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**A.CABRALES<sup>1</sup>**

**O.GOSSNER<sup>2</sup>**

**R.SERRANO<sup>3</sup>**

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<sup>1</sup> Department of Economics, University College London, UK. E-mail : a.cabrales@ucl.ac.uk

<sup>2</sup> CNRS; École Polytechnique; LSE. E-mail : olivier.gossner@polytechnique.edu

<sup>3</sup> Brown university. E-mail : roberto\_serrano@brown.edu

# A normalized value for information purchases <sup>☆</sup>

Antonio Cabrales <sup>a,\*</sup>, Olivier Gossner <sup>b,c</sup>, Roberto Serrano <sup>d</sup>

<sup>a</sup> *Department of Economics, University College London, UK*

<sup>b</sup> *CNRS – École Polytechnique, France*

<sup>c</sup> *Department of Mathematics, London School of Economics, UK*

<sup>d</sup> *Department of Economics, Brown University, USA*

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## Abstract

Consider agents who are heterogeneous in their preferences and wealth levels. These agents may acquire information prior to choosing an investment that has a property of no-arbitrage, and each piece of information bears a corresponding cost. We associate a numeric index to each information purchase (information-cost pair). This index describes the normalized value of the information purchase: it is the risk-aversion level of the unique CARA agent who is indifferent between accepting and rejecting the purchase, and it is characterized by a “duality” principle that states that agents with a stronger preference for information should engage more often in information purchases. No agent more risk-averse than the index finds it profitable to acquire the information, whereas all agents less risk-averse than the index do. Given an empirically measured range of degrees of risk aversion in a competitive economy with no-arbitrage investments, our model therefore comes close to describing an inverse demand for information, by predicting what pieces of information are acquired by agents and which ones are not. Among several desirable properties, the normalized-value formula induces a complete ranking of information structures that extends Blackwell’s classic ordering.

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\* Corresponding author.

*E-mail addresses:* [a.cabrales@ucl.ac.uk](mailto:a.cabrales@ucl.ac.uk) (A. Cabrales), [olivier.gossner@polytechnique.edu](mailto:olivier.gossner@polytechnique.edu) (O. Gossner), [roberto\\_serrano@brown.edu](mailto:roberto_serrano@brown.edu) (R. Serrano).

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## 1. Introduction

We refer to any pair consisting of an information structure and a price for it as an *information purchase*. Such purchases, if they happen, are the manifestation of the demand for information. How many people purchase a piece of information necessarily depends on three components: the quality of that information, the cost of acquiring it, and the agents' primitives given by their wealth and preferences. The current paper aims at answering the following questions. First, given an information purchase, can its normalized value, which captures the information-price tradeoff, be uniquely characterized?<sup>1</sup> Second, in a competitive economy, who are the agents willing to go ahead with a given information purchase?

We answer these questions by analyzing information purchases made by risk-averse agents (or investors) prior to choosing among risky investments. Key to our analysis is the set of investments available which we call *no-arbitrage investments*. In order to study information acquisition based on investment motives, we assume that no-arbitrage investments are not profitable under the agents' prior; the only investors who find investments profitable are the ones who acquire some information. Second, we note that for the value of information to be meaningful and comparable across heterogeneous agents, one needs the set of available investments to be rich enough; in particular, we assume the existence of complete markets.<sup>2</sup> We follow the literature on this subject (such as Kelly, 1956; Arrow, 1971) and take as *no-arbitrage investments* the set of all those with a nonpositive expected return under the prior.<sup>3</sup>

We begin by showing that an agent's demand, or her preference, for information is characterized by her degree of risk aversion. Less risk-averse agents have a stronger preference for information than do more risk-averse agents, in the following sense. We show that an agent  $u_1$  is uniformly less risk-averse than agent  $u_2$  if and only if the fact that agent  $u_2$  acquires some information is enough to guarantee that agent  $u_1$  also acquires that information, independently of the wealth levels considered.<sup>4</sup> Therefore, agents' demand for information in our model is entirely captured by their uniform ranking of risk aversion.

We seek an objective underpinning of normalized values. That is, paralleling the approach of Aumann and Serrano (2008) for ordering riskiness, we apply the following duality principle to define the normalized value of an information purchase<sup>5</sup>: For an information purchase to be considered as objectively more valuable than another one, it must be the case that, whenever

<sup>1</sup> We add the “normalized” qualification because the index will also rely on the price of the purchase, and not be based only on the information structure. The term “normalized informativeness” could also be used to refer to our index.

<sup>2</sup> For instance, some agents find it beneficial to have access to a certain futures market, while others do not value information about this market as much.

<sup>3</sup> No-arbitrage investments, in this sense of not offering any profitable or “arbitrage” opportunity, were also used in Cabrales et al. (2013). “No-speculative” investments could be an alternative name to express the same idea.

<sup>4</sup> To be precise,  $u_1$  is uniformly less risk-averse than  $u_2$  when, for all wealth levels  $w_1$  and  $w_2$ , the coefficient of absolute risk aversion of  $u_1$  at  $w_1$  is not greater than the coefficient of risk aversion of  $u_2$  at  $w_2$ .

<sup>5</sup> In Aumann and Serrano (2008), riskiness is conceived as “dual” to risk aversion, while here the value of information is “dual” to preference for information.

an agent is willing to accept the latter, every agent with a stronger preference for information must, a fortiori, accept the former. To be more precise, when we say that “ $u_1$  likes information better than  $u_2$ ,” we mean “If  $u_2$  accepts a purchase at some wealth level, then  $u_1$  accepts it at any wealth level” (uniform comparison). For this ordering, we introduce a suitable corresponding ordering of information purchases according to the duality principle described above: “If  $u_1$  likes information better than  $u_2$ , and if  $a_1$  is more valuable than  $a_2$ , then  $u_1$  should accept  $a_1$  if  $u_2$  accepts  $a_2$ .” We show that this yields a complete ordering of information purchases, which is characterized by our normalized-value formula. The normalized value of the purchase turns out to be equal to the risk aversion of the unique CARA (constant absolute risk aversion) agent who is indifferent between accepting and rejecting it. Such a critical level of risk aversion is expressed as a specific function of relative entropies and the price of the purchase, where the function is increasing with respect to the former and decreasing with respect to the latter.<sup>6</sup>

The fact that a purchase has a higher normalized value than another one does not mean that more agents will accept it. Rather, a purchase having a higher normalized value is equivalent to that purchase having a larger set of agents who accept it regardless of their initial wealth. Thus, for CARA agents, the more valuable a purchase is, the more of them accept it. But more importantly, given our results connecting preferences for information and risk aversion in our settings, any agent (CARA or not) whose risk aversion always exceeds the normalized value will reject the purchase, while any agent whose risk aversion is less than the normalized value for all wealth levels, will accept it. The remaining agents, whose risk aversion for some initial wealth level is below the normalized value but above it for other wealth levels, may accept or reject the purchase; in other words, their decision is “subject to wealth effects.” Were it not for this latter set of agents, the entire set of agents who accept an information purchase would be monotonically increasing in the normalized value of the purchase. Having said that, if we assume that agents are sufficiently heterogeneous in their wealth, and that we can know the degree of risk aversion of agents, then the normalized value of information purchases is a useful tool that delivers a clear prediction of what pieces of information will be acquired in a competitive economy (i.e., in which each investor views herself as so small that her information purchase decision does not signal any relevant information to the rest of the economy).<sup>7</sup>

Our normalized-value measure provides some interesting insights on the demand for information. For instance, a decrease by a certain percentage in the cost of information translates into an increase in the same proportion in the normalized value of the corresponding information purchase. This means that, whenever the price of information drops by half, agents who are twice more risk-averse than those who initially bought a piece of information, become now willing to acquire that piece of information. Another insight is obtained by examining the least and most valuable purchases. Quite intuitively, the least valuable purchases are the ones associated to those with null informational content. Perhaps surprisingly, the most valuable

<sup>6</sup> In the appendix, we provide two alternative definitions of preference for information: (i) “If  $u_2$  accepts a purchase at some wealth level, then  $u_1$  accepts it at some wealth level” (minimal comparison); and (ii) “If  $u_2$  accepts at some wealth level  $w$ , then  $u_1$  also accepts at  $w$ , for every  $w$ ” (wealth-wise comparison). We formulate the corresponding orderings based on the duality principle. Strikingly, all three orderings of information purchases coincide: all three are represented by our normalized-value formula. As we also show in the appendix, another characterization of the normalized value is expressed in terms of the group of agents who are willing to accept a given information purchase. This parallels the work of Hart (2011), who provides this comparison of orderings for indices of riskiness.

