rigidity flattens the Phillips curve at low rates of inflation, consistently with the empirical evidence provided by Akerlof, Dickens, and Perry (1996, 2000).

2.4 Government

Let B_0 denote the initial level of nominal government debt. At time t, the government collects lump-sum taxes τ_t per capita and purchases a quantity g_t of goods per capita.

[12] A more general specification for the downward nominal wage rigidity would be:

$$(1+\pi_t dt)W_t \geq \left(1+\pi^R dt\right)W_t + \alpha dt\left[\frac{P_t v'\left(L_t^d\right)}{u'(c_t)} - W_t\right].$$

In equilibrium, an increase in the wage flexibility parameter α exacerbates under-employment $l_t^s - L_t^d$ within the secular stagnation steady state. This paradox of flexibility shows that the downward wage rigidity is not the fundamental cause of secular stagnation (Ono 1994, 2001, Michau 2018). Empirically, the Phillips curve is very flat at low rates of inflation, suggesting that α is close to zero.

Real debt per capita $b_t = B_t/(P_t N_t)$ therefore evolves according to:

$$\dot{b}_t = (r_t - n)\, b_t + g_t - \tau_t. \tag{15}$$

The primary fiscal surplus at time t is simply equal to $\tau_t - g_t$. Let Φ_t denote the present value of primary surpluses from time t onwards:

$$\Phi_t = \int_t^{\infty} e^{-\int_t^s (r_u - n)du} \left[\tau_s - g_s\right] ds. \tag{16}$$

At time t, the government's no-Ponzi condition is given by:

$$\lim_{T\to\infty} e^{-\int_t^T (r_t - n)ds} b_T \leq 0, \tag{17}$$

or, equivalently using equation (15), by:

$$b_t \leq \Phi_t. \tag{18}$$

We can therefore define the magnitude of a Ponzi scheme at time t as the difference between government liabilities b_t and the present value of primary surpluses Φ_t:

$$\Delta_t = b_t - \Phi_t. \tag{19}$$

The no-Ponzi condition (17) or (18) can be written as $\Delta_t \leq 0$. Throughout my analysis, I consider that the government's no-Ponzi condition is either binding or violated, i.e. $\Delta_t \geq 0$.

Finally, by the government liability accumulation equation (15) and the definition of the present value of primary surpluses (16), we have:

$$\dot{\Delta}_t = (r_t - n)\, \Delta_t, \tag{20}$$

regardless of monetary and fiscal policy.

Different paths of lump-sum taxes that result in the same value of Δ_0 have exactly the same effect on the equilibrium of the economy. I henceforth consider that the government's policy consists in setting the initial magnitude of the Ponzi scheme Δ_0, the path of government purchases g_t, and the path of the nominal interest rate i_t.

3 Equilibrium

Let us now characterize the equilibrium of the economy. In the absence of capital, the wealth of the representative household a_t must be exclusively composed of government bonds b_t. This yields the asset market clearing condition:

$$a_t = b_t. \tag{21}$$

Hence, by definition of the Ponzi scheme (19), the household's transversality condition (6) can be written as $\lim_{t\to\infty} e^{-(\rho-n)t}\, u'(c_t)\,[\Delta_t + \Phi_t] = 0$. The present value of primary surpluses Φ_t does not appear in any of the other equilibrium conditions of the economy. By the following lemma, which is proved in appendix A, it can also be eliminated from the transversality condition.

Lemma 1 *If Φ_t is not finite, then an equilibrium cannot exist. If Φ_t is finite, then $\lim_{t\to\infty} e^{-(\rho-n)t} u'(c_t)\,\Phi_t = 0$.*

The household's transversality condition can therefore be simplified to:

$$\lim_{t\to\infty} e^{-(\rho-n)t} u'(c_t)\,\Delta_t = 0. \tag{22}$$

For a given governmental policy, determined by $(g_t, i_t)_{t=0}^{\infty}$ and Δ_0, the equilibrium of the economy, $(c_t, L_t^d, l_t^s, \Delta_t, \pi_t, \pi_t^A)_{t=0}^{\infty}$, is fully characterized by the household's optimality conditions:[13]

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\varepsilon_u(c_t)}\left[i_t - \pi_t - \rho + \frac{\gamma'(\Delta_t)}{u'(c_t)}\right], \tag{23}$$

$$v'(l_t^s) = f'\left(L_t^d\right) u'(c_t), \tag{24}$$

$$\lim_{t\to\infty} e^{-(\rho-n)t} u'(c_t)\,\Delta_t = 0; \tag{25}$$

the government's Ponzi scheme:

$$\dot{\Delta}_t = [i_t - \pi_t - n]\Delta_t; \tag{26}$$

the goods market clearing condition:

$$c_t + g_t = f\left(L_t^d\right); \tag{27}$$

and the wage sluggishness:

$$\pi_t = \max\left\{\pi_t^A + \beta\left[\frac{v'\left(L_t^d\right)}{v'(l_t^s)} - 1\right], \pi^R\right\}, \tag{28}$$

$$\dot{\pi}_t^A = \theta\left[\pi_t - \pi_t^A\right], \tag{29}$$

[13] I have omitted the household's no-Ponzi condition $\lim_{t\to\infty} e^{-\int_0^t (r_s-n)ds} a_t \geq 0$ as it must always be satisfied. This follows from the fact that $a_t = \Delta_t + \Phi_t$ with $\Delta_t \geq 0$ and, by the proof of Lemma 1, $\lim_{t\to\infty} e^{-\int_0^t (r_s-n)ds} \Phi_t = 0$.

with π_0^A given.

Let $\bar{\Delta}_0$ denote the magnitude of the initial Ponzi scheme that the government is willing to implement. But, even when a Ponzi scheme of size $\bar{\Delta}_0$ is theoretically feasible, it can only be an equilibrium outcome if households believe it to be sustainable. Thus, $\Delta_0 = 0$ is always an equilibrium possibility. More generally, we can subsequently consider that $\Delta_0 \leq \bar{\Delta}_0$. The government can only determine the maximum magnitude of the Ponzi debt scheme (Michau, Ono, and Schlegl 2020).

4 Steady State Equilibria

Let us now characterize the steady state equilibria of the economy. I consider that the nominal interest rate is constant, i.e. $i_t = i$. I also assume that, at each point in time, the government takes private consumption c_t as given and sets public spending g_t opportunistically such as to maximize households' immediate utility $u(c_t) + \chi(g_t) - v\left(L_t^d\right) + \gamma'(\Delta_t)$ subject to the resource constraint (27). This yields:

$$\chi'(g_t) = \frac{v'\left(L_t^d\right)}{f'\left(L_t^d\right)}. \tag{30}$$

When employment is depressed, the marginal disutility of work is low and the marginal product of labor is high, both of which raise the demand for public spending.

In steady state, the downward wage rigidity (28) can either be non-binding $\pi = \pi^A > \pi^R$, resulting in full employment $L^d = l^s$, or binding $\pi = \pi^A = \pi^R$, resulting in under-employment $L^d < l^s$. Also, the dynamics of the Ponzi scheme (26) imply that, in steady state, there must either be no Ponzi scheme $\Delta = 0$ or a real interest rate $i - \pi$ equal to the economic growth rate n. This results in three possible steady state equilibria:

- A *neoclassical steady state* with full employment $L^d = l^s$ and no Ponzi scheme $\Delta = 0$;
- A *secular stagnation steady state* with under-employment $L^d < l^s$, low inflation $\pi = \pi^A = \pi^R$, and no Ponzi scheme $\Delta = 0$;
- A *Ponzi steady state* with full employment $L^d = l^s$ and a Ponzi scheme of constant size $\Delta > 0$ thanks to a real interest rate equal to the economic growth rate $i - \pi = n$.

The fourth possibility, combining under-employment and a Ponzi scheme, simultaneously requires $\pi = \pi^R$ and $i - \pi = n$, which is generically impossible for a given nominal interest rate i.[14]

[14] Moreover, underemployment should induce the government to set $i = 0$, which rules out the fourth possibility except in the knife-edge case where $-\pi^R = n$.

4.1 Neoclassical Steady State

The neoclassical steady state $\left(c^n, g^n, L_n^d, l_n^s, \Delta^n, \pi^n, r^n\right)$ is uniquely characterized by the labor supply function (24), the goods market clearing condition (27), the demand for public spending (30), full-employment $L_n^d = l_n^s$, and a binding government budget constraint $\Delta^n = 0$.[15] The real interest rate is determined by the consumption Euler equation (23):

$$r^n = \rho - \frac{\gamma'(0)}{u'(c^n)}. \tag{31}$$

Note that r^n corresponds to the natural real interest rate of the economy. A weak level of aggregate demand, induced by a strong preference for wealth $\gamma'(0)$, entails a low natural real interest rate r^n.

Finally, for any given nominal interest rate i, the corresponding rate of inflation π^n is simply determined by the Fisher identity $\pi^n = i - r^n$. Importantly, if the central bank does not allow inflation to exceed $-r^n$ then, by the zero lower bound, the neoclassical steady state is not feasible.[16]

4.2 Secular Stagnation Steady State

The secular stagnation steady state $\left(c^{ss}, g^{ss}, L_{ss}^d, l_{ss}^s, \Delta^{ss}, \pi^{ss}, r^{ss}\right)$ is uniquely characterized by a binding downward wage rigidity $\pi^{ss} = \pi^R$ and by the absence of Ponzi scheme $\Delta^{ss} = 0$. For any given nominal interest rate i, this determines the real interest rate $r^{ss} = i - \pi^R$. Private demand is determined by the consumption Euler equation (23):

$$\frac{1}{u'(c^{ss})} = \frac{\rho - r^{ss}}{\gamma'(0)}. \tag{32}$$

Public demand g^{ss} and employment L_{ss}^d are then jointly determined from the government's opportunistic behavior (30) and by the goods market clearing condition (27). Finally, the corresponding labor supply l_{ss}^s is given by the household's labor supply function (24).

The secular stagnation steady state exists if and only if the corresponding labor demand L_{ss}^d is smaller than labor supply l_{ss}^s. This is equivalent to requiring $r^{ss} = i - \pi^R >$

[15] Uniqueness is straightforward to prove. Substituting (27) into (30) yields $\chi'\left(f\left(L_n^d\right) - c^n\right) = v'\left(L_n^d\right)/f'\left(L_n^d\right)$, which implies $dc^n/dL_n^d > 0$. Substituting $L_n^d = l_n^s$ into (24) yields $v'\left(L_n^d\right) = f'\left(L_n^d\right)u'(c^n)$. Uniqueness immediately follows from the fact that $v'\left(L_n^d\right)$ is increasing in L_n^d, while $f'\left(L_n^d\right)u'(c^n)$ is decreasing in L_n^d.

[16] It can easily be shown that, under a fixed nominal interest rate i, opportunistic government spending given by (30), and in the absence of Ponzi scheme, the neoclassical steady state is locally stable if and only if:

$$\rho - r^n < \beta\left[1 + \frac{c^n}{\varepsilon_u(c^n)}\left(\frac{L_n^d f'\left(L_n^d\right)}{\varepsilon_v\left(L_n^d\right) + \varepsilon_f\left(L_n^d\right)} + \frac{g^n}{\varepsilon_\chi(g^n)}\right)^{-1}\right],$$

where $\varepsilon_v\left(L_n^d\right) = L_n^d v''\left(L_n^d\right)/v'\left(L_n^d\right)$, $\varepsilon_f\left(L_n^d\right) = -L_n^d f''\left(L_n^d\right)/f'\left(L_n^d\right)$, and $\varepsilon_\chi(g^n) = g^n\chi''(g^n)/\chi'(g^n)$.

r^n.[17] Thus, the secular stagnation steady state exists if and only if aggregate demand is so depressed that the natural real interest rate is smaller than the real interest rate implied by a binding downward wage rigidity.

In the secular stagnation steady state, aggregate demand c^{ss} is a decreasing function of the nominal interest rate i. The government should therefore set $i = 0$. Note that the neoclassical steady state is neo-Fisherian, i.e. an increase in i raises π one-for-one, while the secular stagnation steady state is not. I henceforth assume $-\pi^R > r^n$ so that, even with a zero nominal interest rate, the secular stagnation steady state exists.

4.3 Ponzi Steady State

The Ponzi steady state $\left(c^p, g^p, L_p^d, l_p^s, \Delta^p, \pi^p, r^p\right)$ is uniquely characterized by the labor supply function (24), the goods market clearing condition (27), the demand for public spending (30), and full-employment $L_p^d = l_p^s$. The consumption Euler equation (23) determines the size of the Ponzi scheme Δ^p such that the real interest rate is equal to the growth rate of the economy n:

$$\gamma'\left(\Delta^p\right) = \left(\rho - n\right) u'\left(c^p\right). \tag{33}$$

Finally, for any given nominal interest rate i, the corresponding inflation rate π^p can be deduced from the Fisher identity $\pi^p = i - n$.

For the Ponzi steady state to exist, we must have $\Delta^p > 0$ or equivalently, by (31) and (33), $r^n < n$. By the Euler equation (23), any Ponzi scheme raises the real interest rate. Hence, if $r^n > n$, a Ponzi scheme must grow faster than the economy, be explosive, and therefore violate the transversality condition (25). This explains why the existence condition is $r^n < n$.

Note that the real allocation of resources is identical as in the neoclassical steady state, i.e. $c^p = c^n$, $g^p = g^n$, $L_p^d = L_n^d$, and $l_p^s = l_n^s$. However, this will no longer be the case once I endogenize the capital stock. The Ponzi scheme will raise the real interest rate, reduce the capital stock to its golden rule level, and therefore raise consumption (as in Michau, Ono, and Schlegl 2020).

Finally, the existence of a Ponzi steady state requires that the government is willing to implement a Ponzi scheme of sufficient magnitude, i.e. it requires $\bar{\Delta}_0$ to be sufficiently large.

Before turning to the policy analysis, note that my simple model structure, fully summarized by equations (23) to (29), allows for a secular stagnation steady state with

[17] This is easy to prove. As r^{ss} increases, consumption c^{ss} and labor demand L_{ss}^d fall, by (27) and (30), while labor supply l_{ss}^s increases, by (24). Moreover, when $r^{ss} = r^n$, we have $L_{ss}^d = l_{ss}^s$. Hence, $L_{ss}^d < l_{ss}^s$ is equivalent to $r^{ss} = i - \pi^R > r^n$.

Keynesian properties,[18] a neoclassical steady state with classical properties, the possibility of sustainable Ponzi schemes, as well as a non-trivial Phillips curve that will be critical to the pump priming policy.

5 Calibration

My analysis of fiscal policy relies on numerical simulations. In this section, I therefore calibrate my model.

I assume that households display a constant elasticity of intertemporal substitution for private consumption:

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \tag{34}$$

and for public consumption:

$$\chi(g) = k_G \frac{g^{1-\sigma_G} - 1}{1-\sigma_G}. \tag{35}$$

They have a constant Frisch elasticity of labor supply:

$$v(L) = k_L \frac{L^{1+1/\xi}}{1+1/\xi}. \tag{36}$$

The production function implies a constant labor share:

$$f(L) = L^{1-\alpha}. \tag{37}$$

Regarding the parameters, I set $\sigma = \sigma_G = 2$ and determine k_G such that, in the neoclassical steady state, private consumption c^n is three times larger than public consumption g^n, which yields $k_G = 0.111$. I set the Frisch elasticity of labor supply ξ equal to 0.5 and determine k_L such that the steady state labor supply L_n^d is normalized to one, which yields $k_L = 1.244$. A labor share equal to 70% implies $\alpha = 0.3$. I set the discount factor ρ equal to 4% per year and the population growth rate n to 0%. The secular stagnation rate of inflation π^R, which pins down the initial value of the inflation anchor π_0^A, is set equal to 1%, consistently with the recent experience of Japan or the eurozone. The equilibrium marginal utility of wealth $\gamma'(0)$ is determined such that consumption under secular stagnation is 10% below consumption in the neoclassical steady state, $c^{ss} = (1-0.1)c^n$. This yields $\gamma'(0) = 0.110$. This calibration implies that the natural rate r^n is equal to -2.17%.[19] While consumption under stagnation is 10% below

[18] Relying on a similar structure, Michau (2018) derives the paradox of flexibility, of thrift, and of toil, as well as a fiscal multiplier above one.

[19] Note that a lower elasticity of intertemporal substitution makes steady state consumption less sensitive to the steady state real interest rate. Thus, with $\sigma = \sigma_G = 3$, the natural rate would be equal to -2.86%.

its neoclassical level, opportunistic spending is 5.4% higher, resulting in a 6.2% shortfall in output per capita. Secular stagnation generates a consumption equivalent welfare loss equal to 3.2%, i.e. welfare under stagnation is equal to welfare in the neoclassical steady state with consumption decreased by 3.2%.

