Aggregation of Heterogenous Beliefs and Asset Pricing in Complete Financial Markets*

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Abstract

We propose a method to aggregate heterogenous individual beliefs, given a competitive equilibrium in complete asset markets, into a single "market probability" such that it generates, if commonly shared by all investors, the same marginal valuation of assets by the market (the same equilibrium prices) as well as by each individual investor. As a result of the aggregation process, the market portfolio may have to be scalarly adjusted, upward or downward, a reflection of an "aggregation bias" due to the diversity of beliefs. From a "dual" viewpoint, the standard construction of an "expected utility maximizing aggregate investor" designed to "represent" the economy in equilibrium, is shown to be also valid in the case of heterogenous beliefs, modulo the above scalar adjustment of the market portfolio, thereby generating an "Adjusted" version of the "Consumption based Capital Asset Pricing Model" (ACCAPM). Heterogeneity of individual consumptions, or of the allocation of aggregate risks to individuals, is then analyzed in relation to deviations of individual beliefs from the aggregate "market probability". It is shown further that an upward ajustment of the market portfolio due to the heterogeneity of beliefs, may contribute to explaining such challenges as the so-called "equity premium puzzle" whenever aggregate relative risk aversion is decreasing with aggregate income.

JEL Classification numbers : D50, D80, G11, G12
Keywords : Asset pricing, risk sharing, heterogeneity, beliefs, aggregation, representative agent, general equilibrium, equity premium.

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Aggrégation de croyances hétérogènes et valuation d'actifs dans les marchés financiers complets

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Résumé

Nous proposons une méthode pour agréger des croyances individuelles
hétérogènes, étant donné un équilibre concurrentiel sur des marchés financiers
complets, en une seule "probabilité de marché" de telle sorte qu'elle engen-
dre, si elle est partagée par tous les agents, la même valuation à la marge des
actifs par le marché (les mêmes prix d'équilibre) ainsi que par chaque investis-
seur individuel. Cette procédure d'agrégation peut nécessiter un ajustement
scalaire, à la hausse ou à la baisse, du portefeuille de marché, qui reflète un
"biais d'agrégation" due à l'hétérogénéité des croyances. D'un point de vue
"dual", on montre que la construction standard d'un agent aggrégé, doté
de préférences décrites par une espérance d'utilité, qui "représenterait' l'é-
conomie en équilibre, peut s'étendre au cas des croyances hétérogènes, mod-
ulo l'ajustement scalaire ci-dessus du portefeuille de marché : on engendre
ainsi une version "Ajustée" du modèle de valorisation des actifs fondée sur
la consommation (ACCAPM). L'hétérogénéité des consommations individu-
elles, ou de l'allocation des risques agrégés aux individus, est alors analysée
en relation avec les déviations des croyances individuelles par rapport à la
"probabilité de marché" agrégée. On montre en outre qu'un ajustement
scalaire à la hausse du portefeuille de marché dû à l'hétérogénéité des croy-
ances, peut contribuer à expliquer certaines questions comme la "prime de
risque des actions", lorsque l'aversion relative agrégée pour le risque décroît
avec le revenu agrégé.

Classification JEL : D50, D80, G11, G12
Mots-Clés : Valorisation d'actifs, partage des risques, hétérogénéité, croy-
ances, agrégation, agent représentatif, équilibre général, prime de risque.

1 Introduction

The main purpose of the present paper is to attempt to fill a methodological gap in the existing literature on competitive equilibrium under uncertainty and asset markets, by analyzing how one could extend, and modify, the traditional "expected utility maximizing representative agent approach" in order to cover the case, which appears to be empirically most relevant, of heterogenous beliefs.

While modern general treatments of competitive equilibrium under uncertainty do not require particular assumptions on the beliefs of economic agents about the occurrence of "states of nature" (Arrow (1953), Debreu (1959)), many applications do rest on the specification that agents are expected utility maximizers, forecast correctly equilibrium contingent prices or asset returns, and assign the same subjective probabilities to states of nature (*homogenous subjective probabilities*). Under the assumption of complete markets, equilibrium prices are then identical to those that would arise in the (no trade) equilibrium of "a model economy" composed of a single, aggregate "representative agent" who would get the aggregate endowment and would maximize an appropriately defined expected utility (Negishi (1960), Wilson (1968), M. Rubinstein (1974), Breeden and Litzenberger (1978), Constantinides (1982). This expected utility maximizing representative agent approach has been since the basis for many developments in finance and so-called "consumption based" capital asset pricing (Ingersoll (1987), Huang and Litzenberger (1988), Duffie (1996)). It has also become a significant cornerstone of theoretical and applied macroeconomics (R.E. Lucas (1978)).[1]

This framework has been fruitful, owing in particular to its simplicity of use, despite persistent doubts about the empirical relevance of some of its key features, notably about expectation formation. It has been in particular repeatedly argued that diversity of investors' forecasts (due possibly but not exclusively, to differences of information and/or of priors) is an important part of any proper understanding of the workings of asset markets (Lintner (1969), M. Rubinstein (1975, 1976), Gonedes (1976), E. Miller (1977), J. Williams (1977), Jarrow (1980), Mayshar (1981, 1983), Cragg and Malkiel (1982), Varian (1985, 1989), Detemple and Murthy (1994)). In the same vein, it has been advocated that consideration of "noise traders" whose beliefs and stategies are not completely determined by fundamentals but influenced by gurus, imitation, fads, technical analysis and other "popular models", may help in understanding asset markets "excess volatility" or "irrational exhuberance" (Shiller (1981, 1989, 2000), Black (1986), Shleifer and

Summers (1990)).[2] A related strand of research emphasizes similarly that learning along sequences of temporary equilibria may be sluggish, never converge and that "bounded rationality" may be an important fact of life (see, e.g. Brock and Hommes (1997), Grandmont (1998), Kurz (1997), Sargent (1993)). Analogous "evolutionist" arguments suggest that, while "boundedly rational" agents would be presumably driven eventually out of the market in the ideal case where capital markets are perfect (Araujo and Sandroni (1999), Sandroni (2000)), they are likely to have a persistent and significant influence in the real world situation where arbitrage is limited and risky due to capital market imperfections (De Long, Shleifer, Summers and Waldman (1989, 1990, 1991), Blume and Easley (1992)).

As a particular example that is relevant to the topics that will occupy us specifically here, we note also that researchers working on some empirical challenges such as the "equity premium puzzle" (Mehra and Prescott (1985), Weil (1989)), have been increasingly led to amend the framework of a complete markets, expected utility maximizing, fully rational representative agent, within which the "puzzle" was initially formulated. Beyond the introduction of incomplete markets and uninsurable heterogenous individual risks (Weil (1992), Constantinides and Duffie (1996), Angeletos and Calvet (2001)), of habit persistence (Abel (1990), Constantinides (1990), Campbell and Cochrane (1999)), researchers have in particular been led to introduce "distorted" and/or "noisy" beliefs (subjective probabilities), either directly postulated (Abel (1997), Cecchetti, Lam and Mark (2000)), or associated to "cautious" nonexpected utility behavior (Epstein and Wang (1994), Chauveau and Nalpas (1998), Hansen, Sargent and Tallarini (1999)).

There seems accordingly to exist compelling, both empirical and theoretical, reasons to incorporate in our representations of the economy, some significant and persistent doses of "boundedly rational", "noisy" expectations. The issue of heterogeneity of beliefs is then unescapable : although "bounded rationality" may involve some systematic patterns among economic agents, it is most likely to be associated also with some dispersion of individual beliefs. Our aim in the present paper is to analyze the consequences of facing the issue of *heterogenous individual subjective probabilities* in an otherwise standard competitive, complete markets economy operating under uncertainty (while keeping at this stage the assumption that traders do forecast correctly contingent equilibrium prices or asset returns). Among the issues we investigate are : Is it possible to define a "market probability" that would "aggregate" heterogenous individual subjective probabilities and could be used to "explain" (mimic) equilibrium asset prices ? Is it still possible in

such a context to define a version of an "expected utility maximizing aggregate investor" that would "represent" an equilibrium of this economy, i.e. generate the same equilibrium asset prices and mimic equilibrium pricing of assets by individuals, although this approach, recalled at the beginning of this introduction, fails flatly as soon as there is *any* degree of diversity of individual beliefs ? Is it possible to trace part of the observed heterogeneity of equilibrium individual portfolios, back to the dispersion of individual beliefs, together with the heterogeneity of other individual characteristics (risk aversion, income) ? Under which circumstances and to which extent taking into account heterogeneity of individual beliefs ("noisy" expectations) may or may not contribute to a better understanding of such challenges as the "equity premium puzzle" ?

We address these issues in the simplest framework, described in Section 2, of a static exchange economy where individual investors trade today among themselves portfolios of assets generating (positive) income for tomorrow (with the hope that it may not be too difficult to extend progress made in that simple framework to more sophisticated intertemporal setups). We present in Section 3 a method to aggregate heterogenous individual subjective probabilities into a single "market probability", that relies on three criteria. Given an observed equilibrium with heterogenous individual probabilities, we propose to define an "*equivalent equilibrium*", where all investors would share the single "aggregate market probability", by : 1) the equivalent equilibrium generates the same valuation of assets by the market (the same equilibrium asset prices), 2) every investor is indifferent at the margin between investing one additional unit of income in the observed equilibrium (using his own subjective probability) and in the equivalent equilibrium (using the "aggregate probability"), i.e. every investor's marginal expected utility valuations of an asset remain the same in both equilibria, 3) the aggregate probability is equal to the investors' subjective probabilities when they happen to share initially the same beliefs. We show in Section 3 that these three conditions determine uniquely in the present framework an "aggregate market probability" corresponding to a given observed equilibrium. Part of the individual heterogeneity displayed by equilibrium behaviours can then be traced back to the diversity of beliefs by looking at the changes of individual equilibrium portfolios (and incomes) occurring when going from the equivalent equilibrium (from the commonly held equivalent "aggregate probability") to the observed equilibrium (to the investor's own subjective probability). It will be shown that the above marginal indifference condition 2) implies that such changes of individual equilibrium portfolios are *monotone* in the underlying changes of individual probabilities (the two conditions being in fact essen-

tially equivalent for practical purposes). Finally, an important feature of the proposed aggregation procedure is that *the aggregate market portfolio (consumption) may have to be scalarly adjusted, upward or downward, in the equivalent common probability equilibrium, a reflection of an "aggregation bias" due to the diversity of beliefs.* Such a scalar adjustment modifies the apparent mean and variability of aggregate income (consumption). One of the topics that occupies us in this paper is to assess the direction and size of this scalar adjustment, and to evaluate its implications for "aggegate risk aversion" and asset pricing.

The proposed aggregation procedure is formulated in Section 3 in terms of invariance conditions on marginal asset pricing by the market and by every individual investor, that borrow intentionally little from the assumption of complete markets (in the hope to keep the door open to a possible extension to the case of incomplete markets). We take in Section 4 a "dual" viewpoint that exploits fully the complete markets structure. We show there that the standard construction of an "expected utility maximizing aggregate investor", who is designed so as to generate the observed equilibrium asset prices when endowed with the market portfolio, and to value then assets at the margin as does every individual investor in equilibrium, does carry over to the case of heterogenous subjective probabilities, provided that 1) this aggregate investor is assigned the same aggegate "market probability" as was found in the previous section, and that 2) the market portfolio (aggregate consumption) is scalarly adjusted upwardly or downwardly as in section 3. The proposed aggregation procedure generates accordingly, in the case of complete markets, an "*Adjusted*" version of the standard "*Consumption based Capital Asset Pricing Model*" (ACCAPM).

We focus in Section 5 on the relative contribution to the observed heterogeneity of individual equilibrium portfolios, of the underlying diversity of beliefs, in conjunction with heterogeneity of other individual characteristics (risk aversion, incomes) and the presence of aggregate risk. We show in particular that the scalar adjustment of the market portfolio required to aggregate individual beliefs, has to be made upward (resp. downward) whenever individual utilities are independent of the states of nature and aggregate risk is small, provided that individual absolute risk tolerance does not increases too fast (resp. increases fast enough) with income and that dispersion of beliefs is significant. The same result goes through, irrespectively of the size of aggregate risk, in economies where a "two funds separation property" (each investor holds a mixture of the risky market portfolio and of the riskless asset) would be approximately valid in the "common probability

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equivalent equilibrium". This two funds separation property is in particular true (exactly) in the specific configuration, often considered in the finance literature, where the coefficient of absolute risk tolerance of each investor a is independent of the state and linear in his income y_a, i.e. $T_a(y_a) = \theta_a + \eta y_a$, with a derivative (marginal risk tolerance) η that is common to all investors ("Hyperbolic Absolute Risk Aversion" (HARA) family) : the adjustment coefficient r° of the market portfolio required to aggregate diverse individual beliefs then exceeds 1 if and only if the common marginal risk tolerance η is less than 1. In all cases, the relative size of the scalar adjustment of the market portfolio $\mid r^\circ - 1 \mid / r^\circ$ appears to increase, ceteris paribus, with the dispersion of beliefs. Our analysis fits accordingly in some respects, and generalizes, earlier studies of aggregation of diverse beliefs. In the specific configuration where all investors have logarithmic utilities (with a marginal risk tolerance $\eta = 1$ in the HARA family), our analysis implies that there is no "aggregation bias" ($r^\circ = 1$, no scalar adjustment of the market portfolio) due to individual beliefs heterogeneity (and in fact no individual income adjustment as well). Aggregation of diverse individual subjective probabilities becomes in that case extremely simple and can be done exactly, which was among the arguments put forward early by M. Rubinstein (1976) in favor of "the logarithmic utility model as the premier model of finance". Our study of the Constant Absolute Risk Aversion (CARA) configuration (with $\eta = 0$ in the HARA family) generates in our case an upward adjustment coefficient $r^\circ > 1$ of the market portfolio and an aggregate market probability that is generated through a weighted harmonic mean of individual subjective probabilities (similarly to an analogous calculation made in the CARA configuration by Huang and Litzenberger (1988, section 5.26), without however any scalar adjustment of the market portfolio). Also, the condition that the derivative of individual absolute risk tolerance with respect to income (the common marginal risk tolerance η in the HARA family) is less than 1, which is instrumental in our analysis to get an upward scalar adjustment $r^\circ > 1$ of the market portfolio to aggregate diverse individual beliefs, props up also in Varian's study (1985, 1989) of asset prices in a similar setup (see also the exposition in Ingersoll (1987, chap. 9).[3]

One standard view of the competitive mechanism is that it generates an allocation of aggregate risks (of variations accross states of aggregate income or consumption). In the case of homogenous beliefs and state independent utilities, the corresponding risk sharing rule displays the appealing property to be *monotone*, i.e. individual consumption depends only on, and is an increasing function of, aggregate consumption. We show also in section 5 how our results can be reinterpreted as generalizing the standard approach to the

case of heterogenous beliefs, by requiring in particular that the allocation of "residual risks" due to heterogeneity of beliefs be also *monotone* in individual beliefs deviations from the aggregate "market probability".

Finally, section 6 is devoted to an evaluation of the consequences of beliefs diversity on "aggregate risk aversion" and asset pricing, in particular in relation with the so-called "equity premium puzzle". To fix ideas, we focus attention on the case where individual subjective probabilities are the result of some "noise" around an hypothetical "true" probability (set by convention equal to the arithmetic mean, in the population, of individual beliefs). We study when such "noisy" expectations may generate a *positive "risk premium aggregation bias"* by *lowering* the evaluation of the risk premium associated to the market portfolio (aggregate income or consumption) by the corresponding "aggregate representative investor" using the "market probability" arising from our aggregation procedure, by comparison to the evaluation of that same risk premium by an outside observer using the "true" probability : the observer would have then to assume "too much" risk aversion when trying to fit a standard CCAPM as a consequence of his overestimation by comparison to "the market". We shall speak of a "negative bias" in the opposite configuration where the representative investor's risk premium evaluation is larger than that of the observer. We shall find that there may be indeed systematic "distorsions" in the distribution of individual beliefs around its mean in the population that may contribute to a positive risk premium aggregation bias (in the spirit of Abel (1997), Chauveau and Nalpas (1998), Ceccheti, Lam and Mark (2000)). That will be the case, under the mild condition that absolute risk aversion decreases with income, when investors with lower absolute risk aversion and/or higher incomes tend to be more "pessimistic", i.e. assign larger subjective probabilities to "bad states" (with low aggregate income or consumption) than other agents. Or, under the assumption that the derivatives of individual absolute risk tolerances do not increase too fast (are less than 1), when the dispersion of beliefs in the population ("doubt") is more significant for "good states" with larger aggregate consumption or income. More importantly, *there is a mechanism contributing to a positive risk premium aggregation bias that will always operate* even though there may be no systematic association between distorted beliefs and individual risk aversion or income as above, *and that results from the fact that the market portfolio assigned to the "representative investor" has to be scalarly adjusted.*

Specifically, when the adjustment coefficient r° exceeds 1, aggregate incomes (consumptions) assigned to the "representative investor" are larger,

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which should tend to lower his evaluation of the risk premium of the market porfolio (if risk aversion goes down with income), but their variability increases also, which should have the opposite effect. We show in section 6 that the net effect is indeed toward lowering the aggregate representative investor's evaluation of the risk premium associated to the market portfolio. As a consequence, *an upward adjustment of the market portfolio $r^\circ > 1$ due to heterogeneity of beliefs will contribute to a "positive risk premium aggregation bias", when utilities are independent of the state, if relative risk aversion of the representative investor decreases with income.* We also show the elementary but apparently little known property that *whenever individual absolute risk tolerances are increasing with income, microeconomic heterogeneity generates an aggregation bias toward decreasing aggregate relative risk aversion, even though that property may be weak or even absent at the microeconomic level.*

In the particular case of individual investors with CARA utilities ($\eta = 0$ in the HARA family), the adjustment coefficient r° of the market portfolio is larger than 1, but this contributes to a negative risk premium aggregation bias (by increasing the representative investor's evaluation) because individual and aggregate relative risk aversion goes up with income. By contrast, the adjustment coefficient r° of the market portfolio exceeds 1 and contributes to a positive risk premium aggregation bias when, for instance, investors have Constant Relative Risk Aversion (CRRA) utilities with *heterogenous* relative degrees of risk aversion $\rho_a > 1$, because this implies an aggregate representative investor who has *decreasing* relative risk aversion.

Concluding remarks and discussion of possible topics for future research are gathered in section 7.

2   Equilibrium Portfolio Selection

We consider a collection of individual investors of different "types" indexed by a. Each individual investor solves a standard one-period portfolio selection problem : he has a current income $b_a \geqq 0$ that he wishes to invest in financial assets (available on the market to all) indexed by $j = 1, ...n$. A unit of asset j generates income d_{hj} (in units of account or in kind) tomorrow in various states of the world h. If x_{aj} is the number of units of asset j purchased, and p_j the unit price of that asset, the investor's current budget constraint is $\sum_j p_j x_{aj} = b_a$. A portfolio $x_a = (x_{aj})$ generates the income $y_{ah} = \sum_j d_{hj} x_{aj}$ in each state. We impose the constraint that income in each state h has

to be nonnegative, i.e. $y_{ah} \geqq 0$, and assume that the investor maximizes his preferences among random income streams $y_a = (y_{ah}) \geqq 0$ represented (up to an increasing affine transformation) by the expected utility function $\sum_h \pi_{ah} u_{ah}(y_{ah}) = E_{\pi_a}[u_{ah}(y_{ah})]$, where $\pi_{ah} > 0$ is the subjective probability he attaches to state h, with $\sum_h \pi_{ah} = 1$. Although we shall interpret primarily the model in terms of a standard portfolio selection problem where utility is usually supposed to be independent of the state, we allow for state dependent utilities because the analysis can also be applied to insurance problems where the realization of some events (e.g. disease) may affect directly individual welfare. We assume throughout

(**2**.*a*) *Each (possibly state dependent) von Neumann-Morgenstern utility* $u_{ah}(y_{ah})$ *is defined and continuous for* $y_{ah} \geqq 0$, *continuously differentiable up to order 3 for* $y_{ah} > 0$, *with* $u'_{ah}(y_{ah}) > 0$, $u''_{ah}(y_{ah}) < 0$.

We shall also focus on the case of interior solutions where each individual investor has a positive income in every state. In particular, we shall assume when needed

(**2**.*b*) *Marginal utilities of income go to* $+\infty$ *as income goes to 0, and to 0 when income goes to* $+\infty$, *i.e.* $\lim_{y \to 0} u'_{ah}(y) = +\infty$ *and* $\lim_{y \to \infty} u'_{ah}(y) = 0$.

We suppose that all investors face the same price system (p_j) for the traded assets (markets are competitive) and that *they all anticipate the same payoff matrix* $D = (d_{hj})$. We assume also complete markets, i.e. the payoff matrix $D = (d_{hj})$ is $n \times n$ and has full rank with $n \geq 2$. In the absence of arbitrage opportunities (a condition that will have to be satisfied in equilibrium), this means that all investors face the same (and unique) implicit system of state prices $q = (q_h)$, with $q_h > 0$, such that each asset j is valued according to $\sum_h q_h d_{hj} = p_j$. Then for any arbitrary income $b_a \geqq 0$, and any such price system p (or equivalently q), the income $y_{ah} = \sum_j d_{hj} x_{aj} \geqq 0$ generated in each state by the choice of an optimal portfolio $x_a = (x_{aj})$ can be viewed equivalently as a demand $y_{ah}(q, b_a, \pi_a)$ for the corresponding Arrow-Debreu security, which yields one unit of income in state h and none otherwise. The choice of a portfolio is then equivalent to choosing a vector of demands $y_a(q, b_a, \pi_a) = (y_{ah}(q, b_a, \pi_a))$ for Arrow-Debreu securities so as to maximize expected utility under the budget constraint $q \cdot y_a = \sum_h q_h y_{ah} = b_a$. In what follows, we shall always work directly with the markets for Arrow-Debreu (AD) securities. With this convention, the income of an investor of type a is seen as implicitly derived from an *initial portforlio* of AD assets, $\omega_a \geqq 0$, $\omega_a \neq 0$, so that $b_a = q \cdot \omega_a > 0$.

