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An Equilibrium Model of the Market for Bitcoin Mining

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Abstract

We propose a model which uses the Bitcoin/US dollar exchange rate to predict the computing power of the Bitcoin network. We show that free entry places an upper-bound on mining revenues and we devise a structural framework to measure its value. Calibrating the model's parameters allows us to accurately forecast the evolution of the network computing power over time. We establish the accuracy of the model through out-of-sample tests and investigation of the entry rule.

Keywords: Bitcoin, Blockchain, Miners, Industry Dynamics.

1 Introduction

Bitcoin is the first viable currency that operates without any central authority or trusted third party. It enables merchants and customers to transact at a lower cost and almost as securely as they would using the traditional banking system. Along with an ever-growing number of cryptocurrencies, many second layer protocols and applications

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have emerged,[1] making Bitcoin the backbone of a new ecosystem of financial technologies.

Bitcoin's security model relies on a hybrid approach that combines the robustness of its cryptographic primitives with the economic incentives of the agents participating in the execution of its protocol. In particular, miners play a central role as they stack transactions into blocks and timestamp those in a cryptographically robust way by adding a "proof-of-work".[2] The cost of attacking Bitcoin is proportional to the computing power deployed by miners because it determines the difficulty of the cryptographic puzzles included in the proofs-of-work.

For the system to remain secure, the cost of an attack had to follow the exponential increase in the value of Bitcoin, and so did the resources devoted to mining. What started as a hobby for a few miners using their personal computers, eventually blossomed into an industry that consumes nearly 0.15% of the world's electricity through its network of mining farms,[3] each one of them operating thousands of machines specially designed for mining.

In spite of the well-known importance of the market for mining for the viability of Bitcoin, our paper is the first to propose an equilibrium model characterizing its evolution over time. We show that miners' investment in computing power can be accurately forecasted using only the Bitcoin/US dollar (₿/$) exchange rate. Investment in mining hardware has two important characteristics. First, it cannot easily be reversed: machines have no resale value outside of the market for mining because they have been optimized for mining only. Second, there is a lot of uncertainty about future revenues due to the tremendous volatility of the ₿/$ exchange rate. This combination generates a range of inaction where expected revenues are too low to justify entry but high enough to prevent incumbent miners from exiting the market.

The main challenge for our analysis is that we cannot consider the problem of each miner in isolation or treat revenues as exogenous. Instead, we have to take into account how returns are endogenously determined by the number of active miners. A key insight of our approach is that Bitcoin's protocol generates a downward slopping demand for mining power, thereby ensuring that the market for mining behaves as a

[1] Pegged sidechains (Back et al., 2014), RSK (Lerner, 2015), colored coins (Assia et al., 2013) and the lightning network (Poon and Thaddeus, 2016) are among the most prominent examples.

[2] See Section 2 for a description of the tasks accomplished by miners.

[3] See, among other sources, digiconomist.net/bitcoin-energy-consumption .

competitive industry.

Combining the ₿/$ exchange rate with the total computing power of the Bitcoin network, we construct a new variable that measures miners' payoffs. Our model predicts that they buy new hardware only when our payoff measure reaches a reflecting barrier. Payoffs never exceed this threshold because new entries trigger increases in the difficulty of mining which push revenues down. The characterization of the equilibrium is complicated by the fact that mining hardware benefits from a high rate of embodied technological progress. We show how one can adapt the canonical model of Caballero and Pyndick (1996) to account for this trend and prove that the entry barrier decays at the rate of technological progress.

Then we calibrate the model and find that it forecasts remarkably well how miners respond to changes in the ₿/$ exchange rate. The accuracy of its prediction is a testament to the fact that miners operate in a market where perfect competition is a good approximation of reality. Its structure verifies many properties that are often assumed but rarely verified in practice. First, free entry holds because mining is not prevented by any regulation and does not require any specific skill. To enter the mining race, one simply has to buy the appropriate hardware and download the mining software. Second, there is very little heterogeneity among miners since they all face the same problem and earn the same rewards. In other words, the only relevant uncertainty occurs at the aggregate level. Third, as explained below, the mining technology exhibits returns to scale that are constant by nature. Fourth, the elasticity of demand for computing power is commonly known because it does not stem from the hidden preferences of consumers, but is instead encoded in Bitcoin's protocol and is therefore observable by all parties. Finally, we have access to perfectly clean and exhaustive data since all Bitcoin transactions are public. The conjunction of all these features is extremely rare, if not unique, thus making the market for Bitcoin mining a perfect laboratory for models of industry dynamics.

Related literature.—Bitcoin was created almost a decade ago when Nakamoto's paper (Nakamoto, 2008) was made public on October 31st 2008. It did not immediately attract much attention and it took a few years for Bitcoin to become the focus of academic research. Early works analyze the reliability of the Bitcoin network (Karame et al. (2012) and Decker and Wattenhoffer (2013)). Reid and Harrigan (2012) questions the anonymity of users, enabling Athey et al. (2017) to quantify the different ways bitcoins are used. It is only recently that papers studying the economic implications of cryptocurrencies have started to emerge. A few articles rely on monetary economics for their analysis. Observing the plethora of existing cryptocurrencies, Fernández-Villaverde and Sanches (2016) wonder under which conditions competition between currencies is economically efficient and how those currencies should be regulated. Hong et al. (2017) try to evaluate how cryptocurrencies may affect fiat currencies. Chiu and Koeppl (2017) assess the choice of values for the parameters that underlie Bitcoin's design and Gandal et al. (2017) analyze exchange rate manipulations. Rosenfeld (2011), Houy (2016) and Biais et al. (2017) are more closely related to our research since they investigate miners' incentives to behave cooperatively, as expected in Bitcoin's protocol, or to play "selfish". However, they do not characterize entry into the market for Bitcoin mining as our paper is, to the best of our knowledge, the first to adopt an equilibrium perspective.

Structure of the paper.—The article is organized as follows. Section 2 briefly explains how Bitcoin works by focusing on the market for mining. Section 3 introduces our baseline model, which yields the computing power of the network as a function of the ฿/$ exchange rate. Section 4 presents the data and explains how to calibrate the model. Section 5 concludes. The proofs of the Propositions and some additional results are relegated to the Appendix.

2 Bitcoin and the market for mining

This section describes the tasks accomplished by miners and the rewards they get in return. Since it is beyond the scope of this paper to explain the overall architecture of Bitcoin, we only cover the elements that are required for the understanding of our model, and refer readers interested in a more comprehensive treatment to Nakamoto (2008) and Antonopoulos (2014).

The function of miners.— Bitcoin is a decentralized cryptocurrency which operates without a central authority. Decentralization is achieved through the recording of transactions in a public ledger called the blockchain. The main challenge for a decentralized currency is to maintain consensus among all participants on the state of the blockchain (who owns what) in order to prevent double spending of the same coin. A user spends a coin twice when one of her payments is accepted because the recipient is not aware of a previous payment spending the same coin. That is why transactions are added to the blockchain by blocks and producing a valid block is made arbitrarily difficult so that the time it takes to build a block is, on average, much longer than the time it takes for a block to propagate across the network. This ensures that, in most instances, the whole network agrees on which transactions are part of the blockchain.

Blocks are cryptographically chained according to their dates of creation. This incremental process implies that the information contained in a given block cannot be modified without updating all subsequent blocks. Nakamoto's groundbreaking insight was to recognize that the cost of manipulation would increase dramatically in the number of modified blocks, thus ensuring that tampering with a given block becomes prohibitively expensive as more blocks are added on top of it.

