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J-D.Fermanian[1]
C.Florentin[2]

[1] ENSAE-CREST. E-mail: jean-david.fermanian@ensae.fr

[2] ENSAE: E-mail: clement.florentin@ensae.fr

Multi-factor Granularity Adjustments for Market and Counterparty Risks

Jean-David Fermanian*, Clément Florentin †

Abstract

We propose several multi-factor families of models for large portfolios of financial assets. The goal is to evaluate their market risk and/or their counterparty risk quantitatively. Explicit closed-form formulas of granularity adjustments are provided, to approximate their value-at-risks. We prove the relevance of such analytic approximations through simulations.

Keywords: Granularity adjustments, value-at-risk, counterparty risk, market risk, elliptical distributions.

1 Introduction

Risk measures, especially value-at-risk (VaR), provide the foundations of financial risk managements and regulation, in finance (Basel 3) as well as in insurance (Solvency 2). In particular, these measures are required to calculate minimum amounts of regulatory capital for banks. Besides, most financial institutions calculate the value-at-risk and/or expected shortfall associated to (some of) their portfolios on a regular basis, for different internal purposes: risk monitoring, asset allocation, economic capital allocation, etc. Therefore, the ability of calculating such risk measures quickly and efficiently has been recognized as a highly technical and strategic challenge, especially for the largest institutions.

Some "brute-force" solutions, as Monte-Carlo value-at-risk methods [1] are too time-consuming and cannot be called "on the fly" in practice. Parametric VaR models are based on the strong Gaussian assumption of asset returns and are not relevant for credit risk purpose. Historical VaR techniques can be good candidates but their relevance depends strongly on the relevance/choice of the reference historical period of time. Hopefully, approximated analytical calculations of value-at-risks are most often possible with

*CREST-ENSAE, J120, 3, avenue Pierre-Larousse, 92245 Malakoff cedex, France. jean-david.fermanian@ensae.fr

†ENSAE, clement.florentin@ensae.fr

[1] i.e. numerous Monte-Carlo simulations of realizations of the joint vector of asset returns, followed by full reevaluations of the portfolio values

factor models, the usual situation by far: when a portfolio becomes more fine-grained, i.e. when the largest individual exposures account for a negligible share of the total portfolio exposure, idiosyncratic risk is diversified away at the portfolio level. Therefore, the portfolio loss distribution is close to the distribution of its expected loss given the underlying factors. In other terms, portfolio losses depend only on systematic risk, as a first approximation. Typically, the latter distribution is a lot simpler than the initial loss distribution and can be obtained analytically for a lot of market and/or credit portfolio models.

Unfortunately, the risk associated to real portfolios depend on a significant amount of undiversified idiosyncratic risk most often. Indeed, except for large portfolios held by investment banks for which the previous approximation is more or less valid, the majority of portfolios do not diversify away all their idiosyncratic risk. Thus, the "first-order" approximation is clearly worse for medium-sized institutions or specialized institutions. Hopefully, the previous approximation can be refined by some so-called granularity adjustment (GA) techniques. Potentially, they can be applied to any risk-factor model. They provide additional "idiosyncratic" terms in the asymptotic expansion of portfolio loss distributions and of their associated risk measures. This is similar to calculating a second-order Taylor expansion when the infinitely-granular approximations corresponds to first-order approximations only. The additional term is proportional to $1/n$, n being the number of portfolio exposures and it is often not negligible for realistic size portfolios ($n \leq 100$ to fix the ideas).

Most available GA formulas are related to one-factor models: historically, Wilde (2001a), Martin and Wilde (2002), and Gordy (2003) introduced the technique and applied it to the "Basel 2" model; Wilde (2001b) provided the formulas for a single-factor version of CreditRisk+; Emmer and Tasche (2005), refined by Gordy and Lütkebohmert (2012), made the same task for CreditMetrics, etc. Actually, since Gordy (2003), it was well-known that the GA techniques can be applied in models with multiple systematic factors, at least in theory. But only very few papers have provided *explicit* GA formulas. Tasche (2006) pointed out the difficulty and detailed loss distribution in the case of two gaussian factors, but without an analytical VaR approximation. Gagliardini and Gouriéroux (2013) proposed such a formula for a simple two-factor stochastic volatility model. Recently, Fermanian (2015) proposed other simple examples of multifactorial systematic variables models, particularly in the case of CDO pricing with random recoveries.

Note that Pykhtin (2004) proposed to solve the multi-factor problem by building a comparable one-factor portfolio whose loss distribution is close to the original multi-factor loss distribution. This intuition has been extended and refined by Voropaev (2011). In the same spirit, Garcia Cespedes et al. (2006) multiplied stand-alone capital charges by some multi-factor adjustments to reflect diversification effects. But such ideas, even valuable, do not contend with the technical difficulties of well-grounded asymptotic GA formulas that would result from considering several systemic factors.

Clearly, the majority of portfolio models depend on several systematic factors, by far.

For instance, some famous credit portfolio models as CreditMetrics or Moody's KMV Portfolio Manager invoke dozens of industry/country systematic factors. The current standard way of pricing some structured credit products as CDOs is to rely on at least two correlated systematic factors, to drive simultaneously default events and recovery levels. A lot of ABS products are priced and risk managed by assuming several global "market" factors (Libor rates, house price indices, GDP growth rates, etc.) induce the main trends in the market. Therefore, it is highly desirable to obtain GA formulas for a large range of useful and realistic models. Unfortunately, this is not so easy to exhibit closed-form GA formulas. We will explain and illustrate the successive obstacles that make this task difficult.

Moreover, almost all the GA literature has adopted an actuarial point of view and focus on credit risk only, when most models in risk management are "mark-to-market". This has been pointed out by Gordy and Marrone (2012). They have extended the GA methodology to random exposures, mainly rating-based. Nonetheless, their approach is limited to univariate systematic factors. Here, we will consider tractable multi-factor models where risks may be due to default events, recoveries, and other financial factors that drive exposures.

In this paper, we propose several families of models in which granularity adjustments can be calculated in practice, when the systematic variables are multivariate. Granularity adjustment formulas are recalled and discussed in Section 2. Section 3 deals with portfolios that are exposed to counterparty risks, a mix of default risks and random exposures. In Section 4, we reconsider the market risk of a portfolio of assets. At the end of every section, we evaluate the relevance of our GAs by simulation.

2 Multi-factor Granularity adjustments

Let us set the framework. We will study a portfolio with n risky exposures. Every exposure i depends on its own risk (market risk, credit risk, or both) but all these risks are not independent, obviously. The key assumption is the mutual independence of the n underlying individual risky exposures, given a vector of systematic random factors $\mathbf{X} \in \mathbb{R}^m$. This systematic vector $\mathbf{X}$ summarizes the market trends that will occur between now and our time horizon T. Typically, $\mathbf{X}$ reflects the realizations of future macro-economic hazards, of financial variables (interest rates moves, some global indices, e.g.) and/or all the exogenous factors that can influence systematic risk in financial markets: natural catastrophes, pandemic, wars, etc. Note that some exposures may be related to the same counterparty formally, but this is not very realistic under the latter conditional independence assumption.

Formally, the portfolio loss between today ($t = 0$) and our given time horizon T will be written as

$$L_n = \sum_{i=1}^{n} A_{in} Z_i,$$

where the scalar A_{in} denotes the share of i-th value in the total portfolio value at $t = 0$. Thus, by construction, $\sum_{i=1}^{n} A_{in} = 1$. Moreover, the random variables Z_i, $i = 1, \dots, n$ are mutually independent knowing $\mathbf{X}$. They measure the random loss associated to the i-th risky position between $t = 0$ and $t = T$, as a percentage of the current exposure. Note that the total current value of the portfolio is not specified. Implicitly and w.l.o.g., it will be equal to one.

In the literature, a portfolio is called infinitely granular (or fine-grained) when its size n tends towards the infinity and when the portion of every individual exposure i is negligible compared with the total size of the portfolio, i.e. $\lim_{n\to\infty} \sup_{i=1,\dots,n} |A_{in}| = 0$. It is well-known that under the hypothesis of infinite granularity, the law of L_n is asymptotically the same as the law of $\mathbb{E}[L_n|\mathbf{X}]$. Given that the second random variable is much more manageable, it is attractive to approximate the quantiles of L_n by those of $\mathbb{E}[L_n|\mathbf{X}]$ to calculate value-at-risks. In other words, when the portfolio is infinitely granular, we can approximate the portfolio value-at-risk

$$VaR_\alpha(L_n) =: VaR_{n,\alpha} = \inf\{x \,|\, P(L_n \leq x) \geq 1 - \alpha\},\ \alpha \in (0,1),$$

by the value-at-risk that is associated to the expected loss $\mu(\mathbf{X}) = \mathbb{E}[L_n|\mathbf{X}]$, the latter one being denoted by $EVaR_\alpha(L_n)$, or simpler $EVaR_{n,\alpha}$. Roughly, it can be proved that $VaR_{n,\alpha} \simeq EVaR_{n,\alpha}$ when n tends to the infinity when the portfolio is infinitely granular. See Gordy (2003) for details, e.g.

Actually, the latter approximation can (and sometimes must) be refined, i.e. some amount of idiosyncratic risk can be put in an approximated formula. Assume that the random variable $\mu(\mathbf{X})$ has a continuous density with regard to the Lebesgue measure on the real line, denoted by f_μ. Denote $\mathbb{V}(Z_i|\mathbf{X} = \mathbf{x})$ the conditional variance of Z_i knowing $\mathbf{X} = \mathbf{x}$. For every real number y and every $i = 1, \dots, n$, define

$$\kappa_i(y) = \mathbb{E}\left[\mathbb{V}(Z_i|\mathbf{X})|\mu(\mathbf{X}) = y\right] f_\mu(y),\ \text{and}\ T_{n,\infty}(y) = \frac{1}{2}\sum_{i=1}^{n} A_{in}^2 \kappa_i'(y).$$

Under certain technical conditions and when n tends to the infinity, it can be proved that the cdf of L_n is arbitrarily close to the cdf of $\mu(\mathbf{X})$, plus the function $T_{n,\infty}(\cdot)$: see Gordy (2004) or Fermanian (2014), among others. Therefore, in this case, the portfolio value-at-risk can be approximated by a so-called granularity adjustment formula, i.e.

$$VaR_\alpha(L_n) \simeq VaRGA_{n,\alpha} := EVaR_{n,\alpha} - \frac{T_{n,\infty}(EVaR_{n,\alpha})}{f_\mu(EVaR_{n,\alpha})}. \tag{1}$$

Fermanian (2014) has provided some upper bounds of the previous approximation error, and explained how similar formulas can be obtained to approximate the expected shortfall of L_n.

Thus far, most applications of GA formulas assume a univariate systemic factor $\mathbf{X} := X \in \mathbb{R}$. In this case, it is very convenient to calculate κ_i because a single realization of

X induces the event $\{\mu(X) = y\}$. Then we get $\kappa_i(y) = \mathbb{V}(Z_i|X = \mu^{-1}(y))f_\mu(y)$, that can be derived analytically in a lot of models.

When $\mathbf{X}$ is a vector, i.e. $m \geq 2$, things are significantly more difficult in practice because the event $\{\mu(\mathbf{X}) = y\}$ is related to a lot of $\mathbf{X}$ values in general. Typically, when the law of $\mathbf{X}$ is continuous, such values belong to some complex manifolds, that cannot be described easily. Therefore, the calculation of κ_i becomes very complicated quickly, what has discouraged most authors and modelers. Actually, "difficult" does not mean "impossible". Our goal will be to exhibit some flexible families of models for which such GA analytical formulas can be obtained explicitly.

In practical terms, we have to fufill several requirements to have a chance of obtaining closed-form GA formulas in a multi-factor setting:

(i) The conditional expected loss $\mu(\mathbf{x})$ has to admit a simple analytical expression and its density f_μ can be calculated, even if the latter point is not always mandatory (see below). Since the expected loss of the portfolio given $\mathbf{X}$ is the sum of individual expected losses, we need to work with families of distributions that are "stable by aggregation" in general.

(ii) The conditional variance $\mathbb{V}(Z_j|\mathbf{X} = \mathbf{x})$ also has to admit a rather simple expression.

(iii) The last but not the least, we have to calculate $\mathbb{E}[\mathbb{V}(Z_i|\mathbf{X})|\mu(\mathbf{X}) = y]$ and its derivative analytically. Without any model restriction and/or some well-chosen $\mathbf{X}$-laws, this is unfeasible in general.

3 Granularity Adjustment formulas for counterparty risk

3.1 Model specifications

Since the last financial crisis, the risk management of counterparty risk has become a strategic topic for most financial institutions. By definition, this is the risk of mark-to-market losses due to the default of some counterparties. It is related to any position in the banking or trading book. In general, its evaluation is sensitive because exposures are random. In theory, this task would necessitate multivariate dynamic models driving credit spread/rating risk and other random factors (equity, interest rates, FX, etc.), even in the case of a simple vanilla stock option. Therefore, it is tempting to rely on some approximated models, that aggregate default risks and risky exposures. This is the logic behind the models we introduce now.

