

Série des Documents de Travail

n° 2017-27

**Local Asymptotic Equivalence of
Pure States Ensembles and Quantum
Gaussian White Noise**

C.BUTUCEA¹

M.GUTA²

M.NUSSBAUM³

Les documents de travail ne reflètent pas la position du CREST et n'engagent que leurs auteurs.
Working papers do not reflect the position of CREST but only the views of the authors.

¹ CREST; ENSAE. E-mail: cristina.butucea@ensae.fr

² University of Nottingham. E-mail : madalin.guta@nottingham.ac.uk

³ Cornell University. E-mail : nussbaum@math.cornell.edu

LOCAL ASYMPTOTIC EQUIVALENCE OF PURE STATES ENSEMBLES AND QUANTUM GAUSSIAN WHITE NOISE

BY CRISTINA BUTUCEA, MĂDĂLIN GUȚĂ AND MICHAEL NUSSBAUM *

CREST, ENSAE, Université Paris-Saclay; University of Nottingham and Cornell University

Dedicated to Richard D. Gill on the occasion of his 66th birthday.

Abstract Quantum technology is increasingly relying on specialised statistical inference methods for analysing quantum measurement data. This motivates the development of “quantum statistics”, a field that is shaping up at the overlap of quantum physics and “classical” statistics. One of the less investigated topics to date is that of statistical inference for infinite dimensional quantum systems, which can be seen as quantum counterpart of non-parametric statistics. In this paper we analyse the asymptotic theory of quantum statistical models consisting of ensembles of quantum systems which are identically prepared in a pure state. In the limit of large ensembles we establish the local asymptotic equivalence (LAE) of this i.i.d. model to a quantum Gaussian white noise model. We use the LAE result in order to establish minimax rates for the estimation of pure states belonging to Hermite-Sobolev classes of wave functions. Moreover, for quadratic functional estimation of the same states we note an elbow effect in the rates, whereas for testing a pure state a sharp parametric rate is attained over the nonparametric Hermite-Sobolev class.

1. Introduction. A striking insight of quantum mechanics is that randomness is a fundamental feature of the physical world at the microscopic level. Any observation made on a quantum system such as an atom or a light pulse, results in a non-deterministic, stochastic outcome. The study of the *direct map* from the system’s state or preparation to the probability distribution of the measurement outcomes, has been one of the core topics in traditional quantum theory. In recent decades the focus of research has shifted from fundamental physics towards applications at the interface with information theory, computer science, and metrology, sharing the paradigm that individual quantum systems are carriers of a new type of information [53].

In many quantum protocols, the experimenter has incomplete knowledge and control of the system and its environment, or is interested in estimating an external field parameter which affects the system dynamics. In this case one deals with a *statistical inverse problem* of inferring unknown state parameters from the measurement data obtained by probing a large number of individual quantum systems. The theory and practice arising from tackling such questions is shaping up into the field of *quantum statistics*, which lies at the intersection of quantum theory and statistical inference [40, 38, 37, 57, 6, 1].

*Supported in part by NSF Grant DMS-1407600

MSC 2010 subject classifications: Primary 62B15, ; secondary 62G05, 62G10, 81P50

Keywords and phrases: Le Cam distance, local asymptotic equivalence, quantum Gaussian process, quantum Gaussian sequence, quantum states ensemble, nonparametric estimation, quadratic functionals, nonparametric sharp testing rates

One of the central problems in quantum statistics is state estimation: given an ensemble of identically prepared, independent systems with unknown state, the task is to estimate the state by performing appropriate measurements and devising estimators based on the measurement data. A landmark experiment aimed at creating multipartite entangled states [35] highlighted the direct practical relevance of efficient estimation techniques for large dimensional systems, the complexity of estimating large dimensional states, and the need for solid statistical methodology in computing reliable “error bars”. This has motivated the development of new methods such as compressed sensing and matrix ℓ_1 -minimisation [30, 29, 23], spectral thresholding for low rank states [14], confidence regions [18, 19, 68, 65, 22].

Another important research direction is towards developing a *quantum decision theory* as the overall mathematical framework for inference involving quantum systems seen as a form of “statistical data”. Typically, the route to finding the building blocks of this theory starts with a decision problem (e.g. testing between two states, or estimating certain parameters of a state) and the problem of finding optimal measurement settings and statistical procedures for treating the (classical, random) measurement data. For instance, in the context of asymptotic binary hypothesis testing, two key results are the quantum Stein lemma [39, 56] and the quantum Chernoff bound [2, 55, 3, 51]. As in the classical case, they describe the exponential decay of appropriate error probabilities for optimal measurements, and they provide operational interpretations for quantum relative entropy, and respectively quantum Chernoff distance. Similarly, an important problem in state estimation is to identify measurements which allow for the smallest possible estimation error. A traditional approach has been to establish a “quantum Cramér-Rao bound” (QCRB) [40, 38, 10] for the covariance of unbiased estimators, where the right side is the inverse of the “quantum Fisher information matrix”, the latter depending only on the structure of the quantum statistical model. However, while the QCRB is achievable asymptotically for one-dimensional parameters, this is not the case for multi-parameter models due to the fact that the measurements which are optimal for different one-dimensional components, are generally incompatible with each other.

These difficulties can be overcome by developing a fundamental theory of comparison and convergence of quantum statistical models, as an extension of its classical counterpart [66, 49]. While classical “data processing” is described by randomisations, physical transformations of quantum systems are described by *quantum channels* [53]. Following up on this idea, Petz and Jencova [59] have obtained a general characterisation of *equivalent models*, as families of states that are related by quantum channels in both directions. This naturally leads to the notion of *Le Cam distance* between quantum statistical models as the least trace-norm error incurred when trying to map one model into another via quantum channels [44]. In this framework, the asymptotic theory of state estimation can be investigated by adopting ideas from the classical *local asymptotic normality* (LAN) theory [49]. Quantum LAN theory [33, 32, 44] shows that the sequence of models describing large samples of identically prepared systems can be approximated by a simpler *quantum Gaussian shift model*, in the neighbourhood of an interior point of the parameter space. The original optimal state estimation problem is then solved by combining LAN theory with known procedures for estimation of Gaussian states [31, 34, 25].

In this paper we extend the scope of the quantum LAN theory to cover *non-parametric* quantum models; more precisely we will be interested in the set of *pure states* (one-dimensional projections) on *infinite dimensional Hilbert spaces*. Infinite dimensional systems such as light pulses, free particles, are commonly encountered in quantum physics, and their estimation is an important topic in quantum optics [50]. The minimax results derived in this paper can

serve as a benchmark for the performance of specific methods such as for instance quantum homodyne tomography [1, 13], by comparing their risk with the minimax risk derived here.

The paper is organised as follows. In Section 2 we review the basic notions of quantum mechanics needed for understanding the physical context of our investigation. In particular, we define the concepts of state, measurement and quantum channel which can loosely be seen as quantum analogs of probability distribution and Markov kernels, respectively. We further introduce the formalism of quantum Gaussian states, the Fock spaces and second quantisation, which establish the quantum analogs of Gaussian distributions, Gaussian sequences and Gaussian processes in continuous time. In Section 3.1 we review results in classical statistics on non-parametric asymptotic equivalence which serve as motivation and comparison to our work. In Section 3.2 we introduce the general notion of a quantum statistical model and the Le Cam distance between two models. In particular, in Section 3.3 we define the i.i.d. and Gaussian quantum models which are analysed in the remainder of the paper.

One of the main results is Theorem 4.1 giving the *local asymptotic equivalence* (LAE) between the non-parametric i.i.d. pure states model and the Gaussian shift model. This extends the existing local asymptotic normality theory from parametric to non-parametric (infinite dimensional) models. Section 5 details three applications of the LAE result in Theorem 4.1. In Section 5.1 we derive the asymptotic minimax rates and provide concrete estimation procedures for state estimation with respect to the trace-norm and Bures distances, which are analogues of the norm-one and Hellinger distances respectively. The main results are Theorems 5.1 and 5.3 which deal with the upper and respectively lower bound for a model consisting of an ensemble of n independent identically prepared systems in a pure state belonging to a Hermite-Sobolev class $S^\alpha(L)$ of wave functions. In Theorem 5.1 we describe a specific measurement procedure which provides an estimator whose risk attains the nonparametric rate $n^{-\alpha/(2\alpha+1)}$. The lower bound follows by using the LAE result to approximate the model with a Gaussian one, combined with the lower bound for the corresponding quantum Gaussian model derived in Theorem 5.2. In Section 5.2 we consider the estimation of a state functional corresponding to the expectation of a power $N^{2\beta}$ of the number operator. Theorems 5.4 and 5.5 establish the upper and lower bounds for functional estimation for the Hermite-Sobolev class $S^\alpha(L)$. The minimax rates are $n^{-1/2}$ (parametric) if $\alpha \geq 2\beta$, and $n^{-1+\beta/\alpha}$ if $\beta < \alpha < 2\beta$. In Section 5.3 we investigate non-parametric testing between a single state and a composite hypothesis consisting of all states outside a ball of shrinking radius. Surprisingly, we find that the minimax testing rates are parametric, in contrast to the non-parametric estimation rates. This fact is closely related to the fact that the optimal estimation and testing measurements are incompatible with each other, so that no single measurement strategy can allow for minimax estimation and testing in the same time. Results on the minimax optimal rate for testing and the sharp asymptotics are given in Theorems 5.6 and 5.7 respectively. Proofs are given in [15].

Notation. Following physics convention, the vectors of a Hilbert space \mathcal{H} will be denoted by the “ket” $|v\rangle$, so that the inner product of two vectors is the “bra-ket” $\langle u|v\rangle \in \mathbb{C}$ which is linear with respect to the right entry and anti-linear with respect to the left entry. Similarly, $M := |u\rangle\langle v|$ is the rank one operator acting as $M : |w\rangle \mapsto M|w\rangle = \langle v|w\rangle|u\rangle$. We denote by $\mathcal{L}(\mathcal{H})$ the space of bounded linear operators on \mathcal{H} which is a C^* -algebra with respect to the operator norm $\|A\| := \sup_{\psi \neq 0} \|A\psi\|/\|\psi\|$. Additionally, $\mathcal{T}_1(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ is the space of Hilbert-Schmidt (or trace-class) operators equipped with the norm-one $\|\tau\|_1 := \text{Tr}(|\tau|)$, where the operator $|\tau| := (\tau^*\tau)^{1/2}$ is the absolute value of τ , and τ^* is the adjoint of τ . Finally, we denote by $\mathcal{T}_2(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ the space of Hilbert-Schmidt operators equipped with the norm-two

$\|\tau\|_2^2 := \text{Tr}(|\tau|^2)$, which is a Hilbert space with respect to the inner product $(\tau, \sigma) := \text{Tr}(\tau^* \sigma)$.

2. Quantum mechanics background. In this section we review some basic notions of quantum mechanics (QM), in as much as it is required for understanding the subsequent results of the paper. Since QM is a probabilistic theory of quantum phenomena, it is helpful to approach the formalism from the perspective of analogies and differences with “classical” probability. We refer to [53] for more details on the quantum formalism.

2.1. States, measurements, channels. The QM formalism assigns to each quantum mechanical system (e.g. an atom, light pulse, quantum spin) a complex Hilbert space \mathcal{H} , called the space of states. For instance, the finite dimensional space \mathbb{C}^d is the Hilbert space of a system with d “energy levels”, while $L^2(\mathbb{R})$ is the space of “wave functions” of a particle moving in one dimension, or of a monochromatic light pulse. The *state* of a quantum system is represented mathematically by a density matrix.

DEFINITION 1. *Let \mathcal{H} be the Hilbert space of a quantum system. A density matrix (or state) on \mathcal{H} is a linear operator $\rho : \mathcal{H} \rightarrow \mathcal{H}$ which is positive (i.e. it is selfadjoint and has non-negative eigenvalues), and has trace one.*

We denote by $\mathcal{S}(\mathcal{H})$ the convex space of states on \mathcal{H} . Its linear span is the space of trace class operators $\mathcal{T}_1(\mathcal{H})$, which is the non-commutative analogue of the space of absolutely integrable functions on a probability space $L_1(\Omega, \Sigma, \mathbb{P})$. For any states ρ_1 or ρ_2 , the convex combination $\lambda\rho_1 + (1 - \lambda)\rho_2$ is also a state which corresponds to randomly preparing the system in either the state ρ_1 or ρ_2 with probabilities λ and respectively $1 - \lambda$. The extremal elements of the convex set $\mathcal{S}(\mathcal{H})$ are the one dimensional projections $P_\psi = |\psi\rangle\langle\psi|$ where $|\psi\rangle$ is a normalised vector, i.e. $\|\psi\| = 1$. Such states are called *pure* (as opposed to mixed states which are convex combinations of pure ones), and are uniquely determined by the vector $|\psi\rangle$. Conversely, the vector $|\psi\rangle$ is fixed by the state up to a complex phase factor, i.e. $|\psi\rangle$ and $|\psi'\rangle := e^{i\phi}|\psi\rangle$ represent the same state.

Although the quantum state encodes all information about the preparation of the system, it is not a directly observable property. Instead, any measurement produces a *random outcome* whose distribution depends on the state, and thus reveals in a probabilistic way a certain aspect of the system’s preparation. The simplest type of measurement is determined by an orthonormal basis (ONB) $\{|i\rangle\}_{i=1}^{\dim\mathcal{H}}$ and a set of possible outcomes $\{\lambda_i\}_{i=1}^{\dim\mathcal{H}}$ in the following way: the outcome is a random variable X taking the value λ_i with probability given by the diagonal elements of ρ in this particular basis

$$\mathbb{P}_\rho([X = \lambda_i]) = \rho_{ii} = \langle i|\rho|i\rangle.$$

More generally, a measurement M with outcomes in a measurable space (Ω, Σ) is determined by a positive operator valued measure.

DEFINITION 2. *A positive operator valued measure (POVM) is a map $M : \Sigma \rightarrow \mathcal{L}(\mathcal{H})$ having the following properties*

- 1) *positivity:* $M(E) \geq 0$ for all events $E \in \Sigma$
- 2) *σ -additivity:* $M(\cup_i E_i) = \sum_i M(E_i)$ for any countable set of mutually disjoint events E_i
- 3) *normalization:* $M(\Omega) = \mathbf{1}$.

The outcome of the corresponding measurement associated to M has probability distribution

$$\mathbb{P}_\rho(E) = \text{Tr}(\rho M(E)), \quad E \in \Sigma.$$

The most important example of a POVM, is that associated to the measurement of an observable, the latter being represented mathematically by a selfadjoint operator $A : \mathcal{H} \rightarrow \mathcal{H}$. The Spectral Theorem shows that such operators can be “diagonalised”, i.e. they have a spectral decomposition

$$A = \int_{\sigma(A)} xP(dx)$$

where $\sigma(A)$ is the spectrum of A , and $\{P(E) : E \in \Sigma\}$ is the collection of spectral projections of A . The corresponding measurement has outcome $a \in \sigma(A)$ with probability distribution $\mathbb{P}_\rho[a \in E] = \text{Tr}(\rho P(E))$.

Unlike “classical” systems which can be observed without disturbing their state, quantum systems are typically perturbed by the measurement, so the system needs to be reprepared in order to obtain more information about the state. In this sense, the system can be seen as a “quantum sample” which it can be converted into a “classical” sample only by performing a measurement. Thus, a measurement can be seen as a “quantum-to-classical randomisation”, i.e. a linear map \mathcal{M} which sends a state ρ to the probability density $\mathcal{M}(\rho) \equiv p_\rho := \frac{d\mathbb{P}_\rho}{d\mathbb{P}}$ with respect to a reference measure \mathbb{P} . The latter can be taken to be \mathbb{P}_{ρ_0} for a strictly positive density matrix ρ_0 . The following lemma summarises this perspective on measurements.

LEMMA 2.1. *Let \mathcal{H} be a Hilbert space, and let (Ω, Σ) be a measurable space. For any fixed state $\rho_0 > 0$ on \mathcal{H} , there is a one-to-one correspondence between POVMs M over (Ω, Σ) and quantum-to-classical randomisations, i.e. linear maps*

$$\mathcal{M} : \mathcal{T}_1(\mathcal{H}) \rightarrow L_1(\Omega, \Sigma, \mathbb{P})$$

which are positive and normalised (maps states into probability densities). The correspondence is given by

$$\mathbb{P}_\rho(E) = \text{Tr}(M(E)\rho) = \int_A p_\rho(\omega)\mathbb{P}_{\rho_0}(d\omega), \quad \mathcal{M}(\rho) \equiv p_\rho := \frac{d\mathbb{P}_\rho}{d\mathbb{P}}.$$

For comparison, recall that a linear map $R : L_1(\Omega', \Sigma', \mathbb{P}') \rightarrow L_1(\Omega, \Sigma, \mathbb{P})$ is a stochastic operator if it maps probability densities into probability densities [64]. Typically such maps arise from Markov kernels and describe randomizations of dominated statistical experiments (models).

While a measurement is a quantum-to-classical randomization, a “quantum-to-quantum randomization” describes how the system’s state changes as a result of time evolution or interaction with other systems. The maps describing such transformations are called quantum channels.

DEFINITION 3. *A quantum channel between systems with Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 is a trace preserving, completely positive linear map $T : \mathcal{T}_1(\mathcal{H}_1) \rightarrow \mathcal{T}_1(\mathcal{H}_2)$.*

The two properties mentioned above are similar to those of a classical randomization, so in particular T maps states into states. However, unlike the classical case, T is required to

satisfy a stronger positivity property: T is *completely positive* if $\text{Id}_m \otimes T$ is positive for all $m \geq 1$, where Id_m is the identity map on the space of m dimensional matrices. This ensures that when the system is correlated with an ancillary system \mathbb{C}^m , and the latter undergoes the identity transformation, the final joint state is still positive, as expected on physical grounds.

The simplest example of a quantum channel is a unitary transformation $\rho \mapsto U\rho U^*$, where U is a unitary operator on \mathcal{H} . More generally, if $|\varphi\rangle \in \mathcal{K}$ is a pure state of an ancillary system, and V is a unitary on $\mathcal{H} \otimes \mathcal{K}$, then

$$\rho \mapsto T(\rho) := \text{Tr}_{\mathcal{K}}(V(\rho \otimes |\varphi\rangle\langle\varphi|)V^*)$$

is a quantum channel describing the system state after interacting with the ancilla. By computing the partial trace $\text{Tr}_{\mathcal{K}}$ over \mathcal{K} with respect to an orthonormal basis $\{|f_i\rangle\}_{i=1}^{\dim\mathcal{K}}$ we obtain the following expression

$$(1) \quad T(\rho) = \sum_i K_i \rho K_i^*$$

where K_i are operators on \mathcal{H} defined by $\langle\psi|K_i|\psi'\rangle := \langle\psi \otimes f_i|U|\psi' \otimes \varphi\rangle$. Note that by definition, these operators satisfy the normalisation condition $\sum_i K_i^* K_i = \mathbf{1}$. Conversely, the Kraus Theorem shows that any quantum channel is of the form (1) with operators K_i respecting the normalisation condition.

2.2. Continuous variables, Fock spaces and Gaussian states. In this section we look at the class of “continuous variables” (cv) systems, which model a variety of physical systems such as light pulses, or free particles. Such systems play an important role in this work as “carriers” of quantum Gaussian states, and in particular in the local asymptotic equivalence result. We refer to [50] for further reading.

2.2.1. One mode systems. We start with the simplest case of a “one-mode” cv system, after which we show how this construction can be extended to more general “multi-mode” cv systems. The Hilbert space of a one-mode system is $L_2(\mathbb{R})$, i.e. the space of square integrable wave functions on the real line. On this we define the selfadjoint operators acting on appropriately defined domains as

$$(Q\psi)(q) = q\psi(q), \quad (P\psi)(q) = -i \frac{d\psi(q)}{dq}$$

which satisfy the “canonical commutation relations” $QP - PQ = i\mathbf{1}$. To better understand the meaning of the observable Q , let us consider its measurement for a pure state $\rho = |\psi\rangle\langle\psi|$ with wave function $|\psi\rangle$. The outcome takes values in \mathbb{R} , and its probability distribution has density with respect to the Lebesgue measure $p_\rho^Q(x) = |\psi(x)|^2$. Similarly, the probability density of the observable P is given by $p_\rho^P(x) = |\psi(x)|^2$, where $\psi \in L^2(\mathbb{R})$ is the Fourier transform of the function $\psi(\cdot)$. When the system under consideration is the free particle, Q and P are usually associated to the position and momentum observables, while for a monochromatic light mode they correspond to the electric and magnetic fields. Note that the distributions of P and Q are not sufficient to identify the state, even in the case of a pure state. However, it turns out that the state is uniquely determined by the collection of probability distributions of all *quadrature* observables $X_\phi := \cos(\phi) \cdot Q + \sin(\phi) \cdot P$ for angles $\phi \in [0, 2\pi]$. To understand this, it is helpful to think of the state of the one-mode cv system as a quantum analogue of a joint distribution of two real valued variables, i.e. a 2D distribution. Indeed, in the latter case,

the distribution is determined by collection of marginals along all directions in the plane (its Radon transform); this fact is exploited in PET tomography which aims at estimating the 2D distribution from samples of its Radon transform. In the quantum case, since Q and P do not commute with each other, they cannot be measured simultaneously and cannot be assigned a joint distribution in a meaningful way. However, the “quasi-distribution” defined below has some of the desired properties, and is very helpful in visualising the quantum state.

DEFINITION 4. For any state $\rho \in \mathcal{T}_1(L_2(\mathbb{R}))$ we define the quantum characteristic function of ρ

$$\widetilde{W}_\rho(u, v) := \text{Tr}(\exp(-iuQ - ivP)\rho).$$

The inverse Fourier transform of \widetilde{W}_ρ with respect to both variables is called Wigner function W_ρ , or quasi-distribution associated to ρ :

$$W_\rho(q, p) = \frac{1}{(2\pi)^2} \iint \exp(iuq + ivp) \widetilde{W}_\rho(u, v) dudv.$$

A consequence of this definition is that the marginal of $W_\rho(q, p)$ along an arbitrary direction with angle ϕ is the probability density of the quadrature X_ϕ introduced above. This is the basis of a quantum state estimation scheme called “quantum homodyne tomography” [50, 1], where the Wigner function plays the role of the 2D distribution from “classical” PET tomography. One of the important differences however, is that the Wigner functions need not be positive in general, and satisfy other constraints which are specific to the quantum setting and can be exploited in the estimation procedure.

The Wigner function representation offers an intuitive route to defining the notion of Gaussian state.

DEFINITION 5. A state ρ of a one-mode cv system is called Gaussian if its Wigner function W_ρ is a Gaussian probability density, or equivalently if it has the quantum characteristic function

$$\widetilde{W}_\rho(u, v) = \exp\left(- (u, v) \frac{V}{2} (u, v)^T\right) \cdot \exp(iuq_0 + ivp_0).$$

where $(q_0, p_0) \in \mathbb{R}^2$ and V (a real positive 2×2 matrix) are the mean and variance of W_ρ , respectively.

In particular, all the quadratures X_ϕ of a Gaussian state have Gaussian distribution. As consequence of the commutation relation $QP - PQ = i\mathbf{1}$ the observables Q and P cannot have arbitrarily small variance simultaneously; in particular, the covariance matrix V must satisfy the “uncertainty principle” $\text{Det}(V) \geq 1/4$, where the equality is achieved if and only if the state is a pure Gaussian state.

