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# Homogeneous Besov and Triebel–Lizorkin spaces associated tonon-negative self-adjoint operators


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# Homogeneous Besov and Triebel–Lizorkin spaces associated to non-negative self-adjoint operators

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A B S T R A C T

Homogeneous Besov and Triebel–Lizorkin spaces with complete set of indices are introduced in the general setting of a doubling metric measure space in the presence of a non-negative self-adjoint operator whose heat kernel has Gaussian localization and the Markov property. The main step in this theory is the development of distributions modulo generalized polynomials. Some basic properties of the general homogeneous Besov and Triebel–Lizorkin spaces are established, in particular, a discrete (frame) decomposition of these spaces is obtained.

*Keywords:* Heat kernel, Besov spaces, Triebel–Lizorkin spaces, Homogeneous spaces, Distributions Generalized polynomials

## 1. Introduction

The Littlewood–Paley theory of classical Besov and Triebel–Lizorkin spaces on $\mathbb{R}^d$ has been developed primarily by J. Peetre, H. Triebel, M. Frazier, and B. Jawerth, see [7,10,11,2–4]. This theory has been generalized and extended in all sorts of directions and settings. In [1,5], inhomogeneous Besov and Triebel–Lizorkin spaces have been developed in the general setting of a metric measure space with the doubling property and in the presence of a non-negative self-adjoint operator whose heat kernel has Gaussian localization and the Markov property. These spaces have been further generalized in [6]. Surprisingly this general theory develops in almost complete generality as in the classical setting on $\mathbb{R}^d$.

In this article we focus on the homogeneous version of these spaces. More explicitly, we shall develop various aspects of the theory of homogeneous Besov and Triebel–Lizorkin spaces in the general setting

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described below, including generalized polynomials associated to operators and the class of distributions modulo such polynomials, and the frame decomposition of distribution spaces.

We shall operate in the setting put forward in [1,5], which we describe next:

I. We assume that $(M, \rho, \mu)$ is a metric measure space satisfying the conditions: $(M, \rho)$ is a locally compact metric space with distance $\rho(\cdot, \cdot)$ and $\mu$ is a positive Radon measure such that the following *volume doubling condition* is valid

$$0 < \mu(B(x, 2r)) \le c_0 \mu(B(x, r)) < \infty \quad \text{for all } x \in M \text{ and } r > 0, \tag{1.1}$$

where $B(x, r)$ is the open ball centered at $x$ of radius $r$ and $c_0 > 1$ is a constant. From above it follows that

$$\mu(B(x, \lambda r)) \le c_0 \lambda^d \mu(B(x, r)) \quad \text{for } x \in M,\ r > 0, \text{ and } \lambda > 1, \tag{1.2}$$

where $d = \log_2 c_0 > 0$ is a constant playing the role of a dimension.

We also assume that $\mu(M) = \infty$.

II. The main assumption is that the geometry of the space $(M, \rho, \mu)$ is related to an essentially self-adjoint non-negative operator $L$ on $L^2(M, d\mu)$, mapping real-valued to real-valued functions, such that the associated semigroup $P_t = e^{-tL}$ consists of integral operators with (heat) kernel $p_t(x, y)$ obeying the conditions:

(a) *Gaussian upper bound:*

$$|p_t(x, y)| \le \frac{C^\star \exp\{-\frac{c^\star \rho^2(x,y)}{t}\}}{\big[\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))\big]^{1/2}} \quad \text{for } x, y \in M,\ t > 0. \tag{1.3}$$

(b) *Hölder continuity:* There exists a constant $\alpha > 0$ such that

$$\big|p_t(x, y) - p_t(x, y')\big| \le C^\star \Big(\frac{\rho(y, y')}{\sqrt{t}}\Big)^\alpha \frac{\exp\{-\frac{c^\star \rho^2(x,y)}{t}\}}{\big[\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))\big]^{1/2}} \tag{1.4}$$

for $x, y, y' \in M$ and $t > 0$, whenever $\rho(y, y') \le \sqrt{t}$.

(c) *Markov property:*

$$\int_M p_t(x, y) d\mu(y) = 1 \quad \text{for } x \in M \text{ and } t > 0. \tag{1.5}$$

Above $C^\star, c^\star > 0$ are structural constants.

We also stipulate the following additional conditions on the geometry of $M$:

(d) *Noncollapsing condition:* There exists a constant $c_1 > 0$ such that

$$\inf_{x \in M} \mu(B(x, 1)) \ge c_1. \tag{1.6}$$

(e) *Reverse doubling condition:* There exists a constant $c_2 > 1$ such that

$$\mu(B(x, 2r)) \ge c_2 \mu(B(x, r)) \quad \text{for } x \in M \text{ and } r > 0. \tag{1.7}$$

This condition readily implies

$$\mu(B(x, \lambda r)) \ge c_3 \lambda^{d^*} \mu(B(x, r)) \quad \text{for } x \in M,\ r > 0, \text{ and } \lambda > 1, \tag{1.8}$$

where $d^* := \log_2 c_2 \le d$ and $c_3 = c_2^{-1}$.

Observe that as shown in [1, Proposition 2.2] if $M$ is connected, then the reverse doubling condition (1.7) follows by the doubling condition (1.1). Therefore, the reverse doubling condition is not restrictive.

The above setting finds a natural realization in the general framework of strictly local regular Dirichlet spaces with a complete intrinsic metric, where it suffices to only verify the local Poincaré inequality and the global doubling condition on the measure and then the above general setting applies in full. In particular, this setting covers the cases of Lie groups or homogeneous spaces with polynomial volume growth, complete Riemannian manifolds with Ricci curvature bounded from below and satisfying the volume doubling condition. Naturally, it contains the classical setting on $\mathbb{R}^n$. For more details, see [1].

In this article we advance in several directions. In the general setting described above, we introduce spaces of distributions modulo generalized polynomials $\mathcal{S}'/\mathcal{P}$ and establish basic convergence results (§3). This is only possible in the noncompact case ($\mu(M) = \infty$). It is also shown that $\mathcal{S}'/\mathcal{P}$ is a natural generalization of the tempered distributions modulo polynomials in the classical case on $\mathbb{R}^d$. As a next step we show how the construction of frames from [1,5] can be adapted to the homogeneous setting (§4). In Section 5 we introduce two types of homogeneous Besov spaces $\dot{B}^s_{pq}$, $\dot{\tilde{B}}^s_{pq}$ and Triebel–Lizorkin spaces $\dot{F}^s_{pq}$, $\dot{\tilde{F}}^s_{pq}$ with full sets of indices: $s \in \mathbb{R}$, $0 < p, q < \infty$ ($q \le \infty$ in the case of Besov spaces) and establish their frame decomposition using respective sequence spaces $\dot{b}^s_{pq}$, $\dot{\tilde{b}}^s_{pq}$, and $\dot{f}^s_{pq}$, $\dot{\tilde{f}}^s_{pq}$, by adaptation of the construction from the inhomogeneous case, developed in [1,5]. In Section 6 we present without proof some additional results on homogeneous Besov and Triebel–Lizorkin spaces that are easy to prove or straightforward adaptation of results in the inhomogeneous setting. To streamline our presentation we place the proofs of some assertions in Section 7.

The main purpose of the present article is to show that in the general setting described above it is possible to develop the theory of homogeneous Besov and Triebel–Lizorkin spaces, including their discrete (frame) decomposition, in almost complete generality as in the classical case on $\mathbb{R}^n$. This allows to cover new settings such as the ones on Lie groups and Riemannian manifolds.

*Notation*: Throughout we shall denote $|E| := \mu(E)$ and $\mathbb{1}_E$ will stand for the characteristic function of $E \subset M$, $\|\cdot\|_p = \|\cdot\|_{L^p} := \|\cdot\|_{L^p(M,d\mu)}$. $\mathcal{S}(\mathbb{R})$ will stand for the Schwartz class on $\mathbb{R}$. Positive constants will be denoted by $c$, $C$, $c_1$, $c'$, ... and will be allowed to vary at every occurrence. The notation $a \sim b$ will stand for $c_1 \le a/b \le c_2$. We shall also use the standard notation $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

## 2. Background

In this section we provide some basic ingredients for our theory, mainly developed in [1,5].

### 2.1. Some properties related to the geometry of the underlying space

To compare the volumes of balls with different centers $x, y \in M$ and the same radius $r$ we will use the inequality

$$|B(x,r)| \le c_0\Big(1 + \frac{\rho(x,y)}{r}\Big)^d |B(y,r)|, \quad x, y \in M,\ r > 0. \tag{2.1}$$

As $B(x,r) \subset B(y, \rho(y,x) + r)$ the above inequality is immediate from (1.2).

The following simple inequalities are established in [5, Lemma 2.1]: If $\sigma > d$ and $\delta > 0$, then for any $x, y \in M$

$$\int_M \big(1 + \delta^{-1}\rho(x,u)\big)^{-\sigma} d\mu(u) \le c|B(x,\delta)|, \tag{2.2}$$

$$\int_M \big(1+\delta^{-1}\rho(x,u)\big)^{-\sigma}\big(1+\delta^{-1}\rho(u,y)\big)^{-\sigma}d\mu(u) \le c|B(x,\delta)|\big(1+\delta^{-1}\rho(x,y)\big)^{-\sigma+d}. \tag{2.3}$$

The above inequality follows readily by the following more general assertion:

**Lemma 2.1.** *Let $\sigma_1, \sigma_2 > d$ and $\delta_1, \delta_2 > 0$, and*

$$I := \int_M \frac{d\mu(u)}{(1+\delta_1^{-1}\rho(x,u))^{\sigma_1}(1+\delta_2^{-1}\rho(y,u))^{\sigma_2}} \tag{2.4}$$

*Then for any $x, y \in M$*

$$I \le \frac{c|B(x,\delta_1)|}{(1+\delta_2^{-1}\rho(x,y))^{\sigma_2}} + \frac{c|B(y,\delta_2)|}{(1+\delta_1^{-1}\rho(x,y))^{\sigma_1}} \tag{2.5}$$

*and consequently*

$$I \le \frac{c|B(x,\delta_1)|}{(1+\delta_{\max}^{-1}\rho(x,y))^{(\sigma_1-d)\wedge\sigma_2}} \quad \text{and} \quad I \le \frac{c|B(y,\delta_2)|}{(1+\delta_{\max}^{-1}\rho(x,y))^{\sigma_1\wedge(\sigma_2-d)}}. \tag{2.6}$$

*Here $\delta_{\max} := \delta_1 \vee \delta_2$ and the constant $c > 0$ depends only on $\sigma_1$, $\sigma_2$, $d$, and $c_0$.*

The proof of this lemma is given in Section 7.

## 2.2. Functional calculus

First, observe that as $L$ is a non-negative self-adjoint operator that maps real-valued to real-valued functions, then for any real-valued, measurable and bounded function $f$ on $\mathbb{R}_+$ the operator $f(L)$, defined by $f(L) := \int_0^\infty f(\lambda)dE_\lambda$, where $E_\lambda$, $\lambda \ge 0$, is the spectral resolution associated with $L$, is bounded on $L^2$, self-adjoint, and maps real-valued functions to real-valued functions. Furthermore, if $f(L)$ is an integral operator, then its kernel $f(L)(x,y)$ is real-valued and $f(L)(y,x) = f(L)(x,y)$, in particular, $p_t(x,y) \in \mathbb{R}$ and $p_t(y,x) = p_t(x,y)$.

We shall need the following result from the smooth functional calculus induced by the heat kernel, developed in [5].

**Theorem 2.2.** *Suppose $f \in C^N(\mathbb{R})$, $N \ge d+1$, $f$ is real-valued and even, and*

$$|f^{(\nu)}(\lambda)| \le A_N(1+|\lambda|)^{-r} \text{ for } \lambda \in \mathbb{R} \text{ and } 0 \le \nu \le N, \text{ where } r > N+d.$$

*Then $f(\delta\sqrt{L})$, $\delta > 0$, is an integral operator with kernel $f(\delta\sqrt{L})(x,y)$ satisfying*

$$\big|f(\delta\sqrt{L})(x,y)\big| \le \frac{cA_N\big(1+\delta^{-1}\rho(x,y)\big)^{-N}}{\big(|B(x,\delta)||B(y,\delta)|\big)^{1/2}} \le \frac{c'A_N\big(1+\delta^{-1}\rho(x,y)\big)^{-N+d/2}}{|B(x,\delta)|} \tag{2.7}$$

*and*

$$\big|f(\delta\sqrt{L})(x,y) - f(\delta\sqrt{L})(x,y')\big| \le \frac{cA_N\big(\frac{\rho(y,y')}{\delta}\big)^{\alpha}\big(1+\delta^{-1}\rho(x,y)\big)^{-N}}{\big(|B(x,\delta)||B(y,\delta)|\big)^{1/2}} \tag{2.8}$$

*whenever $\rho(y,y') \le \delta$. Here $\alpha > 0$ is from (1.4) and $c, c' > 0$ are constants depending only on $r$, $N$, and the structural constants $c_0, C^\star, c^\star$, $\alpha$.*

*Moreover, $\int_M f(\delta\sqrt{L})(x,y)d\mu(y) = f(0)$.*

**Remark 2.3.** Theorem 2.2 is established in [5, Theorem 3.4] in the case when $0 < \delta \le 1$. However, the same proof applies also to the case $0 < \delta < \infty$.

In the construction of frames we will utilize operators of the form $\varphi(\delta\sqrt{L})$ generated by cutoff functions $\varphi$ specified in the following

**Definition 2.4.** A real-valued function $\varphi \in C^\infty(\mathbb{R}_+)$ is said to be an admissible cutoff function if $\varphi \neq 0$, $\operatorname{supp}\varphi \subset [0,2]$, and $\varphi^{(m)}(0) = 0$ for $m \ge 1$. Furthermore, $\varphi$ is said to be admissible of type $(a)$, $(b)$ or $(c)$ if $\varphi$ is admissible and in addition obeys the respective condition:
$(a)$ $\varphi(t) = 1$, $t \in [0,1]$,
$(b)$ $\operatorname{supp}\varphi \subset [1/2, 2]$ or
$(c)$ $\operatorname{supp}\varphi \subset [1/2, 2]$ and $\sum_{j\in\mathbb{Z}} |\varphi(2^{-j}t)|^2 = 1$ for $t \in (0,\infty)$.

Observe that the even extension of any admissible function belongs to $C^\infty(\mathbb{R})$.

The kernels of operators $\varphi(\delta\sqrt{L})$ with sub-exponential space localization will be the main building blocks in constructing frames.

**Theorem 2.5.** *([5]) For any $0 < \varepsilon < 1$ there exists an admissible cutoff function $\varphi$ of any type, $(a)$ or $(b)$ or $(c)$, such that for any $\delta > 0$*

$$|\varphi(\delta\sqrt{L})(x,y)| \le \frac{c_1 \exp\left\{-\kappa\left(\frac{\rho(x,y)}{\delta}\right)^{1-\varepsilon}\right\}}{\left(|B(x,\delta)||B(y,\delta)|\right)^{1/2}}, \quad x, y \in M, \tag{2.9}$$

*where $c_1, \kappa > 0$ depend only on $\varepsilon$ and the constants $c_0, C^\star, c^\star$ from (1.1)–(1.4). Furthermore, for every $m \in \mathbb{N}$,*

$$|[L^m\varphi(\delta\sqrt{L})](x,y)| \le \frac{c_2\delta^{-2m} \exp\left\{-\kappa\left(\frac{\rho(x,y)}{\delta}\right)^{1-\varepsilon}\right\}}{\left(|B(x,\delta)||B(y,\delta)|\right)^{1/2}}, \quad x, y \in M, \tag{2.10}$$

*with $c_2 > 0$ depending on $\varepsilon, c_0, C^\star, c^\star$, and $m$.*

**Remark 2.6.** Observe that $[L^m\varphi(\delta\sqrt{L})](x,y)$ in (2.10) is the kernel of the operator $L^m\varphi(\delta\sqrt{L})$, however, it can be considered as $L^m$ acting on the kernel $\varphi(\delta\sqrt{L})(\cdot,y)$ or $L^m$ acting on $\varphi(\delta\sqrt{L})(x,\cdot)$ as well. In fact the result is the same: For any $x, y \in M$

$$[L^m\varphi(\delta\sqrt{L})](x,y) = L^m[\varphi(\delta\sqrt{L})(\cdot,y)](x) = L^m[\varphi(\delta\sqrt{L})(x,\cdot)](y). \tag{2.11}$$

This claim is immediate from the following more general result.

