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Helicopter Drops of Money under Secular Stagnation

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Helicopter Drops of Money under Secular Stagnation*

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Abstract

What are the effects of helicopter drops of money under secular stagnation? This paper shows that, if the government cannot sustain a Ponzi debt scheme under full employment, then helicopter drops of money cannot transfer real wealth to households under secular stagnation. In that case, despite being in a permanent liquidity trap, a one-off helicopter drop triggers an upward jump in the price level, without any real effect on the economy. Conversely, if a Ponzi scheme can be sustained, then the helicopter drop can stimulate aggregate demand by raising household wealth. If the stagnation real interest rate is larger than the economic growth rate, the economy converges to full employment and a sustainable Ponzi scheme and, otherwise, it gradually reverts back to stagnation. Finally, continuous helicopter drops of money under stagnation must induce the economy to reach a full employment steady state, with or without a Ponzi scheme.

Keywords: Helicopter drops of money, Liquidity trap, Ponzi scheme, Secular stagnation

JEL Classification: E12, E31, E63, H63

1 Introduction

Japan has now spent two decades at the zero lower bound, with no end in sight. The liquidity trap has become a permanent state of affairs. In this context, there seems to be no limit to the monetary financing of public expenditures. The money supply exceeds 100% of GDP and the debt-to-GDP ratio 250%. Inflation nonetheless remains close to zero and the yield curve completely flat, with a slightly negative 10-year real yield.

But, is high inflation just around the corner? Or, can public debt keep rising without bounds? Can the government lift the economy out of stagnation by going broke? As

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nominal bonds are always backed by the printing press, these are fundamentally questions about the consequences of helicopter drops of money.[1]

Under a temporary liquidity trap, the limits to the monetary financing of budget deficits are trivially determined by the economic forces at work after the crisis is over. Hence, in that context, helicopter drops are essentially an instrument of forward guidance. By contrast, under a permanent liquidity trap, forward guidance is powerless and helicopter drops raise fundamental questions about the limits to government indebtedness. In this paper, I therefore analyze the consequences of helicopter drops of money under secular stagnation, where the economy is permanently liquidity trapped due to a lack of demand.

Following Michau (2018), I introduce a preference for wealth and a downward wage rigidity within a neoclassical economy. This results in the coexistence of up to three different steady states: (i) a neoclassical steady state, with a very low real interest rate and full employment, (ii) a secular stagnation steady state, with a binding zero lower bound, a binding downward wage rigidity, and under-employment, and (iii) a Ponzi steady state, with a Ponzi scheme of constant magnitude and full employment. As in Michau, Ono, and Schlegl (2018), a Ponzi steady state can exist if and only if the natural real interest rate of the neoclassical steady state is smaller than the population growth rate.

I assume that the economy is initially in the secular stagnation steady state and first consider a one-off helicopter drop of money, which consists of a lump-sum transfer of newly printed money to households. There always exists an equilibrium where this triggers an upward jump in the price level, such as to reduce the real value of public liabilities and avoid the existence of a Ponzi scheme, without any real effect on the economy. In fact, this is the only equilibrium possibility whenever a Ponzi steady state does not exist, as any Ponzi scheme would necessarily be explosive. So, even though the economy is permanently liquidity trapped, it is impossible to finance public expenditures from an unbacked increase in the money supply.

Conversely, when a Ponzi steady state exists, there are equilibrium paths without a jump in the initial price level. The helicopter drop therefore sets off a Ponzi scheme. The increase in government liabilities, without a corresponding increase in the present value of future taxes, raises household wealth. This reduces the marginal utility of wealth and raises households' demand for consumption. So, the helicopter drop stimulates aggregate demand. If the real interest rate under stagnation is larger than the growth rate of the economy, the Ponzi scheme swells over time until the economy reaches the Ponzi steady state; otherwise the Ponzi scheme shrinks and the economy converges back to the secular stagnation steady state.

[1] These issues are closely related to "Modern Monetary Theory", which has lately agitated the policy debate, but which still lacks a precise formulation.

There is a maximum feasible magnitude for Ponzi schemes and the economy must be producing at full capacity when this limit is reached. Any further helicopter drop of money must trigger an upward jump in the initial price level.

Finally, I consider a more radical strategy, whereby the government makes continuous helicopter drops of money while the economy fails to produce at full capacity. In that case, the economy cannot remain in the secular stagnation steady state forever. Indeed, continuous drops cannot be offset by a single upward jump in the initial price level. The economy must therefore either reach the neoclassical steady state or wait for the Ponzi scheme to become sufficiently large before ending up in the Ponzi steady state.

Related Literature. In seminal speeches on the Japanese conundrum, Bernanke (2000, 2002, 2003) has advocated for a money-financed tax cut, i.e. for an helicopter drop of money, to bring persistent deflation to an end.[2] However, his proposal was largely ahead of the academic literature on the topic.

A few papers have investigated helicopter drops under a temporary liquidity trap. Auerbach and Obstfeld (2005) have shown that open-market operations can stimulate the economy at zero lower bound, provided that the increase in the money supply is believed to be permanent. This runs into a time consistency problem. But Eggertsson (2006) has pointed out that, in the presence of distortionary taxes, the accumulation of government debt or helicopter drops of money are perfect commitment devices for a higher future price level. More recently, Gali (2018) has compared a money-financed to a debt-financed fiscal stimulus at the zero lower bound within a simple new Keynesian economy. He found money-financing to be more powerful as it commits the government to a lower path of future nominal interest rates. This shows that, under a temporary liquidity trap, helicopter drops can be a powerful instrument of forward guidance. However, forward guidance alone is powerless under secular stagnation.

Benigno and Nistico (2017) carefully derived the conditions for the neutrality of open-market operations. In particular, neutrality breaks down when the treasury does not cover the central bank's losses, in which case the resulting wealth transfer to the private sector must be inflationary. Under my secular stagnation framework, helicopter drops can generate sustainable Ponzi schemes and are therefore not necessarily inflationary. This is closely related to Bassetto and Cui's (2018) demonstration that, in a dynamically inefficient OLG economy, the fiscal theory of the price level cannot uniquely determine the price level.

[2] Interestingly, Bernanke (2000) wrote: *"I think most economists would agree that a large enough helicopter drop* must *raise the price level. Suppose it did not, so that the price level remained unchanged. Then the real wealth of the population would grow without bound, as they are flooded with gifts of money from the government. Surely at some point the public would attempt to convert its increased real wealth into goods and services, spending that would increase aggregate demand and prices."* My aim in this paper is to offer a formal investigation of this conjecture.

Buiter (2014) argues that, if money is valuable and irredeemable, then helicopter drops of money can always stimulate aggregate demand. However, his analysis does not rely on a complete general equilibrium model, which leaves prices indeterminate. My analysis shows that a one-off helicopter drop can always be fully offset by a jump in the initial price level. However, continuous drops are indeed always able to stimulate aggregate demand.[3]

Rachel and Summers (2019) have emphasized the extent to which government debt, pay-as-you-go pension systems, or social insurance programs raise the natural real interest rate and therefore help overcome secular stagnation. By generating a sustainable Ponzi scheme, helicopter drops can be seen as an alternative way of raising the real interest rate consistent with full employment. Also, Blanchard (2019) has pointed out that, historically, the U.S. interest rate has mostly been smaller than the rate of economic growth, implying that debt can be rolled over at no fiscal cost.

Finally, there is a small literature on the stimulative effect of public debt at the zero lower bound in the presence of credit constraints, which my framework does not allow for. Relying on an OLG model of secular stagnation with borrowing constraints, Eggertsson, Mehrotra, and Robbins (2019) have found that increases in public debt raise the natural real interest rate and therefore stimulate aggregate demand. This is due to the non-Ricardian nature of their model, not to the creation of a Ponzi scheme.[4] Caballero and Farhi (2018) have shown that secular stagnation can result from a shortage of safe assets, which can be remedied by helicopter drops. Acharya and Dogra (2018) relied on an OLG model with incomplete markets to show that, at the zero lower bound, public debt can satiate the demand for safe assets, which raises the natural real interest rate and restores full employment, but it crowds out physical capital. Raising the inflation target is preferable, but can generate harmful bubbles. Also, Ragot (2017) has found that, when insufficient demand is due to binding credit constraints, open-market operations can be more efficient than helicopter drops as they target the distribution of newly created money to the market participants who need it the most.

Building on Michau (2018), my model of secular stagnation assumes that the representative household has a preference for wealth.[5] Alternatively, demand-driven secular

[3]Buiter (2014) emphasizes that irredeemable money is an asset for households, but not a liability for the government. In my analysis, the government's no-Ponzi condition is not imposed as an equilibrium condition. The scope for Ponzi scheme is determined by households' willingness to hold Ponzi money or debt.

[4]In my model, the Ricardian equivalence holds whenever the government's no-Ponzi condition is binding. See Michau, Ono, and Schlegl (2018) for a careful discussion of the relationship between Ponzi schemes and the Ricardian equivalence.

[5]The first micro-founded model of demand-driven secular stagnation was offered by Ono (1994, 2001), who assumed an insatiable preference for liquidity. Michaillat and Saez (2015) have also built a model of the business cycle with matching frictions, where a preference for wealth can generate a permanent liquidity trap.

stagnation can be obtained within an OLG economy, as shown by Eggertsson, Mehrotra, and Robbins (2019). However, despite very different micro-foundations, the properties of the secular stagnation equilibrium are identical under the two model structures, both of which allow for the emergence of Ponzi schemes.[6] Hence, it can be safely conjectured that the results derived in this paper could also be obtained under an OLG model of secular stagnation.

Section 2 exposes the setup of the economy and characterizes the corresponding equilibria. Helicopter drops of money are analyzed in Section 3, starting with a one-off helicopter drop before turning to continuous drops. The paper ends with a conclusion. All the proofs are relegated to the appendix.

2 Economy

2.1 Households

Time is continuous. There is a mass 1 of infinitely lived households. Population within each household grows at rate n. The total population of the economy is equal to $N_t = e^{nt}$.