In our counterparty risk models, the future market values (exposures) at the time horizon T will be random and independent given the systematic factor $\mathbf{X} \in \mathbb{R}^m$, as said before. The individual default probability of obligor i given $\mathbf{X}$ is denoted by $p_i(\mathbf{X})$. This means that, in a depressing environment (as indicated by some subset of $\mathbf{X}$-values), the default probabilities $p_i(\mathbf{X})$ tend to become higher. Moreover, we assume that exposures and default events are mutually independent, knowing $\mathbf{X}$. This is weaker than the commonly assumed assumption of unconditional independence between both random quantities. In particular, some wrong-way risks can be taken into account, when some exposures and default likelihoods both depend on $\mathbf{X}$.

Let us illustrate the ideas with a simple and intuitive framework.

Assume that the position i is related to a "bullet" bond. If the corresponding obligor has defaulted, then the associated loss is the (random) value of the loss-given-default (LGD). It is well-known that recovery rates depend strongly on the economic environment. In particular, they tend to decrease during depressions: see Altman et al. 2005, for instance. When macro-factors are included in $\mathbf{X}$, this justifies the following specification:

$$Z_i|\mathbf{X} \sim \begin{cases} \mu_i(\mathbf{X}) & \text{with the probability} \quad p_i(\mathbf{X}), \\ 0 & \text{with the probability} \quad 1 - p_i(\mathbf{X}). \end{cases} \tag{2}$$

Therefore, $\mu_i(\mathbf{X})$ is simply the loss-given-default of bond i, knowing $\mathbf{X}$ [2]. Obviously, $\mu_i(\mathbf{X})$ is nonnegative. Moreover, we consider in (2) that the bond LGD is entirely known, once $\mathbf{X}$ is known (no idiosyncratic LGD risk). Therefore, we get easily

$$\mathbb{E}[Z_i|\mathbf{X} = \mathbf{x}] = p_i(\mathbf{x})\mu_i(\mathbf{x}), \;\; \mathbb{V}(Z_i|\mathbf{X} = \mathbf{x}) = p_i(\mathbf{x})\mu_i^2(\mathbf{x})(1 - p_i(\mathbf{x})).$$

In the case of other securities, particularly derivatives, the laws of future valuations could be approximated reasonably by mixtures of gaussian random variables. Conditional exposures are then truncated gaussian variables. This means

$$Z_i|\mathbf{X} \sim \begin{cases} \max\left(\mathcal{N}(\mu_i(\mathbf{X}), \sigma_i^2(\mathbf{X})), 0\right) & \text{with the probability} \quad p_i(\mathbf{X}), \\ 0 & \text{with the probability} \quad 1 - p_i(\mathbf{X}). \end{cases} \tag{3}$$

Due to the assumed independence between the random variables Z_i knowing $\mathbf{X}$, $i = 1, \ldots, n$, the model is fully specified in the latter case.

Denoting by Φ (resp. ϕ) the cumulative distribution function (resp. density) of a standardized random gaussian variable, simple calculations provide:

Lemma 1 *Under (3),*

$$\mathbb{E}[Z_i|\mathbf{X} = \mathbf{x}] = p_i(\mathbf{x})\mu_i(\mathbf{x})\Phi\left(\frac{\mu_i(\mathbf{x})}{\sigma_i(\mathbf{x})}\right) + p_i(\mathbf{x})\sigma_i(\mathbf{x})\phi\left(\frac{\mu_i(\mathbf{x})}{\sigma_i(\mathbf{x})}\right), \text{ and}$$

$$\begin{aligned}\mathbb{V}(Z_i|\mathbf{X} = \mathbf{x}) &= p_i(\mathbf{x})\left[(\sigma_i(\mathbf{x})^2 + \mu_i(\mathbf{x})^2)\Phi\left(\frac{\mu_i(\mathbf{x})}{\sigma_i(\mathbf{x})}\right) + \mu_i(\mathbf{x})\sigma_i(\mathbf{x})\phi\left(\frac{\mu_i(\mathbf{x})}{\sigma_i(\mathbf{x})}\right)\right] \\ &- \;\; p_i(\mathbf{x})^2\left[\mu_i(\mathbf{x})\Phi\left(\frac{\mu_i(\mathbf{x})}{\sigma_i(\mathbf{x})}\right) + \sigma_i(\mathbf{x})\phi\left(\frac{\mu_i(\mathbf{x})}{\sigma_i(\mathbf{x})}\right)\right]^2.\end{aligned}$$

[2] measured as a percentage of the bond notional or of the bond price today, depending on the convention that has been adopted to define A_{in}.

Note that both cases (bonds or other securities) could be encompassed in the same model specification (3). Indeed, by considering degenerated gaussian variables, for which $\sigma_i(\mathbf{x}) = 0$, we recover model (2). When the bond LGDs depend on some idiosyncratic factors, for instance linked to the debt structure of the firm or to the management of collateral, it makes sense to assume that the bond LGD knowing $\mathbf{X}$ is a "true" random variable, and not a constant. In this case, the model specification (3) can be invoked to deal with such bond models too.

To get analytical formulas of GAs for model (3), we cannot keep the nonlinearities induced by Φ and ϕ in their full generality. A reasonable simplifying assumption is provided by:

Assumption (A). The law of the risky position Z_i given $\mathbf{X}$ is given by (3) for all $i = 1, \ldots, n$. Moreover, for every i, there exists a nonnegative constant a_i s.t. $\sigma_i(\mathbf{x}) = a_i|\mu_i(\mathbf{x})|$ for almost every $\mathbf{x} \in \mathbb{R}^m$.

Let us discuss the realism of the latter assumption.

- If the i-th exposure is related to a long stock or bond position, then $\mu_i(\mathbf{X})$ is the stock/bond level at default knowing $\mathbf{X}$, but divided by the initial stock/bond value. In other words, it is comparable to a LGD. We expect that $\mu_i(\mathbf{X})$ tends to be smaller when $p_i(\mathbf{X})$ is higher, in line with multiple empirical observations. Assumption (A) is reasonable, because the level of uncertainty around LGDs is higher intuitively for high LGDs, in average.

- If the position i is related to a derivative as a long call, the same arguments apply. The single annoying situation occurs when $\mu_i(\mathbf{X})$ is close to zero, in which case there will be small recorded loss, under (A). We could correct such situations by adding to $Z_i|\mathbf{X}$ a small fixed amount of losses in every case. Alternatively, when the derivative value can become positive or negative between $t = 0$ and the time horizon T (think of a swap), $\mu_i(\mathbf{X})$ may have an arbitrary sign, the exposure definition in (3) is able to manage such situations.

- It is not necessary to include a loss-given-default random variable explicitly in the specification (3). Indeed, if a conditional loss-given-default percentage $(1 - R_i)(\mathbf{X})$ is added into the model, it should appear as a multiplicative factor of $\mu_i(\mathbf{X})$ and $\sigma_i(\mathbf{X})$. Formally, this would not change the model specification. Therefore, under (A), the absolute value of $\mu_i(\mathbf{x})$ can be interpreted as the expected loss of the defaulted security i given $\mathbf{X} = \mathbf{x}$ and that i *is defaulted*, once multiplied by the constant A_{in} and a scaling factor depending on a_i.

As a consequence, under the assumptions above and for most financial products (bonds, loans or derivatives), there exists a deterministic function $b_i(\mathbf{x})$ s.t. $\mathbb{E}[Z_i|\mathbf{X} = \mathbf{x}] = b_i(\mathbf{x})p_i(\mathbf{x})\mu_i(\mathbf{x})$. In the case of bonds and model (2), set $b_i = 1$ obviously and $a_i = 0$. Otherwise, $b_i(\mathbf{x}) = \Phi(s_i(\mathbf{x})/a_i) + a_i s_i(\mathbf{x})\phi(s_i(\mathbf{x})/a_i)$ in the case of model (3), where $s_i(\mathbf{x}) \in \{1, -1\}$ is the sign of $\mu_i(\mathbf{x})$ [3]. We deduce

$$\mu(\mathbf{x}) = \mathbb{E}[L_n|\mathbf{X} = \mathbf{x}] = \sum_{i=1}^{n} A_{in} b_i(\mathbf{x}) p_i(\mathbf{x}) \mu_i(\mathbf{x}), \text{ and} \tag{4}$$

$$\mathbb{V}(Z_i|\mathbf{X} = \mathbf{x}) = e_i(\mathbf{x}) p_i(\mathbf{x}) \mu_i(\mathbf{x})^2 - b_i(\mathbf{x})^2 p_i(\mathbf{x})^2 \mu_i(\mathbf{x})^2, \tag{5}$$

by setting

$$e_i(\mathbf{x}) := \left[(a_i^2 + 1)\Phi\left(\frac{s_i(\mathbf{x})}{a_i}\right) + a_i s_i(\mathbf{x}) \phi\left(\frac{s_i(\mathbf{x})}{a_i}\right) \right].$$

To go on and beside (A), we have to make an additional assumption concerning the heterogeneity among individual positions and/or default probabilities.

3.2 Linkage between conditional probabilities and individual exposures

The simplest way of getting closed-form GA formulas under (A) is to assume

Assumption (B.1). For every $i = 1, \ldots, n$, there exists a constant c_i s.t. $p_i(\mathbf{x})\mu_i(\mathbf{x}) = c_i x_1$ for every $\mathbf{x} \in \mathbb{R}^m$.

Without a lack of generality, we have particularized the first component of the systematic vector $\mathbf{X}$. Obviously, the particular role of X_1 could be played by any arbitrary univariate function of $\mathbf{X}$. The latter assumption is connecting conditional default probabilities and exposures for every individual position. It can be interpreted as the existence of a common driver X_1 for all the "individual expected losses" knowing $\mathbf{X}$. Assumption (B.1) shares the same spirit as the usual model for CDO pricing with random recoveries (see Amraoui et al., 2012), in which conditional default probabilities multiplied by conditional LGD are constrained to obtain tractable formulas and easy calibrations w.r.t. CDS quotes.

In the case of our model (3) and under Assumption (A) and (B.1), we get the simple expressions

$$\mathbb{E}[L_n|\mathbf{X}] = \left(\sum_{i=1}^{n} A_{in} b_i(\mathbf{X}) c_i \right) X_1 := \beta(\mathbf{X}).X_1, \text{ and} \tag{6}$$

$$\mathbb{V}(Z_i|\mathbf{X}) = c_i^2 X_1^2 \left[\frac{e_i(\mathbf{X})}{p_i(\mathbf{X})} - b_i^2(\mathbf{X}) \right]. \tag{7}$$

When the law of X_1 is continuous, the event $\mu_i(\mathbf{X}) = 0$ is of measure zero. Let us assume this is the case in this paper. Moreover, the sign of $\mu_i(\mathbf{x})$ is entirely determined by the signs of X_1 and c_i. Then $\beta(\mathbf{X})$, $e_i(\mathbf{X})$ and $b_i(\mathbf{X})$ take only two values: for every $i = 1, \ldots, n$ and a.e., there are constants s.t.

$$\beta(\mathbf{X}) = \beta_1 \mathbf{1}(X_1 > 0) + \beta_2 \mathbf{1}(X_1 < 0),$$

$$e_i(\mathbf{X}) = e_{i,1} \mathbf{1}(X_1 > 0) + e_{i,2} \mathbf{1}(X_1 < 0), \text{ and}$$

[3] We neglect the case $\mu_i(\mathbf{X}) = 0$, whose probability is assumed to be zero.

$$b_i(\mathbf{X}) = b_{i,1}\mathbf{1}(X_1 > 0) + b_{i,2}\mathbf{1}(X_1 < 0).$$

We will denote $\beta(X_1)$, $e_i(X_1)$ and $b_i(X_1)$ from now on.

Set $\mathbb{E}[1/p_i(\mathbf{X})|X_1 = x_1] := \zeta_i(x_1)$. We deduce that, when $X_1 > 0$,

$$\mathbb{E}\left[\mathbb{V}(Z_i|\mathbf{X}) \,|\, \mu(\mathbf{X}) = y\right] = \mathbb{E}\left[\mathbb{V}(Z_i|\mathbf{X}) \,|\, X_1 = \frac{y}{\beta_1}\right] = \frac{c_i^2 y^2}{\beta_1^2}\left[e_{i,1}\zeta_i\left(\frac{y}{\beta_1}\right) - b_{i,1}^2\right]. \tag{8}$$

Similarly, if $X_1 < 0$, then

$$\mathbb{E}\left[\mathbb{V}(Z_i|\mathbf{X}) \,|\, \mu(\mathbf{X}) = y\right] = \frac{c_i^2 y^2}{\beta_2^2}\left[e_{i,2}\zeta_i\left(\frac{y}{\beta_2}\right) - b_{i,2}^2\right]. \tag{9}$$

Note that, when X_1, and then the $\mu_i(\mathbf{X})$ and $\beta(X_1)$, have a constant sign almost surely. But, in every case, $EVaR_{n,\alpha}$ is proportional to the α-quantile of X_1, as usual in the literature:

$$EVaR_{n,\alpha} = F_{X_1}^{-1}(\alpha).\left(\beta_1\mathbf{1}(X_1 > 0) + \beta_2\mathbf{1}(X_1 > 0)\right),$$

where F_{X_1} denotes the cdf of X_1.

Moreover, if f_{X_1} (the density of X_1 w.r.t. the Lebesgue measure) exists, then the density of $E[L_n|\mathbf{X}]$ is

$$f_\mu(y) = \frac{\mathbf{1}(y < 0)}{\beta_1} f_{X_1}\left(\frac{y}{\beta_1}\right) + \frac{\mathbf{1}(y \geq 0)}{\beta_2} f_{X_1}\left(\frac{y}{\beta_2}\right). \tag{10}$$

We get GA formulas under (B.2) because the functions κ_i can be obtained by (8), (9) and (10).

