Série des Documents de Travail

n° 2017-74
Consistent Utility of Investment and Consumption : a forward/backward SPDE viewpoint

N. El Karoui[1]
C. Hillairet[2]
M. Mrad[3]

[1] LPMA; UMR CNRS 6632; Université Pierre et Marie Curie, Email: elkaroui@cmapx.polytechnique.fr
[2] CREST; UMR CNRS 9194; Ensae Paristech; Université Paris Saclay, E-mail : caroline.hillairet@ensae.fr
[3] LAGA; UMR CNRS 7539; Université Paris 13, E-mail : mrad@cmapx.polytechnique.fr

Consistent Utility of Investment and Consumption : a forward/backward SPDE viewpoint *

El Karoui Nicole, † Hillairet Caroline, ‡ Mrad Mohamed §

January 26, 2017

Abstract

This paper provides an extension of the notion of consistent progressive utilities $\mathbf{U}$ to consistent progressive utilities of investment and consumption $(\mathbf{U}, \mathbf{V})$. It discusses the notion of market consistency in this forward framework, compared to the classic backward setting with a given terminal utility, and whose value function is an example of such consistent forward utility. To ensure the consistency with the market model or a given set of test processes, we establish a stochastic partial differential equation (SPDE) of Hamilton-Jacobi-Bellman (HJB)-type that $\mathbf{U}$ has to satisfy. This SPDE highlights the link between the utility of wealth $\mathbf{U}$ and the utility of consumption $\mathbf{V}$, and between the drift and the volatility characteristics of the utility $\mathbf{U}$. By associating with the HJB-SPDE two SDEs, we discuss the existence and the uniqueness of a concave solution. Finally, we provide explicit regularity conditions and characterize the consistent pairs of consistent utilities of investment and consumption. Some examples, such as power utilities, illustrate the theory.

Keywords: Market-consistent progressive utility of investment and consumption, Forward/backward stochastic partial differential equation (SPDE) .

MSC 2010: 60H15, 91B16, 91B70, 91G10.

1 Introduction

In the economic modeling, the consumption rate is usually a key process. Numerous economic issues involve the optimization of the utility of the consumption rate, without a utility of terminal wealth. It is the case for example in the CIR factor model of

*With the financial support of the "Chaire Risque Financier of the Fondation du Risque"

†LPMA, UMR CNRS 6632, Université Pierre et Marie Curie

‡CREST, UMR CNRS 9194, Ensae Paristech, Université Paris Saclay,

§LAGA, UMR CNRS 7539, Université Paris 13

Cox Ingersoll Ross (1985) [5, 4] or in endowment equilibrium models of Björk (2012) [3], among many others. The value function naturally brings to light the wealth process in the optimization framework. Besides, among these economic literature involving the optimization of the utility of the consumption, many papers focus on long term issues. In such frameworks with long horizon, economists agree on the necessity of a sequential decision scheme that allows to revise the first decisions and preferences in the light of new knowledge and direct experiences. The utility criterion must be adaptive and adjusted to the information flow. Indeed, in a dynamic and stochastic environment, the standard notion of utility function is not flexible enough to help us to make good choices in the long run. Musiela and Zariphopoulou (2007,2010) [22, 23] were the first to suggest to use instead of the classic criterion the concept of progressive dynamic utility, that gives an adaptive way to model possible changes over the time of individual preferences of an agent. Obviously the dynamic utility must be consistent with respect to a given investment universe. In the general setting, the questions of the existence and the characterization of consistent dynamic utility has been studied from a PDE point of view in El Karoui and Mrad (2013,2014) [13, 15].

The present paper extends the notion of market-consistent progressive utility with consumption. In this forward setting, the progressive utilities are calibrated to a given learning set. The progressive utilities of investment and consumption were considered at first by Berrier and Tehranchi (2011) [2]. In this work, the authors establish first order consistency conditions and give an explicit characterization of consistent stochastic utilities of investment and consumption only in the restrictive case of smoothness in time of the utility of investment (without volatility vector for the utility process). Here we extend this characterization in a general semimartingale setting for the utility process, and show how the utility of investment and the utility of consumption must be linked in order to ensure the consistency. This link is explicitly given by the SPDE satisfied by $\mathbf{U}$. In particular, the presence of consumption induces a non-linearity in this SPDE.

The present paper puts in light the intuition of the methodology developed in [13] and proposes differential regularity conditions on utilities characteristics ensuring the existence of consistent utilities and extremal policies (similar to the optimal processes of the backward approach). It provides a thorough study of the similarities and the differences between progressive utilities and the value function of a backward standard utility optimization problem. Although the value function of the backward setting is an example of a consistent progressive utility satisfying the same HJB-SPDE (cf. Mania and Tevzadze (2010) [21] for the case without consumption), the way the standard optimization problem is posed is very different from the progressive utility problem. In the standard approach, the optimal processes are computed through a backward analysis, emphasizing their dependency on the horizon of the optimization problem, and leading to inter-temporality issues. The progressive approach relies on a calibration viewpoint, given a set of test processes. The problem is posed forward, leading to time-coherent extremal processes and putting emphasis on their monotonicity with respect to their initial values.

The paper is organized as follows, with a concern for finding a workable accommodation between intuition and technical results. Section 2 defines the investment universe and recalls the framework of the standard backward optimization problem with consumption, underlining the main properties of the value function, such as the market consistency. Those properties emphasize the proximity with the forward viewpoint, although differences exist in the interpretation and in the mathematical treatment. In particular, the time-coherence issue of the backward framework is addressed. Guided by the insights of the backward approach in Section 2, Section 3 studies progressive utilities of investment and consumption, consistent with a learning set of portfolios with consumptions. From this consistency property we derive formally a SPDE of Hamilton-Jacobi Bellman (HJB) type satisfied by the utility of wealth $\mathbf{U}$. The presence of a consumption process impacts this SPDE in a non-linear way, the non-linear factor involving the utility of consumption $\mathbf{V}$. Besides, the SPDE highlights the link between the drift and the volatility characteristics of the utility $\mathbf{U}$. In order to awaken the reader's intuition without too much technicalities, all the results of Section 3 are computed formally, avoiding the technical regularity assumptions, that are postponed in the next section. Section 4 provides explicit regularity assumptions under which the HJB-SPDE is studied, leading to existence results of such consistent progressive utility, and to a closed formula in term of the inverse flow of the extremal processes. This forward/backward analysis is illustrated on the example of power consistent utility, where the SPDE can be reduced to a forward backward SDE. To achieve the study, precise assumptions are given on the solution of the forward SPDE to guarantee the strong existence of the monotonic extremal processes; the key point consists in decomposing the marginal utility SPDE in terms of these two extremal SDEs.

2 Backward standard utility optimization problem of consumption and terminal wealth

2.1 Dynamic investment opportunity set with consumption.

Throughout the paper, we consider an incomplete Itô market, defined on a filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$ driven by a n-standard Brownian motion W. As with every vector, W is a column vector, and the prime denotes transposition. A process is by definition a stochastic process that is progressively measurable with respect to $\mathbb{F} = (\mathcal{F}_t)$.
As usual, the market is characterized by the short rate (r_t), the n-dimensional risk premium vector (η_t), and by the $d \times n$ volatility matrix (σ_t) of the risky assets. We assume that $\int_0^T (|r_t| + \|\eta_t\|^2)dt < \infty$, for any $T > 0$, a.s. The agent may invest in this financial market a fraction π_t of his wealth X_t in the risky assets and is allowed to consume a part of his non-negative wealth at the progressive rate $c_t = \rho_t X_t \geq 0$.

To be short, we give the mathematical definition of the class of admissible strategies

in terms of (κ_t, ρ_t) where $\kappa_t = \sigma_t \pi_t, c_t = \rho_t X_t$. The incompleteness of the market is expressed by restrictions on the risky portfolios κ_t constrained to live in a given progressive vector space $\mathcal{R}_t$. For example, if the incompleteness follows only from the fact that the number of assets is less than the dimension n of the Brownian motion, then typically $\mathcal{R}_t = \sigma_t(\mathbb{R}^n)$. For instance, if the investor only invests in a risky asset with volatility $(\bar{\kappa}_t)$, then $\mathcal{R}_t = \bar{\kappa}_t.\mathbb{R}$. For an Itô market, good references are Karatzas et al. (1987) [8] or Karatzas and Shreve (2001) [10], Skiadas (2008) [25].
To avoid technicalities, we assume throughout the paper that all the processes satisfy the necessary measurability and integrability conditions such that the following formal manipulations and statements are meaningful. The following short notations will be used extensively. Let $\mathcal{R}$ be a vector subspace of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, $x^{\mathcal{R}}$ is the orthogonal projection of the vector x onto $\mathcal{R}$ and $x^{\perp}$ is the orthogonal projection onto $\mathcal{R}^{\perp}$. We are following the presentation in [25].

Definition 2.1 (Admissible consumption plan).
(i) *The self-financing dynamics of a positive wealth process with risky portfolio κ and relative consumption rate $\rho \geq 0$ is given by*

$$dX_t^{\kappa,\rho} = X_t^{\kappa,\rho}[(r_t - \rho_t)\,dt + \kappa_t.(dW_t + \eta_t dt)]\,dt, \qquad X_0^{\kappa,\rho} = X_0 \tag{2.1}$$

with $\int_0^T (\rho_t + \|\kappa_t\|^2)dt < \infty$, for any $T > 0$ a.s.
A consumption plan c is financed by (X_0, κ_t, ρ_t) if $c_t = \rho_t X_t^{\kappa,\rho}$.
(i) *Let $(\mathcal{R}_t)$ be a given progressive family of vector spaces. By assumption, the admissible investment strategies κ are living in the vector spaces $(\mathcal{R}_t)$ a.s.. Then, the consumption plan is called* admissible.
(ii) *The set of the wealth processes financing admissible consumption plan (κ_t, ρ_t) (called admissible wealth processes) is a convex cone denoted by $\mathscr{X}^c$. When the portfolios are starting at a stopping time τ from the initial wealth $\xi \in \mathcal{F}_\tau$, the set is denoted $\mathscr{X}^c(\tau, \xi)$.*

The existence of a multivariate risk premium η (without additional integrability assumption) is a *weak form* of absence of arbitrage opportunity. Since from (2.1), the impact of the risk premium on the wealth dynamics only appears through the term $\kappa_t.\eta_t$ for $\kappa_t \in \mathcal{R}_t$, there is a "minimal" risk premium $(\eta_t^{\mathcal{R}})$, the projection of η_t on the space $\mathcal{R}_t$ $(\kappa_t.\eta_t = \kappa_t.\eta_t^{\mathcal{R}})$, to which we refer in the sequel.
In the following definition, we are interested in the class of the so-called state price density processes Y^ν (taking into account the discount factor) that plays the role of the "orthogonal cone" $\mathscr{Y}$ of the cone of admissible wealth processes $\mathscr{X}^c$ in the "martingale" sense. The main point is that $\mathscr{Y}$ does not depend on the presence of the consumption process, and is uniquely characterized by the admissible financial market.

Definition 2.2 (State price density process).
(i) a) *A non-negative Itô semimartingale Y^ν is called an admissible state price density process if for any admissible (wealth, consumption)-processes $(X^{\kappa,\rho}, c)$ with $c = \rho X^{\kappa,\rho}$,*

$$\textit{the process} \quad \mathcal{H}_t^{\kappa,\rho,\nu} = X_t^{\kappa,\rho} Y_t^\nu + \int_0^t Y_s^\nu c_s\, ds \quad \textit{is a local martingale.} \tag{2.2}$$

b) *It follows that the differential decomposition of Y^ν does not depend on c and*

$$dY_t^\nu = Y_t^\nu[-r_t dt + (\nu_t - \eta_t^{\mathcal{R}}).dW_t], \quad \nu_t \in \mathcal{R}_t^\perp, \quad Y_0^\nu = Y_0. \tag{2.3}$$

(ii) *We denote by $\mathscr{Y}$ the convex family of all state density processes Y^ν where $\nu \in \mathcal{R}^\perp$ and by $\mathscr{Y}(\tau,\psi)$ the subfamily of the processes starting from $\psi \in \mathcal{F}_\tau$ at time τ.*

Observe that any process $Y_t^\nu(y)$, starting from y at time 0, is the product of

$$yY_t^0 = y\exp\big(-\textstyle\int_0^t r_s ds - \int_0^t \eta_s^{\mathcal{R}}.dW_s - \frac{1}{2}\int_0^t ||\eta_s^{\mathcal{R}}||^2 ds\big),$$

by the exponential local martingale $L_t^\nu = \exp\big(\int_0^t \nu_s.dW_s - \frac{1}{2}\int_0^t ||\nu_s||^2 ds\big)$, $(L_0^\nu = 1)$.

2.2 Value function of backward standard utility optimization problem

In this subsection, we recall fundamental results of the theory of consumption-portfolio choice of a risk adverse agent, where as in the seminal Merton's work, the investor optimizes time-additive expected utility, expressed in a backward formulation. We follow the presentation of Kramkov and Schachermayer [19] for the pure investment problem, and Karatzas and Žitković [11] for the consumption-portfolio problem.