We will be particularly interested in *coherent* states $|G(z)\rangle$ which are *pure* Gaussian states whose Wigner functions have covariance matrix $V = I_2/2$, where I_2 is the 2×2 identity matrix. To give a concrete Hilbert space representation, it is convenient to introduce a special orthonormal basis of $L_2(\mathbb{R})$, consisting of the eigenvectors $\{|0\rangle, |1\rangle, \dots\}$ of the *number operator* $N = a^*a$, with $N|k\rangle = k|k\rangle$. Here, the operators $a^* = (Q - iP)/\sqrt{2}$ and $a = (Q + iP)/\sqrt{2}$ are called *creation and annihilation operators* and act as “ladder operators” on the number basis vectors (or Fock states)

$$a|k\rangle = \sqrt{k}|k-1\rangle, \quad a^*|k\rangle = \sqrt{k+1}|k+1\rangle.$$

The coherent states denoted by $|G(z)\rangle$ are obtained by applying the unitary Weyl (displacement) operators to the vacuum state $|0\rangle$

$$(2) \quad |G(z)\rangle = \exp(za^* - \bar{z}a)|0\rangle = \exp(-|z|^2/2) \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}} |k\rangle,$$

where $z \in \mathbb{C}$ is the eigenvalue of the annihilation operator $a|G(z)\rangle = z|G(z)\rangle$; in particular, the quadrature means are $\langle G(z)|Q|G(z)\rangle = \sqrt{2}\operatorname{Re}(z)$ and $\langle G(z)|P|G(z)\rangle = \sqrt{2}\operatorname{Im}(z)$, and the Wigner function is given by

$$(3) \quad W_{|z\rangle}(q, p) = \frac{1}{\pi} \exp\left(-\frac{(q - \sqrt{2}x)^2}{2} - \frac{(p - \sqrt{2}y)^2}{2}\right), \quad q, p \in \mathbb{R}.$$

Equation (2) implies that the number operator N has a Poisson distribution with mean $|z|^2$. Additionally, it can be seen from the Fourier expansion in the second equality that the unitary $\Gamma(\phi) = \exp(i\phi N)$ acts by rotating the coherent states by an angle ϕ in the complex plane, i.e. $\Gamma(\phi)|G(z)\rangle = |G(e^{i\phi}z)\rangle$.

Another important class of Gaussian states are the mixed diagonal states

$$(4) \quad \Phi(r) = (1 - r) \sum_{k=0}^{\infty} r^k |k\rangle\langle k|, \quad 0 < r < 1$$

which are also called thermal states, cf. section 3.3 in [50]. The corresponding Wigner function is a centred Gaussian

$$(5) \quad W_{\Phi(r)}(q, p) = \frac{1}{2\pi\sigma^2(r)} \exp\left(-\frac{q^2 + p^2}{2\sigma^2(r)}\right).$$

with covariance matrix $V = \sigma^2(r) \cdot I_2$ where $\sigma^2(r) = \frac{1}{2} \frac{1+r}{1-r}$.

PROPOSITION 2.2. *Consider the family of coherent states $\{|G(z)\rangle\langle G(z)|, z \in \mathbb{C}\}$, with random displacement (location) z distributed according to $\Pi(dz)$, having a Gaussian law with covariance matrix $\sigma^2 \cdot I_2$. Then, the mixed state $\Phi = \int |G(z)\rangle\langle G(z)|\Pi(dz)$ is the thermal state $\Phi(r)$ with $r = \frac{2\sigma^2}{2\sigma^2+1}$.*

PROOF. Consider the corresponding Wigner function

$$(6) \quad \begin{aligned} W_{\Phi}(q, p) &= \int W_{|G(z)\rangle}(q, p) \exp\left(-\frac{1}{2\sigma^2}(x^2 + y^2)\right) \frac{1}{2\pi\sigma^2} dx dy \\ &= \frac{1}{\pi\sigma^2} \int \exp\left(-\frac{(q - \sqrt{2}x)^2}{2} - \frac{x^2}{2\sigma^2}\right) \frac{dx}{\sqrt{2\pi}} \cdot \int \exp\left(-\frac{(p - \sqrt{2}y)^2}{2} - \frac{y^2}{2\sigma^2}\right) \frac{dy}{\sqrt{2\pi}} \\ &= \frac{1}{\pi(4\sigma^2 + 1)} \exp\left(-\frac{q^2 + p^2}{2(2\sigma^2 + 1/2)}\right). \end{aligned}$$

Therefore, the state Φ is identical to the thermal state $\Phi(r)$ with $2\sigma^2 + \frac{1}{2} = \frac{1}{2} \frac{1+r}{1-r}$, or equivalently $r = \frac{2\sigma^2}{1+2\sigma^2}$. \square

This fact will be used later on in in section 5 in applications to functional estimation and testing.

2.2.2. *Fock spaces and second quantisation.* The above construction can be generalised to multimode systems by tensoring several one-mode systems. Thus, the Hilbert space of a k -mode system is $L_2(\mathbb{R})^{\otimes k} \cong L_2(\mathbb{R}^k)$, upon which we define “canonical pairs” (Q_i, P_i) acting on the i -th tensor as above, and as identity on the other tensors. Similarly we define the one-mode operators a_i, a_i^*, N_i . The number basis consists now of tensor products $|\mathbf{n}\rangle := \otimes_{i=1}^k |n_i\rangle$ indexed by the sequences of integers $\mathbf{n} = (n_1, \dots, n_k)$. A multimode coherent state is a tensor product of one-mode coherent states

$$(7) \quad |G(\mathbf{z})\rangle = \otimes_{i=1}^k |G(z_i)\rangle = \exp(\mathbf{z}\mathbf{a}^\dagger - \mathbf{a}\mathbf{z}^\dagger) |0\rangle = \exp(-|z|^2/2) \sum_{\mathbf{n}=0}^{\infty} \left(\prod_{i=1}^k \frac{z_i^{n_i}}{\sqrt{n_i!}} \right) |\mathbf{n}\rangle \in L_2(\mathbb{R})^{\otimes k}$$

where $\mathbf{z} = (z_1, \dots, z_k)$ is the vector of means, $\mathbf{a} = (a_1, \dots, a_k)$, and \dagger denotes the transposition and adjoint (complex conjugation) of individual entries.

We will now extend this construction to systems with infinitely many modes. One way to do this is by defining an infinite tensor product of one-mode spaces, as completion of the space spanned by tensors in which all but a finite number of modes are in the vacuum state. Instead, we will present an equivalent but more elegant construction called *second quantisation* which will be useful for later considerations.

DEFINITION 6. *Let \mathcal{K} be a Hilbert space. The Fock space over \mathcal{K} is the Hilbert space*

$$(8) \quad \mathcal{F}(\mathcal{K}) = \bigoplus_{n \geq 0} \mathcal{K}^{\otimes_s n}$$

where $\mathcal{K}^{\otimes_s n}$ denotes the n -fold symmetric tensor product, i.e. the subspace of $\mathcal{K}^{\otimes n}$ consisting of vectors which are symmetric under permutations of the tensors. The term $\mathcal{K}^{\otimes_s 0} =: \mathbb{C}|0\rangle$ is called the vacuum state.

In this definition the space \mathcal{K} should be regarded as the “space of modes” rather than physical states. As we will see below, by fixing an orthonormal basis in \mathcal{K} , we can establish an isomorphism between the Fock space $\mathcal{F}(\mathcal{K})$ and a tensor product of one-mode cv spaces, one for each basis vector. In particular, if $\mathcal{K} = \mathbb{C}$, then $\mathcal{F}(\mathbb{C}) \cong L^2(\mathbb{R})$ so that the one-dimensional subspaces in the direct sum in (8) correspond to the number basis vectors $|0\rangle, |1\rangle, \dots \in L^2(\mathbb{R})$ of a one-mode cv system.

We now introduce the general notion of coherent state on a Fock space.

DEFINITION 7. *Let $\mathcal{F}(\mathcal{K})$ be the Fock space over \mathcal{K} . For each $|v\rangle \in \mathcal{K}$ we define an associated coherent state*

$$|G(v)\rangle := e^{-\|v\|^2/2} \bigoplus_{n \geq 0} \frac{1}{\sqrt{n!}} |v\rangle^{\otimes n} \in \mathcal{F}(\mathcal{K}).$$

The coherent vectors form a dense subspace of $\mathcal{F}(\mathcal{K})$. This fact can be used to prove the following factorisation property, and to define the annihilation operators below. Let $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ be a direct sum decomposition of \mathcal{K} into orthogonal subspaces, and let $|v\rangle = |v_0\rangle \oplus |v_1\rangle$ be the decomposition of a generic vector $|v\rangle \in \mathcal{K}$. Then the map

$$\begin{aligned} U : \mathcal{F}(\mathcal{K}) &\rightarrow \mathcal{F}(\mathcal{K}_0) \otimes \mathcal{F}(\mathcal{K}_1) \\ U : |G(v)\rangle &\mapsto |G(v_0)\rangle \otimes |G(v_1)\rangle \end{aligned}$$

is unitary. We will use this correspondence to identify $\mathcal{F}(\mathcal{K})$ with the tensor product $\mathcal{F}(\mathcal{K}_0) \otimes \mathcal{F}(\mathcal{K}_1)$. By the same argument, for any orthonormal basis $\{|e_1\rangle, |e_2\rangle, \dots\}$ of \mathcal{K} , the Fock space $\mathcal{F}(\mathcal{K})$ is isomorphic with the tensor product of one mode spaces $\mathcal{F}_i := \mathcal{F}(\mathbb{C}|e_i\rangle)$ and the coherent states factorise as

$$(9) \quad \begin{aligned} \mathcal{F}(\mathcal{K}) &\cong \bigotimes_i \mathcal{F}_i \\ |G(u)\rangle &\cong \bigotimes_i |G(u_i)\rangle, \quad u_i = \langle e_i | u \rangle. \end{aligned}$$

so that we recover the formula (7).

We define the *annihilation operators* through their action on coherent states as follows: for each mode $|u\rangle \in \mathcal{K}$ the associated annihilator $a(u) : \mathcal{F}(\mathcal{K}) \rightarrow \mathcal{F}(\mathcal{K})$ is given by

$$a(u) : |G(v)\rangle = \langle u | v \rangle |G(v)\rangle, \quad |v\rangle \in \mathcal{K}.$$

Then the annihilation and (their adjoint) the creation operators satisfy the commutation relations

$$a(u)a^*(w) - a^*(w)a(u) = \langle u | w \rangle \mathbf{1}.$$

For each mode we can also define the canonical operators $Q(u), P(u)$ and the number operator $N(u)$ in terms of $a(u), a^*(u)$ as in the one-mode case. Moreover, if $|u\rangle = |u_0\rangle \oplus |u_1\rangle$ is the decomposition of $|u\rangle$ as above, then $a(u_0)$ acts as $a(u_0) \otimes \mathbf{1}_{\mathcal{F}(\mathcal{K}_1)}$, when the Fock space is represented in the tensor product form. Similar decompositions hold for $a^*(u_0), N(u_0), a(u_1), a^*(u_1), N(u_1)$.

The second quantisation has the following functorial properties which will be used later on.

DEFINITION 8. *Let $W : \mathcal{K} \rightarrow \mathcal{K}$ be a unitary operator. The quantisation operator $\Gamma(W)$ is the unitary defined by $\Gamma(W) : \mathcal{F}(\mathcal{K}) \rightarrow \mathcal{F}(\mathcal{K})$ by*

$$(10) \quad \Gamma(W) := \bigoplus_{n \geq 0} W^{\otimes n}$$

where $W^{\otimes n}$ acts on the n -th level of the Fock space $\mathcal{K}^{\otimes n}$.

From the definition it follows that the action of $\Gamma(W)$ on coherent states is covariant in the sense that

$$\begin{aligned} \Gamma(W) : \mathcal{F}(\mathcal{K}) &\rightarrow \mathcal{F}(\mathcal{K}) \\ \Gamma(W) : |G(v)\rangle &\mapsto |G(Wv)\rangle. \end{aligned}$$

In particular, it follows from the definitions that $\Gamma(e^{i\phi}\mathbf{1}) = \exp(i\phi N)$, where N is the total number operator, whose action on the n -th level of the Fock space is $N|v\rangle^{\otimes n} = n|v\rangle^{\otimes n}$. Note that while $|v\rangle$ and $e^{i\phi}|v\rangle$ differ only by a phase and hence represent the same state, the corresponding coherent states $|G(v)\rangle$ and $\Gamma(e^{i\phi})|G(v)\rangle = |G(e^{i\phi}v)\rangle$ are linearly independent and represent different states.

As in the single mode case, the coherent states can be obtained by acting with the unitary displacement (or Weyl) operators onto the vacuum

$$|G(u)\rangle = \exp(a^*(u) - a(u))|0\rangle$$

Moreover, the coherent states $|G(u)\rangle$ are Gaussian with respect to all coordinates. The means of annihilation operators are given by $\langle G(u)|a(w)|G(u)\rangle = \langle w|v\rangle$, from which we can deduce that the coordinates $(Q(w), P(w))$ have means $(\sqrt{2}\text{Re}\langle w|u\rangle, \sqrt{2}\text{Im}\langle w|u\rangle)$. The covariance of coherent states is constant (independent of the displacement u), and is given by $\langle 0|a(w)a^*(v)|0\rangle = \langle w|v\rangle$. This implies that orthogonal modes (i.e. $\langle w|v\rangle = 0$) have independent pairs of coordinates.

2.3. Metrics on the space of states. For future reference we review here the states space metrics used in the paper. Recall that the space of states $\mathcal{S}(\mathcal{H})$ on a Hilbert space \mathcal{H} is the cone of positive, trace one operators in $\mathcal{T}_1(\mathcal{H})$. The *norm-one* (or trace-norm) distance between two states $\rho_0, \rho_1 \in \mathcal{S}(\mathcal{H})$ is given by

$$\|\rho_0 - \rho_1\|_1 := \text{Tr}(|\rho_0 - \rho_1|)$$

where $|\tau| := \sqrt{\tau^*\tau}$ denotes the absolute value of τ . The norm-one distance can be interpreted in terms of the maximum difference between expectations of bounded observables

$$\|\rho_0 - \rho_1\|_1 = 2 \sup_{A: \|A\| \leq 1} |\text{Tr}(\rho_0 A) - \text{Tr}(\rho_1 A)|.$$

Another interpretation is in terms of quantum testing. Let $M = (M_0, M_1)$ be a binary POVM used to test between hypotheses $H_0 := \{\text{measured state is } \rho_0\}$ and $H_1 := \{\text{measured state is } \rho_1\}$. The sum of error probabilities is

$$\mathbb{P}_e^M = \text{Tr}(M_0 \rho_1) + \text{Tr}(M_1 \rho_0).$$

By optimizing over all possible POVM we obtain [38] the optimal error probability sum

$$(11) \quad \mathbb{P}_e^* := \inf_M \mathbb{P}_e^M = 1 - \frac{1}{2} \|\rho_0 - \rho_1\|_1.$$

In the special case of pure states, the norm-one distance is given by

$$(12) \quad \||\psi_0\rangle\langle\psi_0| - |\psi_1\rangle\langle\psi_1|\|_1 = 2\sqrt{1 - |\langle\psi_0|\psi_1\rangle|^2},$$

as proven e.g. in [45]. The previous formula becomes for coherent states

$$\||G(\psi_0)\rangle\langle G(\psi_0)| - |G(\psi_1)\rangle\langle G(\psi_1)|\|_1 = 2\sqrt{1 - \exp(-\|\psi_0 - \psi_1\|^2)}.$$

The second important metric is the *Bures distance* whose square is given by

$$d_b^2(\rho_0, \rho_1) := 2(1 - \text{Tr}(\sqrt{\sqrt{\rho_0}\rho_1\sqrt{\rho_0}}))$$

and is a quantum extension of the Hellinger distance. In the case of pure states the Bures distance becomes

$$(13) \quad d_b^2(|\psi_0\rangle\langle\psi_0|, |\psi_1\rangle\langle\psi_1|) = 2(1 - |\langle\psi_0|\psi_1\rangle|)$$

so for coherent states it is given by

$$d_b^2(|G(\psi_0)\rangle\langle G(\psi_0)|, |G(\psi_1)\rangle\langle G(\psi_1)|) := 2 \left(1 - \exp\left(-\frac{1}{2}\|\psi_0 - \psi_1\|^2\right) \right).$$

Similarly to the classical case, the following inequality holds for arbitrary states [24]

$$(14) \quad d_b^2(\rho_0, \rho_1) \leq \|\rho_0 - \rho_1\|_1 \leq 2d_b(\rho_0, \rho_1).$$

Moreover, since $|\langle \psi_0 | \psi_1 \rangle|^2 \leq |\langle \psi_0 | \psi_1 \rangle|$, the additional inequality holds for *pure* states

$$(15) \quad \|\rho_0 - \rho_1\|_1 \geq \sqrt{2}d_b(\rho_0, \rho_1).$$

This means that for pure states, the trace and Bures distances are equivalent (up to constants).

Finally, we will be using the fact that both the norm-one and the Bures distance are contractive under quantum channels. $T : \mathcal{T}_1(\mathcal{H}) \rightarrow \mathcal{T}_1(\mathcal{H}')$, i.e.

$$\|T(\rho_0) - T(\rho_1)\|_1 \leq \|\rho_0 - \rho_1\|_1, \quad d_b^2(T(\rho_0), T(\rho_1)) \leq d_b^2(\rho_0, \rho_1).$$

3. Classical and quantum statistical models. In this section we review key elements of quantum statistics, and introduce the quantum statistical models which will be analysed later on. For comparison, we review certain asymptotic equivalence results for related classical statistical models.

3.1. Classical models. Here we review several asymptotic normality results for classical models which are analogous to the quantum models investigated in the paper.

A *classical* statistical model is defined as a family of probability distributions $\mathcal{Q} = \{\mathbb{P}_f : f \in \mathcal{W}\}$ on a measurable space $(\mathcal{X}, \mathcal{A})$, indexed by an unknown, possibly infinite dimensional parameter f to be estimated, which belongs to a parameter space \mathcal{W} . In the asymptotic framework considered here we assume that we are given a (large) number n of independent, identically distributed samples X_1, \dots, X_n from \mathbb{P}_f , from which we would like to estimate f . If $d : \mathcal{W} \times \mathcal{W} \rightarrow \mathbb{R}_+$ is a chosen loss function, then the risk of an estimator $\hat{f}_n = \hat{f}_n(X_1, \dots, X_n)$ is

$$R(\hat{f}_n, f) = \mathbb{E}_f \left[d(\hat{f}_n, f)^2 \right].$$

In nonparametric statistics, the parameter of the model f is often a function that belongs to a smoothness class. We consider two classes \mathcal{W} : the periodic Sobolev class $\mathcal{S}^\alpha(L)$ of functions on $[0, 1]$ with smoothness $\alpha > 1/2$, and the Hölder class $\Lambda^\alpha(L)$, with smoothness $\alpha > 0$. For any $f \in \mathbb{L}_2[0, 1]$, let $\{f_j, j \in \mathbb{Z}\}$ be the set of Fourier coefficients with respect to the standard trigonometric basis. The classes are defined as

$$\mathcal{S}^\alpha(L) := \left\{ f : [0, 1] \rightarrow \mathbb{R} : \sum_{j \in \mathbb{Z}} \int |f_j|^2 |j|^{2\alpha} du \leq L \right\}.$$

and

$$\Lambda^\alpha(L) := \{f : [0, 1] \rightarrow \mathbb{R} : |f(x) - f(y)| \leq L|x - y|^\alpha, x, y \in [0, 1]\}.$$

In addition, when densities f are considered, we will assume that \mathcal{W} includes an additional restriction to a class

$$\mathcal{D}_\varepsilon = \left\{ f : [0, 1] \rightarrow [\varepsilon, \infty) : \int_{[0,1]} f(x) dx = 1 \right\}$$

for some $\varepsilon > 0$.

Density model. The classical density model consists of n observations X_1, \dots, X_n which are independent, identically distributed (i.i.d.) with common probability density f

$$\mathcal{P}_n = \left\{ \mathbb{P}_f^{\otimes n} : f \in \mathcal{W} \right\}.$$

Gaussian regression model with fixed equidistant design. In this model, we observe Y_1, \dots, Y_n such that

$$Y_i = f^{1/2} \left(\frac{i}{n} \right) + \xi_i, \quad i = 1, \dots, n,$$

where the errors ξ_1, \dots, ξ_n are i.i.d., standard Gaussian variables. Denote the Gaussian regression model by

$$\mathcal{R}_n = \left\{ \bigotimes_{i=1}^n \mathcal{N} \left(f^{1/2} \left(\frac{i}{n} \right), 1 \right) : f \in \mathcal{W} \right\}.$$

Gaussian white noise model. In this model the square-root density $f^{1/2}$ is observed with Gaussian white noise of variance n^{-1} , i.e.

$$(16) \quad dY_t = f^{1/2}(t)dt + \frac{1}{\sqrt{n}}dW_t, \quad t \in [0, 1].$$

If we denote by \mathbb{Q}_f the probability distribution of $\{Y(t) : t \in [0, 1]\}$, the corresponding model is

$$\mathcal{F}_n := \{ \mathbb{Q}_f : f \in \mathcal{W} \}.$$

Gaussian sequence model. In this model we observe a sequence of Gaussian random variables with means equal to the coefficients of $f^{1/2}$ in some orthonormal basis of $\mathbb{L}_2[0, 1]$ for $f \in \mathcal{F}$

$$(17) \quad y_j = \theta_j(f^{1/2}) + \frac{1}{\sqrt{n}}\xi_j, \quad i = 1, 2, \dots$$

where $\{\xi_i\}_{i \geq 1}$ are Gaussian i.i.d. random variables. We denote this model

$$\mathcal{N}_n = \left\{ \bigotimes_{j \geq 1} \mathcal{N} \left(\theta_j \left(f^{1/2} \right), \frac{1}{n} \right) : f \in \mathcal{W} \right\}.$$

In [54] it was shown that the sequences of models \mathcal{P}_n and \mathcal{F}_n are asymptotically equivalent in the sense that their Le Cam distance converges to zero as $n \rightarrow \infty$ when $\mathcal{W} = \Lambda^\alpha(L) \cap \mathcal{D}_\varepsilon$ with $\alpha > 1/2$; in [12], a similar result was established for \mathcal{R}_n and \mathcal{F}_n (more precisely, with $f^{1/2}$ any real valued function $f^{1/2} \in \Lambda^\alpha(L)$). Later, [63] showed that models \mathcal{F}_n and \mathcal{N}_n are asymptotically equivalent over periodic Sobolev classes $f^{1/2} \in \mathcal{S}^\alpha(L)$ with smoothness $\alpha > 1/2$. Among many other results [28] considered generalized linear models, [11] regression models with random design and [61] multivariate and random design, [27] compared the stationary Gaussian process with the Gaussian white noise model \mathcal{F}_n . In [60] sharp rates of convergence are obtained for the equivalence of \mathcal{P}_n and \mathcal{F}_n , including also Poisson process models.

In all classical results, the underlying nonparametric function was assumed to belong to a smoothness class in order to establish asymptotic equivalence of models. In the quantum setup of pure states and Gaussian states that we discuss later on, no such smoothness assumption is needed.

3.2. *Quantum models, randomisations and convergence.* In this subsection we introduce the basic notions of a theory of quantum statistical models which is currently still in its early stages, cf. [32, 25] for more details. We will focus on the notions of quantum-to-classical randomisation carried out through measurements, and quantum-to-quantum randomisations implemented by quantum channels, which allow us to define the equivalence and the Le Cam distance between models.

In analogy to the classical case, we make the following definition.

DEFINITION 9. *A quantum statistical model over a parameter space Θ consists of a family of quantum states $\mathcal{Q} = \{\rho_\theta : \theta \in \Theta\}$ on a Hilbert space \mathcal{H} , indexed by an unknown parameter $\theta \in \Theta$.*

A simple example is a family of pure states $\{\rho_\theta = |\psi_\theta\rangle\langle\psi_\theta| : \theta \in \mathbb{R}\}$ with $|\psi_\theta\rangle := \exp(i\theta H)|\psi\rangle$, where H is a selfadjoint operator generating the one-dimensional family of unitaries $\exp(i\theta H)$, and $|\psi\rangle \in \mathcal{H}$ is a fixed vector. Physically, the parameter θ could be for instance time, a phase, or an external magnetic field. Another example is that of a completely unknown state of a finite dimensional system, which can be parametrised in terms of its density matrix elements, or the eigenvalues and eigenvectors. In order to increase the estimation precision one typically prepares a number n of identical and independent copies of the state ρ_θ , in which case the corresponding model is $\mathcal{Q}_n := \{\rho_\theta^{\otimes n} : \theta \in \Theta\}$. Our work deals with *non-parametric* quantum statistical models for which the underlying Hilbert space is infinite dimensional, as we will detail below.