**Proposition 2.7.** *Assume that $F$ and $G$ satisfy the hypotheses of Theorem 2.2 with $m \ge 3d/2 + 1$ and let $H$ be a real-valued measurable function on $\mathbb{R}_+$ such that*

$$F(\lambda) = H(\lambda)G(\lambda) \quad \text{for almost all } \lambda \in \mathbb{R}_+. \tag{2.12}$$

*Then $F(\sqrt{L})$ and $G(\sqrt{L})$ are self-adjoint bounded on $L^2(M)$ operators, and $H(\sqrt{L})$ is a self-adjoint operator (defined densely in $M$) such that for all $x \in M$ we have $G(\sqrt{L})(x,\cdot) \in D(H(\sqrt{L}))$ and*

$$F(\sqrt{L})(x,y) = H(\sqrt{L})\big[G(\sqrt{L})(x,\cdot)\big](y) \quad \text{for a.a. } y \in M. \tag{2.13}$$

*Above $D(H(\sqrt{L}))$ stands for the domain of $H(\sqrt{L})$ and $F(\sqrt{L})(x,y)$, $G(\sqrt{L})(x,y)$ are the kernels of the operators $F(\sqrt{L})$, $G(\sqrt{L})$.*

**Proof.** We shall use the notation $F := F(\sqrt{L})$, $G := G(\sqrt{L})$, and $H := H(\sqrt{L})$, and for the kernels $F(x,y) := F(\sqrt{L})(x,y)$ and $G(x,y) := G(\sqrt{L})(x,y)$.

Observe that from that fact that the functions $F$, $G$, and $H$ are real-valued and measurable it follows that (see e.g. [9]) the operators $F$, $G$, and $H$ are self-adjoint. As $F$ and $G$ satisfy the hypotheses of Theorem 2.2, where $m \geq 3d/2+1$, we have $F(x,y) = F(y,x)$ and $G(x,y) = G(y,x)$ for $x, y \in M$, and using (2.2)

$$\sup_{x\in M} \int_M |F(x,y)| d\mu(y) < \infty, \quad \sup_{x\in M} \int_M |G(x,y)| d\mu(y) < \infty. \tag{2.14}$$

Hence the operators $F$ and $G$ are bounded on $L^2(M)$. Also, by Theorem 2.2 and (2.2) it follows that

$$\|F(x,\cdot)\|_2^2 = \int_M |F(x,y)|^2 d\mu(y) \leq c|B(x,1)|^{-1}, \quad \forall x \in M. \tag{2.15}$$

Furthermore, by Theorem 2.2, $F(x,y)$ and $G(x,y)$ are Hölder continuous as functions of $x$ and $y$, that is,

$$\big|F(x,y) - F(x',y)\big| \leq c|B(x,\delta)|^{-1} \rho(x,x')^{\alpha} \big(1+\rho(x,y)\big)^{-N+d/2},$$

whenever $\rho(x,x') \leq 1$, and a similar estimate holds for $G(x,y)$. This readily implies that $F$ and $G$ map $L^2(M)$ into $C(M)$, the space of all continuous functions on $M$.

We claim that

$$G(x,\cdot) \in D(H^*) = D(H), \quad \forall x \in M. \tag{2.16}$$

To prove this we first observe that as is well known (see e.g. [9]) $f \in D(H^*)$ if

$$\Big|\int_M (Hg)(y)\overline{f(y)} d\mu(y)\Big| \leq c\|g\|_2, \quad \forall g \in \tilde{D}(H),$$

for some constant $c > 0$, where $\tilde{D}(H)$ is a dense subspace of $D(H)$. By (2.12) we have $Fg = (GH)g$ for all $g \in D(H)$. From this and the fact that $F$ and $G$ map $L^2(M)$ into $C(M)$ it follows that for every $g \in D(H)$

$$\int_M F(x,y)g(y) d\mu(y) = \int_M G(x,y)(Hg)(y) d\mu(y), \quad \forall x \in M. \tag{2.17}$$

In turn, this and (2.15) yield

$$\Big|\int_M (Hg)(y)G(x,y) d\mu(y)\Big| \leq \|F(x,\cdot)\|_2 \|g\|_2 \leq c|B(x,1)|^{-1/2}\|g\|_2, \quad \forall x \in M.$$

Therefore, (2.16) holds true.

Using that the operator $H$ is self-adjoint, (2.16), and the fact that $G(x,y)$ is real-valued we obtain for every $f \in D(H)$ and all $x \in M$

$$(GH)f(x) = \int_M G(x,y)Hf(y)d\mu(y) = \int_M Hf(y)G(x,y)d\mu(y)$$
$$= \int_M f(y)\overline{H^*\big(\overline{G(x,\cdot)}\big)}(y)d\mu(y) = \int_M H[G(x,\cdot)](y)f(y)d\mu(y).$$

This and (2.17) imply that $F(x,\cdot) = H[G(x,\cdot)](\cdot)$ almost everywhere for all $x \in M$, as claimed. □

*Maximal δ-nets*

The construction of frames in our setting relies on a sequence of $\delta$-nets. By definition $\mathcal{X} \subset M$ is a $\delta$-net on $M$ $(\delta > 0)$ if $\rho(\xi,\eta) \geq \delta$, $\forall \xi,\eta \in \mathcal{X}$, and $\mathcal{X} \subset M$ is a *maximal* $\delta$*-net* on $M$ if $\mathcal{X}$ is a $\delta$-net on $M$ that cannot be enlarged.

Some basic properties of maximal $\delta$-nets will be needed (see [1, Proposition 2.5]): *A maximal* $\delta$*-net on* $M$ *always exists and if* $\mathcal{X}$ *is a maximal* $\delta$*-net on* $M$*, then*

$$M = \cup_{\xi\in\mathcal{X}} B(\xi,\delta) \quad \text{and} \quad B(\xi,\delta/2) \cap B(\eta,\delta/2) = \emptyset \quad \text{if } \xi \neq \eta,\ \xi,\eta \in \mathcal{X}. \tag{2.18}$$

*Furthermore,* $\mathcal{X}$ *is countable and there exists a disjoint partition* $\{A_\xi\}_{\xi\in\mathcal{X}}$ *of* $M$ *consisting of measurable sets such that*

$$B(\xi,\delta/2) \subset A_\xi \subset B(\xi,\delta), \quad \forall \xi \in \mathcal{X}. \tag{2.19}$$

For future use we introduce the following notation for a given maximal $\delta$-net $\mathcal{X}$ on $M$:

$$B_\xi := B(\xi,\delta), \quad \xi \in \mathcal{X}. \tag{2.20}$$

### *2.3. Spectral spaces*

Let $E_\lambda$, $\lambda \geq 0$, be the spectral resolution associated with the self-adjoint positive operator $L$ on $L^2 := L^2(M,d\mu)$. We let $F_\lambda$, $\lambda \geq 0$, denote the spectral resolution associated with $\sqrt{L}$, i.e. $F_\lambda = E_{\lambda^2}$. Then for any measurable and bounded function $f$ on $\mathbb{R}_+$ the operator $f(\sqrt{L})$ is defined by $f(\sqrt{L}) = \int_0^\infty f(\lambda)dF_\lambda$ on $L^2$. For the spectral projectors we have $E_\lambda = \mathbb{1}_{[0,\lambda]}(L) := \int_0^\infty \mathbb{1}_{[0,\lambda]}(u)dE_u$ and

$$F_\lambda = \mathbb{1}_{[0,\lambda]}(\sqrt{L}) := \int_0^\infty \mathbb{1}_{[0,\lambda]}(u)dF_u = \int_0^\infty \mathbb{1}_{[0,\lambda]}(\sqrt{u})dE_u.$$

For any compact $K \subset [0,\infty)$ the spectral space $\Sigma_K^p$ is defined by

$$\Sigma_K^p := \{f \in L^p : \theta(\sqrt{L})f = f \text{ for all } \theta \in C_0^\infty(\mathbb{R}_+),\ \theta \equiv 1 \text{ on } K\}.$$

In general, given a space $Y$ of measurable functions on $M$ we set

$$\Sigma_\lambda = \Sigma_\lambda(Y) := \{f \in Y : \theta(\sqrt{L})f = f \text{ for all } \theta \in C_0^\infty(\mathbb{R}_+),\ \theta \equiv 1 \text{ on } [0,\lambda]\}.$$

The next assertion relates different weighted $L^p$-norms of spectral functions.

**Proposition 2.8.** (*[5]*) *Let* $0 < p \leq q \leq \infty$ *and* $\gamma \in \mathbb{R}$*. Then there exists a constant* $c > 0$ *such that*

$$\||B(\cdot,\lambda^{-1})|^\gamma g(\cdot)\|_q \leq c\||B(\cdot,\lambda^{-1})|^{\gamma+1/q-1/p} g(\cdot)\|_p \quad \textit{for} \quad g \in \Sigma_\lambda,\ \lambda > 0. \tag{2.21}$$

## 3. Distributions

The Besov and Triebel–Lizorkin spaces associated with the operator $L$ are in general spaces of distributions.

### 3.1. Basic facts

In the setting of this article the class of test functions $\mathcal{S} = \mathcal{S}(L)$ is defined (see [5]) as the set of all complex-valued functions $\phi \in \cap_{m\geq 1} D(L^m)$ such that

$$\mathcal{P}_m(\phi) := \sup_{x\in M} (1+\rho(x,x_0))^m \max_{0\leq\nu\leq m} |L^\nu \phi(x)| < \infty, \quad \forall m \geq 0. \tag{3.1}$$

Here $x_0 \in M$ is selected arbitrarily and fixed once and for all. Note that $\mathcal{S}$ is a complete locally convex space with topology generated by the above sequence of norms, i.e. $\mathcal{S}$ is a Fréchet space, see [8].

Observe also that if $\phi \in \mathcal{S}$, then $\overline{\phi} \in \mathcal{S}$, which follows from the fact that $\overline{L\phi} = L\overline{\phi}$, for $L$ maps real-valued to real-valued functions.

As usual the space $\mathcal{S}'$ of distributions on $M$ is defined as the set of all continuous linear functionals on $\mathcal{S}$ and the action of $f \in \mathcal{S}'$ on $\overline{\phi} \in \mathcal{S}$ will be denoted by $\langle f, \phi\rangle := f(\overline{\phi})$, which is consistent with the inner product on $L^2(M)$. Clearly, for any $f \in \mathcal{S}'$ there exist constants $m \in \mathbb{Z}_+$ and $c > 0$ such that

$$|\langle f, \phi\rangle| \leq c\mathcal{P}_m(\phi), \quad \forall \phi \in \mathcal{S}. \tag{3.2}$$

It is important to clarify the action of operators of the form $\varphi(\sqrt{L})$ on $\mathcal{S}'$. Observe that if the function $\varphi \in \mathcal{S}(\mathbb{R})$ is real-valued and even, then from Theorem 2.2 it follows that $\varphi(\sqrt{L})(x,\cdot) \in \mathcal{S}$ and $\varphi(\sqrt{L})(\cdot,y) \in \mathcal{S}$. Moreover, it is easy to see that $\varphi(\sqrt{L})$ maps continuously $\mathcal{S}$ into $\mathcal{S}$.

**Definition 3.1.** We define $\varphi(\sqrt{L})f$ for any $f \in \mathcal{S}'$ by

$$\langle \varphi(\sqrt{L})f, \phi\rangle := \langle f, \varphi(\sqrt{L})\phi\rangle \quad \text{for} \quad \phi \in \mathcal{S}. \tag{3.3}$$

From above it follows that, $\varphi(\sqrt{L})$ maps continuously $\mathcal{S}'$ into $\mathcal{S}'$. Furthermore, if $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ are real-valued and even, then

$$\varphi(\sqrt{L})\psi(\sqrt{L})f = \psi(\sqrt{L})\varphi(\sqrt{L})f, \quad \forall f \in \mathcal{S}'. \tag{3.4}$$

**Proposition 3.2.** *Suppose $\varphi \in \mathcal{S}(\mathbb{R})$ is real-valued and even and let $f \in \mathcal{S}'$. Then*

$$\varphi(\sqrt{L})f(x) = \langle f, \varphi(\sqrt{L})(x,\cdot)\rangle, \quad x \in M. \tag{3.5}$$

*Moreover, $\varphi(\sqrt{L})f$ is a continuous and slowly growing function, namely, there exist constants $m \in \mathbb{Z}_+$ and $c > 0$, depending on $f$, such that*

$$|\varphi(\sqrt{L})f(x)| \leq c(1+\rho(x,x_0))^m, \quad x \in M, \quad \textit{and} \tag{3.6}$$

$$|\varphi(\sqrt{L})f(x) - \varphi(\sqrt{L})f(x')| \leq c(1+\rho(x,x_0))^m \rho(x,x')^\alpha, \quad \textit{if } \rho(x,x') \leq 1. \tag{3.7}$$

*Here $\alpha > 0$ is the constant from (1.4).*

The proof of this proposition is deferred to Section 7.

We refer the reader to [5] for more details on distributions in the general setting of this article.

### 3.2. Distributions modulo generalized polynomials

The homogeneous Besov and Triebel–Lizorkin spaces we consider in this article will be distributions modulo generalized polynomials.

**Generalized polynomials.** In the setting of this article, we define the set $\mathcal{P}_m$ of "generalized polynomials" of degree $m$ ($m \geq 1$) by

$$\mathcal{P}_m := \{g \in \mathcal{S}' : L^m g = 0\} \tag{3.8}$$

and set $\mathcal{P} := \cup_{m\geq 1}\mathcal{P}_m$. Clearly, $g \in \mathcal{P}_m$ if and only if $\langle g, L^m\phi\rangle = 0$ for all $\phi \in \mathcal{S}$.

We define an equivalence $f \sim g$ on $\mathcal{S}'$ by

$$f \sim g \Longleftrightarrow f - g \in \mathcal{P}.$$

We denote by $\mathcal{S}'/\mathcal{P}$ the set of all equivalence classes in $\mathcal{S}'$. To avoid unnecessary complicated notation we will make no difference between any two elements $f_1, f_2$ belonging to one and the same equivalence class in $\mathcal{S}'/\mathcal{P}$.

It will be important that the null space of $L$ contains no nontrivial test functions:

**Proposition 3.3.** *Let $\mathcal{N}(L) := \{f \in D(L) : Lf = 0\}$. Then*

$$\mathcal{N}(L) \cap L^2(M) = \{0\} \quad \textit{and hence} \quad \mathcal{N}(L^k) \cap L^2(M) = \{0\}, \ \ \forall k \in \mathbb{N}.$$

**Proof.** Clearly, $e^{-tu} - 1 = -\int_0^t ue^{-su}ds$ and, therefore, by functional calculus this implies $e^{-tL} - \mathrm{Id} = -\int_0^t Le^{-sL}ds$, $t \geq 0$. In turn, from this it readily follows that $e^{-tL}f = f$, $\forall f \in \mathcal{N}(L) \cap L^2(M)$. Therefore, for any $f \in \mathcal{N}(L) \cap L^2(M)$, $x \in M$, and $t > 0$

$$|f(x)| = |e^{-tL}f(x)| \leq \int\limits_M |f(y)||p_t(x,y)|d\mu(y) \leq \|f\|_2\Big(\int\limits_M |p_t(x,y)|^2 d\mu(y)\Big)^{1/2},$$

where we applied the Cauchy–Schwarz inequality. However, from (1.3) and (2.1) it readily follows that

$$|p_t(x,y)| \leq c_\sigma |B(x,t)|^{-1}\big(1 + t^{-1}\rho(x,y)\big)^{-\sigma} \quad \text{for any } \sigma > 0.$$

This estimate, applied with $\sigma > d/2$, and (2.2) yield

$$\Big(\int\limits_M |p_t(x,y)|^2 d\mu(y)\Big)^{1/2} \leq c|B(x,t)|^{-1/2}.$$

On the other hand, from (1.8) and the noncollapsing condition (1.6) it follows that $|B(x,t)| \geq ct^{d^*}$, $t > 1$. Putting the above together, we obtain $\|f\|_\infty \leq ct^{-d^*/2}$. Finally, letting $t \to \infty$ this implies $f = 0$. $\square$

**The classes $\mathcal{S}_\infty$ and $\mathcal{S}'_\infty$.** Denote by $\mathcal{S}_\infty$ the set of all functions $\phi \in \mathcal{S}$ such that for every $k \geq 1$ there exists $\omega_k \in \mathcal{S}$ such that $\phi = L^k\omega_k$, that is, $L^{-k}\phi \in \mathcal{S}$ for all $k \geq 1$. Note that from Proposition 3.3 it follows that $\omega_k$ above is unique and hence $L^{-k}\phi$ is well defined on $\mathcal{S}_\infty$.