2.2.1 Standard consumption-portfolio optimization problem

We recall that an *utility function* u is a strictly concave, strictly increasing, and nonnegative function defined on $\mathbb{R}^+$, with continuous *marginal utility* u_x, satisfying the Inada conditions $\lim_{x\mapsto\infty} u_x(x) = 0$ and $\lim_{x\mapsto 0} u_x(x) = \infty$. The risk aversion coefficient is measured by the ratio $-u_{xx}(x)/u_x(x)$. The asymptotic elasticity $AE(u) = \limsup_{x\mapsto\infty} xu_x(x)/u(x)$ is a key parameter in the optimization problem. Throughout the paper, we adopt the convention of small letters for deterministic utilities and capital letters for stochastic utilities.

The usual problem of optimizing expected utility of consumption and terminal wealth on a given horizon T_H, is based on two deterministic utility functions $u(.)$ and $v(t,.)$ and the class of admissible wealth processes $\mathscr{X}^c$. It is formulated as the following optimization program, with admissible consumption plan $c_t = \rho_t X_t^{\kappa,\rho}$,

$$\mathcal{U}(x) = \sup_{(\kappa,\rho)\in\mathscr{X}^c(x)} \mathbb{E}\Big(u(X_{T_H}^{\kappa,\rho}) + \int_0^{T_H} v(t,c_t)dt\Big). \tag{2.4}$$

The standard approach for studying the optimization problem (2.5) relies in the use of duality relationships in the spaces of convex functions and semimartingales, together with analysis tools. It requires the assumptions that the asymptotic elasticity of u is strictly less than one and that the value function $\mathcal{U}(x)$ is finite for at least one x (that is guaranteed for instance as soon as the utility function $u(x) \leq C(1+x^\alpha)$ with α in $(0,1)$).

(i) In the problem without consumption ($v \equiv 0$), Kramkov and Schachermayer [18, 19] prove that the value function $\mathcal{U}$ is also a utility function with $AE(\mathcal{U}) < AE(u)$, together with the existence of an unique family of optimal processes denoted $X^{*,H}_{.}(x)$. These results are extended to the framework with consumption, ($v \neq 0$) by Karatzas and Shreve (2001) [10] and Karatzas and Žitković [11]. The pair of optimal processes is then denoted $(X^{*,H}_{.}(x), c^{*,H}_{.}(x))$. [11] considers random utilities satisfying asymptotic elasticity condition, and whose second derivative $U_{xx}(t,x)$ are assumed to be bounded by above (respectively by below) by a nonrandom function $K_2(x)$ (respectively $K_1(x)$) such that $\limsup_{x\mapsto\infty} \frac{K_2(x)}{K_1(x)} < \infty$.

(ii) Under the same assumptions, the problem can be generalized to a random initial condition (τ,ξ) where τ is a stopping time smaller than T_H and $\xi \geq 0$ is a $\mathcal{F}_\tau$-random variable. Recall that $\mathscr{X}^c(\tau,\xi)$ denotes the set of admissible strategies starting from (τ,ξ). The corresponding value system (that is a family of random variables indexed by (τ,ξ)) is defined by its terminal value $\mathcal{U}(T_H,\xi_T) = u(T_H,\xi_T)$ and by

$$\mathcal{U}(\tau,\xi) = \operatorname*{ess\,sup}_{(\kappa,\rho)\in\mathscr{X}^c(\tau,\xi)} \mathbb{E}\Big(u(X^{\kappa,\rho}_{T_H}(\tau,\xi)) + \int_\tau^{T_H} v(s,c_s)ds|\mathcal{F}_\tau\Big),\ a.s. \tag{2.5}$$

As it is usual in stochastic control problems, when the class of admissible strategies is stable by concatenation in time, (El Karoui [12] recently republished in [16]), the *dynamic programming principle*, also called *time-coherence property*, consists in considering a random horizon ϑ shorter than T_H, and the stochastic utility system $\mathcal{U}$ in place of u as criterion at ϑ. It reads as follows: for any pair $\tau \leq \vartheta \leq T_H$ of stopping times,

$$\mathcal{U}(\tau,\xi) = \operatorname*{ess\,sup}_{(\kappa,\rho)\in\mathscr{X}^c(\tau,\xi)} \mathbb{E}\big(\mathcal{U}(\vartheta, X^{\kappa,\rho}_{\vartheta}(\tau,\xi)) + \int_\tau^{\vartheta} v(s,c_s)ds|\mathcal{F}_\tau\big)\ a.s. \tag{2.6}$$

Using regularization results of Dellacherie and Lenglart [6], the previous utility family can be aggregated into a progressive stochastic utility process, still denoted $\mathcal{U}(t,x)$. Then, the previous optimality results can be expressed in terms of processes, which will make it possible to use stochastic calculus.

Proposition 2.3 (Market consistency)**.**

(i) *For any admissible consumption plan (κ,ρ,c) with $c_s = \rho_s X^{\kappa,\rho}_s$, and any initial condition (τ,ξ), on $[\tau,T_H]$, the preference process $\mathcal{Z}^{\kappa,\rho}$ is a supermartingale, where*

$$\mathcal{Z}^{\kappa,\rho}_t(\tau,\xi) = \mathcal{U}(t, X^{\kappa,\rho}_t(\tau,\xi)) + \int_\tau^t v(s,c_s)ds. \tag{2.7}$$

Under the asymptotic elasticity condition on u and v and under regularity and integrability conditions on the optimization space $\mathscr{X}^c$, there exists an optimal strategy $(\kappa^{*,H}_{.}(\tau,\xi), c^{*,H}_{.}(\tau,\xi), X^{*,H}_{.}(\tau,\xi))$, such that the optimal preference process $\mathcal{Z}^{*,H}_t(\tau,\xi)$ is a

martingale

$$\mathcal{Z}_t^{*,H}(\tau,\xi) = \mathcal{U}(t, X_t^{*,H}(\tau,\xi)) + \int_\tau^t v(t, c_t^{*,H}(\tau,\xi))ds. \tag{2.8}$$

(ii) *From the maximum principle, the optimal marginal utility processes* $\mathcal{U}_x(t, X_t^{*,H}(\tau,\xi))$ *and* $v_c(t, c_t^{*,H}(\tau,\xi))$ *coincide.*

(iii) *The optimal wealth processes are time-coherent, since*

$$\textit{for } t \geq \tau,\ X_t^{*,H}(\tau,\xi) = X_t^{*,H}(0, X_0) \ \textit{when } \xi = X_\tau^{*,H}(0, X_0).$$

Such concave random field system $(\mathcal{U}, v)$ is said to belong to the family of *market consistent dynamics utility system* (with terminal condition), defined in the next section.

2.2.2 The standard state price density conjugate problem

As usual, the dual problem highlights some different aspects of the optimization problem. It is based on the Fenchel-Legendre convex conjugate transformation $\tilde{u}(y)$ of a utility function u, where the system $(u, \tilde{u})$ satisfies $\left(\tilde{u}(y) = \sup_{x>0}\big(u(x) - yx\big), u(x) = \inf_{y>0}\big(\tilde{u}(y) + yx\big)\right)$. In particular, $\tilde{u}(y) \geq u(x) - yx$ and the maximum is attained at $u'(x) = y$.

The same transformation can be applied to the stochastic value system $(\mathcal{U}(t,x), v(t,c))$ with concave dependency in x and c, to define the conjugate random field system $(\widetilde{\mathcal{U}}(t,y), \tilde{v}(t,z))$,

$$(S) \quad \begin{cases} \widetilde{\mathcal{U}}(t,y) = \mathcal{U}(t, \mathcal{U}_x^{-1}(t,y)) - y\,\mathcal{U}_x^{-1}(t,y), & \mathcal{U}_x^{-1} = -\widetilde{\mathcal{U}}_y \\ \tilde{v}(t,z) = v(t, v_c^{-1}(t,z)) - c\, v_c^{-1}(t,z), & v_c^{-1} = -\tilde{v}_z. \end{cases}$$

From Proposition 2.3 and in particular the characterization of the optimal processes, the conjugate system $(\widetilde{\mathcal{U}}(t,y), \tilde{v}(t,z))$ is also associated with an optimization problem consistent with the family of the state price density $\mathcal{Y}$:

$$\widetilde{\mathcal{U}}(\tau,\psi) = \operatorname{ess\,inf}_{Y^\nu \in \mathscr{Y}(\tau,\psi)} \mathbb{E}\Big(\tilde{u}(T_H, Y_{T_H}^\nu) + \int_\tau^{T_H} \tilde{v}(s, Y_s^\nu) ds | \mathcal{F}_\tau\Big), \ a.s. \tag{2.9}$$

The main advantage of the conjugate optimization problem relies on the fact that the dual formulation (2.9) does not involve the consumption process and the optimization is done on a single control parameter.

Proposition 2.4 (Market dual consistency)**.**

(i) *For any admissible dual process* $\nu \in \mathcal{R}^\perp$*, and any initial condition* (τ,ψ)*, on* $[\tau, T_H]$*,*

$$\textit{The process} \quad \widetilde{Z}_t^\nu(\tau,\psi) = \widetilde{\mathcal{U}}(t, Y_t^\nu(\tau,\psi)) + \int_\tau^t \tilde{v}(s, Y_s^\nu(\tau,\psi))ds \ \textit{is a submartingale.} \tag{2.10}$$

(ii) *Under the asymptotic elasticity condition, starting from* (τ,ψ)*, there exists an optimal* *process $\nu^{*,H} \in \mathcal{R}^{\perp}$, generating an optimal state price process $Y^{*,H}_{\cdot}(\tau,\psi)$, such that*

$$\text{on } [\tau, T_H], \text{ the process } \widetilde{Z}^{*,H}_t = \widetilde{\mathcal{U}}(t, Y^{*,H}_t(\tau,\psi)) + \int_\tau^t \tilde{v}(t, Y^{*,H}_s(\tau,\psi))ds \text{ is a martingale.} \tag{2.11}$$

(iii) *Assume (ξ,ψ) to be linked by $\mathcal{U}_x(\tau,\xi)=\psi$. Then, the optimal processes are linked by*

$$Y^{*,H}_t(\tau,\psi) = \mathcal{U}_x(X^{*,H}_t(\tau,\xi)) = v_c(c^{*,H}_t(\tau,\xi)). \tag{2.12}$$

The proof relies on the properties of the primal optimal process.

Proof. As a consequence of the market consistency in the primal problem for which $\mathcal{Z}^{*,H}$ is a martingale, for any admissible state price density Y^ν, the process

$$\mathcal{Z}^{*,H}_t - \mathcal{H}^{*,\nu}_t = \mathcal{U}(t, X^{*,H}_t) - Y^\nu_t X^{*,H}_t + \int_0^t (v(s, c^{*,H}_s) - Y^\nu_s c^{*,H}_s)ds \tag{2.13}$$

is a submartingale and a martingale if $\mathcal{H}^{*,\nu}_t = Y^\nu_t X^{*,H}_t + \int_\tau^t Y^\nu_s c^{*,H}_s ds$ is a martingale.
By the definition of the conjugate utility, this submartingale is dominated by the process $\widetilde{Z}^\nu_t = \widetilde{\mathcal{U}}(t, Y^\nu_t(\tau,\psi)) + \int_\tau^t \tilde{v}(s, Y^\nu_s(\tau,\psi))ds$; it is easy to deduce that the process $\widetilde{Z}^\nu_t$ itself is a submartingale.
Assume that $\mathcal{U}_x(\tau,\xi)=\psi$ and define $Y^{*,H}_t(\tau,\psi) = \mathcal{U}_x(X^{*,H}_t(\tau,\xi)) = v_c(c^{*,H}_t(\tau,\xi))$. Following Karatzas-Lehoczky-Shreve(2004)[9], by the properties of the optimal processes $(X^{*,H}, c^{*,H})$, there exists a process $\nu^* \in \mathcal{R}^\perp$ such that $Y^{*,H}_t(\tau,\psi) = Y^{\nu^*,H}_t(\tau,\psi)$, and $\mathcal{H}^{*,\nu^*}_{\cdot}(\tau,\xi)$ is a martingale. The process $\mathcal{Z}^{*,H}_t(\tau,\xi) - \mathcal{H}^{*,\nu^*}_t(\tau,\xi)$ is a martingale which is exactly the process $\widetilde{Z}^{\nu^*,H}_t$. Then, the process $(\widetilde{\mathcal{U}}(t,y))$ is the value function of an optimization problem consistent with the set of admissible processes $\mathcal{Y}$, and objective criterion associated with $(\tilde{u},\tilde{v})$. □

2.2.3 Regularity and time-coherence issues

Regularity. Although the backward primal and dual optimization problems provide a tractable framework to prove the existence of optimal processes, in which comparison arguments are used to justify martingale properties, it is nevertheless very complicated to show the regularity of the value functions. Obtaining closed formula and explicit construction for these value functions and their optimal strategies is a difficult task, except for a few cases like exponential or power utilities.
In the Markovian case, the supermartingale (martingale) properties induced by the market consistency are used to associate a well-known HJB-PDE, with terminal condition, whose resolution uses the viscosity solution point of view to compensate for the lack of regularity. In the same spirit, Mania and Tevzadze (2010) [21], assuming strong differential regularity on the stochastic utility considered as a semimartingale field, make the links with a backward "SPDE" of HJB-type. The main difficulty is to find conditions on u and v such that these random fields regularity is satisfied.