The representative household discounts the future at rate ρ, with $\rho > n$. Let c_t denote consumption per capita and m_t real money holdings per capita. At any point in time, the household derives utility $u(c_t)$ from consuming c_t, with $u'(\cdot) > 0$, $u''(\cdot) < 0$, and $\lim_{c\to 0} u'(c) = \infty$. It also derives utility $h(m_t)$ from the transaction convenience of real money balances m_t, with $h'(\cdot) > 0$, $h''(\cdot) < 0$, $\lim_{m\to 0} h'(m) = \infty$, and $h'(m) = 0$ for all $m \geq \bar{m}$. At $\bar{m}$, the household is satiated with real money balances and does not derive any utility from holding additional money for transaction purposes.

The household also derive utility from holding wealth a_t. However, it knows that it faces future tax liabilities. Let φ_t denote the present value of taxes from time t onwards. The household's net wealth is therefore equal to $a_t - \varphi_t$. The household derives utility $\gamma(a_t - \varphi_t)$ from holding net wealth $a_t - \varphi_t$, with $\gamma'(\cdot) > 0$, $\gamma''(\cdot) < 0$, $\gamma'(0) < \infty$, $\lim_{k\to\infty} \gamma'(k) = 0$, and $\int_0^\infty \gamma'(e^{\lambda t})dt < \infty$ for any $\lambda > 0$.[7] Note that, without the adjustment for the present value of taxes, the government could artificially increase the household's welfare by making a large lump-sum payment that is eventually offset by a large lump-sum tax. In other words, I assume that households are Ricardian and that they do not suffer from any wealth illusion from government transfers.

[6] More specifically, Michau, Ono, and Schlegl (2018) have shown that the characterization of rational bubbles or Ponzi schemes under a preference for wealth is exactly the same as under an OLG structure.

[7] This last technical condition is very mild and, under the other conditions that we have imposed on $\gamma(\cdot)$, it must be satisfied for any polynomial specification of $\gamma(\cdot)$.

The household's intertemporal utility function is given by:[8]

$$\int_0^{\infty} e^{-(\rho-n)t} \left[u\left(c_t\right)+h\left(m_t\right)+\gamma\left(a_t-\varphi_t\right)\right] dt. \tag{1}$$

At each point in time, the household inelastically supplies N_t units of labor. However, labor demand is only equal to $N_t L_t$ with $L_t \in [0,1]$. As we shall see, in a stagnation equilibrium, $L_t < 1$. The real wage is w_t. The household receives dividends from firm ownership equal to ξ_t per capita and must pay a lump-sum tax τ_t per capita. Wealth yields a real return r_t. However, population growth within the household results in a dilution of wealth. The net return on wealth per capita is therefore equal to $r_t - n$. Finally, the real opportunity cost of holding money is equal to the forgone nominal interest rate i_t that could have been earned by holding bonds instead of money. Household wealth per capita therefore evolves according to:[9]

$$\dot{a}_t = \left(r_t - n\right) a_t - i_t m_t + w_t L_t + \xi_t - \tau_t - c_t. \tag{2}$$

By the Fisher identity, we must have $r_t = i_t - \pi_t$, where π_t denotes the inflation rate. Finally, the household is subject to an intertemporal budget constraint that prevents it from running Ponzi schemes:

$$\lim_{t \to \infty} e^{-\int_0^t (r_s - n) ds} a_t \geq 0. \tag{3}$$

The household maximizes its intertemporal utility (1) subject to its budget constraint (2) and (3) with a_0 given. By the maximum principle, the solution to the household's problem is characterized by a consumption Euler equation:

$$\frac{\dot{c}_t}{c_t} = \left[r_t - \rho + \frac{\gamma'\left(a_t-\varphi_t\right)}{u'\left(c_t\right)}\right] \frac{u'\left(c_t\right)}{-u''\left(c_t\right) c_t}, \tag{4}$$

a money demand equation:

$$h'\left(m_t\right) = i_t u'\left(c_t\right), \tag{5}$$

[8] All the results of the paper would hold under a rate g of technical progress, provided that the household has balanced growth preferences:

$$\int_0^{\infty} e^{-(\rho-n)t} \left[\ln\left(c_t\right) + h\left(\frac{m_t}{y_t}\right) + \gamma\left(\frac{a_t-\varphi_t}{y_t}\right)\right] dt,$$

where y_t denotes output per capita (or alternatively, to obtain exactly the same formulae as in the paper, $y_t = e^{gt}$). Under all steady states (including the secular stagnation steady state), the economy would grow at rate $n+g$ instead of n.

[9] A similar wealth accumulation equation is formally derived in a nominal economy without population growth in Michau (2018) and in a real economy with population growth in Michau, Ono, and Schlegl (2018).

and a transversality condition:

$$\lim_{t\to\infty} e^{-(\rho-n)t}u'(c_t)\,a_t = 0. \tag{6}$$

Note that the money demand equation (5) imposes a zero lower bound on the nominal interest rate. This follows from the fact that money always yields a zero nominal return.

2.2 Firms

For simplicity, I assume that labor is the only factor of production.[10] Total population, and hence aggregate labor supply, is equal to N_t. A representative firm employs a fraction L_t of this supply, where $L_t \in [0,1]$. The aggregate production function is $N_t f(L_t)$ with $f'(\cdot) > 0$, $f''(\cdot) \leq 0$, and $f(0) = 0$. Thus, the economy displays constant returns to scale with respect to the total population of the economy, but non-increasing returns with respect to the fraction of the labor force in employment.

The aggregate production of consumption goods satisfies $c_t N_t = N_t f(L_t)$ or, equivalently:

$$c_t = f(L_t). \tag{7}$$

The firms chooses labor demand L_t such as to maximize profits $N_t f(L_t) - w_t N_t L_t$, which implies that the equilibrium real wage must always be equal to marginal product of labor:

$$w_t = f'(L_t). \tag{8}$$

Aggregate profits $\xi_t N_t$ are therefore equal to $N_t f(L_t) - f'(L_t)\,N_t L_t$ or, equivalently:

$$\xi_t = f(L_t) - f'(L_t)\,L_t. \tag{9}$$

Profits are strictly positive whenever the production function is characterized by decreasing returns to scale.

2.3 Downward Wage Rigidity

The profit maximizing behavior of firms implies that the nominal wage W_t is always equal to the marginal product of labor $P_t f'(L_t)$, as implied by (8). If the wage rate was completely flexible, then labor demand L_t would be equal to labor supply 1. The nominal wage would therefore be equal to $P_t f'(1)$.

[10] Allowing for capital would require introducing adjustment costs in order to slow down the response of investment to consumption shocks under secular stagnation. To obtain a complete analytical characterization of equilibrium possibilities under helicopter drops, I have chosen to focus on a simpler framework with labor as the only input.

For a given employment level L_t, the nominal wage $P_t f'(L_t)$ grows at rate π_t. I now assume that, for a given L_t, workers do not accept nominal wage growth to fall below a reference rate of inflation π^R. We must therefore have $(1+\pi_t dt) W_t \geq \left(1+\pi^R dt\right) W_t$ and $W_t \geq P_t f'(1)$ with complementary slackness.[11] Equivalently, we must have:

$$\pi_t \geq \pi^R \text{ and } L_t \leq 1 \text{ with complementary slackness.} \tag{10}$$

If labor demand L_t is smaller than labor supply 1, then the downward wage rigidity constraint must be binding, i.e. $\pi_t = \pi^R$. Conversely, if the constraint is not binding, then the economy must be at full employment, i.e. $L_t = 1$.

When there is under-employment, the downward wage rigidity constraint is binding and the real interest rate satisfies $r_t = i_t - \pi^R \geq -\pi^R$. The consumption Euler equation (4) imposes that, under any steady state, we must have $\rho > r_t$. I therefore assume throughout my analysis that $\pi^R > -\rho$ as, otherwise, we cannot have a secular stagnation steady state with under-employment.

2.4 Government

Let M_0 denote the initial money supply and ω_t the nominal money growth rate at time t. The nominal quantity of money at t is therefore equal to:

$$M_t = e^{\int_0^t \omega_s ds} M_0. \tag{11}$$

The price level P_t is related to inflation through $\pi_t = \dot{P}_t / P_t$. The real supply of money per capita m_t is given by:

$$\begin{aligned} m_t &= \frac{M_t}{P_t N_t}, \\ &= e^{\int_0^t (\omega_s - \pi_s - n) ds} \frac{M_0}{P_0}. \end{aligned} \tag{12}$$

Thus, the evolution of the money supply over time satisfies:

$$\dot{m}_t = (\omega_t - \pi_t - n)\, m_t. \tag{13}$$

Let B_t denote the outstanding supply of nominal government bonds. Real government indebtedness per capita b_t is therefore equal to $B_t/(P_t N_t)$. Newly supplied money is

[11] This corresponds to the downward wage rigidity constraint of Michau (2018) in the special case where the wage flexibility parameter α is equal to zero. In Michau (2018), when $\alpha > 0$, the gap between the wage rate $P_t f'(L_t)$ and the marginal product of labor at full employment $P_t f'(1)$ determines the extent to which inflation can fall below π^R. Empirically, the Phillips curve is very flat at low rates of inflation, which suggests a value of α close to zero.

revenue to the government, whose debt level therefore evolves according to:

$$\dot{b}_t = (r_t - n)\, b_t - \tau_t - \omega_t m_t. \tag{14}$$

Combining the previous two equations yields:

$$\dot{b}_t + \dot{m}_t = (r_t - n)\, [b_t + m_t] - \tau_t - i_t m_t. \tag{15}$$

By issuing money rather than bonds, the government economizes the nominal interest rate, which is the difference between the real returns r_t on bonds and $-\pi_t$ on money. Thus, from a fiscal point of view, the nominal interest rate is a tax on households' money holdings. So, the present value of taxes φ_t from t onwards is equal to:

$$\varphi_t = \int_t^{\infty} e^{-\int_t^s (r_u - n) du}\, (\tau_s + i_s m_s)\, ds. \tag{16}$$

Importantly, when solving the household's problem, I have taken the present value of taxes φ_t as exogenously given. Hence, the household does not internalize the impact of future real money holdings on the present value of future taxes. While I could easily solve the alternative case, where the household reduces its future money holdings such as to feel wealthier today, it seems behaviorally implausible.