In order to obtain information about the parameter θ , we need to perform measurements on the system prepared in ρ_θ . Using the random measurement data, we then employ statistical methods to solve specific decision problems. For instance, the task of estimating an unknown quantum state (also known as quantum tomography) is a key component of quantum engineering experiments [35]. In particular, the estimation of large dimensional states has received significant attention in the context of compressed sensing [30, 23], and estimation of low rank states [14]. Suppose that we perform a measurement M on the system in state ρ_θ , and obtain a random outcome $O \in \Omega$ with distribution $\mathbb{P}_\theta^M(E) := \text{Tr}(\rho_\theta M(E))$, cf. section 2. The measurement data is therefore described by the classical model $\mathcal{P}^M := \{\mathbb{P}_\theta^M : \theta \in \Theta\}$, and the estimation problem can be treated using “classical” statistical methods. The measurement map

$$\begin{aligned} \mathcal{M} &: \mathcal{T}_1 \rightarrow L_1(\Omega, \Sigma, \mathbb{P}) \\ \mathcal{M} &: \rho_\theta \mapsto p_\theta := \frac{d\mathbb{P}_\theta}{d\mathbb{P}} \end{aligned}$$

can be seen as a randomisation from a quantum to a classical model, which intuitively means that \mathcal{Q} is more informative than \mathcal{P}^M for any measurement M . Here \mathbb{P} can be chosen to be the distribution corresponding to an arbitrary full rank (strictly positive) state ρ which insures the existence of all probability densities p_θ . One of the distinguishing features of quantum statistics is the possibility to choose appropriate measurements for specific statistical problems (e.g. estimation, testing) and the fact that optimal measurements for different problems may be incompatible with each other. In the applications section we will discuss specific instances of this phenomenon.

Beside measurements, the quantum model \mathcal{Q} can be transformed into another quantum model $\mathcal{Q}' := \{\rho'_\theta : \theta \in \Theta\}$ on a Hilbert space \mathcal{H}' by means of a *quantum randomisation*, i.e.

by applying a quantum channel

$$\begin{aligned} T &: \mathcal{T}_1(\mathcal{H}) \rightarrow \mathcal{T}_1(\mathcal{H}') \\ T &: \rho_\theta \mapsto \rho'_\theta. \end{aligned}$$

The model \mathcal{Q}' is less informative than \mathcal{Q} in the sense that for any measurement M' on \mathcal{H}' one can construct the measurement $M := M' \circ T$ on \mathcal{H} such that $\mathbb{P}_\theta^{M'} = \mathbb{P}_\theta^M$ for all θ . If there exists another channel S such that $S(\rho'_\theta) = \rho_\theta$ for all θ we say (in analogy to the classical case) that the models \mathcal{Q} and \mathcal{Q}' are *equivalent*; in particular, for any statistical decision problem, one can match a procedure for one model with a procedure with the same risk, for the other model. A closely related concept is that of *quantum sufficiency* whose theory was developed in [59]. More generally, we define the Le Cam distance in analogy to the classical case [49].

DEFINITION 10. *Let \mathcal{Q} and \mathcal{Q}' be two quantum models over Θ . The deficiency between \mathcal{Q} and \mathcal{Q}' is defined by*

$$\delta(\mathcal{Q}, \mathcal{Q}') := \inf_T \sup_{\theta \in \Theta} \|T(\rho_\theta) - \rho'_\theta\|_1$$

where the infimum is taken over all channels T . The Le Cam distance between \mathcal{Q} and \mathcal{Q}' is defined as

$$(18) \quad \Delta(\mathcal{Q}, \mathcal{Q}') := \max(\delta(\mathcal{Q}, \mathcal{Q}'), \delta(\mathcal{Q}', \mathcal{Q})).$$

Its interpretation is that models which are “close” in the Le Cam distance have similar statistical properties. In practice, this metric is often used to approximate a sequence of models by another sequence of simpler models, providing a method to establish asymptotic minimax risks. In particular, the approximation of i.i.d. quantum statistical models by quantum Gaussian ones has been investigated in [33, 32, 44], in the case of finite dimensional systems with arbitrary mixed states. Our goal is to extend these results to non-parametric models consisting of pure states on infinite dimensional Hilbert spaces. The following lemma will be used later on.

LEMMA 3.1. *Let $\mathcal{Q}, \mathcal{Q}'$ be two quantum models as defined above. Let $\rho_i = \sum_j \mu_{i,j} \rho_{\theta_{i,j}}$ be two arbitrary mixtures ($i = 1, 2$) of states in \mathcal{Q} and let $\rho'_i = \sum_j \mu_{i,j} \rho'_{\theta_{i,j}}$ be their counterparts in \mathcal{Q}' . Then*

$$\|\rho'_1 - \rho'_2\|_1 - 2\Delta(\mathcal{Q}, \mathcal{Q}') \leq \|\rho_1 - \rho_2\|_1 \leq \|\rho'_1 - \rho'_2\|_1 + 2\Delta(\mathcal{Q}, \mathcal{Q}').$$

PROOF. Since quantum channels are contractive with respect to the norm-one

$$\|S(\rho'_1) - S(\rho'_2)\|_1 \leq \|\rho'_1 - \rho'_2\|_1$$

and by the triangle inequality we get

$$\|\rho_1 - \rho_2\|_1 \leq \|\rho_1 - S(\rho'_1)\|_1 + \|S(\rho'_1) - S(\rho'_2)\|_1 + \|S(\rho'_2) - \rho_2\|_1 \leq 2\Delta(\mathcal{Q}, \mathcal{Q}') + \|\rho'_1 - \rho'_2\|_1$$

The second inequality can be shown in a similar way. \square

3.3. *The i.i.d. and the quantum white noise models.* We now introduce the non-parametric quantum models investigated in the paper, and discuss the relationship with the classical models described in section 3.1.

Let \mathcal{H} be an infinite dimensional Hilbert space and let $B := \{|e_0\rangle, |e_1\rangle, \dots\}$ be a fixed orthonormal basis in \mathcal{H} . The Fourier decomposition of an arbitrary vector is written as $|\psi\rangle = \sum_{j=0}^{\infty} \psi_j |e_j\rangle$. Since most of the models will consist of pure states, we will sometimes define them in terms of the Hilbert space vectors rather than the density matrices, but keep in mind that the vectors are uniquely defined only up to a complex phase.

Let us consider the general problem of estimating an unknown pure quantum state in \mathcal{H} . For *finite* dimensional systems, the risk with respect to typical rotation invariant loss functions scales linearly with the number of parameters [26], hence with the dimension of the space. Therefore, since \mathcal{H} is infinite dimensional, it is not possible to develop a meaningful estimation theory without any prior information about the state. Motivated by physical principles and statistical methodology we introduce the following *Hermite-Sobolev classes* [9] and [8] of pure states characterised by an appropriate decay of the coefficients with respect to the basis B :

$$(19) \quad S^\alpha(L) := \left\{ |\psi\rangle\langle\psi| : \sum_{j=0}^{\infty} |\psi_j|^2 j^{2\alpha} \leq L, \text{ and } \|\psi\| = 1 \right\}, \quad \alpha > 0, \quad L > 0.$$

To gain some intuition about the meaning of this class, let us assume that B is the Fock basis of a one-mode cv system. Then the constraint translates into the moment condition for the number operator $\langle\psi|N^{2\alpha}|\psi\rangle \leq L$; this is a mild assumption considering that all experimentally feasible states have finite moments to all orders. Even more, the coefficients of typical states such as coherent, squeezed, and Fock states decay exponentially with the photon number.

Our first model describes n identical copies of a pure state belonging to the Sobolev class

$$(20) \quad \mathcal{Q}_n := \{|\psi\rangle\langle\psi|^{\otimes n} : |\psi\rangle\langle\psi| \in S^\alpha(L)\}.$$

In section 5.1 we show that the minimax rate of \mathcal{Q}_n for the norm-one and Bures distance loss functions is $n^{-\alpha/(2\alpha+1)}$. This is identical to the minimax rate of the classical i.i.d. model described in section 3.1.

We now introduce the corresponding quantum Gaussian model. Let $\mathcal{F} := \mathcal{F}(\mathcal{H})$ be the Fock space over \mathcal{H} , and let $|G(\sqrt{n}\psi)\rangle \in \mathcal{F}$ be the coherent state with “displacement” vector $\sqrt{n}\psi$. As discussed in section 2.2.2, the vector $\sqrt{n}\psi$ should be seen now as the *expectation* of the infinite dimensional Gaussian state rather than a quantum state in itself, for which reason we have omitted the ket notation. We define the coherent states model

$$(21) \quad \mathcal{G}_n = \{|G(\sqrt{n}\psi)\rangle\langle G(\sqrt{n}\psi)| : |\psi\rangle \in \mathcal{H}, \text{ such that } |\psi\rangle\langle\psi| \in S^\alpha(L)\}.$$

Using the factorisation property (9) with respect to the orthonormal basis B , we see that the model is equivalent to the product of independent one-mode coherent Gaussian states of mean $\sqrt{n}\psi_i$

$$|G(\sqrt{n}\psi)\rangle \cong \bigotimes_{i=1}^{\infty} |G(\sqrt{n}\psi_i)\rangle$$

which is analogous to the classical Gaussian sequence model \mathcal{N}_n defined in equation (17).

Similarly, we can draw an analogy with the white noise model \mathcal{F}_n by realising \mathcal{H} as $L^2([0, 1])$. Let us define the *quantum stochastic process* [58] on $\mathcal{F}(L^2([0, 1]))$

$$B(t) := a(\chi_{[0,t]}) + a^*(\chi_{[0,t]})$$

and note that $[B(t), B(s)] = 0$ for all $t, s \in [0, 1]$ so that $\{B(t) : t \in [0, 1]\}$ is a commutative family of operators. This implies that $\{B(t) : t \in [0, 1]\}$ have a joint probability distribution which is uniquely determined by the quantum state, and can be regarded as a classical stochastic process. If the state is the vacuum $|0\rangle$, the process is Gaussian and has the same distribution as the Brownian motion. Consider now the process $X(t) := W(\sqrt{n}\psi)^* B(t) W(\sqrt{n}\psi)$. which is obtained by applying a unitary Weyl transformation to $B(t)$. In physics terms we work here in the ‘‘Heisenberg picture’’ where the transformation acts on operators while the state is fixed. Using quantum stochastic calculus one can derive the following differential equation for $X(t)/\sqrt{n}$

$$\frac{1}{\sqrt{n}} dX(t) = \psi(t) dt + \frac{1}{\sqrt{n}} dB(t).$$

Therefore, $X(t)/\sqrt{n}$ is similar to the process (16) with the exception that it has a complex rather than real valued drift function. Note that in this correspondence $\psi(t)$ plays the role of $f^{1/2}$, which agrees with the intuitive interpretation of the wave function as square root of the state $|\psi\rangle\langle\psi|$. Alternatively, one can use the Schrödinger picture, where the state is $|\sqrt{n}\psi\rangle = W(\sqrt{n}\psi)|0\rangle$, such that the process $B(t)$ has the same law as $X(t)$ under the vacuum state.

In section 5.1 we show that the minimax rate of \mathcal{G}_n for loss functions based on the norm-one and the Bures distance, is $n^{-\alpha/(2\alpha+1)}$. Although the rate is identical to that of the corresponding classical model, the result does not follow from the classical case but relies on an explicit measurement strategy for the upper bounds, and on the quantum local asymptotic equivalence Theorem 4.1 for the lower bound. Furthermore, the minimax rate for the estimation of certain quadratic functionals are established in section 5.2, and the minimax testing rates are derived in section 5.3. While the former are similar to the classical ones, the quantum testing rates are parametric as opposed to non-parametric in the classical case. This reflects the fact that in the quantum case, the optimal measurements for different statistical problems are in general incompatible with each other and in some cases they differ significantly from what is expected on classical basis.

4. Local asymptotic equivalence for quantum models. In this section we prove that the sequence (20) of non-parametric pure states models is locally asymptotically equivalent (LAE) with the sequence (21) of quantum Gaussian models, in the sense of the Le Cam distance. This is one of the main results of the paper and will be subsequently used in the applications. Throughout the section $|\psi_0\rangle$ is a fixed but arbitrary state in an infinite dimensional Hilbert space \mathcal{H} . We let $\mathcal{H}_0 := \{|\psi\rangle \in \mathcal{H} : \langle\psi_0|\psi\rangle = 0\}$ denote the orthogonal complement of $\mathbb{C}|\psi_0\rangle$. Any vector state $|\psi\rangle \in \mathcal{H}$ decomposes uniquely as

$$(22) \quad |\psi\rangle = |\psi_u\rangle := \sqrt{1 - \|u\|^2} |\psi_0\rangle + |u\rangle, \quad |u\rangle \in \mathcal{H}_0$$

where the phase has been chosen such that the overlap $\langle\psi|\psi_0\rangle$ is real and positive. Therefore, the pure states are uniquely parametrised by vectors $|u\rangle \in \mathcal{H}_0$.

Further to the i.i.d. and Gaussian models \mathcal{Q}_n and \mathcal{G}_n defined in (20) and respectively (21), we now introduce their local counterparts which are parametrised by the local parameter $|u\rangle$ rather than by $|\psi\rangle$. Let γ_n be a sequence such that $\gamma_n = o(1)$, and define the pure state models

$$(23) \quad \mathcal{Q}_n(\psi_0, \gamma_n) := \{|\psi_u^{\otimes n}\rangle \in \mathcal{H}^{\otimes n} : |u\rangle \in \mathcal{H}_0, \|u\| \leq \gamma_n\}$$

$$(24) \quad \mathcal{G}_n(\psi_0, \gamma_n) := \{|G(\sqrt{n}u)\rangle \in \mathcal{F}(\mathcal{H}_0) : |u\rangle \in \mathcal{H}_0, \|u\| \leq \gamma_n\}.$$

The LAE Theorem below shows that these local models are asymptotically equivalent. An interesting fact is that LAE holds without imposing global restrictions such as defined by the Sobolev classes, rather it suffices that the local balls shrink at an arbitrary slow rate $\gamma_n = o(1)$. This contrasts with the classical case where both types of conditions are needed, as explained in section 3.1. However, since the state cannot be “localised” without any prior knowledge, in applications we need to make additional assumptions which allow us to work in a small neighbourhood and make use of local asymptotic equivalence. In particular, the convergence holds for the restricted models where the Sobolev condition is imposed on top of the local one. This will be used in establishing the estimation, testing, and functional estimation results.

THEOREM 4.1. *Let $\mathcal{Q}_n(\psi_0, \gamma_n)$ and $\mathcal{G}_n(\psi_0, \gamma_n)$ be the models defined in (23) and respectively (24) where $\gamma_n = o(1)$. Then the following convergence holds uniformly over states $|\psi_0\rangle$:*

$$(25) \quad \limsup_{n \rightarrow \infty} \sup_{|\psi_0\rangle \in \mathcal{H}} \Delta(\mathcal{Q}_n(\psi_0, \gamma_n), \mathcal{G}_n(\psi_0, \gamma_n)) = 0$$

where $\Delta(\cdot, \cdot)$ is the quantum Le Cam distance defined in equation (18).

The proof is given in [15].

5. Applications. In this section we discuss three major applications of the local asymptotic equivalence result in Theorem 4.1, namely to the estimation of pure states, estimation of a physically meaningful quadratic functional, and finally to testing between pure states. We stress that local asymptotic equivalence allows us to translate these problems into similar but easier ones involving Gaussian states. This strategy has already been successfully employed [33] in finding asymptotically optimal estimation procedures for *finite dimensional mixed states*, which otherwise appeared to be a difficult problem due to the complexity of the set of possible measurements.

As discussed in section 3.3, we will assume that we are given n independent systems, each prepared in a state $|\psi\rangle \in \mathcal{H}$ belonging to the Sobolev ellipsoid $S^\alpha(L)$ defined in equation (19). The corresponding quantum statistical model \mathcal{Q}_n was defined in equation (20), and the Gaussian counterpart model \mathcal{G}_n was defined in equation (21).

Here is a summary of the results. In Theorem 5.2 we show that the estimation rates over such ellipsoids are $n^{-\alpha/(2\alpha+1)}$; this is similar to the well-known rates, e.g. for density estimation, in nonparametric statistics (see [67]). The estimation of the quadratic functional

$$F(\psi) = \sum_{j \geq 0} |\psi_j|^2 j^{2\beta}, \text{ for some fixed } \beta > 0$$

of the unknown pure state presents two regimes: a parametric rate n^{-1} for the MSE is attained when the unknown state has enough “smoothness” (that is $\alpha \geq 2\beta$), whereas a nonparametric rate $n^{-2(1-\beta/\alpha)}$ is obtained when $\beta < \alpha < 2\beta$. This double regime is known in nonparametric estimation for the density model, with different values for both the rates and the values of the parameters where the phase-transition occurs, cf [17], [46] and references therein.

Parametric rates and sharp asymptotic constants are obtained for the testing problem of a pure state against an alternative described by the Sobolev-type ellipsoid with an L_2 -ball removed. In the classical density model only nonparametric rates for testing of order $n^{-2\alpha/(4\alpha+1)}$ can be obtained for the L_2 norm. In our quantum i.i.d. model, the parametric

rate $n^{-1/2}$ is shown to be minimax for testing $H_0 : \psi = \psi_0$, for some ψ_0 in $S^\alpha(L)$ over the nonparametric set of alternatives:

$$H_1 : \psi \in S^\alpha(L) \text{ is such that } \|\psi\rangle\langle\psi| - |\psi_0\rangle\langle\psi_0|\|_1 \geq cn^{-1/2}.$$

The sharp asymptotic constant we obtain for testing is specific for ensembles of pure states. As we discuss in the sequel, quantum testing of states allows us to optimize over the measurements, and thus to obtain the most distinguishable likelihoods for the underlying unknown quantum state.

5.1. *Estimation.* We consider the problem of estimating an unknown pure state belonging to the Hermite-Sobolev class $S^\alpha(L)$ given an ensemble of n independent, identically prepared systems. The corresponding sequence of statistical models \mathcal{Q}_n was defined in equation (20). We first describe a specific measurement procedure which provides an estimator whose risk attains the nonparametric rate $n^{-2\alpha/(2\alpha+1)}$. We prove the lower bounds for estimating a Gaussian state in the model \mathcal{G}_n defined in (21). Subsequently we use LAE to establish a lower bound showing that the rate is optimal in the i.i.d. model as well.

Before deriving the bounds we briefly review the definitions of the loss functions used here and the relations between them, cf. section 2.3. Recall that the trace norm distance between states ρ and ρ' is given by $\|\rho - \rho'\|_1 := \text{Tr}(|\rho - \rho'|)$, and is the quantum analogue of the norm-one distance between probability densities. The square of the Bures distance is given by $d_b^2 := 2(1 - \text{Tr}(\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}))$, and is a quantum extension of the Hellinger distance. These distances satisfy the inequalities (14).

In the case of pure states (i.e. $\rho = |\psi\rangle\langle\psi|$, and $\rho' = |\psi'\rangle\langle\psi'|$) these metrics become (cf. (12) and (13)),

$$\|\rho - \rho'\|_1 = 2\sqrt{1 - |\langle\psi|\psi'\rangle|^2}, \quad d_b^2(\rho, \rho') = 2(1 - |\langle\psi|\psi'\rangle|).$$

Since vectors are not uniquely defined by the states, the distances cannot be expressed directly in terms of the length $\|\psi - \psi'\|$. However if we consider a reference vector $|\psi_0\rangle$ and define the representative vector $|\psi\rangle$ such that $\langle\psi_0|\psi\rangle \geq 0$, then we can write (as in section 4)

$$|\psi_u\rangle = \sqrt{1 - \|u\|^2}|\psi_0\rangle + |u\rangle, \quad |\psi_{u'}\rangle = \sqrt{1 - \|u'\|^2}|\psi_0\rangle + |u'\rangle, \quad |u\rangle, |u'\rangle \perp |\psi_0\rangle$$

and the distances have the same (up to a constant) quadratic approximation

$$(26) \quad \|\rho_u - \rho_{u'}\|_1^2 = 4\|u - u'\|^2 + O(\max(\|u\|, \|u'\|)^4), \quad d_b^2(\rho_u, \rho_{u'}) = \|u - u'\|^2 + O(\max(\|u\|, \|u'\|)^4),$$

where the correction terms are of order 4 as $\|u\|$ and $\|u'\|$ tend to 0. Below we show that asymptotically with n the estimation risk for norm-one square and Bures distance square will have the same rate as that of estimating the local parameter u with respect to the Hilbert space distance.

5.1.1. *Upper bounds.* We first describe a two steps measurement procedure, which provides an estimator whose risk has rate $n^{-2\alpha/(2\alpha+1)}$.

THEOREM 5.1. *Consider the i.i.d. quantum model \mathcal{Q}_n given by equation (20). There exists an estimator $\hat{\rho}_n := |\hat{\psi}_n\rangle\langle\hat{\psi}_n|$ such that*

$$\limsup_{n \rightarrow \infty} \sup_{|\psi\rangle \in S^\alpha(L)} n^{2\alpha/(2\alpha+1)} \mathbb{E}_\rho [d^2(\hat{\rho}_n, \rho)] \leq C,$$

where $\rho := |\psi\rangle\langle\psi|$, $d(\hat{\rho}_n, \rho)$ denotes either the trace-norm distance, or the Bures distance, and $C > 0$ is a constant depending only on $\alpha > 0$ and $L > 0$.

PROOF OF THEOREM 5.1. According to inequalities (14) and (15) the two distances are equivalent on pure states, so it suffices to prove the upper bound for the trace-norm distance.

Firstly, a projective operation is applied to each of the n copies separately, whose aim is to truncate the state to a finite dimensional subspace of dimension $d_n = \lceil n^{1/(2\alpha+1)} \rceil + 1$. Let P_n be the projection onto the subspace \mathcal{H}_n spanned by the first d_n basis vectors $\{|e_0\rangle, \dots, |e_{d_n-1}\rangle\}$. For a given state $|\psi\rangle$ the operation consists of randomly projecting the state with P_n or $(\mathbf{1} - P_n)$, which produces i.i.d. outcomes $O_i \in \{0, 1\}$ with $\mathbb{P}(O_i = 1) = p_n = \|P_n\psi\|^2$. The posterior state conditioned on the measurement outcome is

$$|\psi\rangle\langle\psi| \mapsto \begin{cases} |\psi^{(n)}\rangle\langle\psi^{(n)}| := \frac{P_n|\psi\rangle\langle\psi|P_n}{p_n} & \text{with probability } p_n \\ \frac{(\mathbf{1}-P_n)|\psi\rangle\langle\psi|(\mathbf{1}-P_n)}{1-p_n} & \text{with probability } 1-p_n \end{cases}$$

Since $|\psi\rangle\langle\psi| \in S^\alpha(L)$, the probability $1-p_n$ is bounded as

$$(27) \quad 1-p_n = \sum_{i=d_n}^{\infty} |\psi_i|^2 = \sum_{i=d_n}^{\infty} i^{-2\alpha} i^{2\alpha} |\psi_i|^2 \leq d_n^{-2\alpha} \sum_{i=1}^{\infty} i^{2\alpha} |\psi_i|^2 = n^{-2\alpha/(2\alpha+1)} L.$$

Let $\tilde{n} = \sum_{i=1}^n O_i$ be the number of systems for which the outcome was equal to 1, so that \tilde{n} has binomial distribution $\text{Bin}(n, p_n)$. Then $\mathbb{E}(\tilde{n}/n) = p_n$ and $\text{Var}(\tilde{n}/n) = p_n(1-p_n)/n = O(1/n)$. Therefore $\tilde{n}/n \rightarrow 1$ in probability.