Time-coherence issue. Optimal processes are highly dependent on the horizon T_H, which leads to inter-temporal issues, as mentioned in Tehranchi et al. [24].
a) *Infinite horizon.* One may then argue that it suffices to take $T_H = \infty$ to be time-coherent, and consider for example a time separable utility $v(t,x) = e^{-\beta t} v(x)$ with an infinite horizon, as it is usually formulated in the economic literature. This is equivalent (in expectation) to consider the utility v and an independent random horizon τ_H exponentially distributed with parameter β. Then the dependency in T_H is transposed into a dependency in β.
b) *Intertemporal Incoherence.* To illustrate the time-coherence issue, let us consider an intermediate horizon T between 0 and T_H and the following two scenarios.

- In the first one, the investor determines his optimal strategy, denoted $\big(X_t^{*,H}, c_t^{*,H}\big)_{t\in[0,T_H]}$, directly for the horizon T_H and the utility functions (u,v).
- In the second one, the investor (starting from the same wealth) first determines his optimal strategy $(\bar{X}_t^{*,T}, \bar{c}_t^{*,T})$ for the horizon T and the utility functions $(\bar{u}, v)$; his wealth $\bar{X}_T^{*,T}$ is then reinvested at time T, optimally between the dates (T, T_H) using now the same utility system than the first investor (u,v), leading to an optimal strategy given by $(\hat{X}_t^{*,H}(T,\bar{X}_T^*), \hat{c}_t^{*,H}(T,\bar{X}_T^*))$. But, by the dynamic programming principle and the uniqueness of the optimal process, on $[T,T_H]$ these optimal processes are exactly the processes $(X_t^{*,H}(T,\bar{X}_T^*), c_t^{*,H}(T,\bar{X}_T^*))$.

This shows that the time-coherence implies that the agent should have used as intermediate utility for the horizon T the stochastic utility $(\mathcal{U}^H(T,x), v)$, which takes into account the information available up to time T.

To summarize Under the asymptotic elasticity assumption on the two utility functions (u,v), the value function $\mathcal{U}(t,x)$ of the backward primal program is a $\mathscr{X}^c$-consistent dynamic utility and the value function of backward dual program $\widetilde{\mathcal{U}}(t,y)$ is a $\mathscr{Y}$-consistent dynamic conjugate utility. This backward point of view is well adapted to comparison problems for instance, but induces strongly horizon-dependent strategies.
This *time-coherence* issue of standard (backward) utility optimization problem was the first motivation to consider consistent progressive utilities in the financial literature as in Musiela and Zariphopolou (2007) [22, 23], or Berrier and Tehranchi [2] which were the first to consider progressive utilities of investment and consumption.
In the economic literature, Lecocq and Hourcade [7] have already argued in favor of a utility criterion that may be adjusted along time. Indeed, how a deterministic utility function (fixed at time 0) may be supposed to model the preferences an investor in a distant future? It is obviously more accurate to consider a decision criterion that is adapted to the financial/economic information flow and thus allows to revise the preferences according to the financial market evolution, to possible future crises. The problem becomes then: what is the "optimal rule" for the revision of the preferences?
In the light of all this discussion on the backward framework, and inspired by the in-

teresting properties of the value function of a backward optimization problem, the next section introduces the notion of a progressive (also called forward) utility system, consistent with a learning set of portfolios with consumption, with a specific care on the issues of regularity and time-coherence that have been raised by the backward setting.

3 Progressive utility system consistent with a given learning set

3.1 Progressive utility system

Progressive utilities are an alternative way to address the market consistency and the time-coherence issue. A subcone of the market strategies $\mathscr{X}^c$, describing the financial landscape in Subsection 2.1, is considered in the forward setting as a (portfolio, consumption) family of *test processes*. Indeed, the *time-coherence* is obtained from a dynamic decision criterion adjusted progressively over the time in reference to this set $\mathscr{X}^c$ of test processes. Thus the role played by the utility criteria and the market constraints are deeply different from before, since now the problem is no more an optimization problem but a kind of calibration problem. As in statistical learning, the utility criteria are dynamically adjusted given the family of test processes, also called the learning set. More precisely we will consider in the sequel two different admissible learning sets, that are variants of the learning set without consumption. In a simple extension (called in the sequel "first test problem with fixed ρ"), the test processes are associated with any admissible portfolio strategy and a **given** relative consumption rate process ρ_t. The tests set is denoted $\mathscr{X}^\rho$. The second extension (called in the sequel "second test problem with consumption") is the case where the relative consumption rate ρ is no more given but also a control parameter. It is the classic situation studied in the backward consumption-portfolio problems considered in the previous section. The set of test processes is then denoted $\mathscr{X}^c$.

3.1.1 Consistent progressive utility system

We start with progressive utilities $(\mathbf{U}, \mathbf{V})$, defined as a family of stochastic utility processes such that for any t, $(U(t,x), V(t,c))$ are some utility functions as defined in [11], adjusted to a learning set in the sense given in Section 2.2, Definition 2.3.
Using the same notation, the learning set is the cone $\mathscr{X}$ (denoting indifferently $\mathscr{X}^c$ or $\mathscr{X}^\rho$) and its "orthogonal" is the set of state price density processes $\mathscr{Y}$ which does not depend on the consumption. The satisfaction associated with a test process $X^{\kappa,\rho} \in \mathscr{X}$ $(c = \rho X^{\kappa,\rho})$ is measured with the help of the *utility system* $(U(t,x), V(t,c))$ and the preference criterion $Z_t^{\kappa,\rho} = U(t, X_t^{\kappa,\rho}) + \int_0^t V(s, c_s)ds$. Since $\mathscr{X}$ is a learning set, there is no satisfaction to invest in the set $\mathscr{X}$, in other words in mean the future is less preferable than the present. From the mathematical point of view, it is equivalent to the *supermartingale property* of

the dynamic preference process $(Z_t^{\kappa,\rho})$. Moreover, to ensure that the system of stochastic utilities $(U(t,x), V(t,c))$ is the best choice, we make the additional assumption that the previous supermartingale constraint is binded by some extremal process (κ^e, ρ^e) whose performance criterion Z^e is a martingale.

Definition 3.1 (Consistent progressive utility system).
Let $(\mathbf{U}, \mathbf{V})$ *be a progressive utility system with learning set* $\mathscr{X}$*.*
(i) *The utility system* $(\mathbf{U}, \mathbf{V})$ *is said to be* $\mathscr{X}$-consistent, *if for any admissible test process* $X^{\kappa,\rho} \in \mathscr{X}$*, the preference process*

$$Z_t^{\kappa,\rho} = U(t, X_t^{\kappa,\rho}) + \int_0^t V(s, c_s)ds \text{ is a non-negative supermartingale.} \tag{3.1}$$

(ii) *The consistent utility system* $(\mathbf{U}, \mathbf{V})$ *is said to be* $\mathscr{X}$-strongly consistent *if there exists an* extremal *system in* $\mathscr{X}$*,* (X^e, κ^e, c^e)*,* $(c^e = \rho^e X^e)$*, binding the constraint, that is*

$$\text{the extremal preference process} \quad Z_t^e = U(t, X_t^e) + \int_0^t V(s, c_s^e)ds \quad \text{ is a martingale.} \tag{3.2}$$

When there is no ambiguity, we refer to this last property as the strongly consistency.

The value function system $(\mathcal{U}(t,x), v(t,c))$ of the classic consumption optimization problem developed in the first section (see in particular Proposition 2.3) is an example of strongly consistent system (with respect to $\mathscr{X}^c$), defined from its terminal condition $\mathcal{U}(T_H, x) = u(x)$. Conversely, given $\mathscr{X}^c$, and given an initial utility system $(U(0,x), V(0,c))$, a strongly consistent system $(\mathbf{U}, \mathbf{V})$ is the value function system of some investment-consumption problem, with stochastic terminal condition $U(T_H, x)$ for any time horizon T_H. In the forward approach, the utility process $\mathbf{U}$ considered as value function is the same for any time horizon T_H.
The two problems differ by their boundary conditions, backward in the classic case (cf. Section 2) and forward in the present case. Furthermore, although the $\mathscr{X}^c$-consistent constraints are the same, this point induces major differences in the interpretation and in the mathematical treatment of their characterization, apart from the issue of time-coherence.

3.1.2 Differential point of view for Itô consistent utility system

In the standard (backward) framework, the initial value of the value function $\mathcal{U}$ is usually not explicit and is computed through a backward analysis, starting from its given terminal utility (possibly random) $u(x)$ at T_H. From a "practical" point of view, the Markov property is strategic for the resolution of the backward framework. For consistent progressive utilities, the initial value is given and the problem is solved forward, without any reference time-horizon T_H; the emphasis is placed on the monotonicity of extremal processes with respect to the initial condition.
In the forward case, in the absence of Markov property, stochastic calculus can be used to characterize $\mathscr{X}$-consistent forward utility system, via a stochastic generalization of

the deterministic backward HJB-PDE. Such random HJB generalization may be found in Musiela and Zariphopoulo [22, 23] and Berrier and Tehranchi [2] who have restricted themselves to the case of decreasing in time forward utility $\mathbf{U}$. In the problem without consumption, El Karoui and Mrad [13] obtained a non linear HJB-SPDE under the more general assumption that the utility random field $\mathbf{U}$ is a "regular" Itô random field with differential decomposition,

$$dU(t,x) = \beta(t,x)dt + \gamma(t,x).dW_t, \tag{3.3}$$

where $\beta(t,x)$ is the drift random field and $\gamma(t,x)$ is the multivariate diffusion random field. The decreasing case ([22], [2]) corresponds to $\gamma(t,x) \equiv 0$ and $\beta(t,x) = \partial_t U(t,x) \leq 0$.
In a backward problem $(\mathcal{U}, v)$, it is not easy to find sufficiently general conditions on the data (u,v) and the class $\mathscr{X}^c$ for $\mathcal{U}$ to be a regular Itô random field. For this reason, Mania and Tevzadze (2010) [21] introduced the regularity of the random field $\mathcal{U}$ as an additional assumption to develop a quite similar stochastic calculus for the value function. This point is not an issue for forward utility, in return it is not easy to read directly on its local characteristics (β,γ) that the process $U(t,x) = U(0,x) + \int_0^t \beta(s,x)ds + \int_0^t \gamma(s,x).dW_s$ is a utility random field (increasing and concave), in absence of general comparison results for stochastic integrals. An exception is given by the solution of stochastic differential equation (SDE) whose monotonicity with respect to the initial condition is obtained under regularity assumption on the coefficients, as we will see below ([13], [20]).
In the following, we reformulate in this new framework the theoretical results of [13] concerning a learning set without consumption, in order to specify the consequence of the additional consumption optimization.

3.2 Itô-Ventzel's formula and applications

To express the supermartingale property implied by the consistency condition in terms of local characteristics, we need the differential decomposition of any compound process $U(t, X_t^{\kappa,\rho})$, where $\mathbf{U}$ is now a dynamic random field. Obviously, if $U(t,x)$ were a deterministic regular function, we could use the Itô formula. The right tool in this more stochastic context is the so-called Itô-Ventzel's formula given in the reference book [20] of Kunita.

3.2.1 Itô-Ventzel's formula

Let us consider a "regular" random field $(G(t,x))$ with local characteristics $\phi(t,x)$ and $\psi(t,x)$. As for Itô's formula, we need the local characteristics of the first (second) differentials of the progressive random field $\mathbf{G}$ assumed to be at least of class $\mathcal{C}^2$. Formally, they are obtained by differentiating the local characteristics of $\mathbf{G}$, but as in the deterministic case, additional assumptions of Sobolev type (defined in Subsection 4.1.1) are necessary to justify to differentiate the stochastic integrals. All these questions yield to technical assumptions detailed in [13] and briefly recalled in Section 4.1. The reader who wants to skip technicalities can read here the notion of "regular" random field as meaning "we can apply Itô-Ventzel's formula".

The *Itô-Ventzel formula* gives the decomposition of the compound random field $G(t, X_t)$ for any Itô semimartingale X as the sum of three terms: the first one is the "differential in t" of G, the second one is the classic Itô's formula (without differentiation in time) and the third one is the infinitesimal covariation between the martingale part of G_x and the martingale part of X, all these terms being taken in X_t.

$$\begin{aligned} dG(t,X_t) &= \big(\phi(t,X_t)\,dt + \psi(t,X_t).dW_t\big) \\ &+ \big(G_x(t,X_t)dX_t + \frac{1}{2}G_{xx}(t,X_t)d<X,X>_t\big) + \big(<\psi_x(t,X_t)dW_t, dX_t>\big). \end{aligned} \tag{3.4}$$

When G has only finite variation, the formula is reduced to a classic Itô formula, since in this case $\psi(t,x) \equiv 0$, $\phi(t,X_t) = \partial_t G_t(t,X_t)$.