If the government does not run a Ponzi scheme, then it must be able to redeem both its supply of bonds and of money. The government's no-Ponzi condition is therefore given by:

$$\lim_{t \to \infty} e^{-\int_0^t (r_s - n) ds}\, (b_t + m_t) \leq 0, \tag{17}$$

or, equivalently, by:

$$b_t + m_t \leq \varphi_t. \tag{18}$$

This implies that the magnitude of a Ponzi scheme is equal to:

$$\Delta_t = b_t + m_t - \varphi_t. \tag{19}$$

By the government liability accumulation equation (15) and the definition of the present value of taxes (16), we have:

$$\dot{\Delta}_t = (r_t - n)\, \Delta_t, \tag{20}$$

regardless of monetary and fiscal policy. Throughout my analysis, I exclusively focus on cases where the government's no-Ponzi condition is either binding or violated, i.e. $\Delta_t \geq 0$.

2.5 Equilibrium

Household wealth a_t is composed of bonds b_t and of real money holdings m_t. This yields the asset market clearing condition:

$$a_t = b_t + m_t. \tag{21}$$

The household's transversality condition (6) can therefore be written as $\lim_{t\to\infty} e^{-(\rho-n)t}u'(c_t)[\Delta_t + \varphi_t] = 0$. The present value of taxes φ_t does not appear in any of the other equilibrium conditions of the economy. But, by the following lemma, it can be eliminated from the transversality condition.

Lemma 1 *If φ_t is not finite, then an equilibrium cannot exist. If φ_t is finite, then $\lim_{t\to\infty} e^{-(\rho-n)t}u'(c_t)\varphi_t = 0$.*

The household's transversality condition can therefore be simplified to:

$$\lim_{t\to\infty} e^{-(\rho-n)t}u'(c_t)\Delta_t = 0. \tag{22}$$

The equilibrium of the economy, $(c_t, L_t, \Delta_t, r_t, m_t, \pi_t)_{t=0}^{\infty}$ and P_0, is fully characterized by the household's optimality conditions for consumption:[12]

$$\frac{\dot{c}_t}{c_t} = \left[r_t - \rho + \frac{\gamma'(\Delta_t)}{u'(c_t)}\right]\frac{u'(c_t)}{-u''(c_t)\,c_t}, \tag{23}$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}u'(c_t)\Delta_t = 0; \tag{24}$$

the government's Ponzi scheme:

$$\dot{\Delta}_t = [r_t - n]\Delta_t; \tag{25}$$

the goods market clearing condition:

$$c_t = f(L_t); \tag{26}$$

the downward wage rigidity constraint:

$$\pi_t \geq \pi^R \text{ and } L_t \leq 1 \text{ with complementary slackness;} \tag{27}$$

[12] We have omitted the household's no-Ponzi condition $\lim_{t\to\infty} e^{-\int_0^t (r_s-n)ds}a_t \geq 0$ as it must always be satisfied. This follows from the fact that $a_t = \Delta_t + \varphi_t$ with $\Delta_t \geq 0$ and, by the proof of Lemma 1, $\lim_{t\to\infty} e^{-\int_0^t (r_s-n)ds}\varphi_t = 0$.

the money supply equation:

$$m_t = e^{\int_0^t (\omega_s - \pi_s - n) ds} \frac{M_0}{P_0}, \tag{28}$$

where M_0 is exogenously given; and, finally, the money demand equation:

$$h'(m_t) = [r_t + \pi_t] \, u'(c_t), \tag{29}$$

which imposes that the nominal interest rate $r_t + \pi_t$ cannot be negative.

The initial size of the Ponzi scheme Δ_0 is not exogenously given, but it is constrained by the government's policy. Following Michau, Ono, and Schlegl (2018), I assume that the government determines the maximum size of the Ponzi scheme it is willing to implement, denoted by $\bar{\Delta}_0$. However, even when a Ponzi scheme of magnitude $\bar{\Delta}_0$ is theoretically feasible, it can only be an equilibrium outcome if households believe it to be sustainable. Thus, even when $\bar{\Delta}_0 > 0$, the government cannot prevent the occurrence of an equilibrium with $\Delta_0 = 0$, where a Ponzi scheme does not occur since households do not believe it to be sustainable. In equilibrium, we must always have $\Delta_0 \leq \bar{\Delta}_0$. By choosing $\bar{\Delta}_0$, the government only prevents the occurrence of Ponzi schemes larger than $\bar{\Delta}_0$.

When the nominal interest rate is at the zero lower bound, real money balances m_t can exceed the transaction demand for money $\bar{m}$, the difference $m_t - \bar{m}$ being held for savings. This corresponds to a liquidity trap situation. Note that an equilibrium with both $L_t = 1$ and $m_t > \bar{m}$ cannot be stable. Indeed, if a strictly positive mass of households decide to spend the money they hold for savings when $L_t = 1$, then the price level must jump upwards such as to bring real money balances below $\bar{m}$. Hence, throughout my analysis, I consider that the economy cannot be liquidity trapped when it produces at full capacity, i.e. $m_t \leq \bar{m}$ whenever $L_t = 1$.

2.6 Steady State Equilibria

Let us now characterize the steady state equilibria $(c, L, \Delta, r, \pi, i)$ of our economy, assuming a constant growth rate ω of the money supply. From the Ponzi equation (25) we must either have $r = n$ or $\Delta = 0$; while from the downward wage rigidity (27) we must either have $r = i - \pi^R$ or $L = 1$. This gives us three different types of steady state equilibria:[13]

- A neoclassical steady state with $\Delta = 0$ and $L = 1$;
- A secular stagnation steady state with $r = i - \pi^R$, $\Delta = 0$, and $L < 1$;

[13] A fourth possibility is to have $\Delta > 0$, $L < 1$, and $i = n + \pi^R$. However, from the money demand (29) and money supply (28) equations, this either requires $\omega = n + \pi^R \geq 0$ or $\omega \geq n + \pi^R = 0$, both of which are generically not satisfied.

- And a Ponzi steady state with $r = n$, $L = 1$, and $\Delta > 0$.

All three types of steady state equilibria must satisfy the consumption Euler equation:

$$\gamma' (\Delta) = (\rho - r) \, u' (c) \, , \tag{30}$$

and the goods market clearing equations:

$$c = f (L) \, . \tag{31}$$

I now characterize the three types of steady state equilibria and provide the corresponding existence conditions.

2.6.1 Neoclassical Steady State

The neoclassical steady state $(c^n, L^n, \Delta^n, r^n, \pi^n, i^n)$ is characterized by full employment $L^n = 1$ and no Ponzi scheme $\Delta^n = 0$. Thus, the consumption level is given by the market clearing equation (31):

$$c^n = f (1) \, , \tag{32}$$

while the real interest rate is determined by the consumption Euler equation (30):

$$r^n = \rho - \frac{\gamma' (0)}{u' (f (1))} . \tag{33}$$

Importantly, r^n is the natural real interest rate of the economy. A weak level of aggregate demand, induced by a strong preference for wealth $\gamma' (0)$, entails a low natural real interest rate r^n.

As discussed above, I consider that the economy is not liquidity trapped, i.e. $m_t \leq \bar{m}$, whenever the economy produces at full capacity. By the money demand equation (29), m_t must therefore be constant over time and, by the money supply equation (28), inflation π^n must be equal to $\omega - n$. The nominal interest rate i^n is therefore equal to $r^n + \omega - n$, but it must be non-negative. Thus, a neoclassical steady state exists, and is unique, if and only if $\omega \geq n - r^n$.[14]

2.6.2 Secular Stagnation Steady State

The secular stagnation steady state $(c^s, L^s, \Delta^s, r^s, \pi^s, i^s)$ is characterized by under-employment $L^s < 1$. The downward wage rigidity constraint must therefore be binding, which forces

[14] For simplicity, throughout my analysis, I ignore the possibility of speculative hyperinflation. It can formally be ruled out by assuming either government backing of currency or $\lim_{m \to 0+} m h'(m) > 0$ (Obstfeld and Rogoff 1983, 2017). Speculative deflation would eventually lead to $m_t > \bar{m}$, which is inconsistent with $L_t = 1$.

inflation π^s to be equal to π^R.

In steady state, the nominal interest rate i^s cannot be strictly positive. If such was the case, then, by the money demand equation (29), real money balances would have to be constant over time and, by the money supply equation (28), this would require $\omega = n + \pi^R$, which generically fails to hold.[15] We must therefore have $i^s = 0$ and, hence, $r^s = -\pi^R$. To have a steady state with zero nominal interest rate, real money balances must not shrink over time, which requires $\omega \geq n + \pi^R$.

As the real interest rate $-\pi^R$ is generically different from n, secular stagnation is inconsistent with the existence of a steady state Ponzi scheme. Hence, $\Delta^s = 0$. The level of aggregate demand is determined by the consumption Euler equation (30):

$$u'(c^s) = \frac{\gamma'(0)}{\rho + \pi^R}, \tag{34}$$

while the corresponding employment level follows from the market clearing equation (31):

$$f(L^s) = c^s. \tag{35}$$

For the economy to be in stagnation, we must have $L^s < 1$. From the previous three equations, (33)-(35), $L^s < 1$ if and only if $r^n < -\pi^R$. In other words, the possibility of secular stagnation requires aggregate demand to be so depressed that the natural real interest rate is smaller than $-\pi^R$.

We have shown that a secular stagnation steady states exists, and is unique, if and only if $r^n < -\pi^R$ and $\omega \geq n + \pi^R$.

2.6.3 Ponzi Steady State

The Ponzi steady state $(c^p, L^p, \Delta^p, r^p, \pi^p, i^p)$ is characterized by $\Delta^p > 0$ and $r^p = n$. We have just established that in general, whenever there is less than full employment, the real interest rate is equal to $-\pi^R$, which is generically different from n. We must therefore have full employment $L^p = 1$, while the consumption level is the same as in the neoclassical steady state:[16]

$$c^p = f(1). \tag{36}$$

The equilibrium magnitude of the Ponzi scheme is given by the consumption Euler equation (30):

$$\gamma'(\Delta^p) = (\rho - n) u'(f(1)). \tag{37}$$

[15] In the special case where $\omega = n + \pi^R$, any level of i^s can be an equilibrium. But, even in that case, we can consider that the government would select the zero nominal interest rate equilibrium, by intervening in the bonds market, such as to minimize the real interest rate.