<sup>7</sup> Even nonatomic agents are often able to avoid detection of informed trading in financial markets by, for example, hiding the order size. See, e.g. Bessembinder et al. (2009).

purchases include not only those that always allow the purchaser to learn the true fundamental state, but also all those that always allow the purchaser to exclude one fundamental state from the set of all possible states that will be realized. More generally, our ordering of information purchases is monotonic with respect to Blackwell (1951)'s ordering of information structures.

As with any index, it is important to understand what our formula is telling us. Ours is the first index to directly capture the information-price tradeoff in an information purchase. It does so by separating that tradeoff from potential wealth effects. We assume a competitive economy in which only no-arbitrage investments are available before the arrival of new information. In such an economy, the higher the index of an information purchase, the larger set of risk-averse investors who would accept the purchase regardless of their wealth. Also, a smaller set of risk-averse investors would reject it regardless of their wealth. Since it is a numeric index, our normalized-value formula orders *all* information purchases. And hence, for a fixed price, it orders *all* information structures.

It is important to compare our results here with those of Cabrales et al. (2013), where we have provided another information index. In that paper, the informativeness of an information structure is characterized by the reduction of entropy from the prior to the posteriors. That entropy informativeness index is silent regarding the information-price tradeoff in an information purchase. Nevertheless, with a uniform prior, and for small amounts of information, that index is close to the index proposed here when the purchase price is kept constant. But they differ significantly when the amount of information in the signals is larger.

The methodology we use here is very different, and it complements the one we used in Cabrales et al. (2013). In that paper, one piece of information is more valuable than another one if the maximal price any agent in the economy is willing to pay for the former is larger than the maximal willingness to pay for the latter. The comparison is made while all agents have the same wealth level. Also, the class of utility functions considered there excluded the possibility of ruin. Therefore, that index is more relevant when agents are homogeneous in wealth and wish to end up with nonnegative wealth with probability 1. In contrast, in this paper, we allow wealth levels to differ across agents, who consider the contingency of negative wealth. This is either because of a high price in the information purchase or because of losses in the investment chosen. Importantly, our index is a vehicle to reveal wealth-independent demand for information, defined as the set of agents who accept the purchase regardless of their wealth, rather than on the maximal price an agent in the economy is willing to pay for it. Thus, we view both indices as providing different useful ways to evaluate a given information structure. As an additional insight, we have shown that while our “maximum price dominance” led to a well-known object – entropy reduction – in our previous paper, thereby endowing entropy with new meaning, our more axiomatic approach in the current study, based on duality, leads to a formula that is new to information theory.

The paper is organized as follows. Section 2 describes the model. Section 3 relates the value of information and risk aversion. Section 4 introduces the “uniformly more valuable” ordering, the normalized-value formula, and establishes our main result. Section 5 presents our results connecting the normalized value of an information purchase to levels of risk aversion in the economy. Section 6 examines a number of properties of the normalized value and presents several examples. Section 7 is devoted to related literature, and Section 8 concludes. Some of the more technical proofs and additional justifications and properties of the index are collected in an appendix.

## 2. The model

We consider an agent who, prior to making an investment decision, may acquire some information at a cost. In this section we define the conditions under which this agent acquires a piece of information.

### 2.1. Utility for wealth

We consider an investor with initial wealth  $w$  and a monetary utility function  $u$  defined on  $\mathbb{R}$ . We assume that  $u$  is nondecreasing, strictly concave,<sup>8</sup> and twice differentiable. We let  $\mathcal{U}$  be the set of such monetary utility functions. We identify agents by their monetary utility functions; thus, the term “agent  $u$ ” refers to an agent with utility function  $u$ .

Given  $u \in \mathcal{U}$  and  $w \in \mathbb{R}$ , let  $\rho_u(w) = -\frac{u''(w)}{u'(w)}$  be the Arrow–Pratt coefficient of absolute risk aversion of agent  $u$  at wealth  $w$ . We also let  $\bar{R}(u) = \sup_w \rho_u(w)$ , and  $\underline{R}(u) = \inf_w \rho_u(w)$ . We say that agent  $u_2$  is uniformly less risk-averse than agent  $u_1$  whenever  $\bar{R}(u_2) \leq \underline{R}(u_1)$ . It is sometimes necessary to assume that  $u$  has decreasing absolute risk aversion in wealth. We thus let  $\mathcal{U}_{DA}$  be the subset of agents in  $\mathcal{U}$  for whom  $\rho_u$  is nonincreasing.

### 2.2. Investments

There is a finite set  $K$  of states of nature, about which the agent is uncertain. The agent’s prior on  $K$  is  $p \in \Delta(K)$ , assumed to have full support. The set of investment opportunities consists of all no-arbitrage assets given  $p$ , that is, assets with a nonpositive expected return:  $B^* = \{x \in \mathbb{R}^K, \sum_k p_k x_k \leq 0\}$ .<sup>9</sup> When (i) the agent’s initial wealth is  $w$ , (ii)  $x \in B^*$  is chosen, and (iii) state  $k$  is realized, the agent’s final wealth is  $w + x_k$ .

Some features of the set  $B^*$  of available investments are worth emphasizing. First,  $0 \in B^*$ , that is, not investing is feasible. Second, no agent in our class prefers to invest in the absence of new information, although this may change if some new information arrives. With “no-arbitrage” assets, the only motive for investing is the arrival of new information, and for this reason “no-arbitrage” assets provide a useful framework to measure the value of information. Finally,  $B^*$  is unbounded, an assumption which, although common in the literature on investment under limited information (see, e.g., Kelly, 1956; Arrow, 1971), may appear incompatible with real-life markets. Note, however, that all agents whose posteriors are in the interior of  $\Delta(K)$  choose investments that are bounded. This implies in particular that replacing  $B^*$  by a sufficiently large compact subset spanning all dimensions of uncertainty (complete markets) would not change the analysis when posteriors are interior. See also Section 6.6 for more on this point.

<sup>8</sup> In our framework, risk-neutral agents would sometimes make unbounded optimal investments, which creates technical problems while adding little to content. We therefore exclude them from the analysis.

<sup>9</sup> The vector  $p = (p_1, \dots, p_k, \dots, p_K)$  in the definition of no-arbitrage assets corresponds to the price vector of Arrow securities, where  $p_k$  can be interpreted as the price of an asset that pays 1 in state  $k$  and 0 in all other states. The fact that this vector coincides with the agent’s prior  $p$  means that no-arbitrage assets cannot yield a positive expected return. We disentangle the two roles of  $p$  – price and priors – in subsection E.3 in the appendix.

### 2.3. Information purchases

Before choosing an investment, the agent has the opportunity to engage in an information purchase  $a = (\mu, \alpha)$ . Here,  $\mu > 0$  represents the cost of the information purchase, paid up front, and  $\alpha$  is the information structure representing the information obtained from  $a$ . That is,  $\alpha$  is given by a finite set of signals  $S_\alpha$ , together with probabilities  $\alpha_k \in \Delta(S_\alpha)$  for every  $k$ .<sup>10</sup> When the state of nature is  $k$ ,  $\alpha_k(s)$  is the probability that the signal observed by the agent is  $s$ . Signal  $s$  has an ex-ante probability  $p_\alpha(s) = \sum_k p_k \alpha_k(s)$  of being realized, and we assume, without loss of generality, that  $p_\alpha(s) > 0$  for every  $s$ . For each signal  $s \in S_\alpha$ , we let  $q_\alpha^s = (q_k^s)_k \in \Delta(K)$  be the posterior probability distribution conditional on  $s$ , and derived from Bayes' rule.

We say that  $a$  is *excluding*, if for every signal  $s$ , there exists  $k$  such that  $q_k^s = 0$ . It is *nonexcluding* otherwise. Excluding information purchases are such that, for every received signal, there exists a state of nature that the agent can exclude.