Typically, in a lot of credit portfolio models (CreditMetrics, Moody's KMV, etc), the systematic factor $\mathbf{X}$ is a Gaussian random vector. Its components can often be chosen as independent, possibly after a re-parametrization [4]. Thus, it is easy to evaluate the law of $\mathbf{X}$ conditional to X_1 in such a gaussian situation, and sometimes to calculate the function ζ_i analytically.

Nonetheless, Assumption (B.1) may appear as not very realistic. Indeed, most of the time, the (joint) law of the default events is specified independently of the law of exposures. The latter ones can often be seen as "exogenously" specified. Then, it is difficult to consider that an upward impact on $p_i(\mathbf{X})$ will be perfectly counter-balanced by a downward shift of $\mu_i(\mathbf{X})$, when X_1 is kept constant [5]. The following specification is an attempt to solve this lack of realism.

[4] The most commonly used credit portfolio models share a lot of characteristics and can be "mapped" to each other, at least in some special cases: see Koyluoglu and Hickman (1998) or Gordy (2000), for instance.

[5] For instance, imagine that X_1 represents the moves of the SP500 index, considered as the "market index". Moreover, X_2 may be the French CAC40 stock index. In some circumstances, the CAC40 will move significantly when the SP500 will stay roughly flat. If the i-th position is related to a French company, $p_i(\mathbf{X})$ can increase significantly due to an increasing country risk, but $\mu_i(\mathbf{X})$ can stay constant if the exposure is related to US rates, for instance.

3.3 Linkage of conditional probabilities/exposures among positions

Here, let us come back one step backwards, by still working under (A) but by leaving (B.1) out. Now, we will assume a certain amount of similarity among the individual default probabilities and among the individual random exposures. This will provide an alternative family of counterparty risk models.

Assumption (B.2). For every $i = 1, \dots, n$ and $\mathbf{x} \in \mathbb{R}^m$, $p_i(\mathbf{x}) = \pi_i p(\mathbf{x})$ and $\mu_i(\mathbf{x}) = \nu_i + \omega_i q(\mathbf{x})$, for some given functions p and q and some constants π_i, ν_i and ω_i.

Note that Assumption (B.2) is equivalent to impose a two (systematic) factor model, that is driven by $(p(\mathbf{X}), q(\mathbf{X}))$. Indeed, the joint law of $(Z_1, \dots, Z_n)$ given $\mathbf{X}$ is the same given $(p(\mathbf{X}), q(\mathbf{X}))$.

Under (B.2), we are able to manage the case of long/short positions, credit derivatives, bonds, etc., in the same framework, playing with the constants ν_i and ω_i and their signs particularly. For instance, if $q(\mathbf{x})$ is high during stressed periods in the credit market (high ITraxx or CDX levels, typically), a protection buyer (resp. seller) Credit Default Swap position will be associated with $\omega_i > 0$ (resp. $\omega_i < 0$). Concerning default probabilities given $\mathbf{X}$, it makes sense to assume that an aggregated factor $p(\cdot)$ drives the conditional default likelihoods all every name. It reflects the likelihood of future states of the credit cycle. The coefficients π_i can be seen as rating-based scaling factors.

Actually, Assumption (B.2) is rather natural and realistic, especially for homogenous portfolio. In this case, the way the systematic factors $\mathbf{X}$ drive the individual risky exposures could be summarized similarly across the names in the portfolio.

Recalling (4) and (5), (B.2) implies

$$\mu(\mathbf{x}) = \mathbb{E}[L_n|\mathbf{X} = \mathbf{x}] = \sum_{i=1}^n A_{in} b_i(\mathbf{x}) \pi_i p(\mathbf{x})[\nu_i + \omega_i q(\mathbf{x})] := p(\mathbf{x})[A(\mathbf{x}) + B(\mathbf{x}) q(\mathbf{x})],$$

$$A(\mathbf{x}) := \sum_{i=1}^n A_{in} b_i(\mathbf{x}) \pi_i \nu_i, \; B(\mathbf{x}) := \sum_{i=1}^n A_{in} b_i(\mathbf{x}) \pi_i \omega_i, \text{ and}$$

$$\mathbb{V}(Z_i|\mathbf{X} = \mathbf{x}) = e_i(\mathbf{x}) \pi_i p(\mathbf{x})[\nu_i + \omega_i q(\mathbf{x})]^2 - b_i(\mathbf{x})^2 \pi_i^2 p(\mathbf{x})^2 [\nu_i + \omega_i q(\mathbf{x})]^2.$$

For convenience and to simplify calculations, let us assume in this subsection that $\mu_i(\mathbf{X}) = \nu_i + \omega_i q(\mathbf{X})$ (and then the $b_i(\mathbf{X})$ constants) has a constant sign for almost every $\mathbf{X}$-realization. This is natural for a lot of securities. In the case of derivatives, this constraint can be seen as a lack of generality, but extended formulas can be written nonetheless. They are left to the reader.

Therefore, set $A := A(\mathbf{x})$, $B := B(\mathbf{x})$, and denote by g the joint density of $(p(\mathbf{X}), q(\mathbf{X}))$. By definition, $EVaR_{n,\alpha}$ is the root of an implicit equation:

$$\alpha = \int \mathbf{1}(t \leq EVaR_{n,\alpha}) g\left(\frac{t}{A + By}, y\right) \frac{dt\, dy}{A + By}. \tag{11}$$

The latter equation can be solved numerically in general. And some closed-form formulas of $EVaR_{n,\alpha}$ could be found surely under some particular distributions g [6].

To calculate GAs (the point (iii) in Section 2, to be specific), we need to evaluate analytically the quantities

$$\mathcal{I}_{a,b}(y) := \mathbb{E}[p(\mathbf{X})^a q(\mathbf{X})^b \,|\, \mu(\mathbf{X}) = y],$$

for several couples of integers (a, b). With our notations, we have

$$\mathcal{I}_{a,b}(y) = \int \frac{y^a t^b}{(A+Bt)^{a+1}} g\left(\frac{y}{A+Bt}, t\right) dt / f_\mu(y). \tag{12}$$

Deduce

$$\mathbb{E}\left[\mathbb{V}(Z_i|\mathbf{X}) \,|\, \mu(\mathbf{X}) = y\right] = \sum_{k=1}^{2}\sum_{l=0}^{2} \gamma_{i,k,l}\mathcal{I}_{k,l}(y), \tag{13}$$

$$\gamma_{i,1,0} = \pi_i \nu_i^2 e_i,\ \gamma_{i,1,1} = 2\pi_i\nu_i\omega_i e_i,\ \gamma_{i,1,2} = \pi_i\omega_i^2 e_i,$$
$$\gamma_{i,2,0} = -\pi_i^2\nu_i^2 b_i^2,\ \gamma_{i,2,1} = -2\pi_i^2\nu_i\omega_i b_i^2,\ \gamma_{i,2,2} = -\pi_i^2\omega_i^2 b_i^2.$$

Note that the calculations of κ_i and its derivatives are a lot simplified by the fact the density of $\mu(\mathbf{X})$ disappears:

$$\kappa_i(y) = \sum_{k=1}^{2}\sum_{l=0}^{2} \gamma_{i,k,l} y^k \int \frac{t^l}{(A+Bt)^{k+1}} g\left(\frac{y}{A+Bt}, t\right) dt. \tag{14}$$

GA formulas are obtained through (1), once we have differentiated the latter functions $\kappa_i(\cdot)$ and once we have calculated the density of $\mu(\mathbf{X})$. This will be detailed for some particular models in Subsection 3.5.2.

We have considered some functions $p(\mathbf{x})$ and $q(\mathbf{x})$ that summarize the effect a lot of (observable or not) systematic factors potentially, for instance through two univariate indices. This idea can be extended as $\mu_i(\mathbf{x}) = \nu_i + \sum_{j=1}^{\bar{m}} \omega_{i,j} q_j(\mathbf{x})$, introducing several functions $q_j(\cdot)$. The same methodology applies as long as we are able to evaluate the joint density of $(p(\mathbf{X}), q_1(\mathbf{X}), \ldots, q_{\bar{m}}(\mathbf{X}))$, without the need of calculating the density of $\mathbb{E}[L_n|\mathbf{X}]$ explicitly.

Under a slightly different specification, it is possible to propose a more flexible model in the same vein. Now, let us still work under (A) but replace (B.2) by the following assumption.

Assumption (B.3): For every $i = 1, \ldots, n$ and $\mathbf{x} \in \mathbb{R}^m$, $p_i(\mathbf{x}) = \pi_i p(\mathbf{x})$ and $\mu_i(\mathbf{x}) = \nu_i + w_i'\mathbf{x}$, for some given function p and some constants π_i, ν_i and some constant vectors $w_i \in \mathbb{R}^m$.

[6] For instance, this is the case when $p(\mathbf{X})$ and $q(\mathbf{X})$ are independent and the law of $p(\mathbf{X})$ is uniform.

In this case, the random exposure of every position i depends on its own index $w_i'\mathbf{x}$, and no longer the same random variable. Then, with the previous notations, we get

$$\mu(\mathbf{x}) = \mathbb{E}[L_n|\mathbf{X}=\mathbf{x}] = \sum_{i=1}^{n} A_{in} b_i \pi_i p(\mathbf{x})[\nu_i + w_i'\mathbf{x}] := p(\mathbf{x})[A + C'\mathbf{x}],$$

$$C := \sum_{i=1}^{n} A_{in} b_i \pi_i w_i,$$

$$\mathbb{V}(Z_i|\mathbf{X}=\mathbf{x}) = e_i \pi_i p(\mathbf{x})[\nu_i + w_i'\mathbf{x}]^2 - b_i^2 \pi_i^2 p(\mathbf{x})^2 [\nu_i + w_i'\mathbf{x}]^2.$$

To calculate κ_i analytically, it is necessary to know the joint law of $(p(\mathbf{X}), C'\mathbf{X}, w_i'\mathbf{X})$ for any i. In general, this involves tedious calculations. This is even the case when $\mathbf{X}$ is a gaussian vector and $p(\mathbf{X})$ depends on an index only [7]. Therefore, we do not try to provide more results towards this direction.

3.4 Model extensions

It is well-known that asset returns and loss distributions exhibit fat tails and/or skewed distributions. Such features may induce larger VaR or expected shortfall values than expected (by a naive model), particularly at high levels. Our initial model assumptions (2) and (3) about random exposures may be seen rather restrictive, because they are based on conditional gaussian exposures implicitly. Actually, this is not really true. Indeed, in a factor model and under the conditional independence property, we are essentially free to specify the laws of $\mathbf{X}$ *and* the laws of the idiosyncratic noises given $\mathbf{X}$. The unconditional laws of exposures or losses are given by mixture models, that can generate fat tails easily, for instance.

The previous framework allows a high degree of flexibility by choosing different distributions of the systematic random factors, possibly fat-tailed or skewed. Through the specification of the first two conditional moments of individual losses, we should build realistic financial (one-period) models. Another more direct way of getting such features is to replace the (truncated) gaussian conditional distributions of individual losses in Section 3.1 by other distributions. In other words, this would mean trying several laws of losses given $\mathbf{X}$, the latter random vector keeping the same distribution.

For instance, instead of assuming (2) in the case of bonds, we could assume that Z_i knowing $\mathbf{X}=\mathbf{x}$ is drawn following a Beta distribution $Z_i|\mathbf{X}=\mathbf{x} \sim B(\alpha_i(\mathbf{x}), \beta_i(\mathbf{x}))$, in such a way that $\mathbb{E}[Z_i|\mathbf{X}=\mathbf{x}] = \alpha_i(\mathbf{x})/(\alpha_i(\mathbf{x}) + \beta_i(\mathbf{x})) = \mu_i(\mathbf{x})$. The functional forms of $\alpha_i(\cdot)$ and $\beta_i(\cdot)$ have to be specified, obviously. Beta distributions are particularly well-suited for LGDs, as several empirical studies have shown: see Calabrese and Zenga (2010), Bellotti and Crook (2012), among others. With Beta distributions, we are to generate a significant percentage of LGDs that are closed to zero or to one, even given $\mathbf{X}$. This is

[7] For instance, as in usual structural-type credit portfolio models, we could assume there exists constants δ and ℓ s.t. $p(\mathbf{x}) = \Phi(\ell - \delta\mathbf{x})$. But there are no closed-form formulas in this case, unfortunately.

in line with some stylized empirical facts, that show the large amount of heterogeneity among corporate bond recovery rates, even after controlling for the situation inside the credit cycle. Some authors have linked this feature to differences in terms of defaulted firm debt structures (Carey and Gordy, 2005), or to fire sales in some distressed industries (Acharya et al., 2007), particularly.

Additionally, let us reconsider (3). Instead of "gaussian-type" exposures, we could assume we live in the larger and more flexible class of elliptical distributions (see Section A in the appendix). If the random variable Y follows an $\mathcal{E}_1(\theta, \sigma^2, g)$, then the law of $(Y-\theta)/\sigma$ is entirely specified by the generator g. We denote by F_g and f_g its cdf and its density respectively. Therefore, with obvious notations, we could replace (3) by

$$Z_i|\mathbf{X} \sim \begin{cases} \max\left(\mathcal{E}_1(\mu_i(\mathbf{X}), \sigma_i(\mathbf{X})^2, g_i(\mathbf{X})), 0\right) & \text{with the probability} \quad p_i(\mathbf{X}), \\ 0 & \text{with the probability} \quad 1-p_i(\mathbf{X}). \end{cases} \tag{15}$$

The calculations above apply similarly, replacing Φ (resp. ϕ) by $F_{g_i(\mathbf{X})}$ (resp. $f_{g_i(\mathbf{X})}$). But, to get nice simple formulas, it is necessary to assume that the generator $g_i(\mathbf{X})$ does not depend on $\mathbf{X}$.