The topology in $\mathcal{S}_\infty$ is defined by the sequence of norms

$$\mathcal{P}^\star_m(\phi) := \sup_{x\in M}(1 + \rho(x, x_0))^m \max_{-m\leq\nu\leq m} |L^\nu\phi(x)|, \quad m \geq 0. \tag{3.9}$$

We denote by $\mathcal{S}'_\infty$ the set of all continuous linear functional on $\mathcal{S}_\infty$. As before the action of $f \in \mathcal{S}'_\infty$ on $\overline{\phi} \in \mathcal{S}_\infty$ will be denoted by $\langle f, \phi\rangle$. Apparently, for any $f \in \mathcal{S}'_\infty$ there exist constants $m \in \mathbb{Z}_+$ and $c > 0$ such that

$$|\langle f, \phi\rangle| \le c\mathcal{P}^\star_m(\phi), \quad \forall \phi \in \mathcal{S}_\infty. \tag{3.10}$$

Several remarks are in order:

(1) The following example shows that the class $\mathcal{S}_\infty$ is sufficiently rich. Let $\theta \in \mathcal{S}(\mathbb{R})$ be real-valued and even, and $\theta^{(\nu)}(0) = 0$ for $\nu = 0, 1, \dots$. Then for any $k \ge 1$ we have $\lambda^{-2k}\theta(\lambda) \in \mathcal{S}(\mathbb{R})$, which implies that $L^{-k}\theta(\sqrt{L})\phi \in \mathcal{S}$ for each $\phi \in \mathcal{S}$ and hence $\theta(\sqrt{L})\phi \in \mathcal{S}_\infty$, $\forall \phi \in \mathcal{S}$.

(2) Clearly, if $\phi \in \mathcal{S}_\infty$, then $L^k\phi \in \mathcal{S}_\infty$, $\forall k \in \mathbb{Z}$.

(3) It is important to note that $\mathcal{S}_\infty$ is a Fréchet space.

The latter assertion follows by the following

**Proposition 3.4.** *The space $\mathcal{S}_\infty$ is complete.*

**Proof.** This proof is quite similar to the proof of the completeness of $\mathcal{S}$, given in [5, Proposition 5.3]. We shall sketch it indicating only the differences.

Let $\{\phi_j\}_{j\ge1}$ be a Cauchy sequence in $\mathcal{S}_\infty$, that is, $\mathcal{P}^\star_m(\phi_j - \phi_n) \to 0$ as $j, n \to \infty$ for all $m \ge 0$. Just as in the proof of Proposition 5.3 in [5], it follows that

$$\|L^\nu\phi_j - L^\nu\phi_n\|_2 \to 0 \quad \text{as } j, n \to \infty,\ \forall \nu \in \mathbb{Z},$$

and by the completeness of $L^2$ there exist $\Psi_\nu \in L^2$ such that $\|L^\nu\phi_j - \Psi_\nu\|_2 \to 0$ as $j \to \infty$, $\nu \in \mathbb{Z}$.

Write $\phi := \Psi_0$. From the proof Proposition 5.3 in [5], it follows that $\phi \in \mathcal{S}$ and $\mathcal{P}_m(\phi_j - \phi) \to 0$, $\forall m \ge 0$, where $\mathcal{P}_m$ is from (3.1).

From $\|L^{-1}\phi_j - \Psi_{-1}\|_2 \to 0$, $\|LL^{-1}\phi_j - \phi\|_2 = \|\phi_j - \phi\|_2 \to 0$, and the fact that $L$ being a self-adjoint operator is closed [8] it follows that $\Psi_{-1} \in D(L)$ and $L\Psi_{-1} = \phi$. Hence, $\Psi_{-1} = L^{-1}\phi$. By the same token inductively it follows that $\Psi_{-\nu} = L^{-\nu}\phi$, $\forall \nu \ge 1$. Furthermore, just as in [5] we obtain

$$\|L^{-\nu}\phi_j - L^{-\nu}\phi\|_\infty \to 0 \quad \text{as } j \to \infty, \quad \forall \nu \ge 1.$$

In turn, this along with the fact that $\mathcal{P}^\star_m(\phi_j - \phi_n) \to 0$ as $j, n \to \infty$, and $\mathcal{P}_m(\phi_j - \phi) \to 0$ as $j \to \infty$, $\forall m \ge 0$, leads to $\mathcal{P}^\star_m(\phi_j - \phi) \to 0$ as $j \to \infty$, $\forall m \ge 0$, which confirms the completeness of $\mathcal{S}_\infty$. □

(4) Let $\varphi \in \mathcal{S}(\mathbb{R})$ be even and real-valued. Then

$$L^{-k}\varphi(\sqrt{L})\phi = \varphi(\sqrt{L})L^{-k}\phi, \quad \forall \phi \in \mathcal{S}_\infty,\ \forall k \ge 1, \tag{3.11}$$

and hence

$$\varphi(\sqrt{L})\phi \in \mathcal{S}_\infty, \quad \forall \phi \in \mathcal{S}_\infty. \tag{3.12}$$

Moreover, $\varphi(\sqrt{L})$ maps $\mathcal{S}_\infty$ into $\mathcal{S}_\infty$ continuously.

Indeed, assuming that $\phi \in \mathcal{S}_\infty$ we know from above that $L^{-k}\phi$ is well defined for any $k \ge 1$ and $L^{-k}\phi \in \mathcal{S}$. Hence

$$\varphi(\sqrt{L})\phi = \varphi(\sqrt{L})L^kL^{-k}\phi = L^k\varphi(\sqrt{L})L^{-k}\phi.$$

However, as was alluded to earlier $\varphi(\sqrt{L})$ maps continuously the class $\mathcal{S}$ into $\mathcal{S}$ and hence $\varphi(\sqrt{L})L^{-k}\phi \in \mathcal{S}$. Therefore, (3.11) holds and as a consequence (3.12) holds as well. The almost exponential localization of the kernel $\varphi(\sqrt{L})(x,y)$ and (3.11) readily imply that the map $\varphi(\sqrt{L}) : \mathcal{S}_\infty \to \mathcal{S}_\infty$ is continuous.

(5) The action of operators of the form $\varphi(\sqrt{L})$ on $\mathcal{S}'_\infty$, where $\varphi \in \mathcal{S}(\mathbb{R})$ is real-valued and even, needs some further clarification. Here the situation is somewhat similar to the one in the case of $\mathcal{S}'$ (see Definition 3.1 and Proposition 3.2).

**Definition 3.5.** We define $\varphi(\sqrt{L})f$ for any $f \in \mathcal{S}'_\infty$ by

$$\langle \varphi(\sqrt{L})f, \phi\rangle := \langle f, \varphi(\sqrt{L})\phi\rangle \quad \text{for} \quad \phi \in \mathcal{S}_\infty. \tag{3.13}$$

From (4) above it follows that, $\varphi(\sqrt{L})$ maps continuously $\mathcal{S}'_\infty$ into $\mathcal{S}'_\infty$. In addition, if $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ are real-valued and even, then

$$\varphi(\sqrt{L})\psi(\sqrt{L})f = \psi(\sqrt{L})\varphi(\sqrt{L})f, \quad \forall f \in \mathcal{S}'_\infty. \tag{3.14}$$

**Proposition 3.6.** *Let $\varphi \in \mathcal{S}(\mathbb{R})$ be real-valued and even and $\varphi^{(\nu)}(0) = 0$ for $\nu = 0, 1, \dots$. Then for any $f \in \mathcal{S}'_\infty$*

$$\varphi(\sqrt{L})f(x) = \langle f, \varphi(\sqrt{L})(x,\cdot)\rangle, \quad x \in M. \tag{3.15}$$

*Moreover, $\varphi(\sqrt{L})f$ is a continuous and slowly growing function, namely, there exist constants $m \in \mathbb{Z}_+$ and $c > 0$, depending on $f$, such that*

$$|\varphi(\sqrt{L})f(x)| \le c(1 + \rho(x, x_0))^m, \quad x \in M, \quad \textit{and} \tag{3.16}$$

$$|\varphi(\sqrt{L})f(x) - \varphi(\sqrt{L})f(x')| \le c(1 + \rho(x, x_0))^m \rho(x, x')^\alpha, \quad \textit{if } \rho(x, x') \le 1. \tag{3.17}$$

*Here $\alpha > 0$ is the constant from (1.4).*

The proof of this proposition is deferred to Section 7.

**Proposition 3.7.** *The following identification is valid:*

$$\mathcal{S}'/\mathcal{P} = \mathcal{S}'_\infty. \tag{3.18}$$

**Proof.** Let $F \in \mathcal{S}'/\mathcal{P}$ and assume $f_1, f_2 \in F$. Then $f_1 = f_2 + g$ for some $g \in \mathcal{P}$. Hence $g \in \mathcal{P}_m$ for some $m \ge 1$, yielding $\langle g, L^m\omega\rangle = 0$ for every $\omega \in \mathcal{S}$. Therefore, $\langle g, \phi\rangle = 0$ for every $\phi \in \mathcal{S}_\infty$ and hence $\langle f_1, \phi\rangle = \langle f_2, \phi\rangle$ for every $\phi \in \mathcal{S}_\infty$. Thus, we can associate with $F$ a unique bounded linear functional in $\mathcal{S}'_\infty$ defined by $\langle f, \phi\rangle$ $\forall \phi \in \mathcal{S}_\infty$, using an arbitrary $f \in F$. Consequently, $\mathcal{S}'/\mathcal{P} \subset \mathcal{S}'_\infty$.

For the other direction, observe that by the Hahn–Banach theorem (see [8, Theorem V.3]) every linear functional $f \in \mathcal{S}'_\infty$ can be extended to a bounded linear functional on $\mathcal{S}$ and the equivalence class, say, $F \in \mathcal{S}'/\mathcal{P}$ that contains this extension of $f$ is the class we associate with $f$. Therefore, $\mathcal{S}'_\infty \subset \mathcal{S}'/\mathcal{P}$. $\square$

From Proposition 3.7 it follows that for a sequence $\{f_j\} \subset \mathcal{S}'/\mathcal{P}$ and $f \in \mathcal{S}'/\mathcal{P}$ we have

$$f_j \to f \ \text{ in } \ \mathcal{S}'/\mathcal{P} \quad \text{if and only if} \quad \langle f_j, \phi\rangle \to \langle f, \phi\rangle, \quad \forall \phi \in \mathcal{S}_\infty. \tag{3.19}$$

A basic convergence result is given in the following

**Proposition 3.8.** *Suppose $\varphi \in \mathcal{S}(\mathbb{R})$, $\varphi$ is real-valued and even, and $\varphi(0) = 1$. Then for every $\phi \in \mathcal{S}_\infty$*

$$\phi = \lim_{t\to 0} \varphi(t\sqrt{L})\phi \quad \text{in } \mathcal{S}_\infty, \tag{3.20}$$

*and for every $f \in \mathcal{S}'_\infty$*

$$f = \lim_{t\to 0} \varphi(t\sqrt{L})f \quad \text{in } \mathcal{S}'_\infty. \tag{3.21}$$

The proofs of Proposition 3.8 is deferred to Section 7.

We now come to the main assertion in this section.

**Theorem 3.9.** *Let $\Psi \in C^\infty(\mathbb{R}_+)$, $\operatorname{supp}\Psi \subset [b^{-1}, b]$ with $b > 1$, $\Psi$ be real-valued, and*

$$\sum_{j\in\mathbb{Z}} \Psi(b^{-j}\lambda) = 1 \quad \text{for} \quad \lambda \in (0,\infty). \tag{3.22}$$

*Then for any $f \in \mathcal{S}'/\mathcal{P}$*

$$f = \sum_{j\in\mathbb{Z}} \Psi(b^{-j}\sqrt{L})f \quad \text{in } \mathcal{S}'/\mathcal{P}, \tag{3.23}$$

*that is, for any $f \in \mathcal{S}'_\infty$*

$$\lim_{n,m\to\infty} \sum_{j=-n}^{m} \big\langle \Psi(b^{-j}\sqrt{L})f, \phi \big\rangle = \langle f, \phi\rangle, \quad \forall \phi \in \mathcal{S}_\infty. \tag{3.24}$$

**Proof.** By duality (see (3.3)) it suffices to prove that for any $\phi \in \mathcal{S}_\infty$

$$\lim_{n,m\to\infty} \sum_{j=-n}^{m} \Psi(b^{-j}\sqrt{L})\phi = \phi \quad \text{in} \quad \mathcal{S}_\infty.$$

Write

$$\varphi(\lambda) := \begin{cases} 1, & 0 \le \lambda \le 1, \\ \Psi(\lambda), & \lambda > 1. \end{cases}$$

From the properties of $\Psi$ it readily follows that $\varphi \in C^\infty(\mathbb{R}_+)$, $\operatorname{supp}\varphi \subset [0, b]$, and

$$\sum_{j=-n}^{m} \Psi(b^{-j}\lambda) = \varphi(b^{-m}\lambda) - \varphi(b^{n+1}\lambda), \quad \lambda \in [0,\infty), \ \ n, m \ge 0.$$

Hence,

$$\sum_{j=-n}^{m} \Psi(b^{-j}\sqrt{L}) = \varphi(b^{-m}\sqrt{L}) - \varphi(b^{n+1}\sqrt{L}).$$

However, by Proposition 3.8 it follows that $\varphi(b^{-m}\sqrt{L})\phi \to \phi$ in $\mathcal{S}_\infty$ as $m \to \infty$ for every $\phi \in \mathcal{S}_\infty$ and it remains to show that

$$\varphi(b^n\sqrt{L})\phi \to 0 \text{ in } \mathcal{S}_\infty \text{ as } n\to\infty \text{ for every } \phi\in\mathcal{S}_\infty.$$

We have to show that for any $m\ge 0$ and $r\in\mathbb{Z}$

$$\lim_{n\to\infty}\sup_{x\in M}(1+\rho(x,x_0))^m\big|L^r\varphi(b^n\sqrt{L})\phi(x)\big| = 0, \quad \forall\phi\in\mathcal{S}_\infty.$$

Choose $k\ge(m+1)/2$ and fix $\phi\in\mathcal{S}_\infty$. Denote $\omega := L^{r-k}\phi$, $\omega\in\mathcal{S}_\infty$. Therefore, it suffices to show that

$$\lim_{n\to\infty}\sup_{x\in M}(1+\rho(x,x_0))^m\big|L^k\varphi(b^n\sqrt{L})\omega(x)\big| = 0. \tag{3.25}$$

Set $\eta(\lambda) := \lambda^{2k}\varphi(\lambda)$. Assume $n\ge 0$. Clearly, $\eta(b^n\sqrt{L}) = b^{2nk}L^k\varphi(b^n\sqrt{L})$, implying the identity $L^k\varphi(b^n\sqrt{L}) = b^{-2nk}\eta(b^n\sqrt{L})$. Using the fact that $\eta\in C^\infty(\mathbb{R}_+)$, $\operatorname{supp}\eta\subset[0,b]$, and $\eta^{(2\nu+1)}(0)=0$ for $\nu\ge 0$ we apply Theorem 2.2 with $N := m+3d/2+1$ to obtain

$$\begin{aligned}
|L^k\varphi(b^n\sqrt{L})\omega(x)| &\le cb^{-2nk}\int_M \frac{|\omega(y)|d\mu(y)}{|B(x,b^n)|(1+b^{-n}\rho(x,y))^{N-d/2}}\\
&\le cb^{-2nk}\sup_{y\in M}(1+\rho(y,x_0))^{m+d+1}|\omega(y)|\\
&\qquad \times|B(x,b^n)|^{-1}\int_M \frac{d\mu(y)}{(1+b^{-n}\rho(x,y))^{m+d+1}(1+\rho(y,x_0))^{m+d+1}}\\
&\le cb^{-2nk}\sup_{y\in M}(1+\rho(y,x_0))^{m+d+1}|\omega(y)||B(x,b^n)|^{-1}\frac{|B(x,b^n)|}{(1+b^{-n}\rho(x,x_0))^{m+1}}\\
&\le cb^{-2nk}\sup_{y\in M}(1+\rho(y,x_0))^{m+d+1}|\omega(y)|\frac{b^{nm}}{(1+\rho(x,x_0))^m}.
\end{aligned}$$

Here for the former inequality we used Lemma 2.1. Consequently,

$$\begin{aligned}
|L^k\varphi(b^n\sqrt{L})\omega(x)| &\le cb^{-n}\mathcal{P}^\star_{m+d+1}(\omega)(1+\rho(x,x_0))^{-m}\\
&\le cb^{-n}\mathcal{P}^\star_{m+d+1+|r-k|}(\phi)(1+\rho(x,x_0))^{-m},
\end{aligned}$$

where we used that $k\ge(m+1)/2$ and $n\ge 0$. The above implies (3.25). □

### *3.3. Tempered distributions on $\mathbb{R}^d$ associated with $L=-\Delta$*

We next show that in the case when $M=\mathbb{R}^d$ and $L=-\Delta$, with $\Delta$ being the Laplacian, the distributions modulo generalized polynomials $\mathcal{S}'/\mathcal{P}$ introduced in §3.2 are just the classical tempered distributions modulo polynomials on $\mathbb{R}^d$. Therefore, our general setting covers the classical case on $\mathbb{R}^d$.

