Série des Documents de Travail

n° 2017-11

Behavioral Heterogeneity : Pareto Distributions of Homothetic Preference Scales and Aggregate Expenditures Income Elasticities

J-M. GRANDMONT[1]

[1] ICEF, Department of Economics; University Ca' Foscari di Venezia; CREST; RIEB Fellow, University of Kobe. E-mail: Jean-Michel.Grandmont@ensae.fr

Behavioral Heterogeneity : Pareto Distributions of Homothetic Preference Scales and Aggregate Expenditures Income Elasticities *

J.M. Grandmont [(1)]

September, 2017

* I have benefited from conversations with Frédérique Bec, Xavier d'Haultfoeuille, Francis Kramarz and Laurent Linnemer. I am very grateful to Fanda TRAORE for her efficient typing. As usual, I am sole responsible for any remaining shortcomings of the paper.

[1]ICEF, Department of Economics, University Ca' Foscari di Venezia at San Giobbe, Italy, CREST (EXCESS, UMR CNRS 9194), University of Paris-Saclay, France, and RIEB Fellow, University of Kobe, Japan.

Behavioral Heterogeneity : Pareto Distributions of Homothetic Preference Scales and Aggregate Expenditures Income Elasticities

Abstract

We evaluate the income elasticity of the aggregate budget share spent on a *sub-group* of commodities, in a competitive framework, by a continuum of agents having the same income, but heterogeneous behavior described by an "homothetic preferences scaling factor" having a bounded Pareto distribution in the population. If individual budget share increases *globally* significantly in the limit from low to large incomes, aggregate budget share is *locally* increasing with medium range incomes when the logarithm of the heterogeneity factor has an increasing (exponential) density with a large support. Aggregate income elasticity converges to that exponential density parameter when its support becomes infinitely large. Symmetric results hold in the decreasing case. Applications are made to market expenditures, wealth effects on portfolio choice with many risky assets, concave expenditures, that are compatible with standard (expected) utility maximization or other "behavioral" decision making processes.

Keywords : Behavioral heterogeneity, aggregation, preference scales, aggregate income elasticity, power law, Pareto distribution, exponential distribution, market demand, wealth effect on aggregate portfolio choices.

1 Introduction

This work seeks to make theoretical progress on the analysis of macroeconomic aggregates by relying significantly on a *distributional* approach, i.e. on specified properties of distributions of heterogeneous individual *preferences* characteristics. Such an approach has been initiated in particular by W. Hildenbrand (1983, 1994) and the author (1987, 1992). It aims at coping with the observed behavioral diversity of economic agents, which has been well documented empirically for long in households demand analysis (see Barten (1964, 1977), Calvet and Comon (2003), Deaton and Muellbauer (1980), Jorgenson and Selsnik (1987), Lewbel (1997), Lewbel and Pendakur (2017), Matzkin (2007), McFadden (2001), Muellbauer (1977, 1980), Prais and Houthakker (1971)), or in households heterogeneous portfolio choices (see Calvet *et alii* (2007, 2009, 2014), Chan and Kogan (2002), Curcuru *et alii* (2010)): for useful recent surveys, see Blundell and Stoker (2005), Lewbel (2007). A distinguishing feature of the approach adopted here is that we take as a reguirement that the introduced heterogeneity should be compatible with individual agents behaving "rationally", i.e. through (expected) utility maximization, as well as with various kinds of "behavioral" biases. This is of course to be contrasted with a large part of (micro or macro) economic theorizing, that is often based more or less implicitly (and misleadingly) on a "representative agent" approach, in which all agents are supposed to behave similarly, whether in markets for goods and services, or in their portfolio choices in finance.

The distributional approach employed in this work, as in previous contributions of the author on this topic (1987, 1992, 1993), seeks to account for behavioral heterogeneity through "*preference scaling factors*" (the word "preference" being there as a reminder of our requirement of individual behaviors being compatible with "rationality"). These are essentially (possible commodity specific) versions of the well known "*households equivalence scales*", due to Prais and Houthakker (1971), Barten (1964, 1977), that have been subsequently much used in econometric studies (Blundell and Lewbel (1991), Deaton and Muellbauer (1980), Jorgenson and Selsnik (1987), Lewbel (1997), Lewbel and Pendakur (2008), Muellbauer (1977, 1980)). Such a structure was also used long ago in general equilibrium analysis to show that suitably dispersed distributions on the space of consumers' characteristics led to a nice "smoothing" of competitive aggregate demand (Mas-Colell and

Neuefind (1977), E. Dierker, H. Dierker and Trockel (1984)). Specifically, one considers a collection (continuum) of agents who have the same income and the same individual (possibly (expected) utility maximizing) "base" behavior, but who behave actually differently, i.e. as if the prices they face were multiplied by these (possibly commodity specific) heterogeneity scaling factors. Previous analysis showed that when the distributions of such commodity specific scaling factors (actually, of their logarithms), were "flat enough", aggregate expenditures became less sensitive to prices (Grandmont (1992), Quah (1997, 2001), Giraud and Maret (2002)). Such aggregate insensitivity property has been further shown to occur as well for non-parametric definitions of "behavioral heterogeneity" (see Kneip (1999), Jerison (1999), Hildenbrand and Kniep (2005) and their references). These "behavioral heterogeneity" concepts were however mostly defined directly as properties of the distributions of individual demand functions themselves, without any explicit guarantee that there were compatible with standard individual "rational" utility maximization, in the spirit of the early "irrational" approach of Becker (1962).

The aim of this paper is to show that one can actually get sharper aggregate properties by exploiting further the convenient parametric nature of the distributions of such heterogeneous preference scaling factors. We focus here on the simple case where such scaling factors are "homothetic", i.e. the same for all commodities exchanged in the market, as in Grandmont (1987), Quah (1997). Such a formulation is indeed particularly adapted to an application we wish to make, i.e. portfolio choices in financial markets, if only to be compatible with standard individual decision making specifications, including VNM expected utility maximization, ambiguity, loss aversion, or whatever. We shall show that focusing, for the distributions of such homothetic scaling factors, on *power laws*, which do have important applications in economics and finance (Gabaix (2009), generates fruitful implications also here. We consider accordingly a continuum of agents who have the same income and the same "base" behavior, but whose actual behavior is indexed by such a multiplicative heterogeneity scaling factor, that is distributed among the agents following a *bounded Pareto distribution*. We focus on the income elasticity of the aggregate budget share spent by these agents on a *sub-group* of commodities. We derive then sharp results on the aggregate income elasticity from rather weak limiting properties of individual "base" behavior for quite low and large income levels, and by relying significantly on the specification of a Pareto distribution for the scaling factor. While our analysis is compatible with standard "rational" (expected) utility maximization over the entire income range, as mentioned earlier, it is compatible as well with any other "behavioral", more or less "irrational" departures in the medium income range, as we do not require any specific properties of the individual "base" behavior there. In the "increasing" case, one requires that the individual "base" budget share spent on the subgroup of commodities, or assets, under consideration, is very small for low incomes, and quite large for high incomes. Then if the logarithm of the homothetic scaling factor has an increasing (exponential) density with a large support, the aggregate budget share is locally increasing for medium range incomes. Further, the aggregate income elasticity converges to the (positive) parameter of this exponential density when its support becomes infinitely large. Symmetric results are valid in the "decreasing" case.

A detailed application is made to expenditures on a subgroup of commodities in a competitive framework. There, low budget shares allocated to this subgroup for low incomes can be viewed as resulting from low marginal rates of substitution for these commodities at low consumption levels. And conversely, large budget shares spent on the subgroup for large incomes, are resulting from large marginal rates of substitution in favor of these commodities at large consumption levels. Another detailed application is made to aggregate portfolio choice in financial markets involving several risky securities and a single riskfree asset. There, a low budget share invested in the riskless asset for low incomes, at the individual level, is then the consequence of very large degrees of relative risk aversion (or of loss aversion) in that income range. Symmetrically, a very low degree of risk aversion (i.e. risk neutrality) for large incomes implies a vanishing investment, at the individual level, in the risk free asset. Our approach enables us then to get the *aggregate* budget share invested in the riskfree asset to be locally decreasing with medium range wealth levels. Such a property has been known to be valid at the individual level since the contribution of Arrow (1970), when individual relative degrees of risk aversion are decreasing everywhere with income, provided however that there is a single risky asset to be invested in. By contrast, when there are several risky securities as here, such a property has been well known to be impossible to obtain at the individual level, without imposing extremely particular specifications, e.g. of the agent's VNM expected utility (Cass and Stiglitz (1970, 1972), Hart (1974)). Our results show that going to the aggregate level does improve significantly the perspective. Detailed applications to higher order derivatives, i.e. concavity or convexity of aggregate expenditures, are also presented, a topic that has attracted a significant attenion in the literature (see Caroll and Kimball (1996)).

The paper is organized as follows. We present the basic framework and assumptions in section 2. We state the core formal results on aggregate income elasticities in section 3 when the homothetic heterogeneity scaling factor has a bounded Pareto distribution, i.e. when its logarithm has an exponential density. The case where that density is uniform is considered in section 3.1, and monotonically increasing or decreasing in section 3.2. Detailed applications to market demand are given in section 4.1, to aggregate portfolio choice in section 4.2, to concavity of aggregate expenditures in section 4.3. We conclude briefly in section 5. Proofs are gathered in an appendix.

2 Framework

We study aggregate expenditures for a particular *group* of commodities (consumption goods, services, physical or financial assets) of a collection (continuum) of agents in a given period, in a competitive set up. Their common "base" behavior is described by an *individual expenditure function* $w^*(p, \beta)$, where $p = (p_1, \ldots, p_n) \gg 0$ is a vector of positive market prices for all commodities exchanged in the period under consideration and $\beta > 0$ is income. It satisfies $0 \leqq w^*(p, \beta) \leqq \beta$ and is homogenous of degree 1. The corresponding *individual budget share function* $s^*(p, \beta) = w^*(p, \beta)/\beta$ lies accordingly in [0,1] and is homogenous of degree 0. Both expenditure functions are continuously differentiable in $(p, \beta) \gg 0$ up to any required order.

The agents' actual behavior is governed by these expenditure functions up to a *"behavioral heterogeneity" scaling factor* λ, an arbitrary positive real number. An agent corresponding to the scaling factor λ has "tastes" for all commodities that are homothetically divided by λ : she behaves accordingly "as if" the market price vector were λp instead of p. We shall work for analytical convenience with the heterogeneity parameter $\alpha = \log \lambda$. If the agent's income is $b > 0$, the resulting expenditures functions are given by

$$
\begin{aligned}
w(\alpha,p,b) &= w^*(e^\alpha p,b) = e^\alpha w^*(p,e^{-\alpha}b), & (2.1)\\
s(\alpha,p,b) &= s^*(e^\alpha p,b) = s^*(p,e^{-\alpha}b). & (2.2)
\end{aligned}
$$

There is a continuum of such individual agents, normalized to 1, that is described by a probability distribution over the behavioral heterogeneity parameter, generated by a density function $g(\alpha) \geqq 0$. It is concentrated on an interval $[\alpha_0, \alpha_1]$ with $\alpha_0 < \alpha_1$, i.e. $g(\alpha) = 0$ for $\alpha \notin [\alpha_0, \alpha_1]$, and continuously differentiable up to any required order on that interval. Under the assumption that all these agents face the same current price system $p \gg 0$ and have the same current income $b > 0$, aggregate expenditures and budget shares for the group of commodities under consideration, are given by

$$
\begin{aligned}
W(p,b) &= \int_{\alpha_0}^{\alpha_1} w(\alpha,p,b)g(\alpha)d\alpha, & (2.3)\\
S(p,b) &= W(p,b)/b = \int_{\alpha_0}^{\alpha_1} s(\alpha,p,b)g(\alpha)d\alpha. & (2.4)
\end{aligned}
$$

Our aim is to analyze how these aggregate expenditures vary with income, namely to evaluate $\partial W(p,b)/\partial b$ and $\partial S(p,b)/\partial b$, in relation with the properties of individual expenditure functions and of the behavioral heterogeneity scaling factor density distribution $g(\alpha)$. It is clear from (2.2) that $b\dfrac{\partial s}{\partial b}(\alpha,p,b) = -\dfrac{\partial s}{\partial \alpha}(\alpha,p,b)$, so that by differentiation of (2.4)

$$
b\frac{\partial S}{\partial b}(p,b) = -\int_{\alpha_0}^{\alpha_1} \frac{\partial s}{\partial \alpha}(\alpha,p,b)g(\alpha)d\alpha. \qquad (2.5)
$$