[16] In the presence of capital, the consumption level would be higher in the Ponzi steady state than in the neoclassical steady state (Michau, Ono, and Schlegl 2018).

For the Ponzi steady state to exist, we must have $\Delta^p > 0$. Hence, from (33) and (37), the Ponzi steady state exists if and only if $r^n < n$.

The intuition for the $r^n < n$ condition is straightforward. By the Ponzi dynamics (25), a Ponzi scheme of constant magnitude requires the interest rate to be equal to the population growth rate. But, by the consumption Euler equation (30), a Ponzi scheme reduces the marginal utility of wealth, which raises the real interest rate. Hence, the existence of a Ponzi steady state requires the real interest rate without a Ponzi scheme, r^n, to be smaller to the population growth rate, n.

As in the neoclassical steady state, the fact that the economy produces at full capacity implies that $\pi^p = \omega - n$. We must therefore have $i^p = r^p + \pi^p = \omega$, which must be nonnegative. Thus, a Ponzi steady state exists, and is unique, if and only if $r^n < n$ and $\omega \geq 0$.

Finally, while we have characterized the equilibria of the economy for any value of Δ_0, recall that a Ponzi steady state can only arise if the government is willing to implement a Ponzi scheme of sufficient magnitude, i.e. if $\bar{\Delta}_0$ is sufficiently large.

3 Helicopter Drops of Money

Fundamentally, I now want to investigate whether the government can lift the economy out of secular stagnation by violating its no-Ponzi condition. Hence, the type of government spending that I consider consists of transfers to households, not of public expenditures. In our Ricardian economy, these transfers are neutral if the government's no-Ponzi condition is always binding. Hence, they can only have real effects by breaking the government budget constraint. For simplicity, I assume that these budget-breaking transfers are financed through helicopter drops of money.

I assume that the economy is initially in the secular stagnation steady state. This implies that the condition $r^n < -\pi^R$ is satisfied. The outstanding supply of government bonds is nonnegative, $B_0 \geq 0$, and the government's no-Ponzi condition is binding, $\Delta_0 = \bar{\Delta}_0 = 0$. Real money balances M_0/P_0 are greater or equal to $\bar{m}$. The growth rate ω of the money supply is constant, with $\omega \geq n - r^n$. This implies that, even though the economy is initially in stagnation, the money growth rate is sufficiently large to make the neoclassical steady state feasible. It also implies that, if $r^n < n$, then $\omega > 0$ and the Ponzi steady state is also feasible.

The Ponzi steady state can only be reached if the government runs a Ponzi scheme of sufficient magnitude. By contrast, the economy can always jump to the neoclassical steady state, independently of any helicopter drop. I therefore assume throughout my analysis that the economy does not move to the neoclassical steady state, unless it is forced to escape secular stagnation. This assumption reflects the fact that secular stagnation is a severe economic problem precisely because raising the growth rate of the money supply above a certain threshold is not sufficient to escape it.

I first consider a one-off helicopter drop of money, before turning to continuous helicopter drops.

3.1 One-Off Helicopter Drop of Money

At time 0, a surprise one-off helicopter drop of money of magnitude H is implemented. The level of government liabilities therefore increases from $B_0 + M_0$ to $B_0 + M_0 + H$.

In addition, there are two instances where the government makes discrete adjustments to the money supply through open-market operations. First, when the economy starts producing at full capacity, i.e. when L_t becomes equal to 1, the government redeems a fraction of the money supply such as to prevent an upward jump in the price level. Second, whenever $L_t < 1$, the government imposes $i_t = 0$ by offering to buy an arbitrarily large quantity of nominal bonds at zero interest rate.[17] Crucially, these monetary policy interventions, unlike the helicopter drop of money, are performed through open-market operations and therefore cannot affect the level of (nominal) government liabilities.

If, following the helicopter drop, the economy remains in secular stagnation, then the present value of taxes φ_0, defined by (16), remains unchanged.[18] However, if the economy moves away from secular stagnation, then changes in the real and nominal interest rates can modify the present value of tax revenue φ_0. In order to eliminate these indirect fiscal repercussions, I make the following assumption.

Assumption 1 *Following the helicopter drop of money H, for any equilibrium path, lump-sum taxes τ_t adjust such as to leave the initial present value of tax revenue φ_0 unchanged.*

Let P_0', Δ_0', and $\bar{\Delta}_0'$ denote the price level, the magnitude of the Ponzi scheme, and the maximum magnitude of the Ponzi scheme immediately after the implementation of the helicopter drop. By Assumption 1, the level of government liabilities increases from $(B_0 + M_0)/P_0$ to $(B_0 + M_0 + H)/P_0'$. We therefore have:

$$\Delta_0' = \frac{B_0 + M_0 + H}{P_0'} - \frac{B_0 + M_0}{P_0}. \tag{38}$$

The downward wage rigidity implies that the price level cannot jump downward, i.e. we

[17] Helicopter drops of money increase real money balances beyond $\bar{m}$. Hence, this second possibility can only be relevant along an equilibrium path where initially $L_t = 1$, triggering a reduction in the money supply, followed by $L_t < 1$. Note that setting $i_t = 0$ requires at most a single open market operation at the point in time when L_t falls below 1 since, once $L_t < 1$, we have $\dot{m}_t/m_t = \omega - \pi^R - n = (\omega - n + r^n) + (-\pi^R - r^n) > 0$.

[18] I exclude the fiscal transfer H at time 0 from the definition of φ_0.

must always have $P_0' \geq P_0$. Hence, the maximum Ponzi scheme occurs when $P_0' = P_0$, which gives $\bar{\Delta}_0' = H/P_0$.

Assumption 1 is in the spirit of the hypothesis of monetary dominance whereby, when economic circumstances change, the government modifies the present value of lump-sum taxes such as to keep its intertemporal budget constraint binding. Here, it conveniently implies that the maximum magnitude of the Ponzi scheme $\bar{\Delta}_0' = H/P_0$ is exogenously determined by H.

The equilibrium responses to a helicopter drop of money are fully characterized by the following proposition.[19]

Proposition 1 *Following a one-off helicopter drop of money:*

- *There always exists an equilibrium with $\Delta_0' = 0$, where the price level increases by:*

$$\frac{P_0'}{P_0} = 1 + \frac{H}{B_0 + M_0}, \tag{39}$$

  without any real effect on the economy, which therefore remains in the secular stagnation steady state.

- *The equilibrium possibilities with $\Delta_0' \in (0, H/P_0]$ are:*

  - *If $r^n < n$, for $\Delta_0' = \Delta^p$, there exists an equilibrium where the economy immediately jumps to the Ponzi steady state, where it then remains.*
  - *If $r^n < n < -\pi^R$, for any given $\Delta_0' \in (0, \Delta^p)$, there exists an equilibrium path where the economy converges to the Ponzi steady state in finite time. The economy operates below full capacity until it reaches the Ponzi steady state.*
  - *If $r^n < -\pi^R < n$, for any given $\Delta_0' \in (0, \tilde{\Delta}]$ for some parameter $\tilde{\Delta} > 0$,[20] there exists an equilibrium path where the economy asymptotically converges to the secular stagnation steady state. The economy operates below full capacity (except at time 0 when $\Delta_0' = \tilde{\Delta}$) and shrinks over time along this path.[21]*

[19] We maintain the assumption that the economy does not converge to the neoclassical steady state unless it is forced to escape secular stagnation.

[20] The parameter $\tilde{\Delta} \in (0, \infty)$ is the solution to:

$$\frac{\gamma'(\Delta^p)}{\rho - n} = \int_0^\infty e^{-\left(\pi^R + \rho\right)t} \gamma' \left(e^{-\left(\pi^R + n\right)t} \tilde{\Delta}\right) dt.$$

This definition implies that, if $-x\gamma''(x)/\gamma'(x)$ is independent of x, then $\tilde{\Delta} < \Delta^p$ if and only if $-x\gamma''(x)/\gamma'(x) < 1$.

[21] In addition, it is possible for the economy to produce at full capacity for some time before reaching $\Delta_t = \tilde{\Delta}$, at which point the economy starts converging to the secular stagnation steady state. If $\tilde{\Delta} > \Delta^p$, we must have $\dot{\Delta}_t > 0$ and $\Delta_t > \Delta^p$ while $c_t = f(1)$. And, if $\tilde{\Delta} < \Delta^p$, we must have $\dot{\Delta}_t < 0$ and $\Delta_t < \Delta^p$ while $c_t = f(1)$. Thus, Δ_t never exceeds $\max\{\Delta^p, \tilde{\Delta}\}$.

It is always possible that a one-off helicopter drop of money triggers an upward jump in the price level without affecting the real allocation of resources. The helicopter drop nonetheless raises real money balances, provided that $B_0 > 0$.

When $r^n < n$, it is also possible for the helicopter drop to have real effects on the economy. In such cases, the increase in government liabilities is not fully offset by the upward jump in the price level, which generates a Ponzi scheme that raises household wealth. This reduces the marginal utility of wealth and stimulates households' demand for consumption. In particular, when H/P_0 is not too large, the price level can remain constant at time 0, in which case the government must be running a Ponzi scheme of magnitude $\Delta_0' = \bar{\Delta}_0' = H/P_0 > 0$. So, if the price level does not jump, the helicopter drop of money must stimulate aggregate demand.

Two cases must be distinguished depending on whether the stagnation real interest rate $-\pi^R$ is below or above the population growth rate n. If $-\pi^R > n$, then government liabilities increase at a faster rate than the economy. Hence, the magnitude of the Ponzi scheme increases over time until the economy reaches the Ponzi steady state. This possibility requires $r^n < n$ as, otherwise, the Ponzi steady state does not exist. Alternatively, if $-\pi^R < n$, then the Ponzi scheme shrinks over time and the economy gradually reverts back to the secular stagnation steady state. Note that, as $r^n < -\pi^R$, even in this case we must have $r^n < n$.