Finally, I need to calibrate the two parameters that determine the inertia of the inflation anchor, θ and β. I consider the half-life of the inflation anchor to be equal to two years, which implies $\theta = 0.347$. Empirically, the slope of the Phillips curve is such that (for a given real interest rate) a 1% increase in employment raises inflation by 0.3% (Levy 2019). But, in the neoclassical steady state, from (28) and (36), this elasticity $d\pi_t/d\ln\left(L_t^d/l_t^s\right)$ is equal to β/ξ. This yields $\beta = 0.15$.[20] The calibration of the model is summarized in Table 1.[21]

Parameter	Calibrated value	Moment
Discount rate	$\rho = 4\%$	·
Population growth	$n = 0\%$	·
Non-labor share	$\alpha = 0.3$	·
CRRA for private consumption	$\sigma = 2$	·
Frisch elasticity of labor supply	$\xi = 0.5$	·
Scale parameter of disutility of labor supply	$k_L = 1.244$	$L_n^d = 1$
CRRA for public consumption	$\sigma_G = 2$	·
Scale parameter of utility of public consumption	$k_G = 0.111$	$g^n = c^n/3$
Equilibrium marginal utility of wealth	$\gamma'(0) = 0.110$	$c^{ss} = (1-0.1)c^n$
Reference rate of inflation for wage bargaining	$\pi^R = 1\%$	·
Speed of adjustment of inflation anchor	$\theta = 0.347$	Half-life of $\pi_t^A = 2$
Wage sluggishness	$\beta = 0.15$	Phillips curve slope = 0.3

Table 1: Calibration of the model

While the choice of some of these parameters is somewhat arbitrary, my analysis mostly focuses on qualitative insights, which are robust to plausible changes to this calibration.

[20]Following my model structure, I am implicitly assuming that there is no unemployment in the neoclassical steady state. More generally, if $L_t^d/l_t^s = 1 - u_t$, the Phillips curve can be written as $\pi_t = \pi_t^A + \beta[(1-u_t)^{1/\xi} - 1]$. This implies that $d\pi_t/d(1-u_t) = \beta(1-u_t)^{1/\xi - 1}/\xi$, which hardly affects the calibrated value of β for u_t below 10%.

[21]Under this calibration, by the condition of footnote 16, the neoclassical steady state is locally stable under passive monetary and fiscal policy if and only if $\beta > 0.101$. However, whether or not this condition is satisfied does not affect the nature of the optimal policy.

6 Fiscal Policy

Assuming that the economy is initially in the secular stagnation steady state, I now characterize the optimal monetary and fiscal policy under commitment. In this section, government spending is financed from lump-sum taxes. Ponzi schemes therefore never arise, i.e. $\Delta_t = 0$ for all t.

For a given path of government spending and of the nominal interest rate, there may be multiple equilibria. In particular, there is an optimal path leading to the neoclassical state; but there may also be another path bringing the economy back to secular stagnation. In this section, I first assume that the government can rely on a state-contingent spending plan to eliminate the sub-optimal equilibrium path. I then solve for the optimal policy when such state-contingent plans are not possible for government spending.

6.1 State-Contingent Fiscal Policy

To characterize the optimal policy, I first solve for the welfare maximizing path of government spending and of the nominal interest rate leading to the neoclassical steady state. I then specify an inflation-contingent government spending plan that eliminates the path leading to secular stagnation.

The objective of the government is to maximize the welfare of the representative household:

$$\int_0^\infty e^{-(\rho-n)t}\left[u\left(c_t\right)+\chi\left(g_t\right)-v\left(L_t^d\right)+\gamma\left(0\right)\right]dt, \tag{38}$$

where the actual quantity of labor supplied is equal to L_t^d rather than to l_t^s. Note that, in the absence of Ponzi schemes, the net wealth of the representative household must always be equal to zero.

Initially, the economy is in the secular stagnation steady state, with inflation anchored at its lower bound, i.e. $\pi_0^A = \pi^R$. The government sets $(g_t, i_t)_{t=0}^\infty$ such as to maximize its objective (38) subject to the behavior of the private sector, which is characterized by the consumption Euler equation (23) with $\Delta_t = 0$, the labor supply function (24), the goods market clearing condition (27), the wage sluggishness equation (28), the dynamics of the inflation anchor (29) with $\pi_0^A = \pi^R$, and the zero lower bound on the nominal interest rate. The optimal policy problem is solved in appendix B.

Following Werning (2012), I decompose government spending g_t into an opportunistic and a stimulus component. The opportunistic component g_t^o naively maximizes welfare at any point in time, i.e. it maximizes $u\left(c_t\right)+\chi\left(g_t^o\right)-v\left(L_t^d\right)+\gamma\left(0\right)$ with respect to g_t^o and L_t^d subject to $f\left(L_t^d\right)=c_t+g_t^o$, which yields $\chi'\left(g_t^o\right)=v'\left(L_t^d\right)/f'\left(L_t^d\right)$ as in equation (30). The stimulus component g_t^s, defined as $g_t^s = g_t - g_t^o$, corresponds to the public spending that is realized to stimulate private demand through dynamic general equilibrium effects:

the fiscal stimulus spurs inflation, which boosts private demand.

Figure 1 displays the paths of total and opportunistic government spending, g_t and g_t^o. Stimulus spending, which is the difference between the two, remains positive until the economy reaches the neoclassical steady state 10.8 years after the launch of the policy. The fiscal stimulus is rather small, and never exceeds 0.33% of GDP.[22] The inertia of the inflation anchor makes it heavily front-loaded. This helps the economy overheat such as to raise the inflation anchor, which is necessary for private demand to permanently recover. The resulting employment boom depresses opportunistic spending throughout the transition period.

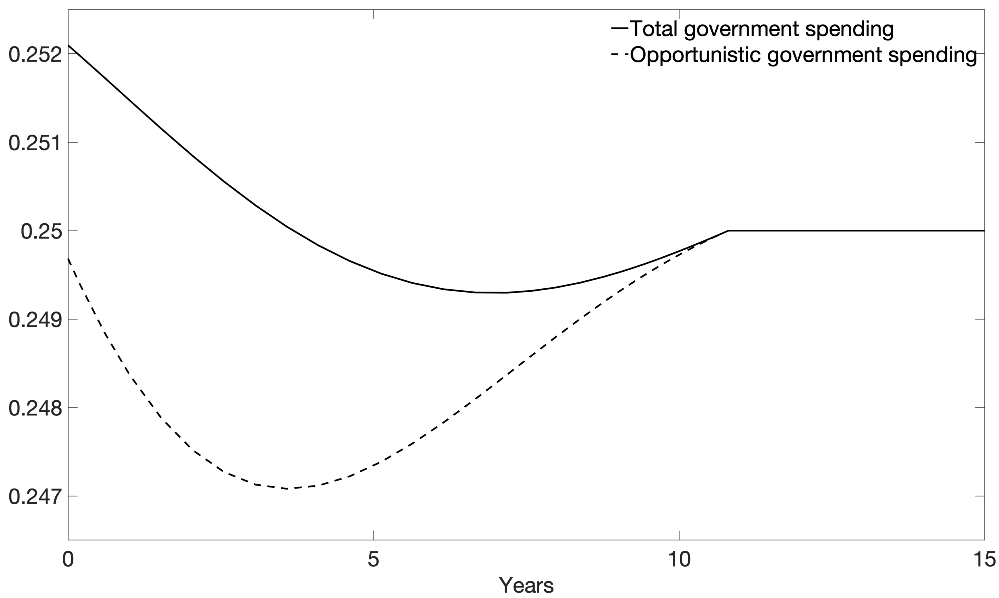

Figure 1: Optimal state-contingent policy

Figure 2 displays the paths of π_t, π_t^A, i_t, and r_t under the optimal policy. Recall that, with the natural interest rate equal to -2.17%, the inflation anchor only needs to rise to 2.17% for the neoclassical steady state to become feasible. Perhaps surprisingly, the optimal policy eventually raises the anchor to 2.45%, significantly above 2.17%. The nominal interest rate remains at the zero lower bound, even once the anchor starts exceeding 2.17%. This commitment to an excessively low interest rate from time 6.9 to 10.8 is a forward guidance policy. This maintains the real interest rate depressed below its natural counterpart, which generates a consumption boom that helps spur inflation throughout the transition to the neoclassical steady state. This shows that, even under secular stagnation, forward guidance is a useful tool.[23] This monetary and fiscal policy

[22] Recall that output in the neoclassical steady state was normalized to one.

[23] In fact, even if government spending is set opportunistically at each point in time, forward guidance alone can bring the economy to the neoclassical steady state (regardless of whether the stability condition

generates a consumption equivalent welfare loss, relative to the neoclassical steady state, of only 0.004%.

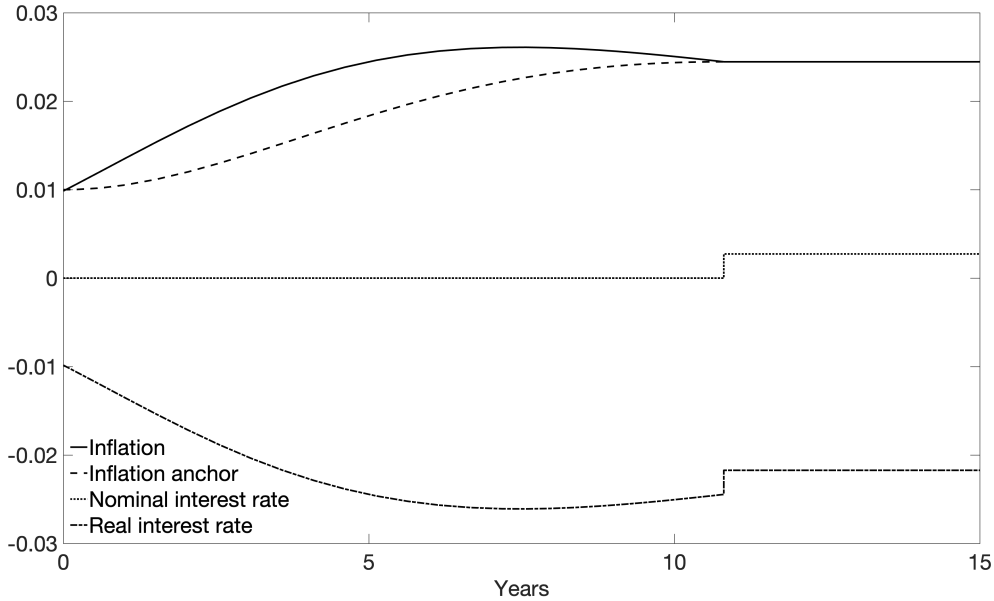

Figure 2: Optimal state-contingent policy

If the optimal policy $(g_t, i_t)_{t=0}^{\infty}$ is implemented deterministically, then the optimal path leading to the neoclassical steady state is not the unique equilibrium. There also exists a possibility that the economy remains in secular stagnation. In this alternative equilibrium, the downward wage rigidity keeps binding, setting inflation equal to π^R. After 10.8 years, steady state consumption is given by $1/u'(c) = \left(\rho - i + \pi^R\right)/\gamma'(0)$, which is even lower than c^{SS} due to $i > 0$.

To make the good equilibrium unique, the government needs to implement a state-contingent fiscal plan. Let π_t^* and π_t^{A*} denote inflation and the inflation anchor along the optimal path, as displayed in Figure 2. If private demand is excessively weak, the government needs to commit to spend sufficiently to raise inflation to π_t^*, which by the updating rule (29) naturally entails π_t^{A*}. From the consumption Euler equation (23), this uniquely implements the optimal consumption path, converging to the neoclassical steady state. Thus, by the resource constraint (27) and the Phillips curve (28), public spending g_t at any time t must be determined as a function of private consumption c_t by:

$$\pi_t^* = \pi_t^{A*} + \beta \left[\frac{v'\left(f^{-1}\left(c_t + g_t\right)\right)}{f'\left(f^{-1}\left(c_t + g_t\right)\right) u'\left(c_t\right)} - 1 \right]. \tag{39}$$

Fiscal policy (off the equilibrium path) at time t is determined such as to target the inflation rate π_t^*. This is reminiscent of Modern Monetary Theory (Kocherlakota 2020).

of footnote 16 is satisfied).

The dotted line of Figure 3 displays the path of consumption under the secular stagnation equilibrium, which exists if the optimal policy $(g_t, i_t)_{t=0}^{\infty}$ (shown in Figure 1 and 2) is implemented deterministically. The solid line of Figure 3 shows the state-contingent level of government spending that, according to (39), would be required to hit the inflation target π_t^*. This entails a large output and employment level, which would depress the magnitude of opportunistic government spending. The stimulus component, given by the difference between total and opportunistic spending, would therefore amount to a whopping 21% of output! Crucially, this level government spending is an off-the-equilibrium threat that destroys the secular stagnation equilibrium. It never needs to be implemented along the equilibrium path.

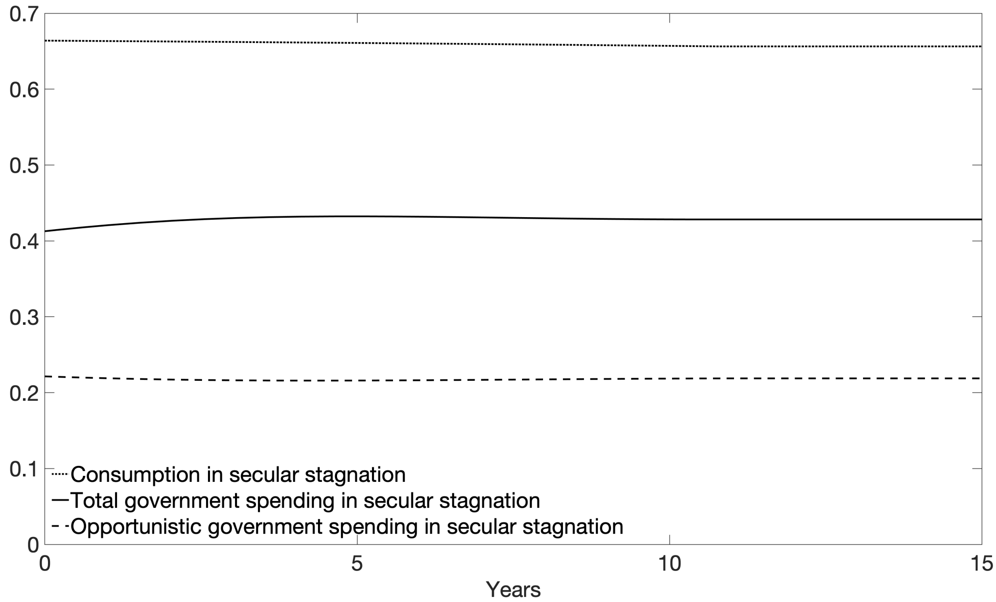

Figure 3: Optimal state-contingent policy

The job of lifting the economy out of stagnation is not realized by the modest stimulus spending on the equilibrium path, but by the threat of massive spending off the equilibrium path. This threat leaves households with no choice, but to coordinate on self-fulfilling inflation expectations. While this insight is theoretically interesting, in practice any government would have a hard time implementing such a massive state-contingent spending plan. Moreover, an infinite commitment horizon is not plausible. In the following section, I therefore characterize the optimal fiscal policy assuming that the government can only commit to a deterministic path of public spending over a fixed period of time.

6.2 Non-Contingent Fiscal Policy

The government is now assumed to commit to a deterministic path of government spending g_t from time 0 to T. After time T, government spending is set opportunistically, in accordance with equation (30). Monetary policy is also set with commitment up to time T. I shall investigate both non-contingent and state-contingent monetary policy. After time T, the nominal interest rate is set such as to get as close as possible to the natural real interest rate, i.e. $i_t = \max\{\pi_t + r^n, 0\}$. These policies result in two equilibrium possibilities: an equilibrium path $(c_t, L_t^d, l_t^s, \pi_t, \pi_t^A)_{t=0}^{\infty}$ leading to the neoclassical steady state and another path $(\bar{c}_t, \bar{L}_t^d, \bar{l}_t^s, \bar{\pi}_t, \bar{\pi}_t^A)_{t=0}^{\infty}$ leading back to the secular stagnation steady state.

How much government spending is needed to kill the secular stagnation equilibrium? I assume that households coordinate on the neoclassical equilibrium path if and only if, by the end of the commitment horizon T, the inflation anchor reaches a threshold level $\hat{\pi}$, *even under the secular stagnation equilibrium path.* In other words, if government spending is sufficiently massive to raise the inflation anchor to $\hat{\pi}$, despite depressed private demand from pessimistic households expecting to remain in stagnation forever, then households realize that stagnation is over. Formally, raising the anchor to at least $\hat{\pi}$ under both paths by time T, i.e. $\pi_T^A \geq \hat{\pi}$ and $\bar{\pi}_T^A \geq \hat{\pi}$, is necessary and sufficient to "prime the pump".

Two cases must be considered. First, households can be naive, in which case the economy remains on the secular stagnation equilibrium path $(\bar{c}_t, \bar{L}_t^d, \bar{l}_t^s, \bar{\pi}_t, \bar{\pi}_t^A)_{t=0}^{\infty}$ until $\bar{\pi}_t^A$ reaches $\hat{\pi}$ at time T, at which point the economy jumps to the neoclassical path $(c_t, L_t^d, l_t^s, \pi_t, \pi_t^A)_{t=0}^{\infty}$. Alternatively, households can form rational expectations, which induces the economy to jump to the neoclassical path as soon as the government announces a policy that would raise the inflation anchor under stagnation to $\hat{\pi}$. Let us now investigate each of these two possibilities in turn.