An elementary integration by parts yields accordingly

Lemma 1. *Aggregate expenditures' income derivatives satisfy*

$$
\begin{aligned}
b\frac{\partial S}{\partial b}(p,b) &= \int_{\alpha_0}^{\alpha_1} s(\alpha,p,b)g'(\alpha)d\alpha - [g(\alpha)s(\alpha,p,b)]_{\alpha_0}^{\alpha_1} & (2.6)\\
\text{where } [g(\alpha)s(\alpha,p,b)]_{\alpha_0}^{\alpha_1} &= g(\alpha_1)s^*(p,e^{-\alpha_1}b) - g(\alpha_0)s^*(p,e^{-\alpha_0}b), &\\
\frac{\partial W}{\partial b}(p,b) &= S(p,b) + b\frac{\partial S}{\partial b}(p,b). & (2.7)
\end{aligned}
$$

The above suggests a potentially fruitful research strategy. Aggregate expenditures' income derivatives, or elasticities, depend on two interacting features : *a)* the relative sizes of individual "base" budget shares for low and large incomes at the endpoints of the distribution, namely $s^*(p, e^{-\alpha_1}b)$ compared to $s^*(p, e^{-\alpha_0}b)$, together with *b)* the shape of the scaling parameter density, through its derivative $g'(\alpha)$ and its values at the endpoints of the distribution, $g(\alpha_0)$ compared to $g(\alpha_1)$. As a quick simple example, one is sure to get a locally increasing aggregate budget share, i.e. $b\partial S(p,b)/\partial b > 0$, if the scaling parameter density is non-decreasing, $g'(\alpha) \geqq 0$ (in which case $g(\alpha_1) \geqq g(\alpha_0)$), whenever the individual budget share $s^*(p, \beta)$ is much higher for large incomes β than for low ones, so that,

$$g(\alpha_0)s^*(p, e^{-\alpha_0}b) - g(\alpha_1)s^*(p, e^{-\alpha_1}b) > 0. \tag{2.8}$$

In such a configuration, a significant increase of individual base behavior "in the large" (for large incomes β compared to low ones) translates into a locally increasing behavior for intermediate incomes b through a smoothing effect of the distribution of the behavioral heterogeneity scaling parameter. In the sequel, we shall rely on the two following alternative configurations where base individual budget shares increase (assumption *(H.0)*), or decrease (assumption *(H.1)*), more or less significantly when comparing large incomes β to low ones.

(H.0) *(Increasing individual budget share "in the large") Given* $p \gg 0$, *there exist* $0 < \beta_0^* < \beta_1^*$ *and* $0 \leqq s_0^* < s_1^* \leqq 1$ *such that* $s^*(p, \beta) \leqq s_0^*$ *for* $0 < \beta \leqq \beta_0^*$ *and* $s_1^* \leqq s^*(p, \beta)$ *for* $\beta_1^* \leqq \beta$.

(H.1) *(Decreasing individual budget share "in the large") Given* $p \gg 0$, *there exist* $0 < \beta_0^* < \beta_1^*$ *and* $1 \geqq s_0^* > s_1^* \geqq 0$ *such that* $s^*(p, \beta) \geqq s_0^*$ *for* $0 < \beta \leqq \beta_0^*$ *and* $s_1^* \geqq s^*(p, \beta)$ *for* $\beta_1^* \leqq \beta$.

In order to ensure the possibility that these increasing or decreasing properties of individual behavior, "in the large", may induce corresponding local monotonicity properties in the aggregate, we consider distributions of the scaling parameter that have a large enough support. In addition to either *(H.0)* or *(H.1)*, we shall assume throughout

(H) *Given* β_0^* *and* β_1^* *in either (H.0) or (H.1), the support of the distribution of the heterogeneity scaling parameter* α *is large, so that* $e^{\alpha_0}\beta_1^* < e^{\alpha_1}\beta_0^*$. *The agents' common income* b *lies in the open interval* $(e^{\alpha_0}\beta_1^*, e^{\alpha_1}\beta_0^*)$, *so that the scaled incomes* $e^{-\alpha}b$ *at the endpoints of the density distribution* $g(\alpha)$ *satisfy*

$$e^{-\alpha_1}b < \beta_0^* \textit{ and } \beta_1^* < e^{-\alpha_0}b. \tag{2.9}$$

This strategy will enable us to get approximate relative evaluations of the individual "base" budget shares $s^*(p, e^{-\alpha}b)$ at the endpoints of the density distribution α_0, α_1, as for instance in (2.8), by direct comparison of $s^*(p, \beta_0^*)$ and $s^*(p, \beta_1^*)$. As for the distribution of the behavioral heterogeneity parameter itself, we shall focus on monotone distributions, and actually *bounded exponential distributions for the indexing parameter* $\alpha = \log\lambda$, which corresponds to the case of *bounded Pareto distributions for the heterogeneity scaling factor* λ itself. That will enable us to derive sharp results on the income elasticities of aggregate expenditures for large heterogeneity distributions, i.e. when $|\alpha_1 - \alpha_0|$ tends to $+\infty$.

3 Exponential distributions of $\alpha = \log\lambda$

We consider bounded exponential distributions for the indexing parameter α, i.e. $g'(\alpha) = \varepsilon g(\alpha)$ for α in $[\alpha_0, \alpha_1]$ in which case

$$g(\alpha) = \gamma_0 e^{\alpha\varepsilon} \textit{ for } \alpha \in [\alpha_0, \alpha_1], g(\alpha) = 0 \textit{ for } \alpha \notin [\alpha_0, \alpha_1], \textit{ with}$$
$$\gamma_0 = \varepsilon/(e^{\alpha_1\varepsilon} - e^{\alpha_0\varepsilon}) \textit{ when } \varepsilon \neq 0, \textit{ and } \gamma_0 = 1/(\alpha_1 - \alpha_0) \textit{ when } \varepsilon = 0. \tag{3.1}$$

Equivalently, the heterogeneity scaling factor $\lambda = e^\alpha$ follows a *bounded Pareto distribution* on $[\lambda_0, \lambda_1]$, where $\lambda_0 = e^{\alpha_0}, \lambda_1 = e^{\alpha_1}$, with a density $f(\lambda)$ given by

$$f(\lambda) = \gamma_0 \lambda^{\varepsilon - 1} \textit{ for } \lambda \in [\lambda_0, \lambda_1], f(\lambda) = 0 \textit{ for } \lambda \notin [\lambda_0, \lambda_1] \textit{ with also here}$$
$$\gamma_0 = \varepsilon/(\lambda_1^\varepsilon - \lambda_0^\varepsilon) \textit{ when } \varepsilon \neq 0 \textit{ and } \gamma_0 = 1/(\log\lambda_1 - \log\lambda_0) \textit{ when } \varepsilon = 0.$$

3.1 Uniform distributions $(\varepsilon = 0)$

In this benchmark configuration, we get immediately from (2.6)

$$b\frac{\partial S}{\partial b}(p, b) = \frac{s^*(p, e^{-\alpha_0}b) - s^*(\beta, e^{-\alpha_1}b)}{\alpha_1 - \alpha_0} \tag{3.2}$$

It is then clear that one gets a locally increasing aggregate budget share, i.e. $b\dfrac{\partial S}{\partial b}(p, b) > 0$, when individual budget shares $s^*(p, \beta)$ are increasing "in the large", i.e. under assumptions *(H.0)* and *(H)*. Symmetrically, one gets $b\dfrac{\partial S}{\partial b}(p,b) < 0$ under assumptions *(H.1)* and *(H)*. On the other hand, if the support of the distribution, i.e. $|\alpha_1 - \alpha_0|$, becomes increasingly large, given β_0^*, β_1^*, aggregate budget shares becomes less sensitive to income variations. One can in fact show that the aggregate budget share income elasticity converges then uniformly to $\varepsilon = 0$.

Proposition 1. *Under assumption (H), given $p \gg 0$ and the agents' common income $e^{\alpha_0}\beta_1^* < b < e^{\alpha_1}\beta_0^*$,*
a) under assumption (H.0), one has $b\dfrac{\partial S}{\partial b}(p,b) > 0$. Furthermore

$$0 < \frac{b\dfrac{\partial S}{\partial b}(\beta,b)}{S(p,b)} \leqq \frac{1}{s_1^*(\log(b/\beta_1^*) - \alpha_0)} \tag{3.3}$$

converges to 0 when α_0 tends to $-\infty$, uniformly on compact income subintervals $[b_m, b_M]$ of $(e^{\alpha_0}\beta_1^*, e^{\alpha_1}\beta_0^*)$.
b) Under assumption (H.1), one has $b\dfrac{\partial S}{\partial b}(p,b) < 0$. Furthermore

$$\frac{1}{s_0^*(\log(b/\beta_0^*) - \alpha_1)} \leqq \frac{b\dfrac{\partial S}{\partial b}(p,b)}{S(p,b)} < 0 \tag{3.4}$$

converges to 0 when α_1 tends to $+\infty$, uniformly on compact subintervals $[b_m, b_M]$ of $(e^{\alpha_0}\beta_1^*, e^{\alpha_1}\beta_0^*)$.

A proof is given in the appendix. The implications for aggregate expenditures $W(p,b)$ are immediate through (2.7). Details are left to the reader.

3.2 Strictly monotonic exponential distributions

We focus now on exponential distributions of the heterogeneity parameter, $g'(\alpha) = \varepsilon g(\alpha)$ for α in $[\alpha_0, \alpha_1]$ with $\varepsilon \neq 0$. We get from (2.6) and (3.1)

$$\begin{aligned} &b\frac{\partial S}{\partial b}(p,b) = \varepsilon S(p,b) + \gamma_0[e^{\alpha_0\varepsilon}s^*(p, e^{-\alpha_0}b) - e^{\alpha_1\varepsilon}s^*(p, e^{-\alpha_1}b)] \\ &\textit{with } \gamma_0 = \varepsilon/(e^{\alpha_1\varepsilon} - e^{\alpha_0\varepsilon}). \end{aligned} \tag{3.5}$$

When $\varepsilon > 0$, it is straightforward to verify here, as above in our heuristic discussion of (2.8), that the aggregate budget share is locally increasing, in fact $b\dfrac{\partial S}{\partial b}(p,b) - \varepsilon S(p,b) > 0$, if individual budget shares $s^*(p,\beta)$ increase strongly enough when comparing low incomes $\beta \leqq \beta_0^*$ to large incomes $\beta \geqq \beta_1^*$, as in assumption *(H.0)*. One can in fact show the much stronger result that the aggregate income elasticity $b\dfrac{\partial S}{\partial b}(p,b)/S(p,b)$ actually converges to $\varepsilon > 0$, uniformly on compact income intervals, when the support $[\alpha_0, \alpha_1]$ of the heterogeneity density becomes infinitely large, if individual budget shares $s^*(p,\beta)$ converge fast enough to 0 (faster than β^ε) when $\beta \leqq \beta_0^*$ goes to 0.

Symmetrically, when $\varepsilon < 0$, one verifies also easily that $b\dfrac{\partial S}{\partial b}(p,b) - \varepsilon S(p,b) < 0$ if individual budget shares $s^*(p,\beta)$ decrease strongly enough when comparing low incomes $\beta \leqq \beta_0^*$ to large incomes $\beta \geqq \beta_1^*$, as in assumption *(H.1)*. Here also, one can show that the aggregate income elasticity $b\dfrac{\partial S}{\partial b}(\beta,b)/S(p,b)$ converges to $\varepsilon < 0$, uniformly on compact income intervals, when the support $[\alpha_0, \alpha_1]$ of the heterogeneity density becomes infinitely large, if individual budget shares $s^*(p,\beta)$ go fast enough to 0 when $\beta \geqq \beta_1^*$ increases without bound.