In the second step we discard the systems for which the outcome was 0, and we collect those with outcome 1, so that the joint state is $|\psi^{(n)}\rangle\langle\psi^{(n)}|^{\otimes \tilde{n}}$ which is supported by the symmetric subspace $\mathcal{H}_n^{\otimes \tilde{n}}$. In order to estimate the truncated state $|\psi^{(n)}\rangle$ (and by implication $|\psi\rangle$), we perform a *covariant* measurement M_n [36] whose space of outcomes is the space of pure states $\hat{\rho}_n = |\hat{\psi}_n\rangle\langle\hat{\psi}_n|$ over \mathcal{H}_n , and the infinitesimal POVM element is

$$(28) \quad M_n(d\hat{\rho}) = \begin{pmatrix} \tilde{n} + d_n - 1 \\ d_n - 1 \end{pmatrix} \hat{\rho}^{\otimes n} d\hat{\rho}.$$

The covariance property means that the unitary group has a covariant action on states and their corresponding probability distributions

$$\mathbb{P}_{U\rho U^*}^{M_n}(d\hat{\rho}) = \text{Tr}(U\rho U^* \cdot d\hat{\rho}) = \mathbb{P}_\rho^{M_n}(d(U^*\hat{\rho}U)).$$

Recall that the trace-norm distance squared for pure states is given by $d_1^2(\rho, \rho') := \|\rho - \rho'\|_1^2 = 4(1 - |\langle\psi|\psi'\rangle|^2)$. In [36] it has been shown that, conditionally on \tilde{n} , the risk of the estimator $\hat{\rho}$ with respect to the trace-norm square distance is¹

$$\mathbb{E}^{\tilde{n}} \left[d_1^2(\hat{\rho}_n, \rho^{(n)}) \right] = \frac{4(d_n - 1)}{d_n + \tilde{n}}.$$

Using the triangle inequality we have $d_1^2(\hat{\rho}_n, \rho) \leq 2(d_1^2(\hat{\rho}_n, \rho^{(n)}) + d_1^2(\rho, \rho^{(n)}))$. Since $|\psi^{(n)}\rangle = P_n|\psi\rangle/\sqrt{p_n}$, the bias term is $d_1^2(\rho, \rho^{(n)}) = 4(1-p_n)$, which by (27) is bounded by $4n^{-2\alpha/(2\alpha+1)}L$. Therefore

$$\mathbb{E} \left[d_b^2(\hat{\rho}_n, \rho) \right] \leq 8\mathbb{E} \left[\frac{(d_n - 1)}{d_n + \tilde{n}} \right] + 8n^{-2\alpha/(2\alpha+1)}L.$$

¹Reference [36] uses a fidelity distance erroneously called "Bures distance", which for pure states coincides with the trace-norm distance up to a constant

For an arbitrary small $\varepsilon > 0$, we have

$$\mathbb{E} \left[\frac{(d_n - 1)}{d_n + \tilde{n}} \right] \leq P \left[\frac{\tilde{n}}{n} < 1 - \varepsilon \right] + \mathbb{E} \left[\frac{(d_n - 1)}{d_n + n \cdot \tilde{n}/n} \cdot I\left(\frac{\tilde{n}}{n} \geq 1 - \varepsilon\right) \right] \leq O\left(\frac{1}{n}\right) + C\frac{d_n}{n}.$$

Putting together the last two upper bounds concludes the proof. \square

5.1.2. Lower bounds - Unimprovable rates. We will first consider the Gaussian model \mathcal{G}_n given by equation (21) which is indexed by Hilbert space vectors $\psi \in \mathcal{H}$ in the Sobolev class $S^\alpha(L)$, playing the role of means of quantum Gaussian states $|G(\sqrt{n}\psi)\rangle$. In Theorem 5.2 we find a lower bound for the mean square error of any estimator $\hat{\psi}$. This is then used in conjunction with the local asymptotic equivalence Theorem 4.1 to obtain a lower bound for the risk of the i.i.d. model \mathcal{Q}_n , with respect to the norm-one and Bures distances.

THEOREM 5.2. *Consider the quantum Gaussian model \mathcal{G}_n given by equation (21). There exists some constant $c > 0$ depending only on α and L such that*

$$\liminf_{n \rightarrow \infty} \inf_{\hat{\psi}_n} \sup_{\psi \in S^\alpha(L)} n^{2\alpha/(2\alpha+1)} \mathbb{E}_\psi \left[\|\hat{\psi}_n - \psi\|_2^2 \right] \geq c,$$

where the infimum is taken over all estimators $\hat{\psi}_n$, understood as combination of measurements and classical estimators.

The proof is given in [15].

We now proceed to consider the i.i.d. model \mathcal{Q}_n defined in (20). We are given n copies of an unknown pure state $|\psi\rangle\langle\psi|$, with ψ in the Sobolev class $S^\alpha(L)$. The goal is to find an asymptotic lower bound for the estimation risk (with respect to the Bures or norm-one loss functions) which matches the upper bound derived in section 5.1.1. Since both loss functions satisfy the triangle inequality, it can be shown that by choosing estimators which are mixed states, rather than pure states, one can improve the risk by at most a constant factor 2. Therefore we consider estimators which are pure states. In order to fix the phase of the vector representing the true and the estimated state, we will assume that $\langle\psi|e_0\rangle \geq 0$ and $\langle\hat{\psi}|e_0\rangle \geq 0$.

THEOREM 5.3. *Consider the i.i.d. quantum model \mathcal{Q}_n given by equation (20). There exists some constant $c > 0$ depending only on $\alpha > 0$ and $L > 0$ such that*

$$\liminf_{n \rightarrow \infty} \inf_{|\hat{\psi}_n\rangle} \sup_{|\psi\rangle \in S^\alpha(L)} n^{2\alpha/(2\alpha+1)} \mathbb{E}_\rho \left[d^2(\hat{\rho}_n, \rho) \right] \geq c,$$

where $\rho := |\psi\rangle\langle\psi|$, the infimum is taken over all estimators $\hat{\rho}_n := |\hat{\psi}_n\rangle\langle\hat{\psi}_n|$ (defined by a combination of measurement and a classical estimator), and the loss function $d(\hat{\rho}, \rho)$ is either the norm-one or the Bures distance.

The proof is given in [15].

5.2. Quadratic functionals. This section deals with the estimation of the quadratic functional

$$F(\psi) = \sum_{j \geq 0} |\psi_j|^2 \cdot j^{2\beta}, \text{ for some fixed } 0 < \beta < \alpha,$$

which is well defined for all pure states $|\psi\rangle$ in the ellipsoid $S^\alpha(L)$. If the Hilbert space \mathcal{H} is represented as $L_2(\mathbb{R})$ and $\{|j\rangle : j \geq 0\}$ is the Fock basis (cf. section 2.2.1) then $F(\psi)$ is the moment of order 2β of the number operator N :

$$F(\psi) = \text{Tr}(|\psi\rangle\langle\psi| \cdot N^{2\beta}).$$

Below we derive upper and lower bounds for the rate of the quadratic risk for estimating $F(\psi)$, which is of order n^{-1} if $\alpha \geq 2\beta$, and $n^{-2(1-\beta/\alpha)}$ if $\beta < \alpha < 2\beta$.

5.2.1. *Upper bounds.* Let us describe an estimator \hat{F}_n of $F(\psi)$ in the quantum i.i.d. model. We consider the measurement of the number operator with projections $\{|j\rangle\langle j|\}_{j \geq 0}$. For a pure state $|\psi\rangle = \sum_{j \geq 0} \psi_j |j\rangle$, we obtain an outcome X taking values $j \in \mathbb{N}$ with probabilities $p_j := \mathbb{P}_\psi(X = j) = |\psi_j|^2$, for $j \geq 0$. By measuring each quantum sample $|\psi\rangle$ separately, we obtain i.i.d. copies X_1, \dots, X_n of X , allowing us to estimate each p_j empirically, by

$$\hat{p}_j = \frac{1}{n} \sum_{k=1}^n I(X_k = j), \quad j \geq 0.$$

which is an unbiased estimator of p_j with variance $p_j(1-p_j)/n$. The estimator of the quadratic functional is defined as

$$(29) \quad \hat{F}_n = \sum_{j=1}^N \hat{p}_j \cdot j^{2\beta}$$

for an appropriately chosen truncation parameter N defined below. The next theorem, shows that a parametric rate can be attained for estimating the quadratic functional $F(\psi)$ if $\alpha \geq 2\beta$, whereas a nonparametric rate is attained if $\beta < \alpha < 2\beta$.

THEOREM 5.4. *Consider the i.i.d. quantum model \mathcal{Q}_n given by equation (20). Let \hat{F}_n be the estimator (29) of $F(\psi)$ with $N \asymp n^{1/4(\alpha-\beta)}$, for $\alpha \geq 2\beta$, respectively $N \asymp n^{1/2\alpha}$, for $\beta < \alpha < 2\beta$. Then*

$$(30) \quad \sup_{\psi \in S^\alpha(L)} \mathbb{E}_\psi \left(\hat{F}_n - F(\psi) \right)^2 = O(1) \cdot \eta_n^2, \quad \text{where } \eta_n^2 = \begin{cases} n^{-1}, & \text{if } \alpha \geq 2\beta \\ n^{-2(1-\beta/\alpha)}, & \text{if } \beta < \alpha < 2\beta. \end{cases}$$

PROOF OF THEOREM 5.4. The usual bias-variance decomposition yields

$$\mathbb{E}_\psi \left(\hat{F}_n - F(\psi) \right)^2 = \left(\mathbb{E}_\psi \hat{F}_n - F(\psi) \right)^2 + \text{Var}_\psi \left(\hat{F}_n \right).$$

The bias can be upper bounded as

$$\left| F(\psi) - \mathbb{E}_\psi \hat{F}_n \right| = \left| F(\psi) - \sum_{j=1}^N p_j \cdot j^{2\beta} \right| = \sum_{j \geq N+1} p_j \cdot j^{2\beta} \leq N^{-2(\alpha-\beta)} \sum_{j \geq N+1} p_j \cdot j^{2\alpha} \leq LN^{-2(\alpha-\beta)}.$$

For the variance, let us note that the vector

$$\hat{V} = n \cdot (\hat{p}_1, \dots, \hat{p}_N, \hat{p}_{N+1}^*), \quad \text{with } \hat{p}_{N+1}^* = n^{-1} \sum_{k=1}^n I(X_k \geq N+1),$$

has a multinomial distribution with parameters n and probability vector $V := (p_1, \dots, p_N, p_{N+1}^* = \sum_{j \geq N+1} p_j)^\top$. The covariance matrix of a multinomial vector writes $n \cdot (\text{Diag}(V) - V \cdot V^\top)$, where $\text{Diag}(V)$ denotes the diagonal matrix with entries from V . In particular, if $\hat{p} := (\hat{p}_1, \dots, \hat{p}_N)^\top$, $p := (p_1, \dots, p_N)^\top$ and $B := (1, 2^{2\beta}, \dots, N^{2\beta})^\top$ then

$$\text{Cov}_\psi(\hat{F}_n) = \text{Cov}_\psi(B^\top \cdot \hat{p}) = B^\top \cdot \text{Cov}_\psi(\hat{p}) \cdot B = \frac{1}{n} \cdot B^\top \cdot (\text{Diag}(p) - p \cdot p^\top) \cdot B.$$

This gives

$$\text{Cov}_\psi(\hat{F}_n) \leq \frac{1}{n} \cdot B^\top \cdot \text{Diag}(p) \cdot B = \frac{1}{n} \sum_{j=1}^N p_j \cdot j^{4\beta}.$$

The bound of this last term and the resulting bound of the risk is treated separately for the two cases.

a) Case $\alpha \geq 2\beta$. In that case,

$$\sum_{j=1}^N p_j \cdot j^{4\beta} \leq \sum_{j=1}^N p_j \cdot j^{2\alpha} \leq L \text{ implying that } \text{Var}(\hat{F}_n) \leq \frac{L}{n}.$$

The upper bound of the risk is, in this case,

$$\mathbb{E}_\psi \left(\hat{F}_n - F(\psi) \right)^2 \leq L^2 N^{-4(\alpha-\beta)} + \frac{L}{n}.$$

If we choose $N \asymp n^{1/(4(\alpha-\beta))}$ or larger, then the parametric rate is attained for the risk:

$$\mathbb{E}_\psi \left(\hat{F}_n - F(\psi) \right)^2 = O(1) \cdot n^{-1}.$$

b) Case $\beta < \alpha < 2\beta$. Here we have,

$$\text{Cov}_\psi(\hat{F}_n) \leq \frac{1}{n} \sum_{j=1}^N p_j \cdot j^{4\beta} \leq \frac{1}{n} \sum_{j=1}^N p_j \cdot j^{4\beta-2\alpha} j^{2\alpha} p_j \leq \frac{N^{4\beta-2\alpha}}{n} L.$$

The upper bound of the risk becomes

$$\mathbb{E}_\psi \left(\hat{F}_n - F(\psi) \right)^2 \leq L^2 N^{-4(\alpha-\beta)} + \frac{N^{4\beta-2\alpha}}{n} L.$$

The optimal choice of the parameter N that balances the two previous terms is $N \asymp n^{1/(2\alpha)}$, giving the attainable rate for the quadratic risk

$$\mathbb{E}_\psi \left(\hat{F}_n - F(\psi) \right)^2 = O(1) \cdot n^{-2(1-\beta/\alpha)}.$$

Cases **a)** and **b)** together prove that the rate η_n^2 is attainable. \square

5.2.2. *Lower bounds.* The next Theorem proves the optimality of the previously attained rate for the estimation of quadratic functionals.

THEOREM 5.5. *Consider the i.i.d. quantum model \mathcal{Q}_n given by equation (20). Then, there exists some constant $c > 0$ depending only on α, β (with $\alpha > \beta > 0$), and $L > 0$ such that*

$$\liminf_{n \rightarrow \infty} \inf_{\hat{F}_n} \sup_{\psi \in S^\alpha(L)} \eta_n^{-2} \cdot \mathbb{E}_\psi \left(\hat{F}_n - F(\psi) \right)^2 \geq c,$$

where the infimum is taken over all measurements and resulting estimators \hat{F}_n of $F(\psi)$.

The proof is given in [15].

5.3. *Testing.* In the problem of testing for signal in classical Gaussian white noise, over a smoothness class with an L_2 -ball removed, minimax rates of convergences (separation rates) are well known [42]; they are expressed in the rate of the ball radius tending to zero along with noise intensity, such that a nontrivial asymptotic power is possible. We will consider an analogous testing problem here for pure states. Accordingly, let $\rho = |\psi\rangle\langle\psi|$ denote pure states, let $\rho_0 = |\psi_0\rangle\langle\psi_0|$ be a fixed pure state to serve as the null hypothesis, and let

$$(31) \quad B(\varphi) = \{\|\rho - \rho_0\|_1 \geq \varphi\}$$

be the complement of a trace norm ball around ρ_0 . We want to test in the i.i.d. quantum model \mathcal{Q}_n given by equation (20) the following hypotheses about a pure state ρ :

$$(32) \quad \begin{aligned} H_0 : & \quad \rho = \rho_0 \\ H_1(\varphi_n) : & \quad \rho \in S^\alpha(L) \cap B(\varphi_n) \end{aligned}$$

for $\{\varphi_n\}_{n \geq 1}$ a decreasing sequence of positive real numbers. Consider a binary POVM $M = (M_0, M_1)$, acting on the product states $\rho^{\otimes n}$, cf. Definition 2. We denote the testing risk between two fixed hypotheses by the sum of the two error probabilities

$$R_n^T(M) = R_n^T(\rho_0^{\otimes n}, \rho^{\otimes n}, M) = \text{Tr}(\rho_0^{\otimes n} \cdot M_1) + \text{Tr}(\rho^{\otimes n} \cdot M_0).$$

In the minimax α -testing approach which dominates the literature on the classical Gaussian white noise case, one would require $\text{Tr}(\rho_0^{\otimes n} \cdot M_1) \leq \alpha$ while trying to minimize the worst case type 2 error $\sup_{\rho \in S^\alpha(L) \cap B(\varphi_n)} \text{Tr}(\rho^{\otimes n} \cdot M_0)$. However we will consider here the so-called detection problem [41] where the target is the worst case total error probability

$$\mathbb{P}_e^M(\varphi_n) = \sup_{\rho \in S^\alpha(L) \cap B(\varphi_n)} R_n^T(\rho_0^{\otimes n}, \rho^{\otimes n}, M) = \text{Tr}(\rho_0^{\otimes n} \cdot M_1) + \sup_{\rho \in S^\alpha(L) \cap B(\varphi_n)} \text{Tr}(\rho^{\otimes n} \cdot M_0).$$

The minimax total error probability is then obtained by optimizing over T :

$$\mathbb{P}_e^*(\varphi_n) = \inf_{M \text{ binary POVM}} \mathbb{P}_e^M(\varphi_n).$$

5.3.1. *Separation rate.* A sequence $\{\varphi_n^*\}_{n \geq 1}$ is called a *minimax separation rate* if any other sequence $\{\varphi_n\}_{n \geq 1}$ fulfills

$$(33) \quad \mathbb{P}_e^*(\varphi_n) \rightarrow 1 \text{ if } \varphi_n/\varphi_n^* \rightarrow 0 \text{ and } \mathbb{P}_e^*(\varphi_n) \rightarrow 0 \text{ if } \varphi_n/\varphi_n^* \rightarrow \infty.$$

Below we establish that $\varphi_n^* = n^{-1/2}$ is a separation rate in the current problem, even though the alternative $H_1(\cdot)$ in (32) is a nonparametric set of pure states. Recall relations (11), (12) describing the total optimal error for testing between simple hypotheses given by two pure states.

THEOREM 5.6. *Consider the i.i.d. quantum model \mathcal{Q}_n given by equation (20), and the testing problem (32). Assume that ρ_0 is in the interior of $S^\alpha(L)$, i.e. $\rho_0 \in S^\alpha(L')$ for some $L' < L$. Then $\varphi_n^* = n^{-1/2}$ is a minimax separation rate.*

The proof is given in [15].

5.3.2. *Sharp asymptotics.* Having identified the optimal rate of convergence in the testing problem, we will go a step further and aim at a sharp asymptotics for the minimax testing error. We will adopt the approach of [21], extended in [42], where testing analogs of the Pinsker-type sharp risk asymptotics in nonparametric estimation were obtained. The result will be framed as follows: if the radius is chosen $\varphi_n \sim cn^{-1/2}$ for a certain $c > 0$, then the minimax testing error behaves as $\mathbb{P}_e^*(\varphi_n) \sim \exp(-c^2/4)$. Thus the sharp asymptotics is expressed as a type of scaling result: a choice of constant c in the radius implies a certain minimax error asymptotics depending on c .

To outline the problem, consider the upper and lower error bounds obtained in the proof of the separation rate. In (66) we establish the bound

$$(34) \quad \mathbb{P}_e^{M_n}(\varphi_n) \leq \exp(-c_n^2/4)$$

if $\varphi_n = c_n n^{-1/2}$, where M_n is the sequence of projection tests $M_n = (\rho_0^{\otimes n}, I - \rho_0^{\otimes n})$. The lower risk bound obtained in the course of proving Theorem 5.6 is

$$\inf_{M \text{ binary POVM}} \mathbb{P}_e^M(\varphi_n) \geq 1 - \sqrt{1 - (1 - c_n^2 n^{-1}/4)^n}.$$

If $c_n = c$ we can summarize this as

$$1 - \sqrt{1 - \exp(-c^2/4)} + o(1) \leq \mathbb{P}_e^*(\varphi_n) \leq \exp(-c^2/4).$$

Our result will be that the upper bound is sharp and represents the minimax risk asymptotics.

THEOREM 5.7. *Consider the i.i.d. quantum model \mathcal{Q}_n given by equation (20), and the testing problem (32). Assume that $\rho_0 \in S^\alpha(L')$ for some $L' < L$. At the minimax separation rate for the radius, i.e. for $\varphi_n \asymp n^{-1/2}$ we have*

$$\lim_n n^{-1} \varphi_n^{-2} \log \mathbb{P}_e^*(\varphi_n) = -1/4.$$

The proof is given in [15].

5.4. Discussion. State estimation.

Tomography and optimal rates. Consider a model where the Sobolev-type assumption $\rho \in S^\alpha(L)$ about the pure state $\rho = |\psi\rangle\langle\psi|$ (cf. (19)) is replaced by a finite dimensionality assumption: $\rho \in \mathcal{H}_d$ where

$$\mathcal{H}_d = \{|\psi\rangle\langle\psi| : \psi_j = 0, j \geq d\}$$

and d is known. One observes n identical copies of the pure state $\rho = |\psi\rangle\langle\psi|$, with possibly $d = d_n \rightarrow \infty$, i. e. the model \mathcal{Q}_n of (20) is replaced by

$$\mathcal{Q}_n := \{\rho^{\otimes n} : \rho \in \mathcal{H}_d\}.$$

Since \mathcal{H}_d can be written $\mathcal{H}_d = \mathcal{S}_{1,d}$ where

$$\mathcal{S}_{r,d} := \{\rho : \langle e_i | \rho | e_j \rangle = 0, i, j \geq d, \text{rank}(\rho) = r\},$$

the model is effectively a special case of the $d \times d$ density matrices of $\text{rank}(\rho) = r$ considered in [48]. In [48] however, it is not known in advance that $r = 1$ but ρ is a density matrix of possibly low rank r , and the aim is estimation of ρ using quantum state tomography performed on n identical copies of ρ . Data are obtained by defining an observable $\otimes_{i=1}^n E_i$ where E_1, \dots, E_n are i.i.d. uniformly selected elements of the Pauli basis of the linear space of $d \times d$ Hermitian matrices, and applying the corresponding measurement to $\rho^{\otimes n}$. Let $\hat{\rho}_n^*$ denote an arbitrary estimator of ρ based on that measurement. A lower asymptotic risk bound for norm-one risk is established; in the special case $d^2 r^2 = o(n)$ it reads as

$$(35) \quad \inf_{\hat{\rho}_n^*} \sup_{\rho \in \mathcal{S}_{r,d}} \mathbb{E}_\rho \left[\|\hat{\rho}_n^* - \rho\|_1^2 \right] \geq c \frac{r^2 d^2}{n}$$

for some $c > 0$ (Theorem 10 in [48]). It is also shown in [48] that (35) is attained, up to a different constant and logarithmic terms, by an entropy penalized least squares type estimator based on measurement of $\otimes_{i=1}^n E_i$, even when the rank r is unknown. Analogous optimal rates for $d \times d$ mixed states ρ with Pauli measurements, but under sparsity assumptions on the entries of the matrix ρ have been obtained in [16].

Returning to our setting of pure states, where $r = 1$ is known, with an infimum over *all measurements of $\rho^{\otimes n}$* and corresponding estimators $\hat{\rho}_n$, according to [36] one has

$$(36) \quad \inf_{\hat{\rho}_n} \sup_{\rho \in \mathcal{S}_{1,d}} \mathbb{E}_\rho \left[\|\hat{\rho}_n - \rho\|_1^2 \right] = \frac{4(d-1)}{d+n}$$

and the bound is attained by an estimator of the pure state ρ based on the covariant measurement (cf. equation (28)). Comparing (35) for $r = 1$ and $d_n \rightarrow \infty$, $d_n = o(n)$ with (36), we find that the latter bound is of order d_n/n whereas the former is of order d_n^2/n . It means that for estimation of finite dimensional pure states, estimators based on the Pauli type measurement $\otimes_{i=1}^n E_i$ do not attain the optimal rate when $d_n \rightarrow \infty$. It may be conjectured that the same holds for the optimal rate over $\rho \in S^\alpha(L)$, i.e. our rate of Theorem 5.1. We emphasize again that our results establish lower asymptotic risk bounds over all quantum measurements and estimators, whereas lower risk bounds within one specific measurement scheme [47] [48] [16] are essentially results of non-quantum classical statistics.