3.5 Empirical illustrations

Let us evaluate the performances of our GAs for counterparty risks numerically. We consider some simple, but not unrealistic, portfolios. To simplify and unless it is specified differently, we assume balanced portfolios, i.e. $A_{in} = 1/n$ for every i and different n values between 10 and 1000. We will compare the empirically estimated value-at-risk $VaR_{n,\alpha}$ with its first-order approximation $EVaR_{n,\alpha}$ and its granularity adjustment approximation $VaRGA_{n,\alpha}$. The value-at-risk level will be $\alpha = 0.99\%$, and the standard deviation around the approximated $VaR_{n,\alpha}$ will be estimated by the usual nonparametric bootstrap (200 replications). When $n = 1000$, the infinitely granular case should not be faraway and we expect $EVaR_{n,\alpha}$ provide convenient approximations. When n is very small, this is no longer the case: idiosyncratic risk dominates and then, any technique based on analytic approximations is questionable. The most favourable situation for GAs should be related to intermediate portfolio sizes, for which it makes sense to calculate asymptotic expansions of loss distributions and some correcting terms are able to adjust for some remaining significant idiosyncratic risks.

3.5.1 A family of models under (B.1).

Under Assumption (B.1), the main technical remaining point is the calculation of the so-called functions ζ_i, where $\zeta_i(x_1) = \mathbb{E}[1/p_i(\mathbf{X})|X_1 = x_1]$. To keep things simple, let us assume that $\mathbf{X}$ is a random vector in $\mathbb{R}_+^m$. As a consequence, all $\mu_i(\mathbf{x})$ are nonnegative, $s_i(\mathbf{x}) = 1$ and the functions $\beta(\mathbf{x})$, $e_i(\mathbf{x})$, $b_i(\mathbf{x})$ and $\beta(\mathbf{x})$ take unique values.

In this example, we assume that, for every i,

$$p_i(\mathbf{X}) = \frac{X_1}{\xi_{i,0} + \sum_{k=1}^m \xi_{i,k} X_k}, \tag{16}$$

for some nonnegative constants $\xi_{i,k}$, $k = 0, \dots, m$. These constants have to be chosen so that $p_i(\mathbf{X})$ is less than one almost surely. For instance, if X_1 is uniform on $(0,1)$, set $\xi_{i,0} = 1$. In every case, we can chose $\xi_{i,1} \geq 1$ to insure such a condition. We deduce

$$\mu_i(\mathbf{x}) = c_i \frac{x_1}{p_i(\mathbf{x})} = c_i \xi_{i,0} + c_i \sum_{k=1}^{m} \xi_{i,k} x_k, \text{ and } \sigma_i(\mathbf{x}) = a_i \mu_i(\mathbf{x}).$$

This model is well-suited to bond/stock portfolios but not swaps, because $\mu_i(\mathbf{X})$ is always positive by construction. Once the law of $\mathbf{X}$ is stated, the model is fully specified. Indeed, once $\mathbf{X}$ is drawn, we can simulate default events and random exposures independently, and we obtain portfolio loss realizations.

To fix the ideas and w.l.o.g., let us assume that $\mathbf{X}$ is a vector of correlated lognormal distributions: there exists a m-dimensional gaussian random vector $\mathbf{Y} \sim \mathcal{N}(\theta, \Sigma)$, $\Sigma = [\sigma_{i,j}]$, and $X_k = \exp(\nu_k Y_k)$, $k = 1, \dots, m$ for some positive constants ν_k. Therefore,

$$\begin{aligned}
\zeta_i(x_1) &= \frac{1}{x_1} \mathbb{E}\left[\xi_{i,0} + \sum_{k=1}^{m} \xi_{i,k} X_k | X_1 = x_1 \right] \\
&= \frac{\xi_{i,0}}{x_1} + \xi_{i,1} + \sum_{k=2}^{m} \frac{\xi_{i,k}}{x_1} \mathbb{E}\left[\exp(\nu_k Y_k) | Y_1 = \ln(x_1)/\nu_1\right] \\
&= \frac{\xi_{i,0}}{x_1} + \xi_{i,1} + \sum_{k=2}^{m} \frac{\xi_{i,k}}{x_1} \exp\left(\nu_k \theta_k + \frac{\nu_k \sigma_{1,k}}{\sigma_{1,1}} (\frac{\ln(x_1)}{\nu_1} - \theta_1) + \frac{\nu_k^2 \sigma_{k,k}}{2} (1 - \rho_{1,k}^2) \right),
\end{aligned}$$

where $\rho_{1,k} = \sigma_{1,k}/(\sigma_{1,1}\sigma_{k,k})^{1/2}$ is the correlation between Y_1 and Y_k. Note that $\sigma_{k,k}$ is the variance of Y_k (not its volatility).

The density of the portfolio expected loss $\mu(\mathbf{X}) = \beta X_1$ is

$$f_\mu(y) = \frac{1}{y \nu_1 \sqrt{\sigma_{1,1}}} \phi\left(\frac{\ln(y/\beta)/\nu_1 - \theta_1}{\sqrt{\sigma_{1,1}}} \right).$$

We deduce from (6) and (8) that

$$\begin{aligned}
\kappa_i(y) &= \mathbb{E}\left[\mathbb{V}(Z_i|\mathbf{X}) | \mu(\mathbf{X}) = y\right] f_\mu(y) = \mathbb{E}\left[\mathbb{V}(Z_i|X) | X_1 = \frac{y}{\beta} \right] f_\mu(y) \\
&= f_\mu(y) \cdot \left(\frac{c_i y}{\beta} \right)^2 \cdot \left[e_i \left(\frac{\xi_{i,0}\beta}{y} + \xi_{i,1} \right.\right. \\
&+ \left.\left. \sum_{k=2}^{m} \frac{\xi_{i,k}\beta}{y} \exp\left(\nu_k \theta_k + \frac{\nu_k \sigma_{1,k}}{\sigma_{1,1}} (\ln(y/\beta)/\nu_1 - \theta_1) + \frac{\nu_k^2 \sigma_{k,k}}{2} (1 - \rho_{1,k}^2) \right) \right) - b_i^2 \right], \text{ and}
\end{aligned}$$

$$\begin{aligned}
\kappa_i'(y) &= \left(\frac{\theta_1 - \ln(y/\beta)/\nu_1}{y \nu_1 \sigma_{1,1}} + \frac{1}{y} \right) \kappa_i(y) + f_\mu(y) \cdot \left(\frac{c_i^2 e_i}{\beta} \right) \\
&\cdot \left[-\xi_{i,0} + \sum_{k=2}^{m} \xi_{i,k} \exp\left(\nu_k \theta_k + \frac{\nu_k \sigma_{1,k}}{\sigma_{1,1}} (\ln(y/\beta)/\nu_1 - \theta_1) + \frac{\nu_k^2 \sigma_{k,k}}{2} (1 - \rho_{1,k}^2) \right) \cdot \left(\frac{\nu_k \sigma_{1,k}}{\sigma_{1,1}\nu_1} - 1 \right) \right].
\end{aligned}$$

AS usual, the corresponding GA formula is given by

$$VaRGA_{n,\alpha} = EVaR_{n,\alpha} - \frac{T_{n,\infty}(EVaR_{n,\alpha})}{f_\mu(EVaR_{n,\alpha})} = EVaR_{n,\alpha} - \frac{\sum_{i=1}^n \kappa_i'(EVaR_{n,\alpha})}{2n^2 f_\mu(EVaR_{n,\alpha})}. \quad (17)$$

but, under the latter model specification, the denominator of the formula (17) simplifies because $\kappa_i'(\cdot)$ is proportional to $f_\mu(\cdot)$. Finally, we obtain

$$\begin{aligned} VaRGA_{n,\alpha} &= EVaR_{n,\alpha} - \frac{1}{2n^2}\sum_{i=1}^n \left(\frac{c_i y}{\beta}\right)^2 \Bigg\{ \left(\frac{\theta_1 - \ln(y/\beta)/\nu_1}{y\nu_1\sigma_{1,1}} + \frac{1}{y}\right) \\ &\cdot \quad \left[e_i \left(\frac{\xi_{i,0}\beta}{y} + \xi_{i,1} + \sum_{k=2}^m \frac{\xi_{i,k}\beta}{y}\psi_k(y)\right) - b_i^2 \right] \\ &+ \quad \frac{e_i\beta}{y^2}\left[\sum_{k=2}^m \xi_{i,k}\left(\frac{\nu_k\sigma_{1,k}}{\sigma_{1,1}\nu_1} - 1\right)\psi_k(y) - \xi_{i,0}\right]\Bigg\}_{|y=EVaR_{n,\alpha}}, \end{aligned}$$

$$\psi_k(y) := \exp\left(\nu_k\theta_k + \frac{\nu_k\sigma_{1,k}}{\sigma_{1,1}}\left(\frac{\ln(y/\beta)}{\nu_1} - \theta_1\right) + \frac{\nu_k^2\sigma_{k,k}}{2}(1-\rho_{1,k}^2)\right).$$

In this experiment, we will choose the following parameters:

- $m = 2$, i.e. a bivariate systematic random vector $\mathbf{X}$;
- a portfolio size $n = 10$, 50, 100, 500 or 1000;
- $a_i = 1$ and $c_i = 1$ for every i; we deduce $b_i = \beta = 1.0833$ and $e_i = 1.9246$;
- $\theta = (1,1)$ and $\nu_1 = \nu_2 = 1$;
- $\Sigma = \begin{pmatrix} 1 & \rho_{1,2} \\ \rho_{1,2} & 1 \end{pmatrix}$ and we have set $\rho_{1,2} = 0.3$;
- $\xi_i = (0,2,1)$ for all i.

The results appear in Table 1. Clearly, GAs improve the $EVaR$-approximations significantly, as long as the portfolio size is smaller than 500. Surprisingly, even with very small portfolio sizes, $VaRGA_{n,\alpha}$ is pretty close to the right value-at-risk. On the other side, when n is large, the additional terms of $VaRGA_{n,\alpha}$ with respect to $EVaR_{n,\alpha}$ do not deteriorate the analytic approximation, but do not provide improvements, because the portfolios can be considered close to the "infinitely granular" case.

n	VaR (stdev)	EVaR	VaRGA	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	35.97 (0.22)	30.16	37.54	1.62 $\times 10^{-1}$	-4.37 $\times 10^{-2}$
50	31.50 (0.15)	30.16	31.63	4.28$\times 10^{-2}$	-4.08 $\times 10^{-3}$
100	30.84 (0.15)	30.16	30.89	2.22 $\times 10^{-2}$	-1.70$\times 10^{-3}$
500	30.25 (0.14)	30.16	30.30	3.09 $\times 10^{-3}$	-1.79$\times 10^{-3}$
1000	30.19 (0.14)	30.16	30.23	1.11 $\times 10^{-3}$	-1.34$\times 10^{-3}$

Table 1: Comparison of value-at-risk calculations for counterparty risk under (A) and (B.1). The level is $\alpha = 0.99$. The "true" value-at-risk $VaR_{n,\alpha}$ is estimated empirically through $500,000$ simulations of portfolio losses, and its standard deviation through 200 nonparametric bootstrap replications.

3.5.2 A family of models under (B.2)

Under Assumption (B.2), the model specifications depend uniquely on the joint law of the "systematic" driver of default events $p(\mathbf{X})$ and the "systematic" driver of random exposures $q(\mathbf{X})$.

Let us consider a bivariate gaussian random vector (Y_1, Y_2), $\mathbb{E}[Y_1] = \mathbb{E}[Y_2] = 0$, $\mathbb{E}[Y_1^2] = \mathbb{E}[Y_2^2] = 1$, and $\mathbb{E}[Y_1.Y_2] = \rho$. Set $p(\mathbf{X}) = \Phi(\nu_p Y_1 + \pi_p)$, with some constants ν_p and π_p, $\nu_p \geq 0$ by convention. For a book of derivatives, we can set $q(\mathbf{X}) = Y_2$ directly. For a portfolio of bonds and/or stocks, for which market values keep the same sign, i.e. are always positive or always negative, set $q(\mathbf{X}) = \exp(\nu_q Y_2 + \pi_q)$, introducing some constants ν_q and π_q, $\nu_q \geq 0$.

It is easy to calculate g, the joint law of $(p(\mathbf{X}), q(\mathbf{X}))$ in such models.

When $q(\mathbf{X}) = Y_2$, we obtain, for every $u \in (0,1)$ and $v \in \mathbb{R}$,

$$G(u,v) := \mathbb{P}(p(\mathbf{X}) \leq u, q(\mathbf{X}) \leq v) = \Phi_\rho\left((\Phi^{-1}(u) - \pi_p)/\nu_p, v\right), \tag{18}$$

where Φ_ρ is the joint cdf of (Y_1, Y_2).