A typical example of Itô semimartingale X is the solution of "regular" stochastic differential equation $X(t,x) = X_t(x)$, with stochastic coefficients $(\sigma(t,x), \mu(t,x))$ whose regularity is studied in Subsection 4.1.1.

$$dX_t(x) = \mu(t,X_t)dt + \sigma(t,X_t).dW_t, \quad X_0 = x. \tag{3.5}$$

The associated *elliptic generator* is $\quad L^{\sigma,\mu}_{t,x} = L^X_{t,x} = \frac{1}{2}\|\sigma(t,x)\|^2\partial_{xx} + \mu(t,x)\partial_x$.
The local characteristics of the random field $X(t,x) = X_t(x)$ are $\beta^X(t,x) = \mu(t, X(t,x))$ and $\gamma^X(t,x) = \sigma(t, X(t,x))$. Then, the Itô-Ventzel formula reads as

$$\begin{aligned} dG(t,X_t) &= \big(G_x(t,X_t)\sigma(t,X_t) + \psi(t,X_t)\big).dW_t \\ &+ \big(\phi(t,X_t) + \psi_x(t,X_t).\sigma(t,X_t)\big)\,dt + L^{\sigma,\mu}G(t,X_t)dt. \end{aligned} \tag{3.6}$$

3.2.2 SDE and SPDE for regular solution and its inverse

It is well known that any "regular" solution of SDE(μ,σ) (see the regularity class $\mathcal{S}^{m,\delta}$ defined in Subsection 4.1.1) is monotonic with respect to its initial condition, since its derivative $DX_t(x) = \partial_x X(t,x)$ is solution of the linear equation

$$dDX_t(x) = DX_t(x)\big(\mu_x(t,X_t)dt + \sigma_x(t,X_t).dW_t\big), \quad DX_0 = 1. \tag{3.7}$$

The first application concerns the dynamics of the inverse flow $X^{-1}(t,z) = \xi^X(t,z)$ of the monotonic solution of the SDE(μ,σ). We refer to Theorem 4.2 and Proposition 4.3 for technical results on the theory of stochastic flows and SDEs.

Theorem 3.2 (SPDE point of view)**.** *Let $(X(t,x))$ be the monotonic solution of a "regular" SDE(μ,σ), with adjoint operator in divergence form,*

$$\widehat{L}^{\sigma,\mu}_{t,z} = \widehat{L}^X_{t,z} = \frac{1}{2}\partial_z(\|\sigma(t,z)\|^2\partial_z) - \mu(t,z)\partial_z.$$

(i) *The inverse flow $\xi(t,z) = X^{-1}(t,z)$ is "regular" and solution of the SPDE*

$$d\xi(t,z) = -\xi_z(t,z)\sigma(t,z).dW_t + \widehat{L}^{\sigma,\mu}_{t,z}(\xi)dt, \quad \xi(0,z) = z. \tag{3.8}$$

This SPDE is denoted SPDE($\widehat{L}^{\sigma,\mu}_{t,z}$).

(ii) *Let Y be a "regular" solution of the SDE(μ^Y, σ^Y) and ϕ any $\mathcal{C}^2$-function. Then the compound random field $H(t,z) := Y(t, \phi(\xi(t,z)))$ with initial condition $H(0,z) = \phi(z)$ evolves as*

$$\begin{aligned} dH(t,z) &= \big(\sigma^Y(t,H(t,z)) - H_z(t,z)\sigma^X(t,z)\big).dW_t \\ &+ \big(\mu^Y(t,H(t,z)) - H_z(t,z)\sigma^X(t,z).\sigma^Y_y(t,H(t,z)) + \widehat{L}^X_{t,z}(H)(t,z)\big)dt. \end{aligned} \tag{3.9}$$

H appears as a solution of a mixture of SDE and SPDE problems.

Proof. (i) We are looking for a "regular" random field $G(t,z)$ such that $G(t,X_t(x)) = x$. Since $G(t,X_t(x))$ is deterministic, by equations (3.4) and (3.6)

$$G_x(t,x)\sigma(t,x) + \psi(t,x) = 0, \quad a.s. \text{ and } \gamma^G_x(t,x) = -\partial_x(G_x(t,x)\sigma(t,x))$$

$$\text{and } -\partial_x(G_x(t,x)\sigma(t,x)).\sigma(t,x) + \tfrac{1}{2}\|\sigma(t,x)\|^2 G_{xx}(t,x) = -\tfrac{1}{2}\partial_z(\|\sigma(t,z)\|^2 G_z(t,z)).$$

The drift condition together with the γ constraint yields to

$\phi(t,x) = -\psi_x(t,x).\sigma(t,x) - L^{\sigma,\mu}G(t,x) = \frac{1}{2}\partial_z(\|\sigma(t,z)\|^2 G_z(t,z)) - \mu(t,x)G_x(t,x)$.

This last term is exactly the adjoint operator $\widehat{L}^X_{t,z}$ applied to G.

(ii) We are looking for a "regular" random field $H(t,z)$ such that $H(t,X_t(x)) = Y_t(\phi(x))$. By Itô-Ventzel's formula, and the γ constraint, $H_z(t,z)\sigma^X(t,z) + \gamma^H(t,z) = \sigma^Y(t,H(t,z))$, and $\gamma^H_z(t,z) = -\partial_x(H_z(t,z)\sigma^X(t,z)) + \sigma^Y_y(t,H(t,z))H_z(t,z))$.

The drift constraint is $(\beta^H(t,z) + \gamma^H_z(t,z).\sigma(t,z))\,dt + L^{\sigma,\mu}G(t,z) = \mu^Y(t,H(t,z))$.
The same transformation than for the inverse flow yields to

$$\beta^H(t,z) = \mu^Y(t,H(t,z)) - H_z(t,z)\sigma^X(t,z).\sigma^Y_y(t,H(t,z)) + \widehat{L}^X_{t,z}(H)(t,z)$$

which is the expected result. □

3.3 Local characteristics of consistent forward utility with consumption

3.3.1 Consistency constraints for two different sets of test processes

We are concerned with two different learning problems associated with two family of test portfolios with different consumption constraints, $\mathscr{X}^\rho$ and $\mathscr{X}^c$, denoted by the generic notation $\mathscr{X}$. For both learning problems, as in Definition 3.1, the consistency condition is, for any admissible test process $X^{\kappa,\rho} \in \mathscr{X}$,

$$Z_t^{\kappa,\rho} = U(t, X_t^{\kappa,\rho}) + \int_0^t V(s, \rho_s X_s^{\kappa,\rho}) ds \quad \text{is a non-negative supermartingale.} \tag{3.10}$$

The main difference with the problem considered in [13] is in the performance function, that includes a past depending criterion, namely $\int_0^t V(s, \rho_s X_s^{\kappa,\rho}) ds$.
The problem is to transform the global supermartingale property into a local condition on the characteristics (β, γ) of the forward utility $\mathbf{U}$ by using Itô-Ventzel's formula, which requires additional regularity assumption on the utilities random fields $\mathbf{U}$ and $\mathbf{V}$, and their characteristics, that we will make explicit in Subsection 4.1.1.

Recall that a test process is a solution $X^{\kappa,\rho}$ of a linear SDE $dX_t^{\kappa,\rho} = X_t^{\kappa,\rho}[(r_t - \rho_t)\, dt + \kappa_t.(dW_t + \eta_t^{\mathcal{R}}\, dt)]$. The differential decomposition of the performance criterion (3.10) is given by

$$\begin{aligned} dZ_t^{\kappa,\rho} &= \big(U_x(t, X_t^{\kappa,\rho}) X_t^{\kappa,\rho} \kappa_t + \gamma(t, X_t^{\kappa,\rho})\big).dW_t \\ &+ \big(\beta(t, X_t^{\kappa,\rho}) + U_x(t, X_t^{\kappa,\rho}) X_t^{\kappa,\rho}(r_t - \rho_t) + V(t, \rho_t X_t^{\kappa,\rho})\big) dt \\ &+ \big(U_x(t, X_t^{\kappa,\rho}) X_t^{\kappa,\rho} \kappa_t.\eta_t^{\mathcal{R}} + \frac{1}{2} U_{xx}(t, X_t^{\kappa,\rho}) \|X_t^{\kappa,\rho}\, \kappa_t\|^2 + X_t^{\kappa,\rho} \kappa_t.\gamma_x(t, X_t^{\kappa,\rho})\big) dt. \end{aligned} \tag{3.11}$$

The third line is proportional to a quadratic form $\mathcal{Q}(t, x, \kappa)$ in $X_t^{\kappa,\rho} \kappa_t$ whose minimum is $-\|x\kappa_t^e(x)\|^2$, where $x\kappa_t^e(x)$ is described below:

$$\begin{cases} \mathcal{Q}(t, x, \kappa) = \|x\kappa\|^2 + 2\dfrac{U_x(t,x)}{U_{xx}(t,x)} x\kappa.\big(\eta_t^{\mathcal{R}} + \dfrac{\gamma_x^{\mathcal{R}}(t,x)}{U_x(t,x)}\big) \geq -\|x\kappa_t^e(x)\|^2. \\ \sigma^e(t,x) = x\kappa_t^e(x) = -\dfrac{U_x(t,x)}{U_{xx}(t,x)}\big(\eta_t^{\mathcal{R}} + \dfrac{\gamma_x^{\mathcal{R}}(t,x)}{U_x(t,x)}\big). \end{cases} \tag{3.12}$$

Remark The positive ratio $-\frac{U_x(t,x)}{xU_{xx}(t,x)}$ is the classic *relative risk tolerance coefficient*, which is constant for the well-known power utilities (see Paragraph 4.2.2). Recall that $U_{xx}(t,x) \leq 0$. More interestingly is that the diffusion process $\gamma(t,x)$ of the progressive utility introduces a "utility risk premium" given by the ratio $\frac{\gamma_x^{\mathcal{R}}(t,x)}{U_x(t,x)}$ measured in terms of the marginal utility. Here, $\gamma_x^{\mathcal{R}}(t,x)$ is the projection on $\mathcal{R}_t$ of the diffusion coefficient of the marginal utility $\mathbf{U_x}$. Another important point is that the extremal investment strategy $x\kappa_t^e(x)$ does not depend on the consumption context.
It is then easy to give sufficient conditions on the $\mathbf{U}$-characteristics (β, γ) in order to satisfy the $\mathscr{X}$-consistency condition.

Proposition 3.3 (Two test problems). *Let* $(\mathbf{U}, \mathbf{V})$ *be a "regular utility" system and* (β, γ) *the local characteristics of* $\mathbf{U}$*. The extremal diffusion coefficient is defined by*

$$\sigma^e(t,x) = x\kappa_t^e(x) = -\tfrac{U_x(t,x)}{U_{xx}(t,x)}\big(\eta_t^{\mathcal{R}} + \tfrac{\gamma_x^{\mathcal{R}}(t,x)}{U_x(t,x)}\big).$$

a) First test problem with fixed ρ. *The utility system* $(\mathbf{U}, \mathbf{V})$ *is consistent with the family* *of test processes $\mathscr{X}^\rho = \{X^{\kappa,\rho}, | \kappa \in \mathcal{R}, \rho \text{ given}\}$ if*

$$\beta^\rho(t,x) = -U_x(t,x)x(r_t - \rho_t) + \frac{1}{2}U_{xx}(t,x)\|\sigma^e(t,x)\|^2 - V(t,\rho_t x). \tag{3.13}$$

b) Second test problem with consumption. *The utility system* $(\mathbf{U},\mathbf{V})$ *is consistent with the family of test processes $\mathscr{X}^c = \{X^{\kappa,\rho}, | \kappa \in \mathcal{R}, \text{any process } \rho > 0\}$ if*

$$\beta^c(t,x) = -U_x(t,x)xr_t + \frac{1}{2}U_{xx}(t,x)\|\sigma^e(t,x)\|^2 - \tilde{V}(t,U_x(t,x)) \tag{3.14}$$

where $\tilde{V}(t,z) = \sup_{\rho>0}(V(t,\rho) - z\rho)$ is the Fenchel transform of V. Moreover, the extremal consumption $\rho^e(t,x)x$ is given by $\rho^e(t,x)x = V_z^{-1}(t,U_x(t,x)) = -\tilde{V}_z(t,U_x(t,x))$.