### 2.4. Optimal investment after receiving information

Given a belief  $q$ , an agent with wealth  $w$  and utility  $u$  chooses  $x \in B^*$  in order to maximize her expected utility over all states  $k \in K$ . The maximum expected utility is then  $V(u, w, q)$ , given by:

$$V(u, w, q) = \sup_{x \in B^*} \sum_k q_k u(w + x_k).$$

### 2.5. Acceptance of information purchases

The agent with utility function  $u$  and wealth  $w$  *accepts* an information purchase  $a = (\mu, \alpha)$  if and only if paying  $\mu$  upfront to receive information according to  $\alpha$  generates an expected utility greater than or equal to staying with wealth  $w$ . This is the case if and only if:

$$\sum_s p_\alpha(s) V(u, w - \mu, q_\alpha^s) \geq u(w).$$

In particular, the agent is small enough so that her acceptance/rejection decision as well as her investments do not affect prices, or the information available in the economy. This assumption rules out situations in which agents need to take into account the strategic consequences of their own actions. In this sense, our framework fits that of a competitive economy. Also, the right-hand side in the above inequality implies that the risk-free rate is zero, an assumption that could be easily modified.

## 3. Risk aversion and preference for information

In order to arrive at the concept of the normalized value of information purchases, it is useful to first understand which characteristics of an agent's utility function make her demand for information increase or decrease. As it turns out, an agent's preference for information is determined by her risk aversion.

<sup>10</sup> In Cabrales et al. (2012), an earlier working paper version of the current work, we show (in Section 6.6) that this finiteness assumption is not crucial for our results.

Our first task is to define what it means for one agent to like information better than another agent. In general terms, we say that an agent  $u_2$  likes information better than another agent  $u_1$  when  $u_2$  accepts information more often than  $u_1$ . In order to make the concept precise, we need to be careful about the wealth levels at which we compare the acceptances and rejections of information purchases by  $u_1$  and  $u_2$ .

Our concept of *uniform preference for information* requires agent  $u_2$  to accept the information purchase at all wealth levels whenever  $u_1$  accepts it at some wealth level. That is, “agent  $u_2$  uniformly likes information better than agent  $u_1$ ” means that, whenever agent  $u_1$  is interested in purchasing information, it is certain that agent  $u_2$  is also interested.

**Definition 1.** Agent  $u_2$  uniformly likes information better than agent  $u_1$  if, for every pair of wealth levels  $w_1, w_2$ , and for every information purchase  $a$ , it is the case that if  $u_1$  accepts  $a$  at wealth  $w_1$ , then  $u_2$  accepts  $a$  at wealth  $w_2$ .

Alternative concepts of preferences for information and their consequences are studied in the appendix. Our first result follows:

**Theorem 1.** Given  $u_1, u_2 \in \mathcal{U}$ , the following two conditions are equivalent:

1.  $u_2$  is uniformly less risk-averse than  $u_1$ ,
2.  $u_2$  uniformly likes information better than  $u_1$ .

The formal proof of the result is in the appendix. Here we provide a verbal and elementary proof showing that (1) implies (2). As is apparent from this proof, this implication does not rely on the fact that *all* no-arbitrage investments are available, but only on the fact that *all* available investments have the no-arbitrage property. Let us assume here that the set of possible investments is an arbitrary closed set  $B$ .

Assume that  $u_1$  accepts  $a$  at wealth  $w_1$ . For every possible signal  $s$ , let  $x_s$  be an optimal investment for  $u_1$ , given signal  $s$  and posterior beliefs  $q_\alpha^s$ . Starting from wealth level  $w_1$ , making the information purchase  $a$ , and then making corresponding investments, is a risky bet that yields an expected utility to agent  $u_1$  at least equal to  $u_1(w_1)$ . Therefore, the same risky bet, starting from wealth  $w_2$ , yields an expected utility to agent  $u_2$  which is at least equal to  $u_2(w_2)$ , since this agent is uniformly less risk-averse than  $u_1$ . This does not mean, of course, that investment  $q_\alpha^s$  after signal  $s$  is optimal for agent  $u_2$ , but it does imply that, when agent  $u_2$  is allowed to choose investments in  $B$ , she obtains an expected utility at least as large as  $u_2(w_2)$ . Therefore,  $u_2$  accepts  $a$  at  $w_2$  as well.

The proof of the converse part is somewhat more involved than that for the direct part, as one needs to derive a conclusion about the risk-aversion levels of the agents at all wealth levels, and risk aversion is a local property. In order to prove the result in the appendix, we rely on information structures that provide “little” information, on agents who take “small” investments, and thus on situations where only local properties of utility functions matter.

**Theorem 1** establishes the connection between preference for information and risk aversion. Lemma 2 in Cabrales et al. (2013) shows that an agent with logarithmic utility agrees to an information purchase whenever a more risk-averse agent does. **Theorem 1** both extends this result to general pairs of utility functions, and shows that a converse result holds, namely, that an agent whose preference for information is higher than another must be necessarily less risk-averse.

It is appropriate to underscore the limitations of [Theorem 1](#). Although many pairs of agents cannot be ranked uniformly by their risk aversion, many other pairs of agents still can. The result shows that, in our settings, those pairs of agents are exactly those who can be ranked according to their uniform preference for information. Indeed, the result applies in a nonvacuous way only to pairs of utility functions whose risk aversion levels can be separated by a constant: the risk aversion of  $u_1$  for any wealth level always exceeds a constant, while for  $u_2$ , it always falls below that constant. Nonetheless, the economic environments described by our no-arbitrage condition provide a general framework in which willingness to pay for information decreases with risk aversion (see [Eeckhoudt and Godfroid, 2000](#), and our literature review section below).

The following example illustrates the relationship between risk aversion and preference for information. We do this in a set-up in which all available investments have the no-arbitrage property. At the same time, not all investments with that property are available. The direct proof in the example that willingness to pay for information decreases with risk aversion is already involved. This is true even though the case featured by the example we consider is the simplest one that is not entirely trivial. There are only two states of nature and the information structure fully reveals the state. The difficulty in proving something so simple shows the usefulness of [Theorem 1](#), which, of course, applies to general no-arbitrage investment sets and general information structures.

**Example 1.** Suppose that a businessperson with CARA preferences is deciding whether or not to invest an amount to support a technology startup in order to do research on a prototype electric car. There are two states of the world: high (the research is successful) and low (the research fails). The payoff from not investing is 0 in both states. The net payoff from investing is  $G$  in the high state and  $-L$  in the low state, where  $G, L > 0$  are parameters. We consider an information structure  $\alpha$  that perfectly reveals the state. Assume that the prior probability of the state being low, denoted by  $p$ , is such that:

$$G(1 - p) - Lp < 0,$$

or, equivalently,

$$p > \frac{G}{G + L}$$

so that even a risk-neutral agent will not invest in the absence of new information. The maximal amount that the businessperson with initial wealth  $w$  and coefficient of risk-aversion  $r$  is willing to pay for  $\alpha$ , denoted  $I$ , is independent of  $w$  and is given by the relationship:

$$\exp(-rI) = (1 - p) \exp(-rG) + p.$$

Hence, we have:

$$I = -\frac{1}{r} (\ln(1 - p) \exp(-rG) + p).$$

If we let

$$f(r) = -\ln[(1 - p) \exp(-rG) + p],$$

then

$$I = \frac{f(r)}{r},$$

but  $I$  is then a decreasing function of  $r$ . To see this, note first that

$$\frac{\partial \left( \frac{f(r)}{r} \right)}{\partial r} = \frac{f'(r)r - f(r)}{r^2}.$$

According to the mean value theorem, since  $f(0) = 0$ , there is some  $y \in (0, r)$  such that

$$f(r) = f'(y)r;$$

and since  $f(r)$  is strictly concave,  $f'(y)r > f'(r)r$ , and thus

$$\frac{\partial \left( \frac{f(r)}{r} \right)}{\partial r} = \frac{f'(r)r - f'(y)r}{r^2} < 0.$$

In closing the section, we observe that willingness to pay for information may not be decreasing with respect to risk aversion if some available investments do not satisfy the no-arbitrage property.

**Example 2.** Consider a CARA agent with a coefficient of risk aversion  $r$ . The agent has the option not to invest, or to invest \$1 in an asset that pays \$10 with probability 0.9 (the state is good) or \$0 with probability 0.1 (the state is bad). The investor could purchase information at a price  $\mu = \$0.50$  and learn the state for sure before investing.

The payoff to the agent who does not purchase the information and always invests is:

$$1 - 0.9 \exp(-9r) - 0.1 \exp(r);$$

the payoff to an agent who purchases the information and invests in the good state is:

$$1 - 0.9 \exp(-8.5r) - 0.1 \exp(0.5r);$$

and the payoff to an agent who does not invest is 0.