When $q(\mathbf{X}) = \exp(\nu_q Y_2 + \pi_q)$, we have, for every $u \in (0,1)$ and $v \in \mathbb{R}^+$,

$$G(u,v) := \mathbb{P}(p(\mathbf{X}) \leq u, q(\mathbf{X}) \leq v) = \Phi_\rho\left((\Phi^{-1}(u) - \pi_p)/\nu_p, (\ln(v) - \pi_q)/\nu_q\right). \tag{19}$$

As a consequence, we can evaluate EVaRs by solving (14) numerically now.

Note that, in the case of a portfolio of derivatives, $q(\mathbf{X}) = Y_2$ can be positive or negative randomly, contrary to the simplifying assumption we made in Subsection 3.3. Nonetheless, it is easy to extend our formulas when all couples of coefficients (ν_i, ω_i) are the same (the case in our empirical illustration below). Therefore, as in Subsection 3.2, $b_i(\cdot)$ and $e_i(\cdot)$ take only two different values:

$$e_i(\mathbf{X}) = \bar{e}_1 \mathbf{1}(Y_2 > 0) + \bar{e}_2 \mathbf{1}(Y_2 < 0), \text{ and}$$

$$b_i(\mathbf{X}) = \bar{b}_1 \mathbf{1}(Y_2 > 0) + \bar{b}_2 \mathbf{1}(Y_2 < 0).$$

We deduce $\mu(\mathbf{X}) = 2\sum_{i=1}^n A_{in}\pi_i. \left(\bar{b}_1 \mathbf{1}(Y_2 > 0) + \bar{b}_2 \mathbf{1}(Y_2 < 0)\right) Y_2 \Phi(Y_1 - 1)$ a.e.

Note that, since $\alpha > 1/2$, the value of $EVaR_{n,\alpha}$ does not depend on b_2. Therefore, $EVaR_{n,\alpha}$ can be obtained as if $\bar{b}_2 = \bar{b}_1$, through Equation (11) as above.

We get GAs through the derivatives of (14). The calculations of GA formulas are detailed in Subsection B.1 in the appendix.

For this experiment, let us choose the following parameters:

- $n = 10$, 50, 100, 500 or 1000;
- $a_i = 1$ for every i; we deduce $b_i = \beta = 1.0833$ and $e_i = 1.9246$;
- the π_i are chosen randomly in the interval $(0, 1)$; set $\nu_i = 0$ and $\omega_i = 2$ for every i;
- concerning $p(\mathbf{x})$, set $\nu_p = 1$ and $\pi_p = -1$;
- for a book of stocks/bonds, choose $q(\mathbf{X}) = \exp(Y_2)$, i.e. $\nu_q = 1$ and $\pi_q = 0$;
- the correlation parameter ρ of (Y_1, Y_2) is equal to 0.5.

Since $(\nu_i, \omega_i) = (0, 2)$ for every i, the sign of $\mu_i(\mathbf{x})$ is simply the sign of Y_2. Then, for all i, $\bar{b}_1 = 1.0833$, $\bar{b}_2 = 0.4006$, $\bar{e}_1 = 1.9246$ and $\bar{e}_2 = 0.5592$.

The simulation results appear in Tables 2 and 3. Globally, they confirm our previous findings. Granularity adjustment calculations are very relevant for small/medium portfolio sizes, up to $n = 500$. In every case, they never provide a significantly worse work than $EVaR_{n,\alpha}$.

n	VaR (stdev)	EVaR	VaRGA	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	15.07 (0.09)	13.95	15.09	7.43×10^{-2}	-1.33×10^{-3}
50	14.25 (0.11)	13.95	14.17	2.10×10^{-2}	5.61×10^{-3}
100	14.20(0.10)	13.95	14.07	1.76×10^{-2}	9.86×10^{-3}
500	14.01(0.10)	13.95	13.98	3.95×10^{-3}	2.32×10^{-3}
1000	13.87(0.09)	13.95	13.96	-5.50×10^{-3}	-6.32×10^{-3}

Table 2: Comparison of value-at-risk calculations for counterparty risk. We consider a book of stocks and/or bonds under (A) and (B.2). The level is $\alpha = 0.99$. The "true" value-at-risk $VaR_{n,\alpha}$ is estimated empirically through $500,000$ simulations of portfolio losses, and its standard deviation through 200 nonparametric bootstrap replications.

n	VaR (stdev)	EVaR	VaRGA	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	3.88 (0.012)	3.49	3.88	9.82×10^{-2}	-1.27×10^{-3}
50	3.59 (0.014)	3.49	3.57	2.55×10^{-2}	3.97×10^{-3}
100	3.54 (0.013)	3.49	3.53	1.28×10^{-2}	1.88×10^{-3}
500	3.51 (0.016)	3.49	3.50	5.56×10^{-3}	3.36×10^{-3}
1000	3.51(0.014)	3.49	3.50	4.02×10^{-3}	2.92×10^{-3}

Table 3: Comparison of value-at-risk calculations for counterparty risk. We consider a book of derivatives, under (A) and (B.2). The level is $\alpha = 0.99$. The "true" value-at-risk $VaR_{n,\alpha}$ is estimated empirically through $500,000$ simulations of portfolio losses, and its standard deviation through 200 nonparametric bootstrap replications.

4 Granularity Adjustment formulas for market risk

The "market risk" associated to a portfolio is the risk of losses that may result from the fluctuations of the prices of some financial instruments. Traditionally, this risk is related to the randomness of stock prices, interest rates, exchange rates, commodity prices, etc. In theory, the main drivers of market risk are the overall economy fluctuations (GDP growth on a national basis, profits on a firm basis), inflation expectations, etc., and as well as investor sentiment over future developments, profits, etc. Technically, the main difference with default/counterparty risk is the continuous shape of individual loss profiles, while jump-to-default events induce a large and sudden jump in terms of market value. Moreover, exposures are always non-negative by definition in the case of counterparty risk. On the contrary and working in the same framework as in Section 2, the previous loss function L_n can take positive or negative values, when it relates to market risk. Indeed, L_n measures the opposite of the so-called "profit and loss" that will be recorded between $t = 0$ and $t = T$, assuming the underlying portfolio is frozen between both dates [8].

4.1 Granularity adjustments with exponential form conditional volatilities

As usual, the loss variables Z_i will be mutually independent given $\mathbf{X}$. Let us assume that

$$\mathbb{E}[Z_i|\mathbf{X}] = w_i'\mathbf{X} + c_i, \text{ and } \mathbb{V}[Z_i|\mathbf{X}] = \exp(\beta_i'\mathbf{X} + d_i), \tag{20}$$

for some fixed quantities w_i, β_i, c_i and d_i. The portfolio conditional expected loss is then $\mu(\mathbf{x}) := \mathbb{E}[L_n|\mathbf{X} = \mathbf{x}] = \sum_{i=1}^n A_{in}(w_i'\mathbf{x} + c_i) := w'\mathbf{x} + c$. Without a lack of generality, let

[8]Default risk is no longer key, i.e. we imagine that the positions are no longer exposed to default. Nonetheless, this risk could still be included in the models below by adding idiosyncratic individual default risks, and through more or less realistic specifications of Z_i's laws. Since this would induce significant complexities, we will not go towards this direction here.

us assume that $c = 0$. Therefore,

$$\mu(\mathbf{x}) = w'\mathbf{x}, \quad w = \sum_{i=1}^{n} A_{in} w_i.$$

To obtain GA formulas, the key technical question is to calculate $\mathbb{E}[\exp(\beta_i'\mathbf{X})|w'\mathbf{X} = v]$ for any v.

The underlying model is fully specified by the joint law of $(Z_1, \ldots, Z_n)$, or even by this law given $\mathbf{X}$. One of the simplest and most natural specifications in financial econometrics would be to assume

$$Z_i = \mathbb{E}[Z_i|\mathbf{X}] + \mathbb{V}[Z_i|\mathbf{X}]^{1/2}\eta_i, \quad i = 1, \ldots, n, \tag{21}$$

for some random variables η_i such that $\mathbb{E}[\eta_i|\mathbf{X}] = 0$, $\mathbb{E}[\eta_i^2|\mathbf{X}] = 1$. Typically, the systematic factor $\mathbf{X}$ and the variables η_i are mutually independent, as in most GARCH-type models. In this case, note that the law of η_i does not influence $EVaR$ and $VaRGA$ calculations. Indeed, the variables η_i are related to (normalized) idiosyncratic risks only, that are diversified away through our first and/or second order expansions. Therefore, we are free of choosing arbitrarily complex η_i-law, for instance skewed and fat-tailed distributions, as long as the second conditional moment of η_i given $\mathbf{X}$ is finite.

4.1.1 Elliptical systematic random vectors

Under (21), our risk measures can be calculated once the law of $\mathbf{X}$ is specified, and without knowing the law of the idiosyncratic noises η_i, $i = 1, \ldots, n$. Here, we assume that $\mathbf{X}$ is a m-dimensional elliptical vector $\mathcal{E}_m(\theta, \Sigma, g_{\mathbf{X}})$ (see appendix A), in particular gaussian random vector $\mathcal{N}(\theta, \Sigma)$. Then, any couple $(\beta_i'\mathbf{X}, w'\mathbf{X})$ is a bivariate elliptical vector. We stress that, even if $\mathbf{X}$ is gaussian, we do not want to evaluate a parametric gaussian value-at-risk. Indeed, only the *conditional distributions* of losses knowing $\mathbf{X}$ are gaussian/elliptical. But the *"true" underlying* loss distributions can be a lot more complex, involving unusual distributions, fat tails, etc.

To lighten notations, set $(Y_i, Z) := (\beta_i'\mathbf{X}, w'\mathbf{X})$. Its expectation is $[\mu_i, \mu_Z] := [\beta_i'\theta, w'\theta]'$, and its variance-covariance matrix is

$$Cov(Y_i, Z) = Cov(\beta_i'\mathbf{X}, w'\mathbf{X}) = \begin{bmatrix} \beta_i'\Sigma\beta_i \ (:= \sigma_i^2) & \beta_i'\Sigma w \ (:= \rho_i\sigma_i\sigma_Z) \\ w'\Sigma\beta_i \ (:= \rho_i\sigma_i\sigma_Z) & w'\Sigma w \ (:= \sigma_Z^2) \end{bmatrix}.$$

Actually, the law of (Y_i, Z) is elliptical: $(Y_i, Z) \sim \mathcal{E}_2([\mu_i, \mu_Z]', Cov(Y_i, Z), g_i)$, with

$$g_i(v) = \int_0^\infty w^{n/2-2} g_{\mathbf{X}}(v + w)\, dw.$$

We deduce, for any $z \in \mathbb{R}$,

$$Y_i|Z = z \sim \mathcal{E}_1\left(\frac{\rho_i\sigma_i}{\sigma_Z}(z - \mu_Z) + \mu_i, (1 - \rho_i^2)\sigma_i^2, g_{i|z}\right), \tag{22}$$

where $g_{i|z}(v) = g_i(v + (z - \mu_Z)^2/\sigma_Z^2)$. With obvious notations (see appendix A), we deduce

$$\mathbb{E}(\exp(\beta_i^{'}\mathbf{X} + d_i)|w'\mathbf{X} = z) = \exp(d_i)\mathbb{E}[\exp(Y_i)|Z = z]$$
$$= \exp(d_i)\Psi_{i|z}(1) = \exp(d_i)\int_{-\infty}^{+\infty} \exp(ix)g_{i|z}(x^2)\,dx.$$

Let us detail the GA formula in the particular case of a gaussian vector $\mathbf{X} \sim \mathcal{N}(\theta, \Sigma)$. Then, we get

$$Y_i|Z = z \sim \mathcal{N}\left(\frac{\rho_i\sigma_i}{\sigma_Z}(z - \mu_Z) + \mu_i, (1 - \rho_i^2)\sigma_i^2\right), \text{ and} \tag{23}$$

$$\mathbb{E}(\exp(\beta_i^{'}\mathbf{X} + d_i)|w'\mathbf{X} = z) = \exp\left(\frac{\rho_i\sigma_i}{\sigma_Z}(z - \mu_Z) + \mu_i + d_i + \frac{(1 - \rho_i^2)\sigma_i^2}{2}\right).$$

Simple calculations provide

$$\kappa_i(z) = \exp\left(d_i + (1 - \rho_i^2)\frac{\sigma_i^2}{2} + \left(\frac{z - \mu_Z}{\sigma_Z}\right)\rho_i\sigma_i + \mu_i\right) f_{\mathcal{N}(\mu_z, \sigma_z^2)}(z), \text{ and}$$

$$\kappa_i'(z) = \kappa_i(z) \cdot \left(\frac{\sigma_i\rho_i}{\sigma_Z} - \frac{z - \mu_Z}{\sigma_Z^2}\right).$$

Note that $T_{n,\infty}(z) = \frac{1}{2}\sum_{i=1}^n A_{in}^2\kappa_i'(z)$ is proportional to the density of $w'\mathbf{X}$ at z. With obvious notations, we get simply

$$VaRGA_{n,\alpha} = EVaR_{n,\alpha} - \frac{T_{n,\infty}(EVaR_{n,\alpha})}{f_\mu(EVaR_{n,\alpha})}$$
$$= EVaR_{n,\alpha} - \frac{1}{2}\sum_{i=1}^n A_{in}^2 \exp\left(\frac{\rho_i\sigma_i}{\sigma_Z}(EVaR_{n,\alpha} - \mu_Z) + \mu_i + d_i + \frac{(1 - \rho_i^2)\sigma_i^2}{2}\right)$$
$$\cdot \left(\frac{\sigma_i\rho_i}{\sigma_Z} - \frac{EVaR_{n,\alpha} - \mu_Z}{\sigma_Z^2}\right)$$
$$= EVaR_{n,\alpha} - \frac{1}{2}\sum_{i=1}^n A_{in}^2 \exp\left(\frac{\beta_i'\Sigma w}{w'\Sigma w}(EVaR_{n,\alpha} - w'\theta) + \beta_i'\theta + d_i + (1 - \frac{(\beta_i'\sigma w)^2}{\beta_i'\Sigma\beta_i w'\Sigma w})\frac{\beta_i'\Sigma\beta_i}{2}\right)$$
$$\cdot \left(\frac{\beta_i'\Sigma w}{w'\Sigma w} - \frac{EVaR_{n,\alpha} - w'\theta}{w'\Sigma w}\right).$$

4.1.2 Empirical illustration

Now, let us illustrate the relevance of such formulas with a simulation exercise. Rather than invoking uniform exposures A_{in}, let us induce a certain amount of heterogeneity in the portfolio: a proportion h of the considered portfolio exposures will be K times higher than the others. In this experiment, let us choose the following parameters:

- $m = 2$, i.e. a bivariate systematic random vector $\mathbf{X}$;
- $n = 10$, 50, 100, 500 or 1000;
- for every i, $d_i := d = 5$, $w_i := w = (4, 0)$ and $\beta_i := \beta = (0.01, 0.3)$;
- $K = 4$, $h = 20\%$;
- $\theta = (0, 0)$, $\Sigma = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$. We have chosen $\tau_1 = 1$ and $\tau_2 = 1/16$.