6.2.1 Naive Expectations

Under naive expectations, the economy must be on the secular stagnation path until time T. Hence, whether or not monetary policy can be state-contingent from time 0 to T is irrelevant. The optimal policy under naive expectations consists in setting T and $(g_t, i_t)_{t=0}^{T}$ such as to maximize households' welfare:

$$\int_0^T e^{-(\rho-n)t}\left[u\left(\bar{c}_t\right)+\chi\left(g_t\right)-v\left(\bar{L}_t^d\right)+\gamma\left(0\right)\right]dt \qquad (40)$$
$$+\int_T^{\infty} e^{-(\rho-n)t}\left[u\left(c_t\right)+\chi\left(g_t\right)-v\left(L_t^d\right)+\gamma\left(0\right)\right]dt,$$

subject to the usual set of constraints, which must hold for both $(\bar{c}_t, \bar{L}_t^d, \bar{l}_t^s, \bar{\pi}_t, \bar{\pi}_t^A)_{t=0}^{\infty}$ and $(c_t, L_t^d, l_t^s, \pi_t, \pi_t^A)_{t=T}^{\infty}$. These constraints consist of the consumption Euler equation (23) with $\Delta_t = 0$, the labor supply function (24), the goods market clearing condition (27), the wage sluggishness equation (28), the dynamics of the inflation anchor (29), and the zero lower bound. The boundary conditions consist of $\bar{\pi}_0^A = \pi^R$, $\bar{\pi}_T^A = \hat{\pi}$, $\bar{c}_\infty = c^{ss}$, and $\pi_T^A = \hat{\pi}$. This last condition follows from the fact that, under naive expectations, the inflation anchor, which is not defined along the neoclassical path before time T, must naturally be equal to $\bar{\pi}_T^A$ at time T. After T, the policy parameters g_t, $\bar{g}_t$, i_t, and $\bar{\imath}_t$ are determined by $\chi'(g_t) = v'\left(L_t^d\right)/f'\left(L_t^d\right)$, $\chi'(\bar{g}_t) = v'\left(\bar{L}_t^d\right)/f'\left(\bar{L}_t^d\right)$, $i_t = \max\{\pi_t + r^n, 0\}$, and $\bar{\imath}_t = \max\{\bar{\pi}_t + r^n, 0\}$, respectively. The optimal policy problem is solved in appendix C.

Recall that, to have a non-binding zero lower bound in the neoclassical steady state, the inflation rate must be greater or equal to $-r^n$. For my numerical simulation, I assume that the inflation anchor must exceed this minimum by 2% for households to realize that stagnation is really over, i.e. $\hat{\pi} = -r^n + 2\%$. With $r^n = -2.17\%$, this gives $\hat{\pi} = 4.17\%$. One interpretation is that the government targets an inflation rate that is 2% above $-r^n$, such as to eventually raise the nominal rate to 2% above the zero lower bound.[24] Households only expect stagnation to be over once the inflation anchor reaches the government's target. Note that, whenever $\hat{\pi} \geq -r^n$, the economy must reach the neoclassical steady state as soon as time T.

Figure 4 displays the paths of total government spending, opportunistic spending, and consumption. To raise the inflation anchor under stagnation $\bar{\pi}_t^A$ to $\hat{\pi}$, the government implements a massive fiscal stimulus. Government spending nearly doubles, reaching up to 54% of the output level of the neoclassical steady state (which was normalized to one). This raises employment so much that opportunistic spending shrinks by up to a quarter of its steady state level (equal to 25% of output under the neoclassical steady state). Thus, the stimulus component $g_t^s = g_t - g_t^o$ accounts for the bulk of government spending. The inflation anchor under stagnation $\bar{\pi}_t^A$ reaches $\hat{\pi}$ after only 1.55 years. The total amount of extra government spending, as measured by $\int_0^T e^{-\int_0^t (i_u - \pi_u - n)du} (g_t - g^n)\, dt$, adds up to 34.5% of the output level under the neoclassical steady state. Consumption is initially depressed as households naively expect the economy to remain under stagnation forever. At time T, they realize that stagnation is over and consumption jumps upward by 11.1% (from 0.675 to 0.750).

Figure 5 shows the paths of inflation, the inflation anchor, the nominal interest rate, and the real interest rate. The massive stimulus raises output by 10 to 21% of the output level under the neoclassical steady state (16 to 28% of the output level under the stagnation steady state). This economic boom raises inflation by up to 12% per year,

[24] This leaves a bit of room to cut the nominal interest rate in case of temporary downturn.

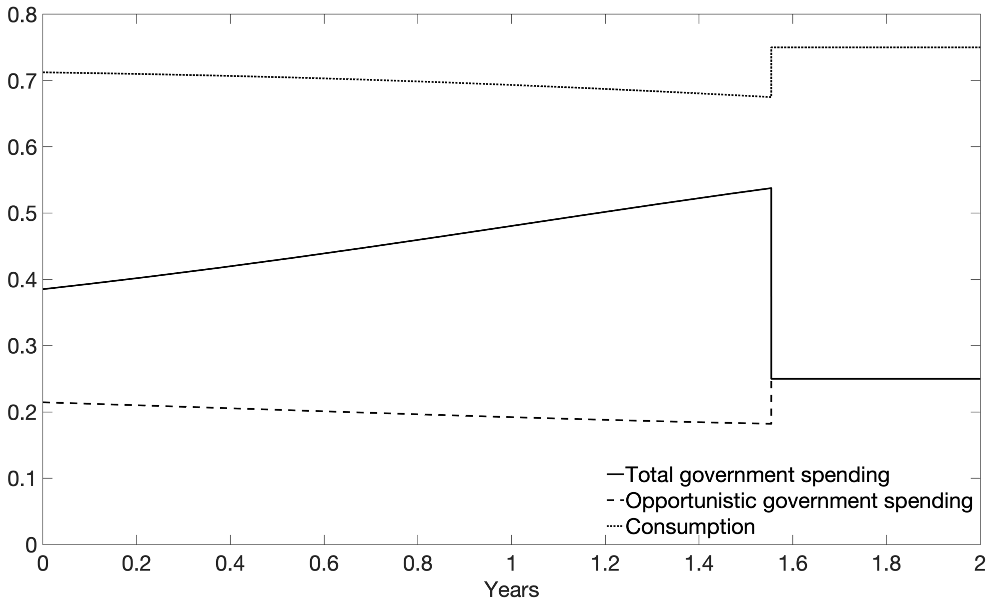

Figure 4: Optimal non-contingent policy under naive expectations

which eventually increases the anchor from $\pi^R = 1\%$ to $\hat{\pi} = 4.17\%$. Once this is achieved, the nominal interest rate rises to 2% and the economy settles in the neoclassical steady state.

The consumption equivalent welfare loss from this optimal policy, relative to the neoclassical steady state, is 1.2%. This is considerably more than under the optimal state-contingent policy of the previous section (where the loss was equal to 0.004%), but much less than the 3.2% loss of remaining under secular stagnation forever, which implies that the optimal fiscal policy is indeed to pump prime the economy.

6.2.2 Rational Expectations

Under rational expectations, the economy jumps on the path leading to the neoclassical steady state as soon as households realize that government spending are sufficiently large to eventually raise the inflation anchor under stagnation $\bar{\pi}_t^A$ to the threshold $\hat{\pi}$. I first consider non-contingent monetary policy, before considering the alternative case of state-contingent monetary policy.

The optimal policy under rational expectations consists in setting T and $(g_t, i_t)_{t=0}^T$ such as to maximize households' welfare:

$$\int_0^\infty e^{-(\rho-n)t} \left[u\left(c_t\right) + \chi\left(g_t\right) - v\left(L_t^d\right) + \gamma\left(0\right)\right] dt \tag{41}$$

subject to the usual set of constraints, which must hold for both the (off-the-equilibrium) path $(\bar{c}_t, \bar{L}_t^d, \bar{l}_t^s, \bar{\pi}_t, \bar{\pi}_t^A)_{t=0}^\infty$ leading to the secular stagnation steady state and the (on-

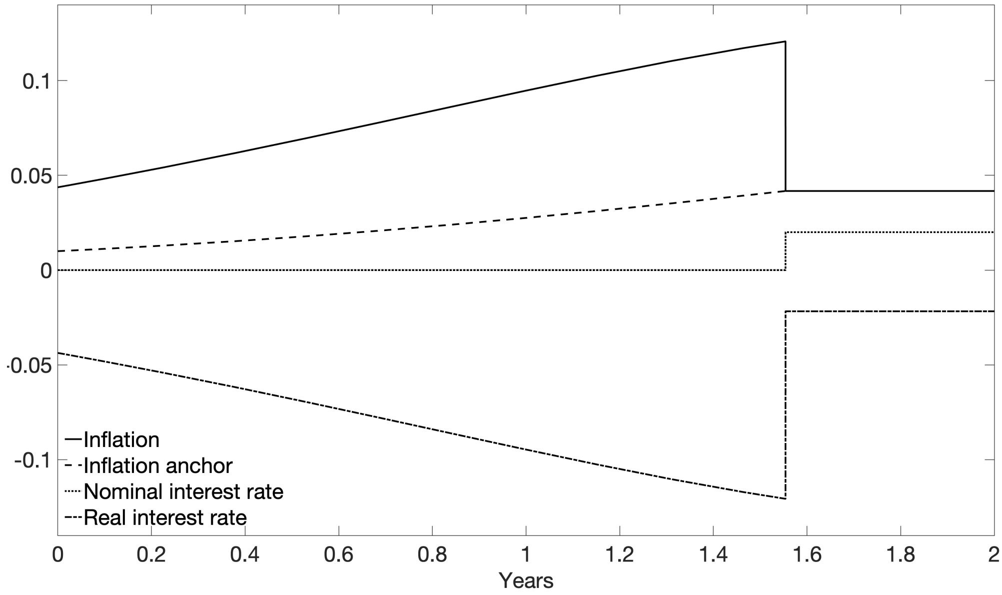

Figure 5: Optimal non-contingent policy under naive expectations

the-equilibrium) path $(c_t, L_t^d, l_t^s, \pi_t, \pi_t^A)_{t=0}^{\infty}$ leading to the neoclassical steady state. The boundary conditions consist of $\bar{\pi}_0^A = \pi^R$, $\bar{\pi}_T^A = \hat{\pi}$, $\bar{c}_\infty = c^{ss}$, and $\pi_0^A = \pi^R$. I ignore the constraint $\pi_T^A \geq \hat{\pi}$, as it is not binding under non-contingent monetary policy and rational expectations. The problem is solved in appendix D.

Not surprisingly, the non-contingent monetary policy is stuck at the zero lower bound throughout the duration of the stimulus, i.e. $i_t = 0$ for all $t \in [0, T]$. Figure 6 displays the paths of total government spending g_t, opportunistic spending g_t^o, consumption c_t, and consumption along the stagnation path $\bar{c}_t$. A massive fiscal stimulus is implemented to raise the inflation anchor under stagnation $\bar{\pi}_t^A$ to $\hat{\pi}$, which is achieved within half a year. As households form rational expectations, they immediately choose a path of consumption leading to the neoclassical steady state. This magnifies the effect of the stimulus, to such an extent that at time T the inflation anchor along the neoclassical path π_t^A reaches 7.9%. From time T onwards, the economy is therefore in the neoclassical steady state with 7.9% inflation. The total amount of extra government spending, as measured by $\int_0^T e^{-\int_0^t (i_u - \pi_u - n) du} (g_t - g^n) \, dt$, equals 22% of the output level under the neoclassical steady state. Despite the stimulus being short-lived, the policy generates a sizeable consumption equivalent welfare loss of 1.7%, relative to the neoclassical steady state. The overheating of the economy from time 0 to T entails a large welfare loss from an inefficiently high labor supply.

Let us now allow for state-contingent monetary policy and solve for T and $(g_t, i_t, \bar{\imath}_t)_{t=0}^{T}$, where i_t and $\bar{\imath}_t$ denote the paths of the nominal rate under the neoclassical and the stagnation path, respectively. Government spending still needs to be massive such as

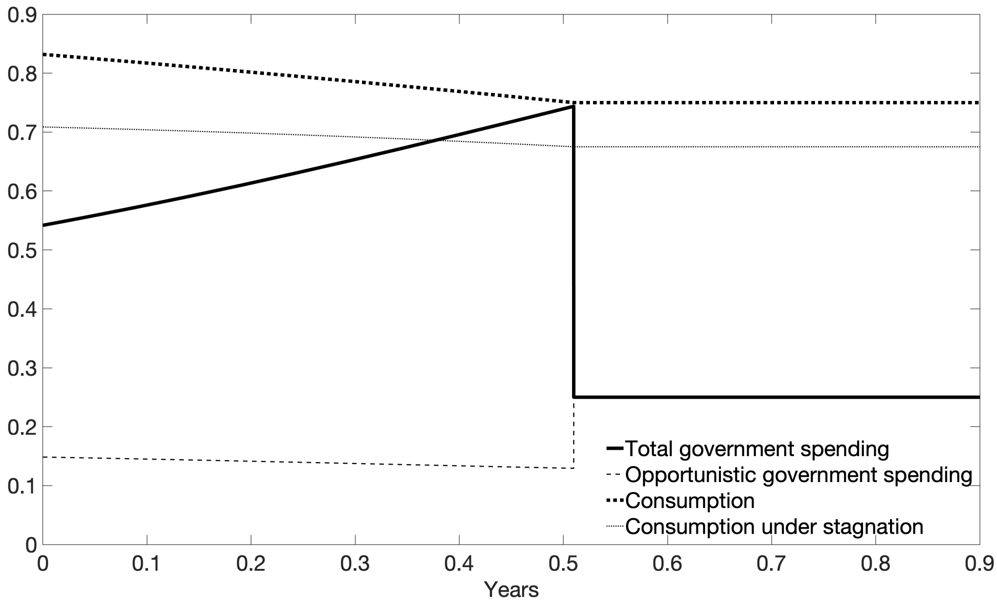

Figure 6: Optimal non-contingent fiscal and monetary policy under rational expectations

to raise the inflation anchor along the stagnation path to $\hat{\pi}$. The nominal interest rate trivially remains equal to zero along that path. However, along the neoclassical path, the monetary authority could be tempted to lean against the wind so much as to prevent labor demand L_t^d from exceeding (desired) labor supply l_t^s, which by (28) and (29) would leave the inflation anchor unchanged. Hence, the condition that, along the neoclassical path, the inflation anchor needs to reach the threshold by time T, i.e. $\pi_T^A \geq \hat{\pi}$, is typically binding. It follows that, under state-contingent monetary policy, we need to impose both $\pi_T^A = \hat{\pi}$ and $\bar{\pi}_T^A = \hat{\pi}$.

Figure 7 shows the paths of g_t, g_t^o, c_t, and $\bar{c}_t$ with a state-contingent monetary policy, where c_t and $\bar{c}_t$ overlap until time T. Consumption along the neoclassical path jumps upwards when government spending drops at time T. This requires an infinitely high nominal interest rate at T along that path.[25] Equivalently, and more precisely, a 23% proportional wealth subsidy can be implemented at time T to induce an 11% jump in consumption.[26] Otherwise, along both paths, the nominal interest rate remains equal to zero from time 0 to T.[27] In sum, the optimal policy under rational expectations simultaneously

[25] If there was an upper bound to the nominal interest rate, it would be binding for some time just before the end of the stimulus episode. The optimal monetary policy can be seen as the limit as this upper bound tends to infinity.

[26] Under a proportional wealth subsidy τ at time T, the Euler equation (23) becomes $u'(c_{T-dt}) = (1+\tau)u'(c_T)$.

[27] It follows that $c_t = \bar{c}_t$ and $\pi_t^A = \bar{\pi}_t^A$ for all $t \in [0, T]$. Indeed, both equilibrium paths are fully characterized by the Euler equation (23) with zero nominal interest rate and the dynamics of the inflation anchor (29) subject to the boundary conditions that the anchor rises from π^R at time 0 to $\hat{\pi}$ at T. Importantly, the observational equivalence between the two paths prevents the implementation of a state-contingent monetary policy. This knife-edge problem can easily be solved by targeting an inflation

consists in stimulating the economy through non-contingent fiscal spending and slowing it down under the (on-the-equilibrium) neoclassical path through state-contingent monetary policy. The flow of government spending g_t is much lower than under non-contingent monetary policy, but the duration of the stimulus is almost three times longer. In the end, this policy raises government spending, $\int_0^T e^{-\int_0^t (i_u - \pi_u - n)du} (g_t - g^n)\, dt$, by 35% of steady state output and generates a welfare loss of 1.2%.

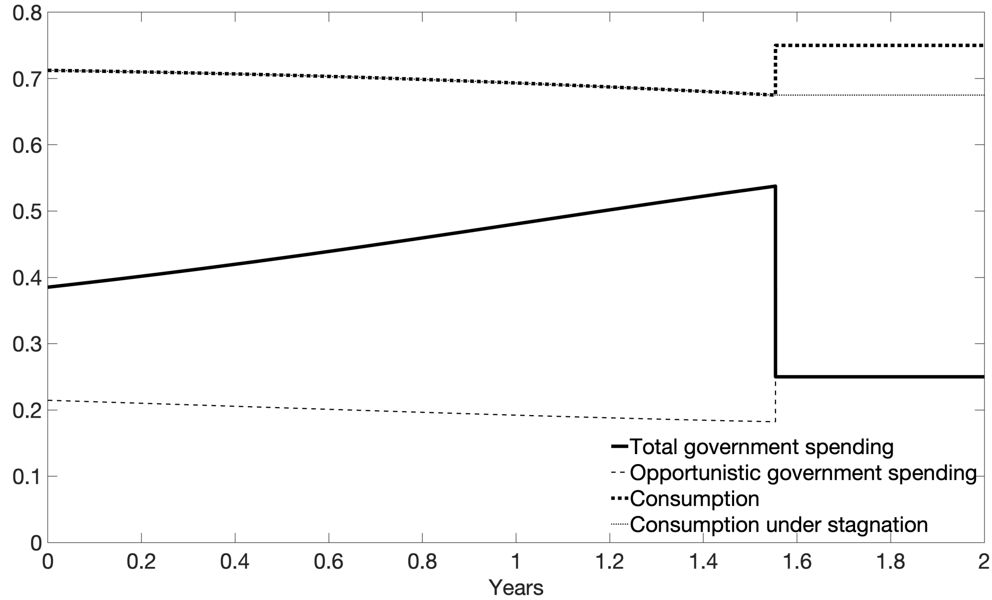

Figure 7: Optimal non-contingent fiscal and state-contingent monetary policy under rational expectations

All these non-contingent fiscal policies require massive levels of government spending. So far, I have assumed that they are financed from lump-sum taxes. But, in practice, fiscal stimulus programs are financed by debt. A common objection to these policies is that the government does not have the necessary fiscal space to pay for them. In the next section, I therefore investigate fiscal policy under debt sustainability concerns.