Proposition 2. *Under assumption (H), given $p \gg 0$ and the agents' common income $e^{\alpha_0}\beta_1^* < b < e^{\alpha_1}\beta_0^*$,*

a) If $\varepsilon > 0$, under assumption (H.0), one has $b\dfrac{\partial S}{\partial b}(p,b) > \varepsilon S(p,b) > 0$ if

$$[\log s_1^* - \log s_0^*]/(\alpha_1 - \alpha_0) > \varepsilon. \tag{3.6}$$

The aggregate income elasticity $b\dfrac{\partial S}{\partial b}(p,b)/S(p,b)$ converges uniformly to ε when $\alpha_0 \to -\infty, \alpha_1 \to +\infty$, if

$$0 \leqq s^*(p,\beta) \leqq a\beta^{\gamma\varepsilon} \text{ for all } \beta \leqq \beta_0^*, \text{ with } a \geqq 0, \gamma > 1. \tag{3.7}$$

b) If $\varepsilon < 0$, under assumption (H.1), one has $b\dfrac{\partial S}{\partial b}(p,b) < \varepsilon S(p,b) < 0$ if

$$[\log s_1^* - \log s_0^*]/(\alpha_1 - \alpha_0) < \varepsilon. \tag{3.8}$$

The aggregate income elasticity $b\dfrac{\partial S}{\partial b}(p,b)/S(p,b)$ converges uniformly to ε when $\alpha_0 \to -\infty, \alpha_1 \to +\infty$, if

$$0 \leqq s^*(p,\beta) \leqq a\beta^{\gamma\varepsilon} \text{for all } \beta \geqq \beta_1^*, \text{ with } a \geqq 0, \gamma > 1. \tag{3.9}$$

A proof is given in the appendix. The above has immediate implications for aggregate expenditures, through (2.7) and (3.5).

Corollary 1. *Under the assumptions of Proposition 2, income derivatives of aggregate expenditures are given by*

$$\frac{\partial W}{\partial b}(p,b) = (1+\varepsilon)S(p,b) + [g(\alpha_0)s^*(p,e^{-\alpha_0}b) - g(\alpha_1)s^*(p,e^{-\alpha_1}b)]. \qquad (3.10)$$

a) When $\varepsilon > 0$, *(H.0) and (3.6) imply* $b\frac{\partial W}{\partial b}(p,b) > (1+\varepsilon)W(p,b) > 0$. *Under (3.7),* $b\frac{\partial W}{\partial b}(p,b)/W(p,b)$ *converges uniformly to* $1+\varepsilon$ *when* $\alpha_0 \to -\infty, \alpha_1 \to +\infty$.

b) When $\varepsilon < 0$, *(H.1) and (3.8) imply* $b\frac{\partial W}{\partial b}(p,b) < (1+\varepsilon)W(p,b)$, *while* $b\frac{\partial W}{\partial b}(p,b)/W(p,b)$ *converges uniformly to* $1+\varepsilon$ *when* $\alpha_0 \to -\infty, \alpha_1 \to +\infty$ *under (3.9). Aggregate expenditures are then decreasing in income if* $1+\varepsilon < 0$.

It may be noted that when $\varepsilon > 0$, the claimed results are valid in the limiting, but nevertheless relevant, configuration where $s^*(p,\beta) = 0$ for all $0 < \beta \leqq \beta_0^*$, which corresponds to the case where $\log s_0^* = -\infty$ in (3.6) and $a = 0$ in (3.7). In that case, convergence of the aggregate income elasticity to $\varepsilon > 0$ obtains when $\alpha_0 \to -\infty$, even when $\alpha_1 > \log(b/\beta_0^*)$ remains bounded above. When $s_0^* > 0$ and thus $a > 0$ in (3.7), one needs both $\alpha_0 \to -\infty$ and $\alpha_1 \to +\infty$. The distribution over the heterogeneity parameter becomes then more concentrated on large α, hence the scaled incomes $e^{-\alpha}b$ on low levels. Both the aggregate budget share $S(p,b)$ and its income derivative become small, while the aggregate income elasticity gets closer to $\varepsilon > 0$. An analogous symmetric argument applies to the case $\varepsilon < 0$.

4 Applications

We apply next the above results and methods to a few specific frameworks : aggregate market demand, portfolio choice, and concave aggregate expenditures.

4.1 Aggregate market demand

We consider the case where "base" behavior is the result of maximizing individual preferences, or utility $U(x)$, over commodity bundles $x = (x_1, \ldots, x_n) \geqq 0$ for low and large incomes. Given the commodity price vector $p = (p_1, \ldots, p_n) \gg 0$, "base" expenditures for each commodity h, $w_h^*(p,\beta), s_h^*(p,\beta)$, are the result of maximizing $U(x)$ subject to $p.x = \beta$ for $\beta < \beta_m$ and for $\beta_M < \beta$, where $0 < \beta_m < \beta_M$ are given thresholds. We assume standard conditions (strict monotonicity and concavity, smoothness) ensuring in particular interior solutions $x_h > 0$ within this admissible income range, that can be characterized by standard FOC relations equating marginal rates of substitution of any pair of commodities h, k with the corresponding price ratios p_h/p_k. Expenditure functions $w_h(\alpha, p, b)$ and $s_h(\alpha, p, b)$ corresponding to the heterogeneity scaling factor $\lambda = e^{\alpha}$, as in (2.1), (2.2), are then obtained by maximizing $U(x/\lambda) = U(e^{-\alpha}x)$ subject to $p.x = b$, whenever $e^{-\alpha}b < \beta_m$ or $\beta_M < e^{-\alpha}b$. *Other (possibly non utility maximizing) behaviors are of course compatible with our analysis for medium range incomes* $\beta_m \leqq \beta \leqq \beta_M$.

We wish to see under which conditions on the utility function $U(x)$ one can get the configurations described in Propositions 1,2 above. For the simplicity of the exposition, we focus on the "increasing" case *(H.0)* and $\varepsilon \geqq 0$. Similar arguments extend easily to the other case *(H.1)* and $\varepsilon \leqq 0$. Intuitively, one should get the individual "base" budget share $s^*(p, \beta)$ of a particular group of commodities $h = (1, \ldots, i)$ to be increasing in the "large" as in *(H.0)* if the marginal rates of substitution $\dfrac{\partial U}{\partial x_h}(x)/\dfrac{\partial U}{\partial x_k}(x)$ between commodities $h = 1, \ldots, i$ in that group and those outside $k = i+1, \ldots, n$, become low for small income commodity bundles, and high for large income commodity bundles, in a sense we wish to make precise. And similarly for limiting conditions such as (3.7) for incomes going to 0 in Proposition 2 for the case $\varepsilon > 0$.

A two commodities CES example

Given $p = (p_1, \ldots, p_n) \gg 0$, let base expenditures be the result of maximizing the CES utility function

$$U(x_1, x_2) = a_1 \frac{x_1^{1-\rho_1}}{1-\rho_1} + a_2 \frac{x_2^{1-\rho_2}}{1-\rho_2}, \quad \rho_1 > 0, \rho_2 > 0 \tag{4.1}$$

with $a_1 > 0, a_2 > 0$, under the budget constraint $p.x = \beta$ for low incomes $0 < \beta < \beta_m$ and large ones $\beta_m < \beta_M < \beta$. The optimum base budget share of commodity $1, s = p_1x_1/\beta = s^*(p, \beta)$ is then the unique solution of the FOC

$$\frac{s^*(p, \beta)^{\rho_1}}{[1 - s^*(p, \beta)]^{\rho_2}} = \beta^{\rho_2 - \rho_1} d \tag{4.2}$$

with $d = a_1 p_1^{\rho_1 - 1}/a_2 p_2^{\rho_2 - 1}$. When $\rho_2 > \rho_1, s^*(p, \beta)$ is increasing within these income ranges $\beta < \beta_m, \beta_M < \beta$. Then *(H.0)* applies : the right hand side of (4.2) *a)* goes to 0 ($s^*(p, \beta)$ decreases to 0) when $\beta < \beta_m$ goes down to 0, and *b)* goes to $+\infty$ ($s^*(p, \beta)$ goes up to 1) when $\beta_M < \beta$ tends to $+\infty$. Therefore, for any arbitrary thresholds $0 < s_0^* < s_1^* < 1$, there exist β_0^* low enough and β_1^* large enough such that *(H.0)* applies.

Then Proposition 1 applies for uniform distributions ($\varepsilon = 0$). For increasing exponential densities $\varepsilon > 0$, Proposition 2 applies as well provided that ρ_2 is significantly greater than ρ_1. Indeed one gets from (4.2) for any $\beta \leqq \beta_0^*$

$$s^*(p,\beta) \leqq \beta^{(\rho_2-\rho_1)/\rho_1} d^{1/\rho_1}, \tag{4.3}$$

which is bounded above by $a\beta^{\gamma\varepsilon}$ for some $\gamma > 1$ as in (3.7) (provided that, without loss of generality, $\beta_0^* < 1$) if $(\rho_2 - \rho_1)/\rho_1 \geqq \gamma\varepsilon > \varepsilon$.

As an illustration of the impact of large supports of the behavioral heterogeneity parameter density $g(\alpha)$, one may note that the limit value $\varepsilon > 0$ in Proposition 2 for the aggregate income elasticity $b\dfrac{\partial S}{\partial b}(p,b)/S(p,b)$, can be somewhat different from the underlying individual base income elasticity of $s^*(p,\beta)$ for low incomes : the latter goes, from (4.2), to $(\rho_2 - \rho_1)/\rho_1 > \varepsilon$ when β goes to 0. This occurs despite the fact that the distributions considered in Proposition 2 give increasing weight on large heterogeneity parameters α, hence on low income $e^{-\alpha}b$, when $\varepsilon > 0$.

Multicommodities markets

The core economic mechanism, in the foregoing two commodities CES example, rests on a simple property of the marginal rate of substitution $MRS_{12}(x_1,x_2) = \dfrac{\partial U}{\partial x_1}(x_1,x_2)/\dfrac{\partial U}{\partial x_2}(x_1,x_2)$ when expenditures are fixed proportions of variable income β, $x_1 = \mu_1\beta, x_2 = \mu_2\beta$, i.e. $\mu_1 = s/p_1, \mu_2 = (1-s)/p_2$ where commodity $1'$s budget share s is fixed. If $\rho_2 > \rho_1$

$$MRS_{12}(\mu_1\beta,\mu_2\beta) = \beta^{\rho_2-\rho_1}(a_1\mu_2^{\rho_2}/a_2\mu_1^{\rho_1}) \tag{4.4}$$

goes to 0 when $\beta < \beta_m$ goes to 0, to $+\infty$ when $\beta_M < \beta$ goes up to $+\infty$. Then for low incomes, one needs to lower commodity $1's$ consumption (decrease s) to reestablish the optimum equality of the marginal rate of substitution MRS_{12} with the given price ratio p_1/p_2, and a similar symmetric argument applies for large incomes. Hence the *(H.0)* pattern that arises in such a case. We consider now a similar mechanism in a multicommodity context.