To simplify matters, we assume that the set of types is *finite*, and let $\mu_a > 0$ be the proportion of investors who belong to type a , with $\sum_a \mu_a = 1$ (the analysis extends without difficulty, modulo some technicalities, to a continuum of types, e.g. when the set of types is a complete separable metric space). In what follows, we shall freely use the notation $E_a[z_a] = \sum_a \mu_a z_a$ to describe the *average* (or *per capita*, *market* or *aggregate value*) in the population of a variable z_a, $\text{var}_a[z_a]$ to represent its variance and so on, although there is no randomness, in our interpretation, in the allocation of investors among types. We shall assume without any loss of generality

(**2**.c) *The market portfolio of AD securities $\overline{\omega} = E_a[\omega_a]$, has all its components positive, i.e. $\overline{\omega}_h > 0$ for every state $h = 1, ..., n$.*

With this notation, a competitive *exchange equilibrium* is a vector of state prices q^*, with $q_h^* > 0$ for every state h such that all markets clear, $E_a[y_a(q^*, q^* \cdot \omega_a, \pi_a)] = \overline{\omega}$. We shall focus exclusively on *interior* equilibria, that satisfy $y_{ah}^* = y_{ah}(q^*, q^* \cdot \omega_a, \pi_a) > 0$ for every investor and every state. It is known that under assumptions (2.a), (2.b) and (2.c), there exists at least one equilibrium, and that all equilibria are interior. An equilibrium price vector q^* is of course defined only up to a positive scalar factor (absence of money illusion).

3 Equilibrium Aggregation of Heterogenous Beliefs

Consider a fixed competitive equilibrium, defined by the system of asset prices q^*. We wish to analyze to which extent part of the heterogeneity of the individual equilibrium portfolios $y_a^* = y_a(q^*, q^* \cdot \omega_a, \pi_a)$ can be traced back to the heterogeneity of individual beliefs π_a. The approach we propose is to define another "equivalent" equilibrium, in which all investors would share the common probability π°, that would "aggregate" accordingly heterogenous individual subjective probalities π_a into a single "market probability" π°. We shall then interpret deviations of individual portfolios y_a^* in the observed equilibrium, from the portfolios y_a° investors would have chosen in the equivalent equilibrium had they shared the common probability π°, in part as the outcome of the diversity of individual beliefs, i.e. of the deviations $\pi_a - \pi^\circ$.

We define such a common probability equivalent equilibrium by three invariance requirements. First, the common probability equivalent equilibrium

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should generate the same equilibrium price system q^* as in the observed equilibrium with heterogenous beliefs, so that every asset gets the same valuation by the market (the same price) in both equilibria. Second, every individual investor should be actually indifferent, at the margin, between investing one additional unit of income in the observed equilibrium (using his own subjective probability π_a) and in the equivalent equilibrium (using the common probability π°), so that every asset gets also the same marginal valuation by each individual investor (in terms of his marginal expected utility) in both equilibria. This second invariance requirement can be alternatively justified from the use to which we wish to put the equivalent common probability π°. From that viewpoint, indeed, one would like it to display the following "monotonicity" property : every investor's demand y_a^* for AD security h, in the observed equilibrium with heterogenous probabilities, should exceed the demand y_{ah}° he would have for that security in the equivalent equilibrium associated to the equivalent common probability π°, if and only if he attaches a higher probability to that state than the equivalent probability, i.e. if and only if $\pi_{ah} \geqq \pi_h^\circ$. We shall see that these two formulations of our second invariance requirement are, for all practical purposes, essentially equivalent, at least in the most relevant case where π_a and π° approximate probabilities having positive continuous densities on a common interval. Our third and final invariance requirement is that the aggregation procedure should be "unbiased", i.e. it should generate the investors' common probability whenever they happen to share the same beliefs initially : one should get $\pi^\circ = \pi$ if and only if $\pi_a = \pi$ for all investors.

We show below that these three invariance requirements determine uniquely the equivalent common probability π°. An important feature of the outcome will be that the market portfolio may have to be scalarly adjusted by a coefficient $r^\circ > 0$, in the common probability equivalent equilibrium, which leads to a modification of its mean and its variability accross states when the adjustment coefficient r° is different from 1. We shall discuss later the implications of that fact for asset pricing and aggregate risk aversion.

As stated above, the *first invariance requirement* we impose is that, given an equilibrium vector q^* of state prices, the "equivalent" probability π° is such that q^* is still an equilibrium when all investors share the common probability π°. The next fact states that this approach is indeed feasible even when one fixes arbitrarily a "reference" market portfolio ω° (that may differ from the actual market portfolio $\overline{\omega}$) and its distribution among investors.

Proposition 3.1. *Suppose that every type statisfies (2.a) and (2.b),*

assume that the market portfolio $\overline{\omega}$ satisfies (2.c), and consider an equilibrium vector of state prices q^*.

Let ω° be an arbitrary reference market portfolio of AD securities, with $\omega_h^\circ > 0$ for every state, and let $b_a^\circ > 0$ be an arbitrary distribution of income among investors, satisfying $E_a[b_a^\circ] = q^* \cdot \omega^\circ$. There is a unique probability π° such that q^* is an equilibrium price vector relative to that reference market portfolio and that income distribution when all investors share the common probability π°, i.e. such that $E_a[y_a(q^*, b_a^\circ, \pi^\circ)] = \omega^\circ$. The common probability π° assigns a positive weight $\pi_h^\circ > 0$ to every state.

Proof. Let Δ be the set of probabilities π defined by $\pi_h \geqq 0, \sum_h \pi_h = 1$. Under the assumption that all investors share such a probability π, the vector of market excess demands for AD securities, corresponding to the reference market portfolio ω° and the income distribution (b_a°), is $z(\pi) = E_a[y_a(q^*, b_a^\circ, \pi)] - \omega^\circ$, where $y_a(q^*, b_a^\circ, \pi)$ is the vector of demands for AD securities resulting from the maximization of the expected utility $E_\pi[u_{ah}(y_{ah})]$ under the budget constraint $q^* \cdot y_a = b_a^\circ$. From the budget identity of each investor, one gets $q^* \cdot z(\pi) = 0$. Note that $z(\pi)$ is well defined and continuous even on the frontier of Δ. The common probability π° we are looking for is caracterized by $z(\pi^\circ) = 0$.

The proof employs routine techniques from general equilibrium analysis (Arrow and Debreu (1954), McKenzie (1954), Debreu (1959)), with the probability π playing here the role of prices there. For every π in Δ, let π' in Δ that minimizes $\pi' \cdot z(\pi)$ (this mimics a "tâtonnement" process, where we try to decrease an initially positive aggregate excess demand by lowering the probability of the corresponding state). The correspondence so defined from Δ into itself satisfies all conditions of the Kakutani's fixed point theorem, so it has a fixed point π°, with $\pi^\circ \cdot z^\circ \leqq \pi \cdot z^\circ$ for all π in Δ, where $z^\circ = z(\pi^\circ)$.

Suppose for a moment that $z^\circ \neq 0$. Since $q^* \cdot z^\circ = 0$, this means that there are states $h \neq k$ such that $z_h^\circ < 0 < z_k^\circ$. But then $\pi_k^\circ = 0$ (otherwise one could decrease slightly π_k° and increase π_h° by the same amount, so as to stay in Δ and decrease $\pi^\circ \cdot z^\circ$, contradicting the fact that $\pi^\circ \cdot z^\circ$ minimizes $\pi \cdot z^\circ$ on Δ). However, if $\pi_k^\circ = 0$, then $y_{ak}(q^*, b_a^\circ, \pi^\circ) = 0$ for every type, hence $z_k^\circ = -\omega_k^\circ < 0$, a contradiction. So it must be that $z^\circ = 0$. Moreover, one has $\pi_h^\circ > 0$ for every state since by the foregoing argument, $\pi_h^\circ = 0$ would imply $z_h^\circ < 0$.

To prove unicity, we remark that the vector of market excess demands $z(\pi)$ is homogenous of degree 0 in the vector π when we relax the constraint $\sum_h \pi_h = 1$, and that it satisfies the gross complementarity property

$\partial z_k / \partial \pi_h < 0$ for $h \neq k$. This can be easily seen by considering the first order condition (FOC) characterizing the individual demands for AD securities $y_{ah}(q^*, b_a^\circ, \pi)$, i.e. $\pi_h u'_{ah}(y_{ah})/q_h^* = \pi_k u'_{ak}(y_{ak})/q_k^*$. If for some $k \neq h, y_{ak}$ increased or stayed constant when the component π_h(of the vector π) goes up, that would be true for every $j \neq h$, with the consequence that y_{ah} itself would increase. But that would contradict the fact that $q^* \cdot y_a = b_a^\circ$ has to stay constant since incomes are fixed. Thus when $h \neq k$, one has $\partial y_{ak}/\partial \pi_h < 0$ for all a, hence $\partial z_k/\partial \pi_h < 0$. Unicity then follows from the argument employed in general equilibrium analysis in the case of gross substitutability (Arrow and Hahn (1971, Theorem 9.7.7)). In short, consider a common probability equilibrium defined by π° such that $z(\pi^\circ) = 0$ and consider another candidate probability $\pi \neq \pi^\circ$. By using the homogeneity of degreee 0 of $z(\pi)$, we may ignore the constraint $\sum_h \pi_h = 1$, say that π is actually a vector (non colinear to π°) and normalize it so as to ensure $\pi \leqq \pi^\circ$ and $\pi_k = \pi_k^\circ$ for some k. Then one can go from the vector π° to the vector π by decreasing sequentially the components h such that $\pi_h < \pi_h^\circ$. From the gross complementarity property, one gets $z_k(\pi) > z_k(\pi^\circ) = 0$. So no probability $\pi \neq \pi^\circ$ can generate a common probability equilibrium with the price system q^* and the income distribution b_a°.[4] $\qquad$ Q.E.D.

What precedes shows that the outcome of our aggregation procedure depends on the arbitrary reference market portfolio ω°, and on the income distribution (b_a°). We consider now our *second invariance requirement*, namely that every asset should get the same marginal valuation by each investor in the observed equilibrium and in the common probability equivalent equilibrium. Specifically, consider the FOC characterizing the interior individual portfolios $y_a^* = y_a(q^*, q^* \cdot \omega_a, \pi_a)$ in the observed equilibrium

$$(3.1) \qquad \pi_{ah} \quad u'_{ah}(y_{ah}^*)/q_h^* = \lambda_a^* \quad ,$$

where $\lambda_a^* = \lambda_a(q^*, q^* \cdot \omega_a, \pi_a)$ is the corresponding marginal expected utility of income. As is well known, the FOC means that the investor is indifferent, at the margin, about which asset to use in order to invest a (virtual) additional piece of income. Indeed, the investor's marginal expected utility of any (virtual) marginal portfolio generating the return R_h in each state (thus satisfying $\sum_h q_h^* R_h = 1$)

$$(3.2) \; \lambda_a^* = E_{\pi_a}[R_h u'_{ah}(y_{ah}^*)] = E_{\pi_a}[R_h] E_{\pi_a}[u'_{ah}(y_{ah}^*)] + cov_{\pi_a}[R_h, u'_{ah}(y_{ah}^*)]$$

is independent of that portfolio. The same marginal indifference holds of course at the common probability equilibrium obtained in Proposition 3.1.

There the interior equilibrium portfolios $y_a^\circ = y_a\left(q^*, b_a^\circ, \pi^\circ\right)$ are characterized by

$$(3.3) \qquad\qquad \pi_h^\circ \quad u_{ah}'\left(y_{ah}^\circ\right) \Big/ q_h^* = \lambda_a^\circ \qquad ,$$

where $\lambda_a^\circ = \lambda_a\left(q^*, b_a^\circ, \pi^\circ\right)$ is again the corresponding marginal expected utility of income, and the investor values equally, at the margin, all assets

$$(3.4)\ \lambda_a^\circ = E_{\pi^\circ}\left[R_h u_{ah}'\left(y_{ah}^\circ\right)\right] = E_{\pi^\circ}\left[R_h\right] E_{\pi^\circ}\left[u_{ah}'\left(y_{ah}^\circ\right)\right] + cov_{\pi^\circ}\left[R_h, u_{ah}'\left(y_{ah}^\circ\right)\right].$$

Our second invariance requirement is that each investor should be indifferent, at the margin, not only between assets within each equilibrium as in (3.2) and (3.4), but also between investing one additional unit of income in the observed equilibrium (using his own subjective probability π_a) and in the equivalent equilibrium (using the common probability π°) :

(3.5) *For every marginal portfolio generating the returns R_h, with $\sum_h q_h^* R_h = 1$*

$$E_{\pi_a}\left[R_h\ u_{ah}'\left(y_{ah}^*\right)\right] = E_{\pi^\circ}\left[R_h\ u_{ah}'\left(y_{ah}^\circ\right)\right].$$

The individual changes of incomes from $q^* \cdot \omega_a$ to b_a° are then precisely those required to compensate the changes of individual probabilities from π_a to π° in order to achieve the marginal indifference condition (3.5).

One potential usefulness of a common probability equivalent equilibrium is that it provides us with a benchmark to analyze to which extent the heterogeneity of individual observed portfolios y_a^* is related to the heterogeneity of individual beliefs π_a (in conjunction with heterogeneity of incomes and of attitudes toward risk). In this respect, a desirable "monotonicity" property would be that for every state h, an investor's observed demand for the corresponding AD security is larger than, equal to or less than his demand of that security in the equivalent equilibrium, if and only if he attaches a subjective probability that is larger than, equal to or less than the corresponding equivalent common probability, respectively.

(3.6) *For every state h, $y_{ah}^* \geqq y_{ah}^\circ$ (resp. $y_{ah}^* \leqq y_{ah}^\circ$) if $\pi_{ah} \geqq \pi_h^\circ$ (resp. $\pi_{ah} \leqq \pi_h^\circ$).*

Clearly such a monotonicity property does obtain under the marginal indifference condition (3.5) since one gets then in every state h

$$(3.7) \qquad\qquad \pi_{ah}\ u_{ah}'\left(y_{ah}^*\right) = \pi_h^\circ\ u_{ah}'\left(y_{ah}^\circ\right).$$

Conversely, imposing directly the monotonicity property (3.6) on an equivalent common probability equilibrium would imply, in view of the individual FOC, i.e. of $q_h^* = (\pi_{ah}/\lambda_a^*) \, u'_{ah}(y_{ah}^*) = (\pi_h^\circ/\lambda_{ah}^\circ) \, u'_{ah}(y_{ah}^\circ)$, that $\lambda_a^* = \lambda_a^\circ$ if there is a state h such that $\pi_{ah} = \pi_h^\circ$, and furthermore that $\lambda_a^*/\lambda_a^\circ < \pi_{ah}/\pi_h^\circ$ when $\pi_{ah} > \pi_h^\circ$, and $\pi_{ah}/\pi_h^\circ < \lambda_a^*/\lambda_a^\circ$ when $\pi_{ah} < \pi_h^\circ$. Therefore

(3.8) *The monotonicity property (3.6) is equivalent to* $l_a < \lambda_a^*/\lambda_a^\circ < L_a$ *when* $l_a < 1 < L_a$, *where* $l_a = Max_h \{\pi_{ah}/\pi_h^\circ \mid \pi_{ah}/\pi_h^\circ \leqq 1\}$ *and* $L_a = Min_h \{\pi_{ah}/\pi_h^\circ \mid \pi_{ah}/\pi_h^\circ \geqq 1\}$, *and to* $\lambda_a^*/\lambda_a^\circ = 1$ *when* $l_a = 1 = L_a$. *The monotonicity (3.6) property will be accordingly almost equivalent to the marginal indifference requirement (3.5) (i.e.* $\lambda_a^* = \lambda_a^\circ$*) in the empirically most relevant case where there is a large number of states and where each investor's belief* π_a, *as well as the common probability* π°, *approximate probabilities with positive continuous densities everywhere on a given common interval, since in that case the distances* $L_a - l_a$ *are bound to be small.*

Intuitively, one should be able to obtain through the aggregation procedure specified in Proposition 3.1, for a given reference market portfolio ω°, *proportionality* of all investors' marginal valuations of assets in each equilibrium, i.e. of $\lambda_a^* = E_{\pi_a}[R_h \, u'_{ah}(y_{ah}^*)]$ and $\lambda_a^\circ = E_{\pi^\circ}[R_h \, u'_{ah}(y_{ah}^\circ)]$, by playing with the income distribution (b_a°) under the constraint $E_a[b_a^\circ] = q^* \cdot \omega^\circ$. To bring about *equality,* one needs generally one additional degree of freedom, namely to fix the composition, but not the scale, of the reference portfolio. The next result states that the outcome of our aggregation procedure is indeed uniquely determined by the requirement that each individual investor's marginal valuation of assets remains invariant when going from the observed equilibrium to the common probability equivalent equilibrium, when one considers reference market portfolios of the form $r^\circ \omega^\circ$, where ω° is a fixed vector of AD securities (satisfying for instance the normalization $\sum_h \omega_h^\circ = \sum_h \overline{\omega}_h$), but where the scalar coefficient r° is free to adjust.

The *final requirement* we impose is that the aggregation procedure should be unbiased, i.e. generate the "correct" result when investors share initially the same beliefs. Specifically, we think of the reference vector ω° and the actual market portfolio $\overline{\omega}$ as fixed (where ω° satisfies the normalization $\sum_h \omega_h^\circ = \sum_h \overline{\omega}_h$), and consider the outcomes of the aggregation procedure when the other characteristics of the economy, in particular the individual subjective probabilities π_a and the equilibrium price vector q^*, are free to vary. The unbiasedness requirement is that when individual probabilities happen to coincide, i.e. $\pi_a = \pi$ for every type, the aggregation procedure should then generate the same probability $\pi^\circ = \pi$. This final requirement imposes, not too surprisingly, $\omega^\circ = \overline{\omega}$.

Theorem 3.2. *Suppose that every type satisfies (2.a) and (2.b), that the market portfolio $\overline{\omega}$ satisfies (2.c), and consider an equilibrium vector q^* of state prices.*

Let ω° be an arbitrary reference market portfolio of AD securities, with positive components, satisfying the normalization $\sum_h \omega_h^\circ = \sum_h \overline{\omega}_h$. There is a unique probability π° with positive components, a unique coefficient of adjustment $r^\circ > 0$ of the reference market portfolio, and a unique distribution of income (b_a°) satisfying $E_a [b_a^\circ] = q^* \cdot (r^\circ \omega^\circ)$, such that

1) q^* is an equilibrium price system relatively to the common probability π°, the adjusted reference market portfolio $r^\circ \omega^\circ$, and the income distribution (b_a°), i.e. $E_a [y_a (q^*, b_a^\circ, \pi^\circ)] = r^\circ \omega^\circ$,

2) Individual marginal valuations of assets remain the same before and after the aggregation procedure, i.e. for every investor and every asset generating the returns R_h, with $\sum_h q_h^* R_h = 1$

$$E_{\pi_a} [R_h \, u'_{ah} (y_{ah}^*)] = E_{\pi^\circ} [R_h \, u'_{ah} (y_{ah}^\circ)],$$

where $y_a^* = y_a (q^*, q^* \cdot \omega_a, \pi_a)$ and $y_a^\circ = y_a (q^*, b_a^\circ, \pi^\circ)$ are the corresponding equilibrium portfolios.

Let the actual market portfolio $\overline{\omega}$ and the reference portfolio ω° be fixed, and consider the outcome of the above aggregation procedure when the other characteristics of the economy are free to vary. The procedure generates $\pi^\circ = \pi$ when individual probabilities π_a coincide with π, if and only if $\omega^\circ = \overline{\omega}$.

Proof. We fix an arbitrary reference market portfolio ω°, with the normalization $\sum_h \omega_h^\circ = \sum_h \overline{\omega}_h$. The proof will be a variation of the fixed point argument used for Proposition 3.1, appropriately modified to take into account that individual incomes are no longer fixed. Let Δ be the set of probabilities $\pi \geqq 0$ such that $\sum_h \pi_h = 1$. For any π in Δ, let $y_{ah} (\pi)$ be given by (3.7), i.e. $\pi_{ah} \, u'_{ah} (y_{ah}^*) = \pi_h \, u'_{ah} (y_{ah})$ when $\pi_h > 0$ and $y_{ah} = 0$ otherwise. By construction, the vector $y_a (\pi)$ stands for the demands of AD securities $y_a (q^*, b_a (\pi), \pi)$ where individual income has been adjusted to keep invariant the investor's marginal valuation of assets, i.e. to achieve $\lambda_a^* = \lambda_a (q^*, q^* \cdot \omega_a, \pi_a) = \lambda_a (q^*, b_a (\pi), \pi)$. From the investors' budget indentities, all $b_a (\pi) = q^* \cdot y_a (\pi) > 0$ vary continuously with, and are positive for every π in Δ. If one adjusts scalarly the reference market portfolio ω° by the coefficient $r (\pi) = E_a [b_a (\pi)] / q^* \cdot \omega^\circ > 0$ and defines accordingly the vector of aggregate excess demands for AD assets as $z (\pi) = E_a [y_a (\pi)] - r (\pi) \omega^\circ$, a common probability equilibrium satisfying (1) and (2) in Theorem 3.2 is by construction given by a probability π° in Δ such that $z (\pi^\circ) = 0$. The

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corresponding coefficient of adjustment and distribution of incomes are then $r^\circ = r(\pi^\circ)$ and $b_a^\circ = b_a(\pi^\circ)$.

Existence of such a probability π°, and the fact that it involves only positive components $\pi_h^\circ > 0$, is proved by exactly the same fixed point argument as in the proof of Proposition 3.1. To prove unicity, fix such a probability π°, thus generating the equilibrium portfolios $y_a^\circ = y_a(\pi^\circ)$, and consider another probability $\pi \neq \pi^\circ$ in Δ. By construction, for every state h, $\pi_h^\circ u'_{ah}(y_{ah}^\circ) = \pi_h$ $u'_{ah}(y_{ah}(\pi))$, so that $\pi_h > \pi_h^\circ$ implies $y_{ah}(\pi) > y_{ah}^\circ$ for all investors, thus $E_a[y_{ah}(\pi)] > r^\circ \omega_h^\circ$, while $\pi_k < \pi_k^\circ$ implies $E_a[y_{ak}(\pi)] < r^\circ \omega_k^\circ$ by the same argument. Therefore $\pi \neq \pi^\circ$ cannot give rise to a common probability equilibrium satisfying (1) and (2), or $z(\pi) = E_a[y_a(\pi)] - r(\pi)\omega^\circ = 0$, since there the ratios $E_a[y_{ah}(\pi)] / \omega_h^\circ$ would have to be equal to the same number $r(\pi)$ for all states h.[5]

To prove the last part of the Theorem, let us fix $\overline{\omega}$ and ω° satisfying the normalization $\sum_h \overline{\omega}_h = \sum_h \omega_h^\circ$, and assume that $\pi_a = \pi$ for all a. Then it is clear that the unique common probability equivalent equilibrium generates $\pi^\circ = \pi$ if and only if $y_a^\circ = y_a^*$ for all a. A necessary and sufficient condition for that is clearly that $\overline{\omega} = E_a[y_a^*] = \omega^\circ$ (implying $r^\circ = 1$).   Q.E.D.