Given the systematic factor $\mathbf{X}$ and the model equations (20), the individual random losses are drawn as $Z_i \sim \mathbb{E}[Z_i|\mathbf{X}] + \mathbb{V}[Z_i|\mathbf{X}]^{1/2} W_i$, where $(W_i)_{i=1,\ldots,n}$ is a sequence of mutually independent "idiosyncratic noises", and $W_i \sim \mathcal{N}(0, 1)$.

The results are detailed in Table 4. Clearly, granularity adjustments provide very significant improvements w.r.t. $EVaR$ approximations, even when the portfolio size if large (1000 names). This is partly due to the reasonable amount of heterogeneity we have introduced. Indeed, more heterogeneity in the portfolio often increases the importance of measuring individual characteristics finely, because the total loss is more sensitive to some idiosyncratic risks. But even without this feature, i.e. with homogeneous portfolios, GAs provide useful and relevant results in every case.

Now, let us discuss the effects of some model parameters on the GA approximations shortly: see Table **??** in the appendix, where the $EVaR$ is kept constant.

- the parameter d measures the amount of idiosyncratic risk. Therefore, the higher d is, the more relevant the Granularity Adjustment term is for reasonable portfolio sizes. Obviously, this parameter has no impact on $EVaR$, that relates to systematic risk only.
- the parameter β does not have any effect on $EVaR$, but large values of β increase the GA term (as well as the simulated VaR) very significantly. Nonetheless, an increase of β up to 2 reduce the value of the GA term, due to offsetting effects between correlations and the size of idiosyncratic risk.
- Concerning the matrix Σ, by increasing the variance of one $\mathbf{X}$ component, we increase the relative relevance of GA terms.

n	VaR	EVaR	VaRGA	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	-14.52 (0.027)	-9.31	-13.55	3.59×10^{-1}	6.64×10^{-2}
50	-10.56 (0.026)	-9.31	-10.15	1.19×10^{-1}	3.84×10^{-2}
100	-9.97 (0.026)	-9.31	-9.73	6.63×10^{-2}	2.38×10^{-2}
500	-9.44 (0.028)	-9.31	-9.39	1.47×10^{-2}	5.69×10^{-3}
1000	-9.41 (0.027)	-9.31	-9.35	1.16×10^{-2}	7.08×10^{-3}

Table 4: Comparison of value-at-risk calculations for market risk and gaussian systemic factors. The level is $\alpha = 0.99$. The "true" value-at-risk $VaR_{n,\alpha}$ is estimated empirically through $500,000$ simulations of portfolio losses, and its standard deviation through 200 nonparametric bootstrap replications.

4.2 Granularity adjustments with "quadratic-type" conditional volatilities

The previous family of models was based on an exponential form of conditional volatilities $\mathbb{V}(Z_i|\mathbf{X})$. Depending on the $\mathbf{X}$-law, the latter assumption could generate too large uncertainties of losses, or too different realized losses among the names in the portfolio (when the coefficients (β_i, d_i) differ from one position to another one). Sometimes, this could be seen as a drawback. Therefore, in this section, we present an alternative family of models of market risk that should not suffer from such feature.

Now, we assume that conditional idiosyncratic variances are quadratic functions of the systematic factor $\mathbf{X}$, instead of an exponential function. In other words, the model specification is

$$\mathbb{E}[Z_k|\mathbf{X}] = w_k'\mathbf{X}, \text{ and } \mathbb{V}(Z_k|\mathbf{X}) = \mathbf{X}'\Omega_k\mathbf{X} = \sum_{i,j=1}^{m} \alpha_{i,j}^{(k)} X_i X_j, \; k = 1, \dots, n, \tag{24}$$

for some positive definite matrices $\Omega_k := [\alpha_{i,j}^{(k)}]$ and deterministic vectors w_k. Therefore, $\mathbb{E}[L_n|\mathbf{X}] = w'\mathbf{X}$, $w := \sum_{k=1}^n A_{k,n} w_k$. And we get explicit GA formulas by calculating $\mathbb{E}[X_i X_j | w'\mathbf{X} = v]$, $1 \leq i, j \leq m$.

4.2.1 GA formulas with elliptically-distributed systematic factors

As previously, let us consider an elliptical vector $\mathbf{X} \sim \mathcal{E}_m(\theta, \Sigma, g_{\mathbf{X}})$, where we impose $\mathbb{E}[\mathbf{X}] = \theta$, $\mathbb{V}(\mathbf{X}) = \Sigma$, and the density generator is $g_{\mathbf{X}}$: see Section A in the appendix.

Note that, for every indices i, j in $\{1, \dots, n\}$,

$$\mathbb{E}[X_i X_j | w'\mathbf{X} = z] = \mathbb{E}\left[\left(\frac{X_i + X_j}{2} \right)^2 - \left(\frac{X_i - X_j}{2} \right)^2 | w'\mathbf{X} = z \right].$$

Set $Y_{ij} = (X_i + X_j)/2$, $\bar{Y}_{ij} = (X_i - X_j)/2$ et $Z = w'\mathbf{X}$. Therefore, to get GA formulas, it will be sufficient to calculate the two quantities of interest $\mathbb{E}(Y_{ij}^2|Z = z)$ et $\mathbb{E}(\bar{Y}_{ij}^2|Z = z)$.

Let us detail these calculations in the case of Y_{ij}. We can lead the same reasoning as in Subsection 4.1.1, replacing the vector β_i by

$$\gamma_{i,j} := (0,\dots,0,1/2,0,\dots,0,1/2,0,\dots,0),$$

when $i \neq j$ (obviously, the coefficients 1/2 appear at the coordinates i and j only), or by

$$\gamma_{i,i} := (0,\dots,0,1,0,\dots,0),$$

with 1 at the i-th position. Moreover, the first two moments of $Y_{i,j}$ given Z are the same as in (23), due to the properties of elliptical vectors (see Theorems 5 and 8 in Gomez et al.): for every couple (i,j), we have

$$\mathbb{E}(Y_{ij}|Z=z) = \rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij} + \mu_{ij},$$

$$\mathbb{E}(Y_{ij}^2|Z=z) = (1-\rho_{ij}^2)\sigma_{ij}^2 + \left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij} + \mu_{ij}\right)^2,$$

where $\sigma_{ij}^2 = \gamma_{i,j}'\Sigma\gamma_{i,j}$, $\sigma_Z^2 = w'\Sigma w$, $\rho_{ij} = \gamma_{i,j}'\Sigma w/(\sigma_{ij}\sigma_Z)$, $\mu_{ij} = \gamma_{i,j}'\theta$ and $\mu_Z = w'\theta$.

The same calculations can be done with $\bar{Y}_{ij}$. The single difference with Y_{ij} comes from a coefficient $-1/2$ instead of $1/2$, for the j-th component of the vectors $\gamma_{i,j}$, $i \neq j$, providing $\bar{\gamma}_{ij}$ and the associated quantities $\bar{\sigma}_{ij}^2 := \bar{\gamma}_{i,j}'\Sigma\bar{\gamma}_{i,j}$, $\bar{\rho}_{ij} := \bar{\gamma}_{i,j}'\Sigma w/(\bar{\sigma}_{ij}\sigma_Z)$, $\bar{\mu}_{ij} := \bar{\gamma}_{i,j}'\theta$. Obviously, we get $\mathbb{E}[\bar{Y}_{i,j}|Z]$ and $\mathbb{E}[\bar{Y}_{i,j}^2|Z]$ as above, replacing $(\sigma_{ij}^2,\rho_{ij},\mu_{ij})$ by $(\bar{\sigma}_{ij}^2,\bar{\rho}_{ij},\bar{\mu}_{ij})$. We obtain, for every $k=1,\dots,n$,

$$\mathbb{E}[\mathbb{V}(Z_k|\mathbf{X})\,|\,w'\mathbf{X}=z] = \sum_{i,j=1}^m \alpha_{i,j}^{(k)}\left\{\mathbb{E}[Y_{ij}^2|w'\mathbf{X}=z] - \mathbb{E}[\bar{Y}_{ij}^2|w'\mathbf{X}=z]\right\}, \tag{25}$$

and the GAs follow relatively easily. Indeed, since $\mu(\mathbf{X}) = w'\mathbf{X}$ is a linear transform of $\mathbf{X}$, it follows an elliptical law $\mathcal{E}_1(w'\theta, w'\Sigma w, g_{\mu(\mathbf{X})})$. where the density generator of $E[L_n|\mathbf{X}]$ is given by

$$g_{\mu(\mathbf{X})}(t) = \int_0^{+\infty} s^{-1/2} g_{\mathbf{X}}(t+s)ds.$$

Therefore, the density of $\mu(\mathbf{X})$ is

$$f_\mu(z) = g_{\mu(\mathbf{X})}\left(\frac{(z-\mu_Z)^2}{\sigma_Z^2}\right)/c_\mu, = g_{\mu(\mathbf{X})}\left(\frac{(z-w'\theta)^2}{w'\Sigma w}\right)/c_\mu \tag{26}$$

$$c_\mu = \frac{\sqrt{\pi w'\Sigma w}}{\Gamma(1/2)}\int_0^{+\infty} v^{-1/2} g_{\mu(\mathbf{X})}(v)\,dv.$$

In general, the latter constant has to be estimated numerically for every particular model.

Thanks to formulas (25) and (26), the associated GA terms are obtained by deriving the functions $\kappa_k(z) = \mathbb{E}[\mathbb{V}(Z_k|\mathbf{X}) \,|\, w'\mathbf{X} = z] f_\mu(z)$, for any $k = 1, \ldots, n$. To be specific, we obtain

$$
\begin{aligned}
\kappa_k(z) = f_\mu(z) \sum_{i,j=1}^m \alpha_{i,j}^{(k)} \Bigg(& (1-\rho_{ij}^2)\sigma_{ij}^2 + \left(\rho_{ij}(\frac{z-\mu_Z}{\sigma_Z})\sigma_{ij} + \mu_{ij} \right)^2 \\
- \quad & (1-\bar{\rho}_{ij}^2)\bar{\sigma}_{ij}^2 - \left(\bar{\rho}_{ij}(\frac{z-\mu_Z}{\sigma_Z})\bar{\sigma}_{ij} + \bar{\mu}_{ij} \right)^2 \Bigg), \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
\kappa_k'(z) = & \sum_{i,j=1}^m \alpha_{i,j}^{(k)} . \Bigg\{ 2 f_\mu(z) \Bigg[\left(\rho_{ij}(\frac{z-\mu_Z}{\sigma_Z})\sigma_{ij} + \mu_{ij} \right) \frac{\rho_{ij}\sigma_{ij}}{\sigma_Z} \\
- \quad & \left(\bar{\rho}_{ij}(\frac{z-\mu_Z}{\sigma_Z})\bar{\sigma}_{ij} + \bar{\mu}_{ij} \right) \frac{\bar{\rho}_{ij}\bar{\sigma}_{ij}}{\sigma_Z} \Bigg] + f_\mu'(z) \Bigg[(1-\rho_{ij}^2)\sigma_{ij}^2 + \left(\rho_{ij}(\frac{z-\mu_Z}{\sigma_Z})\sigma_{ij} + \mu_{ij} \right)^2 \\
- \quad & (1-\bar{\rho}_{ij}^2)\bar{\sigma}_{ij}^2 - \left(\bar{\rho}_{ij}(\frac{z-\mu_Z}{\sigma_Z})\bar{\sigma}_{ij} + \bar{\mu}_{ij} \right)^2 \Bigg] \Bigg\}, \text{ where}
\end{aligned}
$$

$$
f_\mu'(z) = \frac{2(z-\mu_Z)}{c_\mu \sigma_Z^2} g_{\mu(\mathbf{X})}' \left(\frac{(z-\mu_Z)^2}{\sigma_Z^2} \right).
$$

It is difficult to specify these GA formulas further without particularizing some generators $g_{\mathbf{X}}$. In the next subsections, we will study the numerical performances of particular "elliptically-based" model specifications, when $\mathbf{X}$ is bivariate gaussian and when $\mathbf{X}$ follows a non-standard fat-tailed distribution.