We use a simple two steps procedure. Every commodity bundle $x = (x_1,\ldots,x_n) \geqq 0$ is split in two groups $x = (x^1,x^2)$ with $x^1 = (x_1,\ldots,x_i), x^2 = (x_{i+1},\ldots,x_n)$. And similarly for the price vector $p = (p_1,\ldots,p_n) \gg 0$, i.e. $p = (p^1,p^2)$ with $p^1 = (p_1,\ldots,p_i), p^2 = (p_{i+1},\ldots,p_n)$. The base expenditures resulting from maximizing $U(x)$ subject to $p.x = \beta$ for $\beta < \beta_m$ and $\beta_M < \beta$, we focus on the budget share allocated to the first group of commodities, which is noted $s^*(p,\beta) = \sum_{h=1}^{i} s_h^*(p,\beta)$. Let β_1,β_2 be a preliminary spending allocation for the two groups, with $\beta_1 + \beta_2 < \beta_m$

or $\beta_M < \beta_1 + \beta_2$, and define the corresponding "indirect" utility by the constrained maximization

$$V(p, \beta_1, \beta_2) = Max \;\; U(x^1, x^2)/p^1.x^1 = \beta_1, p^2.x^2 = \beta_2. \tag{4.5}$$

Under standard regularity conditions on $U(x)$ (strict monotonicity and concavity, smoothness, interior solutions), the "indirect" utility $V(p, \beta_1, \beta_2)$ will display the same regularity properties. Further, the original optimum allocation of total income β among the two groups will be the result of maximizing $V(p, \beta_1, \beta_2)$ under the total budget constraint $\beta_1 + \beta_2 = \beta$. The corresponding optimum condition is then

$$\frac{\partial V}{\partial \beta_1}(p, \beta_1, \beta_2) = \frac{\partial V}{\partial \beta_2}(p, \beta_1, \beta_2). \tag{4.6}$$

The interpretation of this two steps procedure follows standard lines. The first stage (4.5) allocates expenditures so as to equalize, *within each group of commodities*, marginal utilities of additional spending

$$\begin{aligned} \frac{\partial V}{\partial \beta_1}(p, \beta_1, \beta_2) &= \frac{1}{p_h}\frac{\partial U}{\partial x_h}(x), \quad h = 1, \ldots, i, \\ \frac{\partial V}{\partial \beta_2}(p, \beta_1, \beta_2) &= \frac{1}{p_k}\frac{\partial U}{\partial x_k}(x), \quad k = i+1, \ldots, n, \end{aligned}$$

under the budget constraints $p^1.x^1 = \beta_1, p^2.x^2 = \beta_2$. These marginal utilities are then equalized, *accross the two groups*, in the second stage (4.6), so as to reach the original optimum allocation obtained by maximizing directly $U(x)$ subject to $p.x = \beta$. The outcome of (4.6), under the budget constraint $\beta_1 + \beta_2 = \beta$, yields then the optimum budget share, $\beta_1/\beta = s^*(p, \beta), \beta_2/\beta = 1 - s^*(p, \beta)$.

On expects that, similarly to our two commodities CES example, the *(H.O)* pattern and the conditions of Propositions 1,2 with $\varepsilon \geqq 0$, will obtain if the "indirect" utility's marginal rate of substitution

$$MRS_{12}(p, \beta_1, \beta_2) = \frac{\partial V}{\partial \beta_1}(p, \beta_1, \beta_2)/\frac{\partial V}{\partial \beta_2}(p, \beta_1, \beta_2) \tag{4.7}$$

becomes low for small incomes $\beta_1 + \beta_2 < \beta_m$ and large for high incomes $\beta_M < \beta_1 + \beta_2$, along lines analogous to (4.4).

Proposition 3. *Let the indirect utility function $V(p, \beta_1, \beta_2)$ be defined as in (4.5).*

1) *Assume that for any fixed budget share $0 < s < 1$ of the first group of commodities, the marginal rate of substitution $MRS_{12}(p, \beta_1, \beta_2)$ in (4.7), with $\beta_1 =$*

$s\beta, \beta_2 = (1-s)\beta$, tends : a) to 0 when $\beta < \beta_m$ goes down to 0, b) to $+\infty$ when $\beta_M < \beta$ increases without bound. Then for any thresholds $0 < s_0^* < s_1^* < 1$, there exist $0 < \beta_0^* < \beta_m < \beta_M < \beta_1^*$, with β_0^* small enough and β_1^* large enough, such that (H.0) is satisfied. Then Proposition 1 applies when $\varepsilon = 0$, as well as the first part of Proposition 2. a) when $\varepsilon > 0$, under condition (3.6).

2) *Assume in addition, when $\varepsilon > 0$*

$$MRS_{12}(p, a\beta^{\gamma\varepsilon+1}, \beta) \to 0 \text{ when } \beta \leqq \beta_0^* \text{ tends to } 0 \tag{4.8}$$

for some $a > 0, \gamma > 1$. Then for β_0^* small enough, condition (3.7) is satisfied and the whole Proposition 2.a) applies.

The argument of the proof is straightforward. Details are given in the appendix. A similar symmetric argument can be easily designed to generate the *(H.1)* pattern, to be applied to decreasing densities of the heterogeneity parameter α, i.e. $\varepsilon \leqq 0$.

4.2 Aggregate portfolio choices

We consider the aggregate market behavior of a centinuum of agents who have an identical income b to invest in several risky securities and a single riskfree asset. The issue we investigate is : under which conditions the aggregate amount invested in the risky assets is an increasing proportion of their income. Or equivalently, when the aggregate amount invested in the riskfree asset is a decreasing proportion of their income. It has been well known since the work of Arrow (1970, Ch.3), that in the case of a single risky security and of a single expected utility maximizing investor, such a behavior would obtain if the agent's relative degree of risk aversion was a decreasing function of her wealth. But it has been also known since then that such an investment behavior was no longer true in the case of several risky securities (Cass and Stiglitz (1970, 1972), Hart (1974)). Without mentioning the impact of modern "behavorial" departures from expected utility maximization...

We show that swithching to an aggregate viewpoint with heterogeneous agents allows to make the above portfolio property to be valid even with many risky assets when there are no personal outside income insurance motives (as in the standard frameworks quoted above), and when the available risky securities still imply incomplete markets. We require individual "base" behavior, as measured by the budget shares function $s_h^*(p, \beta)$ of every asset, to be the result of consistent expected utility maximization only for quite low incomes $\beta < \beta_m$, and quite large ones $\beta_M < \beta$, while allowing for any "behavioral" departure (loss aversion, ambiguity or whatever...) in the medium income range $\beta_m \leqq \beta \leqq \beta_M$. Variations of relative degrees of risk aversion are needed only "in the large", i.e. we assume high relative degrees of risk aversion for small incomes β, and a very low one (risk neutrality) for quite large incomes β, while anything can happen in medium income ranges. The proportion of aggregate investment in the riskless security is then shown to be decreasing with the investors' (identical) wealth b for large heterogeneous populations as in Propositions 1,2 above, in the *(H.1)* pattern, with $\varepsilon \leqq 0$.

Formally, there are n risky securities available today. A unit of any such asset $h = 1, \ldots, n$ yields $d_{hj} > 0$ units of income tomorrow, that may depend on the occurrence of some event $j = 1, \ldots, J$. Its price today is $q_h > 0$, the corresponding gross rates of return being $R_{hj} = d_{hj}/q_h > 0$. There is a single riskfree asset, a unit of which generates a sure unit of income tomorrow. Its current unit price is $q_0 > 0$, the corresponding sure gross rate of return being $R_0 = 1/q_0 > 0$. The current asset prices vector is noted $p = (q_0, q_1, \ldots, q_n) \gg 0$. We assume that every portfolio of the risky securities involves some possible loss by comparison to the riskfree asset.

(Incomplete asset markets) *The exists $\Delta R_m < 0$ such that for every $\lambda_h \geqq 0$ with $\sum_{h=1}^{n} \lambda_h = 1$, one has for some $j = 1, \ldots, J$,*

$$\sum_{h=1}^{n} \lambda_h R_{hj} - R_0 < \Delta R_m < 0. \tag{4.9}$$

The "base" behavior of individual investors is described by the non-negative proportions of the variable income $\beta > 0$ invested in the riskless asset, $s = s^*(p, \beta) \geqq 0$, and in the risky ones, $s_h = s_h^*(p, \beta) \geqq 0$, with $s + \sum_{h=1}^{n} s_h = 1$. As in section 2, we consider a continuum of agents with a common income b, whose actual portfolio choices $s(\alpha, p, b), s_h(\alpha, p, b)$, are affected by a behavioral heterogeneity scaling factor $\lambda = e^{\alpha}$, as in (2.2), and look at the proportion of their aggregate investments in the various assets $S(p, b), S_h(p, b)$, defined as in (2.4). We investigate then, within the framework of section 3, conditions ensuring that the aggregate budget share invested in the riskless asset, $S(p, b)$, decreases with income b. To do so, we need to focus on the case where the individual "base" budget share of the riskfree security, $s^*(p, \beta)$, decreases "in the large", as in the *(H.1)* pattern.

We assume that for low incomes, $0 < \beta < \beta_m$, and large ones, $\beta_M < \beta$, where $0 < \beta_m < \beta_M$ are given thresholds, individual "base" choices of a portfolio $s = s^*(p, \beta)$, $s_h = s_h^*(p, \beta)$, are driven by the maximization of a standard VNM expected utility $E[u(y_j)] = \sum_{j=1}^{J} \pi_j u(y_j)$, where $\pi_j > 0$ are the subjective probabilities of occurrence of each state j, with $\sum_{j=1}^{J} \pi_j = 1$, subject to $s + \sum_{h=1}^{n} s_h = 1$. The incomes generated by such a portfolio, are given by, for each state $j = 1, \ldots, J$ (there is no exogenous risky income, so no outside income insurance motive).

$$y_j = (sR_0 + \sum_{h=1}^{n} s_h R_{hj})\beta = R_0\beta + \sum_{h=1}^{n} (R_{hj} - R_0) s_h \beta. \tag{4.10}$$

On the other hand, the corresponding budget shares $s(\alpha, p, b), s_h(\alpha, p, b)$ associated to the heterogeneity scaling factor $\lambda = e^{\alpha}$, as in (2.2), can be viewed as resulting from the maximization of the expected utility $E[u(e^{-\alpha} y_j)]$ under the same constraints in similar income ranges. We make the standard "rationality" assumption for all low income ranges $y_j < y_m$ and large ones $y_M < y_j$ generated by arbitrary portfolios (s, s_h) as in (4.10) with $\beta < \beta_m$ and $\beta_M < \beta$ respectively :

(U) *The VNM utility function $u(y)$ is, on the relevant income ranges $y < y_m$ and $y_M < y$, continuous, twice continously differentiable for $y > 0$, increasing ($u'(y) > 0$, with $\lim_{y \to 0} u'(y) = +\infty$), strictly concave ($u''(y) < 0$).*

As a counterpart of (4.9), the following assumption states that risky securities are nevertheless financially attractive.

There is at least one risky asset h such that

$$E[R_s - R_0] = \sum_{j=1}^{J} \pi_j (R_{hj} - R_0) > 0. \tag{4.11}$$

Proposition 4. *Let the assumptions (U), (4.9) and (4.11) hold.*

1) Assume that investors are risk neutral for high incomes :

$$\textit{There is } y_1^* > y_M \textit{ such that } \rho(y) = -yu''(y)/u'(y) = 0 \textit{ for } y > y_1^*. \tag{4.12}$$

Then there exists $\beta_1^* > \beta_M$ such that $s^*(p, \beta) = 0$ for all $\beta \geqq \beta_1^*$.

2) Assume that investors are very much risk averse for low incomes :

$$\rho(y) \to +\infty \textit{ when } y \to 0. \tag{4.13}$$

Then for any $0 < s_0^* < 1$, there exists $0 < \beta_0^* < \beta_m$ such that $s^*(p, \beta) > s_0^*$ for all $\beta \leqq \beta_0^*$.

3) Under assumptions (4.12), (4.13), the (H.1) pattern applies with $1 > s_0^* > s_1^* = 0$ as above. For any exponential distribution of the heterogeneity scaling parameter α and any investors' common income b satisfying (H), the aggregate budget share invested in the riskless security, $S(p,b)$, is decreasing in b when $\varepsilon \leqq 0$. Its elasticity with respect to b tends to ε when $\alpha_1 \to +\infty$.

A proof is given in the appendix. As a concluding remark, it may be worth re-emphasizing that the standard "rationality" expected utility assumption was used here only for low and large incomes, as in *(U)*. And this only to assume formally risk neutrality for large incomes, as in (4.12), or a relative degree of risk aversion that tends to $+\infty$ (not necessarily monotonically !) for low incomes as in (4.13). As a matter of fact, a cursory look at the argument in the proof suggests strongly that the driving mechanism to derive the above results 1) and 2) (and hence the whole Proposition from Propositions 1.b), 2.b)) is the investors' *degree of loss aversion*, that is absent for high incomes and becomes infinite for low incomes. On the other hand, the whole analysis is compatible with any "behavioral rationality departure" in the medium range incomes.

4.3 Concave/convex aggregate expenditures

We extend the results of section 3 to higher order derivatives, namely concavity/convexity of aggregate budget shares $S(p,b)$ and expenditures $W(p,b)$ with respect to income b. We maintain throughout our assumption (3.1) of exponential distributions for the heterogeneity parameter $\alpha = \log\lambda$ (i.e. Pareto distributions of the scaling factor λ). It is convenient to reformulate the income derivative of aggregate expenditure (3.10) as

$$b\frac{\partial W}{\partial b}(p,b) = W(p,b)(1+\varepsilon) - \gamma_0[e^{\alpha\varepsilon}w(\alpha,p,b)]_{\alpha_0}^{\alpha_1}, \tag{4.14}$$

with the maintained notation $[f(\alpha)]_{\alpha_0}^{\alpha_1} = f(\alpha_1) - f(\alpha_0)$.