Indeed, as is shown in [5, Proposition 5.6] the test functions $\mathcal{S}(L)$ in the setting $M=\mathbb{R}^d$ and $L=-\Delta$ are just the test functions $\mathcal{S}(\mathbb{R}^d)$ in the classical case with the same topology and, therefore, $\mathcal{S}'(L)$ is the set of the classical tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. On the other hand, if $g\in\mathcal{S}'$ and

$$L^m g = (-1)^m\Delta^m g = 0,$$

then applying the Fourier transform, defined by $\hat{\phi}(\xi) := \int_{\mathbb{R}^d}\phi(x)e^{-2\pi i x\cdot\xi}dx$ for $\phi\in\mathcal{S}$, we infer

$$(4\pi^2)^m|\xi|^{2m}\hat{g} = 0.$$

Therefore, the distribution $\hat{g}$ is supported at the origin and hence $g$ is an algebraic polynomial. This leads us to the conclusion that in this setting the set $\mathcal{P}$ defined in §3.2 is the set of all polynomials on $\mathbb{R}^d$ and hence $\mathcal{S}'/\mathcal{P}$ is the set of tempered distributions modulo polynomials on $\mathbb{R}^d$.

## 4. Frames

Frames will play an important rôle in this study. Their construction will be an adaptation of the one in the inhomogeneous case from [5], see also [1]. Therefore, we shall only indicate the needed modifications in the construction from [5].

**Construction of Frame # 1.** We first apply Theorem 2.5 for the construction of a real-valued cutoff function $\Phi$ with the following properties: $\Phi \in C^\infty(\mathbb{R}_+)$, $\Phi(u) = 1$ for $u \in [0,1]$, $0 \le \Phi \le 1$, $\operatorname{supp}\Phi \subset [0,b]$, where $b > 1$ is a constant (see [5]), and such that $\Phi(\delta\sqrt{L})$ is an integral operator with kernel $\Phi(\delta\sqrt{L})(x,y)$ obeying (2.9)–(2.10). Set

$$\Psi(u) := \Phi(u) - \Phi(bu). \tag{4.1}$$

Clearly, $\Psi \in C^\infty(\mathbb{R}_+)$ and $\operatorname{supp}\Psi \subset [b^{-1}, b]$. By the properties of $\Phi(\delta\sqrt{L})(x,y)$ (from Theorem 2.5) it follows that the kernel $\Psi(\delta\sqrt{L})(x,y)$ of the operator $\Psi(\delta\sqrt{L})$ is of sub-exponential localization, that is,

$$|\Psi(\delta\sqrt{L})(x,y)| \le \frac{c_\diamond \exp\big\{-\kappa\big(\frac{\rho(x,y)}{\delta}\big)^\beta\big\}}{\big(|B(x,\delta)||B(y,\delta)|\big)^{1/2}}, \quad \forall x,y \in M, \tag{4.2}$$

and for any $m \ge 1$

$$|[L^m\Psi(\delta\sqrt{L})](x,y)| \le \frac{c_m\delta^{-2m} \exp\big\{-\kappa\big(\frac{\rho(x,y)}{\delta}\big)^\beta\big\}}{\big(|B(x,\delta)||B(y,\delta)|\big)^{1/2}}, \quad \forall x,y \in M. \tag{4.3}$$

Here $0 < \beta < 1$ is an arbitrary constant (as close to 1 as we wish) and $\kappa > 0$, $c_\diamond, c_m > 1$ are constants depending on $\beta$, $b$, and the constants $c_0, C^\star, c^\star$ from (1.1)–(1.4); $c_m$ depends om $m$ as well. Set

$$\Psi_j(u) := \Psi(b^{-j}u), \quad j \in \mathbb{Z}. \tag{4.4}$$

Clearly, $\Psi_j \in C^\infty(\mathbb{R}_+)$, $0 \le \Psi_j \le 1$, $\operatorname{supp}\Psi_j \subset [b^{j-1}, b^{j+1}]$, $j \in \mathbb{Z}$, and

$$\sum_{j\in\mathbb{Z}} \Psi_j(u) = 1 \quad \text{for } u \in (0,\infty).$$

Therefore, by Theorem 3.9 for any $f \in \mathcal{S}'/\mathcal{P}$

$$f = \sum_{j\in\mathbb{Z}} \Psi_j(\sqrt{L})f \quad (\text{convergence in } \mathcal{S}'/\mathcal{P}). \tag{4.5}$$

The sampling Theorem 4.2 from [1] will play an important rôle in this construction. In particular, this theorem yields the following

**Proposition 4.1.** *For any $\varepsilon > 0$ there exists a constant $\gamma$ $(0 < \gamma < 1)$ such that for any maximal $\delta$-net $\mathcal{X}$ on $M$ with $\delta := \gamma\lambda^{-1}$, $\lambda > 0$, and a companion disjoint partition $\{A_\xi\}_{\xi\in\mathcal{X}}$ of $M$ as in Subsection 2.1 consisting of measurable sets such that $B(\xi,\delta/2) \subset A_\xi \subset B(\xi,\delta)$, $\xi \in \mathcal{X}$, we have*

$$(1-\varepsilon)\|f\|_2^2 \le \sum_{\xi\in\mathcal{X}} |A_\xi||f(\xi)|^2 \le (1+\varepsilon)\|f\|_2^2, \quad \forall f \in \Sigma_\lambda^2.$$

At this point, we introduce a constant $0 < \varepsilon < 1$ that will be specified later on. We use the above proposition to produce for each $j \in \mathbb{Z}$ a maximal $\delta_j$-net $\mathcal{X}_j$ on $M$ with $\delta_j := \gamma b^{-j-2}$ and a disjoint partition $\{A_\xi\}_{\xi\in\mathcal{X}_j}$ of $M$ such that

$$(1-\varepsilon)\|f\|_2^2 \le \sum_{\xi\in\mathcal{X}_j} |A_\xi||f(\xi)|^2 \le (1+\varepsilon)\|f\|_2^2, \quad \forall f \in \Sigma^2_{b^{j+2}}. \tag{4.6}$$

Set $\mathcal{X} := \cup_{j\in\mathbb{Z}}\mathcal{X}_j$, where equal points from different sets $\mathcal{X}_j$ will be regarded as distinct elements of $\mathcal{X}$, and hence $\mathcal{X}$ can be used as an index set.

Frame # 1 $\{\psi_\xi\}_{\xi\in\mathcal{X}}$ is defined by

$$\psi_\xi(x) := |A_\xi|^{1/2}\Psi_j(\sqrt{L})(x,\xi), \quad \xi \in \mathcal{X}_j,\ j \in \mathbb{Z}. \tag{4.7}$$

**Construction of Frame # 2.** A dual frame $\{\tilde{\psi}_\xi\}$ will be constructed similarly as in [5] with properties similar to the properties of $\{\psi_\xi\}$.

The first step in this construction is to introduce a cutoff function

$$\Gamma(u) = \Phi(b^{-2}u) - \Phi(bu), \tag{4.8}$$

where $\Phi$ is from the construction of Frame #1. Clearly, $\operatorname{supp}\Gamma \subset [b^{-1}, b^3]$ and $\Gamma = 1$ on $[1, b^2]$, implying $\Gamma(u)\Psi_1(u) = \Psi_1(u)$.

The construction of Frame # 2 hinges on the following

**Lemma 4.2.** *There exists a constant $0 < \varepsilon < 1$ such that the following claim holds true. Given $\lambda > 0$, let $\mathcal{X}$ be a maximal $\delta$-net on $M$, where $\delta := \gamma\lambda^{-1}b^{-3}$ with $\gamma$ the constant from Proposition 4.1, and suppose $\{A_\xi\}_{\xi\in\mathcal{X}}$ is a companion disjoint partition of $M$ consisting of measurable sets such that $B(\xi,\delta/2) \subset A_\xi \subset B(\xi,\delta)$, $\xi \in \mathcal{X}$ (see §2.1). Set $\omega_\xi := (1+\varepsilon)^{-1}|A_\xi|$. Then there exists a linear operator $T_\lambda : L^2(M) \to L^2(M)$ of the form $T_\lambda = \mathrm{Id} + S_\lambda$ such that:*

*(a)*

$$\|f\|_2 \le \|T_\lambda f\|_2 \le \frac{1}{1-2\varepsilon}\|f\|_2, \quad \forall f \in L^2.$$

*(b) $S_\lambda$ is an integral operator with kernel $S_\lambda(x,y)$ verifying*

$$|S_\lambda(x,y)| \le \frac{c\exp\big\{-\frac{\kappa}{2}\big(\lambda\rho(x,y)\big)^\beta\big\}}{\big(|B(x,\lambda^{-1})||B(y,\lambda^{-1})|\big)^{1/2}}, \quad \forall x,y \in M. \tag{4.9}$$

*(c) $S_\lambda(L^2) \subset \Sigma^2_{[\lambda b^{-1},\lambda b^3]}$.*

*(d) For any $f \in L^2(M)$ such that $\Gamma(\lambda^{-1}\sqrt{L})f = f$ we have*

$$f = \sum_{\xi\in\mathcal{X}} \omega_\xi f(\xi) T_\lambda[\Gamma_\lambda(\cdot,\xi)], \tag{4.10}$$

*where $\Gamma_\lambda(\cdot,\cdot)$ is the kernel of the operator $\Gamma_\lambda := \Gamma(\lambda^{-1}\sqrt{L})$ with $\Gamma$ from (4.8).*

This lemma is simply Lemma 4.2 from [1] with the only difference that it is assumed that $\lambda > 0$ instead of $\lambda \ge 1$; the proof is the same and will be omitted.

We use the above lemma to select the constant $\varepsilon$ ($0 < \varepsilon < 1$) that was already used in the construction of Frame #1.

Let $\mathcal{X}_j$ and $\{A_\xi\}_{\xi\in\mathcal{X}_j}$ be as in the definition of Frame #1. Denote $\Gamma_{\lambda_j} := \Gamma(b^{-j+1}\sqrt{L})$ for $j \in \mathbb{Z}$ with $\lambda_j := b^{j-1}$, and let $T_{\lambda_j} = \mathrm{Id} + S_{\lambda_j}$ be the operator from Lemma 4.2, applied with $\lambda = \lambda_j$. The dual frame $\{\tilde{\psi}_\xi\}_{\xi\in\mathcal{X}}$ is defined by

$$\tilde{\psi}_\xi(x) := c_\varepsilon |A_\xi|^{1/2} T_{\lambda_j}\big[\Gamma_{\lambda_j}(\cdot,\xi)\big](x), \quad \xi \in \mathcal{X}_j,\ j \in \mathbb{Z}, \quad c_\varepsilon := (1+\varepsilon)^{-1}. \tag{4.11}$$

In the next theorem we record the main properties of $\{\psi_\xi\}_{\xi\in\mathcal{X}}$ and $\{\tilde{\psi}_\xi\}_{\xi\in\mathcal{X}}$.

**Theorem 4.3.** ($a$) Representation: *For any* $f \in \mathcal{S}'/\mathcal{P}$,

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \tilde{\psi}_\xi\rangle \psi_\xi = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle \tilde{\psi}_\xi \quad \textit{in } \mathcal{S}'/\mathcal{P}. \tag{4.12}$$

($b$) Space localization: *For any* $0 < \hat{\kappa} < \kappa/2$, $m \in \mathbb{Z}$, *and any* $\xi \in \mathcal{X}_j$, $j \in \mathbb{Z}$,

$$|L^m\psi_\xi(x)|, |L^m\tilde{\psi}_\xi(x)| \le c_m b^{2jm}|B(\xi, b^{-j})|^{-1/2} \exp\big\{-\hat{\kappa}(b^j\rho(x,\xi))^\beta\big\}. \tag{4.13}$$

($c$) Spectral localization: $\psi_\xi \in \Sigma^p_{[b^{j-1}, b^{j+1}]}$ *and* $\tilde{\psi}_\xi \in \Sigma^p_{[b^{j-2}, b^{j+2}]}$ *for* $\xi \in \mathcal{X}_j$, $j \in \mathbb{Z}$, $0 < p \le \infty$.

($d$) Norms: *For any* $\xi \in \mathcal{X}_j$, $j \in \mathbb{Z}$,

$$\|\psi_\xi\|_p \sim \|\tilde{\psi}_\xi\|_p \sim |B(\xi, b^{-j})|^{\frac{1}{p}-\frac{1}{2}} \quad \textit{for } 0 < p \le \infty. \tag{4.14}$$

($e$) Frame: *The system* $\{\tilde{\psi}_\xi\}$ *as well as* $\{\psi_\xi\}$ *is a frame for* $L^2$, *namely, there exists a constant* $c > 0$ *such that*

$$c^{-1}\|f\|_2^2 \le \sum_{\xi\in\mathcal{X}} |\langle f, \tilde{\psi}_\xi\rangle|^2 \le c\|f\|_2^2, \quad \forall f \in L^2. \tag{4.15}$$

**Proof.** The proof of parts (b)–(e) of this theorem is carried out just as the proof of the respective claims in Proposition 4.1 and Theorem 4.3 in [1].