It follows that for low degrees of risk aversion such as  $r < .231$ , the agent's optimal strategy is to invest without purchasing the information. For medium degrees of risk aversion,  $r \in [.232, 4.60]$ , the optimal strategy is to purchase the information and invest in the good state. For large degrees of risk aversion,  $r > 4.61$ , it is better for the agent not to purchase the information and never invest.

Thus, for small to medium degrees of risk aversion, a more risk-averse agent exhibits a stronger preference for information. Notice how the proposed investment violates no-arbitrage, as the prior-evaluated expected payoff is positive. The reader can check how, in this example, our conclusion would still hold if one restores the no-arbitrage assumption, for instance, by increasing the loss in the bad state.

The example shows that the conclusion in [Theorem 1](#), while applicable to many settings, is not universal. In particular, in cases in which information may be purchased in order to hedge against risk, for example to hear from experts in order to avoid large losses, if appropriate assets are not available in the market, more risk-averse agents may be more willing to pay for information.

#### 4. Preference for information and a value for information purchases

In this section we propose an objective way to define the normalized value of information purchases. The approach is based on ordering preferences for information. We offer three variants

of the same idea, all of them leading to the same normalized-value index. We present one here and relegate the other two to the appendix, which also includes an additional approach based on total rejections/acceptances.

Based on the uniform preference for information, as presented in the previous section, we move now to define a comparison over information purchases. The definition formalizes the idea that if an information purchase is accepted by a first agent, then any purchase that is deemed objectively more valuable should a fortiori be accepted by an agent who likes information better than the first.<sup>11</sup>

**Definition 2.** Let  $a_1, a_2$  be two information purchases. We say that  $a_1$  is *uniformly more valuable* than  $a_2$  when, given two agents  $u_1, u_2 \in \mathcal{U}$  such that  $u_1$  uniformly likes information better than  $u_2$  and for every two wealth levels  $w_1, w_2$ , if agent  $u_2$  accepts  $a_2$  at wealth level  $w_2$ , then agent  $u_1$  accepts  $a_1$  at wealth level  $w_1$ .

Note that the requirement of two agents being ordered according to their uniform preference for information is extremely demanding. According to [Theorem 1](#), it is equivalent to a situation in which absolute risk aversion – no matter where measured – is always greater for one agent than for the other. Since this strong requirement is assumed about the two agents in the definition, the definition itself actually requires very little. Moreover, what it does require seems very reasonable, namely, we would expect that agents with a stronger preference for information seek out more highly-valued pieces of information.

#### 4.1. Normalized value of information purchases

We now present the cardinal formula that characterizes our information ranking.

For two probability distributions  $p$  and  $q$ , the *relative entropy* from  $p$  to  $q$ , also called their *Kulback–Leibler divergence*, is an asymmetric measure of their discrepancy. It is defined by the formula:

$$d(p\|q) = \sum_k p_k \ln \frac{p_k}{q_k}.$$

It is always nonnegative, and equals zero if and only if  $p = q$ . It is finite, provided the support of  $q$  contains that of  $p$ ; it takes the value  $+\infty$  otherwise. Thus,  $p$  and  $q$  are “maximally different” when  $q$  rules out one possibility that  $p$  does not.<sup>12</sup>

Based on the relative entropy, we define the *normalized value* of an information purchase  $a$  as this quantity:

$$\mathcal{NV}(a) = -\frac{1}{\mu} \ln \left( \sum_s p_\alpha(s) \exp(-d(p\|q_\alpha^s)) \right). \tag{1}$$

In the above formula, and throughout the paper, we use the convention  $\exp(-d(p\|q_\alpha^s)) = 0$  by continuity if  $d(p\|q_\alpha^s) = +\infty$ . The normalized value  $\mathcal{NV}(a)$  of  $a$  is thus well-defined and finite

<sup>11</sup> This principle is referred to as “duality” in [Aumann and Serrano \(2008\)](#).

<sup>12</sup> If  $p$  were the true distribution and  $q$  an approximate hypothesis, information theory would view the relative entropy from  $p$  to  $q$  as giving the expected number of extra bits that would be required to code the information if one were to use  $q$  instead of  $p$ . See, e.g., [Kraft \(1949\)](#) and [McMillan \(1956\)](#), or [Kelly \(1956\)](#) for a betting-market interpretation.

if and only if there exists  $s$  such that  $-d(p\|q_\alpha^s)$  is finite, which is the case if  $a$  is nonexcluding. We let  $\mathcal{NV}(a) = +\infty$  by continuity if  $a$  is excluding.

The normalized value of an information purchase decreases with its price and increases with the relative entropy of the prior to the posterior probabilities. Specifically, the normalized value of an information purchase is measured by the inverse of its price, multiplied by the natural logarithm of the expected exponentials of the negative of relative entropy from the prior to each of the generated posterior probabilities.<sup>13</sup>

As a benchmark for the normalized-value index, note that for all  $t > 0$ ,  $\mathcal{NV} = t$  is the index value for all information purchases which are such that, after every signal  $s$ , the relative entropy  $d(p\|q_\alpha^s)$  between the prior  $p$  and the posterior  $q_\alpha^s$  is (i) equal to a constant, and (ii) priced exactly at  $(1/t)$  times that constant. In particular, if the cost of such a purchase is exactly  $t$ ,  $\mathcal{NV} = 1$  includes all those information purchases that have a relative entropy of  $t$  from the prior to each posterior probability.

#### 4.2. Main result

**Theorem 2.** *Let  $a_1, a_2$  be two information purchases. The following two statements are then equivalent:*

1.  $a_1$  is uniformly more valuable than  $a_2$ .
- 2.

$$\mathcal{NV}(a_1) \geq \mathcal{NV}(a_2).$$

Let us clarify briefly the usefulness of this theorem as a guide for action. Suppose first that an investor has calculated the normalized value of a purchase (hiring an information expert), which turns out to be  $\mathcal{NV}_1$ . She also happens to know that her neighbor, more risk-averse than her, has hired a different information expert. The neighbor's purchase has a normalized value of  $\mathcal{NV}_2 < \mathcal{NV}_1$ . Assume also that she views herself as a small agent, that she is not interested in investing if no new information arrives, and that she has access to developed financial markets. Then she learns from our theorem that she should hire the information expert.

In fact, we show in the next section that the expert hired by the neighbor will be turned down for sure by any person whose risk aversion is higher than  $\mathcal{NV}_2$  everywhere. Also, that expert would be hired by those whose risk aversion is lower than  $\mathcal{NV}_2$  everywhere. The expert with normalized-value  $\mathcal{NV}_1$  that our investor is considering would be turned down by a smaller set of agents in the economy (those whose risk aversion exceeds  $\mathcal{NV}_1$  everywhere). That expert would also be hired by a larger set of investors (with risk aversion lower than  $\mathcal{NV}_1$  everywhere).

This is obviously far from a unanimous evaluation of one expert versus the other. That would be the case were they ordered by Blackwell and were they charging the same price. Nevertheless, our formula and results, which capture the information-price tradeoff for each expert, become

<sup>13</sup> If we ignore the price  $\mu$ , the rest of the expression in the normalized-value formula is, remarkably, referred to as “free energy” in theoretical physics (see, e.g., Landau and Lifshitz, 1980), where relative entropy plays the role of the Hamiltonian of the system. A similar formula appears under the term “stochastic complexity” in machine learning (Hinton and Zemel, 1994). Similar formulas appear also in models of rational inattention, where relative entropies are replaced by utilities of actions less information cost measured by entropy (see, e.g., Stevens, 2016; Matejka and McKay, 2014).

useful benchmarks of analysis for the decision problem of investors in an environment satisfying our assumptions.

**Theorem 2** relies on the relationship between demand for information and risk aversion, as established in **Theorem 1**. As we have already seen, it is necessary for this approach that only no-arbitrage assets be available and that uninformed agents do not make risky bets. On the other hand, when two information purchases are compared, not all assets play a role. This, in turn, implies that the assumption that all no-arbitrage purchases are available is not necessary when comparing a finite number of information purchases. See Section 6.6 for more details on this.

**Proof.** Recall the class of CARA (constant absolute risk aversion) utility functions. Given  $r > 0$ , let  $u_C^r$  be the CARA utility function with parameter  $r$ , given by  $u_C^r(w) = -\exp(-rw)$  for every  $w$ . For a CARA agent with coefficient  $r$  and wealth level  $w$ , we consider the problem of the optimal portfolio choice when the agent’s belief is  $q$ . The next lemma shows that the solution is interior when  $q$  has full support.