Proof. We proceed by verification.
a) From the decomposition (3.11) of $Z^{\kappa,\rho}$, the drift of $dZ_t^{\kappa,\rho}$ is given by a process $\phi(t,X_t^{\kappa,\rho},\kappa_t,\rho_t)$ where $\phi(t,x,\kappa,\rho) = \beta(t,x) + U_x(t,x)x(r_t - \rho_t) + V(t,\rho_t x) - \mathcal{Q}(t,x,\kappa)$. The supermartingale property is satisfied by a non-positive drift. Since $\mathcal{Q}(t,x,\kappa) \geq \mathcal{Q}^e(t,x)$ when the random field β satisfies the relation (3.13), the function $\phi(t,x,\kappa,\rho)$ is negative for any x, and the consistency relation for the first problem (with fixed ρ) holds true.
Verifying the strong consistency consists in showing the existence of an extremal process X^e, solution of the SDE with volatility function $\kappa^e(t,x)$,

$$dX_t^{e,\rho} = X_t^{e,\rho}[(r_t - \rho_t)\,dt + \kappa^e(t,X_t^{e,\rho}).(dW_t + \eta_t^{\mathcal{R}}\,dt)]$$

and then in proving that the process $Z^{e,\rho}$ is a "true martingale". We come back to this point in the next section.
b) The same arguments can be used to justify the equation (3.14), where now we are also concerned with finding bounds for $\phi(t,x,\kappa,\rho)$ that are valid for any $\rho > 0$. Since the dependence in κ is the same, we only have to control the term $V(t,\rho x) - U_x(t,x)x\rho$. The minimal bound is given by the Fenchel-Legendre conjugate $\tilde{V}$ of the concave function V at $U_x(t,x)$. Then, if $\beta(t,x)$ satisfies the equality (3.14), the drift of $Z_t^{\kappa,\rho}$ is non-increasing and the consistency relation for the second problem holds true. The same remark on the extremal process holds true, with $x\rho$ being replaced by $\rho^e(t,x)x = V_z^{-1}(t,U_x(t,x))$ since $\tilde{V}(t,U_x(t,x)) = V(t,\rho^e(t,x)x) - x\rho^e(t,x)U_x(t,x)$. □

In the sequel, we will mainly consider the second test problem with consumption, as it is related to the standard backward optimization problem. Similar results can be proved for the first test problem with fixed ρ.

3.3.2 Marginal utility of consistent forward utility, extremal coefficients

The interpretation of the HJB-consistency constraints (Equations (3.13) and (3.14)) is not easy to do. In the analysis of the backward optimisation problem (corresponding to the second test problem), Proposition 2.4 equation (2.12) indicates that the marginal utility

$\mathcal{U}_x(t,x)$ is a key tool in the study of optimal processes. This suggests to also study the properties of forward marginal utility $\mathbf{U_x}$ under the consistency constraint.

The local characteristics of the marginal utility $\mathbf{U_x}$ are given by $(\beta_x(t,x),\gamma_x(t,x))$.To facilitate the calculation and the analogy with formula (3.9), we put $F(t,x)=U_x(t,x)$ and $\beta_x(t,x)=\beta^F(t,x)$. For the moment, we make only algebraic calculation, in order to write β^F and γ^F as the drift and the volatility coefficients of a compound process. The constraint (3.14) becomes

$$\begin{cases} \beta^F(t,x) &= \frac{1}{2}\partial_x\big(F_x(t,x)\|x\kappa^e(t,x)\|^2\big)-\partial_x(F(t,x)xr_t)-\partial_x(\tilde{V}(t,F(t,x))) \\ \gamma^F(t,x) &= \gamma_x^{\mathcal{R}}(t,x)+\gamma_x^{\perp}(t,x) \end{cases} \tag{3.15}$$

(i) The first term of the consistency constraint on $\beta^F(t,x)$ suggests to introduce the adjoint operator $\widehat{L}^e_{t,x}$ associated with the extremal coefficients

$$\begin{cases} \widehat{L}^e_{t,x} &= \frac{1}{2}\partial_x(\|\sigma^e(t,x)\|^2\partial_x)-\mu^e(t,x)\partial_x, \quad x\rho^e(t,x)=-\widetilde{V}_y(t,F(t,x)) \\ \mu^e(t,x) &:= r_tx+x\kappa_t^e(x).\eta_t^{\mathcal{R}}-x\rho^e(t,x) \\ \sigma^e(t,x) &= x\kappa_t^e(x)=-\frac{F(t,x)}{F_x(t,x)}\big(\eta_t^{\mathcal{R}}+\frac{\gamma_x^{\mathcal{R}}(t,x)}{F(t,x)}\big) \end{cases} \tag{3.16}$$

Then, since $r_tF_x(t,x)-\partial_x(F(t,x)xr_t)=-r_tF(t,x)$, we see that:

$$F_x(t,x)\mu^e(t,x)-\partial_x(F(t,x)xr_t)-\partial_x(\tilde{V}(t,F(t,x)))=F_x(t,x)\kappa_t^e(x).\eta_t^{\mathcal{R}}-r_tF(t,x).$$

The term $\beta^F(t,x)$ becomes $\beta^F(t,x)=\widehat{L}^e_{t,x}F(t,x)+F_x(t,x)(x\kappa^e(t,x).\eta_t^{\mathcal{R}})-r_tF(t,x)$.

(ii) To remain close to formula (3.9) in terms of diffusion random fields, the idea is to use that

$$\gamma^{F,\mathcal{R}}(t,x)=\gamma_x^{\mathcal{R}}(t,x)=-(F_x(t,x)\sigma^e(t,x)+F(t,x)\eta_t^{\mathcal{R}}).$$

Then, the missing volatility $\sigma^Y(t,z)$ has to satisfy $\sigma^Y(t,F(t,x))=\gamma_x^{\perp}(t,x)-F(t,x)\eta_t^{\mathcal{R}}$, and so

$$\sigma^Y(t,y)=\gamma_x^{\perp}(t,F^{-1}(t,y))-y\eta_t^{\mathcal{R}}. \tag{3.17}$$

To recover the drift constraint in (3.9), the main property is that $\sigma^e(t,x).\sigma^Y_y(t,F(t,x))=-\eta^{\mathcal{R}}.\sigma^e(t,x)$. Moreover the equality will be exact if $\mu^Y(t,y)=-r_ty$.
We summarize this important result in the following theorem.

Theorem 3.4. *Let* $(\mathbf{U},\mathbf{V})$ *be a "regular utility" system, with local characteristic* (β,γ). *The HJB-constraint*

$$\beta(t,x)=-U_x(t,x)xr_t+\frac{1}{2}U_{xx}(t,x)\|x\kappa_t^e(x)\|^2-\tilde{V}(t,U_x(t,x)) \tag{3.18}$$

is equivalent to the following property of the marginal utility $F(t,x):=U_x(t,x)$ *with characteristics* (β_x,γ_x)

$$\begin{cases} \beta_x(t,x)=\mu^Y(t,U_x(t,x))-U_{xx}(t,x)\sigma^e(t,x).\sigma^Y_y(t,U_x(t,x))+\widehat{L}^{(\mu^e,\sigma^e)}_{t,x}(U_x) \\ \gamma_x(t,x)=-U_{xx}(t,x)\sigma_t^e(x)+\sigma^Y(t,U_x(t,x)) \end{cases} \tag{3.19}$$

with

$$\begin{cases} \sigma^e(t,x) &= -\frac{U_x(t,x)}{U_{xx}(t,x)}\big(\eta_t^{\mathcal{R}} + \frac{\gamma_x^{\mathcal{R}}(t,x)}{U_x(t,x)}\big),\ \mu_t^e(x) = r_t x + \sigma_t^e(x).\eta_t^{\mathcal{R}} - x\rho^e(t,x). \\ \sigma^Y(t,y) &= \gamma_x^{\perp}(t, U_x^{-1}(t,y)) - y\eta^{\mathcal{R}}, \quad \mu^Y(t,y) = -r_t y. \end{cases} \tag{3.20}$$

Then, the marginal utility $F(t,x) = U_x(t,x)$ has the characteristics of a compound random field generated by the two SDEs, SDE(μ^e, σ^e) and SDE(μ^Y, σ^Y) (see (3.9)).
In particular, if there exists monotonic solutions X^e and Y^e of these regular SDEs, then

$$U_x(t,x) = Y^e(t, u_x((X^e)^{-1}))(t,x), \quad V_c(t,c) = U_x(t,(x\rho^e(t,x))^{-1})(t,c). \tag{3.21}$$

Observe that the coefficient $\sigma^Y(t,y) = \gamma_x^{\perp}(t, F^{-1}(t,y)) - y\eta^{\mathcal{R}}$ depends on $F^{-1} = -\tilde{U}_y$, which makes naturally appear the link with the dual utility $\tilde{U}$ of U. This point will be developed in the sequel.

4 Forward and backward SPDEs interpretation and resolution

Until now, in order to simplify our approach and to guide the intuition, we have deliberately avoided the technical regularity assumptions required to establish our results. This section makes these assumptions explicit and gives a precise framework to apply Itô-Ventzel's formula, to differentiate regular random fields and SDEs solutions. We also state assumptions under which a SDEs admits a local, strong (non-explosive) and regular solution, as well as necessary assumptions under which the inverse of a SDE solution is regular and also a semimartingale (this result being not true in general).

4.1 The different classes of regularity

We specify here the regularity conditions required in the previous section. For that purpose, let us discuss the regularity of an Itô semimartingale random field $G(t,x) = G(0,x) + \int_0^t \phi(s,x)ds + \int_0^t \psi(s,x).dW_s$ in connection with the regularity of its local characteristics (ϕ,ψ) and conversely.

4.1.1 The spaces of regular processes

Let (ϕ,ψ) be continuous $\mathbb{R}^k$-valued progressive random fields and let m be a non-negative integer, and δ a number in $(0,1]$. We need to control the asymptotic behavior in 0 and ∞ of ϕ and ψ, and the regularity of their Hölder derivatives (when they exist). More precisely, let $\phi \in \mathcal{C}^{m,\delta}(]0,+\infty[)$ be (m,δ)-times[1] continuously differentiable in x for any t, a.s.

[1]That is ϕ is m-times continuously differentiable with $\phi^{(m)}$ being δ-Hölder

For any subset $K \subset]0,+\infty[$, we define the family of random (Hölder) K-semi-norms

$$\left\{ \begin{array}{l} \|\phi\|_{m:K}(t,\omega) = \sup_{x\in K} \frac{\|\phi(t,x,\omega)\|}{x} + \sum_{1\le j\le m} \sup_{x\in K} \|\partial_x^j \phi(t,x,\omega)\| \\ \|\psi\|_{m,\delta:K}(t,\omega) = \|\psi\|_{m:K}(t,\omega) + \sup\limits_{x,y\in K} \dfrac{\|\partial_x^m \psi(t,x,\omega) - \partial_x^m \psi(t,y,\omega)\|}{|x-y|^\delta}. \end{array} \right. \tag{4.1}$$

When K is all the domain $]0,+\infty[$, we simply write $\|.\|_m(t,\omega)$, or $\|.\|_{m,\delta}(t,\omega)$.
Calligraphic notation recalls that these semi-norms are random.
a) $\mathcal{K}_{loc}^{m,\delta}$ (resp. $\overline{\mathcal{K}}_{loc}^{m,\delta}$) denotes the set of all $\mathcal{C}^{m,\delta}$-random fields such that for any compact $K \subset]0,+\infty[$, and any T, $\int_0^T \|\phi\|_{m,\delta:K}(t,\omega)dt < \infty$, (resp. $\int_0^T \|\psi\|^2_{m,\delta:K}(t,\omega)dt < \infty$).
b) When these different norms are well-defined on the whole space $]0,+\infty[$, we use the notations $\mathcal{K}_b^m, \overline{\mathcal{K}}_b^m$ or $\mathcal{K}_b^{m,\delta}, \overline{\mathcal{K}}_b^{m,\delta}$.

4.1.2 Differentiability of Itô random fields and SDEs Solutions

We discuss the regularity of an Itô semimartingale random field

$$G(t,x) = G(0,x) + \int_0^t \phi(s,x)ds + \int_0^t \psi(s,x).dW_s \tag{4.2}$$

in connection with the regularity of its local characteristics (ϕ,ψ). An Itô random field $\mathbf{G}$ is said to be a $\mathcal{K}_{loc}^{m,\delta}$**-semimartingale**, whenever $G(0,x)$ is of class $\mathcal{C}^{m,\delta}$, $B^G(t,x) = \int_0^t \phi(s,x)ds$ is of class $\mathcal{K}_{loc}^{m,\delta}$, and $M^G(t,x) = \int_0^t \psi(s,x).dW_s$ is of class $\overline{\mathcal{K}}_{loc}^{m,\delta}$. As in Kunita [20], we are concerned with the regularity of $\mathbf{G}$ (the regularity of its local characteristics (ϕ,ψ) being given) and conversely with the regularity of (ϕ,ψ) (the regularity of $\mathbf{G}$ being given). Those results are useful to differentiate term by term the dynamics of an Itô random field (as in Theorem 3.4) and to apply Itô-Ventzel's formula.