7 Fiscal Policy and Debt Sustainability

Let us now consider that the fiscal stimulus is entirely financed by issuing debt, which initiates a Ponzi debt scheme, i.e. $\Delta_0 > 0$. If households do not believe such a scheme to be sustainable, then this must trigger an upward jump in the initial price level P_0 such that $\Delta_0 = 0$. Alternatively, in the absence of a jump in P_0, the Ponzi scheme interacts with the real allocation of resources and, hence, with the effectiveness of the

anchor at time T along the neoclassical path that is strictly above $\hat{\pi}$.

fiscal stimulus. I now review each of these two possibilities in turn, before investigating how the government can exploit the maturity structure of its debt to pay for the stimulus.

7.1 No Ponzi Scheme

The government cannot force households to buy into a Ponzi scheme. Hence, there always exists an equilibrium with $\Delta_0 = 0$. In that case, the initial price level P_0 jumps upward such as to reduce the real value of nominal liabilities B_0.[28] This is consistent with the fiscal theory of the price level.

The upward jump in P_0 implies an infinitely high rate of inflation at time 0. By the updating rule (29), this can trigger an upward jump in the inflation anchor π_0^A.[29] This can considerably reduce the size of the stimulus needed to escape the secular stagnation equilibrium. Conversely, if π_0^A remains equal to π^R, the jump in P_0, and the corresponding fall in the real value of government liabilities, is equivalent to a lump-sum tax on the representative household, which brings us back to the previous section.

7.2 Ponzi Scheme

In this economy, if $r^n < n$, a Ponzi debt scheme can be sustainable over time (Michau, Ono, and Schlegl 2020, Michau 2020).[30] Let us therefore investigate the effects of a fiscal stimulus financed by debt when P_0, and hence π_0^A, does not jump. To have a sizeable fiscal stimulus, I assume that government spending is non-contingent. The commitment horizon is of length T. As in the previous section, I assume that the government wants to drive the economy to the neoclassical steady state, which requires both $\pi_T^A \geq \hat{\pi}$ and $\bar{\pi}_T^A \geq \hat{\pi}$. I assume rational expectations, which allows me to specify the expected present value of fiscal surpluses at time 0 under both the stagnation and the neoclassical path.

By the Ponzi dynamics (26) with $i_t - \pi_t = -\pi^R$, if a Ponzi scheme exists when $-\pi^R > n$, it must keep growing under secular stagnation. This would prevent the economy from ever returning to the secular stagnation steady state. To focus on the non-trivial case, where the economy can converge back to the secular stagnation steady state, I henceforth assume that $-\pi^R < n$.[31]

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[28] While the nominal wage is sluggish, a surprise one time upward jump is assumed to be possible at the point in time when households lose confidence in the value of money.

[29] In fact, if the updating rule (29) also applies to discrete jumps in the price level, then, as the price level jumps from P_0 to P_{0+dt} at time 0, the inflation anchor must jump from π_0^A to $\pi_{0+dt}^A = \pi_0^A + \theta(P_{0+dt} - P_0)/P_0$.

[30] Relying on an overlapping generation economy, Bassetto and Cui (2018) have shown that, when Ponzi schemes are sustainable, the fiscal theory of the price level does not uniquely pin down the price level.

[31] Recall that, for the secular stagnation steady state to exist, we must have $r^n < -\pi^R$.

There are many different ways to determine the present value of fiscal surpluses, resulting in Ponzi schemes of different magnitudes. For simplicity and clarity of exposition, I assume that, along each equilibrium trajectory, the path of lump-sum taxes is set at time 0 such as to balance the government's intertemporal budget constraint under the steady state level of government spending. Hence, along the neoclassical path, we have:

$$0 = b_0 + m_0 - \int_0^\infty e^{-\int_0^t (i_u - \pi_u - n)du} \left[\tau_t - g^n\right] dt. \tag{42}$$

By definition (18), the size of the Ponzi scheme at time 0 along that path is given by:

$$\Delta_0 = b_0 + m_0 - \int_0^\infty e^{-\int_0^t (i_u - \pi_u - n)du} \left[\tau_t - g_t\right] dt, \tag{43}$$

where g_t is the actual level of government spending. Combining the previous two equations yields:

$$\Delta_0 = \int_0^\infty e^{-\int_0^t (i_u - \pi_u - n)du} \left[g_t - g^n\right] dt. \tag{44}$$

Similarly, along the stagnation path, we must have:

$$\bar{\Delta}_0 = \int_0^\infty e^{-\int_0^t (\bar{\imath}_u - \bar{\pi}_u - n)du} \left[\bar{g}_t - g^{ss}\right] dt. \tag{45}$$

The optimal policy problem is the same as before, except that, for each equilibrium path, we must now add the transversality condition (25), the Ponzi dynamics (26), and the initial size of the Ponzi scheme (44) or (45) as constraints to the optimization problem. This is formalized in appendix E.

The objective of the optimal policy problem is still given by (41). I am therefore assuming that the (paternalistic) planner does not value the utility that households derive from owning Ponzi wealth. This suppresses the mechanical effect of the Ponzi scheme on welfare, which makes the welfare results of this section comparable to those of the previous section.

The preference for wealth now needs to be fully calibrated. Following Kumhof, Rancière, and Winant (2015), I assume constant relative risk aversion relative to some minimum wealth level $\underline{W}$:

$$\gamma(W) = k_W \frac{\left(W - \underline{W}\right)^{1-\sigma_W} - 1}{1 - \sigma_W}. \tag{46}$$

I set $\sigma_W = 1.5$. By the Euler equation (23), this implies that, in steady state, a 1% increase in wealth relative to the reference point $\underline{W}$ raises consumption by $\sigma_W/\sigma = 0.75\%$.[32] I also jointly set k_W and $\underline{W}$ such that $c^{ss} = (1 - 0.1)c^n$ (which, as before, results

[32] For wealthy individuals, the reference point $\underline{W}$ is negligible and σ_W/σ is the steady state elasticity of consumption with respect to wealth. Whenever $\sigma_W < \sigma$, preferences are non-homothetic consistently

in $\gamma'(0) = 0.110$ and $r^n = -2.17\%$) and $\Delta^p = 1.5$, which implies that the maximum sustainable size of a Ponzi scheme is equal to one and a half years of output. This yields $k_W = 1.038$ and $\underline{W} = -4.472$.

The Ponzi schemes do not modify the qualitative features of the optimal policy. The duration of the stimulus tends to be larger, but the level of government spending is smaller than with tax financing. The present value of extra government spending, as measured by (44), is comparable to what we previously had.

More precisely, with non-contingent monetary policy, I obtain $T = 0.71$, $\Delta_0 = 0.24$, and $\bar{\Delta}_0 = 0.15$. The Ponzi scheme raises household wealth, which reduces the marginal utility of wealth. After time T, along the neoclassical path, this is offset by a higher nominal interest rate, inducing the economy to be in steady state, with $c_t = c^n$ and $g_t = g^n$ for all $t \geq T$. However, along the secular stagnation path, the nominal interest rate remains at the zero lower bound and the economy only gradually converges back to the secular stagnation steady state. Thus, after time T, we have $\bar{c}_t > c^{ss}$ and $\bar{L}_t > L^{ss}$, implying $\bar{g}_t < g^{ss}$. By the definitions of Δ_0 and $\bar{\Delta}_0$, given by (44) and (45), this explains why the Ponzi scheme is smaller along the secular stagnation path, i.e. $\bar{\Delta}_0 < \Delta_0$.

With state-contingent monetary policy, I constrain the commitment horizon to be smaller or equal to two years, i.e. $T \leq 2$. Otherwise, the government chooses an implausibly large horizon such as to stimulate the economy, not from government spending, but from the resulting Ponzi scheme $\bar{\Delta}_0$ along the stagnation path (which is increasing in T). I therefore obtain $T = 2$, $\Delta_0 = 0.34$, and $\bar{\Delta}_0 = 0.22$.

To get out of stagnation, the constraint that is costly to satisfy is $\bar{\pi}_T^A \geq \hat{\pi}$. Hence, it is the Ponzi scheme along the stagnation path $\bar{\Delta}_0$ that helps stimulate the economy. By contrast, along the neoclassical path, the Ponzi scheme Δ_0 is either detrimental or neutral. With non-contingent monetary policy, it amplifies the overheating of the economy; while, with state-contingent monetary policy, it is essentially offset through higher nominal interest rates. In fact, in the former case, the inflation anchor π_T^A rises to 8.3%, which is 0.4% higher than under tax financing.

Table 2 gives the consumption-equivalent welfare losses, relative to being in the neoclassical steady state, for our four different scenarios. Financing the stimulus with Ponzi debt, rather than lump-sum taxes, slightly reduces the welfare cost of the optimal policy. Recall that, as the planner does not value Ponzi wealth, this ignores the mechanical impact of wealth on welfare. Instead, the welfare gain is due to a stimulative general equilibrium effect: financing the stimulus through public debt reduces the marginal utility of wealth, which boosts private consumption in the stagnation equilibrium, which helps raise the inflation anchor $\bar{\pi}_T^A$ to $\hat{\pi}$.

with the empirical evidence provided by Straub (2019).

	Monetary policy	
	Non-contingent	State-Contingent
Tax financing	1.75%	1.17%
Ponzi debt financing	1.56%	0.92%

Table 2: Consumption-equivalent welfare loss from optimal policy

So far, the issuance of debt was only a by-product of government spending. Alternatively, the government can make direct transfers to households, which would be equivalent to the implementation of helicopter drops of money (Michau 2020). In fact, a non-paternalistic government, which values the utility that households derive from holding wealth, would try to raise public debt sufficiently to reach the Ponzi steady state.

The problem with such policies is that Ponzi schemes rely on some coordination of expectations across households, resulting in multiple equilibria. If households do not believe the Ponzi scheme to be sustainable, then it must trigger an upward jump in the price level. Governments might therefore be reluctant to rely on debt financing of government expenditures, as this could induce them to lose control of the price level.

However, by reflating the economy, the government permanently modifies the interest rate, which changes financial asset prices. It should therefore be able to design the maturity structure of public debt such as to pay for the fiscal stimulus. Let us now investigate this possibility.

7.3 Maturity Structure of Government Debt

I now introduce a non-trivial maturity structure of government debt. Let D_t^s denote the quantity of nominal debt maturing at time s that the government is liable for at time t, where $s \geq t$. The total quantity of nominal debt at t is given by:

$$B_t = \int_t^{\infty} e^{-\int_t^s i_u du} D_t^s ds, \tag{47}$$

where $e^{-\int_t^s i_u du}$ is the price at time t of a bond yielding one unit of currency at time s. In real terms, we have:

$$b_t = \frac{B_t}{P_t N_t} = \int_t^{\infty} e^{-\int_t^s i_u du} \frac{D_t^s}{P_t N_t} ds. \tag{48}$$

Note that this formulation of the maturity structure of government debt does not modify the formulation of the model. Indeed, differentiating the above expression for b_t yields:

$$\begin{aligned}
\dot{b}_t &= i_t b_t + \int_t^{\infty} e^{-\int_t^s i_u du} \left[\frac{\dot{D}_t^s}{P_t N_t} - \left(\frac{\dot{P}_t}{P_t} + \frac{\dot{N}_t}{N_t} \right) \frac{D_t^s}{P_t N_t} \right] ds - D_t^t, \\
&= \left(i_t - \pi_t - n \right) b_t + \int_t^{\infty} e^{-\int_t^s i_u du} \frac{\dot{D}_t^s}{P_t N_t} ds - D_t^t. \qquad (49)
\end{aligned}$$

But, the government's flow of funds implies the newly issued debt net of maturing debt must be equal to government spending net of fiscal revenue:

$$\int_t^{\infty} e^{-\int_t^s i_u du} \frac{\dot{D}_t^s}{P_t N_t} ds - D_t^t = g_t - \tau_t. \qquad (50)$$

Substituting this equation into the previous one yields the government's debt accumulation equation (15).

Let us consider that, initially, all government debt is of zero maturity. Before the announcement of the fiscal policy, the government chooses a maturity M for a fraction x of its outstanding stock of debt. As households initially expect the economy to remain under secular stagnation forever, with a binding zero lower bound, the price of a bond is independent of its maturity. Thus, total indebtedness remains unchanged.[33]

At time 0, the government announces a non-contingent fiscal policy that raises the inflation anchors under both paths, π_T^A and $\bar{\pi}_T^A$, to at least $\hat{\pi}$. Assuming rational expectations, the economy immediately jumps on the equilibrium path leading to the neoclassical steady state, resulting in a Ponzi scheme of magnitude:

$$\Delta_0 = x e^{-\int_0^M i_u du} b_0 + (1 - x) b_0 + m_0 - \int_0^{\infty} e^{-\int_0^t (i_u - \pi_u - n) du} \left[\tau_t - g_t \right] dt, \qquad (51)$$

where the price of the fraction x of public debt that is of maturity M immediately drops from 1 to $e^{-\int_0^M i_u du}$. As in the previous subsection, I assume that the path of lump-sum taxes at time 0 is set such as to balance the government's intertemporal budget constraint under the steady state level of government spending, resulting in equation (42). Importantly, I consider that the change in the price of government bonds of maturity M does not modify the path of lump-sum taxes. Substituting (42) into the expression for the Ponzi scheme (51) yields:

$$\Delta_0 = x \left[e^{-\int_0^M i_u du} - 1 \right] b_0 + \int_0^{\infty} e^{-\int_0^t (i_u - \pi_u - n) du} \left[g_t - g^n \right] dt. \qquad (52)$$

[33] I am assuming that raising the maturity structure of government debt is not sufficient to induce households to expect the economy to move to the neoclassical equilibrium. Moreover, in practice, governments do not start from a zero duration of public debt.

If the government wants to avoid creating a Ponzi scheme, then it must choose the fraction x of debt with maturity M such as to have $\Delta_0 = 0$. Note that, along the stagnation path, the nominal interest rate remains equal to zero and, hence, the maturity structure of public debt has no impact on total indebtedness following the announcement of the fiscal stimulus. So, $\bar{\Delta}_0$ remains given by (45).

Figure 8 displays the trade-off between M and x such that $\Delta_0 = 0$, as implied by equation (52). The underlying calibration is the same as before, except for the (new) parameter b_0 that I set equal to 1, meaning that the real value of public debt before the announcement of the policy amounts to one year of output under the neoclassical steady state.

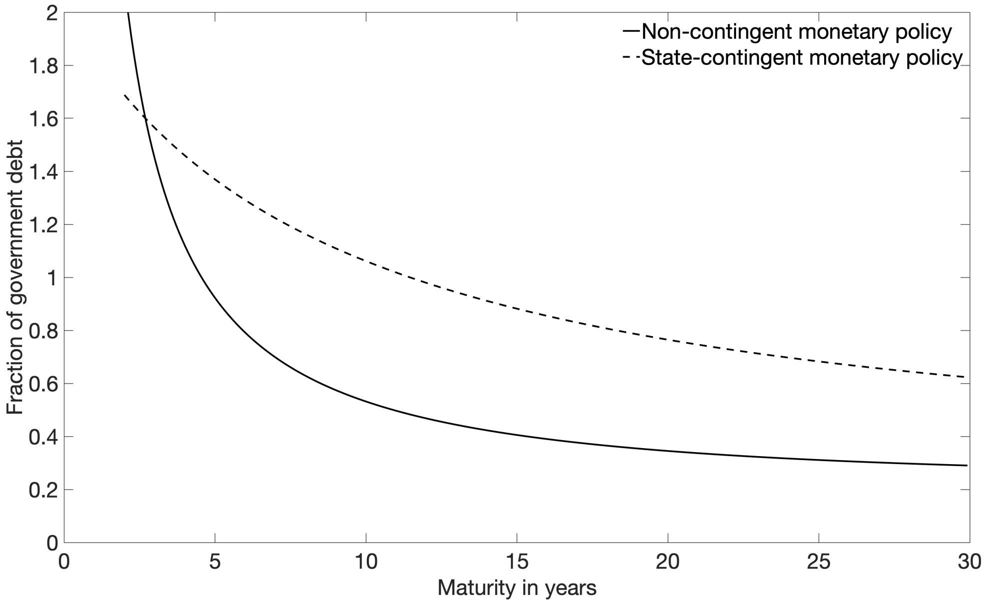

Figure 8: Fraction of debt of non-zero maturity necessary to pay for the fiscal stimulus as a function of that maturity.

Under non-contingent monetary policy, by the end of the stimulus episode, the inflation anchor π_T^A reaches 8.3%. Thus, from time T onwards, the economy is in the neoclassical steady state with the real interest rate equal to $r^n = -2.2\%$ and inflation equal to 8.3%, resulting in a 6.1% nominal interest rate. Such a high nominal rate implies a sharp drop in the value of government debt provided that the corresponding maturity is sufficiently long. For a 7 year maturity, 70% of public debt must be of that maturity to pay for the fiscal stimulus, which amounts to $\int_0^T e^{-\int_0^t (i_u - \pi_u - n)du} (g_t - g^n)\, dt = 24.4\%$ of steady state output. For a 15 year maturity, only 41% of public debt needs to be of that maturity.