**Lemma 1.** For every  $w$  and  $r > 0$ ,

1. the optimal portfolio for agent  $u_C^r$  and belief  $q$  with full support is given by

$$x_k = -\frac{1}{r}(-d(p\|q) + \ln \frac{p_k}{q_k});$$

2. for every  $q$ , the maximum expected utility is:

$$V(u_C^r, w, q) = -\exp(-d(p\|q) - rw).$$

The proof is elementary and provided in the appendix.

We continue with **Lemma 2**, which shows that  $\mathcal{NV}(a)$  can be equivalently defined as the level of risk aversion of a CARA agent who is indifferent between accepting and rejecting the purchase.

**Lemma 2.** Let  $a$  be an information purchase and  $w$  be any wealth level.

1. If  $r > \mathcal{NV}(a)$ , then agent  $u_C^r$  rejects  $a$  at wealth level  $w$ .
2. If  $r \leq \mathcal{NV}(a)$ , then agent  $u_C^r$  accepts  $a$  at wealth level  $w$ .

**Proof.** Agent  $u_C^r$  accepts  $a$  if and only if

$$\sum_s p_\alpha(s) V(u_C^r, w - \mu, q_\alpha^s) \geq u_C^r(w).$$

If  $a$  is excluding, then the left-hand side of the inequality equals 0, and the inequality is satisfied for all  $r$  and  $w$ . If  $a$  is nonexcluding, then the agent accepts  $a$  if and only if

$$-\exp(-r(w - \mu)) \sum_s p_\alpha(s) \exp(-d(p\|q_\alpha^s)) \geq -\exp(-rw).$$

This is equivalent to

$$\exp(-r\mu) \geq \sum_s p_\alpha(s) \exp(-d(p\|q_\alpha^s)),$$

which in turn is equivalent to  $r \leq \mathcal{N}\mathcal{V}(a)$ . Thus, for  $r \leq \mathcal{N}\mathcal{V}(a)$ , the agent accepts  $a$  at every wealth level, whereas for  $r > \mathcal{N}\mathcal{V}(a)$ , the agent rejects  $a$  at every wealth level.  $\square$

Equipped with [Theorem 1](#) and [Lemma 2](#), we now proceed to prove [Theorem 2](#).

First assume that  $a_1$  is uniformly more valuable than  $a_2$ , and that  $\mathcal{N}\mathcal{V}(a_2)$  is finite. By [Lemma 2](#), a CARA agent with a coefficient of risk aversion  $\mathcal{N}\mathcal{V}(a_2)$  accepts  $a_2$  at every wealth level. This agent uniformly likes information better than itself since, by [Lemma 2](#), acceptance or rejection for CARA agents is independent of wealth. Since  $a_1$  is more valuable than  $a_2$ , this CARA agent also accepts  $a_1$  at every wealth level, which implies (also by [Lemma 2](#)) that  $\mathcal{N}\mathcal{V}(a_1) \geq \mathcal{N}\mathcal{V}(a_2)$ .

The case in which  $\mathcal{N}\mathcal{V}(a_2)$  is infinite is dealt with similarly: by [Lemma 2](#) every CARA agent accepts  $a_2$  at every wealth level, which implies that the same agent also accepts  $a_1$  at every wealth level. By [Lemma 2](#) again, this implies that we also have  $\mathcal{N}\mathcal{V}(a_1)$  is also infinite.

Now assume that  $\mathcal{N}\mathcal{V}(a_1) \geq \mathcal{N}\mathcal{V}(a_2)$ . Consider two agents  $u_1$  and  $u_2$  such that  $u_1$  uniformly likes information better than  $u_2$ . Given wealth levels  $w_1$  and  $w_2$ , and assuming that  $u_2$  accepts  $a_2$  at  $w_2$ , we need to prove that  $u_1$  accepts  $a_1$  at  $w_1$ . By [Theorem 1](#) we have  $\overline{R}(u_1) \leq \underline{R}(u_2)$ . Since  $\overline{R}(u_1) > 0$  and  $\underline{R}(u_2)$  is finite,  $\underline{R}(u_2)$  is positive and finite. Let  $r = \underline{R}(u_2)$ . Since  $\overline{R}(u_r^C) = r$ , the agent  $u_r^C$  likes information better than agent  $u_2$  does, by [Theorem 1](#); hence the former accepts  $a_2$  at any wealth level. By [Lemma 2](#) this means that  $r \leq \mathcal{N}\mathcal{V}(a_2)$ , and hence also  $r \leq \mathcal{N}\mathcal{V}(a_1)$ , so that  $u_r^C$  also accepts  $a_1$  at any wealth level. Since  $\overline{R}(u_1) \leq r = \underline{R}(u_r^C)$  and  $u_1$  likes information better than  $u_r^C$  (also by [Theorem 1](#)), it follows that  $u_1$  accepts  $a_1$  at wealth level  $w_1$ .  $\square$

## 5. The demand for information

In this section we show that, in our settings, the normalized value of an information purchase is useful for characterizing the demand for information, namely, the set of agents who are willing to go ahead with any given information purchase. The first result in this section shows that if the minimum coefficient of absolute risk aversion of an agent over all levels of wealth is greater than the normalized value of information, she rejects a purchase independently of her wealth. On the other hand, if the maximum coefficient of absolute risk aversion of an agent over all levels of wealth is smaller than the normalized value of information, she accepts a purchase independently of her wealth. The next theorem follows:

**Theorem 3.** Consider an information purchase  $a$  and an agent  $u \in \mathcal{U}$ .

1. If  $\overline{R}(u) > \mathcal{N}\mathcal{V}(a)$ , then agent  $u$  rejects  $a$  at all wealth levels.
2. If  $\overline{R}(u) \leq \mathcal{N}\mathcal{V}(a)$ , then agent  $u$  accepts  $a$  at all wealth levels.

This simple but important result follows immediately from [Lemma 2](#) and from the first direction of [Theorem 1](#). This is so because an agent with  $\overline{R}(u) \leq r$  is uniformly less risk averse than a CARA agent with risk aversion  $r$ , and an agent with  $\underline{R}(u) \geq r$  is uniformly more risk-averse than a CARA agent with risk aversion  $r$ .

Parts (1) and (2) of [Theorem 3](#) characterize situations in our settings in which one can unequivocally say whether or not  $u$  accepts  $a$ , independent of what one knows about the agent's wealth level. Whenever  $\overline{R}(u) > \mathcal{N}\mathcal{V}(a) \geq \underline{R}(u)$ , it may be the case that agent  $u$  accepts  $a$  for some wealth levels, and rejects it for other wealth levels. This observation makes it clear why the

normalized-value index  $\mathcal{NV}(a)$  is not a universal representation of preferences for information purchases. It is important to note that this is true only because of such wealth effects.

Another way to look at this result is the following. Imagine that it has been estimated econometrically that the agents in this economy have a coefficient of risk aversion  $\rho_u(w) \in [\gamma_1, \gamma_2]$  for all relevant  $w$ . Then, given an information structure  $\alpha$ , one can identify two prices,  $\mu_1$  and  $\mu_2$ , as follows:

$$\gamma_1 = \mathcal{NV}(\mu_1, \alpha), \gamma_2 = \mathcal{NV}(\mu_2, \alpha),$$

where  $\mu_1$  and  $\mu_2$  offer the following interpretation: for prices  $\mu > \mu_1$ , the information purchase  $(\mu, \alpha)$  will be unanimously rejected, whereas for prices  $\mu < \mu_2$ , the purchase will be unanimously accepted. This is the sense in which, for all information structures, the index of normalized value allows one to identify the minimum and maximum prices for individuals within a group whose coefficients of risk aversion are known or at least have been estimated.

More can be said for the case when  $u$  is DARA. For DARA utilities, the next result characterizes utility functions that exhibit unanimous acceptance and unanimous rejection of a purchase. (The proof is also in the appendix).

**Theorem 4.** Consider an information purchase  $a$  and the class of utility functions  $\mathcal{U}_{DA}$ .

1. An agent  $u \in \mathcal{U}_{DA}$  rejects  $a$  at all wealth levels if and only if  $\underline{R}(u) > \mathcal{NV}(a)$ .
2. An agent  $u \in \mathcal{U}_{DA}$  accepts  $a$  at all wealth levels if and only if  $\underline{R}(u) \leq \mathcal{NV}(a)$ .