We now focus on the proof of part (a). By duality to prove (4.12) it suffices show that for any $\phi \in \mathcal{S}_\infty$

$$\phi = \sum_{\xi\in\mathcal{X}} \langle \phi, \tilde{\psi}_\xi\rangle \psi_\xi = \sum_{\xi\in\mathcal{X}} \langle \phi, \psi_\xi\rangle \tilde{\psi}_\xi \quad \text{in } \mathcal{S}_\infty.$$

However, by Theorem 3.9 for any $\phi \in \mathcal{S}_\infty$

$$\phi = \sum_{j\in\mathbb{Z}} \Psi_j(\sqrt{L})\phi \quad (\text{convergence in } \mathcal{S}_\infty)$$

and hence we only have to show that for every $\phi \in \mathcal{S}_\infty$

$$\Psi_j(\sqrt{L})\phi = \sum_{\xi\in\mathcal{X}_j} \langle \phi, \tilde{\psi}_\xi\rangle \psi_\xi = \sum_{\xi\in\mathcal{X}_j} \langle \phi, \psi_\xi\rangle \tilde{\psi}_\xi \quad (\text{convergence in } \mathcal{S}_\infty).$$

The proof of these identities is a straightforward adaptation of the proof of Proposition 5.5 (c) from [5], where now the parameter $m \in \mathbb{Z}$ rather than $m \ge 0$. We omit the details. This completes the proof. □

**Remark 4.4.** The construction of frames with the desired excellent space and spectral localization is particularly simple in the case when the spectral spaces $\Sigma^2_\lambda$ have the polynomial property under multiplication:

*Let $F_\lambda, \lambda \ge 0$, be the spectral resolution associated with the operator $\sqrt{L}$. We say that the associated spectral spaces*

$$\Sigma_\lambda^2 = \{f \in L^2 : F_\lambda f = f\}$$

*have the polynomial property if there exists a constant $a > 1$ such that*

$$\Sigma_\lambda^2 \cdot \Sigma_\lambda^2 \subset \Sigma_{a\lambda}^1, \quad \text{i.e.} \quad f, g \in \Sigma_\lambda^2 \Longrightarrow fg \in \Sigma_{a\lambda}^1. \tag{4.16}$$

In this case, the scheme from [5], §4.4, can be adapted for the construction of a single frame $\{\psi_\xi\}_{\xi\in\mathcal{X}}$ with frame elements having the properties from Theorem 4.3, where in particular, the representation of each $f \in \mathcal{S}'/\mathcal{P}$ takes the form $f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle\psi_\xi$ with convergence in $\mathcal{S}'/\mathcal{P}$. We omit the details.

## 5. Homogeneous Besov (B) and Triebel–Lizorkin (F) spaces

The inhomogeneous Besov and Triebel–Lizorkin spaces in the setting of this article are developed in [5]. Here we focus on the homogeneous version of these spaces.

**Definition of homogeneous Besov and Triebel–Lizorkin spaces.** To deal with possible anisotropic geometries we introduce two types of homogeneous Besov and Triebel–Lizorkin spaces (B- and F-spaces for short):

(i) Classical homogeneous B-spaces $\dot{B}_{pq}^s = \dot{B}_{pq}^s(L)$ and F-spaces $\dot{F}_{pq}^s = \dot{F}_{pq}^s(L)$, and

(ii) Nonclassical homogeneous B-spaces $\dot{\tilde{B}}{}_{pq}^s = \dot{\tilde{B}}{}_{pq}^s(L)$ and F-spaces $\dot{\tilde{F}}{}_{pq}^s = \dot{\tilde{F}}{}_{pq}^s(L)$.

Let the function $\varphi \in C^\infty(\mathbb{R}_+)$ satisfy

$$\operatorname{supp}\varphi \subset [1/2, 2], \quad |\varphi(\lambda)| \ge c > 0 \quad \text{for } \lambda \in [2^{-3/4}, 2^{3/4}]. \tag{5.1}$$

Then $\sum_{j\in\mathbb{Z}} |\varphi(2^{-j}\lambda)| \ge c > 0$ for $\lambda \in \mathbb{R}_+$. Set $\varphi_j(\lambda) := \varphi(2^{-j}\lambda)$ for $j \in \mathbb{Z}$.

**Definition 5.1.** Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$.

($a$) The Besov space $\dot{B}_{pq}^s = \dot{B}_{pq}^s(L)$ is defined as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{B}_{pq}^s} := \Big(\sum_{j\in\mathbb{Z}} \Big(2^{js}\|\varphi_j(\sqrt{L})f(\cdot)\|_{L^p}\Big)^q\Big)^{1/q} < \infty. \tag{5.2}$$

($b$) The Besov space $\dot{\tilde{B}}{}_{pq}^s = \dot{\tilde{B}}{}_{pq}^s(L)$ is defined as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{\tilde{B}}{}_{pq}^s} := \Big(\sum_{j\in\mathbb{Z}} \Big(\||B(\cdot, 2^{-j})|^{-s/d}\varphi_j(\sqrt{L})f(\cdot)\|_{L^p}\Big)^q\Big)^{1/q} < \infty. \tag{5.3}$$

**Definition 5.2.** Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$.

($a$) The Triebel–Lizorkin space $\dot{F}_{pq}^s = \dot{F}_{pq}^s(L)$ is defined as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{F}_{pq}^s} := \Big\|\Big(\sum_{j\in\mathbb{Z}} \Big(2^{js}|\varphi_j(\sqrt{L})f(\cdot)|\Big)^q\Big)^{1/q}\Big\|_{L^p} < \infty. \tag{5.4}$$

($b$) The Triebel–Lizorkin space $\dot{\tilde{F}}{}_{pq}^s = \dot{\tilde{F}}{}_{pq}^s(L)$ is defined as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{\tilde{F}}{}_{pq}^s} := \Big\|\Big(\sum_{j\in\mathbb{Z}} \Big(|B(\cdot, 2^{-j})|^{-s/d}|\varphi_j(\sqrt{L})f(\cdot)|\Big)^q\Big)^{1/q}\Big\|_{L^p} < \infty. \tag{5.5}$$

Above in both definitions the $\ell^q$-norm is replaced by the sup-norm if $q = \infty$.

Several remarks are in order.

(1) Just as in Propositions 6.3 and 7.2 in [5] one shows that the above definitions of the spaces $\dot{B}^s_{pq}$, $\dot{\tilde{B}}^s_{pq}$, $\dot{F}^s_{pq}$, and $\dot{\tilde{F}}^s_{pq}$ are independent of the particular selection of the function $\varphi \in C^\infty(\mathbb{R}_+)$ obeying (5.1).

(2) In the definitions of the $\dot{B}^s_{pq}$, $\dot{\tilde{B}}^s_{pq}$, $\dot{F}^s_{pq}$, and $\dot{\tilde{F}}^s_{pq}$ spaces above the role of the constant 2 can be played by an arbitrary $b > 1$, then e.g. $2^{js}$ in (5.2) and (5.4) will be replaced by $b^{js}$. Similarly as in [5, Proposition 6.3] it can be shown that the resulting norms are equivalent to the ones from Definitions 5.1 and 5.2.

(3) It is easy to show that $\mathcal{S}_\infty$ is continuously embedded in each of the spaces $\dot{B}^s_{pq}$, $\dot{\tilde{B}}^s_{pq}$, $\dot{F}^s_{pq}$, and $\dot{\tilde{F}}^s_{pq}$, that is, there exist constants $m \ge 0$ and $c > 0$, depending on $s, p, q$, such that

$$\|\phi\|_{\dot{B}^s_{pq}} \le c\mathcal{P}^\star_m(\phi), \quad \forall \phi \in \mathcal{S}_\infty, \tag{5.6}$$

and this inequality holds with $\dot{B}^s_{pq}$ replaced by $\dot{\tilde{B}}^s_{pq}$, $\dot{F}^s_{pq}$, or $\dot{\tilde{F}}^s_{pq}$.

(4) The continuous embedding of the homogeneous B- and F-spaces in $\mathcal{S}'/\mathcal{P}$ is more subtle and will be given in the following

**Theorem 5.3.** *Each of the spaces $\dot{B}^s_{pq}$, $\dot{\tilde{B}}^s_{pq}$, $\dot{F}^s_{pq}$, and $\dot{\tilde{F}}^s_{pq}$ is continuously embedded in $\mathcal{S}'/\mathcal{P}$, that is, there exist constants $m \ge 0$ and $c > 0$, depending on $s, p, q$, such that*

$$|\langle f, \phi\rangle| \le c\|f\|_{\dot{B}^s_{pq}}\mathcal{P}^\star_m(\phi), \quad \forall f \in \dot{B}^s_{pq}, \ \forall \phi \in \mathcal{S}_\infty, \tag{5.7}$$

*and similar inequalities hold for $\dot{\tilde{B}}^s_{pq}$, $\dot{F}^s_{pq}$, and $\dot{\tilde{F}}^s_{pq}$.*

(5) By a standard argument the above assertion readily implies that the spaces $\dot{B}^s_{pq}$, $\dot{\tilde{B}}^s_{pq}$, $\dot{F}^s_{pq}$, and $\dot{\tilde{F}}^s_{pq}$ are complete and hence they are quasi-Banach spaces (Banach spaces if $p, q \ge 1$).

We give the proof of the continuous embedding of the B- and F-spaces into $\mathcal{S}'/\mathcal{P}$ (Theorem 5.3) in Section 7 and omit the proofs of the other assertions from above.

*Frame decomposition of homogeneous Besov and Triebel–Lizorkin spaces*

One of the main result in [5] asserts that the inhomogeneous Besov and Triebel–Lizorkin spaces in the setting of this article can be characterized in terms of respective sequence norms of the frame coefficients of distributions.

The primary purpose of this section is to establish similar results for the homogeneous Besov and Triebel–Lizorkin spaces, using the frames $\{\psi_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde{\psi}_\xi\}_{\xi\in\mathcal{X}}$ from §4. As is §4 $\mathcal{X} := \cup_{j\in\mathbb{Z}}\mathcal{X}_j$ will denote the sets of the centers of the frame elements and $\{A_\xi\}_{\xi\in\mathcal{X}_j}$ will be the associated partitions of $M$.

Our first step is to introduce the homogeneous sequence spaces $\dot{b}^s_{pq}$, $\dot{\tilde{b}}^s_{pq}$, and $\dot{f}^s_{pq}$, $\dot{\tilde{f}}^s_{pq}$, associated with the B- and F-spaces.

**Definition 5.4.** Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$.

($a$) The space $\dot{b}^s_{pq}$ is defined as the space of all complex-valued sequences $a = \{a_\xi\}_{\xi\in\mathcal{X}}$ such that

$$\|a\|_{\dot{b}^s_{pq}} := \Big(\sum_{j\in\mathbb{Z}} b^{jsq}\Big[\sum_{\xi\in\mathcal{X}_j}\Big(|B(\xi, b^{-j})|^{1/p-1/2}|a_\xi|\Big)^p\Big]^{q/p}\Big)^{1/q} < \infty. \tag{5.8}$$

(*b*) The space $\dot{\tilde{b}}^s_{pq}$ is defined as the space of all complex-valued sequences $a = \{a_\xi\}_{\xi\in\mathcal{X}}$ such that

$$\|a\|_{\dot{\tilde{b}}^s_{pq}} := \Big(\sum_{j\in\mathbb{Z}}\Big[\sum_{\xi\in\mathcal{X}_j}\Big(|B(\xi,b^{-j})|^{-s/d+1/p-1/2}|a_\xi|\Big)^p\Big]^{q/p}\Big)^{1/q} < \infty. \tag{5.9}$$

**Definition 5.5.** Suppose $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$.

(*a*) The space $\dot{f}^s_{pq}$ is defined as the space of all complex-valued sequences $a = \{a_\xi\}_{\xi\in\mathcal{X}}$ such that

$$\|a\|_{\dot{f}^s_{pq}} := \Big\|\Big(\sum_{j\in\mathbb{Z}} b^{jsq}\sum_{\xi\in\mathcal{X}_j}\big[|a_\xi|\tilde{\mathbb{1}}_{A_\xi}(\cdot)\big]^q\Big)^{1/q}\Big\|_{L^p} < \infty. \tag{5.10}$$

(*b*) The space $\dot{\tilde{f}}^s_{pq}$ is defined as the space of all complex-valued sequences $a = \{a_\xi\}_{\xi\in\mathcal{X}}$ such that

$$\|a\|_{\dot{\tilde{f}}^s_{pq}} := \Big\|\Big(\sum_{\xi\in\mathcal{X}}\big[|A_\xi|^{-s/d}|a_\xi|\tilde{\mathbb{1}}_{A_\xi}(\cdot)\big]^q\Big)^{1/q}\Big\|_{L^p} < \infty. \tag{5.11}$$

Here $\tilde{\mathbb{1}}_{A_\xi} := |A_\xi|^{-1/2}\mathbb{1}_{A_\xi}$ with $\mathbb{1}_{A_\xi}$ being the characteristic function of $A_\xi$.

Above as usual the $\ell^p$ or $\ell^q$ norm is replaced by the sup-norm if $p = \infty$ or $q = \infty$.

In stating our results we shall use the "analysis" and "synthesis" operators defined by

$$S_{\tilde{\psi}} : f \to \{\langle f, \tilde{\psi}_\xi\rangle\}_{\xi\in\mathcal{X}} \quad \text{and} \quad T_\psi : \{a_\xi\}_{\xi\in\mathcal{X}} \to \sum_{\xi\in\mathcal{X}} a_\xi\psi_\xi. \tag{5.12}$$

Here the roles of $\{\psi_\xi\}$ and $\{\tilde{\psi}_\xi\}$ can be interchanged.

**Theorem 5.6.** *Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$. (a) The operators $S_{\tilde{\psi}} : \dot{B}^s_{pq} \to \dot{b}^s_{pq}$ and $T_\psi : \dot{b}^s_{pq} \to \dot{B}^s_{pq}$ are bounded and $T_\psi \circ S_{\tilde{\psi}} = Id$ on $\dot{B}^s_{pq}$. Consequently, for $f \in \mathcal{S}'/\mathcal{P}$ we have $f \in \dot{B}^s_{pq}$ if and only if $\{\langle f, \tilde{\psi}_\xi\rangle\}_{\xi\in\mathcal{X}} \in \dot{b}^s_{pq}$. Moreover, if $f \in \dot{B}^s_{pq}$, then $\|f\|_{\dot{B}^s_{pq}} \sim \|\{\langle f, \tilde{\psi}_\xi\rangle\}\|_{\dot{b}^s_{pq}}$ and*

$$\|f\|_{\dot{B}^s_{pq}} \sim \Big(\sum_{j\in\mathbb{Z}} b^{jsq}\Big[\sum_{\xi\in\mathcal{X}_j}\|\langle f, \tilde{\psi}_\xi\rangle\psi_\xi\|_p^p\Big]^{q/p}\Big)^{1/q}. \tag{5.13}$$

*(b) The operators $S_{\tilde{\psi}} : \dot{\tilde{B}}^s_{pq} \to \dot{\tilde{b}}^s_{pq}$ and $T_\psi : \dot{\tilde{b}}^s_{pq} \to \dot{\tilde{B}}^s_{pq}$ are bounded and $T_\psi \circ S_{\tilde{\psi}} = Id$ on $\dot{\tilde{B}}^s_{pq}$. Hence, $f \in \dot{\tilde{B}}^s_{pq} \Longleftrightarrow \{\langle f, \tilde{\psi}_\xi\rangle\}_{\xi\in\mathcal{X}} \in \dot{\tilde{b}}^s_{pq}$. Furthermore, if $f \in \dot{\tilde{B}}^s_{pq}$, then $\|f\|_{\dot{\tilde{B}}^s_{pq}} \sim \|\{\langle f, \tilde{\psi}_\xi\rangle\}\|_{\dot{\tilde{b}}^s_{pq}}$ and*

$$\|f\|_{\dot{\tilde{B}}^s_{pq}} \sim \Big(\sum_{j\in\mathbb{Z}}\Big[\sum_{\xi\in\mathcal{X}_j}\Big(|B(\xi,b^{-j})|^{-s/d}\|\langle f, \tilde{\psi}_\xi\rangle\psi_\xi\|_p\Big)^p\Big]^{q/p}\Big)^{1/q}. \tag{5.14}$$

*Above in (a) and (b) the roles of $\{\psi_\xi\}$ and $\{\tilde{\psi}_\xi\}$ can be interchanged.*

**Theorem 5.7.** *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. (a) The operators $S_{\tilde{\psi}} : \dot{F}^s_{pq} \to \dot{f}^s_{pq}$ and $T_\psi : \dot{f}^s_{pq} \to \dot{F}^s_{pq}$ are bounded and $T_{\tilde{\psi}} \circ S_\psi = Id$ on $\dot{F}^s_{pq}$. Consequently, $f \in \dot{F}^s_{pq}$ if and only if $\{\langle f, \tilde{\psi}_\xi\rangle\}_{\xi\in\mathcal{X}} \in \dot{f}^s_{pq}$, and if $f \in \dot{F}^s_{pq}$, then $\|f\|_{\dot{F}^s_{pq}} \sim \|\{\langle f, \tilde{\psi}_\xi\rangle\}\|_{\dot{f}^s_{pq}}$. Furthermore,*

$$\|f\|_{\dot{F}^s_{pq}} \sim \Big\|\Big(\sum_{j\in\mathbb{Z}} b^{jsq}\sum_{\xi\in\mathcal{X}_j}\big[|\langle f, \tilde{\psi}_\xi\rangle||\psi_\xi(\cdot)|\big]^q\Big)^{1/q}\Big\|_{L^p}. \tag{5.15}$$

*(b) The operators* $S_{\tilde{\psi}} : \dot{\tilde{F}}^s_{pq} \to \dot{\tilde{f}}^s_{pq}$ *and* $T_\psi : \dot{\tilde{f}}^s_{pq} \to \dot{\tilde{F}}^s_{pq}$ *are bounded and* $T_{\tilde{\psi}} \circ S_\psi = Id$ *on* $\dot{\tilde{F}}^s_{pq}$. *Hence,* $f \in \dot{\tilde{F}}^s_{pq}$ *if and only if* $\{\langle f, \tilde{\psi}_\xi\rangle\}_{\xi\in\mathcal{X}} \in \dot{\tilde{f}}^s_{pq}$, *and if* $f \in \dot{\tilde{F}}^s_{pq}$, *then* $\|f\|_{\dot{\tilde{F}}^s_{pq}} \sim \|\{\langle f, \tilde{\psi}_\xi\rangle\}\|_{\dot{\tilde{f}}^s_{pq}}$. *Furthermore,*

$$\|f\|_{\dot{\tilde{F}}^s_{pq}} \sim \Big\|\Big(\sum_{\xi\in\mathcal{X}} \big[|B(\xi, b^{-j})|^{-s/d}|\langle f, \tilde{\psi}_\xi\rangle||\psi_\xi(\cdot)|\big]^q\Big)^{1/q}\Big\|_{L^p}. \tag{5.16}$$

*As before the roles of* $\psi_\xi$ *and* $\tilde{\psi}_\xi$ *can be interchanged.*

The proofs of Theorems 5.6–5.7 are a straightforward adaptation of the proofs of Theorems 6.10 and 7.7 in [5], where one uses Theorem 3.9 as well. We omit the details.