Lemma 2. *Differentiation with respect to income b, of (3.5) and (4.14), yields*

$$\begin{aligned} b\frac{\partial^2 S}{\partial b^2}(p,b) &= (\varepsilon-1)\frac{\partial S}{\partial b}(p,b) - \gamma_0[e^{\alpha(\varepsilon-1)}\frac{\partial s^*}{\partial \beta}(p,e^{-\alpha}b)]_{\alpha_0}^{\alpha_1}, & (4.15)\\ b\frac{\partial^2 W}{\partial b^2}(p,b) &= \varepsilon\frac{\partial W}{\partial b}(p,b) - \gamma_0[e^{\alpha\varepsilon}\frac{\partial w^*}{\partial \beta}(p,e^{-\alpha}b)]_{\alpha_0}^{\alpha_1}. & (4.16)\end{aligned}$$

In what follows, as in section 3, we let assumption *(H)* hold, while the price system $p \gg 0$ and the agents' common income $e^{\alpha_0}\beta_1^* < b < e^{\alpha_1}\beta_0^*$ are given.

We keep assumption *(H.0)*, or *(H.1)*, that enabled us to evaluate the sign and magnitude of $\dfrac{\partial S}{\partial b}$ from a comparison of individual budget shares $s^*(p,\beta)$ for low and large incomes, i.e. for $\beta \leq \beta_0^*$ versus $\beta_1^* \leq \beta$ for some appropriate thresholds $\beta_0^* < \beta_1^*$. Our strategy is also here to compare the income derivatives of individual "base" budget shares $\dfrac{\partial s^*}{\partial \beta}(p,\beta)$, or expenditures $\dfrac{\partial w^*}{\partial \beta}(p,\beta)$, for low and large incomes β. We focus on budget shares through assumptions *(D.0)*, or *(D.1)* below. The individual "base" income derivative $\dfrac{\partial s^*}{\partial \beta}(p,\beta)$ will play accordingly here, through these conditions, a role similar to that of $s^*(p,\beta)$ in section 3 through *(H.0)*, *(H.1)*. The main technical difference being that, by construction, the budget share $s^*(p,\beta)$ is bounded since it lies in [0,1], whereas the income derivative needs not.

(D.0) *(Increasing individual budget share's income derivative "in the large") Given* $p \gg 0$ *and* $\beta_0^* < \beta_1^*$ *as in (H.0) or (H.1), there exist* $\sigma_0^* < \sigma_1^*$ *such that* $\dfrac{\partial s^*}{\partial \beta}(p,\beta) \leqq \sigma_0^*$ *for* $0 < \beta < \beta_0^*$*, while* $\sigma_1^* \leqq \dfrac{\partial s^*}{\partial \beta}(p,\beta)$ *for* $\beta_1^* \leqq \beta$.

(D.1) *(Decreasing individual budget share's income derivative "in the large") Given* $p \gg 0$ *and* $\beta_0^* < \beta_1^*$ *as in (H.0) or (H.1), there exist* $\sigma_0^* > \sigma_1^*$ *such that* $\dfrac{\partial s^*}{\partial \beta}(p,\beta) \geqq \sigma_0^*$ *for* $0 < \beta \leqq \beta_0^*$*, while* $\sigma_1^* \geqq \dfrac{\partial s^*}{\partial \beta}(p,\beta)$ *for* $\beta_1^* \leqq \beta$.

We focus essentially on one configuration in the main text here, where the aggregate budget share $S(p,b)$ is locally decreasing and convex in income (Proposition 5 below, under assumptions *(H.1)*, *(D.0)* and a decreasing exponential heterogeneity distribution $g(\alpha)$, with $\varepsilon \leqq 0$). Such a configuration seems indeed of specific potential interest, as it can generate aggregate expenditures $W(p,b)$ that are increasing and concave (Corollary 2 below, with $-1 < \varepsilon < 0$), i.e. an aggregate (multicommodities) "concave consumption function", a topic that has attracted a significant attention (see, e.g. Caroll and Kimball (1996), Gourinchas and Parker (2002)). Two other interesting configurations are considered in the appendix, where the aggregate budget share $S(p,b)$ is 1) locally increasing and convex (Proposition 6, under assumptions *(H.0)*, *(D.0)* and a significantly increasing exponential heterogeneity distribution, with $\varepsilon \geqq 1$) and 2) locally increasing and concave (Proposition 7, under assumptions *(H.0)*, *(D.1)* and a moderately increasing exponential heterogeneity distribution with $0 \leqq \varepsilon \leqq 1$).

Proposition 5. *(Locally decreasing and convex aggregate budget share)*
Let (H.1) hold, as well as (D.0) with $\sigma_0^* < \sigma_1^* \leqq 0$*, and consider a decreasing exponential heterogeneity distribution with* $\varepsilon \leqq 0$.

1) One has $b\dfrac{\partial^2 S}{\partial b^2}(p,b) > (\varepsilon - 1)\dfrac{\partial S}{\partial b}(p,b)$ *if* σ_1^* *is close enough to 0, i.e.*

$$[\log|\sigma_1^*| - \log|\sigma_0^*|]/(\alpha_1 - \alpha_0) < \varepsilon - 1. \tag{4.17}$$

The aggregate budget share is then locally strictly decreasing, $\dfrac{\partial S}{\partial b}(p,b) < 0$, *and strictly convex,* $b\dfrac{\partial^2 S}{\partial b^2}(p,b) > 0$, *if either* $\varepsilon = 0$, *or under condition (3.8) when* $\varepsilon < 0$.

2) Assume that the individual "base" budget share is non-increasing with income, $\dfrac{\partial s^*}{\partial \beta}(p,\beta) \leqq 0$, *for all* $\beta > 0$. *Then the aggregate income elasticity,* $b\dfrac{\partial^2 S}{\partial b^2}(p,b)/\dfrac{\partial S}{\partial b}(p,b)$, *converges uniformly to* $\varepsilon - 1 \leqq -1$ *when* $\alpha_0 \to -\infty, \alpha_1 \to +\infty$, *if* $|\dfrac{\partial s^*}{\partial \beta}(p,\beta)|$

a) goes fast enough to 0 when β *tends to* $+\infty$, *and b) is bounded above or does not diverge to* $+\infty$ *too fast when* β *goes down to* 0 :

$$\begin{aligned} a)\ \text{for } \beta \geqq \beta_1^*, |\frac{\partial s^*}{\partial \beta}(p,\beta)| &\leqq a_1\beta^{\nu_1(\varepsilon-1)} \text{ with } a_1 \geqq 0, \nu_1 > 1, && (4.18)\\ b)\ \text{for } 0 \leqq \beta_0^*, |\frac{\partial s^*}{\partial \beta}(p,\beta)| &\leqq a_0\beta^{\nu_0(\varepsilon-1)} \text{ with } a_0 > 0, 0 \leqq \nu_0 < 1. && (4.19)\end{aligned}$$

A detailed proof is given in the appendix. On the other hand, it is immediate to verify that when the aggregate budget share income elasticity $b\dfrac{\partial^2 S}{\partial b^2}(p,b)/\dfrac{\partial S}{\partial b}(p,b)$ converges to δ, the corresponding income elasticity for aggregate expenditure $b\dfrac{\partial^2 W}{\partial b^2}(p,b)/\dfrac{\partial W}{\partial b}(p,b)$ converge to $1+\delta$. One should expect accordingly that a moderately decreasing exponential heterogeneity distribution with $-1 < \varepsilon < 0$, should lead to an increasing and concave (again, multidimentional) aggregate expenditure.

Corollary 2. *(Locally increasing and concave aggregate expenditure)*

Let the assumptions of Proposition 5 above and of Corollary 1 at the end of section 3 hold, with $-1 < \varepsilon < 0$, *and let* $\alpha_0 \to -\infty, \alpha_1 \to +\infty$.

a) Under condition (3.9), the aggregate income elasticity $b\dfrac{\partial W}{\partial b}(p,b)/W(p,b) \to 1 + \varepsilon > 0$.

b) The aggregate income elasticity $b\dfrac{\partial^2 W}{\partial b^2}(p,b)/\dfrac{\partial W}{\partial b}(p,b) \to \varepsilon < 0$.

5 Conclusion

The analysis presented in this paper shows that exploiting further the parametric structure of heterogeneity preference scaling factors, here by assuming Pareto distributions, may be a significantly fruitful research avenue to get sharp monotonicity properties of macroeconomic aggregates. This allowed us to go well beyond earlier results about possible "insensitivity" properties of aggregate expenditures obtained with less specific notions of "behavioral heterogeneity", recalled in the introduction. In particular, the application made here to the study of wealth effects on aggregate portfolio choices, shows that this approach enables us to get neat aggregate monotonicity properties that are usually hard, or even impossible, to get at the individual level from standard microeconomic (expected utility maximizing, ambiguity,...) theory in financial markets.

The theoretical analysis of this paper focussed on a single group of agents displaying heterogeneous behavior generated from a single individual "base" expenditure function and a single homothetic heterogeneity scaling factor Pareto distribution. Clearly, it can be easily extended to encompass other possible sources of heterogeneity, e.g. by considering several groups of such individuals, these groups $i = 1, \ldots, n$ having relative sizes $\mu_i > 0$ with $\sum_{i=1}^{n} \mu_i = 1$, and being endowed with individual incomes $b_i > 0$, individual "base" expenditure functions and heterogeneity factor distributions that are possibly different. For a given income distribution with $b = \Sigma_i \mu_i b_i$, the elasticity of the aggregate budget share with respect to aggregate income b, can be easily deduced from the results of this paper if each group fits our framework (of course, its relation with each group's aggregate elasticity with respect its income b_i will have to take into account covariance, as noted e.g. by Paluch, Kneip and Hildenbrand (2012)). Further, the impact of a (local) change of the income distribution, given aggregate income, can also be easily derived within such an extended framework. Such an extended model is then able to cope with a wide variety of observable behavioral heterogeneities, that have been so much documented in the empirical literature, as reviewed in the introduction, in various frameworks (goods and services, portfolio choices, ...).

Preliminary research work suggests also that the methodological approach of this paper, which focussed as a starting point on aggregate income elasticities, can presumably also be applied fruitfully to price elasticities, by using commodity specific scaling factors, as in Grandmont (1992, 1993). This may open the possibility to extend the analysis to general equilibrium frameworks, with perfect of imperfect competition.

6 References

Arrow, K.J. (1970), *Essays in the Theory of Risk Bearing*, Chap. 3, Markham, Chicago.

Barten, A.P. (1964), "Family Composition, Prices and Expenditure Patterns", in *Econometric Analysis for National Economic Planning : 16th Symposium of the Colston Society*, Hart, P., Mills, G. and J.D. Whitaker (eds.), Butterworth, London, 277-292.

Barten, A.P. (1977), "The Systems of Consumer Demand Functions Approach : A Review", in *Frontiers of Quantitative Economics*, Vol 3, Part A, Intriligator, M.D. (ed.), North-Holland, Amsterdam, 23-58.

Becker, G.S. (1962), "Irrational Behavior and Economic Theory", *Journal of Political Economy, 70,* 1-13.

Blundell, R. W. and A. Lewbel (1991), "The Information Content of Equivalence Scales", *Journal of Econometrics, 50*, 49-68.

Blundell, R. W. and T. M. Stoker (2005), "Heterogeneity and Aggregation", *Journal of Economic Literature, 43 (2)*, 347-391.

Calvet, L. E., Campbell, J.Y. and P. Sodini (2007), "Down or Out : Assessing the Welfare Cost of Households Investment Mistakes", *Journal of Political Economy, 115(5)*, 707-747.

Calvet, L.E., Campbell, J.Y. and P. Sodini (2009), "Measuring the Financial Sophistication of Households", *American Economic Review, Papers and Proceedings, 99 (2)*, 393-398.