### 5.1. Examples and calibrations of the model

This subsection illustrates the results derived for the demand for information. It presents some calibrations of the model in order to gauge the magnitudes implied by the index. Of course, this is meant to be only suggestive, and is far from constituting a careful empirical analysis.

According to [Dohmen et al. \(2011\)](#), “Lottery responses and wealth information imply a distribution of CRRA coefficients mainly between 1 and 10.” People in the lowest quartile of the wealth distribution in most developed countries has zero or negative net worth ([Sierminska et al., 2006](#)) and the wealth of the highest decile ranges from 0.36 (Italy) to 1.81 (Germany) million US\$ (with the US being around 0.95). However, these figures include very young people who have not yet had time to acquire any assets. If we use the median wealth instead, the figures go from about US\$ 20,000 (Sweden) to about US\$ 120,000 (Italy), with the US being at about US\$ 50,000. This means that a large fraction of the risk aversion of the population in the developed world can be characterized with  $\overline{R}(u) = 5 \times 10^{-4}$  and  $\underline{R}(u) = 1.8 \times 10^{-6}$ .

**Example 3.** Let  $a$  be an information purchase about a binary state of the world (e.g., whether or not the US will be in recession in 2020) where the two states are equally likely *a priori*. The information structure  $\alpha$  consists of two signals. Conditional on 1 being received, the probability of a recession is  $\beta$ , and conditional on signal 2 arriving, the probability of recession is  $1 - \beta$ . Then, computation gives:

$$\mathcal{NV}(a) = \frac{1}{\mu} \ln \left( 2 (\beta (1 - \beta))^{1/2} \right).$$

Thus, the information purchase  $a = (\mu, \alpha)$  is accepted by all agents considered, if its price  $\mu$  satisfies

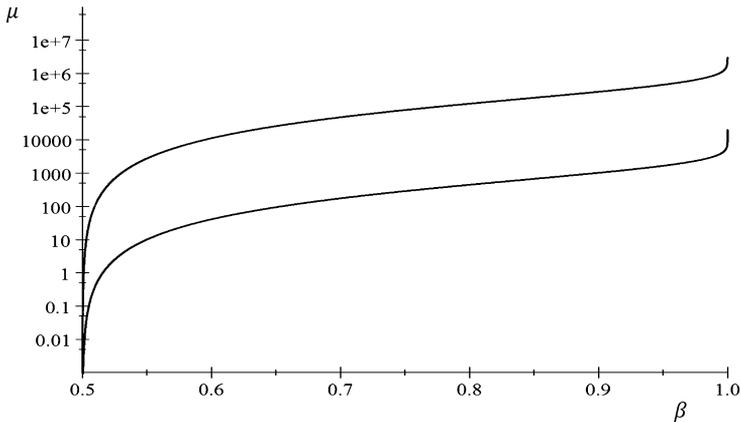


Fig. 1. Maximum and minimum prices at which information is purchased by agents in the economy as a function of signal precision  $\beta$ .

$$\mu \leq -\ln\left(2(\beta(1-\beta))^{1/2}\right) \times 2 \times 10^3.$$

The same information purchase is rejected by all agents considered, if:

$$\mu > -\ln\left(2(\beta(1-\beta))^{1/2}\right) \times 5.5 \times 10^5.$$

In Fig. 1, for every value of  $\beta$ , the information purchase is rejected by all agents considered, if  $\mu$  is above the upper curve, and accepted by all agents considered, if  $\mu$  is below the lower curve.

Of course, both ranges in the previous example are relatively large, since they separate the case in which a large portion of the world population would accept a purchase from the case in which only a few people might. Note, however, that these ranges provide realistic figures, given that it is not hard to think of pieces of information with a higher price than the maximum bound, or with a lower price than the minimum bound. For example, on its website, the Australian Securities and Investments Commission says this about financial advice: “The cost of the advice will depend on its scope. As a guide, expect to pay between \$200 and \$700 for simple advice and between \$2000 and \$4000 for more comprehensive advice.”<sup>14</sup> This fits nicely with the figures in our last example. Future research might be able to provide a more precise range for information prices as a function of risk aversion estimates and of different pieces of information, always under the assumption of frictionless financial markets that allow large investments

## 6. Some properties of the index

We now discuss some properties of our index for the normalized value of information.<sup>15</sup>

<sup>14</sup> Taken from: <https://www.moneysmart.gov.au/investing/financial-advice/financial-advice-costs>.

<sup>15</sup> Some of the proofs and more properties can be found in the appendix. Additional properties and examples can be found in the working paper version available at: <http://ideas.repec.org/p/cte/werepe/we1224.html>.

### 6.1. Continuity

The normalized-value index  $\mathcal{NV}$  is jointly continuous in  $\mu$ , in  $p_\alpha$ , and in  $(q_\alpha^s)_s$  on the domain of nonexcluding information purchases. Continuity is a natural and attractive property: small changes in either the price or the conditional probabilities of signals should translate into small changes in the normalized value of the purchase. By “continuity at infinity,”  $\mathcal{NV}(a)$  is infinite if and only if  $a$  is excluding.

### 6.2. Blackwell monotonicity

The normalized-value index is Blackwell-monotonic, as expressed in the following observation:

**Observation 1.** If an information structure  $\alpha_1$  is more informative than another information structure  $\alpha_2$  in the sense of Blackwell, then for any price  $\mu > 0$ , the information purchase  $(\mu, \alpha_1)$  is more valuable than the information purchase  $(\mu, \alpha_2)$ . Thus, we have:

$$\mathcal{NV}(\mu, \alpha_1) \geq \mathcal{NV}(\mu, \alpha_2).$$

This observation shows that the complete ordering defined by the normalized value is an extension of Blackwell’s ordering over information structures evaluated at the same price. Since the normalized value is a new ordering, this is also a new result. Our simple proof of the observation does not rely on the analytical form of the normalized value function, but rather on its axiomatic underpinning.

### 6.3. Mixtures

A third property concerns what happens when an information structure is constructed by randomizing over two other ones. Given information structures  $\alpha_1, \alpha_2$ , and  $1 > \lambda > 0$ , we let  $\lambda\alpha_1 \oplus (1 - \lambda)\alpha_2$  be the information structure in which (i) a coin toss determines whether the agent’s signal is chosen from  $\alpha_1$  (with probability  $\lambda$ ) or from  $\alpha_2$  (with probability  $1 - \lambda$ ), and (ii) the agent is informed of both the outcome of the coin toss and the signal drawn from the chosen information structure. Formally, the set of signals in  $\lambda\alpha_1 \oplus (1 - \lambda)\alpha_2$  is  $S_{\alpha_1} \cup S_{\alpha_2}$  (where we assume that  $S_{\alpha_1}$  and  $S_{\alpha_2}$  are disjoint), and the probability in state  $k$  that the agent receives signal  $s \in S_{\alpha_1}$  is  $\lambda\alpha_{1,k}(s)$ , whereas the probability of a signal  $s \in S_{\alpha_2}$  is  $(1 - \lambda)\alpha_{2,k}(s)$ .

**Observation 2.** Consider  $\mu > 0$  and  $\alpha_1, \alpha_2$  such that  $\mathcal{NV}(\mu, \alpha_1) \geq \mathcal{NV}(\mu, \alpha_2)$ . For every  $1 > \lambda > 0$ , we have:

$$\mathcal{NV}(\mu, \alpha_1) \geq \mathcal{NV}(\mu, \lambda\alpha_1 \oplus (1 - \lambda)\alpha_2) \geq \mathcal{NV}(\mu, \alpha_2).$$

Thus, quite naturally, the normalized value of the “mixed” information purchase lies between the normalized value of the most valuable one and that of the least valuable one. Here again, the comparison of the values of the normalized index stems naturally from the “more valuable than” ordering.

#### 6.4. Comparison with average relative entropy

So far we have seen that two intuitive properties of the index  $\mathcal{NV}$  are that it is (i) a decreasing function of its price, and (ii) an increasing function of the relative entropy from the prior to each generated posterior. In this light, it is interesting to compare the normalized value with the following index, which simply averages out all generated relative entropies:

$$\hat{A}(a) = \frac{1}{\mu} \sum_s p_\alpha(s) d(p \| q_\alpha^s).$$

It is apparent that this index shares those two properties, and also satisfies separability in the form of price homogeneity. The next example shows that the two indices are, however, fundamentally different. Indeed, the example highlights the essential nature (for the ordering of purchases) of the combined operation of the exponential function and its compensating logarithm as a “blow up/shrink down” of relative entropies. The exponential function with negative exponents, which is bounded above, avoids the problem of infinite relative entropies attached to a single signal. Only when all relative entropies are infinite does the logarithm restore an infinite normalized value. This property is essential in order to satisfy the duality between uniform preferences for information and the proposed function ranking the normalized value of purchases.