Ponzi schemes are impossible whenever $r^n \geq n$. So, paradoxically, public expenditures cannot be financed by just printing money, even though the economy is permanently liquidity trapped. The reason is that, when $r^n \geq n$, any Ponzi scheme would be explosive, which would violate households' transversality condition (24).[22]

Proposition 1 gives us the maximum magnitude that Ponzi schemes can reach. If $r^n \geq n$, then Ponzi schemes cannot occur. If $r^n < n$ and $n < -\pi^R$, then Ponzi schemes are always feasible up to magnitude Δ^p. Finally, if $r^n < n$ and $-\pi^R < n$, Ponzi schemes can never exceed $\max\{\Delta^p, \tilde{\Delta}\}$. If the economy reaches these limits, it must be producing at full capacity and any additional helicopter drop of money must trigger an upward jump in the price level.

In addition to the theoretical possibilities of Proposition 1, the helicopter drop can induce households to coordinate on higher inflation expectations, equal to $\omega - n$, with the economy jumping to the neoclassical steady state. However, any sunspot can have the same effect, which is why I have assumed that the economy does not jump to the neoclassical steady state, unless it is forced to.

When $r^n < n$, the government's intertemporal budget constraint is not necessarily

[22] If $r^n \geq n$, we must indeed have $r_t > n$ whenever $\Delta_t > 0$. If the economy remains in stagnation, the real interest rate $-\pi^R$ is larger than n, since $-\pi^R > r^n$ and $r^n \geq n$. And, if the economy produces at full capacity, by the Euler equation (23) with $c_t = f(1)$, the real interest rate r_t is equal to $\rho - \gamma'(\Delta_t)/u'(f(1))$, which is also larger than n for any $\Delta_t > 0$, since $r_t > \rho - \gamma'(0)/u'(f(1)) = r^n \geq n$.

binding and, hence, the fiscal theory of the price level is not applicable. The same conclusion was reached by Bassetto and Cui (2018) within a dynamically inefficient overlapping generation economy.

3.2 Continuous Helicopter Drops of Money

Let us now investigate a more radical policy option, where the government implements continuous helicopter drops of money whenever the economy fails to produce at full capacity. I still assume that, initially, the economy is in the secular stagnation steady state with $B_0 \geq 0$, $\Delta_0 = \bar{\Delta}_0 = 0$, and $M_0/P_0 \geq \bar{m}$.

From time 0 onwards, whenever $L_t < 1$, the government implements a constant growth rate $\tilde{\omega}$ of the money supply that is positive and that results in growing real money balances, i.e. $\tilde{\omega} > \max\{0, \pi^R + n\}$. All the newly printed money is given lump-sum to households, who therefore receive a sequence of monetary transfers of magnitude $\tilde{\omega} m_t$ at time t.

When the economy reaches full employment, the helicopter drops stop and monetary policy reverts back to the previous situation with a constant growth rate ω of the money supply, satisfying $\omega \geq n - r^n$ (while the monetary transfers drop to zero). In addition, when L_t becomes equal to 1, the government implements a discrete reduction in the money supply, through an open-market operation, such as to prevent an upward jump in the price level. Also, whenever $L_t < 1$, the government is willing to intervene to impose $i_t = 0$.[23] Again, these open-market operations do not affect the level of (nominal) government liabilities.

For simplicity, I shall focus on equilibrium trajectories where $L_t = 1$ is an absorbing state. Let T denote the point in time when the economy starts producing at full capacity. Note that T can be infinite. From time 0 to T, the nominal interest rate is equal to zero and the downward wage rigidity is binding, resulting in inflation being equal to π^R. The real interest rate must therefore be equal to $-\pi^R$. So, the continuous helicopter drops only modify the paths of the nominal and real interest rates after time T. This can alter the present value of tax revenue from time T onwards, φ_T. I eliminate these indirect fiscal repercussions through the following assumption.

Assumption 2 *For any equilibrium path, lump-sum taxes τ_t adjust such as to leave the present value of tax revenue from time T onwards, φ_T, unchanged.*

This assumption, the counterpart to Assumption 1 of the previous subsection, is consistent with the idea that monetary dominance prevails as soon as the helicopter drops

[23] As $\tilde{\omega} > \pi^R + n$, real money balances are growing throughout the continuous helicopter drops. Hence, the government does not need to intervene to impose $i_t = 0$ along any of the equilibrium paths that I characterize below.

end.

From time 0 to T, households receive a sequence of lump-sum transfers of magnitude $\tilde{\omega} m_t$. Also, the announcement of continuous drops can trigger an upward jump in the price level at time 0, which reduces the real value of government liabilities. By Assumption 2, we can ignore the indirect fiscal repercussions occurring after time T. Hence, at time 0, the magnitude of the Ponzi scheme rises from $\Delta_0 = 0$ to:

$$\Delta'_0 = \int_0^T e^{-\int_0^t (r_u - n) du} \tilde{\omega} m_t dt + \frac{B_0 + M_0}{P'_0} - \frac{B_0 + M_0}{P_0}. \tag{40}$$

Using the money supply equation (28), together with $r_t = -\pi^R$ for $t \in [0, T]$, this simplifies to:[24]

$$\begin{aligned} \Delta'_0 &= \frac{M_0}{P'_0} \int_0^T \tilde{\omega} e^{\tilde{\omega} t} dt - \frac{B_0 + M_0}{P_0} \frac{P'_0 - P_0}{P'_0}, \\ &= \frac{M_0}{P'_0} \left[e^{\tilde{\omega} T} - 1 \right] - \frac{B_0 + M_0}{P_0} \frac{P'_0 - P_0}{P'_0}. \end{aligned} \tag{41}$$

This expression readily implies that, if T is not finite, then the magnitude Δ'_0 of the Ponzi scheme is not finite either. But, by Lemma 1, this cannot be an equilibrium possibility.

Lemma 2 *In equilibrium, continuous helicopter drops of money of magnitude $\tilde{\omega} m_t$, with $\tilde{\omega} > 0$, cannot last forever.*

Hence, the economy cannot remain in the secular stagnation steady state forever. By the same token, asymptotic convergence to a full employment steady state is also ruled out. The continuous helicopter drops of money are therefore bound to induce the economy to produce at full capacity in finite time.

The one-off helicopter drop of money can be fully offset by an upward jump in the initial price level. Offsetting continuous drops that last forever would require a continuously rising price level. But, this is not possible in a stagnating economy that is deprived of any inflationary pressure. Hence, the continuous drops are bound to be stimulative.

From the Ponzi dynamics (25), from time 0 to T, the Ponzi scheme grows at rate $-\pi^R - n$. This implies that the magnitude of the Ponzi scheme at time T is given by:

$$\Delta_T = e^{-\left(\pi^R + n\right) T} \Delta'_0, \tag{42}$$

where Δ'_0 is given by (41). As $\tilde{\omega} > \max\{0, \pi^R + n\}$, for any given value of P'_0, this magnitude is strictly increasing in T and diverges as T tends to infinity.

[24] Note that, as T is endogenous, there is no exogenous upper bound $\bar{\Delta}'_0$ to the magnitude of this Ponzi scheme.

A jump in the initial price level can reduce real money balances as well as the real value of government debt. This can be offset by a longer time span T under continuous drops, without any difference for the state of the economy at time T. Hence, for simplicity, I henceforth focus on equilibria without a jump in the initial price level, where $P_0' = P_0$. We must therefore have:

$$\Delta_T = \frac{M_0}{P_0} e^{-\left(\pi^R + n\right)T} \left[e^{\tilde{\omega}T} - 1\right]. \tag{43}$$

This implies a one-to-one relationship between the length T of the depression and the magnitude Δ_T of the Ponzi scheme when the economy starts producing at full capacity. I henceforth focus on Δ_T rather than T.

The equilibrium responses to the continuous helicopter drops of money are characterized by the following proposition.

Proposition 2 *Following the implementation of continuous helicopter drops of money:*

- *There always exists an equilibrium where the economy immediately jumps to the neoclassical steady state, where it then remains.*
- *If $r^n < n$, for any given $\Delta_T \in (0, \Delta^p)$, there exists an equilibrium path where the economy asymptotically converges the neoclassical steady state from time T onwards. After time T, the Ponzi scheme monotonically converges to zero.*
- *If $r^n < n$, for $\Delta_T = \Delta^p$, there exists an equilibrium path where the economy is in the Ponzi steady state from time T onwards.*

The economy can always jump to the neoclassical steady state at time 0. In fact, this is the only possible outcome when $r^n \geq n$. Any Ponzi scheme $\Delta_t > 0$ would raise the real interest rate above r^n, resulting in explosive Ponzi dynamics.[25]

When $r^n < n$, the Ponzi steady state exists and there is an equilibrium path where the economy remains depressed for sufficiently long to generate a Ponzi scheme of magnitude Δ^p. It is also possible for the economy to be on a path where it starts producing at full capacity before the Ponzi scheme reaches Δ^p. In that case, after time T the Ponzi scheme asymptotically converges to zero, leading the economy to the neoclassical steady state.[26]

[25] Formally, by the Euler equation (23), once $c_t = f(1)$, the real interest rate r_t is equal to $\rho - \gamma'(\Delta_t)/u'(f(1))$. Thus, if $\Delta_t > 0$, then $r_t > \rho - \gamma'(0)/u'(f(1)) = r^n$ and, by the Ponzi dynamics (25), Δ_t must be diverging, which violates the transversality condition (24).

[26] Formally, by the Euler equation (23), once $c_t = f(1)$, we have $r_t = \rho - \gamma'(\Delta_t)/u'(f(1))$. Thus, if $\Delta_t < \Delta^p$, then $r_t < \rho - \gamma'(\Delta^p)/u'(f(1)) = n$ and, by the Ponzi dynamics (25), Δ_t asymptotically converges to 0.

4 Conclusion

This paper has uncovered a tight connection between the real effects of helicopter drops of money under secular stagnation and the existence of a Ponzi steady state under full employment. If an helicopter drop of finite magnitude does not induce an upward jump in the price level, then a Ponzi steady state must exist.

Japan has now accumulated a level of public liabilities that will be hard to pay for with future tax revenue. However, this has not triggered any increase in inflation. This suggests that aggregate demand is so depressed in Japan, and the natural real interest rate is so low, that a Ponzi steady state exists. This concurs with Geerolf's (2018) finding that the Japanese economy is dynamically inefficient.