4.2.2 Empirical illustration when X is gaussian

We have particularized the previous model and the corresponding formulas by assuming that $\mathbf{X}$ follows a bivariate gaussian vector: $m = 2$, $\mathbf{X} \sim \mathcal{N}(\theta, \Sigma)$ and $g_{\mathbf{X}}(t) = \exp(-t/2)/\sqrt{2\pi}$.

As previously, given the systematic factor $\mathbf{X}$ and the model equations (24), the individual random losses are drawn as

$$
Z_i \sim \mathbb{E}[Z_i|\mathbf{X}] + \mathbb{V}[Z_i|\mathbf{X}]^{1/2} W_i,
$$

where $(W_i)_{i=1,\ldots,n}$ is a sequence of mutually independent "idiosyncratic noises", $W_i \sim \mathcal{N}(0,1)$. The final GA formula is provided in Subsection B.2 in the appendix.

Let is evaluate the performances of GAs numerically, with a simple example. As in Subsection 4.1.2 and rather than considering uniform asset exposures, some portion h of the portfolio exposures will be K times higher than the others. In this experiment, let us choose the following parameters:

- $n = 10$, 50, 100, 500 or 1000;
- $w_i := w = (1, 0)$;
- $K = 4$, $h = 0.2$;
- $\theta = (2, 2)$, $\Sigma = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$, with $\tau_1 = 64$ and $\tau_2 = 4$;
- $\Omega = \begin{pmatrix} 1.6 & 0.1 \\ 0.1 & 0.4 \end{pmatrix}$.

The results appear in Table 5. GAs perform very well for every portfolio size. Such observations confirm and strengthen our findings in Section 4.1.2.

n	VaR	EVaR	VaRGA	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	-20.91 (0.077)	-16.61	-22.44	2.06×10^{-1}	-7.29×10^{-3}
50	-17.69 (0.055)	-16.61	-17.77	6.12×10^{-2}	-4.68×10^{-3}
100	-17.16 (0.055)	-16.61	-17.19	3.21×10^{-2}	-1.83×10^{-3}
500	-16.72 (0.046)	-16.61	-16.73	6.55×10^{-3}	-4.24×10^{-4}
1000	-16.67 (0.044)	-16.61	-16.67	3.49×10^{-3}	-5.46×10^{-6}

Table 5: Comparison of value-at-risk calculations for market risk and gaussian systematic factors. The level is $\alpha = 0.99$. The "true" value-at-risk $VaR_{n,\alpha}$ is estimated empirically through $500,000$ simulations of portfolio losses, and its standard deviation through 200 nonparametric bootstrap replications.

4.2.3 Application when X is a non-gaussian elliptical vector

To provide complementary results, and to challenge the current framework, we consider a bivariate elliptical vector $\mathbf{X}$ whose density generator is

$$g_{\mathbf{X}}(t) = \frac{1}{\pi(t^2 + 1)}.$$

To be specific, we will assume $\mathbf{X} \sim \mathcal{E}_2(\theta, \Sigma, g_{\mathbf{X}})$, and we will set $\theta = 0$ in the experiment. Note that the second order moments of $\mathbf{X}$ are not finite, due to the fat tails of $\mathbf{X}$. We are interested in checking whether the GA approximations are suffering from such a feature. Indeed, in Fermanian (2014), it has been noticed that fat-tailed loss distributions can disturb GA approximations.

Recall that $\mu(\mathbf{X}) = w'\mathbf{X}$ is a linear transform of $\mathbf{X}$. Following Theorem 5 in Gomez et al. (2003), the density generator of $E[L_n|\mathbf{X}]$ is

$$g_{\mu(\mathbf{X})}(t) = \int_0^{+\infty} s^{-1/2} g_{\mathbf{X}}(t+s) ds = \int_0^{+\infty} \frac{2\, dv}{\pi((t+v^2)^2 + 1)}.$$

Setting $t+i=-r^2\exp(2i\theta)$, we have

$$\frac{1}{\pi((t+v^2)^2+1)}=\frac{\alpha}{v-r\exp(i\theta)}+\frac{\bar{\alpha}}{v-r\exp(-i\theta)}-\frac{\alpha}{v+r\exp(i\theta)}-\frac{\bar{\alpha}}{v+r\exp(-i\theta)},$$

with $\alpha^{-1}=-4ri\exp(i\theta)$. We deduce that a primitive of $v\mapsto((t+v^2)^2+1)^{-1}$ is

$$t\mapsto 2Re\left(\alpha\ln\left(\frac{v-r\exp(i\theta)}{v+r\exp(i\theta)}\right)\right).$$

After some simplifications, we obtain the density generator of $\mu(\mathbf{X})$: for every $t\in\mathbb{R}^+$,

$$g_{\mu(\mathbf{X})}(t)=\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+t^2}+t}\right)^{1/2}\cdot\frac{1}{(1+t^2)^{3/4}}.$$

Therefore, $E[L_n|\mathbf{X}]\sim\mathcal{E}_1(w'\theta,w'\Sigma w,g_{\mu(\mathbf{X})})$, and the density of $\mu(\mathbf{X})$ is

$$f_\mu(z)=g_{\mu(\mathbf{X})}\left(\frac{(z-w'\theta)^2}{w'\Sigma w}\right)/c_\mu,\ c_\mu=\frac{\sqrt{\pi w'\Sigma w}}{\Gamma(1/2)}\int_0^{+\infty}v^{-1/2}g_{\mu(\mathbf{X})}(v)\,dv,$$

the latter constant being estimated numerically. The corresponding GA formulas are detailed in Subsection B.2.

The parameters of this experiment are the following ones:

- $K=4$, $h=0.2$;
- $w_i:=w=(1,0)$;
- $n=10$, 50, 100, 500 or 1000;
- $\theta=(0,0)$ and $\Sigma=\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}$, with $\tau_1=1$ and $\tau_2=1/16$;
- $\Omega=\begin{pmatrix}1.6 & 0.1\\ 0.1 & 0.4\end{pmatrix}$.

n	VaR	EVaR	VaRGA	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	-0.91 (0.0014)	-2.29	-2.83	-1.52 $\times 10^0$	-2.11 $\times 10^0$
50	-0.68(0.0013)	-2.29	-2.40	-2.36 $\times 10^0$	-2.52 $\times 10^0$
100	-0.63 (0.0014)	-2.29	-2.35	-2.64 $\times 10^0$	-2.72 $\times 10^0$
500	-16.72 (0.046)	-16.61	-16.73	6.55$\times 10^{-3}$	-4.24 $\times 10^{-4}$
1000	-16.67 (0.044)	-16.61	-16.67	3.49$\times 10^{-3}$	-5.46 $\times 10^{-6}$

Table 6: Comparison of value-at-risk calculations for market risk and an elliptical systematic factor. The level is $\alpha=0.99$. The "true" value-at-risk $VaR_{n,\alpha}$ is estimated empirically through $500,000$ simulations of portfolio losses, and its standard deviation through 200 nonparametric bootstrap replications.

5 Conclusion

We have explained why granularity adjustment formulas for risk measure calculations were so scarce in a multi-factor framework, by pointing the associated technical difficulties out. We have proposed several flexible families of models to obtain such formulas for some portfolios that are exposed to counterparty and/or market risk. Therefore, we have extended significantly the scope of multi-factor granularity adjustments, particularly for VaR calculations. A complementary work could be to provide the corresponding formulas and empirical illustrations in the case of expected shortfalls.

We have showed the relevance of such multi-factor GAs empirically, for some families of models and some sets of parameters. To check the robustness of our conclusions, we have played with many of our model parameter dimensions: VaRGA provided better approximations than EVaR in all cases virtually, often by a factor of ten. Due to the large number of model parameters, to the calculation times and to space limitations, we have not reported these additional results here. They can be provided under request. But more extensive simulations and some real data experiments are surely necessary to identify under which circumstances such GA techniques reach their limits.

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A A reminder about elliptical distributions

Consider $\theta \in \mathbb{R}^n$, Σ a positive definite matrix of size $n \times n$ and g a nonnegative function on $[0, \infty)$ such that

$$\int_0^\infty t^{\frac{n}{2}-1} g(t) dt < \infty, \tag{27}$$

An absolutely continuous random vector $\mathbf{X} = (X_1, \dots, X_n) \in \mathbb{R}^n$ is said to be a n-dimentional elliptical vector with parameters θ, Σ and g if the $\mathbf{X}$-density (w.r.t. the Lebesgue measure) is given by

$$f_{\mathbf{X}}(\mathbf{x}, \theta, \Sigma, h) := c_n |\Sigma|^{-\frac{1}{2}} g((\mathbf{x} - \theta)' \Sigma^{-1} (\mathbf{x} - \theta)).$$

where

$$c_n = \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}} \int_0^\infty t^{\frac{n}{2}-1} g(t)\, dt}.$$

We denote $\mathbf{X} \sim \mathcal{E}_n(\theta, \Sigma, g)$. Note that gaussian random vectors are particular cases of elliptical distributions, where $g(t) = \exp(-t/2)/\sqrt{2\pi}$.

The definition above implies that $\mathbb{E}[\mathbf{X}] = \theta$. Moreover, Σ and g are not defined uniquely. The couple (Σ, g) can be replaced by $(a\Sigma, g_{a,b})$, with $g_{a,b}(t) = bg(at)$ for every nonnegative t and every couple of positive numbers (a, b). Therefore, when the covariance of $\mathbf{X}$ exists (i.e. when its second-order moments are finite), we will impose $\mathbb{V}(\mathbf{X}) = \Sigma$. This will fix the previous coefficient a. And by imposing that $\int_0^{+\infty} g(t)dt = 1$, we can define g uniquely. Note that, in every case, the value of c_n ensures that $f_{\mathbf{X}}$ is a true density for any couple (a, b): cf Lemma 2 in Gomez et al. (2000).

There is an alternative way of defining an elliptical distribution. Instead of introducing a generator of the density function $f_{\mathbf{X}}$, we can focus on characteristic functions. Therefore, denote $\mathbf{X} \sim \mathcal{E}_n^*(\theta, \Sigma, \Psi)$ when $\mathbb{E}[\mathbf{X}] = \theta$, $\mathbb{V}(\mathbf{X}) = \Sigma$ and $\mathbb{E}[\exp(i\mathbf{t}'\mathbf{X})] = \exp(i\mathbf{t}'\theta)\Psi(\mathbf{t}'\Sigma\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^n$. It is possible to map one definition to the other one through the relation $\Psi(\mathbf{t}'\mathbf{t}) = \int \exp(i\mathbf{t}'\mathbf{x})g_{\mathbf{X}}(\mathbf{x}'\mathbf{x})d\mathbf{x}$. In the case of gaussian distributions, $\Psi(v) = \exp(v/2)$.

Interestingly, and similarly to gaussian distributions, linear transformations of elliptical vectors are still elliptical: if $\mathbf{X} \sim \mathcal{E}_n(\theta, \Sigma, g)$ and $\mathbf{Y} := C\mathbf{X} + b$ for some $(p \times n)$-matrix C and some p-dimensional vector b, then $\mathbf{Y} \sim \mathcal{E}_p(C\theta + b, C\Sigma C', g_Y)$, where $g_Y(t) = \int_0^\infty w^{n-p/2-1} g(t+w)\, dw$: see Theorem 5 in Gomez et al. (2003).

Now, we recall Theorem 8 in Gomez et al. (2000) that is useful to calculate some GA formulas. Let $\mathbf{X} \sim \mathcal{E}_n(\theta, \Sigma, g)$, with $n \geq 2$. Consider the following partition $\mathbf{X} = (\mathbf{X}'_{(1)}, \mathbf{X}'_{(2)})'$, and the respective partition of θ and Σ: $\theta = (\theta'_{(1)}, \theta'_{(2)})'$ and $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$. In this case, for every $\mathbf{x}_{(1)} \in \mathbb{R}^p$, we have

$$(\mathbf{X}_{(2)} | \mathbf{X}_{(1)} = \mathbf{x}_{(1)}) \sim \mathcal{E}_{n-p}(\theta_{(2.1)}, \Sigma_{22.1}, g_{(2.1)}),$$

where

- $\theta_{(2.1)} = \theta_{(2)} + \Sigma_{21}\Sigma_{11}^{-1}(\mathbf{x}_{(1)} - \theta_{(1)})$,
- $\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$, and
- $g_{(2.1)}(t) = g(t + q_{(1)})$, $q_{(1)} = (\mathbf{x}_{(1)} - \theta_{(1)})'\Sigma_{11}^{-1}(\mathbf{x}_{(1)} - \theta_{(1)})$.

The polar representation of an elliptical vector $\mathbf{X} \sim \mathcal{E}_n(\theta, \Sigma, g)$ establishes the identity in law between $\mathbf{X}$ and $\theta + RA'U$, where $A'A = \Sigma$, U is uniformly distributed on the unit sphere in $\mathbb{R}^n$ and R is an absolutely continuous nonnegative random variable that is independent from U (see Theorem 3 in Gomez et al., 2000). Imposing that $\mathbf{X}$ has finite second-order moments implies that the previous radius R satisfies $E[R^2] < \infty$, and this is equivalent to $\int_0^{+\infty} t^{n/2} g(t)\, dt < \infty$.