Under state-contingent monetary policy, as in the previous subsection, I limit the commitment horizon to two years. Inflation from time T onwards is equal to $\hat{\pi} = 4.17\%$

resulting in a nominal interest rate of only 2%. This implies that the value of government debt is much less sensitive to its maturity. However, the state-contingent monetary policy prevents the economy from overheating along the neoclassical path, which requires a sharp rise in the nominal interest rate at time T equivalent to an 18% wealth subsidy. At time 0, this considerably reduces the value of public debt of maturity greater or equal to T. If 169% of public debt is of maturity $T = 2$, this effect fully pays for the fiscal stimulus, which amounts to 18.4% of steady state output. With a 7 year maturity, the share of long maturity debt required for $\Delta_0 = 0$ drops to 122% and, with a 15 year maturity, it drops to 88%.

This shows that, even though Ponzi schemes can be sustainable, the government can choose to pay for a large fiscal stimulus program by adjusting the maturity structure of its debt before implementing the reflation policy.[34]

8 Capital

I now introduce capital into the economy, such as to account for the response of investment to the fiscal policy. Let I_t and K_t denote investment per capita and the capital stock per capita at time t, respectively. Assuming a neoclassical production function, with constant returns to scale, output per capita is given by $F\left(K_t, L_t^d\right)$. To allow for non-trivial dynamics of capital accumulation, I consider that investment entails some adjustment costs. Thus, whenever aggregate investment is equal to I_t, a fraction $\phi\left(I_t/K_t\right)$ of this investment is lost in the adjustment process and does not contribute to the accumulation of capital. The capital accumulation equation is therefore given by:

$$\dot{K}_t = \left[1 - \phi\left(\frac{I_t}{K_t}\right)\right] I_t - (\delta + n) K_t, \tag{53}$$

where δ is the depreciation rate. I assume $\phi''(\cdot) > 0$, to have convex adjustment costs, and $\phi(\delta + n) = \phi'(\delta + n) = 0$, to have no adjustment cost in steady state. The demand for investment is determined by a representative profit maximizing firm. All the details are provided in appendix F. Note that, in the absence of Ponzi scheme, the wealth of the representative household is now an endogenous variable equal to $q_t K_t$, where q_t is the (shadow) price of capital.

[34] Assuming that the government issues nominal debt, it needs to raise the maturity of its debt to take advantage of a higher nominal interest rate under the neoclassical steady state. If the government was instead issuing real debt, it would need to reduce the maturity of its debt to exploit a lower real interest rate under the neoclassical steady state.

To calibrate the model, I assume a quadratic cost of adjustment:

$$\phi\left(\frac{I_t}{K_t}\right) = \frac{k_{I/K}}{2}\left[\frac{I_t}{K_t} - (\delta + n)\right]^2. \tag{54}$$

The parameter $k_{I/K}$ determines the convexity of the adjustment cost function, since $\phi''(I_t/K_t) = k_{I/K}$. It is set such that, from the capital accumulation equation (53), with constant investment, it takes 8 years for capital to close half the gap to the corresponding steady state, starting from 90% of the steady state capital stock. The depreciation rate δ is set such that, at the golden rule level of the capital stock, i.e. in the Ponzi steady state, capital is equal to two and a half years of output. All the other parameters of the model are calibrated matching the same moments as before (see Table F1 from appendix F). Under this calibration, the natural real interest rate r^n is equal to -1.48%. Assuming as before that the inflation threshold is 2% higher than $-r^n$, we have $\hat{\pi} = 3.48\%$.

Let us now simulate the optimal reflation policy with non-contingent government spending, under rational expectations, and with lump-sum taxes, resulting in $\Delta_0 = \bar{\Delta}_0 = 0$. Initially, the economy is in the secular stagnation steady state (with capital K^{ss}). The optimal policy problem is solved in appendix F.

The presence of capital does not modify the main features of the optimal policy, which still consists in a massive amount of public spending over a rather short period of time. Assuming non-contingent monetary policy, Figure 9 displays the paths of government spending g_t, consumption c_t, investment I_t, consumption under stagnation $\bar{c}_t$, and investment under stagnation $\bar{I}_t$. At time 0, households rationally expect the economy to escape stagnation, which boosts consumption and, to a smaller extent, investment. The resulting overheating of the economy raises the inflation anchor at T to 6.7%. The total magnitude of the stimulus amounts to 20.0% of annual output.[35]

Figure 10 shows the paths of g_t, c_t, I_t, $\bar{c}_t$, and $\bar{I}_t$ under state-contingent monetary policy. It remains optimal to keep the nominal interest rate equal to zero at all time, except for an infinitely high rate at time T. This depresses both consumption and investment throughout the stimulus episode thereby preventing the economy from overheating excessively along the neoclassical equilibrium path, leading to $\pi_T^A = \hat{\pi}$. This infinitely high rate at T is equivalent to 23% proportional wealth subsidy to households (such as to discourage consumption before T) and a 23% proportional tax on firms' capital (such as to discourage investment before T). This induces a 23% jump in the price q_t of capital at T. The resulting upward jump in household wealth $q_t K_t$ at time T largely explains why T is much smaller than without capital. The total magnitude of the stimulus amounts

[35] More precisely, the magnitude of the stimulus is measured by $\int_0^T e^{-\int_0^t (i_u - \pi_u - n)du} (g_t - g_t^n)\, dt$, where g_t^n is government spending under the *laissez-faire* equilibrium path leading to the neoclassical steady state, starting from $K_0 = K^{ss}$. The output level at time 0 along this *laissez-faire* equilibrium path was normalized to one.

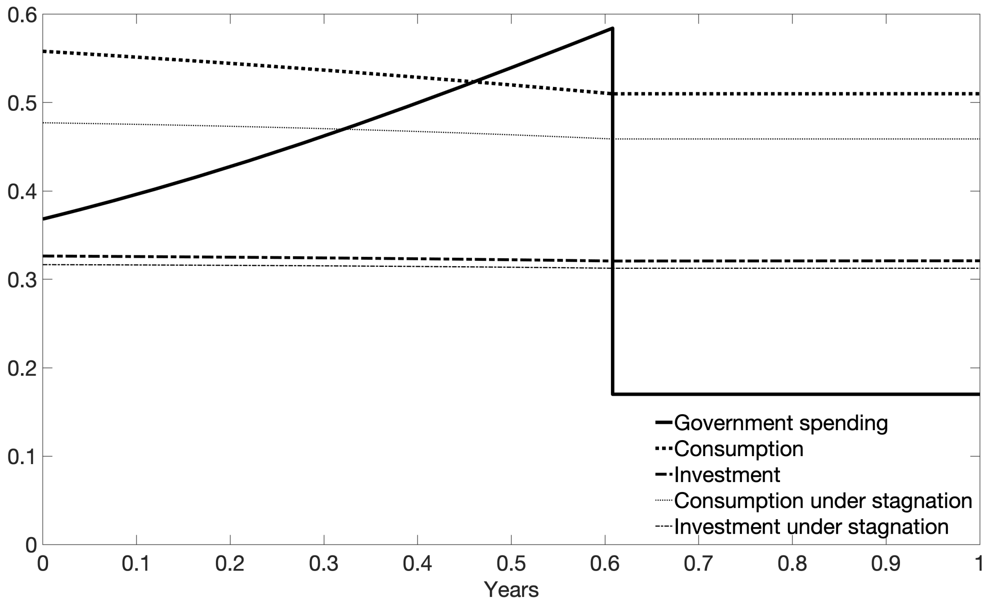

Figure 9: Optimal non-contingent fiscal and monetary policy with capital

to 18.4% of annual output.

9 Conclusion

This paper has shown that, in the context of secular stagnation, the optimal fiscal policy consists in pump priming the economy. This is conceptually different from the usual policy prescription at the zero lower bound, which consists in exploiting the high fiscal multiplier to fill (partly or wholly) the output gap. In fact, a pump priming policy consists in overheating economy, such as to deliver permanently higher inflation.

Throughout my analysis, I have assumed that the government knows the magnitude of the output gap. However, in practice, it is notoriously difficult to measure. The risk is to implement a stimulus package that is too small to prime the pump. This is risky, not because of the accumulation of public debt *per se*, but because there is a large welfare cost from inducing households to work so hard to produce public consumptions goods that no one really needs.

If the natural real interest rate r^n is smaller than the growth rate of the economy n, the stimulus package can be financed through the accumulation of Ponzi debt. This simultaneously fulfils households' preference for wealth and helps stimulate aggregate demand.[36] While this may seem like a free lunch, there is an underlying multiple equilibrium problem: if households to not buy into the Ponzi scheme, this must trigger an upward

[36] Note that, if Ponzi debt is so desirable, this can be extended beyond the level of government spending through the implementation of debt-financed transfers to households.

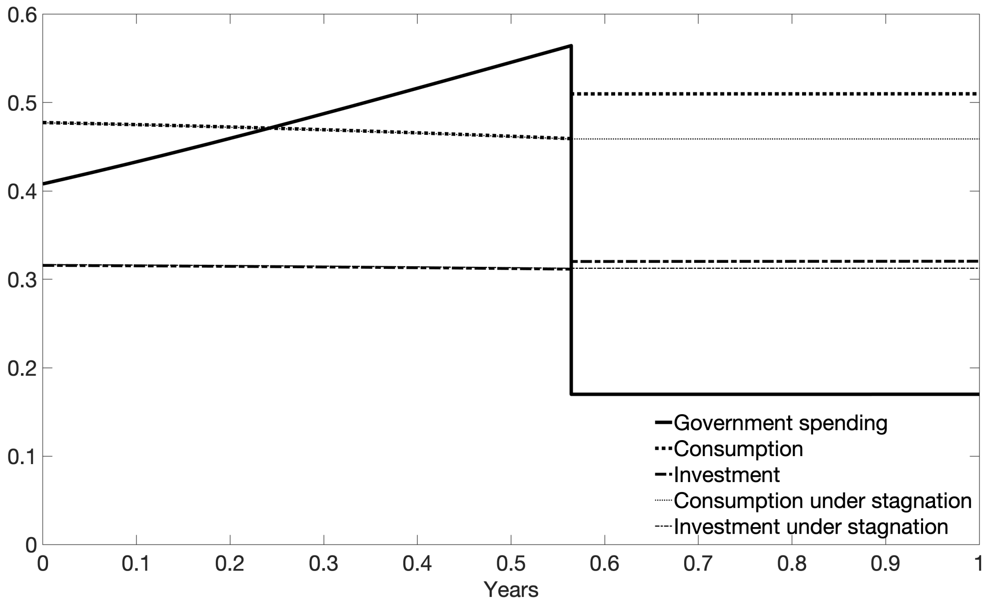

Figure 10: Optimal non-contingent fiscal policy and state-contingent monetary policy with capital

jump in the initial price level. To avoid this possibility, the government can alternatively financed the stimulus through a lump-sum tax on households or through the induced fall in the price of long-term debt, which is effectively a lump-sum tax on the corresponding debt holders.

Pump priming the economy through massive government spending might seem heroic. But, as Milton Friedman (1962) once quipped:

> *[The] basic function [of economists is] to develop alternatives to existing policies, to keep them alive and available until the politically impossible becomes the politically inevitable.*

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A Proof of Lemma 1

By definition of Φ_t, given by (16), we have:

$$\dot{\Phi}_t = (r_t - n)\,\Phi_t - \tau_t + g_t.$$

Integrating this differential equation from time t to infinity yields:

$$\left(\lim_{T\to\infty} e^{-\int_t^T (r_u - n)du}\Phi_T\right) - \Phi_t = -\int_t^\infty e^{-\int_t^s (r_u-n)du}\,(\tau_s - g_s)\,ds.$$

If Φ_t is finite, then by definition of Φ_t in (16) we must have:

$$\lim_{T\to\infty} e^{-\int_t^T (r_u-n)du}\Phi_T = 0.$$

The consumption Euler equation (4) can be written as:

$$\frac{d\ln\left[u'(c_t)\right]}{dt} = -r_t + \rho - \frac{\gamma'(\Delta_t)}{u'(c_t)}.$$

Integrating this differential equation from time zero to t yields:

$$u'(c_t) = u'(c_0)\,e^{\int_0^t\left(\rho - r_u - \frac{\gamma'(\Delta_u)}{u'(c_u)}\right)du}. \tag{A1}$$

Hence:

$$\lim_{t\to\infty} e^{-\int_0^t(\rho - r_u)du}u'(c_t) = u'(c_0)\lim_{t\to\infty} e^{-\int_0^t \frac{\gamma'(\Delta_u)}{u'(c_u)}du} \leq u'(c_0)\,.$$

We must therefore have:

$$\begin{aligned}
\lim_{t\to\infty} e^{-(\rho-n)t}u'(c_t)\,\Phi_t &= \left(\lim_{t\to\infty} e^{-\int_0^t(r_u-n)du}\Phi_t\right)\left(\lim_{t\to\infty} e^{-\int_0^t(\rho-r_u)du}u'(c_t)\right),\\
&= 0\left(u'(c_0)\lim_{t\to\infty} e^{-\int_0^t \frac{\gamma'(\Delta_u)}{u'(c_u)}du}\right),\\
&= 0.
\end{aligned}$$

Let us now show that there cannot be an equilibrium with an infinite value of Φ_t. If $\Phi_t = -\infty$, then $\Delta_t = b_t - \Phi_t = +\infty$.[37] This implies $\gamma'(\Delta_t) = 0$ for all t. From the above consumption Euler equation (A1), we have:

$$u'(c_t) = u'(c_0)\,e^{\int_0^t(\rho - r_u)du}.$$

[37] Recall that, throughout our analysis, we exclusively focus on cases where the no-Ponzi condition is either binding or violated, i.e. $\Delta_t \geq 0$. This rules out $\Phi_t = \infty$.

The household's transversality condition is:

$$\lim_{t\to\infty} e^{-(\rho-n)t} u'\left(c_t\right) b_t = 0.$$

It can therefore be simplified to:

$$\lim_{t\to\infty} e^{-\int_0^t (r_u-n)du} b_t = 0. \tag{A2}$$

Integrating the government liability accumulation equation (15) from t to infinity yields:

$$\lim_{T\to\infty} e^{-\int_t^T (r_u-n)du} b_T = b_t - \Phi_t.$$

Multiplying both sides by $e^{-\int_0^t (r_u-n)du}$ yields:

$$\begin{aligned}
\lim_{T\to\infty} e^{-\int_0^T (r_u-n)du} b_T &= e^{-\int_0^t (r_u-n)du} \Delta_t, \\
&= \infty.
\end{aligned}$$

Hence, the household's transversality condition (A2) cannot be satisfied when $\Phi_t = -\infty$.

B State-Contingent Fiscal Policy

The Lagrangian corresponding to the optimal policy problem over a finite horizon of length S is given by:

$$\begin{aligned}
\mathcal{L} = & \int_0^S e^{-(\rho-n)t} \Bigg[u\left(c_t\right) + \chi\left(g_t\right) - v\left(L_t^d\right) + \gamma\left(0\right) + \lambda_t \left[f\left(L_t^d\right) - c_t - g_t \right] \\
& + \mu_t \left[\dot{c}_t - \frac{c_t}{\varepsilon_u\left(c_t\right)} \left[i_t - \pi_t - \rho + \frac{\gamma'(0)}{u'\left(c_t\right)} \right] \right] + \kappa_t i_t \\
& + \zeta_t \left[\pi_t^A + \beta \left[\frac{v'\left(L_t^d\right)}{f'\left(L_t^d\right) u'\left(c_t\right)} - 1 \right] - \pi_t \right] + \eta_t \left[\theta \left[\pi_t - \pi_t^A \right] - \dot{\pi}_t^A \right] \Bigg] dt \\
& + \omega \left[\pi^R - \pi_0^A \right],
\end{aligned}$$

where I am assuming that, along the optimal path, the downward wage rigidity is never binding, i.e. $\pi_t \geq \pi^R$. Integration by parts yields:

$$\begin{aligned}
\mathcal{L} \;=\; & \int_0^S e^{-(\rho-n)t} \Bigg[u(c_t) + \chi(g_t) - v\left(L_t^d\right) + \gamma(0) + \lambda_t \left[f\left(L_t^d\right) - c_t - g_t \right] \\
& + \mu_t c_t \left[(\rho - n) - \frac{1}{\varepsilon_u(c_t)} \left[i_t - \pi_t - \rho + \frac{\gamma'(0)}{u'(c_t)} \right] \right] - \dot{\mu}_t c_t + \kappa_t i_t \\
& + \zeta_t \left[\pi_t^A + \beta \left[\frac{v'\left(L_t^d\right)}{f'\left(L_t^d\right) u'(c_t)} - 1 \right] - \pi_t \right] + \eta_t \left[\theta \left[\pi_t - \pi_t^A \right] - (\rho - n) \pi_t^A \right] + \dot{\eta}_t \pi_t^A \Bigg] dt \\
& + \omega \left[\pi^R - \pi_0^A \right] + e^{-(\rho-n)S} \mu_S c_S - \mu_0 c_0 - e^{-(\rho-n)S} \eta_S \pi_S^A + \eta_0 \pi_0^A .
\end{aligned}$$