To be accepted by other Bitcoin users, a new block must be stamped with a "proof-of-work". Each block possesses a header, which contains both a "nonce", i.e. an arbitrary integer, and a statistic summing up the transactions of the block, the time the block was built and the header of the previous block. Finding a valid proof-of-work boils down to finding a nonce satisfying the condition $h(header) \leq t$, where h is the SHA-256 hash function applied twice in a row and t is a threshold value. The hash function h has the property of being numerically *non-invertible*. Moreover knowing $h(n)$ for any $n \in \mathbb{N}$ yields no information on $h(m)$ for all $m \neq n$. Hence the only way to find a valid nonce is to randomly hash guesses until the condition above is satisfied. This activity is called mining and, in keeping with the search for gold, it requires no special skill besides the means to spend resources on the mining process. The average time it takes to mine a valid block can be made arbitrarily long by lowering the threshold t. Since the Bitcoin protocol specifies that one valid block should be found every 10 minutes, the threshold is updated every 2016 blocks to account for changes in the computing power, or hashpower, deployed by the miners .

Building a valid block is costly both in terms of hardware and electricity, and so mining must be rewarded. For each block there is a competition between miners. Only the first miner who finds a valid nonce wins the reward: she earns both a predetermined amount of new coins and the sum of the mining fees granted by the transactions included in the block. The amount of new bitcoins for block number B is approximately $50 \times (1/2)^{\lfloor B/210{,}000 \rfloor}$,[4] while fees are freely chosen by users. Note that the amount of new coins halves every 210,000 blocks so as to ensure that the supply of bitcoins converges to a finite limit, namely 21 millions.

[4] We use $\lfloor \cdot \rfloor$ to denote the highest lower-bound in $\mathbb{N}$, i.e. $\lfloor x \rfloor = \max_{n \in \mathbb{N}} \{n \leq x\}$.

The market for Bitcoin mining.— To enter the mining race, one has to buy the right hardware. Free entry prevails because anyone can easily order the machine and download the software. Thus miners all face the same problem, and there is no heterogeneity across miners besides their amounts of hashpower and the price they pay for electricity. Moreover, the technology exhibits constant returns to scale: two pieces of hardware will generate valid blocks exactly twice as often as a single piece of hardware.

The hardware used for mining benefits from constant upgrades. At the beginning, miners used to mine with their own computers. In mid-2010, they realized that Graphical Processing Units (GPU) were much more efficient. One year later, miners started using Field Programmable Gate Arrays (FGPA) and, since 2013, they mostly mine with Application Specific Integrated Circuits (ASIC). Investing in a GPU was a reversible decision since GPUs could serve many other purposes besides mining; should the ₿/$ exchange rate drop, the GPU could easily be sold to some video games addict. By contrast, buying an ASIC is an *irreversible investment* because, as indicated by their names, ASICs can be used for mining only; if the exchange rate collapses, ASICs cannot be resold at a profit because all miners face the same returns. Thus a mining cycle unfolds as follows. A new miner buys a recent piece of hardware and starts mining with it. Little by little, the revenues generated by her machine drop as ever more powerful hardware enters the race. When the flow of income falls below the cost of electricity, the miner turns off her machine and exits the market.

Mining solo is very risky since a miner earns her reward solely when she finds a valid block, which is a very rare event given the number of miners participating in the race.[5] This is why miners pool their resources and share the revenues earned by their pools according to the relative hashpower of each member. Obviously miners have the option but not the obligation to exchange their bitcoins against fiat money. However, since the exchange rate ensures that traders are indifferent between holding fiat money or bitcoins, the value of the reward at the time it is earned is accurately measured by its level in fiat money.[6]

For the sake of completeness, it is worth mentioning that the bitcoins issued with a new block cannot be exchanged straight away. A retention period is imposed because

[5]On July 1st 2017, the best ASIC miner could perform 14 tera hashes per second and cost 2400 dollars. The whole Bitcoin network performs 10 exa hashes (10 millions of tera hashes) per second.

[6]Depending on their locations, some investors may care about USD, some other may care more about RMB, for instance. However, those differences are negligible since from 2009 on, the USD/RMB exchange rate has been far less volatile than the ₿/$ exchange rate.

valid blocks are not always added to the blockchain. The validity of the nonce is not enough to maintain consensus when two blocks are found within a short time lapse by two different miners. Then participants will have different views of the state of the blockchain depending on which of the two blocks was broadcasted to them in the first place. Such conflicts create forks in the blockchain that are eventually resolved as miners coordinate on the branch requesting the greatest amount of hashpower ("longest chain rule"). The blocks that were added to the abandoned branch become orphan blocks. To ensure that the new coins contained in orphan blocks do not contaminate the blockchain, miners have to wait until 100 additional blocks have been added on top of their block before being able to transfer their newly earned coins. In other words, miners have to wait on average 16 hours 40 minutes before transferring their rewards. In practice, this delay is long enough to ensure that the block is indeed included in the blockchain. For our model's purpose, however, forking is a sufficiently rare event[7] that its impact on miners' payoffs can safely be ignored.[8]

3 The Model

We now propose a framework which captures the main features of the market for mining described in the previous section. Our approach takes the demand for bitcoins as given and uses the trajectory of the exchange rate to predict the hashpower of the network. We devise our model in continuous time and normalize the length of a period to 10 minutes because it corresponds to the average duration separating successive blocks. Since returns to scale are constant, we can think of miners as infinitesimal units of hashpower and assume that the total hashrate of the network takes any positive value on the real line.[9]

Miners' payoffs.—We use R_t to denote the block reward in dollars, i.e. the ₿/\$ exchange rate multiplied by the sum of new coins and fees. We also let Π_t denote the Poisson rate at which one miner finds a valid block. Then the *flow payoff* P_t of a miner

[7]Orphan blocks account for less than 0.2% of all mined blocks. The longest chain ever orphaned for a normal reason (not due to a bug) was 4 blocks long, well below 100.

[8]We will also neglect merged mining, i.e. the possibility to mine namecoins together with bitcoins without any additional effort. The reward miners get from namecoins is negligible (not even 0.1%) when compared to the reward in Bitcoins.

[9]Consider, for instance, the following normalization: one miner performs exactly one hash per period, the time interval being 10 minutes. Its relative size is indeed tiny since in mid 2017 the network performed around 10^{19} hashes every second.

is approximately equal to

$$P_t \equiv R_t \times \Pi_t. \tag{1}$$

Π_t is adjusted every 2016 blocks by the Bitcoin protocol. The updating rule takes the overall hashpower of the network over the previous period as given and adjusts the difficulty of the hashing problem until new blocks are created on average every ten minutes. This procedure ensures that monetary creation proceeds at the pace specified in the protocol. Then the complexity of the hashing problem is adjusted on average every two weeks only. Since our model is designed in continuous time, adding this discrete interval makes it impossible to derive tractable solutions. This is why we slightly idealize the actual protocol and assume that Π_t is continuously adjusted.

Assumption 1. *The valid-proof of work threshold is continuously updated according to the actual total hashrate so that* $\Pi_t = 1/Q_t$ *for all* t.

The number of hashes the network needs to perform to find a valid block follows a geometric distribution with parameter Π_t. Since the network computes Q_t hashes in one period (10 minutes), the network expected waiting time is $Q_t/\Pi_t = 1$, as prescribed by the protocol. We show in Appendix 6.2 that, during our period of study, the number of blocks mined every day mostly remains within confidence bounds of the null hypothesis. Hence data do not deviate significantly from the idealized updating state that would prevail under Assumption 1.

Value of hashpower.—Mining is a costly activity. To operate a unit of hashpower bought at time τ, miners incur the flow electricity cost C_τ. The costs vary with the vintages of the machines because they benefit from embodied technological progress, as newer machines are able to perform more hashes with the same amount of energy.[10] We have already stated that investment in hashpower is irreversible in the sense that machines cannot be resold. We now make the simplifying assumption that mining units are never switched-off.