As Japan's real interest rate under stagnation, equal to the negative of the inflation rate, tends to be smaller than the rate of economic growth, Japan keeps converging back to stagnation. It would however be possible to escape secular stagnation by credibly committing to keep implementing helicopter drops until the stagnation is over. However, to reach the neoclassical steady state, the money growth rate under full employment (i.e. the inflation target) must be sufficiently high to prevent a binding zero lower bound.[27]

While I have assumed full commitment throughout my analysis, it should be emphasized that this policy does not require any commitment outside stagnation. So, it remains effective (i.e. Lemma 2 continues to apply), even if households expect past helicopter drops to be reversed once the stagnation is over.

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[27] If, under full employment, the money growth rate satisfies $\omega \in [0, n - r^n)$ with $r^n < n$, then the Ponzi steady state is reachable but the neoclassical steady state is not. In that case, if the government does not run a Ponzi scheme, then a rational bubble must arise on a private asset.

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A Proof of Lemma 1

By definition of φ_t, given by (16), we have:

$$\dot{\varphi}_t = (r_t - n)\,\varphi_t - \tau_t - i_t m_t.$$

Integrating this differential equation from time t to infinity yields:

$$\left(\lim_{T\to\infty} e^{-\int_t^T (r_u - n)du}\varphi_T\right) - \varphi_t = -\int_t^\infty e^{-\int_t^s (r_u-n)du}\left(\tau_s + i_s m_s\right) ds.$$

If φ_t is finite, then by definition of φ_t in (16) we must have:

$$\lim_{T\to\infty} e^{-\int_t^T (r_u-n)du}\varphi_T = 0.$$

The consumption Euler equation (23) can be written as:

$$\frac{d\ln\left[u'\left(c_t\right)\right]}{dt} = -r_t + \rho - \frac{\gamma'(\Delta_t)}{u'\left(c_t\right)}.$$

Integrating this differential equation from time zero to t yields:

$$u'\left(c_t\right) = u'\left(c_0\right) e^{\int_0^t \left(\rho - r_u - \frac{\gamma'(\Delta_u)}{u'(c_u)}\right)du}. \tag{A1}$$

Hence:

$$\lim_{t\to\infty} e^{-\int_0^t(\rho-r_u)du} u'\left(c_t\right) = u'\left(c_0\right)\lim_{t\to\infty} e^{-\int_0^t \frac{\gamma'(\Delta_u)}{u'(c_u)}du} \leq u'\left(c_0\right).$$

We must therefore have:

$$
\begin{aligned}
\lim_{t\to\infty} e^{-(\rho-n)t} u'(c_t)\,\varphi_t &= \left(\lim_{t\to\infty} e^{-\int_0^t (r_u-n)du}\varphi_t\right)\left(\lim_{t\to\infty} e^{-\int_0^t (\rho-r_u)du} u'(c_t)\right), \\
&= 0\left(u'(c_0)\lim_{t\to\infty} e^{-\int_0^t \frac{\gamma'(\Delta_u)}{u'(c_u)}du}\right), \\
&= 0.
\end{aligned}
$$

Let us now show that there cannot be an equilibrium with an infinite value of φ_t. If $\varphi_t = -\infty$, then $\Delta_t = b_t + m_t - \varphi_t = +\infty$.[28] This implies $\gamma'(\Delta_t) = 0$ for all t. From the above consumption Euler equation (A1), we have:

$$u'(c_t) = u'(c_0)\, e^{\int_0^t (\rho-r_u)du}.$$

The household's transversality condition is:

$$\lim_{t\to\infty} e^{-(\rho-n)t} u'(c_t)\,[b_t + m_t] = 0.$$

It can therefore be simplified to:

$$\lim_{t\to\infty} e^{-\int_0^t (r_u-n)du}[b_t + m_t] = 0. \tag{A2}$$

Integrating the government liability accumulation equation (15) from t to infinity yields:

$$\lim_{T\to\infty} e^{-\int_t^T (r_u-n)du}[b_T + m_T] = b_t + m_t - \varphi_t.$$

Multiplying both sides by $e^{-\int_0^t (r_u-n)du}$ yields:

$$
\begin{aligned}
\lim_{T\to\infty} e^{-\int_0^T (r_u-n)du}[b_T + m_T] &= e^{-\int_0^t (r_u-n)du}\Delta_t, \\
&= \infty.
\end{aligned}
$$

This implies that the household's transversality condition (A2) cannot be satisfied when $\varphi_t = -\infty$.

B Proof of Proposition 1

There always exists an equilibrium where households do not believe in the sustainability of the Ponzi scheme. In that case, the helicopter drop of money H triggers an upward jump in the price level from P_0 to P_0' such that the level of government net liabilities

[28] Recall that, throughout our analysis, we exclusively focus on cases where the no-Ponzi condition is either binding or violated, i.e. $\Delta_t \geq 0$. This rules out $\varphi_t = \infty$.

remains unchanged, i.e. $\Delta_0' = \Delta_0 = 0$. By definition of Δ_0' in (38), this requires:

$$\frac{P_0'}{P_0} = 1 + \frac{H}{B_0 + M_0}.$$

Real money balances jump from M_0/P_0 to:

$$\frac{M_0 + H}{P_0'} = \frac{M_0}{P_0}\frac{M_0 + H}{M_0}\frac{B_0 + M_0}{B_0 + M_0 + H} \geq \frac{M_0}{P_0},$$

where the last inequality follows from my assumption that $B_0 \geq 0$. Clearly, in this equilibrium, the helicopter drop of money does not affect the real allocation of resources and the economy remains in the secular stagnation steady state.

I have assumed that the economy does not converge to the neoclassical steady state, unless it is forced to. The fact that the secular stagnation steady state always exists implies that the economy is never forced to reach the neoclassical steady state, which can therefore be ignored.[29] This leaves two possibilities whereby the helicopter drop of money can have real effects: convergence to the Ponzi steady state and converge to the secular stagnation steady state.

Let us first investigate convergence to the Ponzi steady state. This can only occur when $r^n < n$ as, otherwise, the Ponzi steady state does not exist.

When $\bar{\Delta}_0' \geq \Delta^p$, the economy can trivially jump to the Ponzi steady state with $\Delta_0' = \Delta^p$, where it then remains. The economy produces at full capacity, which induces the government to make an open-market operation to redeem a fraction of the money supply (while leaving its nominal liabilities $B_0 + M_0$ unchanged) such as to set the nominal interest rate equal to $\omega > 0$, as required in a Ponzi steady state.

What about convergence to this steady state, starting from $\Delta_0' \neq \Delta^p$?

Lemma 3 *A full employment economy cannot converge to the Ponzi steady state.*

Proof. If the economy is at full employment, then $c_t = f(1)$ and, hence, $\dot{c}_t = 0$. By the Euler equation (23), we must therefore have:

$$r_t = \rho - \frac{\gamma'(\Delta_t)}{u'(f(1))}.$$

Substituting this into the Ponzi dynamics (25) yields:

$$\dot{\Delta}_t = \left[\rho - n - \frac{\gamma'(\Delta_t)}{u'(f(1))}\right]\Delta_t.$$

[29] In theory, the economy can jump to the neoclassical steady state at time 0, which requires $\Delta_0' = 0$. Note that, when $\Delta_0' > 0$, the economy cannot asymptotically converge to the neoclassical steady state while producing below full capacity. This follows from the Euler equation (23) and the Ponzi dynamics (25) with $r_t = -\pi^R$.

Hence, $\dot{\Delta}_t > 0$ if and only if $\Delta_t > \Delta^p$, which implies that Δ_t cannot converge towards Δ^p. ■

Thus, if the economy converges to the Ponzi steady state, it must be operating below full capacity.

Lemma 4 *If $-\pi^R < n$, the economy cannot converge to the Ponzi steady state.*

Proof. From the previous lemma, the only possibility is for the economy to converge to the Ponzi steady state while producing below full capacity. The downward wage rigidity (27) therefore imposes $\pi_t = \pi^R$ while the government prevents the nominal interest rate from being positive. Hence, $r_t = -\pi^R < n$. So, in the neighborhood of the Ponzi steady state, Δ_t must be decreasing towards Δ^p, and c_t must be increasing towards $f(1)$. By the Euler equation (23), it follows that:

$$\frac{-u''(c_t)\,c_t}{u'(c_t)}\frac{\dot{c}_t}{c_t} = -\pi^R - \rho + \frac{\gamma'(\Delta_t)}{u'(c_t)} < n - \rho + \frac{\gamma'(\Delta^p)}{u'(f(1))} = 0,$$

where the last equality follows from the definition of the Ponzi steady state. But, $\dot{c}_t < 0$ with $c_t < f(1)$ is inconsistent with convergence to the Ponzi steady state. ■

There is therefore only one way for the economy to converge to the Ponzi steady state.

Lemma 5 *If $-\pi^R > n$, for any given $\Delta_0' \in (0, \Delta^p)$, there exists a unique equilibrium trajectory where the economy converges to the Ponzi steady state in finite time. The economy operates below full capacity until it reaches the Ponzi steady state.*

Proof. As $L_t < 1$, we must have $r_t = -\pi^R > n$. By the government liability accumulation equation (25):

$$\Delta_t = e^{\left(-\pi^R - n\right)t}\Delta_0'.$$

Let T denote the point in time when the economy reaches the Ponzi steady state. Hence, T is the solution to $\Delta_T = \Delta^p$. As Δ_t is increasing over time, an equilibrium can only exist if $\Delta_0' \in (0, \Delta^p)$. The equilibrium output level c_t is the solution to the Euler equation (23) subject to the boundary condition $c_T = f(1)$. To prove that such an equilibrium exists, we need to solve this differential equation and show that the solution satisfies $c_t \in (0, f(1))$.