Separate measurements. A notable fact is also that $\otimes_{i=1}^n E_i$ is a *separate* (or local) measurement, i.e. produces independent random variables (or random elements) Y_1, \dots, Y_n each based on a measurement of a copy of ρ , whereas the covariant measurement (cp. (28)) we used for attainment our risk bound of Theorem 5.1 is of *collective* (or joint) type with regard to the product $\rho^{\otimes n}$. Separate measurements are of interest from a practical point of view since collective measurements of large quantum systems may be unfeasible in implementations [52]. In [5] it is shown that for fixed $d = 2$, the bound (36) can be attained asymptotically as $n \rightarrow \infty$ (up to a factor $1 + o(1)$) by a separate measurement of $\rho^{\otimes n}$; it is an open question whether in our infinite dimensional setting, the optimal rate of Theorem 5.1 can be attained by a separate measurement. For mixed qubits ($d = 2$), an asymptotic efficiency gap between separate and collective measurements is known to exist [4].

Quadratic functionals.

The elbow phenomenon. The change of regime which occurs in the optimal MSE rate η_n^2 in (30) has been described as the elbow phenomenon in the literature [17]. In the classical Gaussian sequence model, it takes the following shape. Consider observations introduced in (17):

$$y_j = \vartheta_j + n^{-1/2} \xi_j, \quad j = 1, 2, \dots,$$

where $\{\xi_j\}$ are i.i.d. standard normal, and the parameter $\vartheta = (\vartheta_j)_{j=1}^\infty$ satisfies a restriction $\sum_{j=1}^\infty j^{2\alpha} \vartheta_j^2 \leq L$ for some $\alpha > 0$. For estimation of the quadratic functional $\tilde{F}(\vartheta) = \sum_{j=1}^\infty j^{2\beta} \vartheta_j^2$ with $\beta < \alpha$, the minimax MSE rate of convergence is

$$\tilde{\eta}_n^2 = \begin{cases} n^{-1} & \text{if } \alpha \geq 2\beta + 1/4 \\ n^{-2\frac{4(\alpha-\beta)}{4\alpha+1}} & \text{if } \beta < \alpha < 2\beta + 1/4 \end{cases} = n^{-2\tilde{r}} \text{ for } \tilde{r} = \min\left(\frac{1}{2}, \frac{4(\alpha-\beta)}{4\alpha+1}\right)$$

(cf [46] and references cited therein). The same rate holds for estimation of the squared L_2 -norm of the β -th derivative of a density in an α -Hölder class, cf. [7]. Comparing with our rate η_n^2 in (30) which can be written $\eta_n^2 = n^{-2r}$ for $r = \min\left(\frac{1}{2}, \frac{4(\alpha-\beta)}{4\alpha}\right)$, we see that both rates exhibit the elbow phenomenon, but at different critical values for (α, β) , and the rate for the quantum case is slightly faster in the region $\alpha < 2\beta + 1/4$.

A tail functional of a discrete distribution. Our method of proof for the optimal rate $\eta_n^2 = n^{-2r}$ shows that it is also the optimal rate in the following non-quantum problem: suppose $P = \{p_j\}_{j=0}^\infty$ is a probability measure on the nonnegative integers, satisfying a restriction $\sum_{j=0}^\infty j^{2\alpha} p_j \leq L$, and the aim is to estimate the linear functional $F_0(P) = \sum_{j=0}^\infty j^{2\beta} p_j$ on the basis of n i.i.d. observations X_1, \dots, X_n having law P . Indeed, Theorem 5.4 shows that the estimator $\hat{F}_n = \sum_{j=0}^N j^{2\beta} \hat{p}_j$ with $\hat{p}_j = n^{-1} \sum_{i=1}^n I(X_i = j)$ attains the rate η_n^2 for mean square error, for an appropriate choice of N . On the other hand, the observations X_1, \dots, X_n are obtained from one specific measurement in the quantum model (20), in such a way that $p_j = |\psi_j|^2$ for $j \geq 0$ and $F_0(P) = F(\psi)$. If the rate η_n^2 is unimprovable in the quantum model then it certainly is in the present derived (less informative) classical model. In the latter model, we note that since $F_0(P)$ is linear and the law P is restricted to a convex body, optimality of the rate η_n^2 can be confirmed by standard methods, e.g. based on the concept of modulus of continuity [20]. The current problem is thus an example where the elbow phenomenon is present for estimation of a linear functional; a specific feature here is that the probability measure P is discrete.

Fuzzy quantum hypotheses. Our method of proof of the lower bound for quadratic functionals, which works in the approximating quantum Gaussian model, utilizes the well-known idea of setting up two prior distributions and then invoking a testing bound between simple hypotheses. This has been described as the method of fuzzy hypotheses in the literature [67]. A summary of the present quantum variant could be as follows. First, the Gaussian quantum model is represented in a fashion analogous to the classical sequence model (17) where the ϑ_j correspond to the displacement parameter u_j in certain Gaussian pure states (the coherent states). These displacement parameters are then assumed to be random as independent, non-identically distributed normal, for $j = 1, \dots, N$ where $N = o(n)$. Now Gaussian averaging over the displacements u_j leads to certain non-pure Gaussian states, i.e. the thermal states as the alternative, which happen to commute with the vacuum pure state (corresponding to $u_j = 0$) as the null hypothesis. Even though both are again Gaussian states, by commutation the problem is reduced to testing between two ordinary discrete probability distributions, i.e. the point mass at 0 and a certain geometric distribution with parameter r_j , depending on $j = 1, \dots, N$. The combined error probability for this classical testing problem with N independent observations gives the lower risk bound.

Nonparametric testing.

The separation rate $n^{-1/2}$. Recall that for the classical Gaussian sequence model (17), for the testing problem

$$(37) \quad \begin{aligned} H_0 : & \quad \vartheta = 0 \\ H_1(\varphi_n) : & \quad \sum_{j=1}^{\infty} j^{2\alpha} \vartheta_j^2 \leq L \text{ and } \|\vartheta\|_2 \geq \varphi_n \end{aligned}$$

(Sobolev ellipsoid with an L_2 -ball removed), the separation rate is $\varphi_n = n^{-2\alpha/(4\alpha+1)}$ [42]. We established that $\varphi_n = n^{-1/2}$ is the separation rate for the quantum nonparametric testing problem (32) involving a pure state ρ . While this ‘‘parametric’’ rate for a nonparametric problem is somewhat surprising, it should be noted that there also exist testing problems for classical i.i.d. data with nonparametric alternative where that separation rate applies; cf [42], sec. 2.6.2.

In our case, the rate $n^{-1/2}$ appears to be related to the fast rate $\varphi_n^2 = n^{-1}$ in the following nonparametric classical problem: given n i.i.d. observations X_1, \dots, X_n having law $P = \{p_j\}_{j=0}^{\infty}$ on the nonnegative integers, the hypotheses are

$$(38) \quad \begin{aligned} H_0 : & \quad P = \delta_0 \text{ (the degenerate law at 0)} \\ H_1(\varphi_n) : & \quad \|P - \delta_0\|_1 \geq \varphi_n^2. \end{aligned}$$

For that, note first that

$$\|P - \delta_0\|_1 = 1 - p_0 + \sum_{j=1}^{\infty} p_j = 2(1 - p_0).$$

The likelihood ratio test for δ_0 against any $P \in H_1(\varphi_n)$ rejects if $\max_{1 \leq j \leq n} X_j > 0$, thus it does not depend on P . The pertaining sum of error probabilities is

$$P \left(\max_{1 \leq j \leq n} X_j = 0 \right) = p_0^n = \left(1 - \frac{1}{2} \|P - \delta_0\|_1 \right)^n \leq \left(1 - \frac{1}{2} \varphi_n^2 \right)^n$$

and with a supremum over $P \in H_1(\varphi_n)$, the upper bound is attained. This means that for $\varphi_n = cn^{-1/2}$, the minimax sum of error probabilities tends to $\exp(-c^2/2)$, so that $\varphi_n^2 = n^{-1}$ is the separation rate here as claimed.

In fact there is a direct connection to the quantum nonparametric testing problem (32): in the latter, for $n = 1$, consider a measurement defined as follows. Let $\{|\tilde{e}_j\rangle\}_{j=0}^\infty$ be an orthonormal basis in \mathcal{H} such that $\rho_0 = |\tilde{e}_0\rangle\langle\tilde{e}_0|$ and consider the POVM $\{|\tilde{e}_j\rangle\langle\tilde{e}_j|\}_{j=0}^\infty$; the corresponding measurement yields a probability measure P on the nonnegative integers. Here the state ρ_0 is mapped into δ_0 and an alternative state ρ is mapped into $P = \{p_j\}_{j=0}^\infty$ such that $p_0 = \text{Tr}(\rho_0\rho)$. Condition (31) on the distance of the two states implies (cp (12))

$$\varphi_n \leq \|\rho - \rho_0\|_1 = 2\sqrt{1 - \text{Tr}(\rho_0\rho)} = 2\sqrt{1 - p_0} = \sqrt{2}\|P - \delta_0\|_1$$

so that up to a constant, the testing problem (38) is obtained.

In the quantum problem (32), we noted that the optimal test between ρ_0 and a specific alternative ρ depends on ρ , but found that the test (binary POVM) $M_n = \{\rho_0^{\otimes n}, I - \rho_0^{\otimes n}\}$ is minimax optimal in the sense of the rate and also in the sense of a sharp risk asymptotics. The sharp minimax optimality seems to be a specific result for the quantum case. We note that the optimal test M_n can be realized via a measurement $\{|\tilde{e}_j\rangle\langle\tilde{e}_j|\}_{j=0}^\infty$ as described above, applied separately to each component of $\rho^{\otimes n}$, resulting in independent identically distributed r.v.'s X_1, \dots, X_n . The test M_n then amounts to rejecting H_0 if $\max_{1 \leq j \leq n} X_j > 0$. Note that this measurement is incompatible with the one (28) providing the optimal rate for state estimation.

Other separation rates. In our proof of the lower bound for quadratic functionals, we formulated the nonparametric testing problem for pure states (54) where the alternative includes the restriction $\sum_{j \geq 0} |\psi_j|^2 j^{2\beta} \geq \eta_n$, and established that the rate $\eta_n = n^{-1+\beta/\alpha}$ is unimprovable there. Introduce a seminorm

$$\|\psi\|_{2,\beta} = \left(\sum_{j \geq 1} |\psi_j|^2 j^{2\beta} \right)^{1/2}$$

(excluding the term for $j = 0$) and write the restriction as

$$(39) \quad \|\psi\|_{2,\beta} \geq \varphi_n = \eta_n^{1/2};$$

then the case $\beta = 0$ gives (cp (12))

$$\varphi_n^2 \leq \sum_{j \geq 1} |\psi_j|^2 = 1 - |\psi_0|^2 = 1 - |\langle\psi|e_0\rangle|^2 = \frac{1}{4} \|\ |e_0\rangle\langle e_0| - |\psi\rangle\langle\psi|\|_1^2,$$

in other words, for $\rho_0 = |e_0\rangle\langle e_0|$ and $\rho = |\psi\rangle\langle\psi|$, the restriction (39) is equivalent to $\|\rho - \rho_0\|_1 \geq 2\varphi_n$. In that sense, the testing problems (32) and (54) are equivalent up to a constant, if $\beta = 0$ and $\rho_0 = |e_0\rangle\langle e_0|$. For $\beta > 0$, the testing problem (54) is a quantum pure state analog of the generalization of the classical problem (37) where $\|\vartheta\|_2 \geq \varphi_n$ is replaced by $\|\vartheta\|_{2,\beta} \geq \varphi_n$ (α -ellipsoid with a β -ellipsoid removed); the separation rate in the latter is $\varphi_n = n^{-2(\alpha-\beta)/(4\alpha+1)}$, cf. [42], sec. 6.2.1. In (54) the separation rate is $\varphi_n = n^{-1/2+\beta/2\alpha}$, i.e. of the more typical nonparametric form as well.

References.

[1] ARTILES, L, GILL, R. and GUȚĂ, M. (2005). An invitation to quantum tomography. *J. Royal Statist. Soc. B (Methodological)* **67** 109-134.
 [2] AUDENAERT, K. M. R., CALSAMIGLIA, J., MUÑOZ TAPIA, R., BAGAN, E., MASANES, L., ACIN, A. and VERSTRAETE, F. (2007). Discriminating states: the quantum Chernoff bound. *Phys. Rev. Lett.* **98** 160501.

- [3] AUDENAERT, K. M. R., NUSSBAUM, M., SZKOLA, A. and VERSTRAETE, F. (2008). Asymptotic error rates in quantum hypothesis testing. *Commun. Math. Phys.* **279** 251-283.
- [4] BAGAN, E., BALLESTER, M. A., GILL, R. D., MUÑOZ TAPIA, R. and ROMERO-ISART, O. (2006). Separable measurement estimation of density matrices and its fidelity gap with collective protocols. *Phys. Rev. Lett.* **97** 130501.
- [5] BAGAN, E., MONRAS, A. and MUÑOZ TAPIA, R. (2005). Comprehensive analysis of quantum pure-state estimation for two-level systems. *Phys. Rev. A* **71** 062318.
- [6] BARNDORFF-NIELSEN, O. E., GILL, R. and JUPP, P. E. (2003). On quantum statistical inference (with discussion). *J. R. Statist. Soc. B* **65** 775-816.
- [7] BICKEL, P. J. and RITOV, Y. (1988). Estimating integrated squared density derivatives: sharp best order of convergence estimates. *Sankhyā Ser. A* **50** 381-393. MR1065550
- [8] BONGIOANNI, B. (2011). Sobolev spaces diversification. *Rev. Un. Mat. Argentina* **52** 23-34. MR2952949
- [9] BONGIOANNI, B. and TORREA, J. L. (2006). Sobolev spaces associated to the harmonic oscillator. *Proc. Indian Acad. Sci. Math. Sci.* **116** 337-360. MR2256010
- [10] BRAUNSTEIN, S. L. and CAVES C. M. (1994). Statistical distance and the geometry of quantum states. *Phys. Rev. Lett.* **72** 3439-3443.
- [11] BROWN, L. D., CAI, T. T., LOW, M. G. and ZHANG, C.-H. (2002). Asymptotic equivalence theory for nonparametric regression with random design. *Ann. Statist.* **30** 688-707. MR1922538
- [12] BROWN, L. D. and LOW, M. G. (1996). Asymptotic equivalence of nonparametric regression and white noise. *Ann. Statist.* **24** 2384-2398. MR1425958
- [13] BUTUCEA, C., GUȚĂ, M. and ARTILES, L. (2007). Minimax and adaptive estimation of the Wigner function in quantum homodyne tomography with noisy data. *Annals of Statistics* **35** 465-494.
- [14] BUTUCEA, C., GUȚĂ, M. and KYPRAIOS, T. (2015). Spectral thresholding quantum tomography for low rank states. *New Journal of Physics* **17** 113050.
- [15] BUTUCEA, C., GUȚĂ, M. and NUSSBAUM, M. Supplementary material to: Local asymptotic equivalence of pure states ensembles and quantum Gaussian white noise. *Annals of Statistics*.
- [16] CAI, T., KIM, D., WANG, Y., YUAN, M. and ZHOU, H. H. (2016). Optimal large-scale quantum state tomography with Pauli measurements. *Ann. Statist.* **44** 682-712. MR3476614
- [17] CAI, T. T. and LOW, M. G. (2005). Nonquadratic estimators of a quadratic functional. *Ann. Statist.* **33** 2930-2956.
- [18] CARPENTIER, A., EISERT, J., GROSS, D. and NICKL, R. Uncertainty quantification for matrix compressed sensing and quantum tomography problems. arXiv:1504.03234.
- [19] CHRISTANDL, M. and RENNER, R. (2012). Reliable quantum state tomography. *Phys. Rev. Lett.* **109** 120403.
- [20] DONOHO, D. L. and LIU, R. C. (1991). Geometrizing rates of convergence II,III. *Ann. Statist.* **19** 633-667, 668-701.
- [21] ERMAKOV, M. S. (1990). Minimax detection of a signal in Gaussian white noise. *Teor. Veroyatnost. i Primenen.* **35** 704-715. MR1090496
- [22] FAIST, P. and RENNER, R. (2016). Practical and reliable error bars in quantum tomography. *Phys. Rev. Lett.* **117** 010404.
- [23] FLAMMIA, S. T., GROSS, D., LIU, Y. K. and EISERT, J. (2012). Quantum tomography via compressed sensing: error bounds, sample complexity, and efficient estimators. *New. J. Phys.* **14** 095022.
- [24] FUCHS, C. A. and VAN DE GRAAF, J. (1999). Cryptographic distinguishability measures for quantum-mechanical states. *IEEE Transactions Information Theory* **45**.
- [25] GILL, R. D. and GUȚĂ, M. I. (2013). On asymptotic quantum statistical inference. In *From Probability to Statistics and Back: High-Dimensional Models and Processes. Inst. Math. Stat. (IMS) Collect.* **9** 105-127. Inst. Math. Statist., Beachwood, OH. MR3186752
- [26] GILL, R. D. and MASSAR, S. (2000). State estimation for large ensembles. *Phys. Rev. A* **61** 042312.
- [27] GOLUBEV, G. K., NUSSBAUM, M. and ZHOU, H. H. (2010). Asymptotic equivalence of spectral density estimation and Gaussian white noise. *Ann. Statist.* **38** 181-214. MR2589320
- [28] GRAMA, I. and NUSSBAUM, M. (1998). Asymptotic equivalence for nonparametric generalized linear models. *Probab. Theory Related Fields* **111** 167-214. MR1633574
- [29] GROSS, D. (2011). Recovering low-rank matrices from few coefficients in any Basis. *IEEE Transactions on Information Theory* **57** 1548-1566.
- [30] GROSS, D., LIU, Y. K., FLAMMIA, S. T., BECKER, S. and EISERT, J. (2010). Quantum state tomography via compressed sensing. *Physical Review Letters* **105** 150401.
- [31] GUȚĂ, M., JANSSENS, B. and KAHN, J. (2008). Optimal estimation of qubit states with continuous time

- measurements. *Commun. Math. Phys.* **277** 127-160.
- [32] GUȚĂ, M. and JENČOVÁ, A. (2007). Local asymptotic normality in quantum statistics. *Comm. Math. Phys.* **276** 341-379. MR2346393 (2009b:81110)
- [33] GUȚĂ, M. and KAHN, J. (2006). Local asymptotic normality for qubit states. *Phys. Rev. A* **73** 052108.
- [34] GUTA, M. and KAHN, J. (2008). Local asymptotic normality and optimal estimation for d-dimensional quantum systems. In *Quantum Stochastics and Information: Statistics, Filtering and Control* 300-322. World Scientific.
- [35] HÄFFNER, H., HÄNSEL, W., ROOS, C. F., BENHELM, J., CHEK-AL KAR, D., CHWALLA, M., KÖRBER, T., RAPOL, U. D., RIEBE, M., SCHMIDT, P. O., BECHER, C., GÜHNE, O., DÜR, W. and BLATT, R. (2005). Scalable multiparticle entanglement of trapped ions. *Nature* **438** 643.
- [36] HAYASHI, M. (1998). Asymptotic estimation theory for a finite-dimensional pure state model. *Journal of Physics A: Mathematical and General* **31**.
- [37] HAYASHI, M., ed. (2005). *Asymptotic Theory of Quantum Statistical Inference: Selected Papers*. World Scientific.
- [38] HELSTROM, C. W. (1976). *Quantum Detection and Estimation Theory*. Academic Press, New York.
- [39] HIAI, F. and PETZ, D. (1991). The proper formula for relative entropy and its asymptotics in quantum probability. *Commun. Math. Phys.* **143** 99-114.
- [40] HOLEVO, A. S. (1982). *Probabilistic and Statistical Aspects of Quantum Theory*. North-Holland.
- [41] INGSTER, Y. and STEPANOVA, N. (2011). Estimation and detection of functions from anisotropic Sobolev classes. *Electron. J. Stat.* **5** 484-506. MR2813552
- [42] INGSTER, Y. I. and SUSLINA, I. A. (2003). *Nonparametric Goodness-of-Fit Testing under Gaussian Models. Lecture Notes in Statistics* **169**. Springer-Verlag, New York. MR1991446
- [43] JI, P. and NUSSBAUM, M. (2017). Sharp minimax adaptation over Sobolev ellipsoids in nonparametric testing. *Electron. J. Stat.* **11** 4515-4562.
- [44] KAHN, J. and GUȚĂ, M. (2009). Local asymptotic normality for finite dimensional quantum systems. *Commun. Math. Phys.* **289** 597-652.
- [45] KARGIN, V. (2005). On the Chernoff bound for efficiency of quantum hypothesis testing. *Ann. Statist.* **33** 959-976. MR2163165
- [46] KLEMELÄ, J. (2006). Sharp adaptive estimation of quadratic functionals. *Probab. Theory Related Fields* **134** 539-564.
- [47] KOLTCHINSKII, V. (2011). Von Neumann entropy penalization and low-rank matrix estimation. *Ann. Statist.* **39** 2936-2973. MR3012397
- [48] KOLTCHINSKII, V. and XIA, D. (2015). Optimal estimation of low rank density matrices. *J. Mach. Learn. Res.* **16** 1757-1792. MR3417798
- [49] LE CAM, L. (1986). *Asymptotic Methods in Statistical Decision Theory*. Springer Verlag, New York.
- [50] LEONHARDT, U. (1997). *Measuring the Quantum State of Light*. Cambridge University Press.
- [51] LI, K. (2016). Discriminating quantum states: The multiple Chernoff distance. *Ann. Statist.* **44** 1661-1679.
- [52] NAIR, R., GUHA, S. and TAN, S.-H. (2014). Realizable receivers for discriminating coherent and multi-copy quantum states near the quantum limit. *Phys. Rev. A* **89** 032318.
- [53] NIELSEN, M. A. and CHUANG, I. L. (2000). *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge. MR1796805
- [54] NUSSBAUM, M. (1996). Asymptotic equivalence of density estimation and Gaussian white noise. *Ann. Statist.* **24** 2399-2430. MR1425959
- [55] NUSSBAUM, M. and SZKOLA, A. (2006). The Chernoff lower bound for symmetric quantum hypothesis testing. *Ann. Statist.* **37** 1040-1057.
- [56] OGAWA, T. and NAGAOKA, H. (2000). Strong converse and Stein's lemma in quantum hypothesis testing. *IEEE Transactions on Information Theory* **46** 2428-2433.
- [57] PARIS, M. G. A. and ŘEHÁČEK, J., eds. (2004). *Quantum State Estimation. Lecture Notes in Physics* **649**. Springer-Verlag, Berlin, Heidelberg.
- [58] PARTHASARATHY, K. R. (1992). *An Introduction to Quantum Stochastic Calculus. Modern Birkhäuser Classics*. Birkhäuser/Springer Basel AG, Basel.
- [59] PETZ, D. and JENCOVA, A. (2006). Sufficiency in quantum statistical inference. *Commun. Math. Phys.* **263** 259 - 276.
- [60] RAY, K. and SCHMIDT-HIEBER, J. (2017). The Le Cam distance between density estimation, Poisson processes and Gaussian white noise. *arXiv:1608.01824 [math.ST]*.
- [61] REISS, M. (2008). Asymptotic equivalence for nonparametric regression with multivariate and random

- design. *Ann. Statist.* **36** 1957–1982. MR2435461
- [62] REISS, R. D. (1993). *A Course on Point Processes*. Springer-Verlag, New York. MR1199815
- [63] ROHDE, A. (2004). On the asymptotic equivalence and rate of convergence of nonparametric regression and Gaussian white noise. *Statist. Decisions* **22** 235–243. MR2125610
- [64] STRASSER, H. (1985). *Mathematical Theory of Statistics*. Walter de Gruyter & Co., Berlin. MR812467
- [65] TEMME, K. and VERSTRAETE, F. (2015). Quantum chi-squared and goodness of fit testing. *J. Math. Phys.* **56** 012202.
- [66] TORGERSEN, E. (1991). *Comparison of Statistical Experiments*. Cambridge University Press.
- [67] TSYBAKOV, A. B. (2009). *Introduction to Nonparametric Estimation*. Springer, New York.
- [68] WALTER, M. and RENES, J. M. (2014). Lower bounds for quantum parameter estimation. *IEEE Transactions on Information Theory* **60** 8007-8023.