Calvet, L.E. and E. Comon (2003), "Behavioral Heterogeneity and the Income Effect", *Review of Economics and Statistics, 85*, 653-669.

Calvet, L.E. and P. Sodini (2014), "Twin Picks : Disentangling the Determinants of Risk-Taking in Households Portfolio", *Journal of Finance, 69(2)*, 867-906.

Carroll, C.D. and M.S. Kimball (1996), "On the Concavity of the Consumption Function", *Econometrica, 64*, 981-992.

Cass, D. and J.E. Stiglitz (1970), "The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation : A Contribution to the

Pure Theory of Mutual Funds", *Journal of Economic Theory, 2(2)*, 122-160.

Cass, D. and J.E. Stigliz (1972), "Risk Aversion and Wealth Effects on Portfolios with Many Assets", *Review of Economic Studies, 39(3)*, 331-354.

Chan, Y. L. and L. Kogan (2002), "Catching up with the Joneses : Heterogeneous Preferences and the Dynamics of Asset Prices", *Journal of Political Economy, 110(6)*, 1255-1285.

Curcuru, S., Heaton, J., Lucas, D. and D. Moore (2010), "Heterogeneity and Portfolio Choice : Theory and Evidence", *Handbook of Financial Econometrics*, Vol 11, Ch.6, 337-382, Ait Sahalia, Y. and L.P. Hansen (eds.), North Holland, Amsterdam.

Deaton, A. and J. Muellbauer (1980), *Economics and Consumer Behavior*, Cambridge University Press, Cambridge.

Dierker, H., Dierker, E. and W. Trockel (1984), "Price-Dispersed Preferences and C^1 Mean Demand", *Journal of Mathematical Economics, 13*, 11-42.

Gabaix, X. (2009), "Power Law in Economics and Finance", *Annual Review of Economics, 1*, 255-293.

Giraud, G. and I. Maret (2002), "Behavioral Heterogeneity in Large Economies", BETA DP 2002-04, University of Strasbourg.

Grandmont, J.-M. (1987), "Distributions of Preferences and the Law of Demand", *Econometrica, 55*, 155-161.

Grandmont, J.-M. (1992), "Transformations of the Commodity Space, Behavioral Heterogeneity and the Aggregation Problem", *Journal of Economic Theory, 57*, 1-35.

Grandmont, J.-M. (1993), "Behavioral Heterogeneity and Cournot Oligopoly Equilibrium", *Ricerche Economiche, 47*, 167-187.

Hildenbrand, W. (1983), "On the Law of Demand", *Econometrica, 51,* 997-1019.

Hildenbrand (1994), *Market Demand : Theory and Empirical Evidence*, Princeton University Press.

Hildenbrand, W. and A. Kneip (2005), "On Behavioral Heterogeneity", *Eco-*

nomic Theory, 25, 155-169.

Hart, O.D. (1974), "Some Negative Results on the Existence of Comparative Statics Results in Portfolio Theory", *Review of Economic Studies, 42(4),* 615-621.

Jerison, M. (1999), "Dispersed Excess Demands, the Weak Axiom and Uniqueness of Equilibrium", *Journal of Mathematical Economics, 31* 15-48.

Jorgenson, D.W. and D.T. Slesnik (1987), "Aggregate Consumer Behavior and Household Equivalence Scales", *Journal of Business and Economics Statistics, 5,* 219-232.

Kirman, A. (2006), "Heterogeneity in Economics", *Journal of Economic Interactions and Coordination, 1(1),* 89-117.

Kneip, A. (1999), "Behavioral Heterogeneity and Structural Properties of Aggregate Demand", *Journal of Mathematical Economics, 31*, 49-79.

Lewbel, A. (1997), "Consumer Demand Systems and Household Equivalence Scales", *Handbook of Applied Econometrics : Vol. II : Microeconomics,* Pesaran, M.H. and P. Schmidt, eds., Blackwell, Oxford, 167-201.

Lewbel A. (2007), "Modeling Heterogeneity" in Blundell, R., Newey, W. and T. Persson, eds., *Advances in Economics and Econometrics, Theory and Applications : 9th World Congress of the Econometric Society,* Vol. 3, Ch.5, Econometric Society Monographs, 43, Cambridge University Press, 111-119.

Lewbel, A. and K. Pendakur (2008), "Equivalence Scales", Entry for the *New Palgrave Dictionary of Economics*, 2nd Edition, McMillan, Durlauf, S. and L.E. Blume, eds.

Lewbel, A. and K. Pendakur (2017), "Unobserved Preference Heterogeneity in Demand Using Generalized Random Coefficients", *Journal of Political Economy, 125(4),* 1100-1148.

Mas-Colell, A. and W. Neuefind (1977), "Some Generic Properties of Aggregate Excess Demand and an Application", *Econometrica, 45,* 591-599.

Matzkin, R.L. (2007), "Heterogeneous Choice", in Blundell, R., Newey, W. and T. Persson, eds., *Advances in Economics and Econometrics, Theory and Applications : 9th Word Congress of the Econometric Society,* Vol.3, Ch.4, Econometric Society Monographs, 43, Cambridge University Press, 75-110.

McFadden, D. (2001), "Economic Choices", *American Economic Review, 91(3)*, 351-378.

Muellbauer, J. (1977), "Testing the Barten Model of Household Composition Effects and the Cost of Children", *Economic Journal, 87,* 460-487.

Muellbauer, J. (1980), "The Estimation of the Prais-Houthakker Model of Equivalence Scales", *Econometrica, 48,* 153-176.

Prais, S.J. and H.S. Houthakker (1971), *The Analysis of Family Budgets,* 2nd Ed., Cambridge University Press.

Paluch, M., Kneip, A. and W. Hildenbrand (2012), "Individual versus Aggregate Income Elasticities for Heterogeneous Populations", *Journal of Applied Econometrics, 27,* 847-869.

Quah, J. (1997), "The Law of Demand when Income is Price Dependent", *Econometrica*, 65, 1421-1142.

Quah, J. (2001), "Demand is Heterogeneous in Grandmont's Model", Nuffield College, Working Paper, W12.

Appendix

1. *Proof of Proposition 1*

a) Under *(H.0)*, (3.2) implies

$$b\frac{\partial S}{\partial b}(p,b) \geqq [s_1^* - s_0^*]/(\alpha_1 - \alpha_0) > 0.$$

Moreover $S(p,b) \geqq s_1^*(\log(\frac{b}{\beta_1^*}) - \alpha_0)/(\alpha_1 - \alpha_0) > 0$. Combining this inequality with (3.2) yields immediately (3.3), hence the result.

b) Under *(H.1)*, (3.2) implies

$$b\frac{\partial S}{\partial b}(p,b) \leqq [s_1^* - s_0^*]/(\alpha_1 - \alpha_0) < 0.$$

Moreover $S(p,b) \geqq s_0^*(\alpha_1 - \log(\frac{b}{\beta_0^*}))/(\alpha_1 - \alpha_0) > 0$. Combining this inequality with (3.2) yields immediately (3.4), hence the result. Q.E.D.

2. *Proof of Proposition 2*

a) Case $\varepsilon > 0$. Under *(H.0)*, (3.5) implies

$$b\frac{\partial S}{\partial b}(p,b) - \varepsilon S(p,b) \geqq \gamma_0[e^{\alpha_0\varepsilon}s_1^* - e^{\alpha_1\varepsilon}s_0^*].$$

Since $\gamma_0 > 0$, the right hand side will be positive if and only if (3.6) is satisfied (including the limiting case where $s_0^* = 0$, or $\log s_0^* = -\infty$).

When the support $[\alpha_0, \alpha_1]$ becomes large, with $\alpha_0 < \log(b/\beta_1^*), \log(b/\beta_0^*) < \alpha_1$, a lower bound for $S(p,b)$ is given by

$$\begin{aligned} S(p,b) &\geqq \int_{\alpha_0}^{\log(b/\beta_1^*)} s_1^*\gamma_0 e^{\varepsilon\alpha} d\alpha \\ &\geqq s_1^*\frac{\gamma_0}{\varepsilon}[(\frac{b}{\beta_1^*})^\varepsilon - e^{\alpha_0\varepsilon}] > 0. \end{aligned} \tag{A.1}$$

One gets then from (3.5)

$$\left| \frac{b\frac{\partial S}{\partial b}(p,b)}{S(p,b)} - \varepsilon \right| \leqq \varepsilon \frac{|e^{\alpha_0\varepsilon}s^*(p,e^{-\alpha_0}b) - e^{\alpha_1\varepsilon}s^*(p,e^{-\alpha_1}b)|}{s_1^*[(\frac{b}{\beta_1^*})^\varepsilon - e^{\alpha_0\varepsilon}]} \tag{A.2}$$

Since $s^*(p, e^{-\alpha_0}b) \geqq s_1^* > 0$, one needs to have $\alpha_0 \to -\infty$ to make the right hand side of (A.2) go to 0. This condition is actually sufficient when $s_0^* = 0$, hence $s^*(p, e^{-\alpha_1}b) = 0$, even when $\alpha_1 > \log(b/\beta_0^*)$ remains bounded above. Under the more general condition (3.7), where one may have $a > 0$, one gets

$$\left| \frac{\frac{b\partial S}{\partial b}(p,b)}{S(p,b)} - \varepsilon \right| \leqq \varepsilon \frac{e^{\alpha_0\varepsilon} + a e^{\alpha_1\varepsilon(1-\gamma)} b^{\gamma\varepsilon}}{s_1^*[(\frac{b}{\beta_1^*})^\varepsilon - e^{\alpha_0\varepsilon}]} \tag{A.3}$$

which tends uniformly to 0 when $\alpha_0 \to -\infty$ and since $\gamma > 1$, when $\alpha_1 \to +\infty$.

b) Case $\varepsilon < 0$. Under *(H.1)*, (3.5) implies

$$b\frac{\partial S}{\partial b}(p,b) - \varepsilon S(p,b) \leqq \gamma_0[e^{\alpha_0\varepsilon} s_1^* - e^{\alpha_1\varepsilon} s_0^*].$$

Since $\gamma_0 > 0$, the right hand side will be negative if and only if (3.8) is satisfied (including the limiting case where $s_1^* = 0, \text{or } \log s_1^* = -\infty$).

When the support $[\alpha_0, \alpha_1]$ becomes large, with $\alpha_0 < \log(b/\beta_1^*), \log(b/\beta_0^*) < \alpha_1$, a lower bound for $S(p,b)$ is given by

$$\begin{aligned} S(p,b) &\geqq \int_{\log(b/\beta_0^*)}^{\alpha_1} s_0^* \gamma_0 e^{\varepsilon\alpha} d\alpha \\ &\geqq s_0^* \frac{\gamma_0}{\varepsilon}[e^{\alpha_1\varepsilon} - (\frac{b}{\beta_0^*})^\varepsilon] > 0. \end{aligned} \tag{A.4}$$

One gets then from (3.5)

$$\left| \frac{b\frac{\partial S}{\partial b}(p,b)}{S(p,b)} - \varepsilon \right| \leqq |\varepsilon| \frac{|e^{\alpha_1\varepsilon} s^*(p, e^{-\alpha_1}b) - e^{\alpha_0\varepsilon} s^*(p, e^{-\alpha_0}b)|}{s_0^*[(\frac{b}{\beta_0^*})^\varepsilon - e^{\alpha_1\varepsilon}]}. \tag{A.5}$$

Since $s^*(p, e^{-\alpha_1}b) \geqq s_0^* > 0$, one needs to have $\alpha_1 \to +\infty$ to make the right hand side of (A.5) go to 0. This condition is sufficient when $s_1^* = 0$, hence $s^*(p, e^{-\alpha_0}b) = 0$, even when α_0 remains bounded away from $-\infty$. Under the more general condition (3.9), where one may have $a > 0$, one gets

$$\left| \frac{b\frac{\partial S}{\partial b}(p,b)}{S(p,b)} - \varepsilon \right| \leqq |\varepsilon| \frac{e^{\alpha_1\varepsilon} + a e^{\alpha_0\varepsilon(1-\gamma)} b^{\gamma\varepsilon}}{s_0^*[(\frac{b}{\beta_0^*})^\varepsilon - e^{\alpha_1\varepsilon}]} \tag{A.6}$$

which tends uniformly to 0 when $\alpha_1 \to +\infty$ and, since $\varepsilon(1-\gamma) > 0$, when $\alpha_0 \to -\infty$. Q.E.D.