The first-order conditions with respect to c_t, π_t^A, π_t, L_t^d, g_t, and i_t are, respectively, given by:

$$\begin{aligned}
\dot{\mu}_t + \mu_t \left[\left(\frac{1}{\varepsilon_u(c_t)} - \frac{c_t \varepsilon_u'(c_t)}{(\varepsilon_u(c_t))^2} \right) \left[i_t - \pi_t - \rho + \frac{\gamma'(0)}{u'(c_t)} \right] + \frac{\gamma'(0)}{u'(c_t)} - (\rho - n) \right] & \\
= u'(c_t) - \lambda_t + \zeta_t \frac{\varepsilon_u(c_t)}{c_t} \frac{\beta v'\left(L_t^d\right)}{f'\left(L_t^d\right) u'(c_t)}, & \quad \text{(B1)}
\end{aligned}$$

$$\dot{\eta}_t - (\theta + \rho - n) \eta_t + \zeta_t = 0, \tag{B2}$$

$$\zeta_t = \eta_t \theta + \mu_t \frac{c_t}{\varepsilon_u(c_t)}, \tag{B3}$$

$$v'\left(L_t^d\right) = \lambda_t f'\left(L_t^d\right) + \zeta_t \frac{\varepsilon_v\left(L_t^d\right) + \varepsilon_f\left(L_t^d\right)}{L_t^d} \frac{\beta v'\left(L_t^d\right)}{f'\left(L_t^d\right) u'(c_t)}, \tag{B4}$$

$$\lambda_t = \chi'(g_t), \tag{B5}$$

$$\kappa_t = \mu_t \frac{c_t}{\varepsilon_u(c_t)}, \tag{B6}$$

where $\varepsilon_v\left(L_t^d\right) = L_t^d v''\left(L_t^d\right) / v'\left(L_t^d\right)$ and $\varepsilon_f\left(L_t^d\right) = -L_t^d f''\left(L_t^d\right) / f'\left(L_t^d\right)$. In addition, the Kuhn-Tucker conditions associated with the zero lower bound imply:

$$\kappa_t i_t = 0, \ \kappa_t \geq 0, \text{ and } i_t \geq 0. \tag{B7}$$

Finally, the first-order conditions with respect to c_0, c_S, π_0^A, and π_S^A are, respectively, given by:

$$\mu_0 = 0, \ e^{-(\rho-n)S} \mu_S = 0, \ \omega = \eta_0, \text{ and } e^{-(\rho-n)S} \eta_S = 0. \tag{B8}$$

The optimal paths of c_t, π_t^A, μ_t, and η_t are characterized by the four differential equations (23) with $\Delta_t = 0$, (29), (B1), and (B2) subject to four boundary conditions given by $\pi_0^A = \pi^R$, $\mu_0 = 0$, $\mu_S = 0$, and $\eta_S = 0$. The remaining variables g_t, π_t, ζ_t, L_t^d, λ_t, κ_t, and i_t are jointly determined as a function of c_t, π_t^A, μ_t, and η_t by (27), (28) with

$\pi_t \geq \pi^R$ (and l_t^s given by (24)), (B3), (B4), (B5), (B6), and (B7).

C Non-Contingent Fiscal Policy under Naive Expectations

Let us consider a finite horizon of length S, with $S >> T$. Unless $\bar{\pi}_T^A = \hat{\pi}$ is very high, the downward wage rigidity is binding after time T along the stagnation path, i.e. $\bar{\pi}_t = \pi^R$ for all $t \geq T$. Hence, by the Euler equation (23) with $\bar{c}_S = c^{ss}$ and $\bar{\imath}_t - \pi_t = 0 - \pi^R$ for all $t \in [T, S]$, we must have $\bar{c}_t = c^{ss}$ for all $t \in [T, S]$. We can therefore replace $\bar{c}_S = c^{ss}$ by $\bar{c}_T = c^{ss}$. This allows us to ignore the stagnation path $(\bar{c}_t, \bar{L}_t^d, \bar{l}_t^s, \bar{\pi}_t, \bar{\pi}_t^A)_{t=T}^S$ after time T. Also, with $\hat{\pi} \geq -r^n$, the economy must be in the neoclassical steady state as early as time T. Hence, the Lagrangian corresponding to the optimal policy problem is simply given by:[38]

$$\begin{aligned}
\mathcal{L} \;=\; & \int_0^T e^{-(\rho-n)t}\Bigg[u\left(\bar{c}_t\right) + \chi\left(g_t\right) - v\left(\bar{L}_t^d\right) + \gamma\left(0\right) + \bar{\lambda}_t\left[f\left(\bar{L}_t^d\right) - \bar{c}_t - g_t\right] \\
& + \bar{\mu}_t\left[\dot{\bar{c}}_t - \frac{\bar{c}_t}{\varepsilon_u\left(\bar{c}_t\right)}\left[\bar{\imath}_t - \bar{\pi}_t - \rho + \frac{\gamma'(0)}{u'\left(\bar{c}_t\right)}\right]\right] + \kappa_t \bar{\imath}_t \\
& + \bar{\zeta}_t\left[\bar{\pi}_t^A + \beta\left[\frac{v'\left(\bar{L}_t^d\right)}{f'\left(\bar{L}_t^d\right)u'\left(\bar{c}_t\right)} - 1\right] - \bar{\pi}_t\right] + \bar{\eta}_t\left[\theta\left[\bar{\pi}_t - \bar{\pi}_t^A\right] - \dot{\bar{\pi}}_t^A\right]\Bigg]dt \\
& + \int_T^S e^{-(\rho-n)t}\left[u\left(c^n\right) + \chi\left(g^n\right) - v\left(L_n^d\right) + \gamma\left(0\right)\right]dt \\
& + \bar{\omega}\left[\pi^R - \bar{\pi}_0^A\right] + \bar{\nu}\left[\bar{\pi}_T^A - \hat{\pi}\right] + \bar{\psi}\left[c^{ss} - \bar{c}_T\right] + \omega\left[\hat{\pi} - \pi_T^A\right].
\end{aligned}$$

The first-order conditions are given by equations identical to (B1)-(B6). Also, the zero lower bound implies a complementary slackness condition identical to (B7). The first-order conditions with respect to $\bar{c}_0$, $\bar{c}_T$, $\bar{\pi}_0^A$, and $\bar{\pi}_T^A$ are, respectively, given by:

$$\bar{\mu}_0 = 0,\ \bar{\psi} = e^{-(\rho-n)T}\bar{\mu}_T,\ \bar{\omega} = \bar{\eta}_0, \text{ and } \bar{\nu} = e^{-(\rho-n)T}\bar{\eta}_T.$$

Similarly, the first-order conditions with respect to c_T, c_S, π_T^A, and π_S^A are, respectively, given by:

$$e^{-(\rho-n)T}\mu_T = 0,\ e^{-(\rho-n)S}\mu_S = 0,\ \omega = e^{-(\rho-n)T}\eta_T, \text{ and } e^{-(\rho-n)S}\eta_S = 0.$$

[38] If we cannot replace $\bar{c}_S = c^{ss}$ by $\bar{c}_T = c^{ss}$, then we must keep track of of the stagnation path from time T to S, given by $(\bar{c}_t, \bar{L}_t^d, \bar{l}_t^s, \bar{\pi}_t, \bar{\pi}_t^A)_{t=T}^S$, and include the corresponding constraints within the Lagrangian. Similarly, if we do not have $\hat{\pi} \geq -r^n$, then we must include the constraints characterizing $(c_t, L_t^d, l_t^s, \pi_t, \pi_t^A)_{t=T}^S$ within the Lagrangian.

Finally, the first-order condition for the optimal time T is given by:

$$\left[u\left(\bar{c}_T\right)+\chi\left(g_T\right)-v\left(\bar{L}_T^d\right)\right]-\left[u\left(c^n\right)+\chi\left(g^n\right)-v\left(L_n^d\right)\right]=-\bar{\eta}_T\dot{\bar{\pi}}_T^A+\bar{\mu}_T\dot{\bar{c}}_T+\eta_T\dot{\pi}_T^A.$$

The optimal paths of $\bar{c}_t$, $\bar{\pi}_t^A$, $\bar{\mu}_t$, and $\bar{\eta}_t$ from time 0 to T are characterized by the four differential equations (23) with $\Delta_t=0$, (29), (B1), and (B2) subject to the four boundary conditions given by $\bar{\pi}_0^A=\pi^R$, $\bar{\pi}_T^A=\hat{\pi}$, $\bar{c}_T=c^{ss}$, and $\bar{\mu}_0=0$. As in appendix B, the remaining variables are jointly determined by the remaining constraints and first-order conditions.

D Non-Contingent Fiscal Policy under Rational Expectations

Considering, as in appendix C, that the downward wage rigidity constraint is binding after time T along the path leading to the secular stagnation steady state, i.e. $\bar{\pi}_t=\pi^R$ for all $t\geq T$, we can replace $\bar{c}_S=c^{ss}$ by $\bar{c}_T=c^{ss}$. Also, $\hat{\pi}\geq -r^n$ is a sufficient condition for being in the neoclassical steady state from time T onwards. Under these circumstances, the Lagrangian for the optimal policy problem with non-contingent monetary policy is given by:

$$\begin{aligned}\mathcal{L} \;=\; & \int_0^T e^{-(\rho-n)t}\Bigg[u\left(c_t\right)+\chi\left(g_t\right)-v\left(L_t^d\right)+\gamma\left(0\right)+\lambda_t\left[f\left(L_t^d\right)-c_t-g_t\right] \\ & +\mu_t\left[\dot{c}_t-\frac{c_t}{\varepsilon_u\left(c_t\right)}\left[i_t-\pi_t-\rho+\frac{\gamma'(0)}{u'\left(c_t\right)}\right]\right]+\kappa_t i_t \\ & +\zeta_t\left[\pi_t^A+\beta\left[\frac{v'\left(L_t^d\right)}{f'\left(L_t^d\right)u'\left(c_t\right)}-1\right]-\pi_t\right]+\eta_t\left[\theta\left[\pi_t-\pi_t^A\right]-\dot{\pi}_t^A\right] \\ & +\bar{\lambda}_t\left[\bar{c}_t+g_t-f\left(\bar{L}_t^d\right)\right]+\bar{\mu}_t\left[\dot{\bar{c}}_t-\frac{\bar{c}_t}{\varepsilon_u\left(\bar{c}_t\right)}\left[i_t-\bar{\pi}_t-\rho+\frac{\gamma'(0)}{u'\left(\bar{c}_t\right)}\right]\right] \\ & +\bar{\zeta}_t\left[\bar{\pi}_t^A+\beta\left[\frac{v'\left(\bar{L}_t^d\right)}{f'\left(\bar{L}_t^d\right)u'\left(\bar{c}_t\right)}-1\right]-\bar{\pi}_t\right]+\bar{\eta}_t\left[\theta\left[\bar{\pi}_t-\bar{\pi}_t^A\right]-\dot{\bar{\pi}}_t^A\right]\Bigg]dt \\ & +\int_T^S e^{-(\rho-n)t}\left[u\left(c^n\right)+\chi\left(g^n\right)-v\left(L_n^d\right)+\gamma\left(0\right)\right]dt \\ & +\omega\left[\pi^R-\pi_0^A\right]+\bar{\omega}\left[\pi^R-\bar{\pi}_0^A\right]+\bar{\nu}\left[\bar{\pi}_T^A-\hat{\pi}\right]+\psi\left[c^n-c_T\right]+\bar{\psi}\left[c^{ss}-\bar{c}_T\right],\end{aligned}$$

where I am assuming that, along both equilibrium paths, the downward wage rigidity is not binding during the implementation of the fiscal stimulus. From time 0 to T, the equations characterizing each of the two equilibrium paths must now be included within the Lagrangian. Note that the condition $c_T=c^n$ is simply requiring that there is no jump in consumption at time T.

The first-order conditions with respect to c_t, π_t^A, π_t, and L_t^d are still given by (B1), (B2), (B3), (B4), respectively, and the Kuhn-Tucker conditions associated with the zero lower bound on i_t by (B7). Similarly, the first-order conditions with respect to $\bar{c}_t$, $\bar{\pi}_t^A$, $\bar{\pi}_t$, and $\bar{L}_t^d$ are:

$$\dot{\bar{\mu}}_t + \bar{\mu}_t \left[\left(\frac{1}{\varepsilon_u(\bar{c}_t)} - \frac{\bar{c}_t \varepsilon_u'(\bar{c}_t)}{(\varepsilon_u(\bar{c}_t))^2} \right) \left[i_t - \bar{\pi}_t - \rho + \frac{\gamma'(0)}{u'(\bar{c}_t)} \right] + \frac{\gamma'(0)}{u'(\bar{c}_t)} - (\rho - n) \right]$$
$$= \bar{\lambda}_t + \bar{\zeta}_t \frac{\varepsilon_u(\bar{c}_t)}{\bar{c}_t} \frac{\beta v'(\bar{L}_t^d)}{f'(\bar{L}_t^d) u'(\bar{c}_t)},$$

$$\dot{\bar{\eta}}_t - (\theta + \rho - n)\bar{\eta}_t + \bar{\zeta}_t = 0,$$
$$\bar{\zeta}_t = \bar{\eta}_t \theta + \bar{\mu}_t \frac{\bar{c}_t}{\varepsilon_u(\bar{c}_t)},$$
$$\bar{\lambda}_t f'(\bar{L}_t^d) = \bar{\zeta}_t \frac{\varepsilon_v(\bar{L}_t^d) + \varepsilon_f(\bar{L}_t^d)}{\bar{L}_t^d} \frac{\beta v'(\bar{L}_t^d)}{f'(\bar{L}_t^d) u'(\bar{c}_t)},$$

The first-order condition with respect to i_t and g_t are:

$$\kappa_t = \mu_t \frac{c_t}{\varepsilon_u(c_t)} + \bar{\mu}_t \frac{\bar{c}_t}{\varepsilon_u(\bar{c}_t)},$$

$$\chi'(g_t) + \bar{\lambda}_t = \lambda_t.$$

The first-order conditions with respect to c_0, c_T, π_0^A, π_T^A, $\bar{c}_0$, $\bar{c}_T$, $\bar{\pi}_0^A$, and $\bar{\pi}_T^A$ are, respectively, given by:

$$\mu_0 = 0,\ \psi = e^{-(\rho-n)T}\mu_T,\ \omega = \eta_0,\ e^{-(\rho-n)T}\eta_T = 0,$$
$$\bar{\mu}_0 = 0,\ \bar{\psi} = e^{-(\rho-n)T}\bar{\mu}_T,\ \bar{\omega} = \bar{\eta}_0, \text{ and } \bar{\nu} = e^{-(\rho-n)T}\bar{\eta}_T.$$

Finally, the first-order condition for the optimal time T is:

$$\left[u(c^n) + \chi(g_T) - v(L_T^d)\right] - \left[u(c^n) + \chi(g^n) - v(L_n^d)\right] = \mu_T \dot{c}_T - \bar{\eta}_T \dot{\bar{\pi}}_T^A + \bar{\mu}_T \dot{\bar{c}}_T.$$

The optimal paths of c_t, π_t^A, μ_t, η_t, $\bar{c}_t$, $\bar{\pi}_t^A$, $\bar{\mu}_t$, and $\bar{\eta}_t$ are characterized by eight differential equations subject to the eight boundary conditions given by $\pi_0^A = \pi^R$, $c_T = c^n$, $\mu_0 = 0$, $\eta_T = 0$, $\bar{\pi}_0^A = \pi^R$, $\bar{\pi}_T^A = \hat{\pi}$, $\bar{c}_T = c^{ss}$, and $\bar{\mu}_0 = 0$. As in appendix B or C, the remaining variables are jointly determined by the remaining constraints and first-order conditions.

If monetary policy is state-contingent, we need to characterize both the path of the nominal interest rate along the stagnation path $\bar{\imath}_t$ and along the neoclassical path i_t from time 0 to T. A first-order condition and a complementary slackness condition must therefore be satisfied for each nominal interest rate. The possibility of an infinitely high nominal rate at time T along the neoclassical path implies that I no longer impose the constraint $c_T = c^n$. However, I now impose the condition $\pi_T^A \geq \hat{\pi}$, which is typically binding. The first-order conditions remain identical, except for the optimal time T, which is now given by:

$$\left[u\left(c_T\right)+\chi\left(g_T\right)-v\left(L_T^d\right)\right]-\left[u\left(c^n\right)+\chi\left(g^n\right)-v\left(L_n^d\right)\right]=-\eta_T\dot{\pi}_T^A-\bar{\eta}_T\dot{\bar{\pi}}_T^A+\bar{\mu}_T\dot{\bar{c}}_T.$$

Finally, the boundary conditions $c_T = c^n$ and $\eta_T = 0$ need to be replaced by $\pi_T^A = \hat{\pi}$ and $\mu_T = 0$.

E Non-Contingent Fiscal Policy with Ponzi Schemes under Rational Expectations

Let Ψ_t denote the present value of the fiscal stimulus along the neoclassical path at time t:

$$\Psi_t=\int_t^{\infty} e^{-\int_t^s (r_u-n)du}\left[g_s-g^n\right]ds.$$

The initial value of the Ponzi scheme (44) can be written as $\Delta_0 = \Psi_0$ with Ψ_t defined by:

$$\dot{\Psi}_t=\left(r_t-n\right)\Psi_t-g_t+g^n,$$

and:

$$\lim_{T\to\infty} e^{-\int_t^T (r_u-n)du}\Psi_T=0.$$

We can proceed similarly along the stagnation path.