6. Proofs.

PROOF OF THEOREM 4.1. The direct map channel T_n is defined as an isometric embedding

$$\begin{aligned} T_n : \mathcal{T}_1(\mathcal{H}^{\otimes_s n}) &\rightarrow \mathcal{T}_1(\mathcal{F}(\mathcal{H}_0)) \\ \rho &\mapsto V_n \rho V_n^*. \end{aligned}$$

where $V_n : \mathcal{H}^{\otimes_s n} \rightarrow \mathcal{F}(\mathcal{H}_0)$ is an isometry defined below. Since we deal with pure states, it suffices to prove that

$$(40) \quad \limsup_{n \rightarrow \infty} \sup_{|\psi_0\rangle \in \mathcal{H}} \sup_{\|u\| \leq \gamma_n} \|V_n \psi_u^{\otimes n} - G(\sqrt{n}u)\| = 0.$$

We now define the isometric embedding V_n by showing its explicit action on the vectors of an ONB. For any permutation $\sigma \in S_n$, let

$$U_\sigma : |u_1\rangle \otimes \cdots \otimes |u_n\rangle \mapsto |u_{\sigma^{-1}(1)}\rangle \otimes \cdots \otimes |u_{\sigma^{-1}(n)}\rangle$$

be the unitary action on $\mathcal{H}^{\otimes n}$ by tensor permutations. Then $P_s := \frac{1}{n!} \sum_{\sigma \in S_n} U_\sigma$ is the orthogonal projector onto the subspace of symmetric tensors $\mathcal{H}^{\otimes_s n}$. We construct an orthonormal basis in $\mathcal{H}^{\otimes_s n}$ as follows.

Let $B_0 := \{|e_1\rangle, |e_2\rangle, \dots\}$ be an orthonormal basis in \mathcal{H}_0 . Let $\tilde{\mathbf{n}} = (n_0, \mathbf{n}) = (n_0, n_1, \dots)$ be an infinite sequence of integers such that $\sum_{i \geq 0} n_i = n$, and note that only a finite number of n_i s are different from zero. Then the symmetric vectors

$$|\tilde{\mathbf{n}}\rangle = |n_0, n_1, n_2, \dots\rangle := \sqrt{\frac{n!}{n_0! \cdot n_1! \cdot \dots}} P_s \left[|\psi_0\rangle^{\otimes n_0} \otimes \bigotimes_{i \geq 1} |e_i\rangle^{\otimes n_i} \right]$$

form an ONB of $\mathcal{H}^{\otimes_s n}$.

As discussed in section 2.2.2 the Fock space $\mathcal{F}(\mathcal{H}_0)$ can be identified with the infinite tensor product of one-mode Fock spaces $\bigotimes_{i \geq 1} \mathcal{F}(\mathbb{C}|e_i\rangle)$ which has an orthonormal number basis (or Fock basis) consisting of products of number basis vectors of individual modes

$$|\mathbf{n}\rangle := \bigotimes_{i \geq 1} |n_i\rangle$$

where $n_i \neq 0$ only for a finite number of indices. We define $V_n : \mathcal{H}^{\otimes_s n} \rightarrow \mathcal{F}(\mathcal{H}_0)$ as follows

$$V_n : |\tilde{\mathbf{n}}\rangle \mapsto |\mathbf{n}\rangle.$$

Its image consists of states with at most n ‘‘excitations’’, with $|\psi_0\rangle^{\otimes n}$ being mapped to the vacuum state $|\mathbf{0}\rangle$. We would like to show that the embedded state $V_n |\psi_u\rangle^{\otimes n}$ are well approximated by the coherent states $|G(\sqrt{n}u)\rangle$ uniformly over the local neighbourhood $\|u\| \leq \gamma_n$. For this we will make use of the covariance and functorial properties of the second quantisation construction in order to reduce the non-parametric LAE statement to the corresponding one for 2-dimensional systems.

Let $|u\rangle \in \mathcal{H}_0$ be a fixed unit vector. Let $j : \mathbb{C}^2 \mapsto \mathcal{H}$ be the isometric embedding

$$j : |0\rangle \mapsto |\psi_0\rangle, \quad j : |1\rangle \mapsto |u\rangle$$

and let $j_0 : \mathbb{C}|1\rangle \rightarrow \mathcal{H}_0$ be the restriction of j to the one dimensional subspace $\mathbb{C}|1\rangle$. Since second quantisation is functorial under contractive maps, there is a corresponding isometric embedding $J_0 = \Gamma(j_0)$ satisfying

$$(41) \quad \begin{aligned} J_0 : \mathcal{F}(\mathbb{C}|1\rangle) &\rightarrow \mathcal{F}(\mathcal{H}_0) \\ |G(\alpha)\rangle &\mapsto |G(j_0(\alpha))\rangle = |G(\alpha u)\rangle. \end{aligned}$$

Let $\tilde{V}_n : (\mathbb{C}^2)^{\otimes n} \rightarrow \mathcal{F}(\mathbb{C}|1\rangle)$ be the isometry constructed in the same way as V_n , where $|0\rangle$ plays the role of $|\psi_0\rangle$ and $\mathbb{C}|1\rangle$ is the analogue of \mathcal{H}_0 . As before, let $|\tilde{\psi}_\alpha\rangle = \sqrt{1 - |\alpha|^2}|0\rangle + \alpha|1\rangle$, with $|\alpha| \leq 1$. Then by the properties of the embedding map V_n we have

$$(42) \quad J_0 \tilde{V}_n |\tilde{\psi}_\alpha\rangle^{\otimes n} = V_n |\psi_{\alpha u}\rangle^{\otimes n}.$$

From equations (41) and (42) we find

$$\sup_{|\alpha| \leq \gamma_n} \|V_n \psi_{\alpha u}^{\otimes n} - G(\sqrt{n}\alpha u)\| = \sup_{|\alpha| \leq \gamma_n} \|\tilde{V}_n \tilde{\psi}_\alpha^{\otimes n} - G(\sqrt{n}\alpha)\|$$

Since the right-hand side of the above equality is independent of $|u\rangle$ the same equality holds with supremum on the left side taken over all $|u\rangle \in \mathcal{H}_0$ with $\|u\| = 1$, which is the same as the supremum in equation (40). Therefore the LAE for the non-parametric models has been reduced to that of a two-dimensional (qubit) model. Although this has been treated in the more general case of mixed states in [33], we present the calculation for reader's convenience. The product state $|\tilde{\psi}_\alpha\rangle^{\otimes n}$ is mapped into the following pure state on the Fock space $\mathcal{F}(\mathbb{C}|1\rangle)$

$$\tilde{V}_n |\tilde{\psi}_\alpha\rangle^{\otimes n} = \sum_{k=0}^n c_{k,n}(\alpha) |k\rangle, \quad c_{k,n}(\alpha) = \alpha^k (1 - |\alpha|^2)^{(n-k)/2} \sqrt{\binom{n}{k}}.$$

On the other hand, in view of (2) the coherent state can be written as

$$G(\sqrt{n}\alpha) = \sum_k c_k(\sqrt{n}\alpha) |k\rangle, \quad c_k(\sqrt{n}\alpha) := \exp(-n|\alpha|^2/2) \frac{(\sqrt{n}\alpha)^k}{\sqrt{k!}}.$$

Set $\alpha = \phi_\alpha |\alpha|$ where ϕ_α is a phase; then it follows that $c_{k,n}(\alpha) = \phi_\alpha^k c_{k,n}(|\alpha|)$ and $c_k(\sqrt{n}\alpha) = \phi_\alpha^k c_k(\sqrt{n}|\alpha|)$. With this we have

$$(43) \quad \begin{aligned} \left\| \tilde{V}_n \tilde{\psi}_\alpha^{\otimes n} - G(\sqrt{n}\alpha) \right\|^2 &= \sum_{k=0}^{\infty} |c_{k,n}(\alpha) - c_k(\sqrt{n}\alpha)|^2 \\ &= \sum_{k=0}^{\infty} |c_{k,n}(|\alpha|) - c_k(\sqrt{n}|\alpha|)|^2. \end{aligned}$$

Let X be a binomial r.v. with parameters $n, |\alpha|^2$ and Y be a Poisson r.v. with parameter $n|\alpha|^2$. Note that $c_{k,n}(|\alpha|) = P(X = k)^{1/2}$ and $c_k(\sqrt{n}|\alpha|) = P(Y = k)^{1/2}$, and that therefore (43) is the squared Hellinger distance between these two laws. According to Theorem 1.3.1 (ii) in [62] we have

$$\sum_{k=0}^{\infty} |c_{k,n}(|\alpha|) - c_k(\sqrt{n}|\alpha|)|^2 \leq 3|\alpha|^4.$$

Since $|\alpha| \leq \gamma_n = o(1)$, we have shown the first part of LAE in which the i.i.d. and Gaussian models are expressed in terms of the local parameter $|u\rangle$

$$(44) \quad \limsup_{n \rightarrow \infty} \sup_{|\psi_0\rangle \in \mathcal{H}} \sup_{\|u\| \leq \gamma_n} \|V_n \psi_u^{\otimes n} - G(\sqrt{n}u)\| = 0.$$

Conversely, we define the reverse channel $S_n : \mathcal{T}_1(\mathcal{F}(\mathcal{H}_0)) \rightarrow \mathcal{T}_1(\mathcal{H}^{\otimes s_n})$ as follows. Let P_n denote the orthogonal projection in $\mathcal{F}(\mathcal{H}_0)$ onto the image space of V_n , i.e. the subspace with total excitation number at most n

$$\mathcal{F}_{\leq n}(\mathcal{H}_0) := \text{Lin}\{|n_1, n_2, \dots\rangle : \sum_{i \geq 1} n_i \leq n\}.$$

Let $R_n : \mathcal{F}(\mathcal{H}_0) \rightarrow \mathcal{H}^{\otimes s_n}$ be a right inverse of V_n , i.e. $R_n V_n = \mathbf{1}$. Then the reverse channel is defined as

$$S_n(\rho) = R_n P_n \rho P_n R_n^* + \text{Tr}(\rho(1 - P_n)) |\psi_0\rangle \langle \psi_0|^{\otimes n}.$$

Operationally, the action of S_n consists of two steps. We first perform a projection measurement with projections P_n and $(1 - P_n)$; if the first outcome occurs the conditional state of the system is $P_n \rho P_n / \text{Tr}(P_n \rho)$, while if the second outcome occurs the state is $(1 - P_n) \rho (1 - P_n) / \text{Tr}((1 - P_n) \rho)$. In the second stage, if the first outcome was obtained we map the projected state through the map R_n into a state in $\mathcal{H}^{\otimes s_n}$, while if the second outcome was obtained, we prepare the fixed state $|\psi_0\rangle \langle \psi_0|^{\otimes n}$.

When applied to the pure Gaussian states $|G(\sqrt{n}u)\rangle$, the output of S_n is the mixed state

$$S_n(|G(\sqrt{n}u)\rangle \langle G(\sqrt{n}u)|) = p_u^n |\phi_u^n\rangle \langle \phi_u^n| + (1 - p_u^n) |\psi_0\rangle \langle \psi_0|^{\otimes n}$$

where

$$|\phi_u^n\rangle := \frac{R_n P_n |G(\sqrt{n}u)\rangle}{\sqrt{p_u^n}}, \quad p_u^n = \|P_n G(\sqrt{n}u)\|^2.$$

The key observation is that the Gaussian states are almost completely supported by the subspace $\mathcal{F}_{\leq n}(\mathcal{H}_0)$, uniformly with respect to the ball $\|u\| \leq \gamma_n$. Indeed, since $V_n \psi_u^{\otimes n}$ is in $\mathcal{F}_{\leq n}(\mathcal{H}_0)$, from (44) and the properties of projections it follows

$$\limsup_{n \rightarrow \infty} \sup_{|\psi_0\rangle} \sup_{\|u\| \leq \gamma_n} \|P_n G(\sqrt{n}u) - G(\sqrt{n}u)\| = 0,$$

so that

$$(45) \quad \limsup_{n \rightarrow \infty} \sup_{|\psi_0\rangle} \sup_{\|u\| \leq \gamma_n} (1 - p_u^n) = 0.$$

Now again from (44) and the fact that R_n is the inverse of V_n it follows

$$\limsup_{n \rightarrow \infty} \sup_{|\psi_0\rangle} \sup_{\|u\| \leq \gamma_n} \|\psi_u^{\otimes n} - R_n P_n G(\sqrt{n}u)\| = 0,$$

which in conjunction with (45) implies

$$\limsup_{n \rightarrow \infty} \sup_{|\psi_0\rangle} \sup_{\|u\| \leq \gamma_n} \|S_n(|G(\sqrt{n}u)\rangle \langle G(\sqrt{n}u)|) - |\psi_u\rangle \langle \psi_u|^{\otimes n}\|_1 = 0.$$

This completes the proof of (25). \square

PROOF OF THEOREM 5.2. Let us denote by $R_n^E = \inf_{\widehat{\psi}_n} \sup_{\psi \in S^\alpha(L)} \mathbb{E}_\psi \left[\|\widehat{\psi}_n - \psi\|_2^2 \right]$ the minimax risk.

The first step is to reduce the set of states $S^\alpha(L)$ to a finite hypercube denoted $S_{1:N}^\alpha(L)$ consisting of certain “truncated” vectors $|\psi\rangle = \sum_{1 \leq i \leq N} \psi_i |e_i\rangle$ which have $N \asymp n^{1/(2\alpha+1)}$ non-zero coefficients with respect to the standard basis. This will provide a lower bound to the minimax risk. The coefficients are chosen as

$$\psi_j = \pm \frac{\sigma_j}{\sqrt{n}}, \quad \sigma_j^2 = \lambda(1 - (j/N)^{2\alpha}), \quad j = 1, \dots, N, \text{ for some fixed } \lambda > 0$$

and we check that they satisfy the ellipsoid constraint

$$\sum_{j \geq 1} |\psi_j|^2 j^{2\alpha} = \frac{\lambda}{n} \sum_{j=1}^N (j^{2\alpha} - j^{4\alpha} N^{-2\alpha}) \leq \frac{N^{2\alpha+1}}{n} \frac{2\alpha\lambda}{(2\alpha+1)(4\alpha+1)} (1 + o(1)) \leq L$$

for an appropriate choice of $\lambda > 0$.

Using the factorisation property (9) we can identify the corresponding Gaussian states with the N -mode state defined by $|\phi\rangle = \otimes_{j=1}^N |G(\sqrt{n}\psi_j)\rangle$, where the remaining modes are in the vacuum state and can be ignored.

Thus

$$\begin{aligned} R_n^E &\geq \inf_{\widehat{\psi}} \sup_{\psi \in S_{1:N}^\alpha(L)} \mathbb{E}_\psi \left[\|\widehat{\psi} - \psi\|_2^2 \right] \\ &= \inf_{\widehat{\psi}} \sup_{\psi \in S_{1:N}^\alpha(L)} \mathbb{E}_\psi \left[\sum_{j=1}^N |\widehat{\psi}_j - \psi_j|^2 \right]. \end{aligned}$$

The supremum over the finite hypercube $S_{1:N}^\alpha(L)$ is bounded from below by the average over all its elements. This turns the previous maximal risk into a Bayesian risk, that we can further bound from below as follows:

$$\begin{aligned} R_n^E &\geq \inf_{\widehat{\psi}} \frac{1}{2^N} \sum_{\psi \in S_{1:N}^\alpha(L)} \sum_{j=1}^N \mathbb{E}_\psi \left[|\widehat{\psi}_j - \psi_j|^2 \right] \\ &= \inf_{\widehat{\psi}} \sum_{j=1}^N \frac{1}{2^N} \sum_{\psi \in S_{1:N}^\alpha(L)} \mathbb{E}_\psi \left[|\widehat{\psi}_j - \psi_j|^2 \right] \\ (46) \quad &\geq \sum_{j=1}^N \inf_{\widehat{\psi}_j} \frac{1}{2^N} \sum_{\psi \in S_{1:N}^\alpha(L)} \mathbb{E}_\psi \left[|\widehat{\psi}_j - \psi_j|^2 \right]. \end{aligned}$$

In the second line $\widehat{\psi}$ is the result of an arbitrary measurement and estimation procedure of the state $|G(\sqrt{n}\psi)\rangle$. In the third line each infimum is over procedures for estimating the component ψ_j only; since such procedure may not be compatible with a single measurement, the third line is upper bounded by the second.

The second major step in the proof of the lower bounds is to reduce the risk over all measurements, to testing two simple hypotheses. Let us bound from below the term (46) for

arbitrary fixed j between 1 and N :

$$\begin{aligned}
& \frac{1}{2^N} \sum_{\psi \in S_{1:N}^\alpha(L)} \mathbb{E}_\psi \left[|\widehat{\psi}_j - \psi_j|^2 \right] \\
&= \frac{1}{2} \left\{ \frac{1}{2^{N-1}} \sum_{\psi \in S_{(j+)}^\alpha(L)} \mathbb{E}_\psi \left[|\widehat{\psi}_j - \sigma_j/\sqrt{n}|^2 \right] + \frac{1}{2^{N-1}} \sum_{\psi \in S_{(j-)}^\alpha(L)} \mathbb{E}_\psi \left[|\widehat{\psi}_j - (-\sigma_j/\sqrt{n})|^2 \right] \right\} \\
(47) &= \frac{1}{2} \left\{ \mathbb{E}_{\rho_j^+} \left[|\widehat{\psi}_j - \sigma_j/\sqrt{n}|^2 \right] + \mathbb{E}_{\rho_j^-} \left[|\widehat{\psi}_j - (-\sigma_j/\sqrt{n})|^2 \right] \right\},
\end{aligned}$$

where the sum over $\psi \in S_{(j\pm)}^\alpha(L)$ means that the j^{th} coordinate is fixed to $\pm\sigma_j/\sqrt{n}$ and all k^{th} coordinates, for $k \neq j$, take values in $\{\sigma_k/\sqrt{n}, -\sigma_k/\sqrt{n}\}$. In the third line, we denote by ρ_j^\pm the average state over states in $S_{(j\pm)}^\alpha(L)$.

Let us define the testing problem of the two hypotheses $H_0 : \rho = \rho_j^+$ against $H_1 : \rho = \rho_j^-$. For a given estimator $\widehat{\psi}_j$ we construct the test

$$\Delta = I \left(\left| \widehat{\psi}_j - \frac{\sigma_j}{\sqrt{n}} \right| > \left| \widehat{\psi}_j - \left(-\frac{\sigma_j}{\sqrt{n}} \right) \right| \right),$$

and decide H_1 or H_0 , if Δ equals 1 or 0, respectively. By the Markov inequality, we get that

$$\mathbb{E}_{\rho_j^\pm} \left[\left| \widehat{\psi}_j - \left(\pm \frac{\sigma_j}{\sqrt{n}} \right) \right|^2 \right] \geq \frac{\sigma_j^2}{n} \mathbb{P}_{\rho_j^\pm} \left(\left| \widehat{\psi}_j - \left(\pm \frac{\sigma_j}{\sqrt{n}} \right) \right| \geq \frac{\sigma_j}{\sqrt{n}} \right).$$

On the one hand,

$$(48) \quad \mathbb{P}_{\rho_j^+} \left(\left| \widehat{\psi}_j - \sigma_j/\sqrt{n} \right| \geq \frac{\sigma_j}{\sqrt{n}} \right) \geq \mathbb{P}_{\rho_j^+} (\Delta = 1).$$

Indeed, under $\mathbb{P}_{\rho_j^+}$, the event $\Delta = 1$ implies that $|\widehat{\psi}_j - \frac{\sigma_j}{\sqrt{n}}| > |\widehat{\psi}_j + \frac{\sigma_j}{\sqrt{n}}|$, which further implies by the triangular inequality that

$$\left| \widehat{\psi}_j - \frac{\sigma_j}{\sqrt{n}} \right| \geq \frac{2\sigma_j}{\sqrt{n}} - \left| \widehat{\psi}_j + \frac{\sigma_j}{\sqrt{n}} \right| \geq \frac{2\sigma_j}{\sqrt{n}} - \left| \widehat{\psi}_j - \frac{\sigma_j}{\sqrt{n}} \right|,$$

giving $|\widehat{\psi}_j - \psi_j| \geq \frac{\sigma_j}{\sqrt{n}}$. By a similar reasoning for the $\mathbb{P}_{\rho_j^-}$ distribution we get

$$(49) \quad \mathbb{P}_{\rho_j^-} \left(\left| \widehat{\psi}_j + \sigma_j/\sqrt{n} \right| \geq \frac{\sigma_j}{\sqrt{n}} \right) \geq \mathbb{P}_{\rho_j^-} (\Delta = 0).$$

By using (48) and (49) in (47)

$$\frac{1}{2} \left\{ \mathbb{E}_{\rho_j^+} \left[\left| \widehat{\psi}_j - \sigma_j/\sqrt{n} \right|^2 \right] + \mathbb{E}_{\rho_j^-} \left[\left| \widehat{\psi}_j - (-\sigma_j/\sqrt{n}) \right|^2 \right] \right\} \geq \frac{\sigma_j^2}{2n} \left(\mathbb{P}_{\rho_j^+} (\Delta = 1) + \mathbb{P}_{\rho_j^-} (\Delta = 0) \right).$$

To summarise, we have lower bounded the MSE by the probability of error for testing between the states ρ_j^\pm . At closer inspection, these states are of the form $|G(\sigma_j)\rangle\langle G(\sigma_j)| \otimes \rho$ and $|G(-\sigma_j)\rangle\langle G(-\sigma_j)| \otimes \rho$ where ρ is a fixed state obtained by averaging the coherent states of all

the modes except j . Recall that the optimal testing error in (11) gives a further bound from below

$$\mathbb{P}_{\rho_j^+}(\Delta = 1) + \mathbb{P}_{\rho_j^-}(\Delta = 0) \geq 1 - \frac{1}{2} \|\rho_j^+ - \rho_j^-\|_1.$$

Moreover, the state ρ can be dropped without changing the optimal testing error

$$\|\rho_j^+ - \rho_j^-\|_1 = \| |G(\sigma_j)\rangle\langle G(\sigma_j)| - |G(-\sigma_j)\rangle\langle G(-\sigma_j)| \|_1 = 2(1 - \exp(-2\sigma_j^2)).$$

We conclude that

$$\inf_{\hat{\psi}_j} \frac{1}{2} \left\{ \mathbb{E}_{\rho_j^+} \left[\left| \hat{\psi}_j - \sigma_j/\sqrt{n} \right|^2 \right] + \mathbb{E}_{\rho_j^-} \left[\left| \hat{\psi}_j - (-\sigma_j/\sqrt{n}) \right|^2 \right] \right\} \geq \frac{\sigma_j^2}{2n} \cdot \exp(-2\sigma_j^2)$$

and we further use this in (47) to get

$$R_n^E \geq \sum_{j=1}^N \frac{\sigma_j^2}{2n} \cdot \exp(-2\sigma_j^2) = \frac{N}{n} \cdot \frac{\lambda}{2N} \sum_{j=1}^N \left(1 - \left(\frac{j}{N}\right)^{2\alpha} \right) \exp\left(-2 \cdot \lambda \left(1 - \left(\frac{j}{N}\right)^{2\alpha}\right)\right) \geq c \frac{N}{n}.$$

Indeed, the average over j is the Riemann sum associated to the integral of a positive function and can be bounded from below by some constant $c > 0$ depending on α . Moreover, $N/n \asymp n^{-2\alpha/(2\alpha+1)}$ and thus we finish the proof of the theorem. \square

PROOF OF THEOREM 5.3. Let $\tilde{R}_n^E = \inf_{|\hat{\psi}_n\rangle} \sup_{|\psi\rangle \in S^\alpha(L)} \mathbb{E}_\rho [d(\hat{\rho}_n, \rho)^2]$ be the minimax risk for \mathcal{Q}_n .