## 6. Further results

In this section we present without proof some additional results on homogeneous Besov and Triebel–Lizorkin spaces in the setting of this article. Most of them have analogs for the respective inhomogeneous spaces with proofs that are straightforward adaptation of the ones in the inhomogeneous case.

### *6.1. Heat kernel characterization of homogeneous B- and F-spaces*

Homogeneous Besov and Triebel–Lizorkin spaces can be equivalently defined in terms of the semi-group $e^{-tL}$, $t > 0$, similarly as in the case of inhomogeneous spaces (see [5]).

**Definition 6.1.** Let $s \in \mathbb{R}$ and $m$ be the smallest positive integer grater than $s$.

(*i*) Let $1 \le p \le \infty$ and $0 < q \le \infty$ and $f \in \mathcal{S}'/\mathcal{P}$. We set

$$\|f\|_{\dot{B}^s_{pq}(H)} := \Big(\int_0^\infty \Big[t^{-s/2}\|(tL)^{m/2}e^{-tL}f\|_p\Big]^q \frac{dt}{t}\Big)^{1/q}, \tag{6.1}$$

$$\|f\|_{\dot{\tilde{B}}^s_{pq}(H)} := \Big(\int_0^\infty \||B(\cdot, t^{1/2})|^{-s/d}(tL)^{m/2}e^{-tL}f\|_p^q \frac{dt}{t}\Big)^{1/q}, \tag{6.2}$$

with the standard modification when $q = \infty$.

(*ii*) Let $1 < p < \infty$ and $1 < q \le \infty$ and $f \in \mathcal{S}'/\mathcal{P}$. We set

$$\|f\|_{\dot{F}^s_{pq}(H)} := \Big\|\Big(\int_0^\infty \big[t^{-s/2}|(tL)^{m/2}e^{-tL}f(\cdot)|\big]^q \frac{dt}{t}\Big)^{1/q}\Big\|_q, \tag{6.3}$$

$$\|f\|_{\dot{\tilde{F}}^s_{pq}(H)} := \Big\|\Big(\int_0^\infty \big[|B(\cdot, t^{1/2})|^{-s/d}|(tL)^{m/2}e^{-tL}f(\cdot)|\big]^q \frac{dt}{t}\Big)^{1/q}\Big\|_q, \tag{6.4}$$

with the standard modification when $q = \infty$.

The following characterization of homogeneous $B$- and $F$-spaces is valid:

**Theorem 6.2.** *Suppose* $s \in \mathbb{R}$ *and* $m > s$, *as above.*

(*a*) *If* $1 \le p \le \infty$ *and* $0 < q \le \infty$, *then* $\|\cdot\|_{\dot{B}^s_{pq}(H)}$ *and* $\|\cdot\|_{\dot{\tilde{B}}^s_{pq}(H)}$ *are equivalent (quasi-)norms on* $\dot{B}^s_{pq}$ *and* $\dot{\tilde{B}}^s_{pq}$ *respectively.*

(*b*) *If* $1 < p < \infty$ *and* $1 < q \le \infty$, *then* $\|\cdot\|_{\dot F^s_{pq}(H)}$ *and* $\|\cdot\|_{\dot{\tilde F}{}^s_{pq}(H)}$ *are equivalent (quasi-)norms on* $\dot F^s_{pq}$ *and* $\dot{\tilde F}{}^s_{pq}$ *respectively.*

See Theorems 6.7 and 7.5 in [5].

*6.2. Relationship between homogeneous and inhomogeneous B- and F-spaces*

The inhomogeneous Besov and Triebel–Lizorkin spaces $B^s_{pq}$ and $F^s_{pq}$ are introduced by Definitions 6.1 and 7.1 in [5].

Just as in the classical case (see [4]), the following identification is valid:

**Proposition 6.3.** (*i*) *Let* $s > 0$, $1 \le p \le \infty$ *and* $0 < q \le \infty$. *Then*

$$B^s_{pq} = L^p \cap \dot B^s_{pq}.$$

(*ii*) *Let* $s > 0$, $1 \le p < \infty$ *and* $0 < q \le \infty$. *Then*

$$F^s_{pq} = L^p \cap \dot F^s_{pq}.$$

*Furthermore, claims similar to (i)–(ii) above also hold for the nonclassical Besov and Triebel–Lizorkin spaces under the additional assumption* $\sup_{x\in M} |B(x,1)| < \infty$.

The proof of this proposition is straightforward.

*6.3. Potential spaces*

As in the classical case on $\mathbb{R}^d$ (see [4]), there is a natural identification of Potential spaces associated to the operator $L$ with respective Triebel–Lizorkin spaces.

**Definition 6.4.** Let $s \in \mathbb{R}$ and $1 \le p \le \infty$. The potential space $\dot L^p_s$ is defined as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot L^p_s} := \|L^{s/2} f\|_{L^p} < \infty. \tag{6.5}$$

**Theorem 6.5.** *The following identification holds:*

$$\dot L^p_s = \dot F^s_{p2}, \quad s \in \mathbb{R}, \quad 1 < p < \infty,$$

*with equivalent norms, and in particular*

$$L^p = \dot F^0_{p2}, \quad 1 < p < \infty.$$

See [5, Theorem 7.8].

## 7. Proofs

*7.1. Proof of Lemma 2.1*

To prove inequality (2.5) we set

$$M' := \{u \in M : \rho(x,u) \ge \rho(x,y)/2\} \text{ and } M'' := \{u \in M : \rho(y,u) \ge \rho(x,y)/2\}.$$

Evidently, $M \subset M' \cup M''$ and hence $I := \int_M \cdots \le \int_{M'} \cdots + \int_{M''} \cdots =: I' + I''$.

In estimating the first integral we use (2.2) and obtain

$$I' \le \frac{2^{\sigma_1}}{(1+\delta_1^{-1}\rho(x,y))^{\sigma_1}} \int_M \frac{d\mu(u)}{(1+\delta_2^{-1}\rho(y,u))^{\sigma_2}} \le \frac{c|B(y,\delta_2)|}{(1+\delta_1^{-1}\rho(x,y))^{\sigma_1}}.$$

Just in the same way we get

$$I'' \le \frac{c|B(x,\delta_1)|}{(1+\delta_2^{-1}\rho(x,y))^{\sigma_2}}$$

and inequality (2.5) follows.

With no loss of generality we may assume that $\delta_1 \le \delta_2$, implying $\delta_{\max} = \delta_2$. By (2.1) we have that $|B(x,\delta_2)| \le c_0(1+\delta_2^{-1}\rho(x,y))^d|B(y,\delta_2)|$. This coupled with (2.5) yields

$$\begin{aligned} I &\le \frac{c|B(x,\delta_2)|}{(1+\delta_2^{-1}\rho(x,y))^{\sigma_2}} + \frac{c|B(y,\delta_2)|}{(1+\delta_2^{-1}\rho(x,y))^{\sigma_1}} \\ &\le \frac{c|B(y,\delta_2)|}{(1+\delta_2^{-1}\rho(x,y))^{\sigma_2-d}} + \frac{c|B(y,\delta_2)|}{(1+\delta_2^{-1}\rho(x,y))^{\sigma_1}}, \end{aligned}$$

which implies the right-hand side inequality in (2.6).

To prove the left-hand side inequality in (2.6) we consider two cases:

*Case 1:* $\delta_2^{-1}\rho(x,y) \ge 1$. Using (1.1) and (1.2) we get

$$|B(y,\delta_2)| \le c_0(1+\delta_2^{-1}\rho(x,y))^d|B(x,\delta_2)| \le c_0(1+\delta_2^{-1}\rho(x,y))^d(\delta_2/\delta_1)^d|B(x,\delta_1)|$$

implying

$$\begin{aligned} \frac{|B(y,\delta_2)|}{(1+\delta_1^{-1}\rho(x,y))^{\sigma_1}} &\le \frac{c(1+\delta_2^{-1}\rho(x,y))^d(\delta_2/\delta_1)^d|B(x,\delta_1)|}{(\delta_1^{-1}\rho(x,y))^{\sigma_1}} \\ &= \frac{c(1+\delta_2^{-1}\rho(x,y))^d|B(x,\delta_1)|}{(\delta_2^{-1}\rho(x,y))^d(\delta_1^{-1}\rho(x,y))^{\sigma_1-d}} \le \frac{c2^d|B(x,\delta_1)|}{(1+\delta_1^{-1}\rho(x,y))^{\sigma_1-d}}. \end{aligned}$$

This along with (2.5) yield the left-hand side inequality in (2.6).

*Case 2:* $\delta_2^{-1}\rho(x,y) < 1$. Then using (2.2)

$$I \le \int_M \frac{d\mu(u)}{(1+\delta_1^{-1}\rho(x,u))^{\sigma_1}} \le c|B(x,\delta_1)| \le \frac{c2^{\sigma_1}|B(x,\delta_1)|}{(1+\delta_2^{-1}\rho(x,y))^{\sigma_2}},$$

which implies the left-hand side inequality in (2.6). □

### *7.2. Proof of Proposition 3.2 and Proposition 3.6*

Since these two propositions are quite similar will only prove Proposition 3.6.

Let $\varphi \in \mathcal{S}(\mathbb{R})$ be real-valued and even and $\varphi^{(\nu)}(0) = 0$ for $\nu = 0, 1, \dots$. Assume $f \in \mathcal{S}'_\infty$. Then there exist constants $m \in \mathbb{Z}_+$ and $c > 0$ such that (3.10) holds. Let $\phi \in \mathcal{S}_\infty$. We have

$$\varphi(\sqrt{L})\phi(x) = \int_M \varphi(\sqrt{L})(x,y)\phi(y)d\mu(y), \quad x \in M.$$

To prove (3.15) we will interpret the above integral as a Bochner integral over the Banach space

$$V_m := \{g \in \cap_{-m\le\nu\le m} D(L^\nu) : \|g\|_{V_m} := \mathcal{P}^\star_m(g) < \infty\}, \quad m > d/2,$$

with $\mathcal{P}^\star_m$ defined in (3.9), see e.g. [12], pp. 131-133. By Proposition 3.3 it readily follows that $V_m$ is well defined. The completeness of the space $V_m$ follows (just as in the proof of Proposition 3.4) by the fact that $L$ being a self-adjoint operator is also closed. By the Hahn–Banach theorem the continuous linear functional $f$ can be extended to $V_m$ with the same norm.

Denote $F(y) := \varphi(\sqrt{L})(\cdot, y)\phi(y)$. We have

$$\|F(y)\|_{V_m} = \max_{-m\le\nu\le m} \sup_{x\in M} (1+\rho(x,x_0))^m |[L^\nu\varphi(\sqrt{L})](x,y)\phi(y)|.$$

Set $f(\lambda) := \lambda^{2\nu}\varphi(\lambda)$, $-m \le \nu \le m$. From the properties of $\varphi$ it follows that $f \in \mathcal{S}(\mathbb{R})$ and $f$ is even. Then appealing to Theorem 2.2 we conclude that $L^\nu\varphi(\sqrt{L})$ is an integral operator with a kernel satisfying the following inequality for any $\sigma > 0$

$$|[L^\nu\varphi(\sqrt{L})](x,y)| \le c_\sigma |B(y,1)|^{-1}(1+\rho(x,y))^{-\sigma}, \quad -m \le \nu \le m.$$

We choose $\sigma = m$.

On the other hand, as $\phi \in \mathcal{S}_\infty$ in light of (3.9) we have $|\phi(y)| \le \mathcal{P}^\star_\ell(\phi)(1+\rho(y,x_0))^{-\ell}$ for any $\ell \ge 0$. We choose $\ell \ge m + 2d + 1$. Putting these estimates together we get

$$\|F(y)\|_{V_m} \le c \max_{-m\le\nu\le m} \sup_{x\in M} \frac{\mathcal{P}^\star_\ell(\phi)(1+\rho(x,x_0))^m}{|B(y,1)|(1+\rho(x,y))^m(1+\rho(y,x_0))^{m+2d+1}}$$

and using the obvious inequality $1+\rho(x,x_0) \le (1+\rho(x,y))(1+\rho(y,x_0))$ we obtain

$$\begin{aligned}\|F(y)\|_{V_m} &\le c\mathcal{P}^\star_\ell(\phi)|B(y,1)|^{-1}(1+\rho(y,x_0))^{-2d-1}\\ &\le c\mathcal{P}^\star_\ell(\phi)|B(x_0,1)|^{-1}(1+\rho(y,x_0))^{-d-1},\end{aligned}$$

where for the last inequality we used (2.1). From the above and (2.2) it follows that $\int_M \|F(y)\|_{V_m} d\mu(y) \le c\mathcal{P}^\star_\ell(\phi) < \infty$. Now, applying the theory of Bochner's integral we obtain

$$\Big\langle f, \int_M \varphi(\sqrt{L})(\cdot,y)\phi(y)d\mu(y)\Big\rangle = \int_M \langle f, \varphi(\sqrt{L})(\cdot,y)\rangle\overline{\phi(y)}d\mu(y).$$

This coupled with (3.13) implies (3.15).