The three invariance requirements (invariance of the equilibrium price vector, invariance of individual marginal valuations of assets, unbiasedness) pin down accordingly the outcome of the aggregation procedure. We shall call an equilibrium defining a common probability π° by conditions 1) and 2) in Theorem 3.2 with an arbitrary reference portfolio ω°, *the common probability equivalent equilibrium corresponding to the reference portfolio ω°*. When $\omega^\circ = \overline{\omega}$ and if there is no risk of confusion, we shall drop any mention of the reference portfolio and speak simply of "the" *common probability equivalent equilibrium*.

4 The Adjusted Consumption Based Capital Asset Pricing Model (ACCAPM)

The aggregation procedure presented in the preceding section was deliberately couched in terms of invariance conditions on marginal asset pricing by the market and by every individual investor, that borrowed little from the assumption of complete markets (with the hope that the construction might be transposed to incomplete markets as well). We adopt now a "dual" viewpoint that exploits fully the complete markets structure, by looking at the

standard construction of an equilibrium "representative" investor, which is valid in the case of homogenous subjective probabilities, and analyze how the associated Consumption based Capital Asset Pricing Model (CCAPM) must be modified to account for heterogenous individual probabilities, modulo a possible scalar "adjustment" of the market portfolio. Our analysis will lead to another justification of the notion of a common probability equivalent equilibrium introduced in the previous section, through an equivalent, "dual", marginal asset valuation invariance condition, involving this time the equilibrium "representative" aggregate investor.

Consider an equilibrium with heterogenous beliefs π_a defined by the system of state prices q^* and let $y_a^* = y_a(q^*, q^* \cdot \omega_a, \pi_a)$ be the corresponding individual optimum portfolios. It is convenient to introduce the following *normalization* of individual von Neumann Morgensten (VNM) utilities

$$(4.1) \qquad v_{ah}(y_{ah}) = u_{ah}(y_{ah}) \Big/ E_{\pi_a}[u'_{ah}(y^*_{ah})].$$

The normalization (4.1) generates a unique (up to the addition of arbitrary constants) representation of the underlying preferences by the condition $E_{\pi_a}[v'_{ah}(y^*_{ah})] = 1$, or equivalently by the property that every individual investor's marginal valuation, in the observed equilibrium, of an asset with returns R_h satisfying $\sum_h q^*_h R_h = 1$, is not only independent of that asset, but is actually equal to the equilibrium gross rate of return of the riskless asset giving one unit of income in every state, i.e. to $R_o^* = 1 \Big/ \sum_h q^*_h$

$$(4.2)\ E_{\pi_a}[R_h v'_{ah}(y^*_{ah})] = E_{\pi_a}[R_h] + cov_{\pi_a}[R_h, v'_{ah}(y^*_{ah})] = R_o^*.$$

We recall first the construction of an equilibrium "representative" aggregate investor involved in the standard CCAPM when all individual subjective probabilities happen to coincide, i.e. $\pi_a = \pi$ for all a.[6] In that construction, the preferences of the equilibrium "representative" aggregate investor are described by the VNM utilities, for every state h

$$(4.3) \qquad U_h(y_h) = Max\, E_a[v_{ah}(y_{ah})] \text{ subject to } E_a[y_{ah}] = y_h.$$

Under the hypothesis that the above leads to an interior solution $y_{ah} > 0$ for all a (this will be guaranted for every $y_h > 0$ under assumption (2.b)), it will be characterized by $U'_h(y_h) = v'_{ah}(y_{ah})$ for all a. The fact that this procedure defines an equilibrium "representative" investor when $\pi_a = \pi$ for every a, comes then from the property that the solutions to (4.3) when $y_h = \overline{\omega}_h > 0$, generate an allocation $y_a = (y_{ah})$ that coincides with the observed equilibrium individual portfolios y^*_{ah} (for both $v'_{ah}(y_{ah}) = U'_h(\overline{\omega}_h)$

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and $v'_{ah}\left(y^*_{ah}\right) = q^*_h R^*_o \big/ \pi_h$ are then independent of a for every state h. Since marginal utilities v'_{ah} are decreasing, the equalities $y_{ah} = y^*_{ah}$ for all a, h, follow from the common equilibrium conditions $E_a\left[y_{ah}\right] = \overline{\omega}_h = E_a\left[y^*_{ah}\right]$. Therefore the aggregate investor defined in (4.3) does "represent" the observed market equilibrium when $\pi_a = \pi$ for all a, not only in the usual sense that the market portfolio $\overline{\omega}$ maximizes his preferences, or his expected utility $E_\pi\left[U_h\left(y_h\right)\right]$ under the market budget constraint $q^* \cdot y = q^* \cdot \overline{\omega}$, but actually in the stronger sense that the aggregate investor's and all individual investors' (normalized through (4.1)) marginal valuations of an arbitrary asset in the observed equilibrium, are identical.

(4.4) *(The standard CCAPM for homogenous beliefs) When $\pi_a = \pi$ for all a, under the individual normalizations (4.1) and the specification (4.3) of the aggregate VNM utilities, for every asset generating the returns R_h with $\sum_h q^*_h R_h = 1$*

$$E_\pi\left[R_h U'_h\left(\overline{\omega}_h\right)\right] = E_{\pi_a}\left[R_h v'_{ah}\left(y^*_{ah}\right)\right] = R^*_o.$$

Therefore, the specification (4.3) involves in the case of homogenous beliefs, the normalization $E_\pi\left[U'_h\left(\overline{\omega}_h\right)\right] = 1$ or equivalently, the equilibrium marginal asset valuation $E_\pi\left[R_h U'_h\left(\overline{\omega}_h\right)\right] = R^*_o$ that is identical to the individual normalizations (4.1). It defines accordingly a *normalized representation* of the aggregate investor's underlying preferences that is not only independent of the particular choices of the individual VNM utilities (u_{ah}), but is in fact (again similarly to the individual normalized utilities (v_{ah}) defined in (4.1)) unique up to the addition of arbitrary constants.

We show now that the same CCAPM construction does apply to the case of heterogenous beliefs π_a, modulo a possible scalar "adjustment" of the aggregate portfolio.

Let us go back to an observed market equilibrium with heterogenous beliefs π_a, described by the system of state prices q^* and the corresponding individual optimum portfolios $y^*_a = y_a(q^*, q^* \cdot \omega_a, \pi_a)$. We keep the individual normalizations (4.1) and still define by (4.3) the VNM utilities of a (potentially) "representative" equilibrium investor. Given an arbitrary reference market portfolio ω° satisfying the normalization $\sum_h \omega^\circ_h = \sum_h \overline{\omega}_h$ as in Theorem 3.2, a natural extension of the above CCAPM construction is to say that the aggregate investor does indeed "represent" at the margin all individual investors, when endowed with the probability π° and with the possibly scalarly adjusted reference market portfolio $r^\circ \omega^\circ$, not only in the sense that the adjusted portfolio $r^\circ \omega^\circ$ maximizes his preferences under the

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market budget constraint $q^* \cdot y = q^* \cdot (r^\circ \omega^\circ)$, but under the stronger sense that the aggregate investor, under the specification (4.3), does give then the same marginal evaluation of every arbitrary asset as each normalized individual investor in the observed equilibrium (exactly as in (4.4) above, but with π replaced by π° and $\overline{\omega}$ by $r^\circ \omega^\circ$)

(4.5) *For every asset generating the returns R_h with $\sum_h q_h^* R_h = 1$,*

$$E_{\pi^\circ}\left[R_h U_h'\left(r^\circ \omega_h^\circ\right)\right] = E_{\pi_a}\left[R_h v_{ah}'\left(y_{ah}^*\right)\right] = R_o^*.$$

Here again, (4.5) involves the equilibrium normalization $E_{\pi^\circ}\left[U_h'\left(r^\circ \omega_h^\circ\right)\right] = 1$. The specification (4.3) defines accordingly also here in the case of heterogenous beliefs, a *normalized representation* of the aggregate investor's underlying preferences that is unique up to the addition of arbitrary constants.

It is easily seen that this extension of the standard CCAPM construction to the case of heterogenous beliefs does coincide with the notion of a common probability equilibrium relative to a reference market portfolio ω° described in the preceding section (Theorem 3.2). Let indeed $y_a^\circ = (y_{ah}^\circ)$ be the portfolios determined by the solutions of (4.3) when $y_h = r^\circ \omega_h^\circ > 0$. Under the maintained hypothesis of interior solutions, these portfolios are characterized by $U_h'\left(r^\circ \omega_h^\circ\right) = v_{ah}'\left(y_{ah}^\circ\right)$. Rewriting (4.5) by using that fact, in terms of the original individual utilities u_{ah} so as to facilitate a direct comparison with the analysis of the previous section, generates the characterization

(4.6) For any asset with returns R_h satisfying $\sum_h q_h^* R_h = 1$,

$$E_{\pi^\circ}\left[R_h u_{ah}'\left(y_{ah}^\circ\right)\right] = E_{\pi_a}\left[R_h u_{ah}'\left(y_{ah}^*\right)\right] = R_\circ^* E_{\pi_a}\left[u_{ah}'\left(y_{ah}^*\right)\right],$$

together with $E_a\left[y_{ah}^\circ\right] = r^\circ \omega_h^\circ$. This is equivalent to the facts that (1) the price system q^* is an equilibrium relative to the adjusted reference market portfolio $r^\circ \omega^\circ$ and the income distribution $b_a^\circ = q^* \cdot y_a^\circ$ when all investors share the common probability π°, for we have $y_a^\circ = y_a\left(q^*, b_a^\circ, \pi^\circ\right)$ and $E_a\left[y_a\left(q^*, b_a^\circ, \pi^\circ\right)\right] = r^\circ \omega^\circ$, and that (2) individual marginal valuations of assets are the same in both equilibria. That is, the proposed construction of a representative equilibrium investor generates exactly the common probability equivalent equilibrium relative to the reference portfolio ω°, as specified in (1), (2) of Theorem 3.2.

This analysis also provides us with an alternative marginal asset evaluation invariance requirement, involving the aggregate representative investor, to characterize the common probability equivalent equilibrium introduced in Theorem 3.2. We know that the standard CCAPM construction applies to

a common probability equilibrium since there all investors share the probability π°. The normalized VNM utilities of the corresponding representative investor are thus defined (up to the addition of arbitrary constants) by

(4.7) $U_h^\circ(y_h) = Max E_a\left[u_{ah}(y_{ah}) \diagup E_{\pi^\circ}[u'_{ah}(y_{ah}^\circ)]\right]$ subject to $E_a[y_{ah}] = y_h,$

where $y_a^\circ = y_a(q^*, b_a^\circ, \pi^\circ)$ are the corresponding equilibrium portfolios. Then the fact that (π°, r°) determines a common probability equivalent equilibrium relatively to the reference market portfolio ω°, in the sense of Theorem 3.2, is equivalent to the property that the representative investor's normalized VNM utilities obtained through (4.7) are identical (again, up to the addition of arbitrary constants) to those obtained by the application of the same construction, through (4.3), to the observed equilibrium.

The next Proposition summarizes the above discussion.

Proposition 4.1 *(The Adjusted CCAPM). Suppose that every type satisfies (2.a), that the market portfolio $\overline{\omega}$ satisfies (2.c), and consider an equilibrium vector of state prices q^*, with the corresponding $y_a^* = y_a(q^*, q^* \cdot \omega_a, \pi_a)$ individual optimum portfolios.*

Let ω° be an arbitrary reference portfolio of AD securities, with positive components, satisfying the normalization $\sum_h \omega_h^\circ = \sum_h \overline{\omega}_h$. Under the hypothesis that all portfolios under consideration are interior, the following statements are equivalent :

A) π° and r° are the equivalent common probability and the adjustment coefficient of the reference portfolio, defined by the invariance requirements 1) and 2) of Theorem 3.2, with $y_a^\circ = y_a(q^*, b_a^\circ, \pi^\circ)$ being the corresponding individual equilibrium portfolios.

B) The aggregate investor defined by (up to the addition of arbitrary constants) the normalized VNM utilities

(4.8) $U_h(y_h) = Max\ E_a\left[v_{ah}(y_{ah})\right]$ *subject to* $E_a[y_{ah}] = y_h,$

where the individual VNM utilities $v_{ah}(y_{ah}) = u_{ah}(y_{ah}) \diagup E_{\pi_a}[u'_{ah}(y_{ah}^*)]$ have been normalized, is an equilibrium representative investor, when endowed with the common probability π° and the adjusted portfolio $r^\circ\omega^\circ$, in the sense that the portfolio $r^\circ\omega^\circ$ maximizes his expected utility $E_{\pi^\circ}[U_h(y_h)]$ under the market budget constraint $q^* \cdot y = q^* \cdot (r^\circ\omega^\circ)$, and that he evaluates every asset, at the margin, as does every individual investor in the observed equilibrium

(4.9) *For every asset generating the returns R_h with $\sum_h q_h^* R_h = 1,$*

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$$E_{\pi^\circ} \left[R_h U'_h \left(r^\circ \omega^\circ_h \right) \right] = E_{\pi_a} \left[R_h v'_{ah} \left(y^*_{ah} \right) \right] = R^*_o,$$

where $R^*_o = 1 / \sum_h q^*_h$ is the gross rate of return of the riskless asset giving one unit of income in every state. The equilibrium portfolios y°_a are then the solutions of (4.8) for $y_h = r^\circ \omega^\circ_h$.

C) The representative investor's normalized VNM utilities obtained by application of the standard CCAPM to the equivalent common probability equilibrium

(4.10) $U^\circ_h (y_h) = Max E_a \left[u_{ah} \left(y_{ah} \right) \Big/ E_{\pi^\circ} \left[u'_{ah} \left(y^\circ_{ah} \right) \right] \right]$ subject to $E_a \left[y_{ah} \right] = y_h,$

coincide (up to the addition of arbitrary constants) with those obtained, through (4.8), by application of the same construction to the observed equilibrium.[7]

The Adjusted CCAPM obtains when one adds the unbiasedness requirement, i.e. $\pi_a = \pi$ for all a implies $\pi^\circ = \pi$, that is when the reference portfolio is equal to the market portfolio $\omega^\circ = \overline{\omega}$.

The analysis of this section leads also to an alternative simple "dual" argument to demonstrate the existence, and unicity, of a common probability equivalent equilibrium relative to a given reference market portfolio ω° that was stated in Theorem 3.2. We outline it now, as it is instructive on its own right. From B) of the foregoing Proposition, we know that the corresponding probability π° and adjustment coefficient r° are characterized by (4.9) which, when applied to AD securities, yields the FOC : $\pi^\circ_h U'_h \left(r^\circ \omega^\circ_h \right) / q^*_h = R^*_o$. This suggests the following constructive argument. Fix an arbitrary adjustment coefficient $r > 0$, and compute a corresponding probability $\pi^\circ (r)$ from $\pi^\circ_h (r) = \lambda^\circ (r) q^*_h / U'_h (r \omega^\circ_h)$, where $\lambda^\circ (r) = 1 / \sum_k \left(q^*_k / U'_k (r \omega^\circ_k) \right)$. If we consider the portfolios $y_a (r) = (y_{ah} (r))$ (assumed to be interior, e.g. because of assumption (2.b)), solutions of (4.8) for $y_h = r \omega^\circ_h$, hence satisfying $U'_h (r \omega^\circ_h) = v'_{ah} (y^\circ_{ah} (r))$, we have by construction that for every asset generating the returns R_h with $\sum_h q^*_h R_h = 1$

$$E_{\pi^\circ (r)} \left[R_h U'_h \left(r \omega^\circ_h \right) \right] = E_{\pi^\circ (r)} \left[R_h v'_{ah} \left(y^\circ_{ah} (r) \right) \right] = \lambda^\circ (r) .$$

This means that the portfolios $y^\circ_{ah} (r)$ are equilibrium portfolios relative to the price system q^*, the aggregate portfolio $r \omega^\circ$ and the income distribution $b^\circ_a (r) = q^* \cdot y^\circ_a (r)$, when all investors share the common probability $\pi^\circ (r)$, i.e. $y^\circ_a (r) = y_a (q^*, b^\circ_a (r), \pi^\circ (r))$ and $E_a \left[y^\circ_a (r) \right] = r \omega^\circ$. Moreover, in that equilibrium, individual marginal valuations of assets are given by (in terms of the original VNM utilities $u_{ah} (y_{ah})$) $E_{\pi^\circ (r)} \left[R_h u'_{ah} \left(y^\circ_{ah} (r) \right) \right] =$

$\lambda^{\circ}(r) E_{\pi_a} [u'_{ah} (y^*_{ah})]$, and are thus *proportional but not generally equal* to those associated to the observed equilibrium, i.e. to $\lambda^*_a = E_{\pi_a} [R_h u'_{ah} (y^*_{ah})] = R^*_o E_{\pi_a} [u'_{ah} (y^*_{ah})]$. The equilibrium adjustment coefficient r° we are looking for achieves by definition equality of these marginal valuations, and therefore solves $\lambda^{\circ}(r) = 1 / \sum_k (q^*_k / U'_k (r\omega^{\circ}_k)) = R^*_o$. The corresponding equivalent common probability is $\pi^{\circ} = \pi^{\circ}(r^{\circ})$.

Now each utility $U_h (y_h)$ defined in (4.8) is concave, so that $\lambda^{\circ}(r)$ is a decreasing function. In fact, the solution to (4.8) is determined by the FOC: $U'_h (y_h) = u'_{ah} (y_{ah}) / E_{\pi_a} [u'_{ah} (y^*_{ah})]$, together with the material balance condition $E_a [y_{ah}] = y_h$, so one gets immediately by differenciation

(4.11) $T_h (y_h) = E_a [T_{ah} (y_{ah})]$,

where $T_{ah} (y_{ah}) = -u'_{ah} (y_{ah}) / u''_{ah} (y_{ah})$ is the *coefficient of absolute risk tolerance* of the individual VNM utility $u_{ah} (y_{ah})$, $T_h (y_h) = -U'_h (y_h) / U''_h (y_h)$ is similarly the coefficient of absolute risk tolerance of the aggregate VNM utility $U_h (y_h)$, while all $T_{ah} (y_{ah})$ are evaluated at the solution of (4.8). Furthermore, it is easy to see from the FOC: $U'_h (y_h) = u'_{ah} (y_{ah}) / E_{\pi_a} [u'_{ah} (y^*_{ah})]$, that $U_h (y_h)$ satisfies also assumption (2.b) whenever all individual utilities u_{ah} do (if y_h tends to 0, all y_{ah} tend to 0, while if y_h goes to $+\infty$, all y_{ah} must go to $+\infty$). Therefore, under assumption (2.b), $\lambda^{\circ}(r)$ decreases from $+\infty$ to 0 when r increases from 0 to $+\infty$, and there is a unique r° such that $\lambda^{\circ}(r^{\circ}) = R^*_o$, which completes the proof.

Corollary 4.2. *Given an equilibrium with heterogenous beliefs and a reference market portfolio ω°, to determine the corresponding equivalent common probability π° and the adjustment coefficient r° of Proposition 4.1, one can first solve for r° the scalar equation $R^*_o \sum_k q^*_k / U'_k (r^{\circ}\omega^{\circ}_k) = 1$, where the normalized VNM utilities of the representative investor $U_h (y_h)$ are specified by (4.8). The common probability π° is then given by*

$$\pi^{\circ}_h = (q^*_h / U'_h (r^{\circ}\omega^{\circ}_h)) / \sum_k (q^*_k / U'_k (r^{\circ}\omega^{\circ}_k)) = R^*_o q^*_h / U'_h (r^{\circ}\omega^{\circ}_h).$$

To conclude, we know from Theorem 3.2 that adding the unbiasedness requirement ($\pi_a = \pi$ for all a leads to $\pi^{\circ} = \pi$) imposes the equality of the reference market portfolio and the actual one. Then part B of Proposition 4.1 with $\omega^{\circ} = \overline{\omega}$ states that it is possible to find uniquely a probability π° such that the standard CCAPM (4.9) applies to the case of heterogenous individual probabilities π_a, provided that the actual market portfolio $\overline{\omega}$ is scalarly adjusted by an appropriately chosen and uniquely defined adjustment coefficient r°. Alternatively, one may choose to leave unchanged the actual market

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portfolio $\overline{\omega}$, and adjust scalarly instead the aggregate investor's normalized VNM utilities.

Corollary 4.3. *(Alternative formulation of the Adjusted CCAPM). Let the reference market portfolio be equal to the actual market portfolio, $\omega^\circ = \overline{\omega}$. Let (π°, r°) be the common probability and adjustment coefficient corresponding to a given observed equilibrium with heterogenous beliefs $(q^*, (y_a^*))$, as in Proposition 4.1. Consider the normalized VNM utilities of an "adjusted" aggregate investor defined by $V_h(y_h) = U_h(r^\circ y_h) / r^\circ$, where $(U_h(y_h))$ are the VNM utilities specified in (4.8), so that $V_h'(y_h) = U_h'(r^\circ y_h)$. Then, when endowed with the common probability π° and the actual market portfolio $\overline{\omega}$, the "adjusted" aggregate investor represents all individual investors at the observed equilibrium, in the sense that the market portfolio $\overline{\omega}$ maximizes his expected utility $E_{\pi^\circ}[V_h(y_h)]$ under the market budget constraint $q^* \cdot y = q^* \cdot \overline{\omega}$, and that he evaluates there, at the margin, every asset as does every individual investor in the observed equilibrium*

(4.12) *For every asset with returns R_h satisfying $\sum_h q_h^* R_h = 1$,*

$$E_{\pi^\circ}[R_h V_h'(\overline{\omega}_h)] = E_{\pi_a}[R_h v_{ah}'(y_{ah}^*)] = R_o^*.$$

Example 4.4. *The Hyperbolic Absolute Risk Aversion (HARA) family.*

We apply now the foregoing "dual" argument (essentially Corollary 4.2) to the HARA family, that is to the special case where utilities are state independent and display linear absolute risk tolerance, i.e. $T_{ah}(y) = -u_{ah}'(y) / u_{ah}''(y) = \theta_a + \eta y > 0$, where the marginal risk tolerance $T_{ah}'(y) = \eta$ is constant and commonly shared by all investors. This configuration, often considered in the finance literature because it generates a neat aggregation of individual behaviors when all investors share the same beliefs, leads also here to important simplifications. We note in passing that when $\eta \neq 0$, marginal utilities of income are of the form $u_a'(y) = (\theta_a + \eta y)^{-1/\eta}$, while they are $u_a'(y) = e^{-y/\theta_a}$ in the case of Constant Absolute Risk Aversion (CARA), i.e. when $\eta = 0$. The case usually considered empirically most relevant in the literature corresponds to an absolute risk tolerance that is increasing with income $(\eta > 0)$, and to c*oefficients of relative risk aversion* $\rho_a(y) = -y u_a''(y) / u_a'(y)$ that decrease with income $(\theta_a < 0)$. The case of Constant Relative Risk Aversion (CRRA) corresponds to $\theta_a = 0$, in which case $\rho_a(y) = 1/\eta$.