Assumption 2. *Mining units cannot be voluntarily switched-off so as to save on electricity costs.*

[10] Note that we implicitly assume that the price of electricity remains constant. It is easy to relax this restriction by letting C depend on the current date t. However, when compared to the volatility of Bitcoin's exchange rate, changes in electricity costs are so small that they can be ignored in the empirical analysis.

Assumption 2 allows us to express the value of a unit of hashpower of vintage τ as follows

$$V(P_t, \tau) = E_t \left[\int_t^{\infty} e^{-r(s-t)} P_s ds \right] - \frac{C_\tau}{r}, \tag{2}$$

where r is the discount rate.[11] Note that we have assumed that there is no heterogeneity across miners. Apart from the price of electricity, they all face the same problem. Due to free entry, only miners who have access to cheap electricity will find it profitable to invest and so all active miners must face more or less the same operating costs.

Under Assumption 1, the flow payoff is given by

$$P_t = R_t / Q_t. \tag{3}$$

Equation (3) defines an isoelastic demand curve with unitary elasticity. Its microfoundation is rather unique since the decreasing relationship between payoffs P and industry output Q does not stem from the satiation of consumers' demand, but is instead generated by the updating rule encoded in Bitcoin's protocol.

We do not attempt to endogenize the demand for bitcoins and thus take the exchange rate R as given. Following much of the literature on irreversible investment, we assume that $(R_t)_{t\geq 0}$ is a Geometric Brownian Motion (GBM hereafter). We will discuss the accuracy of this assumption when we estimate the model in Section 4.[12]

Assumption 3. $(R_t)_{t\geq 0}$ *follows a Geometric Brownian Motion so there is an* $\alpha \in \mathbb{R}$*, and a* $\sigma \in \mathbb{R}_+$*, such that*

$$dR_t = R_t \left(\alpha dt + \sigma dZ_t \right), \tag{4}$$

where $(Z_t)_{t\geq 0}$ *is a standard Brownian motion.*

Knowing the law-of-motion followed by the exchange rate does not enable us to compute the expected value of payoffs because they also depends on the hashpower of the network Q, whose level is endogenously determined. To solve for the equilibrium, one has to simultaneously derive the process followed by Q and the entry policy of miners.

[11] We could easily let hardware break down at rate δ by adding it to the discount factor. However, we do not observe that the network hashpower decreases in the absence of market entry. This indicates that failures occur at a much slower rate than technological obsolescence and so do not significantly affect expected returns.

[12] Note that this specification neglects halvings of the money creation rate occurring every 2016 blocks.

Market entry.—Entrants that want to join the mining race have to buy a unit of hashpower which price we denote by I_t. Both the entry cost and the cost of electricity decrease over time because machines become ever faster. With the amount needed to buy one unit of hashpower at date 0, a miner can buy A_t units of hashpower at time t. Thus A_t measures technological efficiency at date t. For reasons explained below, we focus on periods where technological improvements accrue at a constant pace, so that $A_t = \exp(at)$ with $a > 0$.

Assumption 4. *Machines get more efficient at the constant rate $a > 0$. Hence the entry costs and the operating costs satisfy $I_t = I_0/A_t = \exp(-at)I_0$ and $C_t = C_0/A_t = \exp(-at)C_0$, respectively.*

Since free-entry ensures that no profits can be made by adding hashpower to the network, the following inequality must hold

$$I_t \geq E_t \left[\int_t^{\infty} e^{-r(s-t)} P_s ds \right] - \frac{C_t}{r} = V(P_t, t) \text{ for all t.} \tag{5}$$

At times where miners enter the market, (5) will hold with equality. Since the exchange rate follows a Markov process, it is natural to conjecture that their decisions will only depend on the current realization of P: whenever payoffs reach some endogenously determined threshold $\overline{P}_t$, a wave of market entries will ensure that the free-entry condition (5) is satisfied.

To see why such a mechanism defines a competitive equilibrium, it is helpful to decompose the law of motion of P. Reinserting (4) into (3) and using Ito's lemma, we find that

$$d\log(P_t) = \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma dZ_t - d\log(Q_t). \tag{6}$$

Payoffs are decreasing in Q because the response of the protocol to an increase in total hashpower is to decrease the valid proof-of-work threshold, thus making it less likely for each miner to earn a reward. This is why free entry places an upper bound on payoffs. Their value can never exceed a threshold $\overline{P}_t$ as more miners would find it profitable to enter the market, which would push payoffs further down.

Industry equilibrium.—So far, the main takeaway from our analysis is that the market for mining can be described as a perfectly competitive industry with irreversible investment because the Bitcoin protocol generates a downward slopping demand for hashpower. Thus we expect to observe equilibria similar to the ones studied by Caballero and Pyndick (1996) in their seminal paper on industry evolution.

Definition 1 (Industry equilibrium)**.**
An industry equilibrium is a payoff process P_t *and an upper barrier* $\overline{P}_t$ *such that:*
(i) $P_t \in [0, \overline{P}_t]$.
(ii) The Free-Entry condition (5) *is satisfied at all points in time, and it holds with equality whenever* $P_t = \overline{P}_t$*.*
(iii) The network hashpower Q_t *increases only when* $P_t = \overline{P}_t$*.*

From a formal standpoint, the only fundamental difference between our model and standard s-S models is that, due to embodied technological progress, entry and variable costs decrease over time. Hence the entry barrier $\overline{P}_t$ cannot remain constant. However, if we impose Assumption 4, so that technology improves at a constant rate, we can solve for the equilibrium in the space of detrended payoffs and recover a flat barrier.

Proposition 2. *Assume that assumptions 1, 2, 3 and 4 hold. Then there is an industry equilibrium* $(P_t, \overline{P}_t)$ *such that* P_t *is a GBM reflected at* $\overline{P}_t = \overline{P}_0 / A_t$ *where*[13]

$$\overline{P}_0 = \frac{\beta(r-\alpha)}{\beta - 1}\left[I_0 + \frac{C_0}{r}\right], \text{ and } \beta = \frac{\frac{\sigma^2}{2} - \alpha - a + \sqrt{\left(\alpha + a - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 (a + r)}}{\sigma^2} > 0. \qquad (7)$$

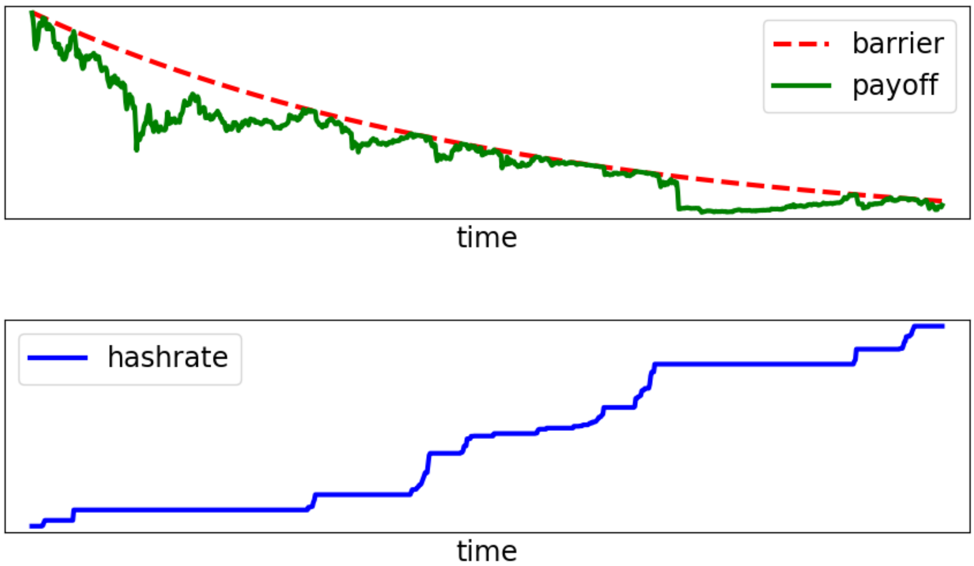
A typical equilibrium is illustrated in Figure 1. The upper-panel reports an arbitrary sample path for the payoff process $(P_t)_{t \geq 0}$. Payoffs follow the changes in the exchange rate and thus behave as a GBM until they hit the reflecting barrier $\overline{P}_t$. Such events trigger market entry, as shown in the lower-panel. The resulting increase in hashpower raises the difficulty of the mining problem and thus pushes payoffs down until market entry is not anymore profitable. The entry barrier decreases at the rate of technological progress because it corresponds to the pace at which both sunk and operating costs fall over time.