We next prove (3.17); the proof of (3.16) is simpler and will be omitted. By the fact that (3.10) holds for the given $f$ for some constants $m \in \mathbb{Z}_+$ and $c > 0$ and using (3.15) we obtain, for $x, x' \in M$,

$$\begin{aligned}|\varphi(\sqrt{L})f(x) - \varphi(\sqrt{L})f(x')| &= |\langle f, \varphi(\sqrt{L})(x,\cdot) - \varphi(\sqrt{L})(x',\cdot)\rangle|\\ &\le c\mathcal{P}^\star_m\big(\varphi(\sqrt{L})(x,\cdot) - \varphi(\sqrt{L})(x',\cdot)\big)\\ &\le c \max_{-m\le\nu\le m}\sup_{y\in M}(1+\rho(y,x_0))^m|[L^\nu\varphi(\sqrt{L})](x,y) - [L^\nu\varphi(\sqrt{L})](x',y)|.\end{aligned} \tag{7.1}$$

As above by Theorem 2.2, applied with $f(\lambda) = \lambda^{2\nu}\varphi(\lambda)$, it follows that for any $\sigma > 0$ and $-m \le \nu \le m$

$$|[L^\nu\varphi(\sqrt{L})](x,y) - [L^\nu\varphi(\sqrt{L})](x',y)| \le c_\sigma|B(x,1)|^{-1}\rho(x,x')^\alpha(1+\rho(x,y))^{-\sigma}$$

provided $\rho(x,x') \le 1$. We choose $\sigma = m$. We insert the above in (7.1) and arrive at (3.17). □

*7.3. Proof of Proposition 3.8*

This proof hinges on the following

**Lemma 7.1.** *Let $\sigma > 0$ and $N \ge \sigma + d + \alpha/2$ with $\alpha > 0$ from (1.4). Then there exists a constant $c > 0$ such that for any $\phi \in \mathcal{S}$ and $x, y \in M$*

$$|\phi(x) - \phi(y)| \le c\rho(x,y)^\alpha \mathcal{P}_N(\phi)\big[(1+\rho(x,x_0))^{-\sigma} + (1+\rho(y,x_0))^{-\sigma}\big]. \tag{7.2}$$

*Here $\mathcal{P}_N(\phi)$ is from (3.1).*

**Proof.** Choose $\varphi_0 \in C^\infty(\mathbb{R}_+)$ so that $0 \le \varphi_0 \le 1$, $\varphi_0(\lambda) = 1$ for $\lambda \in [0,1]$, and $\operatorname{supp}\varphi_0 \subset [0,2]$. Let $\varphi(\lambda) := \varphi_0(\lambda) - \varphi_0(2\lambda)$ and set $\varphi_j(\lambda) := \varphi(2^{-j}\lambda)$, $j \ge 1$. Clearly, $\sum_{j\ge 0}\varphi_j(\lambda) = 1$ for $\lambda \in \mathbb{R}_+$ and hence $\phi = \sum_{j=0}^\infty \varphi_j(\sqrt{L})\phi$ for $\phi \in \mathcal{S}$ with the convergence in $L^\infty$ (see [5, Proposition 5.5]). Therefore,

$$\phi(x) - \phi(y) = \sum_{j=0}^\infty \big(\varphi_j(\sqrt{L})\phi(x) - \varphi_j(\sqrt{L})\phi(y)\big), \quad \forall x, y \in M,\ \forall \phi \in \mathcal{S}.$$

For $j \ge 1$ we have

$$\begin{aligned}\varphi_j(\sqrt{L})\phi(x) - \varphi_j(\sqrt{L})\phi(y) &= L^{-N}\varphi_j(\sqrt{L})L^N\phi(x) - L^{-N}\varphi_j(\sqrt{L})L^N\phi(y)\\ &= \int_M \big([L^{-N}\varphi(2^{-j}\sqrt{L})](x,z) - [L^{-N}\varphi(2^{-j}\sqrt{L})](y,z)\big)L^N\phi(z)d\mu(z).\end{aligned} \tag{7.3}$$

Let $\omega(\lambda) := \lambda^{-2N}\varphi(\lambda)$. Then $L^{-N}\varphi(2^{-j}\sqrt{L}) = 2^{-2jN}\omega(2^{-j}\sqrt{L})$. Clearly, $\omega \in C^\infty$ and $\operatorname{supp}\omega \subset [2^{-1}, 2]$. Hence by Theorem 2.2 it follows that there exists a constant $c_\sigma > 0$ such that

$$\big|[L^{-N}\varphi(2^{-j}\sqrt{L})](x,z)\big| \le \frac{c_\sigma 2^{-2jN}}{|B(x,2^{-j})|\big(1+2^j\rho(x,z)\big)^{\sigma+d}} \quad \text{and} \tag{7.4}$$

$$\big|[L^{-N}\varphi(2^{-j}\sqrt{L})](x,z) - [L^{-N}\varphi(2^{-j}\sqrt{L})](y,z)\big| \le \frac{c_\sigma 2^{-2jN}\big(2^j\rho(x,y)\big)^\alpha}{|B(x,2^{-j})|\big(1+2^j\rho(x,z)\big)^{\sigma+d}}, \tag{7.5}$$

whenever $\rho(x,y) \le 2^{-j}$.

Fix $\phi \in \mathcal{S}$. Then by (3.1) $|L^N\phi(z)| \le \mathcal{P}_N(\phi)(1+\rho(z,x_0))^{-N}$, $z \in M$.

Let $\rho(x,y) \le 2^{-j}$. The above, (7.3), and (7.5) yield

$$\begin{aligned}&|\varphi_j(\sqrt{L})\phi(x) - \varphi_j(\sqrt{L})\phi(y)|\\ &\quad\le c2^{-j(2N-\alpha)}\rho(x,y)^\alpha\mathcal{P}_N(\phi)\int_M \frac{d\mu(z)}{|B(x,2^{-j})|(1+2^j\rho(x,z))^{\sigma+d}(1+\rho(z,x_0))^N}\\ &\quad\le c2^{-j(2N-d-\alpha)}\rho(x,y)^\alpha\mathcal{P}_N(\phi)\int_M \frac{d\mu(z)}{|B(x,1)|(1+\rho(x,z))^{\sigma+d}(1+\rho(z,x_0))^{\sigma+d}}\\ &\quad\le \frac{c2^{-j(2N-d-\alpha)}\rho(x,y)^\alpha\mathcal{P}_N(\phi)}{\big(1+\rho(x,x_0)\big)^\sigma}.\end{aligned}$$

Here we used that $|B(x,1)| \le c_0 2^{jd}|B(x,2^{-j})|$, see (1.2), $N \ge \sigma + d$, and (2.3).

Let $\rho(x,y) > 2^{-j}$. Using (7.4) and some of the ingredients from above we get

$$\Big|\int_M [L^{-N}\varphi(2^{-j}\sqrt{L})](x,z)L^N\phi(z)d\mu(z)\Big|$$
$$\le \int_M \frac{c2^{-2jN}\mathcal{P}_N(\phi)d\mu(z)}{|B(x,2^{-j})|\big(1+2^j\rho(x,z)\big)^{\sigma+d}(1+\rho(z,x_0))^N}$$
$$\le c2^{-j(2N-d-\alpha)}\rho(x,y)^\alpha\mathcal{P}_N(\phi)\int_M \frac{d\mu(z)}{|B(x,1)|(1+\rho(x,z))^{\sigma+d}(1+\rho(z,x_0))^{\sigma+d}}$$
$$\le \frac{c2^{-j(2N-d-\alpha)}\rho(x,y)^\alpha\mathcal{P}_N(\phi)}{\big(1+\rho(x,x_0)\big)^\sigma}.$$

Similarly

$$\Big|\int_M [L^{-N}\varphi(2^{-j}\sqrt{L})](y,z)L^N\phi(z)d\mu(z)\Big| \le \frac{c2^{-j(2N-d-\alpha)}\rho(x,y)^\alpha\mathcal{P}_N(\phi)}{\big(1+\rho(y,x_0)\big)^\sigma}.$$

Putting the above estimates together we get for all $x,y\in M$ and $j\ge 1$

$$|\varphi_j(\sqrt{L})\phi(x)-\varphi_j(\sqrt{L})\phi(y)| \le c2^{-j(2N-d-\alpha)}\rho(x,y)^\alpha\mathcal{P}_N(\phi)\big[\big(1+\rho(x,x_0)\big)^{-\sigma}+\big(1+\rho(y,x_0)\big)^{-\sigma}\big]. \tag{7.6}$$

In the same way, we use that (7.4)–(7.5) hold for $\varphi_0(\sqrt{L})$ with $N=0$ to obtain

$$|\varphi_0(\sqrt{L})\phi(x)-\varphi_0(\sqrt{L})\phi(y)| \le c\rho(x,y)^\alpha\mathcal{P}_N(\phi)\big[\big(1+\rho(x,x_0)\big)^{-\sigma}+\big(1+\rho(y,x_0)\big)^{-\sigma}\big].$$

Summing up this estimate along with the estimates from (7.6) $(2N>d+\alpha)$ we arrive at (7.2). □

We now proceed with the proof of Proposition 3.8. Let $\varphi\in\mathcal{S}(\mathbb{R})$, $\varphi$ be real-valued and even, and $\varphi(0)=1$. It suffices to prove (3.20) only. Then (3.21) follows by duality. To prove (3.20) it suffices to show that for any $m\ge 0$ and $|\nu|\le m$

$$\lim_{\delta\to 0}\sup_{x\in M}(1+\rho(x,x_0))^m|L^\nu[\phi-\varphi(\delta\sqrt{L})\phi](x)| = 0, \quad \forall\phi\in\mathcal{S}_\infty. \tag{7.7}$$

Let $m\ge 0$, $|\nu|\le m$, and $\phi\in\mathcal{S}_\infty$. Choose $\sigma > m+d+\alpha$ and $N\ge\sigma+d+\alpha/2$, where $\alpha>0$ is from (1.4). By Theorem 2.2

$$|\varphi(\delta\sqrt{L})(x,y)| \le c_\sigma|B(x,\delta)|^{-1}(1+\delta^{-1}\rho(x,y))^{-\sigma}$$

and $\int_M \varphi(\delta\sqrt{L})(x,y)d\mu(y) = \varphi(0)=1$. Therefore,

$$(1+\rho(x,x_0))^m|L^\nu[\phi-\varphi(\delta\sqrt{L})\phi](x)| = (1+\rho(x,x_0))^m\Big|\int_M \varphi(\delta\sqrt{L})(x,y)[L^\nu\phi(x)-L^\nu\phi(y)]d\mu(y)\Big|$$

$$\le c_\sigma(1+\rho(x,x_0))^m \int\limits_M \frac{|L^\nu\phi(x)-L^\nu\phi(y)|}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^\sigma}d\mu(y)$$
$$= c_\sigma(1+\rho(x,x_0))^m \Big(\int\limits_{B(x,1)} \cdots + \int\limits_{M\setminus B(x,1)} \cdots\Big).$$

Here we used (3.11) in the case when $\nu<1$. As $\phi\in\mathcal{S}_\infty$, then $L^\nu\phi\in\mathcal{S}$ and applying Lemma 7.1 we obtain

$$(1+\rho(x,x_0))^m \int\limits_{B(x,1)} \frac{|L^\nu\phi(x)-L^\nu\phi(y)|}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^\sigma}d\mu(y)$$
$$\le c(1+\rho(x,x_0))^m \int\limits_{B(x,1)} \frac{\rho(x,y)^\alpha\mathcal{P}^\star_{m+N}(\phi)}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^\sigma(1+\rho(x,x_0))^\sigma}d\mu(y)$$
$$+ c(1+\rho(x,x_0))^m \int\limits_{B(x,1)} \frac{\rho(x,y)^\alpha\mathcal{P}^\star_{m+N}(\phi)}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^\sigma(1+\rho(y,x_0))^\sigma}d\mu(y)$$
$$=: I_1+I_2.$$

Here we used that $\mathcal{P}_N(L^\nu\phi)\le\mathcal{P}^\star_{m+N}(\phi)$ due to $|\nu|\le m$, see (3.1) and (3.9). Now, we use that $\sigma\ge m$, $\sigma-\alpha>d$, and (2.2) to obtain

$$I_1 \le c\mathcal{P}^\star_{m+N}(\phi) \int\limits_{B(x,1)} \frac{\rho(x,y)^\alpha}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^\sigma}d\mu(y)$$
$$\le c\mathcal{P}^\star_{m+N}(\phi) \int\limits_M \frac{\delta^\alpha}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^{\sigma-\alpha}}d\mu(y) \le c\delta^\alpha\mathcal{P}^\star_{m+N}(\phi).$$

Evidently, $1+\rho(x,x_0)\le(1+\rho(x,y))(1+\rho(y,x_0))$ and assuming $\delta\le 1$ we obtain

$$I_2 \le c\mathcal{P}^\star_{m+N}(\phi) \int\limits_{B(x,1)} \frac{\rho(x,y)^\alpha}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^{\sigma-m}}d\mu(y)$$
$$\le c\mathcal{P}^\star_{m+N}(\phi) \int\limits_M \frac{\delta^\alpha}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^{\sigma-m-\alpha}}d\mu(y) \le c\delta^\alpha\mathcal{P}^\star_{m+N}(\phi).$$

Here we also used that $\sigma>m+d+\alpha$ and (2.2). Therefore, for any $x\in M$

$$(1+\rho(x,x_0))^m \int\limits_{B(x,1)} \frac{|L^\nu\phi(x)-L^\nu\phi(y)|}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^\sigma}d\mu(y) \le c\delta^\alpha\mathcal{P}^\star_{m+N}(\phi). \tag{7.8}$$

Since $\phi\in\mathcal{S}_\infty$ we have by (3.9) $|L^\nu\phi(z)|\le\mathcal{P}^\star_{m+N}(\phi)(1+\rho(z,x_0))^{-N}$, $\forall z\in M$. This leads to

$$(1+\rho(x,x_0))^m \int\limits_{M\setminus B(x,1)} \frac{|L^\nu\phi(x)-L^\nu\phi(y)|}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^\sigma}d\mu(y)$$
$$\le c\mathcal{P}^\star_{m+N}(\phi) \int\limits_{M\setminus B(x,1)} \frac{(1+\rho(x,x_0))^m}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^\sigma(1+\rho(x,x_0))^N}d\mu(y)$$

$$+ c\mathcal{P}^\star_{m+N}(\phi) \int\limits_{M\setminus B(x,1)} \frac{(1+\rho(x,x_0))^m}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^\sigma(1+\rho(y,x_0))^N} d\mu(y)$$

$$= J_1 + J_2.$$

Using that $N > \sigma > m$, $\sigma > d + \alpha$, (2.2), and $\rho(x, y) \geq 1$ for $y \in M \setminus B(x, 1)$, we get

$$J_1 \leq c\mathcal{P}^\star_{m+N}(\phi) \int\limits_{M\setminus B(x,1)} \frac{d\mu(y)}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^\sigma}$$

$$\leq c\mathcal{P}^\star_{m+N}(\phi) \int\limits_M \frac{\delta^\alpha d\mu(y)}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^{\sigma-\alpha}} \leq c\delta^\alpha \mathcal{P}^\star_{m+N}(\phi).$$

To estimate $J_2$ we use again that $1 + \rho(x, x_0) \leq (1 + \rho(x, y))(1 + \rho(y, x_0))$ and assuming $\delta \leq 1$ we obtain

$$J_2 \leq c\mathcal{P}^\star_{m+N}(\phi) \int\limits_{M\setminus B(x,1)} \frac{d\mu(y)}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^{\sigma-m}}$$

$$\leq c\mathcal{P}^\star_{m+N}(\phi) \int\limits_M \frac{\delta^\alpha d\mu(y)}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^{\sigma-m-\alpha}} \leq c\delta^\alpha \mathcal{P}^\star_{m+N}(\phi).$$

Consequently,

$$(1+\rho(x,x_0))^m \int\limits_{M\setminus B(x,1)} \frac{|L^\nu\phi(x) - L^\nu\phi(y)|}{|B(x,\delta)|(1+\delta^{-1}\rho(x,y))^\sigma} d\mu(y) \leq c\delta^\alpha \mathcal{P}^\star_{m+N}(\phi).$$

This coupled with (7.8) leads to

$$\sup_{x\in M} (1+\rho(x,x_0))^m |L^\nu[\phi - \varphi(\delta\sqrt{L})\phi](x)| \leq c\delta^\alpha \mathcal{P}^\star_{m+N}(\phi),$$

implying (7.7), which in turn yields (3.20). □

### 7.4. *Proof of Theorem 5.3*

We shall only prove the continuous embedding of $\dot{B}^s_{pq}$ in $\mathcal{S}'/\mathcal{P}$ as stated in (5.7). The proof of the embedding of $\dot{\tilde{B}}^s_{pq}$, $\dot{F}^s_{pq}$, or $\dot{\tilde{F}}^s_{pq}$ in $\mathcal{S}'/\mathcal{P}$ is similar. We shall proceed similarly as in the proof of Proposition 6.5 in [5].