B Detailed calculations of $VaRGA_{n,\alpha}$

B.1 Granularity adjustments under Assumption B.2

Let us detail the exact GA formulas in the case of the models in Subsection 3.5.2.

In the case of our portfolio of derivatives, the joint law of the systematic drivers is given by (18). The chosen model specifications imply that $\mu(\mathbf{X}) = p(\mathbf{X})[A(\mathbf{X}) + B(\mathbf{X})q(\mathbf{X})]$, where $A(\cdot)$ and $B(\cdot)$ takes only two values: almost everywhere,

$$A(\mathbf{X}) = A_1\mathbf{1}(Y_2 \geq 0) + A_2\mathbf{1}(Y_2 < 0),\ B(\mathbf{X}) = B_1\mathbf{1}(Y_2 \geq 0) + B_2\mathbf{1}(Y_2 < 0),$$

$$A_k = \bar{b}_k \sum_{i=1}^{n} A_{in}\pi_i\nu_i,\ B_k = \bar{b}_k \sum_{i=1}^{n} A_{in}\pi_i\omega_i,\ k = 1, 2.$$

Then, the density f_μ of the portfolio expected loss given $\mathbf{X}$ is, for every $y \in \mathbb{R}$,

$$\begin{aligned} f_\mu(y) = & \int_0^1 \mathbf{1}(y \geq A_1 u)\phi_\rho\left(\frac{\Phi^{-1}(u) - \pi_p}{\nu_p}, \frac{y - A_1 u}{B_1 u}\right) \cdot \frac{du}{B_1\nu_p u\, \phi \circ \Phi^{-1}(u)} \\ + & \int_0^1 \mathbf{1}(y < A_2 u)\phi_\rho\left(\frac{\Phi^{-1}(u) - \pi_p}{\nu_p}, \frac{y - A_2 u}{B_2 u}\right) \cdot \frac{du}{B_2\nu_p u\, \phi \circ \Phi^{-1}(u)}. \end{aligned} \tag{28}$$

Note that, in our particular case, $A_1 = A_2 = 0$ since $\nu_i = 0$ for all i. And the density of the conditional expected loss is simply

$$\begin{aligned} f_\mu(y) = & \mathbf{1}(y \geq 0) \int_0^1 \phi_\rho\left(\frac{\Phi^{-1}(u) - \pi_p}{\nu_p}, \frac{y}{B_1 u}\right) \cdot \frac{du}{B_1\nu_p u\, \phi \circ \Phi^{-1}(u)} \\ + & \mathbf{1}(y < 0) \int_0^1 \phi_\rho\left(\frac{\Phi^{-1}(u) - \pi_p}{\nu_p}, \frac{y}{B_2 u}\right) \cdot \frac{du}{B_2\nu_p u\, \phi \circ \Phi^{-1}(u)}. \end{aligned} \tag{29}$$

When the portfolio components are stocks and/or bonds, we have assumed (19) and thinks are simpler. In particular the conditional expected loss density is, for every $y \in \mathbb{R}^+$,

$$f_\mu(y) = \int_0^{+\infty} \phi_\rho\left(\frac{\Phi^{-1}(y/[A + Bv]) - \pi_p}{\nu_p}, \frac{\ln(v) - \pi_q}{\nu_q}\right) \cdot \frac{\mathbf{1}(\max(0, (y - A)/B) \leq v))\, dv}{\nu_p\nu_q v[A + Bv]\, \phi \circ \Phi^{-1}(y/[A + Bv])}. \tag{30}$$

Under (18), the joint density of $(p(\mathbf{X}), q(\mathbf{X}))$ is

$$g(u, v) = \phi_\rho\left((\Phi^{-1}(u) - \pi_p)/\nu_p, v\right) \cdot \frac{\mathbf{1}(u \in (0, 1))}{\phi \circ \Phi^{-1}(u)\nu_p}, \tag{31}$$

and under (19), it is

$$g(u, v) = \phi_\rho\left((\Phi^{-1}(u) - \pi_p)/\nu_p, (\ln(v) - \pi_q)/\nu_q\right) \cdot \frac{\mathbf{1}(u \in (0, 1), v \geq 0)}{v\phi \circ \Phi^{-1}(u)\nu_p\nu_q}. \tag{32}$$

Under (19) (a portfolio of stock/bonds), $\kappa_i(\cdot)$ is obtained by invoking Equations (13) and (14). We get easily

$$\kappa_i'(y) = \sum_{k=1}^{2}\sum_{l=0}^{2}\gamma_{i,k,l}\left\{ky^{k-1}\int \frac{t^l}{(A+Bt)^{k+1}}g\left(\frac{y}{A+Bt},t\right)dt\right.$$
$$+\quad \left. y^k\int \frac{t^l}{(A+Bt)^{k+2}}\partial_1 g\left(\frac{y}{A+Bt},t\right)dt\right\}, \tag{33}$$

$$\partial_1 g(u,v) = g(u,v)\cdot\left(\frac{\rho(\ln(v)-\pi_q)}{\nu_p\nu_q(1-\rho^2)} - \frac{\Phi^{-1}(u)-\pi_p}{(1-\rho^2)\nu_p^2} + \Phi^{-1}(u)\right)\cdot\frac{1}{\phi\circ\Phi^{-1}(u)}. \tag{34}$$

When the sign of $\mu_i(\mathbf{X})$ is arbitrary, as with the portfolio of derivatives, the calculations of the κ_i functions may be tedious. Fortunately, with our model choice, this can be done relatively easily because all coefficients ν_i are zero and then only the sign of Y_2 determines the sign of all the $\mu_i(\mathbf{X})$. Therefore, $\kappa_i(\cdot)$ is obtained by invoking Equations (13) and (14), simply by replacing A, B, b_i and e_i by A_1, B_1, $\bar{b}_1$ and $\bar{e}_1$ respectively. With obvious notations, we deduce that, under (18) and when $y>0$,

$$\kappa_i'(y) = \sum_{k=1}^{2}\sum_{l=0}^{2}\gamma_{i,k,l}\left\{ky^{k-1}\int \frac{t^l}{(A_1+B_1t)^{k+1}}g\left(\frac{y}{A_1+B_1t},t\right)dt\right.$$
$$+\quad \left. y^k\int \frac{t^l}{(A_1+B_1t)^{k+2}}\partial_1 g\left(\frac{y}{A_1+B_1t},t\right)dt\right\}. \tag{35}$$

$$\partial_1 g(u,v) = g(u,v)\cdot\left(\frac{\rho v}{\nu_p(1-\rho^2)} - \frac{\Phi^{-1}(u)-\pi_p}{(1-\rho^2)\nu_p^2} + \Phi^{-1}(u)\right)\cdot\frac{1}{\phi\circ\Phi^{-1}(u)}. \tag{36}$$

Since

$$VaRGA_{n,\alpha} = EVaR_{n,\alpha} - \frac{\sum_{i=1}^n A_{i,n}^2\kappa_i'(EVaR_{n,\alpha})}{2f_\mu(EVaR_{n,\alpha})},$$

we obtain the corresponding GA formula under (18) (resp. under (19)) invoking Equations (35), (36) and (28) (resp. (33), (34) and (30)).

B.2 Granularity adjustments under a "quadratic form" conditional variance

Under (24), let us provide the exact formula when $\mathbf{X}$ is normal. Using the notations of Subsection 4.2.1, the conditional expected loss follows a gaussian distribution $\mathcal{N}(\mu_Z,\sigma_Z^2)$, with $\mu_Z = w'\theta$, $\sigma_Z^2 = w'\Sigma w$. We obtain

$$\kappa_k(z) = f_{\mathcal{N}(\mu_Z,\sigma_Z^2)}(z)\sum_{i,j=1}^{m}\alpha_{i,j}^{(k)}\left\{(1-\rho_{ij}^2)\sigma_{ij}^2 + \left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)^2\right.$$
$$-\quad \left.(1-\bar{\rho}_{ij}^2)\bar{\sigma}_{ij}^2 - \left(\bar{\rho}_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar{\sigma}_{ij}+\bar{\mu}_{ij}\right)^2\right\},\ \text{ and}$$

$$
\begin{aligned}
\frac{\kappa_k'(z)}{f_{\mathcal{N}(\mu_Z,\sigma_Z^2)}(z)} &= \sum_{i,j=1}^{m} \alpha_{i,j}^{(k)} \left\{ 2\left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)\frac{\rho_{ij}\sigma_{ij}}{\sigma_Z} \right. \\
- & \quad 2\bar{\rho}_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar{\sigma}_{ij}+\bar{\mu}_{ij}\Big)\frac{\bar{\rho}_{ij}\bar{\sigma}_{ij}}{\sigma_Z} \\
- & \quad \left(\frac{z-\mu_Z}{\sigma_Z^2}\right)\cdot\left[(1-\rho_{ij}^2)\sigma_{ij}^2+\left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)^2\right. \\
- & \quad \left.\left.(1-\bar{\rho}_{ij}^2)\bar{\sigma}_{ij}^2-\left(\bar{\rho}_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar{\sigma}_{ij}+\bar{\mu}_{ij}\right)^2\right]\right\}.
\end{aligned}
$$

We deduce

$$
\begin{aligned}
VaRGA_{n,\alpha} &= EVaR_{n,\alpha} - \frac{1}{2}\sum_{k=1}^{n} A_{k,n}^2 \sum_{i,j=1}^{m}\alpha_{i,j}^{(k)} \\
& \left\{2\left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)\frac{\rho_{ij}\sigma_{ij}}{\sigma_Z}-2\left(\bar{\rho}_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar{\sigma}_{ij}+\bar{\mu}_{ij}\right)\frac{\bar{\rho}_{ij}\bar{\sigma}_{ij}}{\sigma_Z}\right. \\
- & \quad \left(\frac{z-\mu_Z}{\sigma_Z^2}\right)\cdot\left[(1-\rho_{ij}^2)\sigma_{ij}^2+\left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)^2\right. \\
- & \quad \left.\left.(1-\bar{\rho}_{ij}^2)\bar{\sigma}_{ij}^2-\left(\bar{\rho}_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar{\sigma}_{ij}+\bar{\mu}_{ij}\right)^2\right]\right\}_{|z=EVaR_{n,\alpha}}.
\end{aligned}
$$

Let us lead similar calculations in the case of the elliptical vector $\mathbf{X}$ of Subsection 4.2.3. Given the functional form of the density generator, the expressions of the different functions are as follows:

$$
f_\mu(z) = g_{\mu(\mathbf{X})}\left(\frac{(z-\mu_Z)^2}{\sigma_Z^2}\right)/c_\mu, \quad f_\mu'(z) = \frac{2(z-\mu_Z)}{\sigma_Z^2} g_{\mu(\mathbf{X})}'\left(\frac{(z-\mu_Z)^2}{\sigma_Z^2}\right)/c_\mu,
$$

$$
g_{\mu(\mathbf{X})}(t) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+t^2}+t}\right)^{1/2}\cdot\frac{1}{(1+t^2)^{3/4}},
$$

$$
\begin{aligned}
g_{\mu(\mathbf{X})}'(t) &= \frac{(-1)}{2\sqrt{2}}\left(\frac{1}{\sqrt{1+t^2}+t}\right)^{1/2}\cdot\frac{1}{(1+t^2)^{5/4}} - \frac{3t}{2\sqrt{2}(1+t^2)^{7/4}}\cdot\left(\frac{1}{\sqrt{1+t^2}+t}\right)^{1/2} \\
&= \frac{(-1)}{2\sqrt{1+t^2}}\left[1+\frac{3t}{\sqrt{1+t^2}}\right] g_{\mu(\mathbf{X})}(t).
\end{aligned}
$$

We deduce

$$
\begin{aligned}
VaRGA_{n,\alpha} = EVaR_{n,\alpha} - \frac{1}{2}\sum_{k=1}^{n} A_{k,n}^2 \sum_{i,j=1}^{m} \alpha_{i,j}^{(k)} \\
\left\{ 2\left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)\frac{\rho_{ij}\sigma_{ij}}{\sigma_Z} - 2\left(\bar{\rho}_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar{\sigma}_{ij}+\bar{\mu}_{ij}\right)\frac{\bar{\rho}_{ij}\bar{\sigma}_{ij}}{\sigma_Z}\right. \\
+ \quad \frac{f'_\mu}{f_\mu}(z)\cdot\left[(1-\rho_{ij}^2)\sigma_{ij}^2 + \left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)^2\right. \\
- \quad \left.\left.(1-\bar{\rho}_{ij}^2)\bar{\sigma}_{ij}^2 - \left(\bar{\rho}_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar{\sigma}_{ij}+\bar{\mu}_{ij}\right)^2\right]\right\}_{|z=EVaR_{n,\alpha}}, \quad \text{where}
\end{aligned}
$$

$$
\frac{f'_\mu}{f_\mu}(z) = -\frac{z-\mu_Z}{\sqrt{\sigma_Z^2+(z-\mu_Z)^2}}\left[1+\frac{3(z-\mu_Z)}{\sqrt{\sigma_Z^2+(z-\mu_Z)^2}}\right].
$$

Since we have imposed $\theta = 0$ in the numerical experiment, $\mu_Z = w'\theta = 0$ and $\sigma_Z^2 = w'\Sigma w$.