Comparative statics.—The higher the barrier, the lower the average rate of investment as miners procrastinate further before entering the market. It is therefore instructive to study the effect of the parameters on $\overline{P}_0$. Differentiating its expression in (7), we find that $\partial \overline{P}_0 / \partial a > 0$ and $\partial \overline{P}_0 / \partial r > 0$. If technological progress accelerates, miners' revenues shrink faster because there will be more entries in the future. Hence miners must earn more in the periods following their entries and so the barrier will be higher. A similar mechanism explains the impact of r since the value of future profits

[13] Note that, when $\alpha = r$, $\overline{P}_0 = (I_0 + \frac{C_0}{r})\left(\alpha + a + \frac{\sigma^2}{2}\right)$.

is discounted at a higher rate when r goes up. Not surprisingly, an increase in the average growth rate α of the block reward incentivizes entry as $\partial\overline{P}_0/\partial\alpha < 0$. Finally, the volatility of payoffs σ discourages entry since $\partial\overline{P}_0/\partial\sigma > 0$. Note that this effect is not due to an increase in the value of waiting because the perfectly competitive structure of the industry rules out such an option: competitors would preempt any procrastination beyond the zero expected profit threshold. Instead, the negative impact of σ on $\overline{P}_0$ is mechanical. Given that payoffs are truncated from above by the reflecting barrier, an increase in their spread automatically lowers their expected value.

Figure 1: Industry Equilibrium



4 Calibration

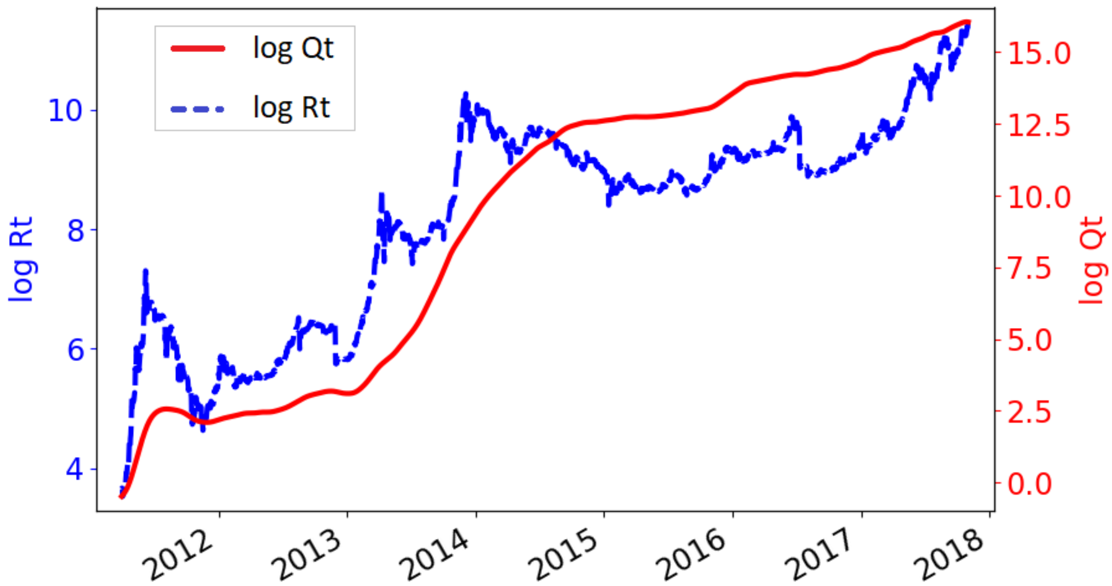
Data.—We now show that feeding our model with exchange rate data allows one to accurately predict the evolution of the network hashrate. For this purpose, we need to infer the miners' payoffs $P_t = R_t/Q_t$. Remember that the numerator, R_t, is equal to the value of new coins plus the transaction fees. The number of created coins per block is encoded in the protocol while the exchange rate is directly available from coindesk.com.[14] The transaction fees are recorded in the blockchain and can easily be retrieved from btc.com. Thus all the components of $(R_t)_{t\geq 0}$ are readily available.

[14]There are many different exchanges and the exchange rates vary a bit across them. We neglect those variations since they are dwarfed by the changes over time of the exchange rate.

This is, however, not the case for the network hashrate $(Q_t)_{t\geq 0}$ whose values must be estimated using the theoretical probability of success and the number of blocks found each day. Since we are not primarily interested in statistical inference, we relegate the description of our estimation procedure to Appendix 6.4 and save on notation by using Q_t to denote our estimate, although its time series only approximates the true hashrate. We show in Appendix 6.4 that the approximation is accurate. We update the value of Q_t on a daily basis and, since there are on average 144 blocks mined every day, the expected payoffs per period are given by $P_t = 144 \times R_t/Q_t$.

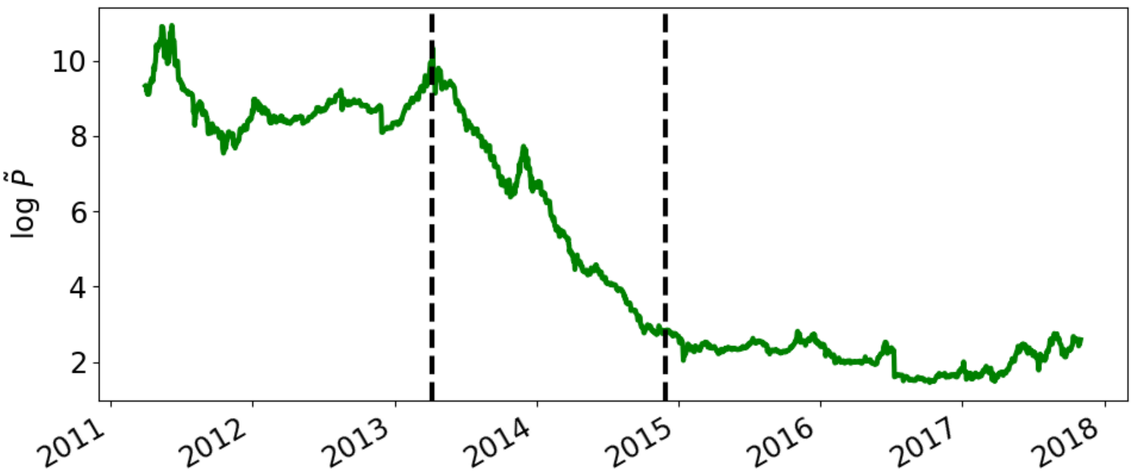
We report the series followed by $(R_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ in Figure 2. There is a clear correlation between the two variables. Our model suggests that their structural relation should become apparent if one takes the ratio of the two series and detrend it at the rate of technological progress a. Then the resulting series should behave as a reflected Brownian motion. A natural guess for the rate of progress is Moore's law according to which processor speed doubles every two years. We actually expect improvements in the mining technology to outpace those in processing speed because miners came up with a series of innovations that allowed them to leverage their computing power. Thus we will refine our guess later on by calibrating the value of a. Yet it is still instructive at this exploratory stage to use Moore's law as a benchmark.