Let $f \in \mathcal{S}'/\mathcal{P}$ and $\phi \in \mathcal{S}_\infty$. Choose a real-valued function $\varphi \in C_0^\infty(\mathbb{R}_+)$ so that $\operatorname{supp}\varphi \subset [2^{-1}, 2]$ and $\sum_{j\in\mathbb{Z}} \varphi^2(2^{-j}\lambda) = 1$ for $\lambda \in \mathbb{R}_+$. Set $\varphi_j(\lambda) := \varphi(2^{-j}\lambda)$, $j \in \mathbb{Z}$. Then $\sum_{j\in\mathbb{Z}} \varphi_j^2(\lambda) = 1$ for $\lambda \in \mathbb{R}_+$ and hence, using Theorem 3.9,

$$f = \sum_{j\in\mathbb{Z}} \varphi_j^2(\sqrt{L})f \quad \text{in } \mathcal{S}'/\mathcal{P}. \tag{7.9}$$

Also, observe that $\{\varphi_j\}_{j\in\mathbb{Z}}$ are just like the functions in the definition of $\dot{B}^s_{pq}$ (see Definition 5.2) and can be used to define an equivalent norm on $\dot{B}^s_{pq}$ as in (5.2). From (7.9) we get

$$\langle f, \phi\rangle = \sum_{j\in\mathbb{Z}} \langle \varphi_j^2(\sqrt{L})f, \phi\rangle = \sum_{j\in\mathbb{Z}} \langle \varphi_j(\sqrt{L})f, \varphi_j(\sqrt{L})\phi\rangle. \tag{7.10}$$

We next estimate $|\langle\varphi_j(\sqrt{L})f, \varphi_j(\sqrt{L})\phi\rangle|$ for $j \in \mathbb{Z}$. We consider two cases.

*Case 1:* $j \ge 0$. Choose $m > |s| + 3d + d/p$. We first estimate $|\varphi_j(\sqrt{L})\phi(x)|$. Set $\omega(\lambda) := \lambda^{-2m}\varphi(\lambda)$. Then $\varphi_j(\sqrt{L}) = 2^{-2mj}\omega(2^{-j}\sqrt{L})L^m$ and hence

$$\varphi_j(\sqrt{L})\phi(x) = 2^{-2mj}\int_M \omega(2^{-j}\sqrt{L})(x,y)L^m\phi(y)d\mu(y).$$

Clearly, $\omega \in C_0^\infty(\mathbb{R}_+)$ and $\operatorname{supp}\omega \subset [1/2, 2]$. Therefore, by Theorem 2.2

$$|\omega(2^{-j}\sqrt{L})(x,y)| \le c|B(y,2^{-j})|^{-1}(1+2^j\rho(x,y))^{-m}.$$

On the other hand, since $\phi \in \mathcal{S}_\infty$ we have by (3.9)

$$|L^m\phi(y)| \le c(1+\rho(y,x_0))^{-m}\mathcal{P}_m^\star(\phi).$$

Putting the above together we obtain

$$|\varphi_j(\sqrt{L})\phi(x)| \le c2^{-2mj}\mathcal{P}_m^\star(\phi)\int_M \frac{d\mu(y)}{|B(y,2^{-j})|(1+2^j\rho(x,y))^m(1+\rho(y,x_0))^m}.$$

By (1.2) and (2.1) it readily follows that

$$|B(x_0,1)| \le c_0(1+\rho(y,x_0))^d|B(y,1)| \le c_0^2 2^{jd}(1+\rho(y,x_0))^d|B(y,2^{-j})|. \tag{7.11}$$

Therefore,

$$\begin{aligned}|\varphi_j(\sqrt{L})\phi(x)| &\le c2^{-j(2m-d)}\mathcal{P}_m^\star(\phi)\int_M \frac{d\mu(y)}{|B(x_0,1)|(1+\rho(x,y))^{m-d}(1+\rho(y,x_0))^{m-d}}\\ &\le c2^{-j(2m-d)}\mathcal{P}_m^\star(\phi)(1+\rho(x,x_0))^{-m+2d}, \quad j \ge 0.\end{aligned} \tag{7.12}$$

Here for the last inequality we used (2.3) and that $m > 2d$.

We are now prepared to estimate the inner products in (7.10). We consider two subcases:

*Case 1* (a): $1 < p \le \infty$. Then applying Hölder's inequality ($1/p + 1/p' = 1$) we get

$$\begin{aligned}|\langle\varphi_j(\sqrt{L})f, \varphi_j(\sqrt{L})\phi\rangle| &\le \int_M |\varphi_j(\sqrt{L})f(x)||\varphi_j(\sqrt{L})\phi(x)|d\mu(x)\\ &\le \|\varphi_j(\sqrt{L})f\|_p\|\varphi_j(\sqrt{L})\phi\|_{p'} \le c2^{-js}\|f\|_{\dot{B}^s_{pq}}\|\varphi_j(\sqrt{L})\phi\|_{p'}.\end{aligned}$$

Here we used that $\|\varphi_j(\sqrt{L})f\|_p \le 2^{-js}\|f\|_{\dot{B}^s_{pq}}$, which follows from (5.2). On the other hand, from (7.12) it follows that

$$\begin{aligned}\|\varphi_j(\sqrt{L})\phi\|_{p'} &\le c2^{-(2m-d)j}\mathcal{P}_m^\star(\phi)\Big(\int_M \frac{d\mu(x)}{(1+\rho(x,x_0))^{(m-2d)p'}}\Big)^{1/p'}\\ &\le c'|B(x_0,1)|^{1/p'}2^{-j(2m-d)}\mathcal{P}_m^\star(\phi),\end{aligned}$$

where we used that $(m-2d)p' > d$ and (2.2). Hence,

$$|\langle\varphi_j(\sqrt{L})f, \varphi_j(\sqrt{L})\phi\rangle| \le c2^{-j(2m-d+s)}|B(x_0,1)|^{1-1/p}\|f\|_{\dot{B}^s_{pq}}\mathcal{P}_m^\star(\phi), \quad j \ge 0.$$

Summing up these estimates we get

$$\sum_{j\ge 0}\big|\big\langle \varphi_j(\sqrt{L})f,\varphi_j(\sqrt{L})\phi\big\rangle\big| \le c|B(x_0,1)|^{1-1/p}\|f\|_{\dot{B}^s_{pq}}\mathcal{P}^\star_m(\phi), \tag{7.13}$$

where we used that $2m > d-s$.

*Case 1* (b): $0<p\le 1$. Setting $\gamma := 1-1/p$ we have for $j\ge 0$

$$\big|\big\langle \varphi_j(\sqrt{L})f,\varphi_j(\sqrt{L})\phi\big\rangle\big| \le \||B(\cdot,2^{-j})|^{-\gamma}\varphi_j(\sqrt{L})f\|_1\||B(\cdot,2^{-j})|^{\gamma}\varphi_j(\sqrt{L})\phi\|_\infty.$$

As $\varphi_j(\sqrt{L})f\in\Sigma_{2^{j+1}}$, Proposition 2.8 yields

$$\begin{aligned}\||B(\cdot,2^{-j})|^{-\gamma}\varphi_j(\sqrt{L})f\|_1 &\le c\||B(\cdot,2^{-j})|^{-\gamma+1-1/p}\varphi_j(\sqrt{L})f\|_p \\ &= c\|\varphi_j(\sqrt{L})f\|_p \le c2^{-js}\|f\|_{\dot{B}^s_{pq}}.\end{aligned} \tag{7.14}$$

On the other hand, by (7.12)

$$\||B(\cdot,2^{-j})|^{\gamma}\varphi_j(\sqrt{L})\phi\|_\infty \le c2^{-j(2m-d)}\mathcal{P}^\star_m(\phi)\sup_{x\in M}\frac{|B(x,2^{-j})|^{1-1/p}}{(1+\rho(x,x_0))^{m-2d}}$$

and from (7.11) $|B(x_0,1)|\le c_0^2 2^{jd}(1+\rho(x,x_0))^d|B(x,2^{-j})|$, implying

$$\||B(\cdot,2^{-j})|^{\gamma}\varphi_j(\sqrt{L})\phi\|_\infty \le c|B(x_0,1)|^{1-1/p}2^{-j(2m-2d+d/p)}\mathcal{P}^\star_m(\phi),$$

where we used that $m-2d\ge d(1/p-1)$. Therefore,

$$\big|\big\langle \varphi_j(\sqrt{L})f,\varphi_j(\sqrt{L})\phi\big\rangle\big| \le c2^{-j(2m+s-2d+d/p)}|B(x_0,1)|^{1-1/p}\|f\|_{\dot{B}^s_{pq}}\mathcal{P}^\star_m(\phi),\ \ j\ge 0.$$

Summing up these estimates we get

$$\sum_{j\ge 0}\big|\big\langle \varphi_j(\sqrt{L})f,\varphi_j(\sqrt{L})\phi\big\rangle\big| \le c|B(x_0,1)|^{1-1/p}\|f\|_{\dot{B}^s_{pq}}\mathcal{P}^\star_m(\phi), \tag{7.15}$$

where we used that $2m > -s+2d-d/p$.

*Case 2:* $j<0$. Choose $m>|s|+3d+d/p$. Set $\omega(\lambda) := \lambda^{2m}\varphi(\lambda)$. Then $\varphi_j(\sqrt{L}) = 2^{2mj}L^{-m}\omega(2^{-j}\sqrt{L})$ and using (3.11)

$$\varphi_j(\sqrt{L})\phi(x) = 2^{2mj}\omega(2^{-j}\sqrt{L})L^{-m}\phi(x) = 2^{2mj}\int_M \omega(2^{-j}\sqrt{L})(x,y)L^{-m}\phi(y)d\mu(y).$$

Clearly, $\omega\in C_0^\infty(\mathbb{R}_+)$ and $\operatorname{supp}\omega\subset[1/2,2]$. Therefore, by Theorem 2.2

$$|\omega(2^{-j}\sqrt{L})(x,y)| \le c|B(y,2^{-j})|^{-1}(1+2^j\rho(x,y))^{-m}.$$

On the other hand, since $\phi\in\mathcal{S}_\infty$ we have

$$|L^{-m}\phi(x)| \le c(1+\rho(x,x_0))^{-m}\mathcal{P}^\star_m(\phi).$$

From the above we obtain

$$|\varphi_j(\sqrt{L})\phi(x)| \le c2^{2mj}\mathcal{P}^\star_m(\phi)\int\limits_M \frac{d\mu(y)}{|B(y,2^{-j})|(1+2^j\rho(x,y))^m(1+\rho(y,x_0))^m}.$$

By (2.1) $|B(x,2^{-j})| \le c_0(1+2^j\rho(x,y))^d|B(y,2^{-j})|$ and hence

$$\begin{aligned}|\varphi_j(\sqrt{L})\phi(x)| &\le c2^{2mj}\mathcal{P}^\star_m(\phi)\int\limits_M \frac{d\mu(y)}{|B(x,2^{-j})|(1+2^j\rho(x,y))^{m-d}(1+2^j\rho(y,x_0))^{m-d}}\\ &\le c2^{2mj}\mathcal{P}^\star_m(\phi)(1+2^j\rho(x,x_0))^{-m+2d}\\ &\le c2^{j(m+2d)}\mathcal{P}^\star_m(\phi)(1+\rho(x,x_0))^{-m+2d}, \quad j<0.\end{aligned} \tag{7.16}$$

Here for the former inequality we used (2.3) and that $m > 2d$.

To estimate the inner products in (7.10) we consider as before two subcases:

*Case 2* (a): $1 < p \le \infty$. Then applying Hölder's inequality ($1/p + 1/p' = 1$) we get

$$\begin{aligned}\big|\langle\varphi_j(\sqrt{L})f,\varphi_j(\sqrt{L})\phi\rangle\big| &\le \int\limits_M |\varphi_j(\sqrt{L})f(x)||\varphi_j(\sqrt{L})\phi(x)|d\mu(x)\\ &\le \|\varphi_j(\sqrt{L})f\|_p\|\varphi_j(\sqrt{L})\phi\|_{p'} \le c2^{-js}\|f\|_{\dot B^s_{pq}}\|\varphi_j(\sqrt{L})\phi\|_{p'}.\end{aligned}$$

Using (7.16) we obtain

$$\begin{aligned}\|\varphi_j(\sqrt{L})\phi\|_{p'} &\le c2^{j(m+d)}\mathcal{P}^\star_m(\phi)\Big(\int\limits_M \frac{d\mu(x)}{(1+\rho(x,x_0))^{(m-2d)p'}}\Big)^{1/p'}\\ &\le c'|B(x_0,1)|^{1/p'}2^{j(m+d)}\mathcal{P}^\star_m(\phi),\end{aligned}$$

where we used that $(m-2d)p' > d$ and (2.2). Hence,

$$\big|\langle\varphi_j(\sqrt{L})f,\varphi_j(\sqrt{L})\phi\rangle\big| \le c2^{j(m+d-s)}|B(x_0,1)|^{1-1/p}\|f\|_{\dot B^s_{pq}}\mathcal{P}^\star_m(\phi), \quad j<0.$$

Summing up these estimates we get

$$\sum_{j<0}\big|\langle\varphi_j(\sqrt{L})f,\varphi_j(\sqrt{L})\phi\rangle\big| \le c|B(x_0,1)|^{1/p-1}\|f\|_{\dot B^s_{pq}}\mathcal{P}^\star_m(\phi), \tag{7.17}$$

where we used that $m > s-d$.

*Case 2* (b): $0 < p \le 1$. Setting $\gamma := 1-1/p$ we have for $j \ge 0$

$$\big|\langle\varphi_j(\sqrt{L})f,\varphi_j(\sqrt{L})\phi\rangle\big| \le \||B(\cdot,2^{-j})|^{-\gamma}\varphi_j(\sqrt{L})f\|_1\||B(\cdot,2^{-j})|^{\gamma}\varphi_j(\sqrt{L})\phi\|_\infty.$$

As $\varphi_j(\sqrt{L})f \in \Sigma_{2^{j+1}}$, Proposition 2.8 yields

$$\begin{aligned}\||B(\cdot,2^{-j})|^{-\gamma}\varphi_j(\sqrt{L})f\|_1 &\le c\||B(\cdot,2^{-j})|^{-\gamma+1-1/p}\varphi_j(\sqrt{L})f\|_p\\ &= c\|\varphi_j(\sqrt{L})f\|_p \le c2^{-js}\|f\|_{\dot B^s_{pq}}.\end{aligned}$$

On the other hand by (7.16)

$$\||B(\cdot,2^{-j})|^{\gamma}\varphi_j(\sqrt{L})\phi\|_\infty \le c2^{j(m+2d)}\mathcal{P}^\star_m(\phi)\sup_{x\in M}\frac{|B(x,2^{-j})|^{1-1/p}}{(1+\rho(x,x_0))^{m-2d}}.$$

By (2.1) and the fact that $j < 0$ we get

$$|B(x_0, 1)| \le c_0(1 + \rho(x, x_0))^d |B(x, 1)| \le c_0(1 + \rho(x, x_0))^d |B(x, 2^{-j})|.$$

Hence,

$$\||B(\cdot, 2^{-j})|^\gamma \varphi_j(\sqrt{L})\phi\|_\infty \le c|B(x_0, 1)|^{1-1/p} 2^{j(m+2d)} \mathcal{P}^\star_m(\phi),$$

where we used that $m - 2d \ge d(1/p - 1)$. Therefore,

$$\big|\big\langle \varphi_j(\sqrt{L})f, \varphi_j(\sqrt{L})\phi \big\rangle\big| \le c2^{j(m+2d-s)} |B(x_0, 1)|^{1-1/p} \|f\|_{\dot{B}^s_{pq}} \mathcal{P}^\star_m(\phi), \quad j < 0.$$

Summing up these estimates we get

$$\sum_{j<0} \big|\big\langle \varphi_j(\sqrt{L})f, \varphi_j(\sqrt{L})\phi \big\rangle\big| \le c|B(x_0, 1)|^{1-1/p} \|f\|_{\dot{B}^s_{pq}} \mathcal{P}^\star_m(\phi), \tag{7.18}$$

where we used that $m + 2d > s$.

Clearly, estimates (7.13), (7.15), (7.17), and (7.18) imply (5.7). □

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