**Example 4.** Let  $K = \{1, 2, 3\}$  and fix a uniform prior. Consider, for instance, two information structures, each of which has two signals:

$$\alpha_1 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 1 - \varepsilon & \varepsilon \\ 1/2 & 1/2 \\ \varepsilon & 1 - \varepsilon \end{bmatrix}$$

Each row represents the probability of each signal under each possible state: 1, 2, 3.

Fix an arbitrary  $\mu > 0$ , and define the purchases  $a_1 = (\mu, \alpha_1)$  and  $a_2 = (\mu, \alpha_2)$ . Note that  $\hat{A}(a_1)$  is infinite because the relative entropy of the prior to the posterior generated by the first signal is infinite. On the other hand, for any  $\varepsilon > 0$ ,  $\hat{A}(a_2)$  is finite. We next argue that the normalized value of the purchases is not well measured by  $\hat{A}$ . Indeed, for a small enough  $\varepsilon > 0$ , the purchase  $a_2$  is almost excluding, and hence, in such a case  $r_1 = \mathcal{NV}(a_1) < \mathcal{NV}(a_2) = r_2$ . Here,  $r_1$  and  $r_2$  are the risk-aversion coefficients of the two CARA individuals who define the two corresponding levels of normalized values. Let  $r = (r_1 + r_2)/2$ . Clearly, the CARA agent  $r$  uniformly likes information better than the CARA agent  $r_2$ : the CARA agent  $r_2$  accepts  $a_2$ , which according to the index  $\hat{A}$  would be less valuable than  $a_1$ ; but agent  $r$ , who likes information more, rejects  $a_1$ . This suggests that  $\hat{A}$  is not measuring value of information well.

#### 6.5. Comparison with entropy informativeness

Several authors (Kelly, 1956; Arrow, 1971; Cabrales et al., 2013) have proposed measuring the informativeness of a statistical experiment by the reduction of uncertainty on the state of nature, quantified by its entropy. The following example compares the rankings generated by our normalized index (when the price of the purchase is kept constant) and entropy informativeness.

**Example 5.** Let  $K = \{1, 2, 3\}$  with a uniform prior. Consider the following two information structures, where rows correspond to states of nature, columns to signals, and cells to the probability of the signal in the corresponding state:

$$\alpha_1 = \begin{bmatrix} 1 - \varepsilon_1 & \varepsilon_1 \\ 1 - \varepsilon_1 & \varepsilon_1 \\ \varepsilon_1 & 1 - \varepsilon_1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 - \varepsilon_2 & \varepsilon_2 \\ 0.1 & 0.9 \\ \varepsilon_2 & 1 - \varepsilon_2 \end{bmatrix}.$$

For  $\varepsilon_1$  and  $\varepsilon_2$  small enough, these information structures are not ranked according to Blackwell. To see this, note that  $\alpha_1$  allows one to separate states 2 and 3 with greater precision than  $\alpha_2$  does, while  $\alpha_2$  allows one to statistically distinguish between states 1 and 2, which  $\alpha_1$  does not.

Now let us compare the entropy informativeness for a normalized price of 1:  $\mathcal{NV}(a_1)$  for  $a_1 = (1, \alpha_1)$  and  $\mathcal{NV}(a_2)$  for  $a_2 = (1, \alpha_2)$ :

$$\mathcal{NV}(a_i) = -\frac{1}{\mu} \ln \left( \sum_s p_{\alpha_i}(s) \exp(-d(p \| q_{\alpha_i}^s)) \right).$$

Taylor approximations give:

$$\sum_s p_{\alpha_1}(s) \exp(-d(p \| q_{\alpha_1}^s)) \simeq \frac{2}{3} \varepsilon_1^{1/3},$$

$$\sum_s p_{\alpha_2}(s) \exp(-d(p \| q_{\alpha_2}^s)) \simeq \varepsilon_2^{1/3}.$$

If  $\varepsilon_1 = \varepsilon_2$  and both are small enough, then  $\mathcal{NV}(a_2) < \mathcal{NV}(a_1)$ . That is, both purchases are becoming almost excluding at the same rate, but  $a_1$  leads to almost eliminate one more state following the second signal, allowing the investor to make very aggressive bets against more states. On the other hand, if  $\varepsilon_2 = \varepsilon_1^2$  and both are small, then  $\mathcal{NV}(a_1) < \mathcal{NV}(a_2)$ . Namely, because of the different rates at which signal probabilities approach 0,  $a_2$  is becoming almost excluding much faster.

Let us now estimate the entropy reduction from the uniform prior, which we denote by  $I_e(\cdot)$ . Straightforward computation yields the following:

$$I_e(\alpha_1) \simeq \ln 3 - \frac{2}{3} (\ln 2) \simeq \ln 3 - 0.46;$$

$$I_e(\alpha_2) \simeq \ln 3 - \frac{1}{3} (1.1 \ln 1.1 - 0.1 \ln 0.1 + 1.9 \ln 1.9 - 0.9 \ln 0.9)$$

$$\simeq \ln 3 - 0.55.$$

This implies that  $I_e(\alpha_1) > I_e(\alpha_2)$  whenever  $\varepsilon_1, \varepsilon_2$  are sufficiently close to zero, as the generated posteriors by  $\alpha_1$  eliminate a greater degree of uncertainty (note the difference between the two information structures in the second state).

To explore somewhat more systematically the difference between the index based on entropy and the one in this paper, we investigate conditions for purchases with small amounts of information that renders them equivalent. Let  $a_i = (\mu, \alpha_i)$ . We then have the following equations:

$$\mathcal{NV}(a_i) = -\frac{1}{\mu} \ln \left( \sum_s p_{\alpha_i}(s) \exp \left( - \sum_k p_k (\ln p_k - \ln q_{\alpha_i}^s(k)) \right) \right)$$

$$I_e(\alpha_i) = \sum_s p_{\alpha_i}(s) \left( - \sum_k p_k (\ln p_k - \ln q_{\alpha_i}^s(k)) - \sum_k (p_k - q_{\alpha_i}^s(k)) \ln q_{\alpha_i}^s(k) \right).$$

The two previous expressions imply that, to a first-order approximation when  $q_{\alpha_i}^s$  is close to  $p$ ,

$$\begin{aligned} \mathcal{NV}(a_i) &\simeq \frac{1}{\mu} \sum_s p_{\alpha_i}(s) \left( \sum_k p_k (\ln p_k - \ln q_{\alpha_i}^s(k)) \right); \\ I_e(\alpha_i) &\simeq \sum_s p_{\alpha_i}(s) \left( - \sum_k (1 + \ln q_{\alpha_i}^s(k)) p_k (\ln p_k - \ln q_{\alpha_i}^s(k)) \right). \end{aligned}$$

As a result, it follows that:

$$\mathcal{NV}(a_i) \simeq \frac{1}{\mu} \sum_s p_{\alpha_i}(s) \left( \sum_k p_k (\ln p_k - \ln q_{\alpha_i}^s(k)) \right) \tag{2}$$

and

$$I_e(\alpha_i) \simeq \sum_s p_{\alpha_i}(s) \left( - \sum_k (1 + \ln q_{\alpha_i}^s(k)) p_k (\ln p_k - \ln q_{\alpha_i}^s(k)) \right). \tag{3}$$

A comparison of expressions (2) and (3) makes it clear that when priors and posteriors are similar, the two indices point in the same direction – as long as it is also true that the  $q_{\alpha_i}^s(k)$  vectors are all parallel to the unit vector and that  $\ln q_{\alpha_i}^s(k) < -1$ , that is, when priors are close to uniform and there are more than two states. Otherwise, cases such as the one provided in Example (5), when posteriors are very informative, are likely to make the indices diverge.

### 6.6. On smaller investment sets

**Theorem 2**, which characterizes demand for information according to our normalized index, makes the assumption that  $B^*$  consists of all no-arbitrage assets. We show here that this assumption is not necessary, as long as the set of available investments is rich enough.

Consider an information purchase  $a = (\mu, \alpha)$ , and assume that all posterior probabilities following  $\alpha$  are in the interior of  $\Delta(K)$ . Depending on the signal  $s \in S_\alpha$  received, let  $x_r(s) \in B^*$  be the optimal investment for a CARA agent with parameter  $r$  given the agent’s posterior belief; also let  $X_r(a) = \{x_r(s), s \in S_\alpha\}$ .