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A first remark is that the aggregate probability π° defined in Proposition 3.1 depends in that case on the reference market portfolio ω°, but not on the distribution of incomes b_a° among investors. The proof of that remark, and its implications for the determination of the equivalent common probability π° and the adjustment coefficient r°, are derived in the following fact.

Corollary 4.5. *Assume that individual VNM utilities are state independent and belong to the HARA family, with $T_{ah}(y) = -u'_{ah}(y) / u''_{ah}(y) = \theta_a + \eta y > 0$. Assume interior solutions throughout.*

1) Let ω° be an arbitrary reference market portfolio. The representative investor who supports the equilibrium with the common probability π° in Proposition 3.1 belongs to the same HARA family, with $T_h(y) = -U'_h(y) / U''_h(y) = \overline{\theta} + \eta y$, and $\overline{\theta} = E_a[\theta_a]$. The corresponding common probability π° is given by

$$\pi_h^\circ = q_h^* \left(\overline{\theta} + \eta \omega_h^\circ\right)^{1/\eta} \Big/ \sum_k q_k^* \left(\overline{\theta} + \eta \omega_k^\circ\right)^{1/\eta}$$

when $\eta \neq 0$, and by $\pi_h^\circ = q_h^* \, e^{\omega_h^\circ/\theta} \Big/ \sum_k q_k^* \, e^{\omega_k^\circ/\overline{\theta}}$ in the CARA configuration $\eta = 0$. It depends upon the reference market portfolio ω°, but not on the income distribution b_a°.

2) Let the reference portfolio be equal to the actual market portfolio, $\omega^\circ = \overline{\omega}$. Let (π°, r°) be the common probability and adjustment coefficient associated to a given equilibrium with heterogenous beliefs $(q^*, (y_a^*))$ as in Proposition 4.1.

When $\eta \neq 0$, the normalized VNM marginal utilities of the aggregate investor (4.8) are

$$(4.13) \qquad U'_h(y_h) = R_o^* \left(\left(\overline{\theta} + \eta y_h\right) / \nu^*\right)^{-1/\eta},$$

with $\nu^* = E_a[\nu_a^*]$ and

$$(4.14) \qquad \nu_a^* = \frac{\theta_a + \eta y_{ah}^*}{(\pi_{ah}/q_h^*)^\eta} = \frac{q_o^* \, \theta_a + \eta q^* \cdot \omega_a}{\sum_k q_k^* \, (\pi_{ak}/q_k^*)^\eta},$$

where $q_o^* = \sum_h q_h^* = 1/R_o^*$ is the price of the riskless asset. The adjustment coefficient r° is then determined as in Corollary 4.2 by

$$(4.15) \qquad \sum_k q_k^* \left(\left(\overline{\theta} + \eta r^\circ \overline{\omega}_k\right) / \nu^*\right)^{1/\eta} = 1.$$

In the CARA configuration $\eta = 0$, the normalized VNM marginal utilities of the aggregate investor (4.8) are

$$(4.16) \qquad U_h'\left(y_h\right) = R_o^* \quad e^{-(y_h-\nu^*)/\overline{\theta}} \quad ,$$

with $\nu^* = E_a\left[\nu_a^*\right]$ and

$$(4.17)\ \nu_a^* = y_{ah}^* - \theta_a\,\mathrm{Log}\,\left(\pi_{ah}/q_h^*\right) = R_o^*\left[q^*\cdot\omega_a - \theta_a\sum_k q_k^*\,\mathrm{Log}\,\left(\pi_{ak}/q_k^*\right)\right].$$

In particular,

$$(4.18) \qquad U_h'\left(y_h\right) = R_o^*\,q_h^* \quad e^{-(y_h-\overline{\omega}_h)/\overline{\theta}} \Big/ e^{E_a\left[\left(\theta_a/\overline{\theta}\right)\,\mathrm{Log}\,\pi_{ah}\right]} \quad ,$$

so that the adjustment coefficient r°, determined as in Corollary 4.2, is obtained directly from the fondamentals through

$$(4.19) \qquad \sum_k \quad e^{(r^\circ-1)\overline{\omega}_k/\overline{\theta}} \quad e^{E_a\left[\left(\theta_a/\overline{\theta}\right)\,\mathrm{Log}\,\pi_{ak}\right]} = 1.$$

Given the adjustment coefficient r°, the common probability π° is in all cases obtained as in Corollary 4.2, with ω° replaced by $\overline{\omega}$. In the CARA case $\eta = 0$, one gets [8]

$$\pi_h^\circ = e^{(r^\circ-1)\overline{\omega}_h/\overline{\theta}} \quad e^{E_a\left[\left(\theta_a/\overline{\theta}\right)\,\mathrm{Log}\,\pi_{ah}\right]} \quad .$$

Proof. 1) The aggregate investor who supports the common probability equilibrium in Proposition 3.1 has VNM utilities defined through the standard CCAPM as in (4.10), with $r^\circ\omega^\circ$ replaced by ω° since there is no adjustment of the reference portfolio in that case. The fact that it belongs to the same HARA family is a direct consequence of (4.11), which is also valid here. Indeed

$$T_h\left(y_h\right) = E_a\left[T_{ah}\left(y_{ah}\right)\right] = E_a\left[\theta_a + \eta y_{ah}\right] = \overline{\theta} + \eta y_h.$$

so the marginal utilities $U_h'\left(y_h\right)$ of that aggregate investor are proportional (up to a common multiplicative normalizing factor) to $\left(\overline{\theta}+\eta y_h\right)^{-1/\eta}$ when $\eta\neq 0$ and to $e^{-y_h/\overline{\theta}}$ in the CARA case $\eta = 0$. The expressions of the common probability in 1) are obtained by remarking that, as in Corollary 4.2, one has the FOC : $\pi_h^\circ U_h'\left(\omega_h^\circ\right) = R_o^*\,q_h^*$, so that $\pi_h^\circ = \left(q_h^*/U_h'\left(\omega_h^\circ\right)\right)\Big/\left(\sum_k q_k^*/U_k'\left(\omega_k^\circ\right)\right)$.

2) Consider the case $\omega^\circ = \overline{\omega}$. When $\eta\neq 0$, it is easily seen from the individual FOC that normalized individual marginal VNM utilities $v_{ah}'\left(y_{ah}\right) = u_{ah}'\left(y_{ah}\right)\Big/E_{\pi_a}\left[u_{ah}'\left(y_{ah}^*\right)\right]$ in the observed equilibrium, are given by $v_{ah}'\left(y_{ah}\right) = R_o^*\left(\left(\theta_a+\eta y_{ah}\right)/\nu_a^*\right)^{-1/\eta}$, where ν_a^* is determined by (4.14) ($\lambda_a^* = \left(\nu_a^*\right)^{-1/\eta}$ is in fact the marginal expected utility of income $\lambda_a^* = \lambda_a\left(q^*, q^*\cdot\omega_a, \pi_a\right)$ corresponding to the specification $u_{ah}'\left(y_{ah}\right) = \left(\theta_a+\eta y_{ah}\right)^{-1/\eta}$). The normalized VNM utilities of the aggregate investor defined in (4.8) satisfy then

$U'_h(y_h) = v'_{ah}(y_{ah})$, hence $\nu^*_a (U'_h(y_h) / R^*_o)^{-\eta} = \theta_a + \eta y_{ah}$ (where the y_{ah} are solutions of (4.8)), which gives (4.13) by aggregation over all types a. The adjustment coefficient is obtained by solving $R^*_o \sum_k (q^*_k / U'_k (r^\circ \overline{\omega}_k)) = 1$ as in Corollary 4.2, which gives (4.15).

When $\eta = 0$, the same procedure shows that normalized individual VNM utilities are given by $v'_{ah}(y_{ah}) = R^*_o\ e^{-y_{ah}/\theta_a} / e^{-\nu^*_a/\theta_a}$ where ν^*_a is determined by (4.17) ($\lambda^*_a = e^{-\nu^*_a/\theta_a}$ is the marginal expected utility of income $\lambda^*_a = \lambda_a(q^*, q^* \cdot \omega_a, \pi_a)$ corresponding to the specification $u'_{ah}(y_{ah}) = e^{-y_{ah}/\theta_a}$), and one gets by aggregation over investors the general expression (4.16) of $U'_h(y_h)$. The specific expression (4.18) is obtained by using the first equality for ν^*_a in (4.17). Then it is immediate from (4.18) that the equation $\sum_k (R^*_o\ q^*_k / U'_k (r^\circ \overline{\omega}_k)) = 1$ to determine r°, as in Corollary 4.2, is (4.19).

It is also clear that given r°, π° is obtained in all cases from the FOC : $\pi^\circ_h = q^*_h R^*_o / U'_h (r^\circ \overline{\omega}_h)$, as in Corollary 4.2.   Q.E.D.

5   Individual Heterogeneity and Risk Sharing

Consider an observed equilibrium price system q^* corresponding to the heterogenous beliefs π_a. We take in this section the actual market portfolio as the reference portfolio, i.e. $\omega^\circ = \overline{\omega}$, and use the associated common probability equivalent equilibrium, as defined by the common probability π° and the adjustment coefficient r°, as a benchmark to study how heterogeneity of individual beliefs may explain observed individual portfolio diversity in conjunction which heterogeneity of incomes and of attitudes toward risk. Our strategy is to evaluate deviations $y^*_a - y^\circ_a$ of the observed equilibrium portfolios y^*_a from the portfolios y°_a that investors would hold at the common probability equivalent equilibrium (assumed both to be interior), as functions of the deviations $\pi_a - \pi^\circ$ of individual beliefs from the equivalent common probability. In the particular case where individual VNM utilities are state independent and when there is no aggregate risk ($\overline{\omega}_h = \overline{\omega}_k$), for instance, the approach allows to quantify departures of observed portfolios from full insurance (which should prevail then in the common probability equivalent equilibrium) as a result of the diversity of individual beliefs. Similarly, in the specific configuration of the HARA family (Example 4.4), the approach permits an evaluation of failures of individual observed portfolios to satisfy the socalled "two funds separation theorem" (i.e. to be combinations of the market portfolio and of the riskless asset), in relation to the heterogeneity of individual beliefs.

In the same vein, we analyze in this section under which conditions diversity of beliefs generates in the aggregate an adjustment coefficient r° that is greater or less, than one. We shall discuss in the next section some implications of such a scalar adjustment, that modifies the mean of aggregate wealth (consumption) as well as its variability accross states of nature, for aggregate risk aversion and asset pricing. We also reinterpret individual heterogeneity at the end of the present section in terms of monotone risk sharing rules.

Our starting point is the FOC conditions

$$(5.1) \qquad \pi_{ah} u'_{ah}\left(y^*_{ah}\right) = \pi^\circ_h u'_{ah}\left(y^\circ_{ah}\right) = \lambda^*_a q^*_h$$

where $\lambda^*_a = \lambda_a\left(q^*, q^* \cdot \omega_a, \pi_a\right)$ is the marginal expected utility of income of investor a in the observed equilibrium (as well as in the common probability equivalent equilibrium, i.e. $\lambda^*_a = \lambda^\circ_a = \lambda_a\left(q^*, b^\circ_a, \pi^\circ\right)$). These conditions determine indeed the variations $y^*_a - y^\circ_a$ of the equilibrium portfolios as functions of the deviations of the individual probabilities $\pi_a - \pi^\circ$, through

$$(5.2) \qquad y^*_{ah} - y^\circ_{ah} = g_{ah}\left(\pi_{ah}\right) - g_{ah}\left(\pi^\circ_h\right), \text{ where}$$

$$g_{ah}\left(\pi_{ah}\right) \overset{def}{\equiv} \left(u'_{ah}\right)^{-1}\left(\frac{\pi^\circ_h}{\pi_{ah}}\, u'_{ah}\left(y^\circ_{ah}\right)\right).$$

It is clear that $g_{ah}\left(\pi_{ah}\right)$ is an increasing function : an individual investor's investment in an AD security will be larger in the observed equilibrium, i.e. $y^*_{ah} \gtreqqless y^\circ_{ah}$, if and only if the investor believes the corresponding state to be more probable than specified in the common equivalent probability, i.e. if and only if $\pi_{ah} \gtreqqless \pi^\circ_h$. We already observed (see (3.6) and (3.7)) that this simple monotonicity property was a direct consequence of, and for all practical purposes, actually almost equivalent to our requirement that marginal individual valuations of assets remain invariant when constructing a common probability equivalent equilibrium.

We wish to gain further insight into the way in which deviations of individual beliefs from the common probability affect individual equilibrium portfolios, by considering exact 2nd order Taylor expansions of $g_{ah}\left(\pi_{ah}\right) - g_{ah}\left(\pi^\circ_h\right)$, where the π_{ah} are considered as "free" variables while π° and y°_a are "fixed". We know indeed that there exist $\widehat{\pi}_{ah}$ in the intervals $[\pi^\circ_h, \pi_{ah}]$ such that

$$(5.3) \qquad y^*_{ah} - y^\circ_{ah} = \left(\pi_{ah} - \pi^\circ_h\right) g'_{ah}\left(\pi^\circ_h\right) + \frac{1}{2}\left(\pi_{ah} - \pi^\circ_h\right)^2 g''_{ah}\left(\widehat{\pi}_{ah}\right).$$

One verifies by direct inspection that $g'_{ah}\left(\pi_{ah}\right) = T_{ah}(g_{ah}\left(\pi_h\right))/\pi_h$, where $T_{ah}\left(y\right) = -u'_{ah}\left(y\right)/u''_{ah}\left(y\right)$ is the individual coefficient of absolute risk tolerance, so that $g'_{ah}\left(\pi^\circ_h\right) = T^\circ_{ah}/\pi^\circ_h$, with $T^\circ_{ah} = T_{ah}\left(y^\circ_{ah}\right)$. The first (linear)

term in the right hand side of (5.3) is therefore positive when π_{ah} exceeds π_h° and negative otherwise. By contrast, the second (nonlinear) term has the sign of $g_{ah}''(\widehat{\pi}_{ah}) = T_{ah}(g_{ah}(\widehat{\pi}_{ah}))(T_{ah}'(g_{ah}(\widehat{\pi}_{ah})) - 1)/\widehat{\pi}_{ah}^2$. It contributes therefore to a negative portfolio deviation $y_{ah}^* - y_{ah}^\circ$ for *any* individual belief $\pi_{ah} \neq \pi_h^\circ$ whenever individual absolute risk tolerance increases with income (the empirically plausible case), but not too fast, that is when $T_{ah}'(y) < 1$, so that $g_{ah}(\pi_h)$ is a concave function, i.e. $g_{ah}''(\pi_h) < 0$. On the other hand this second nonlinear term will contribute to a positive portfolio deviation in the opposite configuration where absolute risk tolerance increases fast enough, $T_{ah}'(y) > 1$, i.e. when $g_{ah}(\pi_h)$ is convex. Even though the two terms in the right hand side of (5.3) may have different signs, the first linear term is bound to dominate since, by construction of the common probability equivalent equilibrium, the function $g_{ah}(\pi_h)$ is increasing : an upward individual belief deviation $\pi_{ah} - \pi_h^\circ \gtreqless 0$ does result in an upward portfolio shift $y_{ah}^* - y_{ah}^\circ \gtreqless 0$ and conversely.

The same approach gives a way to evaluate the impact of individual belief deviations on the income compensating changes $q^* \cdot \omega_a - b_a^\circ = q^* \cdot y_a^* - q^* \cdot y_a^\circ$ needed to keep individual marginal asset valuations invariant when constructing the equivalent common probability equilibrium. Summing over states the expressions (5.3) multiplied by q_h^* gives

$$(5.4) \quad q^* \cdot \omega_a - b_a^\circ = \sum_h (\pi_{ah} - \pi_h^\circ) q_h^* g_{ah}'(\pi_h^\circ) + \frac{1}{2} \sum_h (\pi_{ah} - \pi_h^\circ)^2 q_h^* g_{ah}''(\widehat{\pi}_{ah}).$$

Since both probabilities π_{ah} and π_h° sum to 1, the linear part in the right hand size of (5.4) involves positive and negative terms and therefore does not necessarily dominates the nonlinear terms here. In fact this linear part, being equal to [9]

$$(5.5) \qquad E_{\pi_a}\left[q_h^* \frac{T_{ah}^\circ}{\pi_h^\circ}\right] - E_{\pi^\circ}\left[q_h^* \frac{T_{ah}^\circ}{\pi_h^\circ}\right] = cov_{\pi^\circ}\left[\frac{\pi_{ah}}{\pi_h^\circ}, q_h^* \frac{T_{ah}^\circ}{\pi_h^\circ}\right],$$

has an ambiguous sign : we are going to see that it vanishes in particular when all individual utilities are state independent and if there is no aggregate risk. Indeed in that case the equivalent common probability π° coincides with the socalled "risk adjusted probability" $\pi_h^* = q_h^* R_\circ^*$ (for then aggregate VNM utilities defined by (4.8) are also state independent, so that $U_h'(r^\circ \overline{\omega}_h)$ is independent of h when $\overline{\omega}_h = \overline{\omega}_k$. Corollary 4.2 implies then that π_h° is proportional to q_h^*, hence the result). All individual investors are then fully insured in the equivalent common probability equilibrium, i.e. $y_{ah}^\circ = y_{ak}^\circ$, and the covariance in (5.5) vanishes since the expressions $q_h^* T_{ah}^\circ / \pi_h^\circ$ are independent of the

state h. Compensating income changes in (5.4) will be therefore dominated by the corresponding squared deviations terms in that case. In particular, individual observed income will have to be adjusted upward when constructing the equivalent common probability equilibrium, i.e. $q^* \cdot \omega_a - b_a^\circ < 0$, if absolute risk tolerance does not increase too fast, $T'_{ah}(y) < 1$, so as to ensure that $g_{ah}(\pi_h)$ is concave. By continuity, this picture is unaltered when aggregate risk and/or dependence of VNM utilities on states is "small". To sum up,

Proposition 5.1. *Suppose that every type satisfies (2.a) and that the market portfolio satisfies (2.c). Consider an interior equilibrium vector q^* of state prices and the corresponding interior common probability equivalent equilibrium defined by (π°, r°), with the reference portfolio equal to the market porfolio, $\omega^\circ = \overline{\omega}$, and where $y_a^* = y_a(q^*, q^* \cdot \omega_a, \pi_a)$ and $y_a^\circ = y_a(q^*, b_a^\circ, \pi^\circ)$ are the associated equilibrium individual portfolios. For each investor, let $T_{ah}^\circ = T_{ah}(y_{ah}^\circ)$, where $T_{ah}(y) = -u'_{ah}(y) / u''_{ah}(y)$ are the degrees of absolute risk tolerance.*

There exist $\widehat{\pi}_{ah}$ in the intervals $[\pi_h^\circ, \pi_{ah}]$ such that individual equilibrium portfolio adjustments $y_a^* - y_a^\circ$ are linked to deviations of individual beliefs $\pi_a - \pi^\circ$ through

$$(5.6) \qquad y_{ah}^* - y_{ah}^\circ = \frac{\pi_{ah} - \pi_h^\circ}{\pi_h^\circ} \, T_{ah}^\circ + \frac{1}{2} \left(\frac{\pi_{ah} - \pi_h^\circ}{\widehat{\pi}_{ah}} \right)^2 \widehat{T}_{ah} \left(\widehat{T}'_{ah} - 1 \right),$$

where the degrees of absolute risk tolerance $\widehat{T}_{ah} = T_{ah}(\widehat{y}_{ah})$ and their derivatives $\widehat{T}'_{ah} = T'_{ah}(\widehat{y}_{ah})$ are evaluated at the portfolios defined by $\widehat{\pi}_{ah} u'_{ah}(\widehat{y}_{ah}) = \pi_h^\circ u'_{ah}(y_{ah}^\circ)$. Income shifts needed to maintain individual marginal asset valuations when constructing the equivalent common probability equilibrium are in turn given by

$$(5.7) \; q^* \cdot \omega_a - b_a^\circ = cov_{\pi^\circ} \left[\frac{\pi_{ah}}{\pi_h^\circ}, q_h^* \frac{T_{ah}^\circ}{\pi_h^\circ} \right] + \frac{1}{2} \sum_h \left(\frac{\pi_{ah} - \pi_h^\circ}{\widehat{\pi}_{ah}} \right)^2 q_h^* \widehat{T}_{ah} \left(\widehat{T}'_{ah} - 1 \right).$$

The covariance term vanishes in the case of no aggregate risk, $\overline{\omega}_h = \overline{\omega}_k$, and when all individual VNM utilities are state independent, for then π° coincides with the risk adjusted probability $\pi_h^* = R_\circ^* q_h^*$, while investors are fully insured at the equivalent common probability equilibrium, i.e. $y_{ah}^\circ = y_{ak}^\circ$, hence $T_{ah}^\circ = T_{ak}^\circ$. In that case, when $\pi_a \neq \pi^\circ$, individual income has to be shifted upward, i.e. $q^* \cdot \omega_a - b_a^\circ < 0$ if absolute risk tolerance does not increase too fast with income, i.e. $\widehat{T}'_{ah} < 1$. Income is shifted downward, i.e. $q^* \cdot \omega_a - b_a^\circ > 0$, if absolute risk tolerance increases fast enough, i.e. $\widehat{T}'_{ah} > 1$.

These configurations are qualitatively preserved when aggregate risk and/or state dependence of VNM utilities is not too large. For instance, assume that individual absolute risk tolerance is bounded away from 0, i.e. $0 < \theta_{am} \leqq \widehat{T}_{ah}$ and does not increase too fast, i.e. $\widehat{T}'_{ah} \leqq \eta_a < 1$. Then

$$(5.8) \quad q^* \cdot \omega_a - b_a^\circ \leqq cov_{\pi^\circ} \left[\frac{\pi_{ah}}{\pi_h^\circ}, q_h^* \frac{T_{ah}^\circ}{\pi_h^\circ} \right] + \frac{1}{2} \theta_{am} \left(\eta_a - 1 \right) \sum_h \left(\pi_{ah} - \pi_h^\circ \right)^2 q_h^*$$

is negative if the covariance is not too great while $\eta_a < 1$ is small and the individual belief π_a differs significantly from the common probability π°.