3. *Proof of Proposition 3*

We consider the standard case where $MRS_{12}(p, \beta_1, \beta_2)$ in (4.7) is a decreasing function of β_1 that tends to $+\infty$ when β_1 goes to 0, and an increasing function of β_2 that goes down to 0 when β_2 tends to 0, in the small income configuration $\beta_1 + \beta_2 < \beta_m$. Symmetrically $MRS_{12}(p, \beta_1, \beta_2)$ tends to 0 when $\beta_M < \beta_1$ goes up to $+\infty$, and tends to $+\infty$ when $\beta_M < \beta_2$ increases without bound. For $\beta < \beta_m$ and $\beta_M < \beta$, the optimum individual base budget share $s^*(p, \beta)$ of the first group of commodities is then the unique solution, given β, of

$$MRS_{12}(p, s\beta, (1-s)\beta) = 1, \tag{A.7}$$

where the left hand side of (A.7) is a decreasing function of s that tends to $+\infty$ when s goes down to 0, and tends to 0 when s goes up to 1.

The assumption made in Proposition 3.1) states that $MRS_{12}(p, s\beta, (1-s)\beta)$ is an increasing function of β, given s, that tends to 0 when β goes down to 0. Thus for any $0 < s_0^* < 1$ fixed, there exists β_0^* small enough such that $MRS_{12}(p, s_0^*\beta, (1-s_0^*)\beta)$ is uniformly small, hence less than 1, for all $\beta \leqq \beta_0^*$. Then one needs to lower s to reestablish the optimum of (A.7), i.e. $s^*(p, \beta) < s_0^*$ for all $\beta \leqq \beta_0^*$. A symmetric argument applies for an arbitrary $1 > s_1^* > s_0^*$ and $\beta_1^* > \beta_M$, so that $s_1^* < s^*(p, \beta)$, as in *(H.0)*.

Thus Proposition 1 applies for the case $\varepsilon = 0$. When $\varepsilon > 0$, Proposition 2. *a)* applies as well if condition (3.6) is satisfied. Assume now (4.8) for some $a > 0$ and $\gamma > 1$. One has for $\beta \leqq \beta_0^*$ going to 0

$$MRS_{12}(p, a\beta^{\gamma\varepsilon+1}, (1-a^{\gamma\varepsilon})\beta) < MRS_{12}(p, a\beta^{\gamma\varepsilon+1}, \beta) \to 0. \tag{A.8}$$

So if β_0^* is chosen small enough, the above is small, hence less than 1, which implies that $s^*(p, \beta) < a\beta^{\gamma\varepsilon}$, i.e. (3.7), for all $\beta \leqq \beta_0^*$. Thus Proposition 2. *a)* applies fully for the case $\varepsilon > 0$. Q.E.D.

4. *Proof of Proposition 4*

1) Under (4.12), let $\beta_1^* > \beta_M$ be large enough so that all incomes generated by arbitrary portfolios as in (4.10) satisfy $y_j > y_1^*$. Then for any $\beta \geqq \beta_1^*$, if the "base" budget share invested in the riskfree security at the optimum portfolio $(s, (s_h))$, is positive, $s = s^*(p, \beta) > 0$, one should have $E[(R_{hj} - R_0)u'(y_j)] \leqq 0$ for every risky

asset h, with equality whenever $s_h > 0$, where (y_j) is the resulting optimum income stream (4.10). That would not be possible for the risky security h mentioned in (4.11), since risk neutrality means that $u'(y_j)$ is a constant independent of y_j. Thus $s^*(p, \beta) = 0$ for all $\beta \geqq \beta_1^*$.

2) Let $0 < s_0^* < 1$, an arbitrary "large" budget share invested in the riskfree asset, be fixed. Consider an arbitrarily small income level $0 < \beta < \beta_m$. Let $(s_h(\beta))$ be the budget shares of risky securities that maximize expected utility $E[u(y_j)]$ with $s = s_0^*$ and $\sum_{h=1}^{n} s_h(\beta) = 1 - s_0^*$. Let the resulting income stream $y_j^*(\beta)$ be given by (4.10) :

$$y_j^*(\beta) = R_0\beta + \sum_{h=1}^{n} s_h(\beta)\beta(R_{hj} - R_0) > 0. \tag{A.9}$$

From the corresponding FOC, marginal expected utilities $E[R_{hj}u'(y_j^*(\beta))] > 0$ for additional investment in risky assets h must all be equalized for assets with $s_h(\beta) > 0$, while marginal expected utilities must be lower for those other assets k with $s_k(\beta) = 0$.

Consider now the optimum portfolio budget shares when the riskless security proportion s is free to vary. The necessary and sufficient condition for $s^*(p, \beta) > s_0^*$ is that marginal expected utility of investing in the riskfree asset exceeds that of investing in the risky securities, when these are evaluated at the optimum portfolio $(s_h(\beta))$ obtained when s is fixed at s_0^* :

$$E[\sum_{h=s}^{n} s_h(\beta)\beta(R_{hj} - R_0)u'(y_j^*(\beta))] < 0. \tag{A.10}$$

Taking an exact first order Taylor development of $u'(y_j^*(\beta))$ at $R_0\beta$ and using (A.9) generates the equivalent inequality

$$E[(y_j^*(\beta) - R_0\beta)u'(R_0\beta)] < E[(y_j^*(\beta) - R_0\beta)^2\rho(\theta_j(\beta))u'(\theta_j(\beta))/\theta_j(\beta)]$$

where the income $\theta_j(\beta) > 0$ lies in $[R_0\beta, y_j^*(\beta)]$. Dividing both sides by $\beta u'(R_0\beta) > 0$ and introducing the notation $\Delta R_j(\beta) = \sum_{h=1}^{n} s_h(\beta)(R_{hj} - R_0)$, leads to the equivalent inequality

$$E[\Delta R_j(\beta)] < E[(\Delta R_j(\beta))^2 \frac{\beta}{\theta_j(\beta)} \frac{u'(\theta_j(\beta))}{u'(R_0\beta)} \rho(\theta_j(\beta))]. \tag{A.11}$$

We now make $\beta > 0$ tend to 0 and prove that the right hand side of (A.11) tends to $+\infty$, which implies the existence of a small income level $\beta_0^* > 0$ such that $s^*(p, \beta > s_0^*$ for all $\beta \leqq \beta_0^*$, as claimed. Since $\sum_{h=1}^{n} s_h(\beta) = 1 - s_0^* > 0$, the income $\theta_j(\beta) > 0$ lies in the interval $[R_0\beta, (\sum_{h=1}^{n} s_h(\beta)R_{hj} + s_0^* R_0)\beta]$ and tends thus to 0, but the ratio $\beta/\theta_j(\beta)$ remains bounded away from 0. On the other hand, (4.9) implies the existence of a state j (that may depend on β) such that $\Delta R_j(\beta)(1 - s_0^*) < \Delta R_m < 0$. For that state j, one has $y_j^*(\beta) < R_0\beta$, hence $\theta_j(\beta) \leqq R_0\beta$ in which case $u'(\theta_j(\beta))/u'(R_0\beta) \geqq 1$. If $\pi > 0$ is the minimum of all state probabilities, the right hand side of (A.11) is larger than

$$\pi[\Delta R_m/(1 - s_0^*)]^2 \frac{\beta}{\theta_j(\beta)} \rho(\theta_j(\beta)).$$

Since $\theta_j(\beta)$ goes to 0, $\rho(\theta_j(\beta))$ tends to $+\infty$ by (4.13), so the above goes to $+\infty$ when β goes to 0, which proves 2).

3) The claims made in 3) of Proposition 4 follow immediately, from Proposition 1. *b)* when $\varepsilon = 0$ and Proposition 2. *b)* when $\varepsilon < 0$. Q.E.D.

5. *Proof of Proposition 5*

1) Under *(D.0)*, (4.5) implies

$$b\frac{\partial^2 S}{\partial b^2}(p, b) - (\varepsilon - 1)\frac{\partial S}{\partial b}(p, b) \geqq \gamma_0 \left[e^{\alpha_0(\varepsilon-1)}\sigma_1^* - e^{\alpha_1(\varepsilon-1)}\sigma_0^*\right]. \tag{A.12}$$

If $\sigma_0^* < \sigma_1^* \leqq 0$, since $\gamma_0 > 0$, the right hand side is positive if and only if (4.17) holds (including the limiting case where $\sigma_1^* = 0$, or $\log|\sigma_1^*| = -\infty$). To complete the proof, one needs to prove that $\frac{\partial S}{\partial b}(p, b) < 0$. Under *(H.1)*, this is true if $\varepsilon = 0$ (Proposition 1), and under condition (3.8) when $\varepsilon < 0$ (Proposition 1), and under condition (3.8) when $\varepsilon < 0$ (Proposition 2.*b)*.

2) Let $[\alpha_0, \alpha_1]$ become large, with $\alpha_0 < \log(b/\beta_1^*), \log(b/\beta_0^*) < \alpha_1$. Under the assumption $\frac{\partial s^*}{\partial \beta}(p, \beta) \leqq 0$ for all $\beta > 0$, a lower bound for

$$\left|\frac{\partial S}{\partial b}(p, b)\right| = \int_{\alpha_0}^{\alpha_1} \left|\frac{\partial s^*}{\partial \beta}(p, e^{-\alpha}b)\right| e^{-\alpha} g(\alpha) d\alpha \tag{A.13}$$

is given by

$$
\begin{aligned}
\left|\frac{\partial S}{\partial b}(p,b)\right| &\geqq \int_{\log(b/\beta_0^*)}^{\alpha_1} \gamma_0|\sigma_0^*|e^{\alpha(\varepsilon-1)}d\alpha \\
&\geqq |\sigma_0^*|\frac{\gamma_0}{\varepsilon-1}\left[e^{\alpha_1(\varepsilon-1)}-(b/\beta_0^*)^{\varepsilon-1}\right]. \qquad (A.14)
\end{aligned}
$$

One gets then from (4.15)

$$
\left|\frac{b\dfrac{\partial^2 S}{\partial b^2}(p,b)}{\dfrac{\partial S}{\partial b}(p,b)}-(\varepsilon-1)\right| \leqq \left|\frac{\varepsilon-1}{\sigma_0^*}\right| \frac{\left|e^{\alpha_0(\varepsilon-1)}\dfrac{\partial s^*}{\partial \beta}(p,e^{-\alpha_0}b)-e^{\alpha_1(\varepsilon-1)}\dfrac{\partial s^*}{\partial \beta}(p,e^{-\alpha_1}b)\right|}{(b/\beta_0^*)^{\varepsilon-1}-e^{\alpha_1(\varepsilon-1)}}. \tag{A.15}
$$

Since $\dfrac{\partial s^*}{\partial \beta}(p,e^{-\alpha_1}b) \leqq \sigma_0^* < 0$, one needs to have $\alpha_1 \to +\infty$, together with (4.19), to make $e^{\alpha_1(\varepsilon-1)}\left|\dfrac{\partial s^*}{\partial \beta}(p,e^{-\alpha_1}b)\right|$ go to 0. This condition is actually sufficient to make the right hand side of (A.15) go to 0 when $\sigma_1^*=0$, hence $\dfrac{\partial s^*}{\partial \beta}(p,e^{-\alpha_0}b)=0$, even when α_0 is bounded below. Under the additional more general condition (4.18), where one may have $a_1>0$ when $\sigma_1^*<0$, one verifies that the absolute value of the numerator of the right hand side of (A.15) is bounded above by

$$
a_1 b^{\nu_1(\varepsilon-1)}e^{\alpha_0(\varepsilon-1)(1-\nu_1)}+a_0 b^{\nu_0(\varepsilon-1)}e^{\alpha_1(\varepsilon-1)(1-\nu_0)}, \tag{A.16}
$$

which tends to 0 when $\alpha_0 \to -\infty, \alpha_1 \to +\infty$ under the conditions specified in (4.18), (4.19). Q.E.D.

6. Proposition 6. *(Locally increasing and concave aggregate budget share)*
Let (H.0) hold, as well as (D.O) with $\sigma_1^* > \sigma_0^* \geqq 0$, and consider an increasing exponential heterogeneity distribution with $\varepsilon \geqq 1$.