We bound from below the risk by restricting to (pure) states in a neighbourhood $\Sigma_n(e_0)$ of the basis vector $|e_0\rangle$ defined as follows. As in (22) we write the state and the estimator in terms of their corresponding local vectors

$$|\psi\rangle = \sqrt{1 - \|u\|^2} |e_0\rangle + |u\rangle, \quad |\hat{\psi}\rangle = \sqrt{1 - \|\hat{u}\|^2} |e_0\rangle + |\hat{u}\rangle, \quad |u\rangle, |\hat{u}\rangle \perp |e_0\rangle.$$

Then the neighbourhood is given by $\Sigma_n(e_0) := \{|\psi_u\rangle : \|u\| \leq \gamma_n\}$; we choose $\gamma_n = (\log n)^{-1}$. Such states are described by the local model $\mathcal{Q}_n(e_0, \gamma_n)$, cf. equation (23). The risk is bounded from below by

$$\tilde{R}_n^E \geq \inf_{|\hat{\psi}_n\rangle} \sup_{|\psi\rangle \in S^\alpha(L) \cap \Sigma_n(e_0)} \mathbb{E}_\rho [d(\hat{\rho}_n, \rho)^2].$$

By using the triangle inequality we can assume that $\hat{\psi} \in \Sigma_n(e_0)$, while incurring at most a factor 2 in the risk. By using the quadratic approximation (26) we find that

$$(50) \quad d^2(\hat{\rho}_n, \rho) = k \|u - \hat{u}\|^2 + o(n^{-1})$$

where $k = 1$ or $k = 4$ depending on which distance we use. Since n^{-1} decreases faster than $n^{-2\alpha/(2\alpha+1)}$, the second term does not contribute to the asymptotic rate and can be neglected, so that the problem has been reduced to that of estimating the local parameter u with respect to the Hilbert space distance. To study the latter, we further restrict the set of states to a hypercube similar to the one in the proof of Theorem 5.2, consisting of states $|\psi_u\rangle$ with ‘‘truncated’’ local vectors $|u\rangle = \sum_{1 \leq i \leq N} u_i |e_i\rangle$ belonging to $S_{1:N}^\alpha(L)$. As before, there are $N \asymp n^{1/(2\alpha+1)}$ non-zero coefficients of the form

$$u_j = \pm \frac{\sigma_j}{\sqrt{n}}, \quad \sigma_j^2 = \lambda \left(1 - \left(\frac{j}{N}\right)^{2\alpha}\right), \quad j = 1, \dots, N.$$

It has been already shown that such vectors belong to the ellipsoid $S^\alpha(L)$. Additionally, we show that they also belong to the local ball $\Sigma_n(e_0)$. Indeed

$$\begin{aligned} \|u\|^2 &= \sum_{j=1}^N |u_j|^2 = \frac{1}{n} \sum_{j=1}^N \sigma_j^2 = \frac{1}{n} \sum_{j=1}^N \lambda \left(1 - (j/N)^{2\alpha}\right) \\ &= \frac{N}{n} \left(\frac{1}{N} \sum_{j=1}^N \lambda \left(1 - (j/N)^{2\alpha}\right) \right) \leq C_1 \frac{N}{n}, \end{aligned}$$

where we used that as $N \rightarrow \infty$ the expression between the parentheses tends to a finite integral. As N scales as $n^{1/(2\alpha+1)}$, the upper bound becomes

$$\|e_0 - \psi_u\|^2 \leq C_2 n^{-2\alpha/(2\alpha+1)} = o(\gamma_n^2)$$

and the state $|\psi_u\rangle$ belongs to the local ball $\Sigma_n(e_0)$. Taking into account (50) the risk is therefore lower bounded as

$$\tilde{R}_n^E \geq \inf_{\hat{u}} \sup_{u \in S_{1:N}^\alpha(L)} \mathbb{E}_{\rho_u} [\|u - \hat{u}\|^2] + o(n^{-1}).$$

where $\rho_u = |\psi_u\rangle\langle\psi_u|$, and the infimum is now taken over the local component $|\hat{u}\rangle$ of an estimator $|\hat{\psi}\rangle = \sqrt{1 - \|\hat{u}\|^2}|e_0\rangle + |\hat{u}\rangle$. The first term is further lower bounded by passing to the Bayes risk for the uniform distribution over $S_{1:N}^\alpha(L)$, similarly to the proof of Theorem 5.2

$$\tilde{R}_n^E \geq \sum_{j=1}^N \inf_{\hat{u}_j} \frac{1}{2^N} \sum_{u \in S_{1:N}^\alpha(L)} \mathbb{E}_{\psi_u} [|\hat{u}_j - u_j|^2] + o(n^{-1}).$$

By following the same steps we get

$$\begin{aligned} \frac{1}{2^N} \sum_{u \in S_{1:N}^\alpha(L)} \mathbb{E}_{\rho_u} [|\hat{u}_j - u_j|^2] &= \frac{1}{2} \left\{ \mathbb{E}_{\tau_j^+} [|\hat{\psi}_j - \sigma_j/\sqrt{n}|^2] + \mathbb{E}_{\tau_j^-} [|\hat{\psi}_j - (-\sigma_j/\sqrt{n})|^2] \right\}, \\ (51) \qquad \qquad \qquad &\geq \frac{\sigma_j^2}{2n} \left(\mathbb{P}_{\tau_j^+}(\Delta = 1) + \mathbb{P}_{\tau_j^-}(\Delta = 0) \right) \geq \frac{\sigma_j^2}{2n} \cdot \left(1 - \frac{1}{2} \|\tau_j^+ - \tau_j^-\|_1\right), \end{aligned}$$

where we denote by τ_j^\pm the average state over states $|\psi_u\rangle\langle\psi_u|^{\otimes n}$ with $u \in S_{(j\pm)}^\alpha(L)$, and Δ is a test for the hypotheses $H_0 : \tau = \tau_j^+$ and $H_1 : \tau = \tau_j^-$. In the last inequality we used the Helstrom bound [38] which expresses the optimal average error probability for two states discrimination in terms of the norm-one distance between states.

We now make use of the local asymptotic equivalence result in Theorem 4.1. From (25) we know that there exist quantum channels S_n such that

$$\delta_n := \max_{u \in S_{1:N}^\alpha(L)} \left\| |\psi_u\rangle\langle\psi_u|^{\otimes n} - S_n(|G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)|) \right\|_1 \leq \Delta(\mathcal{Q}_n, \mathcal{G}_n) = o(1).$$

By Lemma 3.1 we get

$$\|\tau_j^+ - \tau_j^-\|_1 \leq \|\rho_j^+ - \rho_j^-\|_1 + 2\delta_n$$

where ρ_j^\pm are the corresponding mixtures in the Gaussian model as defined in the proof of Theorem 5.2. From (51) we then get

$$\frac{1}{2^N} \sum_{u \in S_{1:N}^\alpha(L)} \mathbb{E}_{\rho_u} [|\hat{u}_j - u_j|^2] \geq \frac{\sigma_j^2}{2n} \cdot \left(1 - \frac{1}{2} \|\rho_j^+ - \rho_j^-\|_1 - \delta_n\right) \geq \frac{\sigma_j^2}{2n} \cdot (\exp(-2\sigma_j^2) - \delta_n)$$

The rest of the proof follows as in the proof of Theorem 5.2, with the additional remark that

$$\min_j \exp(-2\sigma_j^2) = \lambda(1 - N^{-\alpha}) \asymp \lambda$$

and infinitely larger than δ_n , for n large enough. \square

PROOF OF THEOREM 5.5. Denote by $R_n^F = \inf_{\widehat{F}_n} \sup_{\psi \in S^\alpha(L)} \eta_n^{-2} \cdot \mathbb{E}_\psi \left(\widehat{F}_n - F(\psi) \right)^2$ the minimax risk.

The case **a**) where $\alpha \geq 2\beta$ reduces to the Cramér-Rao bound that proves that the parametric rate $1/n$ is always a lower bound for the mean square error for estimating $F(\psi)$.

We prove that in the case **b**) where $\beta < \alpha < 2\beta$, this bound from below increases to $n^{-2(1-\beta/\alpha)}$ (up to constants). By the Markov inequality,

$$(52) \quad \eta_n^{-2} \cdot \mathbb{E}_\psi \left(\widehat{F}_n - F(\psi) \right)^2 \geq \frac{1}{4} \cdot \mathbb{P}_\psi \left(|\widehat{F}_n - F(\psi)| \geq \frac{\eta_n}{2} \right).$$

Let us restrict the set of pure states $S^\alpha(L)$ to its intersection with the local model $\mathcal{Q}_n(e_0, \gamma_n)$ (see equation (23)) where $|\psi_u\rangle = \sqrt{1 - \|u\|^2} \cdot |e_0\rangle + |u\rangle$ is such that $\|u\| \leq \gamma_n$, with $\gamma_n = (\log n)^{-1}$. In other words, u belongs to the set

$$s^\alpha(L, \gamma_n) = \left\{ u \in \ell_2(\mathbb{N}^*) : \sum_{j \geq 1} |u_j|^2 j^{2\alpha} \leq L \text{ and } \|u\| \leq \gamma_n \right\}.$$

Using the fact that $F(e_0) = 0$, we have

$$(53) \quad \begin{aligned} & \sup_{\psi \in S^\alpha(L)} \frac{1}{4} \cdot \mathbb{P}_\psi \left(|\widehat{F}_n - F(\psi)| \geq \frac{\eta_n}{2} \right) \\ & \geq \frac{1}{4} \max \left\{ \mathbb{P}_{e_0} \left(|\widehat{F}_n| \geq \frac{\eta_n}{2} \right), \sup_{u \in s^\alpha(L, \gamma_n), F(\psi_u) \geq \eta_n} \mathbb{P}_{\psi_u} \left(|\widehat{F}_n - F(\psi_u)| \geq \frac{\eta_n}{2} \right) \right\} \\ & \geq \frac{1}{8} \left\{ \mathbb{P}_{e_0} \left(|\widehat{F}_n| \geq \frac{\eta_n}{2} \right) + \sup_{u \in s^\alpha(L, \gamma_n), F(\psi_u) \geq \eta_n} \mathbb{P}_{\psi_u} \left(|\widehat{F}_n - F(\psi_u)| \geq \frac{\eta_n}{2} \right) \right\} \\ & \geq \frac{1}{8} \left\{ \mathbb{P}_{e_0} \left(|\widehat{F}_n| \geq \frac{\eta_n}{2} \right) + \sup_{u \in s^\alpha(L, \gamma_n), F(\psi_u) \geq \eta_n} \mathbb{P}_{\psi_u} \left(|\widehat{F}_n| < \frac{\eta_n}{2} \right) \right\} \end{aligned}$$

where in the last inequality we used that $|\widehat{F}_n| < \eta_n/2$ and $F(\psi_u) \geq \eta_n$ imply $|\widehat{F}_n - F(\psi_u)| \geq \eta_n/2$. Note also that $F(\psi_u) = F(u)$ for $|u\rangle \in \mathcal{H}_0$; we now consider the testing problem with hypotheses

$$(54) \quad \begin{cases} H_0 : & |u\rangle = |0\rangle \\ H_1(\alpha, L, \gamma_n, \eta_n) : & |u\rangle, \text{ with } u \in s^\alpha(L, \gamma_n) \text{ and } F(u) \geq \eta_n. \end{cases}$$

Let $\Delta = \Delta(\eta_n) = I(|\widehat{F}_n| \geq \eta_n/2)$ be the test that accepts the null hypothesis when $\Delta = 0$ and rejects the null hypothesis when $\Delta = 1$. Then the right-hand side of (53) is lower bounded by the sum of the error probability of type I and of the maximal error probability of type

II of Δ . We can describe Δ as a binary POVM $M = (M_0, M_1)$, depending on η_n : $M(\eta_n) = (M_0(\eta_n), M_1(\eta_n))$. Thus,

$$(55) \quad \mathbb{P}_{e_0} \left(|\widehat{F}_n| \geq \frac{\eta_n}{2} \right) = \text{Tr}(|e_0\rangle\langle e_0|^{\otimes n} \cdot M_1)$$

and

$$(56) \quad \mathbb{P}_{\psi_u} \left(|\widehat{F}_n| < \frac{\eta_n}{2} \right) = \text{Tr}(|\psi_u\rangle\langle \psi_u|^{\otimes n} \cdot M_0).$$

By putting together (52)-(56), we get that the minimax risk has the lower bound

$$R_n^F \geq \frac{1}{8} \inf_M \left(\langle e_0^{\otimes n} | M_1 | e_0^{\otimes n} \rangle + \sup_{u \in s^\alpha(L, \gamma_n), F(u) \geq \eta_n} \langle \psi_u^{\otimes n} | M_0 | \psi_u^{\otimes n} \rangle \right).$$

Now, using the local asymptotic equivalence Theorem 4.1 with respect to the state $|\psi_0\rangle := |e_0\rangle$ we map the i.i.d. ensemble $|\psi_u\rangle^{\otimes n}$ to the Gaussian state $|G(u)\rangle \in \mathcal{F}(\mathcal{H}_0)$. The lower bound becomes

$$(57) \quad R_n^F \geq \frac{1}{8} \inf_M \left(\langle 0 | M_1 | 0 \rangle + \sup_{u \in s^\alpha(L, \gamma_n), F(u) \geq \eta_n} \langle G(\sqrt{n}u) | M_0 | G(\sqrt{n}u) \rangle \right) + o(1)$$

where the infimum is taken over tests $M = (M_0, M_1)$ and the $o(1)$ terms stems from the vanishing Le Cam distance $\Delta(\mathcal{Q}_n(e_0, \gamma_n), \mathcal{G}_n(e_0, \gamma_n))$. The lower bound has been transformed into a testing problem for the Gaussian model.

In order to bound from below the maximal error probability of type II, we define a prior distribution on the set of alternatives and average over the whole set with respect to this a priori distribution. Similarly to the classical proofs of lower bounds, our construction will lead to a test of simple hypotheses: the former null and the constructed averaged state. Assume that $\{u_j\}_{j \geq 1}$ are all independently distributed, such that u_j has a complex (bivariate) Gaussian distribution $N_2(0, \frac{1}{2}\sigma_j^2 \cdot I_2)$ for all j from 1 to N , and that $u_j = 0$ for all $j > N$, where I_2 is the 2×2 identity matrix. The σ_j^2 are defined as

$$(58) \quad \sigma_j^2 = \lambda \left(1 - \left(\frac{j}{N} \right)^{2\alpha} \right)_+,$$

where $\lambda, N > 0$ are selected such that

$$(59) \quad \sum_{j \geq 1} j^{2\alpha} \sigma_j^2 = L(1 - \varepsilon) \text{ and } \sum_{j \geq 1} j^{2\beta} \sigma_j^2 = n^{-1+\beta/\alpha}(1 + \varepsilon),$$

for an arbitrary $\varepsilon > 0$. Let us denote by Π the joint prior distribution of $\{u_j\}_{j \geq 1}$.

Such a choice of the prior distribution was first introduced by Ermakov [21] for establishing sharp minimax risk bounds for nonparametric testing in the Gaussian white noise model. This construction represents an analog of the prior distribution used in Pinsker's theory for sharp estimation of functions. In our case, using a Gaussian prior as an alternative hypothesis leads to the well-known Gaussian thermal state.

The essence of this construction is that the random vectors $u = \{u_j\}_{j \geq 1}$ concentrate asymptotically, with probability tending to 1, on the spherical segment

$$\{u \in \ell_2(\mathbb{N}) : C n^{-1} \leq \|u\|^2 \leq C n^{-1}(1 + 2\varepsilon')\},$$

for $\varepsilon' > 0$ depending on ε and some constant $C > 0$ depending on α and β described later on, and on the alternative set of hypothesis, $H_1(\alpha, L, \gamma_n, \eta_n)$. Note that the spherical segment is included in the set $\|u\| \leq \gamma_n$, as $\gamma_n = (\log n)^{-1} \gg n^{-1/2}$. The asymptotic concentration is proved by the following lemma.

LEMMA 6.1. *A unique solution (λ, N) of (58), (59), exists for n large enough and admits an asymptotic expansion with respect to n*

$$(60) \quad \begin{aligned} \lambda &\sim n^{-1-1/2\alpha} C_\lambda \frac{(1+\varepsilon)^{(\alpha+1/2)/(\alpha-\beta)}}{(1-\varepsilon)^{(\beta+1/2)/(\alpha-\beta)}}, \quad C_\lambda = \frac{((2\beta+1)(2\beta+2\alpha+1))^{(\alpha+1/2)/(\alpha-\beta)}}{2\alpha(L(2\alpha+1)(4\alpha+1))^{(\beta+1/2)/(\alpha-\beta)}} \\ N &\sim n^{1/2\alpha} C_N \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1/(2(\alpha-\beta))}, \quad C_N = \left(\frac{L(2\alpha+1)(4\alpha+1)}{(2\beta+1)(2\beta+2\alpha+1)}\right)^{1/(2(\alpha-\beta))}. \end{aligned}$$

The independent complex Gaussian random variables $u_j \sim N_2(0, \frac{1}{2}\sigma_j^2 I_2)$, with σ_j 's and (λ, N) given in (58), (59), are such that, for an arbitrary $\varepsilon > 0$,

$$(61) \quad \mathbb{P} \left(C n^{-1} \leq \sum_{j=1}^N |u_j|^2 \leq C n^{-1}(1+2\varepsilon') \right) \rightarrow 1,$$

$$(62) \quad \mathbb{P} \left(\sum_{j=1}^N j^{2\alpha} |u_j|^2 \leq L \right) \rightarrow 1,$$

$$(63) \quad \mathbb{P} \left(\sum_{j=1}^N j^{2\beta} |u_j|^2 \geq n^{-1+\beta/\alpha} \right) \rightarrow 1,$$

where $C = C_\lambda \cdot C_N \cdot 2\alpha/(2\alpha+1)$ is a positive constant depending on α and β , and $\varepsilon' > 0$ depends only on ε .

PROOF OF LEMMA 6.1. The solution of the problem (58), (59) can be found in [21] (see also [43], Lemma A.1) for $\beta = 0$; a similar reasoning applies here. Let us prove that the random variables $\{u_j\}_{j=1, \dots, N}$ satisfy (61) to (63). We have

$$(64) \quad \begin{aligned} \sum_{j=1}^N \sigma_j^2 &= \lambda \sum_{j=1}^N \left(1 - \left(\frac{j}{N} \right)^{2\alpha} \right) \sim \lambda N \frac{2\alpha}{2\alpha+1} \\ &\sim C_\lambda C_N \frac{2\alpha}{2\alpha+1} n^{-1} (1+\varepsilon)^{\alpha/(\alpha-\beta)} (1-\varepsilon)^{-\beta/(\alpha-\beta)} = C n^{-1} (1+\varepsilon'), \end{aligned}$$

where we denote $\varepsilon' = (1+\varepsilon)^{\alpha/(\alpha-\beta)} (1-\varepsilon)^{-\beta/(\alpha-\beta)} - 1$ which is positive for all $\varepsilon \in (0, 1)$.

Note that $E|u_j|^2 = \sigma_j^2$ and $\text{Var}(|u_j|^2) = \sigma_j^4$. We have

$$\mathbb{P} \left(C n^{-1} \leq \sum_{j=1}^N |u_j|^2 \leq C n^{-1}(1+2\varepsilon') \right) = 1 - \mathbb{P} \left(\sum_{j=1}^N |u_j|^2 < C n^{-1} \right) - \mathbb{P} \left(|u_j|^2 > C n^{-1}(1+2\varepsilon') \right).$$

Now, by the Markov inequality,

$$\begin{aligned}
\mathbb{P}\left(\sum_{j=1}^N |u_j|^2 < C n^{-1}\right) &= \mathbb{P}\left(\sum_{j=1}^N (|u_j|^2 - \sigma_j^2) < C n^{-1} - C n^{-1}(1 + \varepsilon' + o(1))\right) \\
&\leq \mathbb{P}\left(\sum_{j=1}^N (\sigma_j^2 - |u_j|^2) > C n^{-1}(\varepsilon' + o(1))\right) \\
&\leq \frac{\sum_{j=1}^N \text{Var}(|u_j|^2)}{C^2 n^{-2} \varepsilon'^2 / 2} \leq \frac{2 \sum_{j=1}^N \sigma_j^4}{C^2 n^{-2} \varepsilon'^2} \\
&\asymp \frac{\lambda^2 N}{C^2 n^{-2} \varepsilon'^2} \asymp n^{-1/2\alpha} = o(1).
\end{aligned}$$

Moreover,

$$\mathbb{P}\left(\sum_{j=1}^N |u_j|^2 > C n^{-1}(1 + 2\varepsilon')\right) = \mathbb{P}\left(\sum_{j=1}^N (|u_j|^2 - \sigma_j^2) > C n^{-1}(\varepsilon' + o(1))\right) = o(1),$$

which finishes the proof of (61).

Also, in view of (59), we have

$$\begin{aligned}
\mathbb{P}\left(\sum_{j=1}^N j^{2\alpha} |u_j|^2 > L\right) &= \mathbb{P}\left(\sum_{j=1}^N j^{2\alpha} (|u_j|^2 - \sigma_j^2) > L\varepsilon\right) \\
&\leq \frac{\sum_{j=1}^N j^{4\alpha} \text{Var}(|u_j|^2)}{L^2 \varepsilon^2} = \frac{\sum_{j=1}^N j^{4\alpha} \sigma_j^4}{L^2 \varepsilon^2} \\
&\asymp \frac{\lambda^2 N^{4\alpha+1}}{L^2 \varepsilon^2} \asymp n^{-1/2\alpha} = o(1),
\end{aligned}$$

proving (62). Also,

$$\begin{aligned}
\mathbb{P}\left(\sum_{j=1}^N j^{2\beta} |u_j|^2 < n^{-1+\beta/\alpha}\right) &\leq \mathbb{P}\left(\sum_{j=1}^N j^{2\beta} (|u_j|^2 - \sigma_j^2) < -n^{-1+\beta/\alpha} \varepsilon\right) \\
&\leq \frac{\sum_{j=1}^N j^{4\beta} \text{Var}(|u_j|^2)}{n^{-2+2\beta/\alpha} \varepsilon^2} = \frac{\sum_{j=1}^N j^{4\beta} \sigma_j^4}{n^{-2+2\beta/\alpha} \varepsilon^2} \\
&\asymp \frac{\lambda^2 N^{4\beta+1}}{n^{-2+2\beta/\alpha} \varepsilon^2} \asymp n^{-1/2\alpha} = o(1),
\end{aligned}$$

proving (63). □

Let us go back to (57) and bound from below the maximal error probability of type II by the averaged risk, with respect to our prior measure Π :

$$\begin{aligned}
\sup_{u \in s^\alpha(L), F(u) \geq \eta_n} \langle G(\sqrt{n}u) | M_0 | G(\sqrt{n}u) \rangle &\geq \int_{H_1(\alpha, L, \gamma_n, \eta_n)} \text{Tr}(|G(\sqrt{n}u)\rangle \langle G(\sqrt{n}u)| \cdot M_0) \Pi(du) \\
&= \text{Tr}\left(\int |G(\sqrt{n}u)\rangle \langle G(\sqrt{n}u)| \Pi(du) \cdot M_0\right) - \int_{H_1(\alpha, L, \gamma_n, \eta_n)^C} \text{Tr}(|G(\sqrt{n}u)\rangle \langle G(\sqrt{n}u)| \cdot M_0) \Pi(du) \\
&\geq \text{Tr}\left(\int |G(\sqrt{n}u)\rangle \langle G(\sqrt{n}u)| \Pi(du) \cdot M_0\right) - \Pi(H_1(\alpha, L, \gamma_n, \eta_n)^C).
\end{aligned}$$

In the last inequality we used that $\text{Tr}(|G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)| \cdot M_0) \leq 1$. By Lemma 6.1, $\Pi(H_1(\alpha, L, \gamma_n, \eta_n)^C) = o(1)$ and thus we deduce from (57) that

$$R_n^F \geq \frac{1}{8} \inf_M \left(\text{Tr}(|G(0)\rangle\langle G(0)| \cdot M_1) + \text{Tr} \left(\int |G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)| \Pi(du) \cdot M_0 \right) \right) + o(1).$$

We recognize in the previous line the sum of error probabilities of type I and II for testing two simple quantum hypotheses, i.e. the underlying state is either $|G(0)\rangle$ or the mixed state

$$\Phi := \int |G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)| \Pi(du).$$

As a last step of the proof, we characterize more precisely the previous mixed Gaussian state as a thermal state and use classical results from quantum testing of two simple hypotheses to give the bound from below of the testing risk. Recall from Section 2.2.2, equation (9) that coherent states $|G(\sqrt{n}u)\rangle$ factorize as tensor product of one-mode coherent states with displacements u_j , i.e. $\otimes_{j \geq 1} |G(\sqrt{n}u_j)\rangle$. A coherent state with displacement $z = x + iy$ with $x, y \in \mathbb{R}$ is fully characterized by its Wigner function given by equation (3). Since the prior is Gaussian, our mixed state Φ is Gaussian and can be written

$$\begin{aligned} \int |G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)| \Pi(du) &= \left(\bigotimes_{j=1}^N \int |G(\sqrt{n}u_j)\rangle\langle G(\sqrt{n}u_j)| \Pi_j(du_j) \right) \otimes \left(\bigotimes_{j \geq N+1} |0\rangle\langle 0| \right) \\ &:= \bigotimes_{j=1}^N \Phi_j \otimes \left(\bigotimes_{j \geq N+1} |0\rangle\langle 0| \right) \end{aligned}$$

where Π_j represents the bivariate centred Gaussian distribution with covariance matrix $\sigma_j^2/2 \cdot I_2$ over the complex plane $u_j = x_j + iy_j$. Using equation (6), and setting $\sigma^2 = n\sigma_j^2/2$ there, we find that the individual modes with index $j \leq N$ are centred Gaussian thermal states $\Phi_j = \Phi(r_j)$ (cf. definition (4)) with $r_j = n\sigma_j^2/(n\sigma_j^2 + 1)$.