As noted earlier, an important issue to ascertain is whether the adjustment coefficient r° is greater or less than 1 : modifying the mean and variability of aggregate wealth has consequences on aggregate risk aversion and asset pricing. The approach taken here should generate some insights into the matter since aggregation of individual portfolios in (5.6) yields $(1 - r^\circ) \, \overline{\omega}_h = E_a \left[y_{ah}^* - y_{ah}^\circ \right]$, while working with incomes in (5.7) gives $(1 - r^\circ) \, q^* \cdot \overline{\omega} = E_a \left[q^* \cdot \omega_a - b_a^\circ \right]$. From the above analysis, one should expect accordingly this coefficient r° to be greater than 1 if, for instance, aggregate risk and state dependence of VNM utilities are not too large, every investor's absolute risk tolerance does not increase too fast with income and when there is a significant dispersion of beliefs in the population.

Aggregation of individual portfolios in (5.6) or (5.3) gives

$$(5.9) \quad (1 - r^\circ) \overline{\omega}_h = E_a \left[\frac{\pi_{ah} - \pi_h^\circ}{\pi_h^\circ} T_{ah}^\circ \right] + \frac{1}{2} \, E_a \left[\left(\pi_{ah} - \pi_h^\circ \right)^2 g_{ah}'' \left(\widehat{\pi}_{ah} \right) \right].$$

We know that the contribution of the second order nonlinear terms is negative, and thus tends to make $r^\circ > 1$ if the functions g_{ah} are concave, i.e. if absolute risk tolerance does not increase too fast with income $(T'_{ah} < 1)$. On the other hand, the contributions of the first order linear terms are ambiguous since they may be positive for some states and negative for others. A convenient, symmetric way to proceed is to aggregate over states the expressions (5.9), premultiplied by π_h° / T_h°, where $T_h^\circ = E_a \left[T_{ah}^\circ \right]$ are the degrees of absolute risk tolerance $T_h(y) = -U_h'(y) / U_h''(y)$ of the representative equilibrium investor defined in (4.8) of Proposition 4.1, evaluated at the adjusted market portfolio, i.e. $T_h^\circ = T_h^\circ (r^\circ \overline{\omega}_h)$. If the representative investor's degrees of relative risk aversion are noted $\rho_h(y) = -y U_h''(y) / U_h'(y)$, with $\rho_h^\circ = \rho_h (r^\circ \overline{\omega}_h) = r^\circ \overline{\omega}_h / T_h^\circ$, one gets in this way

$$(5.10) \frac{1 - r^\circ}{r^\circ} E_{\pi^\circ} \left[\rho_h^\circ \right] = E_a \left[E_{\pi_a} \left[\frac{T_{ah}^\circ}{T_h^\circ} \right] - E_{\pi^\circ} \left[\frac{T_{ah}^\circ}{T_h^\circ} \right] \right] + \frac{1}{2} E_{a,\pi^\circ} \left[\left(\pi_{ah} - \pi_h^\circ \right)^2 \frac{g_{ah}'' \left(\widehat{\pi}_{ah} \right)}{T_h^\circ} \right].$$

The aggregate first order linear term can be thus written (we use here again the change of probability formula of footnote 9)

$$E_a \left[cov_{\pi^\circ} \left[\frac{\pi_{ah}}{\pi_h^\circ}, \frac{T_{ah}^\circ}{T_h^\circ} \right] \right],$$

and measures the average contribution, in the population, of the covariance of the relative deviations of individual beliefs from the equivalent common probability, i.e. $(\pi_{ah} - \pi_h^\circ) \big/ \pi_h^\circ$, with the relative deviations of individual absolute risk tolerance from the average degree of absolute risk tolerance in the market, evaluated at the equivalent common probability equilibrium, i.e. $(T_{ah}^\circ - E_a[T_{ah}^\circ]) \big/ E_a[T_{ah}^\circ]$. The sign of this term is ambiguous, but it will vanish, for instance, if the ratios $T_{ah}^\circ \big/ T_h^\circ$ are, for every investor, independent of the state. We know already that this will be the case if there is no aggregate risk and if all investor's VNM utilities are state independent, since then T_{ah}° and T_h° are actually independent of the state h (individual investors and the representative investor are fully insured in the common probability equilibrium). We shall see shortly that this configuration occurs also in another important special case, namely when VNM utilities are state independent and display linear absolute risk tolerance as in the HARA family considered in Example 4.4. This will be a simple restatement of the socalled "two funds separation theorem". In all these cases, i.e. when the first order linear term in (5.10) vanishes, the sign of $(1 - r^\circ)$ is entirely determined by the second order terms, i.e. by the convexity or concavity of the functions g_{ah}. For instance, one gets in such a case that $r^\circ > 1$ if there is some dispersion of beliefs, when every investor's absolute risk tolerance does not increase too fast with income, i.e. when $T'_{ah} < 1$ for all a, h. If the first order covariance term is not too large, one should expect this picture to be unchanged : for example one should still get $r^\circ > 1$ if $T'_{ah} \leqq \eta < 1$ for all a, h, with η small and if the variance of beliefs in the population is significant. The following result sums up and make precise these intuitions.

Corollary 5.2. *Under the assumptions and notations of Proposition 5.1, let the degrees of absolute risk tolerance of the representative equilibrium investor defined in (4.8) of Proposition 4.1, be given by $T_h(y) = -U'_h(y) \big/ U''_h(y)$, with $T_h^\circ = T_h(r^\circ \overline{\omega}_h) = E_a[T_{ah}^\circ]$. Let also the representative investor's degrees of relative risk aversion be noted $\rho_h(y) = -y U''_h(y) \big/ U'_h(y)$, with $\rho_h^\circ = \rho_h(r^\circ \overline{\omega}_h)$.*

The adjustment coefficient r° of the market portfolio is given by

$$(5.11) \quad \frac{1 - r^\circ}{r^\circ} E_{\pi^\circ}[\rho_h^\circ] = E_a \left[cov_{\pi^\circ} \left[\frac{\pi_{ah}}{\pi_h^\circ}, \frac{T_{ah}^\circ}{T_h^\circ} \right] \right] + \tfrac{1}{2} E_{\pi^\circ, a} \left[\left(\frac{\pi_{ah} - \pi_h^\circ}{\widehat{\pi}_{ah}} \right)^2 \frac{\widehat{T}_{ah}}{T_h^\circ} \left(\widehat{T}'_{ah} - 1 \right) \right]$$

The covariance term vanishes when the ratios T_{ah}° / T_h° are independent of the states for every investor. This happens in particular when all investors have state independent VNM utilities and when there is no aggregate risk. Then $r^\circ > 1$ when $\widehat{T}'_{ah} < 1$ for all a, h, whereas $r^\circ < 1$ when $\widehat{T}'_{ah} > 1$ for all a, h, if there is some dispersion of beliefs.

Assume that individual absolute risk tolerance is bounded above and away from 0, i.e. $0 < \theta_m \leqq \widehat{T}'_{ah} \leqq \theta_M$ for all states and investors.

A) If absolute risk tolerance does not increase fast with income, i.e. $\widehat{T}'_{ah} \leqq \eta < 1$ for all a, h

$$(5.12) \quad \frac{1 - r^\circ}{r^\circ} E_{\pi^\circ} \left[\rho_h^\circ \right] \leqq E_a \left[cov_{\pi^\circ} \left[\frac{\pi_{ah}}{\pi_h^\circ}, \frac{T_{ah}^\circ}{T_h^\circ} \right] \right] - \frac{(1 - \eta)\theta_m}{2\theta_M} E_{\pi^\circ} \left[var_a \left[\pi_{ah} \right] \right],$$

so that $r^\circ > 1$ if the first order covariance term is not too great whereas η is significantly less than 1 and if there is a significant dispersion of beliefs in the population ($var_a \left[\pi_{ah} \right]$ is significant).

B) If absolute risk tolerance increases fast enough with income, i.e. $\widehat{T}'_{ah} \geqq \eta > 1$ for all a, h

$$(5.13) \quad \frac{1 - r^\circ}{r^\circ} E_{\pi^\circ} \left[\rho_h^\circ \right] \geqq E_a \left[cov_{\pi^\circ} \left[\frac{\pi_{ah}}{\pi_h^\circ}, \frac{T_{ah}^\circ}{T_h^\circ} \right] \right] + \frac{(\eta - 1)\theta_m}{2\theta_M} E_{\pi^\circ} \left[var_a \left[\pi_{ah} \right] \right],$$

so that $r^\circ < 1$ if the first order covariance term is not too great while η is large and if there is a significant dispersion of beliefs in the population($var_a \left[\pi_{ah} \right]$ is significant).

Proof. (5.11) is nothing else than (5.10) in the text. Then (5.12) and (5.13) follow directly by bounding the right hand side of (5.11), using $\widehat{\pi}_{ah} \leqq 1$ and the fact that $E_a \left[(\pi_{ah} - \pi_h^\circ)^2 \right] \geqq var_a \left[\pi_{ah} \right]$. Q.E.D.

The above analysis relies upon an *exact* 2nd order Taylor expansion of $y_{ah}^* - y_{ah}^\circ$ around π_h°, that is then aggregated over investors and accross states. One may note, incidentally, that if one is willing to neglect 3rd order terms involving $(\pi_{ah} - \pi_h^\circ)^3$, e.g. in practical applications, one can obtain *approximate* evaluations of individual portfolio deviations in (5.6), of individual income shifts in (5.7), or of the aggregate portfolio adjustment coefficient r° in (5.11), by substituting π_h° to $\widehat{\pi}_{ah}$ in these expressions.

Example 5.3. *The HARA family*

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We focus again on the specific case considered in example 4.4, where individual VNM utilities are state independent and display linear risk tolerance, i.e. $T_{ah}(y) = \theta_a + \eta y > 0$ and where all investors share the same marginal risk tolerance $T'_{ah}(y) = \eta$. This case involves a well known simplification when beliefs are homogenous, called "two funds separation" : every investor holds in equilibrium a portfolio that is a combination of the market portfolio and of the riskless asset. This property holds by construction in the equivalent common probability equilibrium since there all investors are supposed to share the same belief π° : it is actualy expressed, as we shall see shortly, by the fact that the ratios T°_{ah}/T°_h are, for each investor, independent of the state. The approach presented here then generates an evaluation of individual departures from this two funds separation property, in the observed equilibrium, as a function of the deviations of individual heterogenous beliefs π_a from π°. Also the fact that the ratios T°_{ah}/T°_h are independent of the state implies that the covariance term in (5.11) of Corollary 5.2 vanishes, with the consequence that the adjustment coefficient r° is greater than 1 if and only if the common marginal risk tolerance $T'_{ah}(y) = \eta$ is less than 1.

Our point of departure is here again the FOC conditions (the equivalent of (5.1) above) that characterize an observed equilibrium and the corresponding equivalent common probability equilibrium, assumed both to be interior. When $\eta \neq 0$, these FOC are obtained by applying (4.14) in Corollary 4.5 to both equilibrium portfolios y^*_a and y°_a :

$$
\begin{aligned}
(5.14) \qquad \nu^*_a &= \frac{\theta_a + \eta y^*_{ah}}{(\pi_{ah}/q^*_h)^\eta} = \frac{q^*_\circ \theta_a + \eta q^* \cdot \omega_a}{\sum_k q^*_k (\pi_{ak}/q^*_k)^\eta} \\
&= \frac{\theta_a + \eta y^\circ_{ah}}{(\pi^\circ_h/q^*_h)^\eta} = \frac{q^*_\circ \theta_a + \eta b^\circ_a}{\sum_k q^*_k (\pi^\circ_k/q^*_k)^\eta}
\end{aligned}
$$

In the same way, in the CARA configuration ($\eta = 0$) one gets from (4.17) in Corollary 4.5,

$$
(5.15) \quad
\begin{aligned}
\nu^*_a &= y^*_{ah} - \theta_a \mathrm{Log}(\pi_{ah}/q^*_h) = R^*_\circ[q^* \cdot \omega_a - \theta_a \sum_k q^*_k (\pi_{ak}/q^*_k)] \\
&= y^\circ_{ah} - \theta_a \mathrm{Log}(\pi^\circ_h/q^*_h) = R^*_\circ[b^\circ_a - \theta_a \sum_k q^*_k (\pi^\circ_k/q^*_k)].
\end{aligned}
$$

Then a straithtforward manipulation of these FOC conditions generates the following facts.

Corollary 5.4. *Consider the HARA family as in Corollary 4.5. Let an observed equilibrium and the corresponding equivalent common probability equilibrium, assumed both to be interior, with the reference portfolio equal to the market portfolio. Then*

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A) (Two funds separation) In the equivalent common probability equilibrium, investors hold a portfolio y_a° that is a (possibly investor dependent) combination of the market portfolio $\overline{\omega}$ and of the riskless asset that gives one unit of income in every state. When $\eta \neq 0$, this is expressed by the fact that the ratios

$$(5.16) \qquad \frac{T_{ah}^\circ}{T_h^\circ} = \frac{\theta_a + \eta y_{ah}^\circ}{\overline{\theta} + \eta r^\circ \overline{\omega}_h} = \frac{q_\circ^* \theta_a + b_a^\circ}{q_\circ^* \overline{\theta} + \eta r^\circ q^* \cdot \overline{\omega}} = \frac{\nu_a^*}{\nu^*} ,$$

where $T_h^\circ = E_a\left[T_{ah}^\circ\right]$ and $\nu^* = E_a\left[\nu_a^*\right]$, are independent of the state. In the CARA configuration $\eta = 0$

$$(5.17) \qquad \frac{y_{ah}^\circ}{\theta_a} - \frac{r^\circ \overline{\omega}_h}{\overline{\theta}} = R_\circ^* \left[\frac{b_a^\circ}{\theta_a} - r^\circ \frac{q^* \cdot \overline{\omega}}{\overline{\theta}}\right] = \frac{\nu_a^*}{\theta_a} - \frac{\nu^*}{\overline{\theta}} .$$

Departures of individual portfolios y_a^* in the observed equilibrium from this two funds separation property are measured by

$$(5.18) \qquad \frac{\theta_a + \eta y_{ah}^*}{\theta_a + \eta y_{ah}^\circ} = \left(\frac{\pi_{ah}}{\pi_h^\circ}\right)^\eta$$

when $\eta \neq 0$ and by

$$(5.19) \qquad y_{ah}^* - y_{ah}^\circ = \theta_a \left(\mathrm{Log}\pi_{ah} - \mathrm{Log}\pi_h^\circ\right)$$

in the CARA configuration $\eta = 0$.

B) (Compensating income shifts) Income variations needed to keep invariant individual marginal asset valuations in both equilibria are given by

$$(5.20) \qquad \frac{q_\circ^* \theta_a + \eta \; q^* \cdot \omega_a}{q_\circ^* \theta_a + \eta \; b_a^\circ} = \frac{\sum_k q_k^* \left(\pi_{ak}/q_k^*\right)^\eta}{\sum_k q_k^* \left(\pi_k^\circ/q_k^*\right)^\eta}$$

when $\eta \neq 0$ and by

$$(5.21) \qquad q^* \cdot \omega_a - b_a^\circ = \theta_a \sum_k q_k^* \left(\mathrm{Log}\pi_{ak} - \mathrm{Log}\pi_k^\circ\right)$$

in the CARA configuration.

Proof. The two funds separation property (5.16) or (5.17) is obtained by aggregating over all investors the FOC condition in (5.14) or (5.15) that is relative to the equivalent common probability equilibrium, and by taking into account $E_a\left[y_a^\circ\right] = r^\circ \overline{\omega}$ and $E_a\left[b_a^\circ\right] = r^\circ q^* \cdot \overline{\omega}$. Then individual deviations from the two funds separation (5.18) or (5.19), as well as the income compensations

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(5.20) or (5.21), are obtained by division in (5.14) (substraction in (5.15)) of the FOC relative to one equilibrium by the FOC corresponding to the other.

Q.E.D.

A simple consequence of the above result is that there is no compensating income variation, i.e. $q^* \cdot \omega_a = b_a^\circ$, in the case of logarithmic utilities $(\eta = 1, \ u_{ah}(y) = \text{Log}(\theta_a + y))$. Indeed the numerator and denominator of the right hand side of (5.20) are both equal to 1 for any belief π_a and any common probability π° when $\eta = 1$. On the other hand, we know from Proposition 5.1 that when there is no aggregate risk, individual incomes have to be shifted upward if $\eta < 1$ and downward if $\eta > 1$ for all beliefs $\pi_a \neq \pi^\circ$. From the same proposition, we also know that these features are preserved by continuity when aggregate risk remains small, and if η differs from 1, at least when the belief π_a is significantly far from π°. We give in the following a direct proof of these statements that relies on the convexity properties of the function $f(x) = x^\eta$ and that will also imply that these properties should not be expected to prevail for every belief π_a.

Corollary 5.5. *(Compensating income shifts) In the HARA family,*

1) There is no compensating income variation, i.e. $q^* \cdot \omega_a = b_a^\circ$, *in the case of logarithmic VNM utilities* $\eta = 1$.

2) Assume that absolute risk tolerance does not increase too fast, i.e. $\eta < 1$. *If there is no aggregate risk, then* π° *coincides with the risk adjusted probability* $\pi_h^* = R_\circ^* q_h^*$ *and* $q^* \cdot \omega_a < b_a^\circ$ *for all beliefs* $\pi_a \neq \pi^\circ$. *If there is some aggregate risk so that* $\pi^\circ \neq \pi^*$, *one still has* $q^* \cdot \omega_a < b_a^\circ$ *if* π° *is near* π^* *(aggregate risk is small) and if* π_a *differs significantly from* π^*, *but the inequality is reversed i.e.* $q^* \cdot \omega_a > b_a^\circ$, *if* π_a *is equal or close enough to* π^*.

3) The same statements hold with reverse inequalities throughout when $\eta > 1$.

Proof. We detail the proof when $\eta > 1$, the argument is similar when $\eta < 1$. If $\eta > 1$, since the function $f(x) = x^\eta$ is convex and increasing for $x > 0$, the right hand side (RHS) of (5.20) is bounded below by

$$(5.22) \qquad \frac{E_{\pi^*}\left[(\pi_{ah}/\pi_h^*)^\eta\right]}{E_{\pi^*}\left[(\pi_h^\circ/\pi_h^*)^\eta\right]} \geqq \frac{1}{E_{\pi^*}\left[(\pi_h^\circ/\pi_h^*)^\eta\right]} \ ,$$

and the minimum is actually reached if and only if $\pi_a = \pi^*$. So if there is no aggregate risk, one has $\pi^\circ = \pi^*$, the RHS of (5.22) is equal to 1 : the

left hand side (LHS) is thus greater than 1, hence $q^* \cdot \omega_a > b_a^\circ$, for every $\pi_a \neq \pi^* = \pi^\circ$. If there is some aggregate risk, so that $\pi^\circ \neq \pi^*$, the RHS of (5.22) is less than 1. By continuity, the LHS of (5.22) still exceeds 1, hence one still has $q^* \cdot \omega_a > b_a^\circ$, if π^* is close to π° (the RHS is close to 1) and π_a differs significantly from π^*(the LHS is significantly far from the RHS in (5.22)). But if π_a is equal or close enough to π^* and $\pi^\circ \neq \pi^*$, the LHS of (5.22) is equal or close to the RHS and is therefore less than 1, so one gets $q^* \cdot \omega_a < b_a^\circ$. Similar arguments go through when $\eta < 1$ or in the CARA configuration $\eta = 0$.   Q.E.D.

It is remarkable that despite this ambiguity at the individual level about the direction of income shifts between the observed and the equivalent common probability equilibria, the fact that the *aggregate* coefficient of adjustment r° is greater or less than 1 is nevertheless entirely determined, in the HARA family, by whether the marginal risk tolerance η is less or greater than 1, *no matter what is the distribution of individual beliefs π_a*. This property is a consequence of the general result stated in Corollary 5.2, and more particularly in (5.11), since the ratios T_{ah}° / T_h° are independent of the state so that the linear covariance term collapses to 0 in that case (this is implied by (5.16) above when $\eta \neq 0$, and by $T_{ah}^\circ / T_h^\circ = \theta_a / \overline{\theta}$ when $\eta = 0$). Thus in the HARA family, with the notations of Corollary 5.2,

$$(5.23) \qquad \frac{r^\circ - 1}{r^\circ} E_{\pi^\circ}\left[\rho_h^\circ\right] = \frac{1-\eta}{2} E_{a,\pi^\circ}\left[\left(\frac{\pi_{ah} - \pi_h^\circ}{\widehat{\pi}_{ah}}\right)^2 \frac{\widehat{T}_{ah}}{T_h^\circ}\right].$$

The adjustment coefficient r° is therefore greater than, equal to or less than 1 according to whether $\eta < 1, \eta = 1,$ or $\eta > 1$. We give next a direct argument that relies here again on the convexity or concavity of the function $f(x) = x^\eta$ for $x > 0$.[10]

Corollary 5.6. *(Adjustment coefficient). In the HARA family as in Corollary 5.4, there is no adjustment of the market portfolio, i.e. $r^\circ = 1$, in the case of logarithmic utilities $\eta = 1$. The market portfolio is adjusted upward, i.e. $r^\circ > 1$, when $\eta < 1$ and in the CARA configuration $\eta = 0$, and downward, i.e. $r^\circ < 1$, when $\eta > 1$.*

Proof. The fact that $r^\circ = 1$ when $\eta = 1$ is clear since there is then no adjustment of indivual incomes, i.e. $q^* \cdot \omega_a = b_a^\circ$ for every investor. In all cases, from Corollary 4.2, the adjustment coefficient r° is obtained by solving $R_\circ^* \sum_k \left(q_k^* / U_k'\left(r^\circ \overline{\omega}_k\right)\right) = 1$, or equivalently in the case of the HARA family, (4.15) when $\eta \neq 0$ and (4.19) in the CARA configuration $\eta = 0$. Since

marginal utilities of the aggregate investor are decreasing, we shall get $r^\circ < 1$ if and only if $R_\circ^* \sum_k (q_k^* / U_k' (\overline{\omega}_k)) > 1$. When $\eta \neq 0$, this condition can be written in view of (4.15), $\sum_k q_k^* \left(\left(\overline{\theta} + \eta \overline{\omega}_k \right) / \nu^* \right)^{1/\eta} > 1$. But aggregating over investors the FOC (5.14) relative to the observed portfolio y_a^* gives that $q_k^* \left(\overline{\theta} + \eta \overline{\omega}_k \right)^{1/\eta} = \left(E_a \left[(\pi_{ak})^\eta \nu_a^* \right] \right)^{1/\eta}$. So we get $r^\circ < 1$ if and only if

$$ \sum_k \left(E_a \left[(\pi_{ak})^\eta \frac{\nu_a^*}{\nu^*} \right] \right)^{1/\eta} > 1. $$

It is easily seen that this inequality is verified when $\eta > 1$. Indeed since the function $f(x) = x^\eta$ is in that case increasing and convex for $x > 0$, one has that $\left(E_a \left[(\pi_{ak})^\eta (\nu_a^*/\nu^*) \right] \right)^{1/\eta} > E_a \left[\pi_{ak} \nu_a^*/\nu^* \right]$, hence the desired result by summing over k. A similar (symmetric) direct reasonning shows that this inequality is reversed, and that one gets accordingly $r^\circ > 1$, when $\eta < 1$, $\eta \neq 0$ and also for the CARA configuration. Q.E.D.