The Euler equation (23) with $r_t = -\pi^R$ can be written as:

$$\frac{du'(c_t)}{dt} - \left(\pi^R + \rho\right)u'(c_t) = -\gamma'(\Delta_t).$$

Integrating this equation from t to T yields:

$$u'(c_t) = e^{-\left(\pi^R+\rho\right)(T-t)}u'(f(1)) + \int_t^T e^{-\left(\pi^R+\rho\right)(s-t)}\gamma'(\Delta_s)\,ds.$$

This trivially shows that, for any given $\Delta_0' \in (0, \Delta^p)$, there exists a unique candidate equilibrium trajectory, which satisfies $c_t > 0$. We now need to prove that $c_t < f(1)$ for $t < T$. As Δ_t is increasing over time, we know that $\Delta_t < \Delta^p$ for all $t < T$. Hence:

$$u'(c_t) > e^{-\left(\pi^R+\rho\right)(T-t)}u'(f(1)) + \gamma'(\Delta^p)\int_t^T e^{-\left(\pi^R+\rho\right)(s-t)}ds.$$

Recall that, by definition of the Ponzi steady state, $\gamma'(\Delta^p) = (\rho - n)\,u'(f(1))$. Thus:

$$\begin{aligned}
u'(c_t) &> u'(f(1))\left[e^{-\left(\pi^R+\rho\right)(T-t)} + (\rho-n)\int_t^T e^{-\left(\pi^R+\rho\right)(s-t)}ds\right], \\
&> u'(f(1))\left[e^{-\left(\pi^R+\rho\right)(T-t)} + (\rho-n)\frac{1-e^{-\left(\pi^R+\rho\right)(T-t)}}{\pi^R+\rho}\right], \\
&> u'(f(1))\left[e^{-\left(\pi^R+\rho\right)(T-t)}\frac{\pi^R+n}{\pi^R+\rho} + \frac{\rho-n}{\pi^R+\rho}\right].
\end{aligned}$$

But, $\rho > -\pi^R > n$. Hence:

$$u'(c_t) > \left[\frac{\pi^R+n}{\pi^R+\rho} + \frac{\rho-n}{\pi^R+\rho}\right]u'(f(1)) = u'(f(1)),$$

which finally proves that, as required, $c_t < f(1)$ for $t < T$.

At time T, the economy starts producing at full capacity. This induces the government to redeem a fraction of the money supply M_T (while leaving $M_T + B_T$ unchanged) such as to raise the nominal interest rate from zero to $\omega > 0$, as required in a Ponzi steady state. ■

Note that, when $-\pi^R = n$, the only solution is to have $\Delta_0' = \Delta^p$ with the economy immediately jumping to the Ponzi steady state. Thus, if $-\pi^R = n$, the economy cannot converge to the Ponzi steady state starting from $\Delta_0' \neq \Delta^p$.

We now investigate convergence to the secular stagnation steady state. As c_t is a continuous function of time, the economy must produce below full capacity in the neighborhood of the secular stagnation steady state. Thus, $L_t < 1$ and $r_t = -\pi^R$ in that neighborhood. When $-\pi^R \geq n$, the government net liabilities Δ_t must therefore be non-decreasing over time, which is not consistent with convergence to $\Delta^s = 0$. This leaves $-\pi^R < n$ as the only case where an equilibrium with $\Delta_0' > 0$ might converge to the secular stagnation steady state.

Lemma 6 *Let $\tilde{\Delta} > 0$ be defined by:*[30]

$$\frac{\gamma'(\Delta^p)}{\rho - n} = \int_0^\infty e^{-\left(\pi^R+\rho\right)t}\gamma'\left(e^{-\left(\pi^R+n\right)t}\tilde{\Delta}\right)dt.$$

If $-\pi^R < n$, for any given $\Delta_0' \in (0, \tilde{\Delta}]$, there exists a unique equilibrium trajectory converging to the secular stagnation steady state. The economy produces below full capacity (except at time 0 when $\Delta_0' = \tilde{\Delta}$), i.e. $c_t < f(1)$, and the economy shrinks, i.e. $\dot{c}_t < 0$, along this trajectory.

Proof. As c_t is a continuous function of time, the economy must be producing below full capacity, i.e. with $L_t < 1$ and $r_t = -\pi^R$, as it converges to the secular stagnation steady state. The corresponding equilibrium trajectories are characterized by:

$$\begin{aligned}\dot{\Delta}_t &= \left[-\pi^R - n\right]\Delta_t,\\ \frac{\dot{c}_t}{c_t} &= \left[-\pi^R - \rho + \frac{\gamma'(\Delta_t)}{u'(c_t)}\right]\frac{u'(c_t)}{-u''(c_t)\,c_t},\end{aligned}$$

where $\rho + \pi^R > 0$. Relying on a phase diagram, it is straightforward to establish that a trajectory converging to the secular stagnation steady state must always satisfy $\dot{c}_t < 0$.

Integrating the Euler equation from time t to infinity, subject to the boundary condition that c_t converges to c^s, yields:

$$u'(c_t) = \int_t^\infty e^{-\left(\pi^R+\rho\right)(s-t)}\gamma'(\Delta_s)\,ds.$$

As $\Delta_t = e^{\left(-\pi^R-n\right)t}\Delta_0'$ for any $t \geq 0$, this uniquely characterizes the whole path of c_t from time 0 onwards. In particular, we must have:

$$u'(c_0) = \int_0^\infty e^{-\left(\pi^R+\rho\right)s}\gamma'\left(e^{\left(-\pi^R-n\right)s}\Delta_0'\right)ds.$$

It follows that c_0 is an increasing function of Δ_0'. Moreover, c_0 tends to infinity as Δ_0' tends to infinity. There therefore exists a $\tilde{\Delta} > 0$ such that $c_0 = f(1)$ when $\Delta_0' = \tilde{\Delta}$. Using the fact that $u'(f(1)) = \gamma'(\Delta^p)/(\rho - n)$, we can deduce that $\tilde{\Delta}$ is formally defined by:

$$\frac{\gamma'(\Delta^p)}{\rho - n} = \int_0^\infty e^{-\left(\pi^R+\rho\right)t}\gamma'\left(e^{-\left(\pi^R+n\right)t}\tilde{\Delta}\right)dt.$$

We have therefore shown by construction that, for any $\Delta_0' \in (0, \tilde{\Delta}]$, there exists a unique equilibrium trajectory converging to the secular stagnation steady state with $c_t < f(1)$ and $\dot{c}_t < 0$. ■

[30] This definition implies that, if $-x\gamma''(x)/\gamma'(x)$ is independent of x, then $\tilde{\Delta} < \Delta^p$ if and only if $-x\gamma''(x)/\gamma'(x) < 1$.

If $\Delta'_0 = \tilde{\Delta}$, then $c_0 = f(1)$. But, is it possible to have an equilibrium where the economy produces at full capacity for some time before Δ_t becomes equal to $\tilde{\Delta}$?

Lemma 7 *If $-\pi^R < n$ and Δ'_0 is sufficiently large, there exists an equilibrium trajectory where the economy produces at full capacity for some time before moving to the equilibrium trajectory with a Ponzi scheme of magnitude $\tilde{\Delta}$ described in the previous lemma.*

Proof. Let us consider the existence of a time $\tilde{T} > 0$ such that $c_t = f(1)$ for all $t \leq \tilde{T}$ and $\Delta_{\tilde{T}} = \tilde{\Delta}$. Also, after time $\tilde{T}$, we must have $r_t = -\pi^R$, $c_t < f(1)$, $\dot{c}_t < 0$, $\Delta_t < \tilde{\Delta}$, and $\dot{\Delta}_t < 0$ (as described in the previous lemma).

By the Euler equation (23), from time 0 to $\tilde{T}$, we must have $r_t = \rho - \gamma'(\Delta_t)/u'(f(1))$. Substituting this within the Ponzi dynamics (25) yields:

$$\dot{\Delta}_t = \left[\rho - n - \frac{\gamma'(\Delta_t)}{u'(f(1))}\right]\Delta_t,$$

where we must have $\Delta_{\tilde{T}} = \tilde{\Delta}$. Thus, if $\tilde{\Delta} > \Delta^p$, then $\dot{\Delta}_t > 0$ and $\Delta_t > \Delta^p$ for all $t \leq \tilde{T}$. Conversely, if $\tilde{\Delta} < \Delta^p$, then $\dot{\Delta}_t < 0$ and $\Delta_t < \Delta^p$ for all $t \leq \tilde{T}$.

We have $r_{\tilde{T}} = \rho - \gamma'(\tilde{\Delta})/u'(f(1))$. By the Euler equation (23), the real interest rate immediately after time $\tilde{T}$ satisfies:

$$-\pi^R = \rho - \frac{\gamma'(\tilde{\Delta})}{u'(f(1))} + \left(\frac{-f(1)\,u''(f(1))}{u'(f(1))}\right)\frac{\dot{c}_t}{f(1)},$$

with $\dot{c}_t < 0$. Thus, the real interest rate drops at time $\tilde{T}$ from $r_{\tilde{T}}$ to $-\pi^R$. From the downward wage rigidity, we must also have $\pi_{\tilde{T}} \geq \pi^R$. Hence, $i_{\tilde{T}} = r_{\tilde{T}} + \pi_{\tilde{T}} > -\pi^R + \pi^R = 0$. So, at time $\tilde{T}$, as the economy ceases to produce at full capacity, the government raises the money supply through an open-market operation (while leaving $M_{\tilde{T}} + B_{\tilde{T}}$ unchanged) such as to reduce the nominal interest rate to zero.

For any value of $\pi_{\tilde{T}} \geq \pi^R$, there is a corresponding value of $m_{\tilde{T}}$ determined by the money demand equation:

$$h'(m_{\tilde{T}}) = i_{\tilde{T}}u'(f(1)).$$

As $i_{\tilde{T}} = r_{\tilde{T}} + \pi_{\tilde{T}} > 0$, we must clearly have $m_{\tilde{T}} < \bar{m}$. Finally, the equilibrium path of m_t from time 0 to $\tilde{T}$ is determined by the money supply equation:

$$\begin{aligned}
\frac{\dot{m}_t}{m_t} &= \omega - n - \pi_t, \\
&= \omega - n + r_t - i_t, \\
&= \omega - n + \rho - \frac{\gamma'(\Delta_t)}{u'(f(1))} - \frac{h'(m_t)}{u'(f(1))}.
\end{aligned}$$

Thus, for any value of $\pi_{\tilde{T}} \geq \pi^R$, there is a corresponding value of $m_{\tilde{T}}$ and a corresponding solution of m_t for $t \in [0, \tilde{T}]$. This solution must satisfy $m_t \geq 0$, $m_t \leq \bar{m}$ (since $c_t = f(1)$), and $\pi_t = i_t - r_t = h'(m_t)/u'(f(1)) - \rho + \gamma'(\Delta_t)/u'(f(1)) \geq \pi^R$. For $\pi_{\tilde{T}} > \pi^R$, by continuity of m_t and Δ_t, none of these constraints are binding for $\tilde{T}$ sufficiently close to zero.