Let us assume a finite horizon of length S, with $S >> T$. Let $(c_t, L_t^d, l_t^s, \Delta_t, \Psi_t, \pi_t, \pi_t^A)_{t=0}^S$ denote the equilibrium path leading to the neoclassical steady state and $(\bar{c}_t, \bar{L}_t^d, \bar{l}_t^s, \bar{\Delta}_t, \bar{\Psi}_t, \bar{\pi}_t, \bar{\pi}_t^A)_{t=0}^S$ the path leading to the secular stagnation steady state. As in appendix D, $\hat{\pi} \geq -r^n$ is a sufficient condition for the economy to be in the neoclassical steady state from time T onwards.[39] This entails $\Psi_T = 0$. The Ponzi scheme $\bar{\Delta}_t$ implies that the secular stagnation equilibrium only asymptotically reaches its steady state. Thus, the stagnation path from time T onwards is fully characterized (as a function of $\bar{\Delta}_T$) by the Euler equation (23) and the Ponzi dynamics (26) with boundary condition $\bar{c}_S = c^{ss}$. Also, after time T, along the stagnation path, government spending $\bar{g}_t$ is set opportunistically, in accordance with equation (30), while the zero lower bound on the nominal interest rate $\bar{\imath}_t$ is binding.

[39] As the planner does not value "Ponzi wealth", i.e. its utility of wealth is always equal to $\gamma(0)$, the Ponzi scheme Δ_t does not affect the chosen allocation of resources along the neoclassical path after time T.

Incorporating these features, and the Ponzi schemes, within the optimal policy problem of appendix D, for non-contingent monetary policy, yields the following Lagrangian:

$$
\begin{aligned}
\mathcal{L} \;=\; & \int_0^T e^{-(\rho-n)t}\Bigg[u\left(c_t\right)+\chi\left(g_t\right)-v\left(L_t^d\right)+\gamma(0)+\lambda_t\left[f\left(L_t^d\right)-c_t-g_t\right] \\
& +\mu_t\left[\dot{c}_t-\frac{c_t}{\varepsilon_u\left(c_t\right)}\left[i_t-\pi_t-\rho+\frac{\gamma'\left(\Delta_t\right)}{u'\left(c_t\right)}\right]\right]+\kappa_t i_t \\
& +\zeta_t\left[\pi_t^A+\beta\left[\frac{v'\left(L_t^d\right)}{f'\left(L_t^d\right)u'\left(c_t\right)}-1\right]-\pi_t\right]+\eta_t\left[\theta\left[\pi_t-\pi_t^A\right]-\dot{\pi}_t^A\right] \\
& +\xi_t\left[\left(i_t-\pi_t-n\right)\Delta_t-\dot{\Delta}_t\right]+\phi_t\left[\dot{\Psi}_t-\left(i_t-\pi_t-n\right)\Psi_t+g_t-g^n\right] \\
& +\bar{\lambda}_t\left[\bar{c}_t+g_t-f\left(\bar{L}_t^d\right)\right]+\bar{\mu}_t\left[\dot{\bar{c}}_t-\frac{\bar{c}_t}{\varepsilon_u\left(\bar{c}_t\right)}\left[i_t-\bar{\pi}_t-\rho+\frac{\gamma'(\bar{\Delta}_t)}{u'\left(\bar{c}_t\right)}\right]\right] \\
& +\bar{\zeta}_t\left[\bar{\pi}_t^A+\beta\left[\frac{v'\left(\bar{L}_t^d\right)}{f'\left(\bar{L}_t^d\right)u'\left(\bar{c}_t\right)}-1\right]-\bar{\pi}_t\right]+\bar{\eta}_t\left[\theta\left[\bar{\pi}_t-\bar{\pi}_t^A\right]-\dot{\bar{\pi}}_t^A\right] \\
& +\bar{\xi}_t\left[\left(i_t-\bar{\pi}_t-n\right)\bar{\Delta}_t-\dot{\bar{\Delta}}_t\right]+\bar{\phi}_t\left[\dot{\bar{\Psi}}_t-\left(i_t-\bar{\pi}_t-n\right)\bar{\Psi}_t+g_t-g^{ss}\right]\Bigg]dt \\
& +\int_T^S e^{-(\rho-n)t}\Bigg[u\left(c^n\right)+\chi\left(g^n\right)-v\left(L_n^d\right)+\gamma(0)+\bar{\lambda}_t\left[\bar{c}_t+\bar{g}_t-f\left(\bar{L}_t^d\right)\right] \\
& \bar{\mu}_t\left[\dot{\bar{c}}_t-\frac{\bar{c}_t}{\varepsilon_u\left(\bar{c}_t\right)}\left[-\pi^R-\rho+\frac{\gamma'(\bar{\Delta}_t)}{u'\left(\bar{c}_t\right)}\right]\right]+\bar{\upsilon}_t\left[\chi'\left(\bar{g}_t\right)-\frac{v'\left(\bar{L}_t^d\right)}{f'\left(\bar{L}_t^d\right)}\right] \\
& +\bar{\xi}_t\left[\left(-\pi^R-n\right)\bar{\Delta}_t-\dot{\bar{\Delta}}_t\right]+\bar{\phi}_t\left[\dot{\bar{\Psi}}_t-\left(-\pi^R-n\right)\bar{\Psi}_t+\bar{g}_t-g^{ss}\right]\Bigg]dt \\
& +\omega\left[\pi^R-\pi_0^A\right]+\bar{\omega}\left[\pi^R-\bar{\pi}_0^A\right]+\bar{\nu}\left[\bar{\pi}_T^A-\hat{\pi}\right]+\psi\left[c^n-c_T\right]+\bar{\psi}\left[c^{ss}-\bar{c}_S\right] \\
& -\Lambda\Psi_T-\bar{\Lambda}e^{-\int_0^S\left(\bar{\imath}_u-\pi^R-n\right)du}\bar{\Psi}_S+\Omega\left[\Psi_0-\Delta_0\right]+\bar{\Omega}\left[\bar{\Psi}_0-\bar{\Delta}_0\right] \\
& +\bar{\varepsilon}\left[\bar{c}_{T+}-\bar{c}_{T-}\right]+\bar{\Theta}\left[\bar{\Delta}_{T-}-\bar{\Delta}_{T+}\right]+\bar{\Xi}\left[\bar{\Psi}_{T+}-\bar{\Psi}_{T-}\right].
\end{aligned}
$$

As the transversality condition (25) has not been incorporated into the Lagrangian, it much be checked that the solution satisfies this condition along each of the two equilibrium paths.[40]

Proceeding as in appendix D, we obtain the first-order conditions. In particular, the first-order condition with respect to T is given by:

$$
\begin{aligned}
\left[u\left(c^n\right)+\chi\left(g_T\right)-v\left(L_T^d\right)\right]-\left[u\left(c^n\right)+\chi\left(g^n\right)-v\left(L_n^d\right)\right] &= \mu_T\dot{c}_T-\bar{\eta}_T\dot{\bar{\pi}}_T^A \\
&+\phi_T\dot{\Psi}_T-\bar{\mu}_T[\dot{\bar{c}}_{T+}-\dot{\bar{c}}_{T-}]+\xi_T[\dot{\bar{\Delta}}_{T+}-\dot{\bar{\Delta}}_{T-}]-\phi_T[\dot{\bar{\Psi}}_{T+}-\dot{\bar{\Psi}}_{T-}].
\end{aligned}
$$

This results in a system of 16 differential equations determining the paths of c_t, π_t^A, Ψ_t,

[40] Using the Euler equation (23), it can easily be shown that a sufficient condition for the transversality condition (25) to be satisfied, when $\Delta_0 > 0$, is $\lim_{S\to\infty} r_S < n$.

Δ_t, μ_t, η_t, ϕ_t, ξ_t, $\bar{c}_t$, $\bar{\pi}_t^A$, $\bar{\Psi}_t$, $\bar{\Delta}_t$, $\bar{\mu}_t$, $\bar{\eta}_t$, $\bar{\phi}_t$, and $\bar{\xi}_t$. The corresponding 16 boundary conditions are given by $\pi_0^A = \pi^R$, $c_T = c^n$, $\Delta_0 = \Psi_0$, $\Psi_T = 0$, $\mu_0 = 0$, $\eta_T = 0$, $\phi_0 = \xi_0$, $\xi_T = 0$, $\bar{\pi}_0^A = \pi^R$, $\bar{\pi}_T^A = \hat{\pi}$, $\bar{c}_S = c^{ss}$, $\bar{\Delta}_0 = \bar{\Psi}_0$, $\bar{\Psi}_T = 0$, $\bar{\mu}_0 = 0$, $\bar{\phi}_0 = \bar{\xi}_0$, and $\bar{\xi}_T = 0$.

If monetary policy is state-contingent, we can proceed as in appendix D. The first-order conditions remain identical, except for the optimal time T, which is now given by:

$$\left[u\left(c_T\right) + \chi\left(g_T\right) - v\left(L_T^d\right)\right] - \left[u\left(c^n\right) + \chi\left(g^n\right) - v\left(L_n^d\right)\right] = -\eta_T \dot{\pi}_T^A - \bar{\eta}_T \dot{\bar{\pi}}_T^A$$
$$+ \phi_T \dot{\Psi}_T - \bar{\mu}_T[\dot{\bar{c}}_{T+} - \dot{\bar{c}}_{T-}] + \xi_T[\dot{\bar{\Delta}}_{T+} - \dot{\bar{\Delta}}_{T-}] - \phi_T[\dot{\bar{\Psi}}_{T+} - \dot{\bar{\Psi}}_{T-}].$$

For the boundary conditions, $c_T = c^n$ and $\eta_T = 0$ need to be replaced by $\pi_T^A = \hat{\pi}$ and $\mu_T = 0$.

F Capital with Adjustment Costs

In this section of the appendix, I first present the model with capital and adjustment costs. I then calibrate this model, before solving for the optimal monetary and fiscal policy.

F.1 Introducing Capital into the Model

Let I_t and K_t denote investment per capita and the capital stock per capita at time t, respectively. Total investment and capital at t are therefore equal to $K_t N_t$ and $I_t N_t$. Whenever aggregate investment is equal to I_t, a fraction $\phi\left(I_t/K_t\right)$ of this investment is lost in the adjustment process and does not contribute to the accumulation of capital. The capital accumulation equation is therefore given by:

$$\dot{K}_t = \left[1 - \phi\left(\frac{I_t}{K_t}\right)\right] I_t - \left(\delta + n\right) K_t, \tag{F1}$$

where δ is the depreciation rate. I assume $\phi''\left(\cdot\right) > 0$, to have convex adjustment cost, and $\phi(\delta + n) = \phi'(\delta + n) = 0$, to have no adjustment cost in steady state.

Output is produced from capital and labor using a constant returns to scale neoclassical production function $F\left(K_t N_t, L_t^d N_t\right)$ where, as before, L_t^d denotes employment per capita. In intensive form, output per capita is given by:

$$\frac{F\left(K_t N_t, L_t^d N_t\right)}{N_t} = L_t^d f\left(\frac{K_t}{L_t^d}\right), \tag{F2}$$

where $f\left(x\right) = F(x, 1)$.

Let $V(K_t)$ denote the value (per capita) of a firm with capital stock K_t. The corresponding profit maximization problem from time 0 to t is given by:

$$\begin{aligned} V(K_0) &= \max_{(L_s, I_s)_{s=0}^t} \int_0^t e^{-\int_0^s (r_u - n)du} \left[L_s^d f\left(\frac{K_s}{L_s^d}\right) - w_s L_s^d - I_s \right] ds \\ &\quad + e^{-\int_0^t (r_u - n)du} V(K_t), \quad \text{(F3)} \end{aligned}$$

subject to the capital accumulation equation (F1). The first-order conditions with respect to L_s, I_s, K_s, and K_t are given by:

$$w_s = f\left(\frac{K_s}{L_s^d}\right) - \frac{K_s}{L_s^d} f'\left(\frac{K_s}{L_s^d}\right), \quad \text{(F4)}$$

$$q_s = \frac{1}{1 - \phi\left(\frac{I_s}{K_s}\right) - \frac{I_s}{K_s}\phi'\left(\frac{I_s}{K_s}\right)}, \quad \text{(F5)}$$

$$r_s = \frac{1}{q_s} f'\left(\frac{K_s}{L_s^d}\right) + \left(\frac{I_s}{K_s}\right)^2 \phi'\left(\frac{I_s}{K_s}\right) - \delta + \frac{\dot{q}_s}{q_s}, \quad \text{(F6)}$$

$$q_t = V'(K_t), \quad \text{(F7)}$$

where q_s is current-value multiplier on the capital accumulation equation, which corresponds to the shadow price of capital within the firm. Substituting these optimality conditions within the value of the firm yields:

$$V(K_t) = q_t K_t + e^{\int_0^t (r_u - n)du} \left[V(K_0) - q_0 K_0 \right]. \quad \text{(F8)}$$

In the absence of bubble, we have $V(K_t) = q_t K_t$, which is Hayashi's (1982) celebrated result that, under constant returns to scale, the marginal q is equal to the average q.

The asset market clearing equation is now given by:

$$a_t = b_t + q_t K_t, \quad \text{(F9)}$$

and the goods market clearing equation by:

$$L_t^d f\left(\frac{K_t}{L_t^d}\right) = c_t + I_t + g_t. \quad \text{(F10)}$$

For a given governmental policy, determined by $(g_t, i_t)_{t=0}^\infty$ and Δ_0, the equilibrium of the economy, $(c_t, L_t^d, l_t^s, I_t, K_t, \Delta_t, \pi_t, \pi_t^A, q_t, r_t)_{t=0}^\infty$, is fully characterized by:

- The Fisher identity $r_t = i_t - \pi_t$;
- The consumption Euler equation (4), where a_t is given by the asset market clearing equation (F9);

- The labor supply function (5), where w_t is given by marginal product of labor (F4);
- The household's transversality condition which, by Lemma 1, can be written as:

$$\lim_{t\to\infty} e^{-(\rho-n)t} u'(c_t)\left[q_t K_t + \Delta_t\right] = 0; \tag{F11}$$

- The demand for investment given by (F6), where the shadow price of capital is defined by (F5);
- The dynamics of the government's Ponzi scheme (20);
- The goods market clearing condition (F10);
- The capital accumulation equation (F1) with K_0 given;
- The nominal wage sluggishness equation (14);
- The inflation anchor updating equation (12) with π_0^A given.

As before, for a given nominal interest rate i and assuming opportunistic government spending:

$$\chi'(g_t) = \frac{v'\left(L_t^d\right)}{f\left(\frac{K_t}{L_t^d}\right) - \frac{K_t}{L_t^d} f'\left(\frac{K_t}{L_t^d}\right)}, \tag{F12}$$

there are three steady state equilibria: a neoclassical steady state $\left(c^n, g^n, L_n^d, l_n^s, I^n, K^n, \Delta^n, \pi^n, q^n, r^n\right)$ with $L_n^d = l_n^s$ and $\Delta^n = 0$; a secular stagnation steady state $\left(c^{ss}, g^{ss}, L_{ss}^d, l_{ss}^s, I^{ss}, K^{ss}, \Delta^{ss}, \pi^{ss}, q^{ss}, r^{ss}\right)$ with $L_{ss}^d < l_{ss}^s$, $\pi^{ss} = \pi^R$, and $\Delta^{ss} = 0$; and a Ponzi steady state $\left(c^p, g^p, L_p^d, l_p^s, I^p, K^p, \Delta^p, \pi^p, q^p, r^p\right)$ with $L_p^d = l_p^s$, $r^p = n$, and $\Delta^p > 0$.

F.2 Calibration

Following much of the literature, the adjustment cost function is assumed be quadratic with respect to the reference point $\delta + n$ such as to normalize the cost of adjustment to zero in steady state:

$$\phi\left(\frac{I_t}{K_t}\right) = \frac{k_{I/K}}{2}\left[\frac{I_t}{K_t} - (\delta + n)\right]^2. \tag{F13}$$

The parameter $k_{I/K}$ determines the convexity of the adjustment cost function, since $\phi''(I_t/K_t) = k_{I/K}$.

I perform a yearly calibration of the model following exactly the same procedure as before. The two new parameters are calibrated as follows. The depreciation rate δ is set such that, in the Ponzi steady state, i.e. at the golden rule level of the capital stock, capital is equal to two and a half years of output. The scale parameter of the adjustment cost function $k_{I/K}$ is set such that, from the capital accumulation equation (F1), with constant investment, it takes 8 years for capital to close half the gap to the corresponding steady state, starting from 90% of the steady state capital stock.

Let $(c_t^n, I_t^n, g_t^n)_{t=0}^{\infty}$ denote the trajectory of consumption, investment, and government spending along the neoclassical equilibrium under *laissez-faire* (and ignoring the zero lower bound), starting from $K_0 = K^{ss}$. Thus, if the economy was to jump to the neoclassical equilibrium at time 0, it would reach c_0^n, I_0^n, and g_0^n. I therefore calibrate the parameters k_L, k_G, and k_W such as to hit the same moments as before, but at time 0. Hence, the neoclassical output level at time 0, given by $c_0^n+I_0^n+g_0^n$, is normalized to one; consumption c_0^n is three times as large as government spending g_0^n; and consumption under stagnation is 10% below neoclassical consumption, $c^{ss} = (1-0.1)c_0^n$. The calibration of the model is summarized in Table F1.