Remark 5.7. *Monotone risk sharing rules*

In what precedes, heterogeneity of individual equilibrium portfolios y_a^* was studied in relation with heterogeneity of beliefs π_a, by comparison with a common probability equivalent equilibrium that was defined exclusively in terms of marginal asset pricing invariance conditions. We look now at an alternative, essentially equivalent formulation, that is cast in terms of (optimal) risk sharing rules and may be useful, in particular in the focal case where VNM utilities are state independent. One standard approach, in the case of homogenous beliefs, is to view equilibrium consumptions y_{ah}^* as resulting from an allocation to individuals of the macroeconomic risks implied by the variations accross states of aggregate income or consumption $\overline{\omega}_h$. The corresponding risk sharing rule is then, when utilities are state independent, *monotone* : for each investor a, consumption y_{ah}^* is increasing with aggregate consumption $\overline{\omega}_h$. We show briefly here that our results can be reinterpreted as generalizing this approach to the case of heterogenous beliefs, by decomposing the risk sharing rule in two : one that reproduces the standard approach with the commonly shared probability π°, and the other that specifies a "residual risk" sharing rule in a way that is also *monotone* with respect to the individual belief deviations $\pi_{ah} - \pi_h^\circ$.

Specifically, consider an equilibrium $q^*, (y_a^*)$ with heterogenous beliefs π_a. It is known that such an equilibrium is a Pareto optimum given these beliefs, or more specifically that the individual portfolios (y_a^*) are the solutions of the maximization problem

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(5.24)   Max $E_a \left[E_{\pi_a} \left[v_{ah} \left(y_{ah} \right) \right] \right]$ subject to $E_a \left[y_a \right] = y$,

when y is equal to the market portfolio $\overline{\omega}$, where individual VNM utilities $v_{ah}(y_{ah}) = u_{ah}(y_{ah}) \big/ E_{\pi_a} \left[u'_{ah} \left(y^*_a \right) \right]$ have been normalized as in Proposition 4.1. Indeed, this problem splits into independent maximization problems, for each state h

(5.25) $W_h \left(y_h \; ; \; (\pi_{bh}) \right) = $ Max $E_a \left[\pi_{ah} v_{ah}(y_{ah}) \right]$ subject to $E_a \left[y_{ah} \right] = y_h$.

Assuming interior portfolios throughout, the solution to (5.25) is characterized by $\pi_{ah} v'_{ah}(y_{ah}) = W'_h \left(y_h \; ; \; (\pi_{bh}) \right)$, which implies $y_{ah} = y^*_{ah}$ for every a when $y_h = \overline{\omega}_h$ (because $v'_{ah}(y_{ah}) \big/ v'_{ah}(y^*_{ah})$ is then independent of a and $E_a \left[y_{ah} \right] = E_a \left[y^*_{ah} \right]$), and therefore

(5.26) $\pi_{ah} v'_{ah}(y^*_{ah}) = W'_h \left(\overline{\omega}_h \; ; \; (\pi_{bh}) \right) = q^*_h R^*_\circ.$

The aggregate investor with separable (but non VNM expected) utility $\sum_h W_h \left(y_h \; ; \; (\pi_{bh}) \right)$ does accordingly "represent" the economy in equilibrium, in the sense that the market portfolio $\overline{\omega}$ maximizes his preferences under the aggregate budget constraint $q^* \cdot y = q^* \cdot \overline{\omega}$. One may equivalently state that *the competitive equilibrium mechanism generates a risk sharing rule*

$$
\begin{aligned}
(5.27) \quad y^*_{ah} \;&=\; \Phi_{ah} \left(\overline{\omega}_h \; ; \; (\pi_{bh}) \right) \;=\; \left(v'_{ah} \right)^{-1} \left(W'_h \left(\overline{\omega}_h \; ; \; (\pi_{bh}) \right) \big/ \pi_{ah} \right) \\
&=\; \left(v'_{ah} \right)^{-1} \left(q^*_h R^*_\circ \big/ \pi_{ah} \right)
\end{aligned}
$$

that is (Pareto) optimal, conditionnally upon the investors' beliefs (π_b).

Such a Pareto optimal risk sharing rule displays attractive monotonicity properties whenever all individual investors share the same belief, i.e. $\pi_a = \pi$ for all a. In that case, the aggregate utilities $W_h \left(y_h \; ; \; (\pi_h) \right)$ in (5.25) coincide with $\pi_h U_h \left(y_h \right)$ where $U_h \left(y_h \right)$ are the normalized VNM utilities of the aggregate investor involved in the CCAPM of section 4.1. The corresponding equilibrium sharing rule becomes then quite simple and has the property, when VNM utilities are state independent, that an investor's consumption y^*_{ah} depends only on aggregate wealth $\overline{\omega}_h$, and increases with $\overline{\omega}_h$ in proportion of the relative contribution of the individual absolute risk tolerance to aggregate absolute risk tolerance.

(5.28) *(Risk sharing with homogenous beliefs). Consider an interior equilibrium $q^*, (y^*_a)$ with homogenous beliefs, i.e. $\pi_a = \pi$ for all investors a. The corresponding risk sharing rule is given by*

$$
y^*_{ah} = \varphi_{ah} \left(\overline{\omega}_h \right) = \left(v'_{ah} \right)^{-1} \left(U'_h \left(\overline{\omega}_h \right) \right) = \left(v'_{ah} \right)^{-1} \left(q^*_h R^*_\circ \big/ \pi_h \right)
$$

where normalized individual and aggregate investors' VNM utilities (v_{ah}) and (U_h) are defined as in Proposition 4.1. The risk sharing rule $\varphi_{ah}(y) \equiv (v'_{ah})^{-1}(U'_h(y))$ is increasing, with $\varphi'_{ah}(y) = T_{ah}(\varphi_{ah}(y)) / T_h(y) > 0$, where $T_{ah}(y) = -v'_{ah}(y) / v''_{ah}(y)$ and $T_h(y) = -U'_h(y) / U''_h(y) = E_a[T_{ah}(\varphi_{ah}(y))]$ are the individual and aggregate degrees of absolute risk tolerance.

When all VNM utilities are state independent, so is the risk sharing rule $\varphi_{ah}(y) = (v'_a)^{-1}(U'(y))$. In that case, individual consumption $y^*_{ah} = \varphi_a(\overline{\omega}_h)$ depends only on aggregate consumption $\overline{\omega}_h$ and increases with aggregate consumption in the sense that $y^*_{ak} - y^*_{ah} = \varphi_a(\overline{\omega}_k) - \varphi_a(\overline{\omega}_h) \gtreqless 0$ if and only if $\overline{\omega}_k \gtreqless \overline{\omega}_h$ (mutuality principle).

When individual beliefs π_a are heterogenous, the risk sharing rule Φ_{ah} (5.27) is still increasing in $\overline{\omega}_h$, provided that one keeps fixed all investors' beliefs (π_{bh}). So in the particular case where VNM utilities are state independent, one does get a weak form of the mutuality principle (Varian (1985, 1989)), in the sense that if one considers two states, one will get $y^*_{ak} = y^*_{ah}$ (resp. $y^*_{ak} > y^*_{ah}$) when $\overline{\omega}_k = \overline{\omega}_h$ (resp. $\overline{\omega}_k > \overline{\omega}_h$) provided that the distribution of beliefs (π_{bk}) and (π_{bh}) among investors is the same in both states, which is a serious limitation. The construction of an equivalent common probability equilibrium, and its use to study heterogeneity of individual portfolios, can be reinterpreted as a decomposition of the equilibrium risk sharing rule (5.27) in two parts. The first part defines, exactly as in (5.28), a risk sharing rule $y^\circ_{ah} = \varphi^\circ_{ah}(\overline{\omega}_h)$ associated to the equilibrium with the commonly shared probability π° : that part generates by construction the same simple mutuality principles as stated in (5.28). The second part then formulates a rule intended to allocate the "residual risks" due to the deviations $\pi_{ah} - \pi^\circ_h$ of individual beliefs from the common probability, of the form $y^*_{ah} - y^\circ_{ah} = \Phi_{ah}(\overline{\omega}_h ; (\pi_{bh})) - \varphi^\circ_{ah}(\overline{\omega}_h)$. By construction, this residual risk sharing rule has the attractive property, under our aggregation procedure, to be *monotone* in individual subjective probability deviations, i.e. $y^*_{ah} - y^\circ_{ah} \gtreqless 0$ if and only if $\pi_{ah} - \pi^\circ_h \gtreqless 0$, and in fact this monotonicity requirement is essentially equivalent to the formulation of the text (when there is a large number of states, and when individual and aggregate probabilities π_a and π° approximate probabilities with continuous densities on a common interval, see (3.8)). The cost to be paid to get monotonicity of the residual risks sharing rule, being that the allocation of risks under the hypothesis that all investors share the common probability π°, adds up to a scalarly adjusted market portfolio instead of the actual market portfolio, i.e. $E_a[y^\circ_a] = r^\circ \overline{\omega}$, the adjustment coefficient r° reflecting an aggregation bias due to the diversity of beliefs π_a.

Corollary 5.8. *(Monotone risk sharing rules with heterogenous beliefs). Consider an interior equilibrium $q^*, (y_a^*)$ with heterogenous beliefs π_a. There exists a probability π° and an adjustment coefficient r° of the market portfolio $\overline{\omega}$ such that the Pareto optimal risk sharing rule (5.27), $y_{ah}^* = \Phi_{ah}(\overline{\omega}_h \; ; \; (\pi_{bh}))$ can be decomposed in two parts.*

1) A risk sharing rule associated with the common probability π° and the adjusted market portfolio $r^\circ \overline{\omega}$

$$y_{ah}^\circ = \varphi_{ah}(r^\circ \overline{\omega}_h) = (v_{ah}')^{-1}(U_h'(r^\circ \overline{\omega}_h)) = (v_{ah}')^{-1}(q_h^* R_\circ^* / \pi_h^\circ),$$

where individual and aggregate VNM utilities v_{ah} and U_h are normalized as in Proposition 4.1. The common probability sharing rule $\varphi_{ah}(y) = (v_{ah}')^{-1}(U_h'(y))$ satisfies then the mutuality principle properties stated in (5.28).

2) A residual risks sharing rule

$$y_{ah}^* - y_{ah}^\circ = (v_{ah}')^{-1}\left(\frac{\pi_h^\circ}{\pi_{ah}}v_{ah}'(\varphi_{ah}(r^\circ \overline{\omega}_h))\right) - \varphi_{ah}(r^\circ \overline{\omega}_h)$$

satisfying the monotonicity property $y_{ah}^* \gtreqless y_{ah}^\circ$ if and only if $\pi_{ah} \gtreqless \pi_{ah}^\circ$. [11,12]

The results of this section can then be reinterpreted as analyzing the determinants of these risk sharing rules, in particular of their deviations $y_{ah}^* - y_{ah}^\circ$ and $(1 - r^\circ)$ from the common probability configuration, in relation to aggregate risk, and to heterogeneity of beliefs, of attitudes toward risk and of incomes. When specialized to the HARA family where $T_{ah}(y) = \theta_a + \eta y$, it is known that the optimal (state independent) risk sharing rule, in the case of homogenous beliefs, is *linear* (Wilson (1968)). This is easily verified here by differentiating once more the risk sharing rule $\varphi_{ah}(y)$ given in (5.28), or 1) of Corollary 5.8, which generates

$$(5.29) \qquad \varphi_{ah}''(y) = \frac{T_{ah}(\varphi_{ah}(y))}{(T_h(y))^2}(T_{ah}'(\varphi_{ah}(y)) - T_h'(y)),$$

hence $\varphi_{ah}''(y) = 0$ since $T_{ah}'(y) = \eta = T_h'(y)$ in the HARA family. One verifies also by looking at the common probability equivalent equilibrium (π°, r°), that the linearity of the corresponding (state independent) risk sharing rule $y_{ah}^\circ = \varphi_{ah}(r^\circ \overline{\omega}_h)$ is in fact equivalent to the "two funds separation" property that prevails in equilibrium for the HARA family in the case of homogenous beliefs (see (5.16) in Corollary 5.4 for $\eta = 0$ and (5.17) for the CARA configuration $\eta = 0$). Our analysis shows in fact that, *in the case of HARA family, linearity of the equilibrium risk sharing rule with respect to aggregate*

consumption extends also to the residual risk sharing rule 2) of Corollary 5.8 and therefore to the overall risk sharing rule (5.27), $y_{ah}^* = \Phi_{ah}\left(\overline{\omega}_h \; ; \; (\pi_{bh})\right),$ given the investors heterogenous beliefs and (π°, r°). Indeed consideration of (5.16) - (5.19) in Corollary 5.4 above generates

$$(5.30) \; \theta_a + \eta y_{ah}^* = \left(\frac{\pi_{ah}}{\pi_h^\circ}\right)^\eta (\theta_a + \eta y_{ah}^\circ) = \left(\frac{\pi_{ah}}{\pi_h^\circ}\right)^\eta \frac{\nu_a^*}{\nu^*} \left(\overline{\theta} + \eta r^\circ \overline{\omega}_h\right)$$

when $\eta \neq 0$ and

$$(5.31) \; \frac{y_{ah}^*}{\theta_a} = \mathrm{Log}\left(\frac{\pi_{ah}}{\pi_h^\circ}\right) + \frac{y_a^\circ}{\theta_a} = \mathrm{Log}\left(\frac{\pi_{ah}}{\pi_h^\circ}\right) + \left(\frac{\nu_a^*}{\theta_a} - \frac{\nu^*}{\overline{\theta}}\right) + \frac{r^\circ \overline{\omega}_h}{\overline{\theta}}$$

in the CARA configuration $\eta = 0$.

6 Aggregate risk aversion and asset pricing

We focus in this section on a few implications of heterogeneity of individual beliefs and of tastes, for aggregate risk aversion and asset pricing, with the aim, among others, to identify a few channels through which heterogeneity of this sort may contribute toward explaining some phenomena such as the socalled "equity premium puzzle".[13]

Specifically, we consider an oberved equilibrium with heterogenous beliefs π_a, as described by the vector of state prices q^* and the corresponding equilibrium portfolios y_a^*, and apply our aggregation procedure with the reference portfolio equal to the market portfolio, i.e. $\omega^\circ = \overline{\omega}$. We know from the adjusted CCAPM presented in Section 4 that, under the maintained assumption of interior equilibria, there is a representative investor with normalized VNM utilities $U_h(y)$ who, when endowed with the common probability π° and the adjusted market portfolio $r^\circ \overline{\omega}$, has for every asset generating the returns R_h with $\sum_h q_h^* R_h = 1$, a marginal valuation

$$(6.1) \; E_{\pi^\circ}\left[R_h U_h'\left(r^\circ \overline{\omega}_h\right)\right] = E_{\pi^\circ}\left[R_h\right] + cov_{\pi^\circ}\left[R_h, U_h'\left(r^\circ \overline{\omega}_h\right)\right] = R_\circ^*$$

that is by construction identical to the marginal asset valuation of every individual investor in the observed equilibrium

$$(6.2) \; E_{\pi_a}\left[R_h v_{ah}'\left(y_{ah}^*\right)\right] = E_{\pi_a}\left[R_h\right] + cov_{\pi_a}\left[R_h, v_{ah}'\left(y_{ah}^*\right)\right] = R_\circ^*,$$

where individual VNM utilities have been normalized by $v_{ah}(y) = u_{ah}(y) \big/ E_a\left[u_{ah}'\left(y_{ah}^*\right)\right]$. As noted in section 4, one may equivalently leave invariant the

market portfolio and adjust instead the representative investor normalized VNM utilities through $V_h(y) = U_h(r^\circ y) \diagup r^\circ$, or $V_h'(y) = U_h'(r^\circ y)$.

The representative investor's evaluation of risk premia is therefore given by the usual formulation (in view of the normalization $E_{\pi^\circ}[U_h'(r^\circ \overline{\omega}_h)] = 1$)

(6.3) $\quad E_{\pi^\circ}[R_h] - R_\circ^* = -cov_{\pi^\circ}[R_h, U_h'(r^\circ \overline{\omega}_h)] \diagup E_{\pi^\circ}[U_h'(r^\circ \overline{\omega}_h)].$

Heterogeneity of beliefs may thus contribute to explain something like the equity premium puzzle if, when applied to the market portfolio with returns $R_h^M = \overline{\omega}_h \diagup q^* \cdot \overline{\omega}$ (and to the case of state independent VNM utilities), it generates a *lower evaluation* by the representative investor, of the corresponding risk premium $E_{\pi^\circ}[R_h^M] - R_\circ^*$ through (6.3), by comparison to the evaluation of an econometrician using an hypothetical "true" probability $\pi \neq \pi^\circ$: the econometrician would have then to assume "too much" risk aversion while trying to fit a standard CCAPM formulation like

(6.4) $\quad E_\pi[R_h^M] - R_\circ^* = -cov_\pi\left[R_h^M, \widetilde{V}'(\overline{\omega}_h)\right] \diagup E_\pi\left[\widetilde{V}'(\overline{\omega}_h)\right]$

for some specification $\widetilde{V}'(y)$ of the VNM marginal utility of an hypothetical representative investor. We seek to identify in what follows a few possible channels through which diversity of beliefs may indeed generate such a possible "heterogeneity aggregation bias" by decreasing the representative investor's risk premium evaluations (6.3) of the market portfolio, or more generally of assets generating returns that vary positively with aggregate income.

We investigate first how varies the risk evaluation of such an asset as in the right hand side of (6.3), with the scalar adjustment coefficient. When this coefficient goes up, aggregate incomes increase, which should lower the risk premium evaluation if risk aversion goes down with income, while the variability of these aggregate incomes goes up, which should tend to increase the risk premium evaluation. The next elementary fact implies that the first (income) effect should prevail, when utilities are state independent and *relative* risk aversion decreases with income.

Lemma 6.1. *(Scalar adjustment and risk evaluation). Let π be an arbitrary probability with positive components, and let $U_h(y)$ stand for VNM utilities satisfying assumption (2.a), with $\rho_h(y) = -yU_h''(y)\diagup U_h'(y)$ denoting the corresponding degrees of relative risk aversion. Let the market portfolio $\overline{\omega}$ satisfy assumption (2.c). Then for any adjustment coefficient $r > 0$, the evaluation of the risk premium of an asset yielding the returns R_h, given by*

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(6.5) $g\left(r\right) = -cov_{\pi}\left[R_h, U'_h\left(r\,\overline{\omega}_h\right)\right]\big/E_{\pi}\left[U'_h\left(r\,\overline{\omega}_h\right)\right],$

varies with the adjustment coefficient as

(6.6) $\dfrac{dg\left(r\right)}{d\mathrm{Log}r} = cov_{\pi(r)}\left[R_h, \rho_h\left(r\,\overline{\omega}_h\right)\right],$

where the covariance is taken with respect to the probability $\pi\left(r\right)$ described by $\pi_k\left(r\right) = \pi_k U'_k\left(r\,\overline{\omega}_k\right)\big/E_{\pi}\left[U'_h\left(r\,\overline{\omega}_h\right)\right].$ The covariance is negative for every asset, the returns of which vary negatively with the degrees of relative risk aversion $\rho_h\left(r\,\overline{\omega}_h\right).$

In the particular case where VNM utilities are state independent and relative risk aversion $\rho_h\left(y\right) = \rho_k\left(y\right)$ is decreasing with income, and if there is some aggregate risk, an increase of the adjustment coefficient r pushes down the risk premium evaluation (6.5) for any asset with returns R_h that vary positively with aggregate income $\overline{\omega}_h$, and thus for the market portfolio $R_h^M = \overline{\omega}_h\big/q^* \cdot \overline{\omega}.$

Proof. One can rewrite (6.5) as $g\left(r\right) = E_{\pi}\left[R_h\right] - E_{\pi(r)}\left[R_h\right],$ so that

$$\begin{aligned}
\frac{dg\left(r\right)}{d\mathrm{Log}r} &= -E_{\pi(r)}\left[\frac{d\mathrm{Log}\pi_h\left(r\right)}{d\mathrm{Log}r}R_h\right] \\
&= -E_{\pi(r)}\left[\frac{d\mathrm{Log}\pi_h\left(r\right)}{d\mathrm{Log}r}\right]E_{\pi(r)}\left[R_h\right] - cov_{\pi(r)}\left[R_h, \frac{d\mathrm{Log}\pi_h\left(r\right)}{d\mathrm{Log}r}\right].
\end{aligned}$$

One gets (6.6) from the fact that

$$-\frac{d\mathrm{Log}\pi_k\left(r\right)}{d\mathrm{Log}r} = \rho_k\left(r\,\overline{\omega}_k\right) - E_{\pi(r)}\left[\rho_h\left(r\,\overline{\omega}_h\right)\right],$$

which implies $E_{\pi(r)}\left[\dfrac{d\mathrm{Log}\pi_h\left(r\right)}{d\mathrm{Log}r}\right] = 0.$ The other statements are then immediate consequences of (6.6).   Q.E.D.

The foregoing fact, when applied to the equivalent common probability equilibrium $\left(\pi^{\circ}, r^{\circ}\right)$, suggests that *an adjustment coefficient r° exceeding 1 might be associated*, in the configuration where there is some aggregate risk and where VNM utilities are state independent that is usually considered in financial studies, *to a low risk premium evaluation by the representative investor as in* (6.3) *of the market portfolio*, or generally of assets yielding

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returns that vary positively with aggregate income. We know from the previous section that $r^\circ > 1$ does obtain when aggregate risk is small and dispersion of beliefs is significant, provided that *individual absolute risk tolerances do not increase too fast* $(T'_{ah}(y) < 1)$. The above lemma uncovers an additional independent condition sufficient to get the desired outcome, namely that *aggregate relative risk aversion decreases with income.* These two sorts of conditions are in principle independent even when there is no microeconomic heterogeneity, since for a given utility function, the degrees of absolute risk tolerance $T(y)$ and of relative risk aversion $\rho(y)$ are linked by $T(y) = y \diagup \rho(y)$. So the condition that relative risk aversion $\rho(y)$ is decreasing means that the elasticity $\varepsilon_T(y) = yT'(y) \diagup T(y) = T'(y) \rho(y)$ exceeds 1, which implies $T'(y) > 0$ and is compatible with $T'(y) < 1$ as long as relative risk aversion $\rho(y)$ is above 1.