We now consider an arbitrary set  $B$  of no-arbitrage assets, which is not necessarily the full set  $B^*$ . All definitions, including “ $a_1$  is more valuable than  $a_2$ ” extend to the context in which acceptance of an information purchase depends on the set of available assets. The two results that follow provide counterparts to **Theorem 2**, while relaxing the assumptions made on the set of available assets.

**Proposition 1.** Consider two information purchases  $a_1$  and  $a_2$ , and assume that  $\mathcal{NV}(a_1) \geq \mathcal{NV}(a_2)$ . If  $B$  contains  $X_r(a_1)$  for some  $r \in [\mathcal{NV}(a_2), \mathcal{NV}(a_1)]$ , then  $a_1$  is more valuable than  $a_2$  given the investment set  $B$ .

**Proof.** Assume agent  $u_2$  accepts  $a_2$  at wealth  $w_2$ , given the available set  $B$ . Agent  $u_2$  also accepts  $a_2$  at  $w_2$  if  $B^*$  is available; hence,  $\underline{R}(u_2) \leq \mathcal{NV}(a_2) \leq r \leq \mathcal{NV}(a_1)$ . A CARA agent with risk aversion  $r$  accepts  $a_1$ , given either  $B^*$  or  $B$ , since only investments in  $B$  are chosen by this agent. Now, any agent  $u_1$  who likes information better than  $u_2$  satisfies  $\bar{R}(u_1) \leq r$ . Hence, this agent  $u_1$  also accepts  $a_1$  given  $B$ . □

The following is an extension of the converse part of our main result:

**Proposition 2.** Consider two information purchases  $a_1$  and  $a_2$ , and assume that  $a_1$  is more valuable than  $a_2$  given some investment set  $B$ . If  $B$  contains  $B_{\mathcal{N}\mathcal{V}(a_2)}(a_2)$ , then  $\mathcal{N}\mathcal{V}(a_1) \geq \mathcal{N}\mathcal{V}(a_2)$ .

**Proof.** A CARA agent with risk aversion  $\mathcal{N}\mathcal{V}(a_2)$  accepts  $a_2$  under  $B$ . Thus, this same agent accepts  $a_1$  under  $B$ , which implies that she also accepts  $a_1$  under  $B^*$ , which implies that  $\mathcal{N}\mathcal{V}(a_2) \leq \mathcal{N}\mathcal{V}(a_1)$ .  $\square$

Of course, in general, rankings of information purchases may depend on investments available. But, as shown by the previous results, as long as enough investments are available, the ranking between any two information purchases is unambiguous, and is represented by our normalized-value index.

## 7. Related literature

The classical approach to ranking information structures is due to Blackwell (1953).<sup>16</sup> However, this approach does not provide a complete ordering of information structures. For an information structure to be more informative in the sense of Blackwell than another one, it must be the case that agents (weakly) prefer the former over the latter no matter what their preferences are. More recent research has focused on restricting preferences to a particular class. For example, Lehmann (1988) restricts attention to problems with monotone decision rules, and Persico (2000), Athey and Levin (2001), and Jewitt (2007) focus on some more general classes of monotone problems.<sup>17</sup> The main difference between this line of research and our approach is that we provide a complete order through a duality axiom for problems with a restricted set of investment opportunities.<sup>18</sup>

Working independently, several authors (Kelly, 1956; Arrow, 1971; Cabrales et al., 2013) have studied a complete ordering of information structures, indexed by the reduction of uncertainty on the state of nature due to the signal, as measured by entropy. All three papers hinge on the fact that entropy measures the value of information for a logarithmic utility investor. Arrow (1971) simply assumes such an investor. Kelly (1956) shows that when investment opportunities are repeatedly available, the betting strategy that maximizes the growth of long-run wealth is the one that maximizes instantaneous expected logarithmic utility. Cabrales et al. (2013) measure the value of an information source by the maximal price that an agent in the economy is willing to pay to access it, and the logarithmic agent is characterized as having the maximum willingness to pay for information.<sup>19, 20</sup>

The present paper departs from this literature in several ways. First, we offer a way to measure not just information, but an information purchase, thus adding the price of information into the object of study. To the best of our knowledge, ours is the first index capturing this tradeoff.

<sup>16</sup> Veldkamp (2011) shows the many ways in which economists have measured informativeness and their applications.

<sup>17</sup> Measuring information is even harder if several agents interact, as shown, for instance in, Gossner (2000), Gossner and Mertens (2001), and Lehrer and Rosenberg (2006).

<sup>18</sup> Moscarini and Smith (2002); Azrieli (2014), and Ganuza and Penalva (2010) also study partial orderings of information structures in various environments.

<sup>19</sup> This is related to the fact that this is also the least risk-averse agent in the economy considered.

<sup>20</sup> Samuelson (1969) had already discussed Kelly (1956) and the properties of logarithmic utility investing, and Blume and Easley (2002) the potential for the dominance in a market in the long run of “Kelly investors”. A good summary on “Kelly investing” is MacLean et al. (2011). Cesa-Bianchi and Lugosi (2006) note the formal equivalence between sequential gambling and forecasting under the logarithmic loss function.

Second, a perhaps more important argument is that in our methodology, instead of studying the value of information to one particular agent, we obtain an index that captures *demand for* information, in the sense that larger values of the index correspond to larger sets of agents accepting the corresponding purchases.

Prior literature has investigated the relationship between risk aversion and demand for information. Freixas and Kihlstrom (1984) study an environment in which a consumer with CARA preferences decides which variety of a horizontally differentiated good to consume. The consumer can obtain a normally distributed signal that would add precision to her knowledge. The paper shows that the willingness to pay for information in that environment decreases with risk aversion.<sup>21</sup> Eeckhoudt and Godfroid (2000) assert at the beginning of their paper, “It is widely believed among economists and businessmen that increases in risk or in risk aversion should increase the value of information for decision-makers.” Then they provide an example of a decision problem in which the optimal decision in the absence of information is not to invest, and in which the value of information falls with increasing risk aversion. We view our work as providing a general framework for thinking about the demand for information, and thereby extending further the logic of their example.

Finally, our approach to ranking information purchases is based on a ranking of preferences for information. Relatively few papers in the literature deal with the comparison of different agents’ preferences for information. One such study is Grant et al. (1998), which explores intrinsic preferences for information, that is, preferences that are unrelated to the fact that information can lead to more profitable decisions. This is very different from our framework, since our agents like information precisely because it helps them to make better decisions. But, interestingly, just as we found (see Theorem 1) that risk aversion is related to preferences for information, Grant et al. (1998) find that their notion of Information Loving is related to the convexity of preferences.

## 8. Conclusion

There are multiple ways to index information, but ours is the first index that captures the information-price tradeoff, by indexing information purchases. Our normalized-value index is based on a duality principle (as in Aumann and Serrano (2008)) between value and preference for information in settings in which the investment opportunities are described by a no-arbitrage condition. Because no-arbitrage assets provide a clean way to measure the value of information, we have been able to extend their use from such previous studies as (Kelly, 1956; Arrow, 1971; Cabrales et al., 2013). The result we offer here can be viewed as a translation of Aumann and Serrano (2008) to informational settings with no-arbitrage investments. In such settings, the new index captures an aspect of the demand for information in a market economy. Our paper has characterized agents’ demand for information using a simply computable number called the normalized value of the information purchase, which relates to agents’ risk aversion. For practical applications, one can rely on some of the known estimates for risk aversion levels provided in the literature in order to identify prices at which every agent—no agent—will accept that information purchase. In this way we can describe a useful inverse demand curve for information in no-arbitrage investment settings. In our analysis, the asset prices were exogenous. Prices could also be an endogenous function of the information agents gather. This would have an impact on

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<sup>21</sup> Alepuz and Urbano (1995) study the experimentation problem of a monopolist who is trying to learn the slope of her (linear) demand. They show that for small experimentation costs, CARA agents who are more risk-averse experiment less than do less risk-averse agents.

the demand for information. We have ignored such considerations, present in the noisy rational-expectations literature. Combining them with the normalized-value notion developed here could be an interesting future research agenda.

## Appendix A. Supplementary material

Supplementary material related to this article can be found online at <http://dx.doi.org/10.1016/j.jet.2017.05.007>.

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