Figure 2: Miners' Revenues R and Hashrate Q



The detrended series based on Moore's law is reported in Figure 3. It exhibits two stationary regimes, with a break in the middle where payoffs decreased regularly until they reached a lower plateau. At first, this behavior does not seem to square with the model. But if we focus on the date at which the break initiates, we realize that it coincides with the switch to Application Specific Integrated Circuits (ASIC). Since this revolution in the mining technology boosted the rate of progress well above its long-run trend, Assumption 4 does not hold and thus one should not expect the predictions of our model to be verified during this transitory period. Hence we leave aside the lapse of time where miners switched from GPUs and FPGAs to ASICs, and focus instead on the two subperiods where miners used the same technology. More precisely, during the first period, which ranges from 2011/04/01 to 2013/01/31, miners mainly mined with GPUs; while they mostly relied on ASICS from 2014/10/01 onwards. Our second study period ranges from 2014/10/01 to 2017/03/31. Note that the first halving of the monetary creation rate happened on 2012/11/28, towards the end of the first period, while the second halving happened in 2016/07/09, around the middle of the second period.

Figure 3: Detrended Payoff Series



Calibrating the parameters.—We calibrate the parameters for each subperiod. The model is parsimonious enough to rely on six parameters only: the deterministic trend α of rewards and their volatility σ^2, the rate of technological progress a, the discount rate r, the price I_0 of one unit of hashpower bought at time 0 and the electricity cost C_0 of that same unit. The first two parameters can be directly estimated using $(R_t)_{t\geq 0}$ only.

Under assumption 3, the log returns are independent and follow a normal distribution with mean $\mu \equiv \alpha - \sigma^2/2$ and variance σ^2. Hence we estimate them by maximum likelihood.

Assumption 3 is satisfied except for the tails which are too fat (See Appendix 6.3). This is a well-known and usual problem shared by many financial series. In our case, it will only affect the fit of the model in the short run. As discussed below, after one of those extreme returns, miners cannot immediately increase the hashrate as much as the model would predict and payoffs temporarily exceed the barrier.

The rate of technological progress a can be calibrated minimizing a distance between the observed hashrate path and the simulated one. A direct consequence of our Definition 1 of the industry equilibrium is that $Q_t = \max\left(Q_{t-1}, \frac{R_t A_t}{\overline{P}_0}\right)$ for all t. This condition provides us with an easy way to simulate the hashrate:

1. Set the initial value of the simulated hashrate Q_0^{sim} equal to its empirical counterpart, i.e. $Q_0^{sim} := Q_0$.
2. Update the simulated hashrate as follows $Q_t^{sim} := \max\left(Q_{t-1}^{sim}, \frac{R_t A_t}{\overline{P}_0}\right)$, for $t = 1, \ldots, T$.

Since $(R_t)_t$ and Q_0 are observed, this simulation procedure has only two unknown inputs, a and $\overline{P}_0$, which can be calibrated minimizing a (pseudo) distance between the simulated and observed hashrate paths. With obvious notations, we solve:

$$(\hat{a}, \hat{\overline{P}}_0) \in \underset{(a,\overline{P}_0)\in\mathbb{R}\times\mathbb{R}^+}{argmin} \sum_{t=1}^{T} \left(\frac{Q_t - Q_t^{sim}(a, \overline{P}_0)}{Q_t} \right)^2 . \tag{8}$$

Unfortunately, the three other parameters $\{r, I_0, C_0\}$ cannot be disentangled. We therefore fix r and recover the total costs of one terahash per second bought at the beginning of each subperiod, $K_0 \equiv I_0 + C_0/r$, equating the expression for $\overline{P}_0$ obtained in (7) with the estimated $\hat{\overline{P}}_0$. Since the term $\beta(r-\alpha)/(\beta-1)$ in (7) does not change a lot with r, the total costs are also rather inelastic with respect to r, thus rendering its arbitrary choice relatively unimportant. The parameters resulting from the calibration strategy are summarized in Table 1, where all values are expressed as yearly rates.[15]

[15]For example, the estimates for a means that the price of a new machine has been on average divided by $\exp(a)$ every year during each period of study.

Table 1: Calibrated Parameters

Method	Parameter	Interpretation	1st period	2nd period
(fixed)	r	Discount Rate	0.1	0.1
(estimated)	μ	Trend R_t	1.41	0.19
	σ^2	Variance R_t	1.95	0.54
(calibrated)	a	Rate of TP	1.18	0.76
	K_0	Total Costs	5.6×10^6 $	1825 $

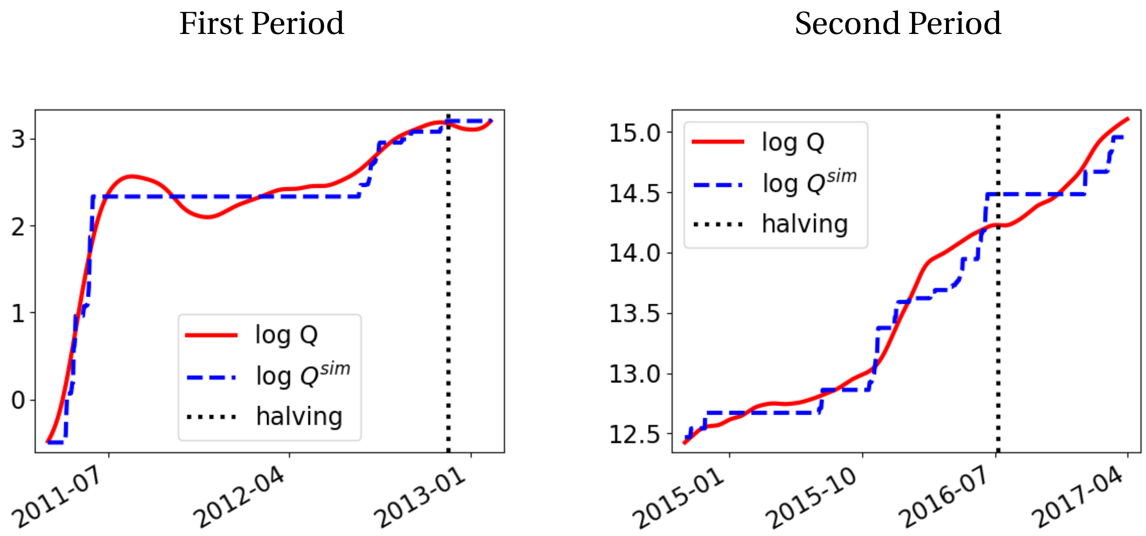
According to Moore's law a should be close to $\log(2)/2 \approx 0.35$ since it predicts that the price of one unit of hashpower is divided by two every two years. Our calibration suggests that the mining technology progressed at a much faster rate although it slowed considerably in the second period. This finding is consistent with the observation that miners were able to implement innovations specific to the hashing problem on top of the raw increase in computing power. But such improvements became harder to unearth as the mining technology matured and the rate of progress gradually converged towards the one predicted by Moore's law.