We investigate next the links between microeconomic and aggregate risk aversion. Our point of departure is the remark we used already that aggregate risk tolerance is an appropriate average of individual risk tolerances in the population (see (4.11)). We show now that *whenever individual absolute risk tolerances are increasing, microeconomic heterogeneity introduces an aggregation bias toward decreasing aggregate relative risk aversion, even though such a property may be weak or even absent at the microeconomic level.* In particular, if all individual investors have Constant Relative Risk Aversion (CRRA) VNM utilities that are different, aggregate relative risk aversion is decreasing. This sort of result is comforting since, while empirical studies appear indeed to point toward microeconomic increasing absolute risk tolerance, the evidence on individual decreasing relative risk aversion seems to be more mixed.[14]

Lemma 6.2. (*Aggregate risk aversion*). *Under assumptions (2.a) and (2.c), consider an equilibrium vector of state prices q^* with individual beliefs π_a and the associated equivalent common probability equilibrium defined by (π°, r°) with the reference portfolio equal to the market portfolio, i.e. $\omega^\circ = \overline{\omega}$, the corresponding individual portfolios y_a^* and y_a° being interior. Let the normalized VNM of the equilibrium representative investor be defined as in Proposition 4.1 by*

(6.7) $U_h(y_h) = Max\ E_a\left[u_{ah}(y_{ah}) \diagup E_{\pi_a}\left[u'_{ak}(y_{ak}^*)\right]\right]$ *subject to* $E_a[y_{ah}] = y_h$.

Individual degrees of absolute risk tolerance and of relative risk aversion are noted as before $T_{ah}(y) = -u'_{ah}(y) \diagup u''_{ah}(y)$ *and* $\rho_{ah}(y) = -yu''_{ah}(y) \diagup u'_{ah}(y)$, *while those of the representative investor are* $T_h(y) = -U'_h(y) \diagup U''_h(y)$ *and*

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$\rho_h(y) = -yU_h''(y) / U_h'(y)$. *It is also convenient to define corresponding degrees of "relative risk tolerance" by* $\tau_{ah}(y) = 1 / \rho_{ah}(y) = T_{ah}(y) / y$ *and* $\tau_h(y) = 1 / \rho_h(y) = T_h(y) / y$. *Interior solutions are assumed throughout.*

Aggregate absolute and relative risk tolerances are (possibly weighted) averages of individual absolute and relative risk tolerances

$$(6.8) \quad T_h(y_h) = E_a[T_{ah}(y_{ah})] \text{ and } \tau_h(y_h) = E_a\left[\tau_{ah}(y_{ah})\frac{y_{ah}}{y_h}\right],$$

where the y_{ah} *are solutions of (6.7).*

Income derivatives of aggregate absolute risk tolerance are weighted averages of the corresponding income derivatives of individual absolute risk tolerances

$$(6.9) \qquad T_h'(y_h) = E_a\left[T_{ah}'(y_{ah})\frac{T_{ah}(y_{ah})}{T_h(y_h)}\right].$$

If all individual absolute risk tolerances are increasing with income, so does aggregate absolute risk tolerance, or more generally $T_{ah}'(y_{ah}) \geqq \eta$ *(resp.* $T_{ah}'(y_{ah}) \leqq \eta$) *for all a implies* $T_h'(y_h) \geqq \eta$ *(resp.* $T_h'(y_h) \leqq \eta$).

By contrast, microeconomic heterogeneity introduces an aggregation bias toward decreasing relative risk aversion (increasing relative risk tolerance), whenever individual absolute risk tolerance is increasing. Elasticities of absolute and relative risk tolerances $\varepsilon_{T_h}(y) = y\,T_h'(y) / T_h(y)$ *and* $\varepsilon_{\tau_h}(y) = \varepsilon_{T_h}(y) - 1$, *are related to the corresponding individual elasticities* $\varepsilon_{T_{ah}}(y) = yT_{ah}'(y)/T_{ah}(y)$ *and* $\varepsilon_{\tau_{ah}}(y) = \varepsilon_{T_{ah}}(y) - 1$ *through*

$$(6.10)\ \varepsilon_{\tau_h}(y_h) = E_a\left[\varepsilon_{\tau_{ah}}(y_{ah})\left(\frac{\tau_{ah}(y_{ah})}{\tau_h(y_h)}\right)^2\frac{y_{ah}}{y_h}\right] + E_a\left[\left(\frac{\tau_{ah}(y_{ah})}{\tau_h(y_h)} - 1\right)^2\frac{y_{ah}}{y_h}\right].$$

If $\varepsilon_{\tau_{ah}}(y_{ah}) \geqq \gamma$ *for all investors, then*

$$(6.11) \qquad \varepsilon_{\tau_h}(y_h) \geqq \gamma + (1+\gamma)E_a\left[\left(\frac{\tau_{ah}(y_{ah})}{\tau_h(y_h)} - 1\right)^2\frac{y_{ah}}{y_h}\right],$$

which exceeds γ *by a positive variance term when individual absolute risk tolerance is increasing, i.e.* $\varepsilon_{T_{ah}}(y_{ah}) \geqq 1 + \gamma > 0$, *and whenever individual degrees of relative risk aversion* $\rho_{ah}(y_{ah}) = 1 / \tau_{ah}(y_{ah})$ *differ at the microeconomic level. In particular, if all investors have different Constant Relative*

Risk Aversion (CRRA) VNM utilities, $\varepsilon_{\tau_{ah}}(y) \equiv \gamma = 0$ for all a, aggregate relative risk aversion is decreasing.

The inequality in (6.11) is reversed when $\varepsilon_{\tau_{ah}}(y_{ah}) \lessgtr \gamma$, and microeconomic heterogeneity generates an opposite bias toward increasing aggregate relative risk aversion when individual absolute risk tolerance is decreasing, i.e. $\varepsilon_{T_{ah}}(y_{ah}) \lessgtr 1 + \gamma < 0$.

Proof. As noted in the text, the first part of (6.8) was already established in (4.11) and results from the differenciation of the FOC of (6.7). The second part of (6.8) is a simple restatement of the first part by using $T(y) = \tau(y)y$. One gets next (6.9) by differenciation of the first part of (6.8) and by noting that the solution (y_{ah}) of (6.7) satisfies $dy_{ah}/dy_h = T_{ah}(y_{ah})/T_h(y_h)$. Inserting the expressions of the elasticities $\varepsilon_{\tau_h}(y) + 1 = yT_h'(y)/T_h(y)$ and $\varepsilon_{\tau_{ah}}(y) + 1 = yT_{ah}'(y)/T_{ah}(y)$ in (6.9) gives

$$\varepsilon_{\tau_h}(y_h) + 1 = E_a\left[(\varepsilon_{\tau_{ah}}(y_{ah}) + 1)\left(\frac{\tau_{ah}(y_{ah})}{\tau_h(y_h)}\right)^2 \frac{y_{ah}}{y_h}\right],$$

which is identical to (6.10) if one remarks that

$$E_a\left[\left(\frac{\tau_{ah}(y_{ah})}{\tau_h(y_h)} - 1\right)^2 \frac{y_{ah}}{y_h}\right] = E_a\left[\left(\frac{\tau_{ah}(y_{ah})}{\tau_h(y_h)}\right)^2 \frac{y_{ah}}{y_h}\right] - 1$$

is in fact, in view of the second part of (6.8), the variance in the population of $\tau_{ah}(y_{ah})/\tau_h(y_h)$ when each investor is given the weight $\mu_a y_{ah}/y_h$ (where μ_a is the proportion of investors of type a in the market). The statements from (6.11) until the end of the lemma are then immediate.   Q.E.D.

The preceding analysis does suggest that in the case of state independent VNM utilities, an adjustment coefficient $r^\circ > 1$ may contribute to a low risk premium evaluation of the market portfolio, by the representative investor as in (6.3), in the plausible configuration of a decreasing aggregate relative risk aversion. Yet the argument is not completely convincing since that adjustment coefficient r° is not an independent parameter : it is jointly determined with the equivalent common probability π°. What we need eventually to do is to identify all the mechanisms that may contribute to a low risk premium evaluation by the representative investor $E_{\pi^\circ}[R_h] - R_\circ^*$, in particular for the market portfolio with returns $R_h^M = \overline{\omega}_h/q^* \cdot \overline{\omega}$, for instance by putting cautiously more weight on states involving lower than average aggregate incomes, than would be justified by an hypothetical "true" reference probability.

Assessing fully such an issue would require an explicit dynamic analysis of the genesis of the distribution of individual beliefs, in particular in terms of differential access to information, processing and learning. We focus here on the possibility of an "aggregation bias" by looking at the specific simple case where the distribution of individual beliefs π_a among investors is the result of the presence of some "noise" around a "true" probability, taken as the average belief in the population $\overline{\pi} = E_a[\pi_a]$. We are thus looking for mechanisms that may contribute to a *positive risk premium aggregation bias*

$$(6.12) \quad E_{\overline{\pi}}[R_h] - E_{\pi^\circ}[R_h] = cov_{\pi^\circ}\left[\frac{\overline{\pi}_h}{\pi_h^\circ}, R_h\right] > 0$$

for the market portfolio and more generally for assets with returns positively related with aggregate income.

We use here again a variation on the second order Taylor expansion introduced in the previous section in particular to analyse the determinants of the adjustment coefficient r°. With the notations of Proposition 5.1 and Corollary 5.2, aggregation of individual portfolios in (5.6) leads to (5.9), which gives after dividing by aggregate risk tolerance $T_h^\circ = E_a[T_{ah}^\circ]$ in the equivalent equilibrium and rearranging :

$$(6.13)\frac{\overline{\pi}_h - \pi_h^\circ}{\pi_h^\circ} = -\frac{r^\circ - 1}{r^\circ}\rho_h^\circ - cov_a\left[\frac{\pi_{ah}}{\pi_h^\circ}, \frac{T_{ah}^\circ}{T_h^\circ}\right] + \tfrac{1}{2}E_a\left[\left(\frac{\pi_{ah} - \pi_h^\circ}{\widehat{\pi}_{ah}}\right)^2 \frac{\widehat{T}_{ah}}{T_h^\circ}\left(1 - \widehat{T'}_{ah}\right)\right].$$

That relation allows to inventory the main channels through which heterogeneity of beliefs may affect the risk premium aggregation bias (6.12). Taking the covariance, according to the common probability π°, of an asset's returns R_h with the relative belief deviations $(\overline{\pi}_h - \pi_h^\circ)/\pi_h^\circ$, as expressed in (6.13), generates three terms. The first one, $A = -((r^\circ - 1)/r^\circ)cov_{\pi^\circ}[R_h, \rho_h^\circ]$ quantifies the influence of heterogeneity of beliefs through the scalar adjustment of the market portfolio, exactly along the lines laid down earlier in expression (6.6) of Lemma 6.1. The two other terms identify two different channels through which systematic "distorsions" in the distribution of beliefs in the population may generate a positive risk premium aggregation bias, namely when more risk tolerant or richer investors assign cautiously more weight to "bad" states ("pessimism"), or when agents have on average beliefs that are relatively more dispersed for "good" states ("doubt").

Proposition 6.3. *(Risk premium aggregation bias). Under the assumptions and notations of Proposition 5.1 and Corollary 5.2, let $\overline{\pi} = E_a[\pi_a]$ be the average belief among investors and consider an arbitrary asset with*

returns R_h satisfying $\sum_h q_h^* R_h = 1$. *The corresponding risk premium aggregation bias is the sum of three terms*

$$E_{\overline{\pi}}[R_h] - E_{\pi^\circ}[R_h] = cov_{\pi^\circ}\left[R_h, \frac{\overline{\pi}_h}{\pi_h^\circ}\right] = A + B + C.$$

1) (Adjustment coefficient) The first term $A = -\left((r^\circ - 1)\big/r^\circ\right)cov_{\pi^\circ}[R_h, \rho_h^\circ]$ measures the direct influence of the heterogeneity of beliefs through the corresponding scalar adjustment of the market portfolio. When there is some aggregate risk, an upward adjustment coefficient $r^\circ > 1$ contributes to a positive risk premium aggregation bias if VNM utilities are state independent and aggregate relative risk aversion $\rho(y)$ is decreasing with income, for every asset with returns that vary positively with aggregate income and in particular for the market portfolio. For small aggregate risks, an approximate evaluation of this term in the case of state independent utilities is $A \simeq -(r^\circ - 1)\rho'(r^\circ E_{\pi^\circ}[\overline{\omega}_h])\, cov_{\pi^\circ}[R_h, \overline{\omega}_h].$

2) ("Pessimism") The second term $B = -cov_{\pi^\circ}\left[R_h, cov_a\left[\frac{\pi_{ah}}{\pi_h^\circ}, \frac{T_{ah}^\circ}{T_h^\circ}\right]\right]$ is the sum of two terms $B_1 + B_2$, where

$$B_1 = -cov_a\left[E_{\pi_a}[R_h], E_{\pi^\circ}\left[\frac{T_{ah}^\circ}{T_h^\circ}\right]\right], \quad B_2 = -E_a\left[cov_{\pi^\circ}\left[\frac{\pi_{ah}}{\pi_h^\circ}(R_h - E_{\pi^\circ}[R_k]), \frac{T_{ah}^\circ}{T_h^\circ}\right]\right].$$

The term B_1 will be positive if people having on average higher absolute risk tolerances T_{ah}° in the equivalent common probability equilibrium, have also a more pessimistic evaluation of the expected return of that asset $E_{\pi_a}[R_h]$. The term B_2 vanishes if the ratios $T_{ah}^\circ\big/T_h^\circ$ are independent of the state h, e.g. when VNM utilities are state independent, if there is no aggregate risk or in the case of the HARA family (two funds separation). So when VNM utilities are state independent and if pessimism is significant, the term $B_1 > 0$ dominates the term B_2 provided that aggregate risk, and/or departure from the HARA family specification, is small.

3) ("Doubt") The third term $C = \frac{1}{2}cov_{\pi^\circ}\left[R_h, E_a\left[\left(\frac{\pi_{ah} - \pi_h^\circ}{\widehat{\pi}_{ah}}\right)^2 \frac{\widehat{T}_{ah}}{T_h^\circ}\left(1 - \widehat{T}'_{ah}\right)\right]\right]$

can be approximated by $C \simeq \frac{1}{2}cov_{\pi^\circ}\left[R_h, E_a\left[\left(\frac{\pi_{ah} - \pi_h^\circ}{\pi_h^\circ}\right)^2 \frac{T_{ah}^\circ}{T_h^\circ}\left(1 - T'_{ah}(y_{ah}^\circ)\right)\right]\right]$ for heterogeneities of beliefs small enough to allow neglecting third order terms. This term then tends to be positive if the relative dispersion of beliefs in the population is on average larger for "good" states (defined by larger

returns R_h) and lower for "bad" states, provided that individual absolute risk tolerance does not increase too fast, i.e. when $T'_{ah}(y^\circ_{ah}) < 1$ for all a, h.

The above statements follow by direct inspection. It may be worth remarking that the effect of heterogeneity of beliefs on the risk premium aggregation bias, through the adjustment coefficient r°, operates via the first term A above, whether or not there is any systematic pattern in the distribution of beliefs among investors. We attempt next to gain more insights into the interactions between this "adjustment coefficient effect" and the two other "pessimism" and "doubt" effects, by looking at some specific examples drawn from the HARA family.

Example 6.4. *The HARA family*

We go back to the HARA family where VNM utilities are state independent with individual absolute risk tolerance given by $T_{ah}(y) = \theta_a + \eta y > 0$ and thus aggregate absolute risk tolerance $T_h(y) = \overline{\theta} + \eta y > 0$ with $\overline{\theta} = E_a[\theta_a]$. The empirically plausible case where absolute risk tolerance increases corresponds to $\eta > 0$. Aggregate relative risk aversion $\rho(y) = y / (\overline{\theta} + \eta y) > 0$ is then decreasing (for $y > -\overline{\theta}/\eta$) if and only if $\overline{\theta} < 0$.

The simpler specification to consider is the case of logarithmic utilities $\eta = 1$. We know that there is then no scalar adjustment of the market portfolio in the common probability equivalent equilibrium $(r^\circ = 1)$, so that the first and third terms in (6.13) and Proposition 6.3 disappear (as well as the therm B_2, that vanishes for the HARA family). In fact, there is no income compensation in that case, i.e. $b^\circ_a = q^* \cdot \omega_a$, so that from (5.16) in Corollary 5.4, $T^\circ_{ah} / T^\circ_h = (q^*_\circ \theta_a + q^* \cdot \omega_a) / (q^*_\circ \overline{\theta} + q^* \cdot \overline{\omega})$. Therefore

$$(6.14) \qquad \frac{\overline{\pi}_h - \pi^\circ_h}{\pi^\circ_h} = -cov_a\left[\frac{\pi_{ah}}{\pi^\circ_h}, \frac{q^*_\circ \theta_a + q^* \cdot \omega_a}{q^*_\circ \overline{\theta} + q^* \cdot \overline{\omega}}\right],$$

which implies

Lemma 6.5. *(Risk premium aggregation bias : logarithmic utilities $\eta = 1$) Consider an asset with returns R_h satisfying $\sum_h q^*_h R_h = 1$. In the case of logarithmic utilities $(\eta = 1)$, the risk premium aggregation bias takes the form*

$$E_{\overline{\pi}}[R_h] - E_{\pi^\circ}[R_h] = -cov_a\left[E_{\pi_a}[R_h], \frac{q^*_\circ \theta_a + q^* \cdot \omega_a}{q^*_\circ \overline{\theta} + q^* \cdot \overline{\omega}}\right].$$

It is equal to 0 for every asset (i.e. $\pi^\circ = \pi$) if the distribution of beliefs π_a in the population is independent of the distribution of risk attitudes θ_a and

of endowments ω_a. For a particular asset with returns R_h, the risk premium aggregation bias will be positive if investors with larger risk tolerance and/or income, i.e. with larger $q_\circ^* \theta_a + q^* \cdot \omega_a$, are more pessimistic about the expected return of the asset.

The case of logarithmic utilities $\eta = 1$ involves only the "pessimism effect" identified in 2) of Proposition 6.3. The other case that is amenable to explicit global calculations, i.e. the CARA configuration, $\eta = 0$, $\theta_a > 0$, has the potential for a richer interaction between the two other effects. We know from Corollary 4.5 that this CARA specification implies an upward scalar adjustment coefficient $r^\circ > 1$ since the common probability π° is given by

$$b_h = \pi_h^\circ \ e^{-(r^\circ - 1)\overline{\omega}_h / \overline{\theta}} = E_a \left[\left(\theta_a / \overline{\theta} \right) \text{Log } \pi_{ah} \right]$$

with $b_h < E_a \left[\left(\theta_a / \overline{\theta} \right) \pi_{ah} \right]$ and therefore $\sum_h b_h < 1$ from the concavity of the Log function. This specification displays the unpleasant feature of an increasing relative risk aversion, with the consequence that the contribution of the adjustment effect to the risk premium aggregation bias is negative for assets with returns positively related to aggregate consumption : for the market portfolio, for instance, the adjustment effect term in Proposition 6.3 is $A = -(r^\circ - 1) \ var_{\pi^\circ} [\overline{\omega}_h] \big/ \left(\overline{\theta} q^* \cdot \overline{\omega} \right) < 0$. On the other hand, the pessimism effect term identified in Proposition 6.3 reduces here to $B = -cov_a \left[E_{\pi_a} [R_h], \theta_a / \overline{\theta} \right]$ and will disappear again if we assume that the distributions of beliefs π_a and of risk tolerance θ_a are independent in the population . The potential for interactions with the third "doubt effect", is nevertheless much richer as the following example is going to show.

To this end, we shall allow in our calculations for a continuum of states and at some point also for a continuum of agents, although our theoretical analysis is not strictly speaking quite valid in these cases. Specifically, we assume that h is any positive real number and set $\overline{\omega} (h) \equiv h > 0$. To take advantage of the fact that the common probability π° is a weighted harmonic mean of individual beliefs, we assume that each individual probability $\pi_a (h)$ is a *two parameters* $(\alpha_a > 0, \ \beta_a > 0)$ *gamma distribution*, with density

$$\pi_a (h) = h^{\alpha_a - 1} \ e^{-h / \beta_a} \big/ (\beta_a^{\alpha_a} \Gamma (\alpha_a)) ,$$

where $\Gamma (\alpha) = \int_0^{+\infty} x^{\alpha - 1} \ e^{-x} dx$ is the "complete gamma function" (see Johnson, Kotz and Lalakrishnan (JKB), 1994, Ch. 17). The equivalent common probability π° is then also a two parameters $(\alpha^\circ, \beta^\circ)$ gamma distribution. The mean and variance of a (α, β) gamma distribution being $\alpha\beta$ and $\alpha\beta^2$

respectively, evaluating the risk premium aggregation bias associated with the market portfolio $\overline{\omega}(h) \equiv h$ means computing the difference between $E_{\overline{\pi}}[h] = E_a E_{\pi_a}[h] = E_a[\alpha_a \beta_a]$ and $E_{\pi^\circ}[h] = \alpha^\circ \beta^\circ$. We are going to see that (under the assumption that β_a is lognormally distributed in the population) increasing the variance of the distribution of α_a and/or of β_a among investors does increase, as expected, the scalar adjustment coefficient $r^\circ > 1$ of the market portfolio. This contributes to a negative, and larger in absolute value, adjustment effect on the risk premium aggregation bias for the market portfolio, as identified in Proposition 6.3. Yet some dispersion in the distribution of the parameter β_a in the population, associated with a much smaller variance of the distribution of α_a, generates an overall *positive* risk premium aggregation bias for the market portfolio, i.e. $E_{\overline{\pi}}[h] - E_{\pi^\circ}[h] > 0$: a positive "doubt effect", overcomes in that case the negative "adjustment effect" implied by an increasing relative risk aversion involved in the CARA specification.