1) One has $b\dfrac{\partial^2 S}{\partial b^2}(p,b) > (\varepsilon-1)\dfrac{\partial S}{\partial b}(p,b)$ if

$$
[\log\sigma_1^* - \log\sigma_0^*]/(\alpha_1-\alpha_0) > \varepsilon - 1 \geqq 0. \tag{A.17}
$$

The aggregate budget share is then locally strictly increasing, $\dfrac{\partial S}{\partial b}(p,b) > 0$, and strictly convex, $b\dfrac{\partial^2 S}{\partial b^2}(p,b) > 0$, under condition (3.6).

2) Assume that the individual "base" budget share is non-decreasing with income, $\dfrac{\partial s^*}{\partial \beta}(p,\beta) \geqq 0$ for all $\beta > 0$. Then the aggregate income elasticity $b\dfrac{\partial^2 S}{\partial b^2}(p,b)/\dfrac{\partial S}{\partial b}(p,b)$ converges uniformly to $\varepsilon - 1 \geqq 0$ when $\alpha_0 \to -\infty, \alpha_1 \to +\infty$, if $\dfrac{\partial s^*}{\partial \beta}(p,b)$ a) is bounded above or does not tend to $+\infty$ too fast when $\beta_1^* \leqq \beta \to +\infty$, and b) goes to 0 fast enough when $\beta_0^* \geqq \beta \to 0$:

$$a)\ 0 < \sigma_1^* \leqq \frac{\partial s^*}{\partial \beta}(p,\beta) \leqq a_1 \beta^{\nu_1(\varepsilon-1)} \text{ for } \beta_1^* \leqq \beta, \text{ with } a_1 > 0, 0 \leqq \nu_1 < 1, \tag{A.18}$$

$$b)\ 0 \leqq \frac{\partial s^*}{\partial \beta}(p,\beta) \leqq a_0 \beta^{\nu_0(\varepsilon-1)} \text{ for } 0 < \beta \leqq \beta_0^*, \text{ with } a_0 > 0, \nu_0 > 1. \tag{A.19}$$

Proof. 1) As seen in the proof of Proposition 5, *(D.0)* implies (A.12). Since $\gamma_0 > 0$ and $0 \leqq \sigma_0^* < \sigma_1^*$ here, the right hand side is positive if and only if (A.17) holds. Since $\dfrac{\partial S}{\partial b}(p,b) > 0$ under condition (3.6) (Proposition 2.*a)*), the proof is complete.

2) Let $[\alpha_0, \alpha_1]$ become large, with $\alpha_0 < \log(b/\beta_1^*), \log(b/\beta_0^*) < \alpha_1$. Under the assumption $\dfrac{\partial s^*}{\partial \beta}(p,\beta) \geqq 0$ for all $\beta > 0$, from (A.13), a lower bound for $\dfrac{\partial S}{\partial b}(p,b)$ is given by

$$\frac{\partial S}{\partial b}(p,b) \geqq \int_{\alpha_0}^{\log(b/\beta_1^*)} \gamma_0 \sigma_1^* e^{\alpha(\varepsilon-1)} d\alpha,$$

which is equal to $\sigma_1^* \dfrac{\gamma_0}{\varepsilon - 1}[(b/\beta_1^*)^{(\varepsilon-1)} - e^{\alpha_0(\varepsilon-1)}] > 0$ when $\varepsilon > 1$, and to $\sigma_1^*\gamma_0[\log(b/\beta_1^*) - \alpha_0] > 0$ when $\varepsilon = 1$.

One gets then from (4.15) when $\varepsilon = 1$

$$\left| \frac{b\dfrac{\partial^2 S}{\partial b^2}(p,b)}{\dfrac{\partial S}{\partial b}(p,b)} \right| \leqq \frac{1}{\sigma_1^*} \frac{\left| \dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_0} b) - \dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_1} b) \right|}{\log(b/\beta_1^*) - \alpha_0}.$$

Assumption (A.18) implies that $\dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_0} b)$ is bounded above by $a_1 > 0$ when $\varepsilon = 1$, while $0 \leqq \dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_1} b) \leqq \sigma_0^*$ from *(D.0)*. So the above income elasticity goes to $\varepsilon - 1 = 0$ when $\alpha_0 \to -\infty$ (even if α_1 is bounded above).

When $\varepsilon > 1$, (4.15) implies

$$\left| \frac{b\dfrac{\partial^2 S}{\partial b^2}(p,b)}{\dfrac{\partial S}{\partial b}(p,b)} - (\varepsilon - 1) \right| \leqq \frac{\varepsilon - 1}{\sigma_1^*} \frac{|e^{\alpha_0(\varepsilon-1)}\dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_0}b) - e^{\alpha_1(\varepsilon-1)}\dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_1}b)|}{(b/\beta_1^*)^{\varepsilon-1} - e^{\alpha_0(\varepsilon-1)}}. \tag{A.20}$$

Since $0 < \sigma_1^* \leqq \dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_0}b)$, one needs to make α_0 go to $-\infty$, together with (A.18), to make $e^{\alpha_0(\varepsilon-1)}\dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_0}b)$ go to 0. The condition is in fact sufficient to make the right hand side of (A.20) go to 0 when $\sigma_0^* = 0$, hence $\dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_1}b) = 0$, even when α_1 is bounded above. Under the additional more general condition (A.19), where one may have $\sigma_0^* > 0$ and $a_0 > 0$, the absolute value of the numerator of the right hand side of (A.20) is bounded above by an expression identical to (A.16). This expression goes to 0 under conditions (A.18), (A.19) when $\alpha_0 \to -\infty, \alpha_1 \to +\infty$. Q.E.D.

7. Proposition 7. *(Locally increasing and concave aggregate budget share)*
Let (H.0) hold, as well as (D.1) with $\sigma_0^* > \sigma_1^* \geqq 0$, and consider an increasing exponential heterogeneity distribution with $0 \leqq \varepsilon \leqq 1$.

1) One has $b\dfrac{\partial^2 S}{\partial b^2}(p,b) < (\varepsilon - 1)\dfrac{\partial S}{\partial b}(p,b)$ if

$$[\log \sigma_1^* - \log \sigma_0^*]/(\alpha_1 - \alpha_0) < \varepsilon - 1 \leqq 0. \tag{A.21}$$

The aggregate budget share is then locally strictly increasing, $\dfrac{\partial S}{\partial b}(p,b) > 0$, and strictly concave, $b\dfrac{\partial^2 S}{\partial b^2}(p,b) < 0$, if either $\varepsilon = 0$, or under condition (3.6) when $\varepsilon > 0$.

2) Assume that the individual "base" budget share is non-decreasing with income, $\dfrac{\partial s^*}{\partial \beta}(p,\beta) \geqq 0$, for all $\beta > 0$. The aggregate income elasticity $b\dfrac{\partial^2 S}{\partial b}(p,b)/\dfrac{\partial S}{\partial b}(p,b)$ converges uniformly to $\varepsilon - 1 \leqq 0$ when $\alpha_0 \to -\infty, \alpha_1 \to +\infty$, if $\dfrac{\partial s^*}{\partial \beta}(p,\beta)$ a) goes to 0 fast enough when $\beta_1^* \leqq \beta$ tends to $+\infty$, and b) is bounded above or does not go too fast to $+\infty$ when $\beta_0^* \geqq \beta$ tends to 0 :

$$a)\ 0 \leqq \frac{\partial s^*}{\partial \beta}(p,\beta) \leqq a_1 \beta^{\nu_1(\varepsilon-1)} \text{ for } \beta \geqq \beta_1^*, \text{ with } a_1 \geqq 0, \nu_1 > 1, \tag{A.22}$$

b) $0 < \sigma_0 \leqq \dfrac{\partial s^*}{\partial \beta}(p,\beta) \leqq a_0 \beta^{\nu_0(\varepsilon-1)}$ *for* $\beta \geqq \beta_0^*$, *with* $a_0 > 0, 0 \leqq \nu_0 < 1$. (A.23)

Proof. 1) Under *(D.1)*, (4.15) implies

$$b\frac{\partial^2 S}{\partial b^2}(p,b) - (\varepsilon-1)\frac{\partial S}{\partial b}(p,b) \leqq \gamma_0 \left[e^{\alpha_0(\varepsilon-1)}\sigma_1^* - e^{\alpha_1(\varepsilon-1)}\sigma_0^*\right].$$

If $\sigma_0^* > \sigma_1^* \geqq 0$, since $\gamma_0 > 0$, the right hand side is negative if and only if (A.21) holds. To complete the proof, one needs to show $\dfrac{\partial S}{\partial b}(p,b) > 0$. Under *(H.0)*, this is true if $\varepsilon = 0$ (Proposition 1), and under condition (3.6) when $\varepsilon > 0$ (Proposition 2.a).

2) Let $[\alpha_0, \alpha_1]$ become large with $\alpha_0 < \log(b/\beta_1^*), \log(b/\beta_0^*) < \alpha_1$. Under the assumption $\dfrac{\partial s^*}{\partial \beta}(p,\beta) \geqq 0$ for all $\beta > 0$, from (A.13), a lower bound for $\dfrac{\partial S}{\partial b}(p,b)$ is given by

$$\frac{\partial S}{\partial b}(p,b) \geqq \sigma_0^* \frac{\gamma_0}{1-\varepsilon}[(b/\beta_0^*)^{\varepsilon-1} - e^{\alpha_1(\varepsilon-1)}]$$

when $0 \leqq \varepsilon < 1$, and by

$$\frac{\partial S}{\partial b}(p,b) \geqq \sigma_0^* \gamma_0[\alpha_1 - \log(b/\beta_0^*)]$$

when $\varepsilon = 1$.

One gets then from (4.15) when $\varepsilon = 1$

$$\left|\frac{b\dfrac{\partial^2 S}{\partial b^2}(p,b)}{\dfrac{\partial S}{\partial b}(p,b)}\right| \leqq \frac{1}{\sigma_0^*}\frac{\left|\dfrac{\partial s^*}{\partial \beta}(p,e^{-\alpha_0}b) - \dfrac{\partial s^*}{\partial \beta}(p,e^{-\alpha_1}b)\right|}{\alpha_1 - \log(b/\beta_0^*)}.$$

Assumption (A.23) implies that $0 < \sigma_0^* \leqq \dfrac{\partial s^*}{\partial \beta}(p,e^{-\alpha_1}b) \leqq a_0$ when $\varepsilon = 1$ while $0 \leq \dfrac{\partial s^*}{\partial \beta}(p,e^{-\alpha_0}b) \leq \sigma_1^*$ from *(D.1)*. So the above income elasticity goes to $0 = \varepsilon - 1$ when α_1 tends to $+\infty$ (even when α_0 is bounded below).

When $\varepsilon < 1$, (4.15) implies

$$\left| \frac{b\dfrac{\partial^2 S}{\partial b^2}(p,b)}{\dfrac{\partial S}{\partial b}(p,b)} - (\varepsilon - 1) \right| \leqq \frac{|\varepsilon - 1|}{\sigma_0^*} \frac{|e^{\alpha_0(\varepsilon-1)}\dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_0}b) - e^{\alpha_1(\varepsilon-1)}\dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_1}b)|}{(b/\beta_0^*)^{\varepsilon-1} - e^{\alpha_1(\varepsilon-1)}}. \tag{A.24}$$

Since $\dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_1}b) \geqq \sigma_0^* > 0$, one needs to make $\alpha_1 \to +\infty$, together with (A.23), to make $e^{\alpha_1(\varepsilon-1)}\dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_1}b)$ go to 0. This condition is infact sufficient to make the right hand side of (A.24) go to 0 when $\sigma_1^* = 0$, hence $\dfrac{\partial s^*}{\partial \beta}(p, e^{-\alpha_0}b) = 0$, even when α_0 is bounded below. Under the additional more general condition (A.22), where one may have $\sigma_1^* > 0, a_1 > 0$, the absolute value of the numerator of the right hand side of (A.24) is bounded above by the same expression stated in (A.16). Under the conditions (A.22), (A.23), this expression goes to 0 when $\alpha_0 \to -\infty, \alpha_1 \to +\infty$. Q.E.D.