Theorem 4.1 (Differential Rules). *Let $\delta \in (0,1]$ and $\mathbf{G}$ be an Itô semimartingale random field with local characteristics (ϕ,ψ), $G(t,x) = G(0,x) + \int_0^t \phi(s,x)ds + \int_0^t \psi(s,x).dW_s$*
(i) *If $\mathbf{G}$ is a $\mathcal{K}_{loc}^{m,\delta}$-semimartingale for some $m \ge 0$, its local characteristics (ϕ,ψ) are of class $\mathcal{K}_{loc}^{m,\varepsilon} \times \overline{\mathcal{K}}_{loc}^{m,\varepsilon}$ for any $\varepsilon < \delta$, and conversely.*
(ii) *For $m \ge 1$, the derivative random field $\mathbf{G_x}$ is an Itô random field with local characteristics (ϕ_x,ψ_x), and for $m \ge 2$ the Itô-Ventzel formula is applicable*
(iii) *Moreover, if $\mathbf{G}$ is a $\mathcal{K}_{loc}^{1,\delta} \cap \mathcal{C}^2$-semimartingale, for any Itô process X, $G(.,X_.)$ is a continuous Itô semimartingale satisfying the Itô-Ventzel formula (3.4).*

As previously mentioned, we also need results on the existence and the regularity of one dimensional random fields which are also solutions of stochastic differential equations (SDE). Such random fields are called stochastic flows and are the main subject (in the multidimensional case) of Kunita's book [20].
The question is now to make assumptions on the coefficients in place of local characteristics. The following result justifies (3.7) and Theorem 3.2.

Theorem 4.2 (Flows property of SDE)**.** *We consider a SDE(μ,σ),*

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t).dW_t, \quad X_0 = x. \tag{4.3}$$

Let $m \geq 1, \delta \in (0,1]$ and $\varepsilon < \delta$.
(i) *Assume uniformly Lipschitz coefficients, that is $(\mu,\sigma) \in \mathcal{K}_{\mathbf{b}}^{0,1} \times \overline{\mathcal{K}}_{\mathbf{b}}^{0,1}$. Then, it admits a unique strong solution X which is strictly monotonic satisfying $X(0) = 0$ and $X(+\infty) := \lim_{x\to+\infty} X(x) = +\infty$.*
(ii) *Assume $\mu \in \mathcal{K}_{\mathbf{b}}^{m,\delta}$ and $\sigma \in \overline{\mathcal{K}}_{\mathbf{b}}^{m,\delta}$.*
a) *Then the solution $\mathbf{X} = (X_t(x), x > 0)$ is a $\mathcal{K}_{loc}^{m,\varepsilon}$ semimartingale the derivatives $\mathbf{X_x}$ and $\mathbf{1/X_x}$ are $\mathcal{K}_{loc}^{m-1,\varepsilon}$-semimartingales. Its inverse $\mathbf{X^{-1}}$ is also of class $\mathcal{C}^m$.*
b) *The local characteristics of $\mathbf{X}$, $\lambda^X(t,x) = \mu(t,X_t(x))$ and $\theta^X(t,x) = \sigma(t,X_t(x))$ have only local properties and belong to $\mathcal{K}_{loc}^{m,\varepsilon} \times \overline{\mathcal{K}}_{loc}^{m,\varepsilon}$.*

When the coefficients are only locally Lipschitz, the solution can explode at the explosion time $\tau(x) = \inf\{t; |X_t(x)| = \infty\}$.

Proposition 4.3 (Flows property of SDE with explosion)**.** *Assume that the coefficients are only locally Lipschitz, $(\mu,\sigma) \in \mathcal{K}_{\mathbf{loc}}^{0,1} \times \overline{\mathcal{K}}_{\mathbf{loc}}^{0,1}$.*
(i) *Then, for any initial condition x, the SDE(μ,σ) (4.3) has a unique maximal monotonic solution $(X_t(x))$ up to the explosion time $\tau(x)$. $(X_t(x))$ is a global solution if and only if the explosion time $\tau(x)$ is equal to ∞ for all $x > 0$ a.s..*
In the sequel, we assume at least that $m \geq 1, \delta \in (0,1]$ and $\varepsilon < \delta$.
(ii) *If $(\mu,\sigma) \in \mathcal{K}_{loc}^{m,\delta} \times \overline{\mathcal{K}}_{loc}^{m,\delta}$, $X_t(.)$ is of class $\mathcal{K}^{m,\varepsilon}$ on $\{\tau(x) > t\}$, and for any semi-martingale Y, Itô-Ventzel's formula (3.4) holds true for the compound process $X(Y)$.*
(iii) *When $m \geq 3$ and X non-explosive, the inverse $\mathbf{X^{-1}}$ of $\mathbf{X}$ is a true semimartingale in the class $\mathcal{K}_{loc}^{m-2,\varepsilon}$ and m-times continuously differentiable on x.*

In view of this result we give the definition of the class $\mathcal{S}^{m,\delta}$ of SDEs used in the sequel:
Class $\mathcal{S}^{m,\delta}$: A SDE(μ,σ) with $(\mu,\sigma) \in \mathcal{K}_{loc}^{m,\delta} \times \overline{\mathcal{K}}_{loc}^{m,\delta}$ whose local solution is non explosive is said to be of **class $\mathcal{S}^{m,\delta}$.**
This technical result shows clearly the interest of using Hölder property: the solution is fractionally less regular than the coefficients (going from δ to $\varepsilon < \delta$). Otherwise, if we are only interested with processes of class $\mathcal{K}^m$ (m integer) without worrying about the Hölder's dimension, then we will lose a whole order in the regularity: instead of a solution of class $\mathcal{K}_{loc}^{m,\varepsilon}$, we will only obtain a solution of class $\mathcal{K}_{loc}^{m-1}$.

Remark 4.1. *Under the regularity assumption $X \in \mathcal{S}^{m,\delta}, m \geq 2,\ \delta \in]0,1]$, the inverse flow $X^{-1}(t,z)$ of X is strictly monotonic and is a semimartingale of class $\mathcal{K}_{loc}^{m-2,\delta} \cap \mathcal{C}^m$. Note the loss of regularity from m to $m-2$.*

4.2 Examples of consistent utilities

As in many concave problems, the conjugate problem gives some useful complementary information as we have seen in the backward framework. The forward point of view is also interesting to make explicit.

4.2.1 Conjugate consistent progressive utility with consumption.

Sometimes, as in the backward case with consumption, it is easier to consider the dual problem and the associated SPDE. The consistency constraint with respect to the $\mathscr{Y}$-test set is given by the following definitions, by analogy with the backward case (cf. Proposition 2.4)

(i) $(\tilde{\mathbf{U}}, \tilde{\mathbf{V}})$ *is* $\mathscr{Y}$*-consistent* if for any admissible state price density process $Y^\nu \in \mathscr{Y}$, $\tilde{U}(t, Y_t^\nu) + \int_0^t \tilde{V}(s, Y_s^\nu)ds$ is a submartingale.

(ii) *Extremal price density process.* An admissible process Y^e is extremal if $\tilde{U}(t, Y_t^e) + \int_0^t \tilde{V}(s, Y_s^e)ds$ is a martingale. The existence of such extremal process guarantees the *strong consistency* of the conjugate utility system $(\tilde{\mathbf{U}}, \tilde{\mathbf{V}})$.

Assuming that $\mathbf{U}$ satisfies regular conditions ($\mathbf{U}$ in $\mathcal{K}^{3,\delta}_{loc}$) to guarantee that $\tilde{\mathbf{U}}$ is also a regular Itô random field with local characteristics $\tilde{\beta}(t, y)$ and $\tilde{\gamma}(t, y)$, it is possible to give the conjugate version of Proposition 3.3, as follows:

Proposition 4.4. *Let* $(\tilde{\mathbf{U}}, \tilde{\mathbf{V}})$ *be a regular progressive conjugate utility system with consumption, with with local characteristics* $(\tilde{\beta}, \tilde{\gamma})$*. Define a dual random field* $\tilde{\sigma}^e(t, y) := y\nu_t^e(y) - y\eta_t^{\mathcal{R}}$ *with* $y\nu_t^e(y) := -\frac{\tilde{\gamma}_y^\perp(t,y)}{\tilde{U}_{yy}(t,y)}$ *and a dual drift process* $\tilde{\mu}^e(t, y) = -r_t\, y$*.*

(i) *Then the* $\mathscr{Y}$*-consistency of* $(\tilde{\mathbf{U}}, \tilde{\mathbf{V}})$ *is implied by the following HJB-constraint*

$$\tilde{\beta}(t, y) = y\tilde{U}_y(t, y)r_t - \tilde{V}(t, y) - \frac{1}{2}\tilde{U}_{yy}(t, y)\|\tilde{\sigma}^e(t, y)\|^2 - \tilde{\sigma}^e(t, y).\tilde{\gamma}_y(t, y). \tag{4.4}$$

(ii) *Any solution of the SDE*$(\tilde{\mu}^e, \tilde{\sigma}^e)$ *is a* $\mathscr{Y}$*-extremal price density process.*

Observe that the extremal coefficients of the dual problem $(\tilde{\mu}^e, \tilde{\sigma}^e)$ are exactly (μ^Y, σ^Y) of Theorem 3.4. The idea for the proof is similar to the one of the backward case (Proposition 2.4) and relies on the fact that $U_x(X^e)$ has the same characteristics than a state price density process in $\mathscr{Y}$. A detailed proof can be found in [14]. The result of Proposition 4.4 completes Proposition 3.3, as it provides the interpretation of the orthogonal part of $\gamma_x^\perp$ that appears in the diffusion coefficient σ^Y characterizing the dual extremal process.

The example of decreasing dynamics utility studied in Berrier and Tehranchi [2] is equivalent to the case $\gamma \equiv 0$; the problem is reduced to the pathwise resolution of a *forward linear elliptic PDE*, $\partial_t\tilde{U}_y(t, y) = y\tilde{U}_y(t, y)r_t - \tilde{V}(t, y) - \frac{1}{2}\tilde{U}_{yy}(t, y)\|y\eta_t^{\mathcal{R}}\|^2$. Under additional regularity in time, the problem has been solved by analytical methods based on Widder's Theorem in [2]. In the backward case, it is not easy to find condition on the market to ensure that the value function $\mathcal{U}(t, x)$ is decreasing in t.

4.2.2 Consumption utilities compatible with coherent power utilities

Power utilities with constant risk aversion are the standard framework in the economic literature. They are a useful example in the framework of progressive utilities for its simplicity and its easy interpretation of the parameters. The previous analysis yields to a nice comparison between the forward and backward point of view. Consistent wealth power utilities have been fully characterized in [14] in a framework without consumption. Here, we characterize the consumption progressive utility $\mathbf{V}$ generating progressive power utilities for the general consumption problem with test processes in $\mathscr{X}^c$. The first problem can be studied exactly in the same way. The SPDE point of view is well adapted to this study.

Characteristics of power progressive utility Let us consider a consistent progressive power utility, with risk aversion coefficient $\alpha < 1$, defined as proportional to the initial power utility $u^{(\alpha)}(x) = \frac{x^{1-\alpha}}{1-\alpha}$, whose useful properties are as follows:
$xu^{(\alpha)}_x(x) = x^{1-\alpha} = (1-\alpha)u^{(\alpha)}(x)$ and $x^2 u^{(\alpha)}_{xx}(x) = -\alpha x u^{(\alpha)}_x(x) = -\alpha(1-\alpha)u^{(\alpha)}(x)$.
The dynamics of $U^{(\alpha)}(t,x) = Z_t u^{(\alpha)}(x)$ is driven by the positive Itô process Z with coefficients $(\mu^Z_t = Z_t\, b^Z_t)$ and $(\sigma^Z_t = Z_t\, \delta^Z_t)$ and the local characteristics of $\mathbf{U}^{(\alpha)}$ are:

$$
\begin{aligned}
dZ_t &= Z_t(\, b^Z_t dt + \delta^Z_t .dW_t), \quad Z_0 = 1\\
\beta^{(\alpha)}(t,x) &= \mu^Z_t u^{(\alpha)}(x) = b^Z_t\, U^{(\alpha)}(t,x),\\
\gamma^{(\alpha)}(t,x) &= \sigma^Z_t\, u^{(\alpha)}(x) = \delta^Z_t\, U^{(\alpha)}(t,x), \quad \gamma^{(\alpha)}_x(t,x) = \delta^Z_t U^{(\alpha)}_x(x).
\end{aligned}
$$

The HJB constraint is based on the process $x\kappa^e_t(x)$ defined in (3.12) by

$$\sigma^e(t,x) = x\kappa^e_t(x) = -\frac{U^{(\alpha)}_x(t,x)}{U^{(\alpha)}_{xx}(t,x)}\big(\eta^{\mathcal{R}}_t + \frac{\gamma^{(\alpha,\mathcal{R})}_x}{U^{(\alpha)}_x(t,x)}\big) = \frac{x}{\alpha}\big(\eta^{\mathcal{R}}_t + \frac{\sigma^{Z,\mathcal{R}}_t}{Z_t}\big) = \frac{x}{\alpha}(\eta^{\mathcal{R}}_t + \delta^{Z,\mathcal{R}}_t).$$

The extremal strategy $\kappa^e_t(x)$ which does not depend on x, is a constant investment equal to the relative risk tolerance coefficient $-\frac{U^{(\alpha)}_x(t,x)}{x\, U^{(\alpha)}_{xx}(t,x)} = 1/\alpha$ in the modified risk premium vector $\eta^{\mathcal{R}}_t + \delta^{Z,\mathcal{R}}_t$. Here, the utility risk premium is the vector $\delta^{Z,\mathcal{R}}_t$ (independent of α). Recall that only the projection of $\delta^{Z,\mathcal{R}}_t$ on the vector $\eta^{\mathcal{R}}_t$ contributes to the return of the extremal process X^e.