Lemma 6.6 (*Risk premium aggregation bias in the CARA - gamma configuration*). *Assume that a state h is any positive real number, that each individual belief $\pi_a(h)$ has a $(\alpha_a > 0, \beta_a > 0)$ gamma distribution and that $\overline{\omega}(h) \equiv h$. Let $\alpha = E_a\left[\left(\theta_a \big/ \overline{\theta}\right)\alpha_a\right]$, $1 \big/ \beta = E_a\left[\left(\theta_a \big/ \overline{\theta}\right)\left(1 \big/ \beta_a\right)\right]$. Then the equivalent common probability $\pi^\circ(h)$ has a $(\alpha^\circ, \beta^\circ)$ gamma distribution with $\alpha^\circ = \alpha$ and*

$$1 \big/ \beta^\circ = e^{E_a\left[(\theta_a \alpha_a) \big/ \left(\overline{\theta}\alpha\right) \, \mathrm{Log}(1/\beta_a)\right]} \; e^{\left(\mathrm{Log}\Gamma(\alpha) - E_a\left[\left(\theta_a \big/ \overline{\theta}\right) \, \mathrm{Log}\Gamma(\alpha_a)\right]\right) \big/ \alpha} \; ,$$

the corresponding adjustment coefficient being given by $r^\circ - 1 = \overline{\theta}\left((1 \big/ \beta) - (1 \big/ \beta^\circ)\right) > 0$.

Assume further that risk tolerance θ_a and the parameters α_a, β_a are independently ditributed in the population, and moreover that β_a has a lognormal distribution with $\mathrm{Log}\beta_a \sim \mathcal{N}\left(m_\beta, v_\beta^2\right)$, so that $\alpha = E_a[\alpha_a]$ and $1 \big/ \beta = E_a[1 \big/ \beta_a] = e^{-m_\beta} \, e^{v_\beta^2 / 2}$.

1) One gets then

$$1 \big/ \beta^\circ = e^{-m_\beta} \; e^{(\mathrm{Log}\Gamma(\alpha) - E_a[\mathrm{Log}\Gamma(\alpha_a)]) \big/ \alpha}, \; r^\circ - 1 = \overline{\theta}\left(e^{-m_\beta} e^{v_\beta^2 / 2} - 1 \big/ \beta^\circ\right).$$

The function $\mathrm{Log}\Gamma(\alpha)$ being strictly convex, the mean $\alpha \beta^\circ$ and the variance $\alpha(\beta^\circ)^2$ of the equivalent common probability π°, as well as the adjustment coefficient $r^\circ > 1$, increase following a mean-preserving spread of the distribution of the parameter α_a. The adjustment coefficient r° also increases with the variance v_β^2 of the distribution among investors of the parameter $\mathrm{Log}\beta_a$.

2) The contribution to the risk premium aggregation bias, of the "pessimism effect" term $B = -\text{cov}_a\left[E_{\pi_a}\left[R_h\right], \theta_a \big/ \overline{\theta}\right]$ in Proposition 6.3 vanishes for all assets when (α_a, β_a) and θ_a are assumed to be independently distributed as here. For the market portfolio (assuming its market value to be normalized to 1), the corresponding adjustment coefficient effect $A = -\left(r^\circ - 1\right)$ $\text{var}_{\pi^\circ}\left[h\right]\big/\overline{\theta} = -\left(r^\circ - 1\right)\alpha\left(\beta^\circ\right)^2\big/\overline{\theta}$ is negative and increases in absolute value with v_β^2 and with a mean-preserving spread of the distribution of α_a.

On the other hand, the overall market portfolio risk premium aggregation bias is

$$E_{\overline{\pi}}\left[h\right] - E_{\pi^\circ}\left[h\right] = \alpha e^{m_\beta}\left(e^{v_\beta^2/2} - e^{-\left(\text{Log}\Gamma(\alpha) - E_a\left[\text{Log}\Gamma(\alpha_a)\right]\right)/\alpha}\right).$$

It decreases with a mean-preserving spread of the distribution of α_a, but increases with v_β^2. In particular, it is positive if there is some dispersion in the distribution of $\beta_a\left(v_\beta^2 > 0\right)$ whereas the variance of the distribution of α_a is small.

Proof. The property that π° has a gamma distribution comes from the fact that

(6.15) $\text{Log}\pi^\circ\left(h\right) = \left(r^\circ - 1\right)\left(h/\overline{\theta}\right) + E_a\left[\left(\theta_a\big/\overline{\theta}\right)\text{Log}\pi_a\left(h\right)\right]$

together with

$$\text{Log}\pi_a\left(h\right) = \left(\alpha_a - 1\right)\text{Log}h - \left(h\big/\beta_a\right) - \text{Log}\left(\beta_a^{\alpha_a}\Gamma\left(\alpha_a\right)\right).$$

The expressions for the parameters $\alpha^\circ, \beta^\circ$ are then obtained by direct inspection.

The properties stated in 1) and 2) use a few elementary facts that we recall now. First, if x is a normal random variable distributed as $\mathcal{N}\left(\mu, \sigma^2\right)$, then for every real number t, $E\left[e^{tx}\right] = e^{t\mu + \frac{t^2\sigma^2}{2}}$ (by direct inspection). Equivalently, if y is lognormal, with $\text{Log}y$ distributed as $\mathcal{N}\left(\mu, \sigma^2\right)$, then again for every real number t, $E\left[y^t\right] = e^{t\mu + \frac{t^2\sigma^2}{2}}$. This implies indeed that $1\big/\beta = E_a\left[1\big/\beta_a\right] = e^{-m_\beta}e^{v_\beta^2/2}$ as stated in the Lemma, and also $E_a\left[\beta_a\right] = e^{m_\beta}e^{v_\beta^2/2}$, which is used in 2).

The results rest essentially on the property that the function $\text{Log}\Gamma\left(\alpha\right)$ is strictly convex. This follows from the fact that the "psi" or "digamma" function

$$\Psi\left(\alpha\right) = \frac{d\text{Log}\Gamma\left(\alpha\right)}{d\alpha} = \int\limits_0^{+\infty}\left[e^{-t} - \frac{1}{\left(1+t\right)^\alpha}\right]\frac{dt}{t}$$

is increasing, or equivalently that the "trigamma" function

$$\Psi'(\alpha) = \int\limits_0^{+\infty} \frac{te^{-\alpha t}}{1 - e^{-t}} dt$$

is positive (Abramowitz and Stegum (1965), Ch. 6, (6.3.21) and (6.4.1)).

The fact that $\text{Log}\Gamma(\alpha)$ is strictly convex implies that $E_a[\text{Log}\Gamma(\alpha_a)]$ exceeds $\text{Log}\Gamma(\alpha)$ where $\alpha = E_a[\alpha_a]$, and that it increases following a mean-preserving spread of the distribution of the α_a (in the sense of Rotschild and Stiglitz (1970), see Mas-Colell, Whinston and Green (1995, section 6.D)). All the comparative statics statements about the consequences of increasing the dispersion of the distributions of the parameters α_a, β_a, follow then by direct inspection.

The fact that when focussing attention on the market portfolio, a significant dispersion of the distribution of the parameter β_a generates a positive "doubt effect" that overcomes in this case the negative "adjustment coefficient effect" when the dispersion of the distribution of α_a is small, may be intuitively understood if one remarks that the corresponding term in Proposition 6.3 may be approximated when neglecting third order terms, by $C \simeq \frac{1}{2}cov_{\pi^\circ}\left[h, E_a\left[\left(\left(\pi_a(h) - \pi^\circ(h)\right)\diagup\pi^\circ(h)\right)^2\right]\right]$, where in fact $\left(\left(\pi_a(h) - \pi^\circ(h)\right)\diagup\pi^\circ(h)^2\right)$ can in turn be approximated by $\left(\text{Log}(\pi_a(h)\diagup\pi^\circ(h))\right)^2$. For large h, the average $E_a[\cdot]$ is dominated by $E_a\left[\left(\left(1\diagup\beta_a\right) - \left(1\diagup\beta^\circ\right)\right)^2\right]h^2$, while for small $h > 0$, this average behaves like $var_a[\alpha_a](\text{Log}h)^2$. Thus if α_a has a small dispersion in the population, the covariance in C will tend to be dominated by the terms involving large $\overline{\omega}(h) = h$ and thus to be positive, if the variance of the distribution of the parameters β_a is significant.   Q.E.D.

The foregoing analysis of a few special cases of the HARA family illustrates the possible interactions between the three effects identified in Proposition 6.3, that may contribute to a positive risk premium aggregation bias, in particular for the market portfolio. It would be useful to supplement it with studies of other configurations, notably in the case where absolute risk tolerance does not increase too fast, $0 < \eta < 1$, and where aggregate relative risk aversion decreases with aggregate income, i.e. $\overline{\theta} < 0$, since we know that the adjustment coefficient $r^\circ > 1$ contributes in that case to a positive risk premium aggregation bias for the market portfolio. As the adjustment coefficient effect vanishes in the HARA family in the case of a CRRA utility ($\theta_a = 0$, individual and aggregate VNM utilities coincide with $\rho_a = \rho = 1/\eta$),

another case of interest would be for instance the configuration where investors have different CRRA utilities ($T_a(y) = \theta_a + \eta_a y$ with $\theta_a = 0$) since we know (from Lemma 6.2) that aggregate relative risk aversion should be then decreasing.[15]

7 Conclusions

It seems most relevant to incorporate heterogenous, "noisy" beliefs in our representations of the workings of actual economies. The methods proposed in this paper show that it is possible indeed to achieve this goal while retaining the analytical simplicity of being able to describe a particular equilibrium through a single, commonly shared "aggregate market probability". In a complete markets framework, the proposed approach allows the standard construction of an "expected utility maximizing representative agent", designed so as to mimic equilibrium prices and marginal asset valuations by individual investors, to be extended to cover the case of diverse beliefs. Heterogeneity of individual portfolios, or of risk sharing, can then be studied in particular in relation with deviations of individual beliefs from the "aggregate market probability" so constructed. The proposed design of an aggregate probability may require a scalar adjustment of the market portfolio, that reflects an aggregation bias due to the heterogeneity of beliefs, and generates accordingly an "Adjusted" version of the "Consumption based Capital Asset Pricing Model" (ACCAPM). We also saw that an upward scalar adjustment could contribute to a positive risk premium aggregation bias, at least when aggregate relative risk aversion is decreasing.

Our study was made in the deliberately oversimplified setup of a static (one period) asset exchange economy with finitely many states of the world, in order to keep the technical apparatus down so as to be able to focus on ideas. It remains to be seen whether the approach developped here can be fruitfully extended to more realistic and more applied frameworks.

In particular, it would appear important to include intertemporal choice (portfolio selection), with a finite or infinite horizon, discrete or continuous time, in order to see if and how our approach of the consequences of heterogenous beliefs, can be related to more traditional theoretical and/or applied models in the finance and the macroeconomics literature, that employ the convenient but presumably counterfactual assumption of homogenous beliefs. Extension to a continuum of states, beyond taking a step toward more realism, would also allow to make tighter parts of our characterization of the aggregation procedure in terms of monotonicity of individual portfolio

deviations (residual risk sharing rules), notably in the case of probabilities with continuous densities. Our results, in particular existence and *unicity* of a common probability equivalent equilibrium, relied also heavily on the explicit assumption that incomes had to be positive, or equivalently that returns were bounded below. Allowing for unbounded returns (above and below) would appear important for practical applications involving for instance normal distributions. A cursory glance at a specific example with CARA utilities and normal distributions shows easily that unicity of a common probability equivalent equilibrium does not survive the incorporation of such unbounded returns.[16] This raises interesting technical and conceptual issues to be studied further (in particular, what is the meaning of the existence of several "equivalent aggregate probabilities" ?). While the construction of an "expected utility maximizing representative agent" is presumably closely tied to the specific assumption of complete markets, our construction of an equivalent aggregate market probability in terms of marginal asset pricing invariance requirements (section 3) may perhaps be fruitfully extended to the case of incomplete markets (but one may lose also unicity there ?). Finally, it should be worth exploring thoroughly the welfare implications of heterogeneity of beliefs, which were only tangentially alluded to here.

Footnotes

1. It should be clear that the notion of a "representative agent" used in this discussion and in fact in the whole paper, holds only in equilibrium. It should not be confused with the more demanding notion of an aggregate agent who would represent the economy even out of equilibrium, i.e. for every price system, as in Gorman (1953).

2. These is actually a sizeable literature that explores conditions implying that it would be in fact "rational" for all agents to coordinate their beliefs and strategies on signals that are indeed not perfectly correlated with fundamentals, generating in this way excess volatility and endogenous business cycles due to "sunspots", "self-fulfilling prophecies", "animal spirits", "market psychology" or "endogenous uncertainty" (Benhabib and Nishimura (1979, 1985), Benhabib and Day (1982), Cass and Shell (1983), Azariadis (1981, 1993), Farmer (1993), Kurz (1997) and among others, various symposia in the Journal of Economic Theory (1986, 1994, 1998a, 1998b, 2001) or Economic Theory (1996)). As many of these models involve a large multiplicity and indeterminacy of deterministic and stochastic intertemporal equilibria, one should then expect a persistent and significant heterogeneity of "noisy" individual beliefs due to exacting expectations coordination problems (Grandmont (1985)).

3. Specifically, Varian shows that within the same fixed equilibrium with heterogenous subjective probabilities, with complete asset markets and when agents have identical tastes, an Arrow-Debreu security will have a lower price for a state of nature h than for another state k, if and only if the dispersion of individual subjective probabilities (weighted by marginal expected utilities), is larger for state h than for state k (ceteris paribus, in particular for an unchanged mean of these weighted individual probabilities). The direction of

this association is reversed when the income derivative of individual absolute risk tolerance (η in the HARA family) exceeds 1.

4. The careful reader will have noted that we did not use actually the property that q^* is an equilibrium price vector. At this stage, q^* can be an arbitrary price system.

5. The careful reader will have noted also here that we have not yet used the property that q^* is an equilibrium vector of state prices. The first part of the Theorem is valid when q^* is an arbitrary price system.

6. The construction of an equilibrium "expected utility maximizing representative investor" in the case of homogenous beliefs is standard, see Huang and Litzenberger (1988, ch. 5), Duffie (1996, ch. 1, section E).

7. Again, the careful reader will have noted that these statements are valid when q^* is an arbitrary price system, not necessarily an equilibrium price vector.

8. As noted in the introduction, aggregation of heterogenous beliefs in the CARA configuration, leading to an aggregate probability having the form of a weighted harmonic mean of individual probabilities as here, was performed along similar lines some time ago by Huang and Litzenberger (1988, section 5.26), without any scalar adjustment of the market portfolio, however.

9. We use here the change of probability formula

$$E_\pi\left[x_h\right] = E_{\pi^\circ}\left[\frac{\pi_h}{\pi_h^\circ}\ x_h\right] = E_{\pi^\circ}\left[x_h\right] + cov_{\pi^\circ}\left[\frac{\pi_h}{\pi_h^\circ}, x_h\right].$$

10. Our results imply that exact aggregation of diverse individual probabilities is possible without any shift of individual income ($q^* \cdot \omega_a = b_a^\circ$) and any scalar adjustment of the market portfolio ($r^\circ = 1$) in the case of logarithmic utilities $\eta = 1$, confirming the results obtained some time ago by M. Rubinstein (1976) in this specific case, as noted in the introduction.

11. The above framework can be used to study equilibrium state prices as a function of aggregate consumption and of heterogenous individual beliefs. Assuming interior portfolios throughout, consideration of the FOC of (5.25) shows that the aggregate utilities $W_h\left(y_h\ ;\ (\pi_{bh})\right)$ are strictly concave in y_h and increasing in π_{bh}. Thus according to (5.26), equilibrium state prices

$q_h^* R_\circ^* = W_h' \left(\overline{\omega}_h \; ; \; (\pi_{bh}) \right)$ are "decreasing functions of aggregate consumption $\overline{\omega}_h$ and increasing functions of anyone individual's probability beliefs" (Varian (1985, 1989)). More precisely, in the particular case of state independent utilities, the equilibrium prices associated with two different states k and h will satisfy $q_k^* = q_h^*$ (resp. $q_k^* < q_h^*$) when $\overline{\omega}_k = \overline{\omega}_h$ (resp. $\overline{\omega}_k > \overline{\omega}_h$) provided that all individual beliefs (π_{bk}) and (π_{bh}) are the same in both states. Similarly, one will get $q_k^* = q_h^*$ (resp. $q_k^* > q_h^*$) when $\pi_{bk} = \pi_{bh}$ (resp. $\pi_{bk} > \pi_{bh}$) for some b provided that aggregate consumtion is the same in both states, $\overline{\omega}_k = \overline{\omega}_h$, and all beliefs other than those of b are also invariant, $\pi_{ak} = \pi_{ah}$ for all investors $a \neq b$.

12. Similarly, by rewriting individual FOC with unnormalized VNM utilities $u_{ah}(y_{ah})$, one gets $y_{ah}^* = (u_{ah}')^{-1} (q_h^* R_\circ^* / \widetilde{\pi}_{ah})$, where individual beliefs $\widetilde{\pi}_{ah} = \pi_{ah} / E_{\pi_a} [u_{ah}' (y_{ah}^*)]$ are "normalized" (weighted by $\lambda_a^* / R_\circ^*$). When utilities are state independent and all investors have identical tastes, one gets by adopting the same VNM utility representation $u_{ah}(y) = u(y)$ for all, $y_{ah}^* = (u')^{-1} (q_h^* R_\circ^* / \widetilde{\pi}_{ah}) = f(q_h^* R_\circ^* / \widetilde{\pi}_{ah})$ hence by aggregation $\overline{\omega}_h = E_a [f(q_h^* R_\circ^* / \widetilde{\pi}_{ah})]$. Therefore a "mean preserving spread", in the sense of Rotschild and Stiglitz (1970), Mas-Colell, Whinston and Green (1995, section 6.D), when comparing state h to state k, of the distribution of weighted individual probabilities from $(\widetilde{\pi}_{ah})$ to $(\widetilde{\pi}_{ak})$, given the same aggregate consumption $\overline{\omega}_k = \overline{\omega}_h$, will decrease the equilibrium state price, i.e. imply $q_k^* < q_h^*$, provided that the function $f(q_h^* R_\circ^* / \widetilde{\pi}_{ah})$ considered as a function of $\widetilde{\pi}_{ah}$, is concave or equivalently if absolute risk tolerance does not increase too fast, i.e. $T'(y) = -u'(y) / u''(y) < 1$ (Varian (1985, 1989), see also the exposition in Ingersoll (1987, Chap. 9)). The reason why the condition $T' < 1$ arises in Varian's analysis as well as in ours, should be clear since both express the concavity of the function $g_{ah}(\pi_{ah})$ defined in (5.2).

13. See Mehra and Prescott (1985), P. Weil (1989), Kocherlakota (1996), Cecchetti, Lam and Mark (2000) among others.

14. Empirical evidence in finance seems to favor decreasing relative risk aversion at the microeconomic level, see e.g. Friend and Blume (1975), Morin and Suarez (1983). Yet evidence coming from other types of data is more mixed, see e.g. the discussion in Peress (2000). Arrow (1970) produces a theoretical argument showing that bounded VNM utilities implies a degree of relative risk aversion below 1 for small wealth and above 1 for large wealth, with the consequence that relative risk aversion, if monotone, should be increasing.

15. One may note an alternative interesting formulation, where for in-

stance the "true" probability π belongs to a particular class indexed by a vector of parameters γ, corresponding actually to a particular vector $\overline{\gamma}$, and where individual beliefs π_a are also all members of that class of probabilities, each being indexed by a vector γ_a with $E_a\left[\gamma_a\right] = \overline{\gamma}$ (with the interpretation that investors receive, say, unbiased signals about the "true" vector of parameters $\overline{\gamma}$). The risk premium aggregation bias would then be $E_\pi\left[R_h\right] - E_{\pi^\circ}\left[R_h\right]$, where π may now differ from the average $\overline{\pi} = E_a\left[\pi_a\right]$. Such a formulation may be interesting to study despite the fact that it is not invariant to a nonlinear change of variables. For instance, in the particular CARA - gamma specification of Lemma 6.6, the market portfolio risk premium aggregation bias would then be $\overline{\alpha}\overline{\beta} - \alpha\beta^\circ$, where $\overline{\alpha}$ and $\overline{\beta}$ are the parameters of the "true" gamma distribution. If agents receive an unbiased signal about $\overline{\alpha}$, then $\overline{\alpha} = E_a\left[\alpha_a\right] = \alpha$. If they get an unbiased signal about β_a, then $\overline{\beta} = E_a\left[\beta_a\right] = e^{m_\beta}e^{v_\beta^2/2}$ and one gets the same results as in the text. If on the other hand the signal is about relative deviations of β_a, i.e. with $E_a\left[\mathrm{Log}\beta_a\right] = \mathrm{Log}\overline{\beta} = m_\beta$, the market portfolio risk premium aggegation bias $\alpha\left(\overline{\beta} - \beta^\circ\right) < 0$ is dominated by the effect of the adjustment coefficient $r^\circ > 1$, which is negative since aggregate relative risk aversion is increasing in the CARA specification.

16. Specifically, consider the CARA specification with the state h being any positive and negative real number and $\overline{\omega}\left(h\right) \equiv h$ as in Lemma 6.6. Let $\pi_a\left(h\right)$ be normally distributed as $\mathcal{N}\left(m_a, \sigma_a^2\right)$. One verifies then by direct inspection from (6.15) that $\pi^\circ\left(h\right)$ is also normally distributed as $\mathcal{N}\left(m^\circ, \sigma^2\right)$, where $1/\sigma^2 = E_a\left[\left(\theta_a/\overline{\theta}\right)\left(1/\sigma_a^2\right)\right]$, m° and r° are related by $m^\circ = m + \left(r^\circ - 1\right)\sigma^2/\overline{\theta}$ with $m = E_a\left[\left(\theta_a/\overline{\theta}\right)\dfrac{1/\sigma_a^2}{1/\sigma^2}m_a\right]$, and m° (or r°) is solution of the second degree equation

$$\left(m^\circ\right)^2 = E_a\left[\left(\theta_a/\overline{\theta}\right)\frac{1/\sigma_a^2}{1/\sigma^2}m_a^2\right] - \sigma^2 E_a\left[\left(\theta_a/\overline{\theta}\right)\mathrm{Log}\left(\frac{1/\sigma_a^2}{1/\sigma^2}\right)\right].$$

The term after the minus sign is less than $\mathrm{Log}[1] = 0$, since the function Log is concave, so $\left(m^\circ\right)^2$ is greater than the first term of the right hand side of the equation, which exceeds itself m^2. Therefore $\left(m^\circ\right)^2 > m^2$. There are two solutions, one involving $m^\circ > m$ and $r^\circ > 1$, the other $m^\circ < -m$ and $r^\circ < 1$. This specific feature is due to the fact that $\overline{\omega}\left(h\right)$ can take any unbounded positive and negative value, while the assumption that $\overline{\omega}\left(h\right)$ was bounded below by 0 played a crucial role to get unicity in the general analysis of the text.

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