Finally, at time 0, as the economy starts producing at full capacity, the government redeems a fraction of the money supply through an open-market operation (while leaving $M_0 + B_0$ unchanged) such as to reduce real money balances below $\bar{m}$. For a given value of $\tilde{T}$ (and therefore of Δ_t from time 0 to $\tilde{T}$), the value of m_0 resulting from the open-market operation can be seen as determining the corresponding $\pi_{\tilde{T}}$. ■

Finally, let us rule out the possibility of an equilibrium path that does not converge to any of our three steady state equilibria. We first consider the possibility of having $c_t = f(1)$ forever, then the possibility of fluctuating between $c_t = f(1)$ and $c_t < f(1)$ forever, and finally of having $c_t < f(1)$ forever.

Lemma 8 *We cannot have an equilibrium eventually satisfying $c_t = f(1)$ forever without the economy either converging to the neoclassical or to the Ponzi steady state.*

Proof. By the proof of Lemma 3, if the economy does not converge to the neoclassical or to the Ponzi steady state, then Δ_t must diverge to infinity. Substituting the Euler equation (23) with $c_t = f(1)$ into the Ponzi dynamics (25) yields $\dot{\Delta}_t/\Delta_t = \rho - n - \gamma'(\Delta_t)/u'(f(1))$. Integrating this differential equation from 0 to t and substituting it within the transversality condition (24) yields:

$$\lim_{t\to\infty} e^{-\int_0^t \frac{\gamma'(\Delta_s)}{u'(f(1))} ds} = 0.$$

We must therefore have:

$$\lim_{t\to\infty} \int_0^t \frac{\gamma'(\Delta_s)}{u'(f(1))} ds = \infty.$$

But, $\Delta_s = \Delta_0 e^{\int_0^s (r_u - n) du}$ and $r_t - n$ tends to $\rho - n > 0$ as Δ_t tends to infinity. Hence, the assumption that $\int_0^\infty \gamma'(e^{\lambda t}) dt < \infty$ for any $\lambda > 0$, implies:

$$\lim_{t\to\infty} \int_0^t \frac{\gamma'(\Delta_s)}{u'(f(1))} ds < \infty.$$

This rules out the possibility of Δ_t diverging to infinity. ■

Lemma 9 *There cannot be a cyclical equilibrium where the economy eventually fluctuates between $c_t = f(1)$ and $c_t < f(1)$ forever.*

Proof. For such a cyclical equilibrium to exist, there must be some time T_1, T_2 and T_3 such that $c_{T_1} = c_{T_2} = c_{T_3} = f(1)$, $c_t < f(1)$ for all $t \in (T_1, T_2)$, $c_t = f(1)$ for all

$t \in [T_2, T_3]$, $\dot{c}_{T_1} < 0$, $\dot{c}_{T_2} > 0$, and $\dot{c}_{T_3} < 0$. By the Euler equation (23) with $r_t = -\pi^R$ from time T_1 to T_2, if $\dot{c}_{T_1} < 0$ and $\dot{c}_{T_2} > 0$, we must have $\Delta_{T_1} > \Delta_{T_2}$. This requires $-\pi^R - n < 0$. From time T_2 to T_3, we have $c_t = f(1)$ and therefore, as in the proof of Lemma 8, $\dot{\Delta}_t/\Delta_t = \rho - n - \gamma'(\Delta_t)/u'(f(1))$. As $\dot{\Delta}_t < 0$ from time T_1 to T_2, we must have $\dot{\Delta}_t > 0$ from time T_2 to T_3 and, hence, $\Delta_t > \Delta^p$ for all t. Let $\hat{\Delta}$ be determined by $\gamma'(\hat{\Delta}) = (\rho+\pi^R)u'(f(1))$. Note that $\gamma'(\hat{\Delta}) = (\rho+\pi^R)u'(f(1)) > (\rho-n)u'(f(1)) = \gamma'(\Delta^p)$, which implies $\hat{\Delta} < \Delta^p$. The Euler equation with $r_t = -\pi^R$ from time T_1 to T_2 can be written as:

$$\begin{aligned}\frac{\dot{c}_t}{c_t} &= \left[-\pi^R - \rho + \frac{\gamma'(\Delta_t)}{u'(c_t)}\right]\frac{u'(c_t)}{-u''(c_t)\,c_t}, \\ &= \left[\frac{\gamma'(\Delta_t)}{u'(c_t)} - \frac{\gamma'(\hat{\Delta})}{u'(f(1))}\right]\frac{u'(c_t)}{-u''(c_t)\,c_t}.\end{aligned}$$

But, as $\Delta_t > \Delta^p$, we must have $\Delta_t > \hat{\Delta}$ and, hence, $\dot{c}_t < 0$. This is inconsistent with $\dot{c}_{T_2} > 0$. ■

Lemma 10 *We cannot have an equilibrium eventually satisfying $c_t < f(1)$ forever without the economy converging to any of our three steady state equilibria.*

Proof. If the equilibrium eventually satisfies $c_t < f(1)$ forever, we must have $r_t = -\pi^R$ with the corresponding equilibrium dynamics characterized by:

$$\begin{aligned}\dot{\Delta}_t &= \left[-\pi^R - n\right]\Delta_t, \\ \frac{\dot{c}_t}{c_t} &= \left[-\pi^R - \rho + \frac{\gamma'(\Delta_t)}{u'(c_t)}\right]\frac{u'(c_t)}{-u''(c_t)\,c_t},\end{aligned}$$

where $\rho + \pi^R > 0$.

We first consider that $\Delta_t > 0$ for all t. Integrating these two differential equations from time 0 to t and substituting them within the transversality condition (24) yields:

$$\lim_{t\to\infty} e^{-\int_0^t \frac{\gamma'(\Delta_s)}{u'(c_s)}ds} = 0.$$

We must therefore have:

$$\lim_{t\to\infty}\int_0^t \frac{\gamma'(\Delta_s)}{u'(c_s)}ds = \infty.$$

If $-\pi^R - n > 0$, then Δ_t tends to infinity and the transversality condition is violated (as in the proof of Lemma 8). If $-\pi^R - n < 0$, then Δ_t tends to zero. Relying on a phase diagram, it is straightforward to establish that, if $c_t < f(1)$ forever while the economy does not converge to the secular stagnation steady state, then c_t must be converging to zero. From the Euler equation, we have $u'(c_t) = u'(c_0)\,e^{\int_0^t\left(\rho+\pi^R-\gamma'(\Delta_u)/u'(c_u)\right)du}$. This implies that:

$$\lim_{t\to\infty}\int_0^t \frac{\gamma'(\Delta_s)}{u'(c_s)}ds < \frac{\gamma'(0)}{u'(c_0)}\lim_{t\to\infty}\int_0^t e^{-\int_0^s\left(\rho+\pi^R-\gamma'(\Delta_u)/u'(c_u)\right)du}ds < \infty,$$

where the last inequality is implied by the fact that $\gamma'(\Delta_t)/u'(c_t)$ asymptotically tends to zero. This candidate equilibrium path therefore violates the transversality condition.

Finally, let us consider that $\Delta_t = 0$ for all t. This implies that the dynamics of c_t are simply determined by the consumption Euler equation with $\Delta_t = 0$. If $c_0 = c^s$, then the economy must be in the secular stagnation steady state. If $c_0 > c^s$, it must reach the neoclassical steady state in finite time. Finally, if $c_0 < c^s$, consumption must be converging to zero. But, we have just proved that this would violate the transversality condition. ■

C Proof of Proposition 2

By Lemma 2, T must be finite. We focus on equilibrium trajectories where $L_t = 1$ is an absorbing state, implying that $L_t = 1$ for all $t \geq T$. Hence, by the Euler equation (23), from time T onwards:

$$r_t = \rho - \frac{\gamma'(\Delta_t)}{u'(f(1))}.$$

Substituting this into the Ponzi dynamics (25) yields:

$$\dot{\Delta}_t = \left[\rho - n - \frac{\gamma'(\Delta_t)}{u'(f(1))}\right]\Delta_t.$$

By (33), we have $r^n = \rho - \gamma'(0)/u'(f(1))$. Hence, if $r^n > n$, then $\dot{\Delta}_t > 0$ whenever $\Delta_t > 0$. Conversely, if $r^n < n$, then the Ponzi steady state exists, i.e. $\Delta^p > 0$, and $\dot{\Delta}_t < 0$ if and only if $\Delta_t < \Delta^p$.

In equilibrium, by the proof of Lemma 8 of the proof of Proposition 1, Δ_t cannot diverge as this would violate the transversality condition (24). So, if $r^n > n$, we must have $\Delta_t = 0$ for all $t \geq T$. And, if $r^n < n$, we must have $\Delta_T \leq \Delta^p$, which leaves two possibilities. We must either have $\Delta_t = \Delta^p$ for all $t \geq T$, with the economy therefore remaining in the Ponzi steady state forever, or $\Delta_T < \Delta^p$ with Δ_t converging to zero and the economy therefore converging to the neoclassical steady state.

Finally, substituting the money demand equation (29) into the money supply equation (28) yields:

$$\frac{\dot{m}_t}{m_t} = \omega + r_t - n - \frac{h'(m_t)}{u'(f(1))},$$

which determines the dynamics of real money balances from time T onwards. Since

$\omega \geq n - r^n$ (and therefore $\omega > 0$ when $r^n < n$), real money balances can be constant in either the neoclassical or the Ponzi steady state. Also, when the economy converges to the neoclassical steady state from $\Delta_T > 0$, real money balances can converge to this constant level. The corresponding dynamics are fully described by the above three equations for r_t, $\dot{\Delta}_t$, and $\dot{m}_t$, with Δ_T and $\lim_{t \to \infty} m_t$ given.