The average growth rate of rewards, μ, also decreased a lot between the two periods of study. As one would expect, early buyers of Bitcoins earned higher returns. Information about their profits pushed the demand for Bitcoins which raised the exchange rate even more. But the extremely high returns observed at the beginning became harder to sustain as the market capitalization grew from a negligible amount to nearly 20 billions $ by the end of our sample. In spite of this cooling process investing in Bitcoin remained extremely profitable, especially if one bears in mind that the values we report for μ take the halvings into account. This tremendous returns have led many observers to brand Bitcoin a giant bubble and to announce its imminent collapse.[16] Whether or not such predictions will eventually be vindicated is beyond the scope of this paper, but our estimates for the volatility coefficient σ indicate that there was no obvious arbitrage opportunity as investors willing to bet on Bitcoin also had to bear a huge risk. Even though the volatility of rewards was divided by three in the second period, its value remained an order of magnitude higher than its counterpart for the S&P 500.[17]

[16] The webpage 99bitcoins.com/bitcoinobituaries/ keeps track of Bitcoin's obituaries. By November 2017, 174 analysts had already published opinion pieces predicting the death of Bitcoin.

[17] We find that, for the S&P 500, $\sigma^2 = 0.053$ for the first period and $\sigma^2 = 0.027$ for the second period

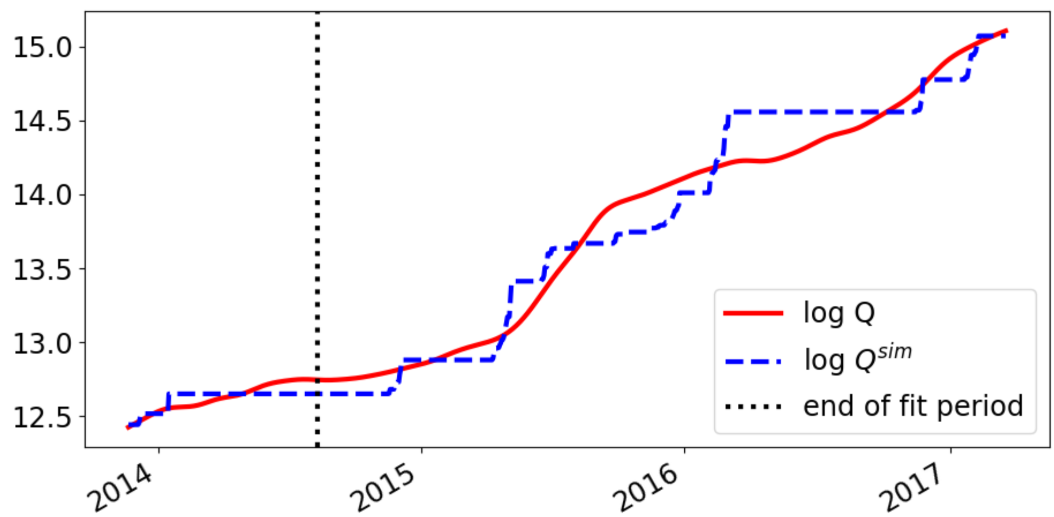
Figure 4: Simulated vs Observed Hashrates



Predicted vs actual hashpower.—The estimation procedure provides us with an estimate for the reflecting barrier, $\overline{P}_0$, as well as for its trend, a. Using these two values, we can run the two-step algorithm described above to simulate the network hashpower $(Q_t^{sim})_{t\geq 0}$. We report the simulated series against its empirical counterpart in Figure 4. In spite of its very parsimonious structure, the model tracks the actual hashpower remarkably well over the long run. Yet we do notice some temporary discrepancies. In particular, during the second period, the model is a bit less accurate around the halving date (2016/07/09). This is not surprising because miners do not anticipate halvings in our model while they certainly do in reality. Hence, it is actually more intriguing that such a disconnect between the simulation and the data is not apparent around the first halving date (2012/11/28). A potential explanation could be that the exchange rate was so volatile during this period that even a 50% drop in the payoff was not such an exceptional event. Another noticeable difference between the actual and the simulated hashrates is that the former sometimes decreases, especially during the first period, while the latter never does. Our model cannot reproduce these drops in haspower because it is based on the premise that investment is totally irreversible.

These discrepancies do not invalidate our approach since the model was devised to capture long-run trends in haspower. Yet one could argue that such a conclusion is too generous as our procedure would reproduce the data fairly well even if the model were misspecified because it minimizes the distance between simulated and observed hashrates. To this argument our first answer is that we optimize on two parameters only, which is not much to fit times series of 608 and 913 data points. Moreover, to perform the simulations we start from the initial hashrate for each subperiod and then let the model run without using intermediate realizations to correct its output. Given that our estimation procedure does not place any additional weight on the final values of the hashpower, any fundamental misspecification would have generated a noticeable gap between the simulation and the data during some subperiod. Thus we view the fact that there is no obvious deterioration of the model's fit over time as a convincing verification of its accuracy. We now provide some support for this interpretation by performing some out-of-sample tests, and by comparing the entry rule predicted by the model with the one prevailing in the data.

Figure 5: Out-of-sample Test

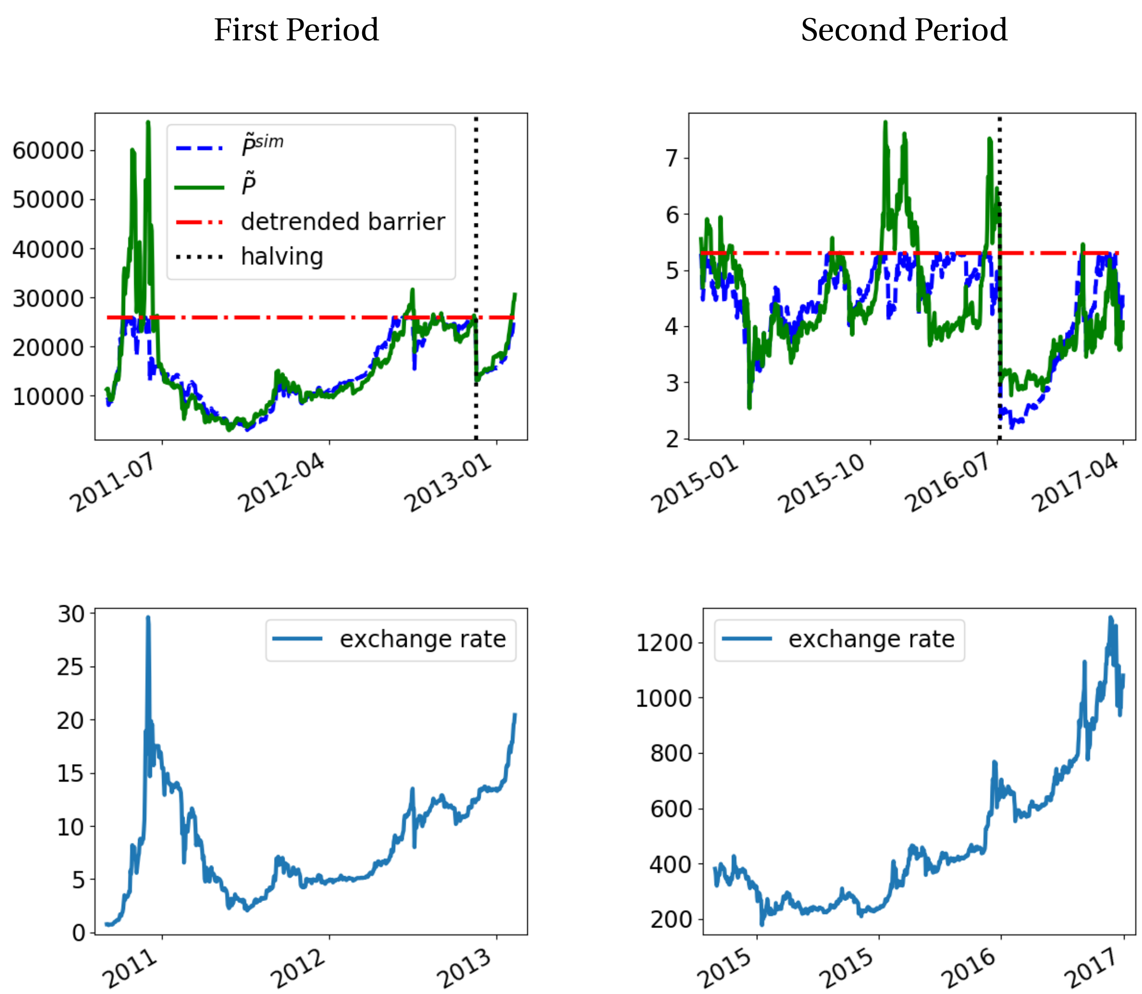


Out-of-sample tests.—We assess the model's ability to match out-of-sample data by dividing the second period into a fit period and a test period. We calibrate a and $\overline{P}_0$ on the fit period only and find that, even when the fit period is pretty small, the calibrated values remain close to the ones based on the full sample. Thus the predicted hashrate remains accurate several years after the end of the fit period, as shown in Figure 5.