In order to bound from below the right-hand side term in (57) we use the theory of quantum testing of two simple hypotheses

$$H_0 : \otimes_{j \geq 1} \Phi(0) \quad \text{against} \quad H_1 : \otimes_{j=1}^N \Phi(r_j) \otimes_{j \geq N+1} \Phi(0).$$

Using (11), it is easy to see that this testing problem is equivalent to

$$H_0 : (\Phi(0))^{\otimes N} \quad \text{against} \quad H_1 : \otimes_{j=1}^N \Phi(r_j).$$

As the vacuum and the thermal state are both diagonalized by the Fock basis, they commute, which reduces the problem to a classical test between the N -fold products of discrete distributions $H_0 : \{\mathcal{G}(0)\}^{\otimes N}$ and $H_1 : \{\otimes_{j=1}^N \mathcal{G}(r_j)\}$. In view of the form (4) of the thermal state, $\mathcal{G}(r_j)$ is the geometric distribution $\left\{ (1-r_j)r_j^k \right\}_{k=0}^{\infty}$ and $\mathcal{G}(0)$ is the degenerate distribution concentrated at 0. The optimal testing error is given by the maximum likelihood test which decides H_0 if and only if all observations are 0. The type I error is 0 and the type II error is

$$\prod_{j=1}^N (1-r_j) = \prod_{j=1}^N \frac{1}{n\sigma_j^2 + 1} \geq \exp \left(-n \sum_{j=1}^N \sigma_j^2 \right) \geq \exp(-c),$$

for some $c > 0$, where in the last inequality we used (64). Using this in (57), we get as a lower bound

$$R_n^F \geq \exp(-c) + o(1) \geq c_0,$$

where $c_0 > 0$ is some constant depending on c . This finishes the proof. \square

PROOF OF THEOREM 5.6. Let $\varphi_n = c_n n^{-1/2}$ for a positive sequence c_n . Let $M_n = (\rho_0^{\otimes n}, I - \rho_0^{\otimes n})$ be the well-known projection test for the problem (32). Then

$$\begin{aligned} R_n^T(M_n) &= \text{Tr}(\rho^{\otimes n} \cdot \rho_0^{\otimes n}) + \text{Tr}(\rho_0^{\otimes n} \cdot (I - \rho_0^{\otimes n})) \\ &= (\text{Tr}(\rho \cdot \rho_0))^n = |\langle \psi | \psi_0 \rangle|^{2n}. \end{aligned}$$

Let us recall that for any pure states $\rho = |\psi\rangle\langle\psi|$ and $\rho_0 = |\psi_0\rangle\langle\psi_0|$, we have

$$(65) \quad \|\rho - \rho_0\|_1 = 2\sqrt{1 - |\langle \psi | \psi_0 \rangle|^2},$$

thus $|\langle \psi | \psi_0 \rangle|^2 = 1 - \frac{1}{4}\|\rho - \rho_0\|_1^2$ and hence

$$R_n^T(M_n) = \left(1 - \frac{1}{4}\|\rho - \rho_0\|_1^2\right)^n.$$

For any ρ satisfying the alternative hypothesis $H_1(\varphi_n)$, we have $\|\rho - \rho_0\|_1 \geq \varphi_n$ and consequently

$$(66) \quad \begin{aligned} \mathbb{P}_e^{M_n}(\varphi_n) &\leq \left(1 - \frac{1}{4}\varphi_n^2\right)^n = \left(1 - \frac{c_n^2}{4}n^{-1}\right)^n \\ &\leq \left(\exp\left(-\frac{c_n^2}{4}n^{-1}\right)\right)^n = \exp\left(-\frac{c_n^2}{4}\right). \end{aligned}$$

If now $\varphi_n/\varphi_n^* \rightarrow \infty$ then $c_n \rightarrow \infty$ and $\mathbb{P}_e^{M_n}(\varphi_n) \rightarrow 0$, so that the second relation in (33) is fulfilled.

Consider now the case $\varphi_n/\varphi_n^* \rightarrow 0$ so that $c_n \rightarrow 0$. For any vector $v \in \mathcal{H}$ define

$$(67) \quad \|v\|_\alpha^2 = \sum_{j=0}^{\infty} |\langle e_j | v \rangle|^2 j^{2\alpha};$$

then $\|v\|_\alpha$ is a seminorm on the space of v fulfilling $\|v\|_\alpha^2 < \infty$. The assumption that $\rho_0 = |\psi_0\rangle\langle\psi_0| \in S^\alpha(L')$ means that $\|\psi_0\|_\alpha^2 \leq L' < L$. For some $N > 0$, consider the linear space

$$\mathcal{H}_{0,N} = \{u \in \mathcal{H} : \langle u | \psi_0 \rangle = 0, \langle u | e_j \rangle = 0, j > N\};$$

it is nonempty if $N \geq 1$. Let $u \in \mathcal{H}_{0,N}$, $\|u\| = 1$ be a unit vector; and for $\varepsilon > 0$ consider

$$(68) \quad \psi_{u,\varepsilon} = \psi_0 \sqrt{1 - \varepsilon^2} + \varepsilon u.$$

Then $\|\psi_{u,\varepsilon}\| = 1$, $\rho_{u,\varepsilon} = |\psi_{u,\varepsilon}\rangle\langle\psi_{u,\varepsilon}|$ is a pure state, and

$$|\langle \psi_{u,\varepsilon} | \psi_0 \rangle|^2 = 1 - \varepsilon^2.$$

According to (65) we then have

$$\|\rho_{u,\varepsilon} - \rho_0\|_1 = 2\sqrt{1 - |\langle \psi_{u,\varepsilon} | \psi_0 \rangle|^2} = 2\varepsilon$$

so for a choice $\varepsilon = c_n n^{-1/2}/2$ it follows $\|\rho_{u,\varepsilon} - \rho_0\|_1 = \varphi_n$ and $\rho_{u,\varepsilon} \in B(\varphi_n)$. On the other hand, by (68) and the triangle inequality

$$\|\psi_{u,\varepsilon}\|_\alpha \leq \sqrt{1 - \varepsilon^2} \|\psi_0\|_\alpha + \varepsilon \|u\|_\alpha.$$

Now $\|u\|_\alpha < \infty$ for $u \in \mathcal{H}_{0,N}$, and by assumption $\|\psi_0\|_\alpha < L^{1/2}$, so for sufficiently large n

$$\|\psi_{u,\varepsilon}\|_\alpha \leq L^{1/2}$$

and thus $\rho_{u,\varepsilon} \in S^\alpha(L)$. Thus $\rho_{u,\varepsilon} \in S^\alpha(L) \cap B(\varphi_n)$ for sufficiently large n . By (11) the optimal error probability for testing between states $\rho_{u,\varepsilon}$ and ρ_0 fulfills

$$\begin{aligned} \inf_{M \text{ binary POVM}} R_n^T(\rho_0^{\otimes n}, \rho_{u,\varepsilon}^{\otimes n}, M) &= 1 - \frac{1}{2} \|\rho_0^{\otimes n} - \rho_{u,\varepsilon}^{\otimes n}\|_1 \\ &= 1 - \sqrt{1 - |\langle \psi_0^{\otimes n} | \psi_{u,\varepsilon}^{\otimes n} \rangle|^2} = 1 - \sqrt{1 - |\langle \psi_0 | \psi_{u,\varepsilon} \rangle|^{2n}} \\ (69) \quad &= 1 - \sqrt{1 - (1 - \varepsilon^2)^n} = 1 - \sqrt{1 - (1 - c_n^2 n^{-1}/4)^n}. \end{aligned}$$

Obviously if $c_n^2 \rightarrow 0$ then $(1 - c_n^2 n^{-1}/4)^n \rightarrow 1$ so that

$$\inf_{M \text{ binary POVM}} R_n^T(\rho_0^{\otimes n}, \rho_{u,\varepsilon}^{\otimes n}, M) \geq 1 + o(1).$$

But since $\rho_{u,\varepsilon} \in S^\alpha(L) \cap B(\varphi_n)$ we have

$$\mathbb{P}_e^*(\varphi_n) \geq \inf_{M \text{ binary POVM}} R_n^T(\rho_0^{\otimes n}, \rho_{u,\varepsilon}^{\otimes n}, M) \geq 1 + o(1),$$

so that the first relation in (33) is shown. \square

PROOF OF THEOREM 5.7. It suffices to prove that if $\varphi_n = c_n n^{-1/2}$ with $c_n \rightarrow c > 0$ then $\mathbb{P}_e^*(\varphi_n) \rightarrow \exp(-c^2/4)$. In view of the upper bound (34), it suffices to prove

$$(70) \quad \mathbb{P}_e^*(\varphi_n) \geq \exp(-c^2/4) (1 + o(1)).$$

Recall (cf. (65)) that for any pure states $\rho = |\psi\rangle\langle\psi|$ and $\rho_0 = |\psi_0\rangle\langle\psi_0|$, the condition $\|\rho - \rho_0\|_1 \geq \varphi_n$ in $H_1(\varphi_n)$ is equivalent to a condition for the fidelity $F^2(\rho, \rho_0) = |\langle \psi | \psi_0 \rangle|^2 \leq 1 - \varphi_n^2/4$.

Let $\mathcal{H}_0 \subset \mathcal{H}$ be the orthogonal complement of $\mathbb{C}|\psi_0\rangle$ in \mathcal{H} . Consider the vector

$$\psi_u = \sqrt{1 - \|u\|^2} \cdot \psi_0 + u, \quad u \in \mathcal{H}_0$$

and the corresponding pure state $|\psi_u\rangle\langle\psi_u|$ defined in terms of the local vector u . We restrict the alternative hypothesis to a smaller set of states such that $\|u\| \leq \gamma_n$, with $\gamma_n = (\log n)^{-1}$.

Since the fidelity is given by $F^2(\rho_0, |\psi_u\rangle\langle\psi_u|) = |\langle\psi_u|\psi_0\rangle|^2 = 1 - \|u\|^2$, the restricted hypothesis is characterised by

$$1 - \gamma_n^2 \leq F^2(\rho_0, |\psi_u\rangle\langle\psi_u|) \leq 1 - \varphi_n^2/4, \quad \text{or} \quad \varphi_n^2/4 \leq \|u\|^2 \leq \gamma_n^2.$$

and additionally by $\|\psi_u\|_\alpha^2 \leq L$ where $\|\cdot\|_\alpha$ is given by (67).

Consider again the linear space $\mathcal{H}_{0,N}$ defined in the proof of Theorem 5.7 for a choice $N = N_n \sim \log \log n$. Since $\mathcal{H}_{0,N} \subset \mathcal{H}_0$, we can further restrict the local vector u to $u \in \mathcal{H}_{0,N}$. Note that for $u \in \mathcal{H}_{0,N}$ and $\|u\| \leq \gamma_n$ we have

$$\begin{aligned} \|u\|_\alpha^2 &= \sum_{j=0}^N |\langle e_j | u \rangle|^2 j^{2\alpha} \leq N^{2\alpha} \|u\|^2 \leq N^{2\alpha} \gamma_n^2 \\ &\sim (\log \log n)^{2\alpha} (\log n)^{-2} = o(1). \end{aligned}$$

It follows that

$$\|\psi_u\|_\alpha \leq \sqrt{1 - \|u\|^2} \|\psi_0\|_\alpha + \|u\|_\alpha \leq L^{1/2}$$

for sufficiently large n , thus $\psi_u \in S^\alpha(L)$. We can now write the test problem with restricted alternative as

$$\begin{aligned} H_0 : & \quad \rho = \rho_0 \\ H_1'(\varphi_n) : & \quad \rho = |\psi_u\rangle\langle\psi_u| : u \in \mathcal{H}_{0,N}, \varphi_n/2 \leq \|u\| \leq \gamma_n. \end{aligned}$$

By the strong approximation proven in Theorem 4.1 we get that the models

$$\{|\psi_u\rangle\langle\psi_u|^{\otimes n}, \|u\| \leq \gamma_n\} \quad \text{and} \quad \{|G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)|, \|u\| \leq \gamma_n\}$$

are asymptotically equivalent, where $G(\sqrt{n}u)$ is the coherent vector in the Fock space $\Gamma_s(\mathcal{H}_0)$ pertaining to $\sqrt{n}u$. Note that this proof is very similar to the previous proofs of lower bounds, with a major difference: the reduced set of states under the alternative hypothesis is defined with respect to ρ_0 given by the null hypothesis H_0 instead of an arbitrary state previously.

In the asymptotically equivalent Gaussian white noise model, the modified hypotheses concern Gaussian states which can be written in terms of their coherent vectors as

$$(71) \quad \begin{aligned} H_0 : & \quad |G(0)\rangle \\ H_1(\varphi_n) : & \quad |G(\sqrt{n}u)\rangle : u \in \mathcal{H}_{0,N}, \varphi_n/2 \leq \|u\| \leq \gamma_n. \end{aligned}$$

In order to prove the theorem it is sufficient to prove that

$$(72) \quad \inf_{M_n} \sup_{\varphi_n/2 \leq \|u\| \leq \gamma_n, u \in \mathcal{H}_{0,N}} R_n^T(|G(0)\rangle\langle G(0)|, |G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)|, M_n)$$

$$(73) \quad \geq \exp(-c^2/4) + o(1)$$

as $n \rightarrow \infty$.

Note that $\dim \mathcal{H}_{0,N} = N$; let $\{g_j, j = 1, \dots, N\}$ be an orthogonal basis of $\mathcal{H}_{0,N}$ and let $|u\rangle = \sum_{j=1}^N u_j |g_j\rangle$. The quantum Gaussian white noise model $\{|G(\sqrt{n}u)\rangle, u \in \mathcal{H}_{0,N}, \|u\| \leq \gamma_n\}$ is then equivalent to the quantum Gaussian sequence model $\{\otimes_{j=1}^N |G(\sqrt{n}u_j)\rangle, \|u\| \leq \gamma_n\}$. From now on $|G(z)\rangle$ denotes the coherent vector in the Fock space $\mathcal{F}(\mathbb{C})$ pertaining to $z := x + iy \in \mathbb{C}$. Recall that such a state is fully characterized by its Wigner function $W_{G(z)}$, which in the case of coherent states is the density function of a bivariate Gaussian distribution.

We shall bound from below the maximal type 2 error probability in the risk $R_n^T(M_n)$ in (72)

$$(74) \quad \sup_{\varphi_n/2 \leq \|u\| \leq \gamma_n, u \in \mathcal{H}_{0,N}} \text{Tr}(|G(\sqrt{nu})\rangle\langle G(\sqrt{nu})| \cdot M_{n,0})$$

by an average over u , where the average is taken with respect to a prior distribution defined as follows. Assume that u_j , $j = 1, \dots, N$ are independently distributed following a complex centered Gaussian law with variance $\frac{\sigma^2}{2}I_2$, where $\sigma^2 = \frac{c^2}{4n} \frac{1+\varepsilon}{N}$, for some fixed and arbitrary small $\varepsilon > 0$, and I_2 is the 2 by 2 identity matrix.

LEMMA 6.2. *Let Π be the distribution of independent complex random variables u_j , for $j = 1, \dots, N$, each one distributed as*

$$N\left(0, \frac{\sigma^2}{2}I_2\right), \quad \sigma^2 = \frac{c^2}{4n} \frac{1+\varepsilon}{N},$$

for fixed $\varepsilon > 0$ and $N \sim \log \log n$. Then as $n \rightarrow \infty$

$$\mathbb{P}\left(\frac{c_n^2}{4n} \leq \|u\|^2 \leq \frac{c_n^2}{4n}(1+\varepsilon)^2\right) \rightarrow 1, \quad \text{as } n \rightarrow \infty,$$

and in particular if $\gamma_n = (\log n)^{-1}$ then $\mathbb{P}(\varphi_n/2 \leq \|u\| \leq \gamma_n) \rightarrow 1$, as $n \rightarrow \infty$.

PROOF. We have

$$\begin{aligned} \mathbb{P}\left(\|u\|^2 < \frac{c_n^2}{4n}\right) &= \mathbb{P}\left(\sum_{j=1}^N (|u_j|^2 - \sigma^2) < \frac{c_n^2}{4n} - N \frac{c^2}{4n} \frac{1+\varepsilon}{N}\right) \\ &\leq \frac{\text{Var}(\sum_{j=1}^N |u_j|^2)}{(c_n^2 - c^2(1+\varepsilon))^2/16n^2} = \frac{N\sigma^4}{(c^2\varepsilon + o(1))^2/16n^2} \\ &= \frac{Nc^4(1+\varepsilon)^2/16n^2N^2}{(c^2\varepsilon + o(1))^2/16n^2} = \left(\frac{1+\varepsilon}{\varepsilon + o(1)}\right)^2 \frac{1}{N} = o(1), \end{aligned}$$

since $N \sim \log \log n \rightarrow \infty$. Similarly, as $(1+\varepsilon)^2 > 1+\varepsilon$, one shows that

$$\mathbb{P}\left(\|u\|^2 > \frac{c_n^2}{4n}(1+\varepsilon)^2\right) \rightarrow 0,$$

as $n \rightarrow \infty$ and thus we get

$$\mathbb{P}\left(\frac{c_n^2}{4n} \leq \|u\|^2 \leq \frac{c_n^2}{4n}(1+\varepsilon)^2\right) \rightarrow 1.$$

As $\gamma_n^2 = (\log n)^{-2}$ decays slower than c_n^2/n , and $\varphi_n/2 = c_n n^{-1/2}/2$, we deduce that

$$\mathbb{P}(\varphi_n/2 \leq \|u\| \leq \gamma_n) \rightarrow 1$$

as $n \rightarrow \infty$ which ends the proof of the lemma. \square

Let us denote by Π the prior distribution introduced in Lemma 6.2. Let us go back to (74) and bound the expression from below as follows:

$$\begin{aligned}
& \sup_{\varphi_n/2 \leq \|u\| \leq \gamma_n, u \in \mathcal{H}_{0,N}} \text{Tr}(|G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)| \cdot M_{n,0}) \\
& \geq \int_{\varphi_n/2 \leq \|u\| \leq \gamma_n} \text{Tr}(|G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)| M_{n,0}) \Pi(du) \\
& \geq \int \text{Tr}(|G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)| M_{n,0}) \Pi(du) - \int_{\{\varphi_n/2 \leq \|u\| \leq \gamma_n\}^c} \text{Tr}(|G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)| M_{n,0}) \Pi(du) \\
& \geq \int \text{Tr}(|G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)| M_{n,0}) \Pi(du) - \Pi(\{\varphi_n/2 \leq \|u\| \leq \gamma_n\}^c).
\end{aligned}$$

By Lemma 6.2, we get for (72)

$$\begin{aligned}
& \sup_{\varphi_n/2 \leq \|u\| \leq \gamma_n, u \in \mathcal{H}_{0,N}} R_n^T(G(0), G(\sqrt{n}u), M_n) \\
(75) \quad & \geq \text{Tr}(|G(0)\rangle\langle G(0)| M_{n,1}) + \text{Tr} \left(\int |G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)| \Pi(du) \cdot M_{n,0} \right) + o(1).
\end{aligned}$$

The integral on the right side is a mixed state which can be written as

$$\Phi := \int |G(\sqrt{n}u)\rangle\langle G(\sqrt{n}u)| \Pi(du) = \bigotimes_{j=1}^N \int |G(\sqrt{n}u_j)\rangle\langle G(\sqrt{n}u_j)| \cdot \Pi_j(du_j).$$

Similarly to the proof of Theorem 5.5 we use equation (6) to show that each of the Gaussian integrals above produces a thermal (Gaussian) state

$$\Phi(r) = (1-r) \sum_{k=0}^{\infty} r^k |k\rangle\langle k|, \quad r = \frac{n\sigma^2}{n\sigma^2 + 1}.$$

Since $|G(0)\rangle\langle G(0)| = \Phi(0)$, the main terms in (75) are the sum of error probabilities for testing two simple hypothesis $H_0 : \Phi(0)^{\otimes N}$ against $H_1 : \Phi(r)^{\otimes N}$. Moreover, we have two commuting product states under the two simple hypotheses, which reduces the problem to a classical test between the N -fold products of discrete distributions $H_0 : \{\mathcal{G}(0)\}^{\otimes N}$ and $H_1 : \{\mathcal{G}(r)\}^{\otimes N}$. Here $\mathcal{G}(r)$ is the geometric distribution $\{(1-r)r^k\}_{k=0}^{\infty}$; in particular $\mathcal{G}(0)$ is the degenerate distribution concentrated at 0. The optimal testing error is given by the maximum likelihood test which decides H_0 if and only if all observations are 0. The type 1 error is 0 and the type 2 error is

$$\begin{aligned}
(1-r)^N &= (n\sigma^2 + 1)^{-N} \geq \exp(-N \cdot n\sigma^2) \\
&= \exp\left(-Nn \frac{c^2}{4n} \frac{1+\varepsilon}{N}\right) = \exp\left(-\frac{c^2(1+\varepsilon)}{4}\right).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this establishes the lower bound (73) and thus (70). \square

CREST, ENSAE, UNIVERSITÉ PARIS-SACLAY
5, AVENUE HENRY LE CHATELIER,
91120 PALAISEAU,
FRANCE
E-MAIL: cristina.butucea@ensae.fr

UNIVERSITY OF NOTTINGHAM
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY PARK
NG7 2RD NOTTINGHAM
UK
E-MAIL: madalin.guta@nottingham.ac.uk

DEPARTMENT OF MATHEMATICS
MALOTT HALL, CORNELL UNIVERSITY
ITHACA, NY 14853
USA
E-MAIL: nussbaum@math.cornell.edu