Parameter	Calibrated value	Moment
Discount rate	$\rho = 4\%$	·
Population growth	$n = 0\%$	·
Capital share	$\alpha = 0.3$	·
CRRA for private consumption	$\sigma = 2$	·
Frisch elasticity of labor supply	$\xi = 0.5$	·
Scale parameter of disutility of labor supply	$k_L = 9.216$	$c_0^n+I_0^n+g_0^n = 1$
CRRA for public consumption	$\sigma_G = 2$	·
Scale parameter of utility of public consumption	$k_G = 0.111$	$g_0^n = c_0^n/3$
CRRA for wealth (relative to reference level)	$\sigma_W = 1.5$	·
Scale parameter of preference for wealth	$k_W = 1.763$	$c^{ss} = (1-0.1)c_0^n$
Reference wealth level	$\underline{W} = -1.201$	$\Delta^p = 1.5(c^p+I^p+g^p)$
Depreciation rate	$\delta = 0.12$	$K^p/(c^p+I^p+g^p) = 2.5$
Reference rate of inflation for wage bargaining	$\pi^R = 1\%$	·
Speed of adjustment of inflation anchor	$\theta = 0.347$	Half-life of $\pi_t^A = 2$
Wage sluggishness	$\beta = 0.15$	Phillips curve slope = 0.3
Investment adjustment cost	$k_{I/K} = 467$	Half-life of $K_t = 8$

Table F1: Calibration of the model with capital

Under this calibration, the natural real interest rate r^n is equal to -1.48%. As before, I set the inflation threshold 2% above $-r^n$, resulting in $\hat{\pi} = 3.48\%$.

F.3 Non-Contingent Fiscal Policy with Capital

Let us now solve for the optimal non-contingent fiscal policy under rational expectations. Government spending is financed from lump-sum taxes, resulting in $\Delta_t = \bar{\Delta}_t = 0$ for all t. At time 0, the economy is in the secular stagnation steady state, with capital equal to K^{ss} and the inflation anchor equal to π^R. As before, to move the economy to the neoclassical equilibrium path, the inflation anchor under both paths must reach the inflation threshold $\hat{\pi}$ by the end of the stimulus episode, i.e. $\pi_T^A \geq \hat{\pi}$ and $\bar{\pi}_T^A \geq \hat{\pi}$. I first consider the case of non-contingent monetary policy.

I consider a finite horizon of length S, with $S >> T$. Let $(c_t, L_t^d, l_t^s, I_t, K_t, \pi_t, \pi_t^A, q_t)_{t=0}^S$ denote the equilibrium path leading to the neoclassical steady state and $(\bar{c}_t, \bar{L}_t^d, \bar{l}_t^s, \bar{I}_t, \bar{K}_t, \bar{\pi}_t, \bar{\pi}_t^A, \bar{q}_t)_{t=0}^S$ the path leading to the secular stagnation steady state. After time T, along both paths, government spending is set opportunistically, in accordance with equation (F12). Also, from time T onwards, along the stagnation path, the zero lower bound and the downward wage rigidity are both binding, resulting in $\bar{\imath}_t = 0$ and $\bar{\pi}_t = \pi^R$; while, along the neoclassical path, the nominal interest rate is set such that labor demand is equal to labor supply, resulting in $v'\left(L_t^d\right) = w_t u'\left(c_t\right)$ (which, by the Phillips curve (14), implies constant inflation equal to π_T^A).[41] Incorporating these features into the optimal policy problem, and using the notation $\tilde{f}(x) = f(x) - x f'(x)$, yields the following Lagrangian:

$$
\begin{aligned}
\mathcal{L} \;=\; & \int_0^T e^{-(\rho-n)t} \Bigg\{ u\left(c_t\right) + \chi\left(g_t\right) - v\left(L_t^d\right) + \gamma\left(q_t K_t\right) + \lambda_t \left[L_t^d f\left(\frac{K_t}{L_t^d}\right) - c_t - I_t - g_t \right] \\
& + \mu_t \left[\frac{c_t}{\varepsilon_u\left(c_t\right)} \left[i_t - \pi_t - \rho + \frac{\gamma'\left(q_t K_t\right)}{u'\left(c_t\right)} \right] - \dot{c}_t \right] + \kappa_t i_t \\
& + \zeta_t \left[\pi_t^A + \beta \left[\frac{v'\left(L_t^d\right)}{\tilde{f}\left(K_t / L_t^d\right) u'\left(c_t\right)} - 1 \right] - \pi_t \right] + \eta_t \left[\theta \left[\pi_t - \pi_t^A \right] - \dot{\pi}_t^A \right] \\
& + \xi_t \left[\dot{q}_t - \left(i_t - \pi_t + \delta\right) q_t + f'\left(\frac{K_t}{L_t^d}\right) + q_t \left(\frac{I_t}{K_t}\right)^2 \phi'\left(\frac{I_t}{K_t}\right) \right] \\
& + \Gamma_t \left[\left[1 - \phi\left(\frac{I_t}{K_t}\right) \right] I_t - \left(\delta + n\right) K_t - \dot{K}_t \right] + \Lambda_t \left[\frac{1}{q_t} - \left[1 - \phi\left(\frac{I_t}{K_t}\right) - \frac{I_t}{K_t} \phi'\left(\frac{I_t}{K_t}\right) \right] \right] \\
& + \bar{\lambda}_t \left[\bar{c}_t + \bar{I}_t + g_t - \bar{L}_t^d f\left(\frac{\bar{K}_t}{\bar{L}_t^d}\right) \right] + \bar{\mu}_t \left[\dot{\bar{c}}_t - \frac{\bar{c}_t}{\varepsilon_u\left(\bar{c}_t\right)} \left[i_t - \bar{\pi}_t - \rho + \frac{\gamma'(\bar{q}_t \bar{K}_t)}{u'\left(\bar{c}_t\right)} \right] \right] \\
& + \bar{\zeta}_t \left[\bar{\pi}_t^A + \beta \left[\frac{v'\left(\bar{L}_t^d\right)}{\tilde{f}\left(\bar{K}_t / \bar{L}_t^d\right) u'\left(\bar{c}_t\right)} - 1 \right] - \bar{\pi}_t \right] + \bar{\eta}_t \left[\theta \left[\bar{\pi}_t - \bar{\pi}_t^A \right] - \dot{\bar{\pi}}_t^A \right] \\
& + \bar{\xi}_t \left[\dot{\bar{q}}_t - \left(i_t - \bar{\pi}_t + \delta\right) \bar{q}_t + f'\left(\frac{\bar{K}_t}{\bar{L}_t^d}\right) + \bar{q}_t \left(\frac{\bar{I}_t}{\bar{K}_t}\right)^2 \phi'\left(\frac{\bar{I}_t}{\bar{K}_t}\right) \right]
\end{aligned}
$$

[41] I consider that the zero lower bound is not binding along the neoclassical path after time T. A sufficient condition for this is $\hat{\pi} \geq -r^n$.

$$
\begin{aligned}
&+\bar{\Gamma}_t\left[\dot{\bar{K}}_t-\left[1-\phi\left(\frac{\bar{I}_t}{\bar{K}_t}\right)\right]\bar{I}_t+(\delta+n)\bar{K}_t\right]+\bar{\Lambda}_t\left[\left[1-\phi\left(\frac{\bar{I}_t}{\bar{K}_t}\right)-\frac{\bar{I}_t}{\bar{K}_t}\phi'\left(\frac{\bar{I}_t}{\bar{K}_t}\right)\right]-\frac{1}{\bar{q}_t}\right]\Bigg\}dt\\
&+\int_T^S e^{-(\rho-n)t}\left\{u(c_t)+\chi(g_t)-v\left(L_t^d\right)+\gamma(q_tK_t)+\Upsilon_t\left[v'\left(L_t^d\right)-\tilde{f}\left(\frac{K_t}{L_t^d}\right)u'(c_t)\right]\right.\\
&+\lambda_t\left[L_t^d f\left(\frac{K_t}{L_t^d}\right)-c_t-I_t-g_t\right]+\mu_t\left[\frac{c_t}{\varepsilon_u(c_t)}\left[i_t-\pi_T^A-\rho+\frac{\gamma'(q_tK_t)}{u'(c_t)}\right]-\dot{c}_t\right]\\
&+\xi_t\left[\dot{q}_t-\left(i_t-\pi_T^A+\delta\right)q_t+f'\left(\frac{K_t}{L_t^d}\right)+q_t\left(\frac{I_t}{K_t}\right)^2\phi'\left(\frac{I_t}{K_t}\right)\right]\\
&+\Gamma_t\left[\left[1-\phi\left(\frac{I_t}{K_t}\right)\right]I_t-(\delta+n)K_t-\dot{K}_t\right]+\Lambda_t\left[\frac{1}{q_t}-\left[1-\phi\left(\frac{I_t}{K_t}\right)-\frac{I_t}{K_t}\phi'\left(\frac{I_t}{K_t}\right)\right]\right]\\
&+\upsilon_t\left[\chi'(g_t)-\frac{v'\left(L_t^d\right)}{\tilde{f}\left(K_t/L_t^d\right)}\right]+\bar{\upsilon}_t\left[\chi'(\bar{g}_t)-\frac{v'\left(\bar{L}_t^d\right)}{\tilde{f}\left(\bar{K}_t/\bar{L}_t^d\right)}\right]\\
&+\bar{\lambda}_t\left[\bar{c}_t+\bar{I}_t+\bar{g}_t-\bar{L}_t^d f\left(\frac{\bar{K}_t}{\bar{L}_t^d}\right)\right]+\bar{\mu}_t\left[\dot{\bar{c}}_t-\frac{\bar{c}_t}{\varepsilon_u(\bar{c}_t)}\left[-\pi^R-\rho+\frac{\gamma'(\bar{q}_t\bar{K}_t)}{u'(\bar{c}_t)}\right]\right]\\
&+\bar{\xi}_t\left[\dot{\bar{q}}_t-\left(-\pi^R+\delta\right)\bar{q}_t+f'\left(\frac{\bar{K}_t}{\bar{L}_t^d}\right)+\bar{q}_t\left(\frac{\bar{I}_t}{\bar{K}_t}\right)^2\phi'\left(\frac{\bar{I}_t}{\bar{K}_t}\right)\right]\\
&+\bar{\Gamma}_t\left[\dot{\bar{K}}_t-\left[1-\phi\left(\frac{\bar{I}_t}{\bar{K}_t}\right)\right]\bar{I}_t+(\delta+n)\bar{K}_t\right]+\bar{\Lambda}_t\left[\left[1-\phi\left(\frac{\bar{I}_t}{\bar{K}_t}\right)-\frac{\bar{I}_t}{\bar{K}_t}\phi'\left(\frac{\bar{I}_t}{\bar{K}_t}\right)\right]-\frac{1}{\bar{q}_t}\right]\Bigg\}dt\\
&+\omega\left[\pi^R-\pi_0^A\right]+\bar{\omega}\left[\pi^R-\bar{\pi}_0^A\right]+\bar{\nu}\left[\bar{\pi}_T^A-\hat{\pi}\right]+\bar{\psi}\left[c^{ss}-\bar{c}_S\right]+\Omega\left[K^{ss}-K_0\right]\\
&+\bar{\Omega}\left[\bar{K}_0-K^{ss}\right]+\Psi\left[1-q_S\right]+\bar{\Psi}\left[1-\bar{q}_S\right]+\varepsilon\left[c_{T-}-c_{T+}\right]+\Theta\left[K_{T-}-K_{T+}\right]\\
&+\Xi\left[q_{T+}-q_{T-}\right]+\bar{\varepsilon}\left[\bar{c}_{T+}-\bar{c}_{T-}\right]+\bar{\Theta}\left[\bar{K}_{T+}-\bar{K}_{T-}\right]+\bar{\Xi}\left[\bar{q}_{T+}-\bar{q}_{T-}\right].
\end{aligned}
$$

Note that, the government only optimizes with respect to g_t and i_t from time 0 to T. The constraints after time T are only included to link the variables of the problem to the boundary conditions at time S.[42]

Proceeding as before, we can derive the first-order conditions. The first-order condition with respect to T, after simplification, is given by:

$$
\begin{aligned}
\left[\chi(g_{T-})-v\left(L_{T-}^d\right)\right]-\left[\chi(g_{T+})-v\left(L_{T+}^d\right)\right]&=-\bar{\eta}_T\dot{\bar{\pi}}_T^A+\mu_T\left[\dot{c}_{T+}-\dot{c}_{T-}\right]\\
&\quad-\xi_T\left[\dot{q}_{T+}-\dot{q}_{T-}\right]-\bar{\mu}_T\left[\dot{\bar{c}}_{T+}-\dot{\bar{c}}_{T-}\right]-\bar{\xi}_T\left[\dot{\bar{q}}_{T+}-\dot{\bar{q}}_{T-}\right].
\end{aligned}
$$

[42] To understand why $c_S=c^n$ has not been imposed as boundary condition, note that after time T, along the neoclassical path, the real interest rate $i_t-\pi_T^A$ is determined such as to have labor market clearing, $v'\left(L_t^d\right)=\tilde{f}(K_t/L_t^d)u'(c_t)$. The resulting system of equations (implicitly) relates q_t and c_t independently of $i_t-\pi_T^A$. Hence, the Euler equation (23) and the asset pricing equation (F6) are not independent from each other and must therefore be combined into a single differential equation for $q_tu'(c_t)$. It follows that we must either impose $q_S=1$ or $c_S=c^n$ (or some combination of both) as boundary condition, the other one being automatically satisfied in the limit as S tends to infinity.

We obtain a system of 16 differential equations determining the paths of c_t, π_t^A, K_t, q_t, μ_t, η_t, Γ_t, ξ_t, $\bar{c}_t$, $\bar{\pi}_t^A$, $\bar{K}_t$, $\bar{q}_t$, $\bar{\mu}_t$, $\bar{\eta}_t$, $\bar{\Gamma}_t$, and $\bar{\xi}_t$. The corresponding 16 boundary conditions are $\pi_0^A = \pi^R$, $K_0 = K^{ss}$, $q_S = 1$, $\mu_0 = 0$, $\eta_T = 0$, $\Gamma_S = 0$, $\xi_0 = 0$, $c_{T+} = c_{T-}$, $\bar{\pi}_0^A = \pi^R$, $\bar{\pi}_T^A = \hat{\pi}$, $\bar{c}_S = c^{ss}$, $\bar{K}_0 = K^{ss}$, $\bar{q}_S = 1$, $\bar{\mu}_0 = 0$, $\bar{\Gamma}_S = 0$, $\bar{\xi}_0 = 0$. Note that there is no discontinuity at time T in any of these variables.[43]

Let us now consider state-contingent monetary policy. It turns out that, as before, the zero lower bound is binding along the neoclassical path until time T, at which point it becomes infinitely high resulting in a jump in c_t and q_t at T. However, combining the Euler equation (23) and the asset pricing equation (F6) reveals that $u'(c_t)\, q_t$ cannot jump. In the Lagrangian, the boundary conditions $c_{T+} = c_{T-}$ and $q_{T+} = q_{T-}$ are therefore replaced by $\pi_T^A = \hat{\pi}$ (since $\pi_T^A \geq \hat{\pi}$ is now binding) and $q_{T+}u'(c_{T+}) = u'(c_{T-})q_{T-}$. The first-order condition with respect to T, after simplification, is given by:

$$\begin{aligned}
&\left[u\left(c_{T-}\right) + \chi\left(g_{T-}\right) - v\left(L_{T-}^d\right) + \gamma\left(q_{T-}K_T\right)\right] \\
&\qquad - \left[u\left(c_{T+}\right) + \chi\left(g_{T+}\right) - v\left(L_{T+}^d\right) + \gamma\left(q_{T+}K_T\right)\right] = -\bar{\eta}_T\dot{\bar{\pi}}_T^A - \eta_T\dot{\pi}_T^A \\
&+ \mu_{T-}\left[\frac{q_{T+}}{q_{T-}}\frac{u''\left(c_{T+}\right)}{u''\left(c_{T-}\right)}\dot{c}_{T+} - \dot{c}_{T-}\right] + \Gamma_T\left[\dot{K}_{T+} - \dot{K}_{T-}\right] - \xi_{T-}\left[\frac{u'\left(c_{T+}\right)}{u'\left(c_{T-}\right)}\dot{q}_{T+} - \dot{q}_{T-}\right] \\
&\qquad\qquad - \bar{\mu}_T[\dot{\bar{c}}_{T+} - \dot{\bar{c}}_{T-}] - \bar{\xi}_T\left[\dot{\bar{q}}_{T+} - \dot{\bar{q}}_{T-}\right].
\end{aligned}$$

The 16 differential equations remain unchanged, while the two boundary conditions $\eta_T = 0$ and $c_{T+} = c_{T-}$ are now replaced by $\pi_T^A = \hat{\pi}$ and $\mu_{T-} = [\varepsilon_u\left(c_{T-}\right)/c_{T-}]q_{T-}\xi_{T-}$.[44]

References

[1] Hayashi, F. (1982), 'Tobin's Marginal q and Average q: A Neoclassical Interpretation', *Econometrica*, 50(1), 213-224.

[43] From time T to S, the differential equations for c_t and q_t are combined into a single differential equation for $q_t u'(c_t)$ (see previous footnote). When solving the differential equations, the initial value of $q_t u'(c_t)$ at T must be set equal to $q_{T-}u'(c_{T-})$, such as to have no discontinuity in $q_t u'(c_t)$ at time T. The boundary condition $c_{T+} = c_{T-}$ must therefore be imposed to ensure that both c_t and q_t are continuous at time T.

[44] When solving the differential equations, the initial value of $q_t u'(c_t)$ at T is set equal to $q_{T-}u'(c_{T-})$, which ensures that the boundary condition $q_{T+}u'(c_{T+}) = u'(c_{T-})q_{T-}$ is satisfied (see the previous two footnotes).