Note, however, that out-of-sample tests are much less conclusive for the first subperiod because the hashrate increases only at the beginning and at the end of the period. Hence, if we split the first data sample into a fit and a test period, the payoffs do not hit the reflecting barrier often enough to deliver reliable calibrations. For instance, the parameters are not identified if the payoffs hit the barrier only once as one cannot pinpoint a line with the help of a single point.

Inspecting the entry rule.—Besides assessing the model's overall fit, we can also check whether the data are in line with the s-S rule predicted by the theory. For this purpose, we report the simulated and observed detrended payoffs in the upper-panels of Figure 6. As forecasted by the model, the observed payoffs remain below the barrier most of the time and tend to reflect downwards when they reach its vicinity. This is remarkable in itself since $\overline{P}_0$ was estimated regardless of this requirement, fitting the hashrate only.

Figure 6: Simulated vs. Observed Payoffs



Although the observed and simulated series are nearly superimposed for most of the dates, there are some time intervals where the two series differ significantly. These divergences occur for two reasons. First, the fit of the model deteriorates significantly around the halving date (2016/07/09) of the second period. But, as explained above, this is precisely what one should expect since the model does not take halvings into account. Second, the model sometimes fails due to extreme realizations of the exchange rate. This can be seen by comparing the upper-panels of Figure 6 with the lower panels where we report the ₿/$ exchange rate. One quickly notices that the periods of divergence between observed and simulated payoffs are clustered around the dates where the exchange rate is extremely volatile. Quite intuitively, when the exchange rate goes up 30% or more in one day,[18] miners cannot enter the market as quickly as the model predict because they are facing, among many other frictions, delivery and manufacturing delays. Devising a model that takes into account such constraints, by introducing investment delays along with potentially convex adjustment costs at the industry level, would probably improve the correspondence between the two series. We leave such refinements to further research since they greatly complicate the solution of the model[19] while our findings suggest that they are not likely to yield significant forecasting gains beyond short-term horizons.[20]

5 Conclusion

We have shown that miners' decisions to invest in hashpower are well approximated by a standard model of industry dynamics with irreversible investment. We believe that our results will be of interest to both Bitcoin practitioners and economists. The model provides practitioners with a tool that accurately predicts the network hashrate and sheds a new light on miners' incentives, which will help to clarify the ongoing debate about Bitcoin's viability. For economists, the market for Bitcoin mining provides an ideal environment to test models of industry evolution. In this respect, our findings are reassuring since they show that the behavior of miners is very much in line with the theory. Further research should strive to improve the realism of our framework by taking into account halvings and by allowing for partially reversible investment.

[18] For example, such extreme gains were observed on 05/10/2011, 06/03/2011, 04/17/2013, 11/18/2013 and 12/18/2013.

[19] See for example the work of Aid et al. (2015) on regulated Brownian motions with delays.

[20] This is confirmed by the fact that when the payoff variable temporarily exceeds the barrier due to a surge in the exchange rate, it always quickly decreases. These corrections are very much in line with our model: they occur because the hashrate catches up and not because the exchange rate decrease.

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6 Appendix

6.1 Proof of Proposition 2

Let $W\left(P_t, \overline{P}_t, A_t\right) \equiv V\left(P_t, t\right) + C_t/r$ denote the value of an entrant net of variable costs as a function of the payoff P_t, the entry barrier $\overline{P}_t$ and the efficiency of the technology A_t. Assumption 4 requires that $dA_t = -aA_t dt$. Assumptions 1 and 3 imply that $dP_t = P_t\left(\alpha dt + \sigma dZ_t\right)$ whenever $P_t < \overline{P}_t$ because Q_t remains constant in that region of the payoff space. Finally, the law-of-motion of the entry barrier $\overline{P}_t$ is endogenous and it is precisely the aim of this proof to show that the market for mining satisfies the equilibrium requirements stated in Definition 1 when $\overline{P}_t$ decreases at the rate of technological progress. Thus we conjecture that $\overline{P}_t = \overline{P}_0/A_t$, with $\overline{P}_0$ as in Proposition 1, and proceed to show that it is indeed optimal for entrants to wait until $P_t = \overline{P}_t$.

Having specified the law of motion of the three state variables allows us to use Ito's Lemma to derive the Hamilton-Jacobi-Bellman equation satisfied by the value function

$$\begin{aligned} rW\left(P_t, \overline{P}_t, A_t\right) &= P_t + \alpha P_t W_1\left(P_t, \overline{P}_t, A_t\right) - a\overline{P}_t W_2\left(P_t, \overline{P}_t, A_t\right) + aA_t W_3\left(P_t, \overline{P}_t, A_t\right) \\ &\quad + \frac{\sigma^2}{2} P_t^2 W_{11}\left(P_t, \overline{P}_t, A_t\right). \end{aligned}$$

Assume that $\alpha \neq r$.[21] Then the general solution of the Hamilton-Jacobi-Bellman equation reads

$$W\left(P_t, \overline{P}_t, A_t\right) = \frac{P_t}{r-\alpha} + \frac{D_1}{A_t}\left(\frac{P_t}{\overline{P}_t}\right)^{\beta_1} + \frac{D_2}{A_t}\left(\frac{P_t}{\overline{P}_t}\right)^{\beta_2},$$

where D_1 and D_2 are constants whose values will be chosen so as to match some boundary conditions, while β_1 and β_2 are the two roots of the following quadratic equation

$$\mathcal{Q}(\beta) \equiv \frac{\sigma^2}{2}\beta(\beta-1) + (\alpha + a)\beta - a - r = 0.$$

Since $Q(0) = -a - r < 0$ and the coefficient associated to the second order term is strictly positive, we know that one root, β_1 for instance, is strictly positive while the other root, β_2, is strictly negative.

[21] As r tend to α, $\overline{P}_0$ converges to $\left(I_0 + \frac{C_0}{r}\right)\left(\alpha + a + \sigma^2/2\right)$ and $W\left(P_t, \overline{P}_t, A_t\right)$ tends to $\frac{I_0 + \frac{C_0}{r}}{A_t}\left(\frac{P_t}{\overline{P}_t}\right)\left[1 - \log\left(\frac{P_t}{\overline{P}_t}\right)\right]$.