IDENTIFICATION OF CONSISTENT CONSUMPTION UTILITY $\mathbf{V}$.

For the *general consumption problem*, the determination of the utility $\mathbf{V}$ follows from the HJB constraint (3.14), $\beta(t,x) = -U_x(t,x)xr_t + \frac{1}{2}U_{xx}(t,x)\|\sigma^e(t,x)\|^2 - \tilde{V}(t, U_x(t,x))$,

$$b^Z_t U^{(\alpha)}(t,x) = U^{(\alpha)}(t,x)\big[-(1-\alpha)r_t - \frac{1-\alpha}{2\alpha}\|\eta^{\mathcal{R}}_t + \delta^{Z,\mathcal{R}}_t\|^2\big] - \tilde{V}(t, U^{(\alpha)}_x(t,x)).$$

The constraint implies that $\tilde{V}(t, U^{(\alpha)}_x(t,x)) = \bar{v}_t\, U^{(\alpha)}(t,x)$, with $\bar{v}_t$

$$\bar{v}_t = -\big[b^Z_t + (1-\alpha)r_t + \frac{1-\alpha}{2\alpha}\|\eta^{\mathcal{R}}_t + \delta^{Z,\mathcal{R}}_t)\|^2\big] \geq 0. \tag{4.5}$$

Differentiating the equality $\tilde{V}(t, U^{(\alpha)}_x(t,x)) = \bar{v}_t\, U^{(\alpha)}(t,x)$, yields

$$-\tilde{V}_y(t, U^{(\alpha)}_x(t,x)) = \bar{v}_t U^{(\alpha)}_x(t,x)/U^{(\alpha)}_{xx}(t,x) = \bar{v}_t x/\alpha \qquad ,$$

Since $V_c(t, -\tilde{V}_y(t,y)) = y$, we obtain that $U_x^{(\alpha)}(t,x) = V_c(t, \bar{v}_t x/\alpha)$, and integrating leads to the characterization of $\mathbf{V}$ as $V(t,c) = \frac{\bar{v}_t}{\alpha} U^{(\alpha)}(t, \frac{\alpha}{\bar{v}_t} c) = (\frac{\bar{v}_t}{\alpha})^\alpha U^{(\alpha)}(t,c)$.

Recall that from Theorem 3.4, the other extremal coefficients are

$$\begin{cases} \mu_t^e(x) & = r_t x + \sigma_t^e(x).\eta_t^{\mathcal{R}} - x\rho^e(t,x), \quad x\rho^e(t,x) = \tilde{V}_y(t, U_x^{(\alpha)}(t,x) = \bar{v}_t x/\alpha. \\ \sigma^Y(t,y) & = \gamma_x^\perp(t, (U_x^{(\alpha)})^{-1}(t,y)) - y\eta^{\mathcal{R}}, \quad \mu^Y(t,y) = -r_t y. \end{cases} \tag{4.6}$$

All of them are linear function of x that implies that the linear extremal processes are linear in their initial condition. It remains to note that the representation of the marginal utility in terms of extremal processes X^e and Y^e takes a very simple form as $Z_t = Y_t^e (X_t^e)^\alpha$. We put together all these results in the following proposition.

Proposition 4.5. *A consumption consistent progressive power utility system is necessarily a pair of power utilities with the same risk aversion coefficient α such that*

$$U^{(\alpha)}(t,x) = Z_t \frac{x^{1-\alpha}}{1-\alpha} = Z_t u^{(\alpha)}(x) \quad \text{and} \quad V^{(\alpha)}(t,c) = (\tfrac{\bar{v}_t}{\alpha})^\alpha U^{(\alpha)}(t,x).$$

whose coefficient Z_t satisfies the HJB drift constraint,

$$dZ_t = -Z_t \big[((1-\alpha) r_t + \frac{1-\alpha}{2\alpha} \|\eta_t^{\mathcal{R}} + \delta_t^{Z,\mathcal{R}}\|^2 + \bar{v}_t) dt - \delta_t^Z . dW_t \big]. \tag{4.7}$$

The extremal processes are linear with respect of their initial condition, i.e.

$$X_t^e(x) = x X_t^e, \quad Y_t^e(y) = y Y_t^e, \text{ and } \quad c_t^e(z) = z\, \rho_t^e = z\, \tfrac{\bar{v}_t}{\alpha}$$

$$dX_t^e = X_t^e \big((r_t - \rho_t^e) dt + \kappa_t^e . (dW_t + \eta_t^{\mathcal{R}}) \big),\ dY_t^e = Y_t^e \big(-r_t dt + (\nu_t^e - \eta_t^{\mathcal{R}}) dW_t \big). \tag{4.8}$$

Moreover $Z_t = Y_t^e (X_t^e)^\alpha = Y_t^e u_x^{(\alpha)}(X_t^e)$.

It is then easy to study the backward case, where the power utility $\mathbf{U}$ is the value function of a consumption-portfolio optimization problem.

Backward consumption-investment problem with power utilities Consider now a backward consumption-investment power problem, whose terminal wealth utility and consumption utility are power utility functions, with the same risk aversion coefficient, since this condition is necessary. Remark that all the previous results still hold, as they are a consequence of the HJB-constraint, valid for both forward and backward cases. There is two main differences: first the function $V(t,x)$ is given as $V(t,x) = \phi_t u^{(\alpha)}(t,x)$, where ϕ_t is a given Itô process, deterministic in the classic case; second, the value function $\mathcal{U}$ is "unknown", but easily identified to a power utility $Z_t u^{(\alpha)}(t,x)$, since this condition is true at maturity T_H, with terminal value Z_{T_H}.
So, the difference between the two points of view lies on the definition of Z: in the forward point of view, Z is given as satisfying HJB constraints and the only free parameter is the volatility δ^Z of Z; in the backward point view only the terminal value of Z is given, and the two processes (Z, δ^Z) has to be identified together, since b^Z is given by the HJB constraint. The first step is to use the consumption constraint induced by the

assumption that $V(t,x) = \phi_t u^{(\alpha)}(t,x)$ where ϕ is given. By Proposition 4.5, $V(t,x)$ must be proportional to the value function, that implied that $\bar{v}_t = \alpha(\frac{Z_t}{\phi_t})^{-1/\alpha}$. The dynamics of Z is given by

$$-\,dZ_t = Z_t\big[((1-\alpha)r_t + \frac{1-\alpha}{2\alpha}\|\eta^{\mathcal{R}}_t + \delta^{Z,\mathcal{R}}_t\|^2 + \alpha(\frac{Z_t}{\phi_t})^{-1/\alpha})dt - \delta^Z_t.dW_t\big]. \qquad (4.9)$$

This kind of relation is typical of backward stochastic equation with given terminal value, where the solution is the pair of processes (Z, δ^Z) satisfying (4.9). A huge literature is dedicated to the application of BSDEs in finance, see for example El Karoui, Peng and Quenez [17]. The backward equation associated with power utility is of quadratic type, since the drift depends in a quadratic form of the volatility δ^Z. An additional difficulty comes from the presence of negative power of Z in the drift. Nevertheless, a solution may be probably found by approximation (see [1]). As a conclusion,

Proposition 4.6. *Consider a consumption-portfolio optimization problem with time horizon T_H, terminal wealth utility $\zeta_{T_H} u^{(\alpha)}(x)$, and consumption utility $v(t,c) = \phi_t u^{(\alpha)}(x)$, where ϕ_t is a given process. Assume the value function $\mathcal{U}(0,x)$ well-defined at time 0. The value system $(\mathcal{U}(t,x), v(t,c))$ is a consistent power utility system $(Z_t u^{(\alpha)}(x), \phi_t u^{(\alpha)}(x))$, if there exists a solution (Z_t, δ^Z_t) of the backward stochastic equation*

$$dZ_t = -Z_t\big[((1-\alpha)r_t + \frac{1-\alpha}{2\alpha}\|\eta^{\mathcal{R}}_t + \delta^{Z,\mathcal{R}}_t\|^2 + \bar{v}_t)dt - \delta^Z_t.dW_t\big], \quad Z_{T_H} = \zeta_{T_H}. \quad (4.10)$$

Then all the properties given in the forward case hold true.

The HJB-SPDE problem is reduced to solve a one-dimensional quadratic BSDE.

4.3 Forward-Backward HJB-SPDE and its resolution

In the power case, the utility criterion is separable in time and (wealth, consumption) and all the uncertainty is supported by the time component Z_t. Then, the HJB-constraint can be reduced into a non linear Forward or Backward SDE for the process Z. The study can be extended to a general framework by using HJB-SPDE instead of SDE.

4.3.1 Definition of the Forward-Backward HJB-SPDE

The consistency conditions (3.13) and (3.14) can be interpreted in terms of stochastic PDEs. The drift operator $\beta(t,x) = F(t,x,U_x,U_{xx},\widetilde{V},\gamma_x)$ of the HJB-type PDE is highly non-linear, but this non-linearity is essentially due to the extremal coefficients, $x\kappa^e$ for the diffusion term and $x\rho^e$ for the consumption term, since

$$\left\{\begin{array}{l} \beta(t,x) = F(t,x,U_x,U_{xx},\widetilde{V},\gamma_x) = -U_x(t,x)xr_t + \frac{1}{2}U_{xx}(t,x)\|\sigma^e(t,x)\|^2 - \tilde{V}(t,U_x(t,x)) \\ \sigma^e(t,x) = x\kappa^e_t(x) = -\frac{U_x(t,x)}{U_{xx}(t,x)}\big(\eta^{\mathcal{R}}_t + \frac{\gamma^{\mathcal{R}}_x(t,x)}{U_x(t,x)}\big) \\ \tilde{V}(t,z) = \sup_{\rho>0}(V(t,\rho) - z\rho),\ \rho^e(t,x)x = -\tilde{V}_y(t,U_x(t,x)) = V_c^{-1}(t,U_x(t,x)). \end{array}\right. \qquad (4.11)$$

A precise definition of the solution requires to distinguish between the forward or backward point of view as for simple SDE.

Definition 4.7 (SPDE solution)**.** *Formally, the "HJB"-stochastic PDE with diffusion random field $\gamma(t,x)$, reads as:*

$$dU(t,x) = F(t,x,U_x,U_{xx},\tilde{V},\gamma_x)(t,x) + \gamma(t,x).dW_t \tag{4.12}$$

FORWARD SPDE SOLUTION: *The data are a diffusion random field $\gamma(t,x)$, an initial utility function u and a consumption progressive utility $V(t,c)$ with conjuguate $\tilde{V}(t,c)$.*
A "regular" solution of the forward *non linear HJB-SPDE (4.12) is a random field* $\mathbf{U}$ *which is a progressive utility random field with initial condition $U(0,x) = u(x)$, and diffusion random field $\gamma(t,x)$.*
BACKWARD SPDE SOLUTION: *The data are an horizon T_H, and a terminal (random) utility at the horizon, $u(T_H,x)$, a (stochastic) consumption utility $v(t,c)$, and an adapted process θ_t. Then, a* backward *solution is a triple $(U(t,x),\theta(t,x),\theta_t)$, solution of the following HJB-SPDE with terminal condition $U(T_H,x) = u(T_H,x)$, whose component $U(t,x)$ is a progressive utility, whose diffusion random field is $\Theta(t,x) = \theta_t + \int_0^x \theta(t,z)dz$*

$$dU(t,x) = F(t,x,U_x,U_{xx},\tilde{V},\theta)(t,x) + \Theta(t,x).dW_t. \tag{4.13}$$

The two points of view yield to stochastic utility consistent with the family of test processes $\mathscr{X}^c$, under additional regularity assumption.

The particular representation given in the backward case shows that the γ-regularity is not necessary as in the forward case. The main information is given by the process γ_x.

The next section is dedicated to define the regularity needed to show existence of both the solution $\mathbf{U}$ and the extremal processes.

4.3.2 Extremal processes and forward utility characterization

We are concerned here with the existence of extremal processes and thus of *strongly consistent* progressive utilities. Following Theorem 3.4 and the decomposition of the HJB-SPDE in terms of the SDE(μ^e,σ^e) and SDE(μ^Y,σ^Y), with

$$\begin{cases} \sigma^e(t,x) &= -\frac{U_x(t,x)}{U_{xx}(t,x)}\big(\eta_t^{\mathcal{R}} + \frac{\gamma_x^{\mathcal{R}}(t,x)}{U_x(t,x)}\big),\ \mu_t^e(x) = r_t x + \sigma_t^e(x).\eta_t^{\mathcal{R}} - x\rho^e(t,x). \\ \sigma^Y(t,y) &= \tilde{\sigma}^e(t,y) = \gamma_x^{\perp}(t,U_x^{-1}(t,y)) - y\eta^{\mathcal{R}}, \quad \mu^Y(t,y) = \tilde{\mu}^e(t,y) = -r_t y \end{cases}$$

the first step consists in providing regularity assumptions under which these "extremal" SDEs admit solutions. Note that in view of Subsection 4.2.1 (Proposition 4.4), the SDE(μ^Y,σ^Y) may be reinterpreted as the extremal dual SDE$(\tilde{\mu}^e,\tilde{\sigma}^e)$. Considering the form of the extremal policies, we see that the regularity of $\mathbf{U}$ or that of $\tilde{\mathbf{U}}$ plays an important role (as the asymptotic elasticity introduced for backward problems). Assuming

$\mathcal{K}^{2,\delta}_{loc} \cap \mathcal{C}^3$-regularity for $\mathbf{U}$ and a linear growth for the consumption rate insures that the wealth SDE(σ^e, μ^e) has locally Lipschitz coefficients and thus a strong solution X^e exists up to an explosion time τ^e. Then observing that $U_x(t, X^e_t)$ is solution of the SDE$(\tilde{\mu}^e, \tilde{\sigma}^e)$, we first show the existence of a regular strong solution of the optimal dual SDE$(\tilde{\mu}^e, \tilde{\sigma}^e)$ whose coefficients are based only on the diffusion characteristics γ_x of $\mathbf{U}_x$. This allows us to give refined regularity assumptions under which there is no explosion for X^e ($\tau^e = +\infty$).