Instead of directly using the boundary conditions to pin down the constants, we first note that W is log-linear in A_t since

$$w\left(\tilde{P}_t\right) \equiv \frac{\tilde{P}_t}{r-\alpha} + D_1\left(\frac{\tilde{P}_t}{\overline{P}_0}\right)^{\beta_1} + D_2\left(\frac{\tilde{P}_t}{\overline{P}_0}\right)^{\beta_2}$$

satisfies

$$w\left(\tilde{P}_t\right) = A_t W\left(P_t, \overline{P}_t, A_t\right) \text{ when } \tilde{P}_t \equiv P_t A_t. \tag{9}$$

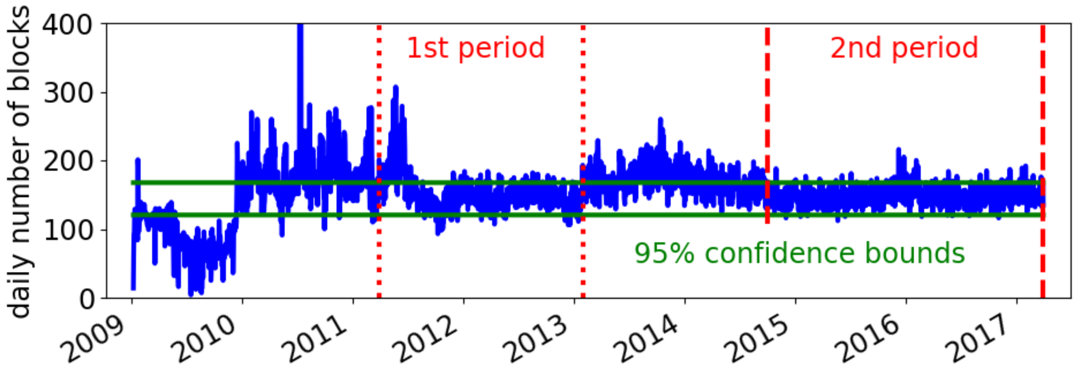
The function w has to satisfy the following three boundary conditions. First, since $\tilde{P}_t = 0$ is an absorbing state, we must have $w(0) = 0$. This implies that $D_2 = 0$, as otherwise the value function would diverge to either minus or plus infinity when P goes to zero. Second, the left continuity of the value function at the entry threshold $\overline{P}_t$ implies that there can be no arbitrage opportunity solely if the value function is flat at the contact point. This requirement, known as the smooth-pasting condition, is satisfied when $w'\left(\overline{P}_0\right) = 0$, i.e. when $D_1 = -\frac{\overline{P}_0}{\beta_1(r-\alpha)}$. Finally, the entry barrier is pinned down by the free entry condition $W\left(\overline{P}_t\right) = I_t + C_t/r$. Equation (9) shows that free entry holds at all dates if $w\left(\overline{P}_0\right) = I_0 + C_0/r$, i.e. if $\overline{P}_0 = \left(I_0 + C_0/r\right)\frac{(r-\alpha)\beta}{\beta-1}$.[22] Thus we have found a solution which satisfies all the requirements laid-out in Definition 1 for the existence of a competitive equilibrium.

6.2 Test of Assumption 1

If Assumption 1 were true, finding a block would always take 10 minutes on average so that the daily number of blocks found would not be statistically different from 144. Figure 7 plots the daily number of blocks found and the two 95 % confidence bounds. For our two periods of interest, the results are satisfying except for the beginning of the first period. According to this graph, it is sensible not to consider the period in between which witnessed the ASICs revolution. Technical progress was so fast that the hashrate increased significantly within only two weeks implying a block-finding rate faster than one block every ten minutes.

[22] Alternatively, we could have solved the planner's problem and used the "super contact" condition $w''\left(\overline{P}_0\right) = 0$.

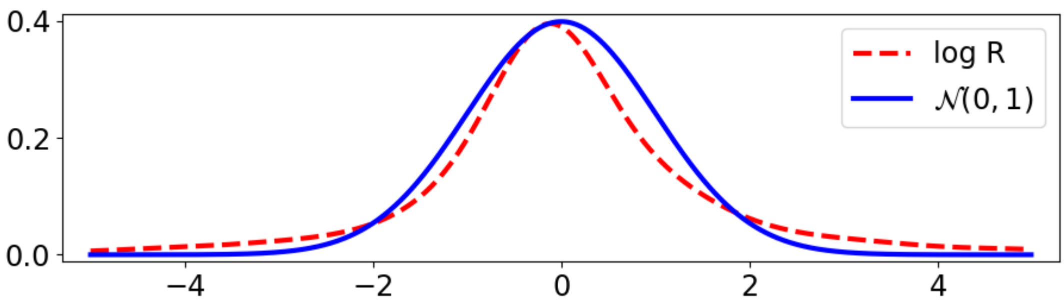
Figure 7: Number of blocks found per day



6.3 Test of Assumption 3

Figure 8 plots the reduced and centered density of the log returns against the density of the normal distribution. For this comparison only, the variance of the log returns has been estimated discarding the 5% most extreme returns on each side. Apart from the tails, which are clearly too fat, assumption 3 seems quite sensible.

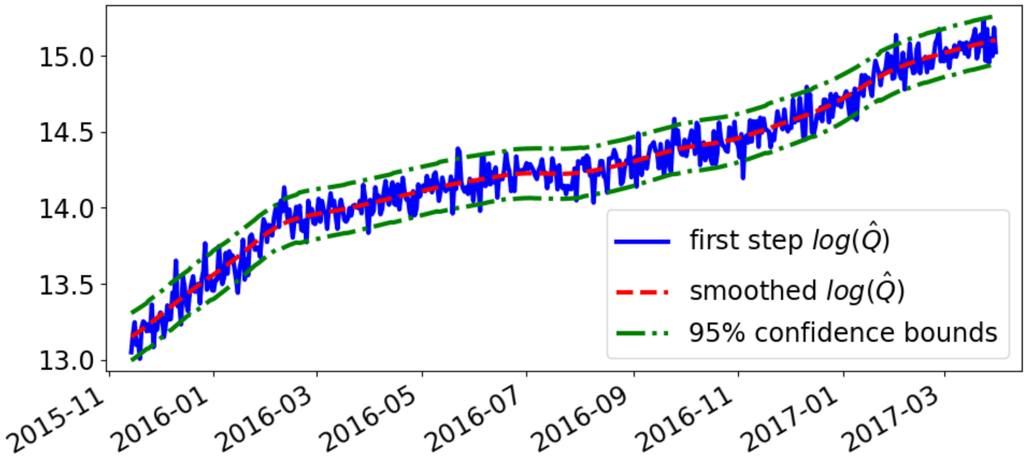
Figure 8: GBM test



6.4 Estimation of Q

$(Q_t)_{t\geq 0}$ is not observable but can be estimated using a two-step procedure. First, for each day t, let $\hat{Q}_t \equiv N_t/\tilde{\Pi}_t$, where N_t is the number of blocks found for day t and $\tilde{\Pi}_t$ is the probability to find a valid block with a single hash. Both are directly observable in the blockchain. Since $N_t \sim \mathcal{B}i\left(Q_t, \tilde{\Pi}_t\right)$, $\hat{Q}_t$ is a very natural estimator of the daily hashrate. This estimator is non biased and it can easily be shown that it is asymptotically equivalent to the maximum likelihood estimator. Of course there is a lot of variation across the daily estimates. We then smooth this new time series using a local linear regression. Figure 9 shows we are not losing much information performing a local linear regression over $\hat{Q}$.

Figure 9: Estimation of Q



The two green curves are confidence bounds for the first step estimation if the true $(log\,(Q)_t)_{t\geq 0}$ were the red curve (the second step estimate). If the erratic variations of the first step estimation captured not only the first step estimation variance but also some real variations of the hashrate not captured by the second step estimation, then its variance should be bigger than the one resulting from the first step estimation error only. Thus it should cross the green bounds much more often than 5% of the time, which does not happen in our data. For the sake of clarity, we do not show the whole series but the test works very well for the whole period.