Theorem 4.8. *Let* $\mathbf{U}$ *be a* $\mathcal{K}^{2,\delta}_{loc} \cap \mathcal{C}^3$*-regular*$(\delta > 0)$ *progressive utility, whose local characteristics* (β, γ) *satisfy the HJB-constraint*

$$\beta(t,x) = -U_x(t,x)xr_t + \frac{1}{2}U_{xx}(t,x)\|\sigma^e(t,x)\|^2 - \tilde{V}(t, U_x(t,x)). \tag{4.14}$$

Existence of Extremal processes: (i) *Suppose the existence of two adapted bounds* $(K^1, K^2) \in L^2(dt)$ *such that the regular random field* $\gamma^\perp_x$ *satisfies*

$$\|\gamma^\perp_x(t,x)\| \leq K^1_t\,|U_x(t,x)|,\ \|\gamma^\perp_{xx}(t,x)\| \leq K^2_t\,|U_{xx}(t,x)|,\ a.s. \tag{4.15}$$

Then the extremal dual SDE$(\tilde{\mu}^e, \tilde{\sigma}^e)$ *is uniformly Lipschitz and its unique strong solution* $Y^e_t(y)$ *is increasing, with range* $[0, \infty)$.
(ii) *Moreover, assume the existence of an adapted bound* K^3 *such that process* $V_c(t, K^3 x) \geq U_x(t,x)$ *a.s. for any* x*. Then the SDE*(μ^e, σ^e) *of Theorem 3.4 admits an unique increasing strong solution* $X^e(x)$ *with range* $[0, \infty)$.
Strong Consistency : *The random field* $(\mathbf{U}, \mathbf{V})$ *is a* **strongly consistent** *utility of consumption and wealth, with extremal process* X^e.

Remark: The existence of extremal processes is induced by the constraint (4.15) and the regularity of $\mathbf{U}$. Therefore, Assumption (4.15) replaces the Asymptotic Elasticity condition needed in the backward case to ensure the existence of optimal processes.

Proof. (i) The $\mathcal{K}^{2,\delta}_{loc} \cap \mathcal{C}^3$-regularity of U and the assumption on $\gamma^\perp_{xx}$ insure that the coefficients (σ^e, μ^e) are locally Lipschitz coefficients and so that the strong solution X^e exists only up to an explosion time τ^e; on the other hand, Assumption (4.15) implies that $(\tilde{\mu}^e, \tilde{\sigma}^e)$ are globally Lipschitz with linear growth and that the SDE$(\tilde{\mu}^e, \tilde{\sigma}^e)$ has a unique monotonic strong solution $Y^e_t(y)$ (Theorem 4.2).
(ii) From regularity of $\mathbf{U}$, the wealth SDE(μ^e, σ^e) has coefficients that are locally Lipschitz, with linear growth since $x\rho^e(t,x) = (V_c)^{-1}(t, U_x(t,x)) \leq K^3_t x$. Then, by Proposition 4.3, a strong solution X^e exists up to a explosion time $\tau^e(x)$. Nevertheless, we can apply Itô-Ventzel's formula to the marginal utility function $\mathbf{U_x}$ and X^e. By the flows composition formula (3.6), it is easy to verify that $U_x(t, X^e_t)$ is solution of the SDE$(\tilde{\mu}^e, \tilde{\sigma}^e)$ and thus is equal to its unique monotonic solution. As a by-product $X^e_t(x)$ is a non explosive monotonic solution of the SDE(μ^e, σ^e). □

A direct consequence of the previous theorem is the following result:

Corollary 4.9. *Under Assumptions of Theorem 4.8, the marginal utility functions* $\mathbf{U_x}$ *and* $\mathbf{V_c}$ *are indistinguishable from*

$$U_x(t,x) = Y_t^e\big(u_x((X_t^e)^{-1}(x))\big), \quad V_c(t,c) = U_x(t,(x\rho^e(t,x))^{-1}(t,c)). \tag{4.16}$$

The optimal consumption along the optimal wealth process is monotonic and is given by

$$c_t^e(x) = X_t^e(x)\rho^e(t,X_t^e(x)) = -\widetilde{V}_y\big(t,U_x(t,X_t^e(x))\big) = -\widetilde{V}_y\big(t,Y_t^e(u_x(x))\big).$$

4.3.3 Reverse Engineering Problem

These ideas were first developped in [13]. Corollary 4.9 and Theorem 3.2 suggest that X^e and Y^e should be replaced by "regular " monotonic processes $X \in \mathscr{X}^c$ and $Y \in \mathcal{Y}$. The composition formula $Y_t\big(u_x((X_t)^{-1}(x))\big) = H(t,x)$ could be interpreted as providing the solution of a reverse engineering problem, since H is a decreasing random field which satisfies the marginal utility SPDE of a consistent utility (see Theorem 3.2). In particular, if Y is regular enough to apply Ito's Ventzel formula and if the inverse ξ of X is a regular semimartingale associated with the well-defined adjoint operator $\widehat{L}_{t,x}^X$, then the problem is solved. The next theorem specifies under which regularity assumption on the SDE's coefficients, these conditions are satisfied. The justification is given in Theorem 4.2 and Proposition 4.3.

Theorem 4.10. *Let* $\kappa \in \mathcal{R}$ *be a volatility vector and* ρ *a positive random field such that* $x\rho(t,x)$ *is increasing. Define* $\sigma(t,x) = x\kappa_t(x)$ *and* $\mu_t(x) = r_t x + \sigma(t,x).\eta_t^{\mathcal{R}} - x\rho(t,x)$, $\tilde{\mu}(t,y) = -yr_t$, *and* $\tilde{\sigma}(t,y) = y(\nu_t(y) - \eta_t^{\mathcal{R}})$, $\nu \in \mathcal{R}^{\perp}$.
a) *Assume the SDE*(μ,σ) *in the class* $\mathcal{S}^{3,\delta}$ *so that the unique solution* $(X_t(x))$ *is monotonic and its inverse* $\mathcal{X}_t(x)$ *is a regular Itô random field.*
b) *Assume the SDE*$(\tilde{\mu},\tilde{\sigma})$ *in the class* $\mathcal{S}^{2,\delta}$ *with monotonic solution* Y.
Main result *For any initial utility function* u, *the stochastic random fields* $\mathbf{F}$ *and* $\mathbf{W}$ *are defined by*

$$F(t,x) = Y_t\big(u_x(\mathcal{X}_t(x))\big), \quad W(t,c) = F(t,(x\rho(t,x))^{-1}(c)). \tag{4.17}$$

If $F(t,x)$ *and* $W(t,c)$ *are integrable near to zero, then* $\big(\mathbf{F},\mathbf{W}\big)$ *is the derivative of a* $\mathscr{X}^c$*- strongly consistent stochastic utility system* $(\mathbf{U},\mathbf{V})$, *and* $U(t,x) = \int_0^x F(t,z)dz$, *and* $V(t,c) = \int_0^c F(t,(x\rho(t,x))^{-1}(z))dz$, *whose extremal processes are* X *and* Y.

Remark The assumptions about X may be weakened: we can only assume the SDE(μ,σ) in the class $\mathcal{S}^{1,\delta}$, and the existence of a solution $\mathcal{X}$ of the SPDE$(\widehat{L}_{t,x}, -\sigma\partial_x)$ associated with the adjoint operator of $\mathcal{X}$.

Conclusion This paper provides an intuitive and tractable framework of market consistent progressive utilities of investment and consumption, emphasizing the similarities and the different viewpoints between the backward and the forward approach. The next step consists in studying the implications, especially concerning long term economic issues, such as for example long term yield curves modeling. Inspired by the economic literature, we can provide a financial interpretation of the Ramsey rule that links endogenous discount rate and marginal utility of aggregate optimal consumption at equilibrium. For such a long term modeling, the possibility of adjusting preferences to new economic information is crucial, as well as been able of identifying the utility associated to given extremal processes. This can be achieved by means of consistent progressive utility.

References

[1] P. Barrieu and N. El Karoui. Monotone stability of quadratic semimartingales with applications to unbounded general quadratic bsdes. *The Annals of Probability*, 41(3B):1831–1863, 2013. 24

[2] F.P. Berrier and M.R. Tehranchi. Forward utility of investment and consumption. *Preprint*, 2011. 2, 9, 12, 21

[3] T. Björk. Equilibrium theory in continuous time. *Lecture Notes*, 2012. 2

[4] J.C. Cox, J.C. Ingersoll, and S.A. Ross. An intertemporal general equilibrium model of asset prices. *Econometrica*, 53(2):363–384, March 1985. 2

[5] J.C. Cox, J.C. Ingersoll, and S.A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–403, March 1985. 2

[6] C. Dellacherie and P.-A. Meyer. *Probablités et Potentiel*, volume Chapitres I à IV. Hermann, Paris, 1975. 6

[7] J.C. Hourcade and F. Lecocq. Leçons à partir du cas des politiques climatiques: le taux d'actualisation contre le principe de précaution? *L'actualité Economique*, 80(1)(1):41–65, January 2004. 9

[8] I. Karatzas, J.P. Lehoczky, and S.E. Shreve. Optimal portfolio and consumption decisions for a small investor on a finite horizon. *SIAM J. Control and Optimization*, 25:1557–1586, 1987. 4

[9] I. Karatzas, J.P. Lehoczky, S.E. Shreve, and G.L.Xu. Martingale and duality methods for utility maximization in an incomplete market. *SIAM J. Control Optim.*, 29(3):702–730, 1991. 8

[10] I. Karatzas and S.E. Shreve. *Methods of Mathematical Finance*. Springer, September 2001. 4, 6

[11] I. Karatzas and G. Žitković. Optimal consumption from investment and random endowment in incomplete semimartingale markets. *The Annals of Probability*, 31(4):1821–1858, 2003. 5, 6, 10

[12] N. El Karoui. Les aspects probabilistes du contrôle stochastique. In *Ninth Saint Flour Probability Summer School—1979 (Saint Flour, 1979)*, volume 876 of *Lecture Notes in Math.*, pages 73–238. Springer, Berlin, 1981. 6

[13] N. El Karoui and M. Mrad. An Exact connection between two Solvable SDEs and a Non Linear Utility Stochastic PDEs. *SIAM Journal on Financial Mathematics,*, 4(1):697–736, 2013. 2, 12, 13, 15, 27

[14] N. El Karoui and M. Mrad. Mixture of consistent stochastic utilities, and a priori randomness. *Preprint.*, 2016. 21, 22

[15] N. El Karoui and M. Mrad. Stochastic utilities with a given optimal portfolio : approach by stochastic flows. *Preprint.*, 2016. 2

[16] N. El Karoui, E. Pardoux, and M. Yor. In *Stochastic Filtering at Saint Flour*, Probability at Saint Flour, chapter Les aspects probabilistes du contrôle stochastique, pages 43–203. Springer, Berlin, 2012. 6

[17] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Mathematical finance*, 7(1):1–71, 1997. 24

[18] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9(3):904–950, 1999. 6

[19] D. Kramkov and W. Schachermayer. Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Annals of Applied Probability*, 13(4):1504–1516, 2003. 5, 6

[20] H. Kunita. *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Reprint of the 1990 original. 12, 19

[21] M. Mania and R. Tevzadze. Backward stochastic pdes related to the utility maximization problem. *Georg. Math.J.*, 17, 2010. 2, 8, 12

[22] M. Musiela and T. Zariphopoulou. Investment and valuation under backward and forward dynamic exponential utilities in a stochastic factor model. In *Advances in mathematical finance*, pages 303–334. Birkhäuser Boston, 2007. 2, 9, 12

[23] M. Musiela and T. Zariphopoulou. Stochastic partial differential equations in portfolio choice. *Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen, C. Chiarella and A. Novikov (Eds)*, pages 161–170, 2010. 2, 9, 12

[24] S. Nadtochiy and M. Tehranchi. Optimal investment for all horizons and Martin boundary. *Mathematical Finance*, (Published online), 2015. 9

[25] C. Skiadas. Dynamic portfolio choice and risk aversion. *Handbooks in Operations Research and Management Science*, 15:789–843, 2007. 4