Centre de recherche en économie et statistique

Série des Documents de Travail

n° 2017-01
Pegging the Interest Rate on Bank Reserves

B. DIBA [1]

O.LOISEL [2]

[1] Department of Economics, Georgetown University. E-mail: dibab@georgetown.edu

[2] CREST (ENSAE). E-mail: olivier.loisel@ensae.fr

Pegging the Interest Rate on Bank Reserves

Behzad Diba and Olivier Loisel*

February 8, 2017

Abstract

We develop a model of monetary policy with a small departure from the basic New Keynesian (NK) model. In this model, the central bank can set the interest rate on bank reserves and the nominal stock of bank reserves independently, because these reserves reduce the costs of banking (i.e., have a convenience yield). The model delivers local-equilibrium determinacy under a permanent interest-rate peg. Consequently, it does not share the puzzling and paradoxical implications of the basic NK model under a temporary peg (e.g., in the context of a liquidity trap). More specifically, it offers a resolution of the "forward guidance puzzle," a related puzzle about fiscal multipliers, and the "paradox of flexibility," even for an arbitrarily small departure from the basic NK model (i.e., arbitrarily small banking costs and convenience yield of reserves).

1 Introduction

In the aftermath of the 2007-2009 financial crisis, major central banks have kept the interest rate on bank reserves (IOR rate) near zero and have occasionally conducted balance-sheet policies. In this paper, we develop a model in which the central bank can set the IOR rate and adjust the size of its balance sheet independently, because bank reserves serve to reduce the costs of banking. We show that this model delivers local-equilibrium determinacy under a permanent IOR-rate peg and, as a consequence, solves several puzzles and paradoxes that arise in New Keynesian (NK) models under a temporary interest-rate peg − e.g., in the context of a liquidity trap during which the interest rate is set to its effective lower bound.

Standard NK models, in particular the basic NK model studied in Woodford (2003), exhibit equilibrium indeterminacy under a permanent interest-rate peg. In these models, following Sargent and Wallace (1975), the central bank is assumed to set the nominal interest rate on a bond that serves only as a store of value (i.e., has no non-pecuniary "convenience yield").[1] Once the central bank sets this interest rate, it commits to buy or sell the bond at the implied

*Diba: Department of Economics, Georgetown University, Intercultural Center 580, 37th and O Streets, N.W., Washington, D.C. 20057, U.S.A., dibab@georgetown.edu. Loisel: CREST (ENSAE), 15 boulevard Gabriel Péri, 92245 Malakoff Cédex, France, olivier.loisel@ensae.fr. We would like to thank Jeffrey Huther and Kalin Nikolov for useful discussions. We gratefully acknowledge the financial support of the grant Investissements d'Avenir (ANR-11-IDEX-0003/Labex Ecodec/ANR-11-LABX-0047).

[1] The assumption that the bond in question has absolutely no convenience yield (such as eligibility as collateral) is important for the determinacy properties of standard models, as Canzoneri and Diba (2005) illustrate.

price. This makes the money supply endogenous, and it makes any arbitrary price level (with the associated nominal money stock) consistent with an equilibrium. In our model, because bank reserves have a convenience yield, setting the IOR rate does not make the supply of reserves endogenous. Our central bank can set the supply of reserves independently, and the indeterminacy problems discussed in Sargent and Wallace (1975) and Woodford (2003) do not arise. The central bank's IOR-rate policy determines the demand for real reserves; and, given the outstanding nominal stock of reserves, this pins down the price level.[2]

As we elaborate below, indeterminacy under a *permanent* interest-rate peg is behind some puzzling and paradoxical implications of NK models under a *temporary* interest-rate peg. In the basic NK model, in particular, the effects of a temporary interest-rate peg on current inflation and output become unboundedly large as the duration of the peg goes to infinity (the so-called "forward-guidance puzzle") or as prices become perfectly flexible (the so-called "paradox of flexibility"). Moreover, the effects on current inflation and output of a given fiscal expansion at the end of the peg also grow explosively as the duration of the peg goes to infinity (what we henceforth call the "fiscal-multiplier puzzle").[3] These implications are perplexing because inflation, output, and fiscal multipliers all take finite values in the limit case of a permanent peg or perfectly flexible prices.[4] There is, thus, a stark discontinuity in the limit as the duration of the peg goes to infinity, or as the degree of price stickiness goes to zero.

A number of recent contributions − most forcefully Carlstrom, Fuerst, and Paustian (2015) and Cochrane (2016a) − view these implications of NK models as unreasonable. Some contributions propose to enrich the basic NK model with features that *quantitatively* tone down the effects of an interest-rate peg for a given duration of the peg and a given degree of price stickiness. In particular, Del Negro, Giannoni, and Patterson (2015) and McKay, Nakamura, and Steinsson (2016) attenuate the forward-guidance puzzle respectively with alternative exit paths for the interest rate and with incomplete markets and borrowing constraints; Wiederholt (2015) attenuates the forward-guidance and fiscal-multiplier puzzles with signalling effects of monetary policy; and Angeletos and Lian (2016) attenuate the forward-guidance puzzle and the paradox of flexibility with lack of common knowledge.

As García-Schmidt and Woodford (2015) note, however, it seems important to explore also model features that solve the puzzles and paradox *qualitatively*, i.e. in the limit as the du-

[2]Our determinacy result echoes the determinacy results obtained by Woodford (2003, Chapters 2 and 4) in the context of a model in which money is household cash on which interest can somehow be paid, and in which the central bank sets both the money supply and the interest rate on money. Adão, Correia, and Teles (2003) also note that letting the central bank somehow set both the interest rate and the money supply would deliver determinacy in their model.

[3]These results can be found in, e.g., Werning (2012), Carlstrom, Fuerst, and Paustian (2015), Farhi and Werning (2016), and Cochrane (2016a). The phrases "forward-guidance puzzle" and "paradox of flexibility" were coined by, respectively, Del Negro, Giannoni, and Patterson (2015), and Eggertsson and Krugman (2012).

[4]A permanent interest-rate peg generates multiple local equilibria in the basic NK model. What we mean is that inflation, output, and fiscal multipliers take finite values in each of these equilibria. Similarly, inflation is not uniquely pinned down under perfectly flexible prices, but takes a finite value.

ration of the peg goes to infinity or as the degree of price stickiness goes to zero. Indeed, a model's quantitative predictions can be questioned if its qualitative implications seem puzzling or paradoxical. In this spirit, Angeletos and Lian (2016), Gabaix (2016), and García-Schmidt and Woodford (2015) propose qualitative resolutions of the forward-guidance puzzle that rely on, respectively, a sufficient degree of lack of common knowledge, a sufficient degree of bounded rationality, and reflective equilibria.[5] And Cochrane (2016a) proposes to address the two puzzles and the paradox with a "backward-stability" or "local-to-frictionless" equilibrium-selection criterion, or by invoking the fiscal theory of the price level.

In this paper, we show that a simple and possibly minimal departure from the basic NK model can qualitatively solve all three puzzles and paradox − the forward-guidance puzzle, the fiscal-multiplier puzzle, and the paradox of flexibility. Our starting point is to note that central banks, during the crisis, have not pegged "the" interest rate that appears in the IS equation of the basic NK model. Instead, the lower bound on nominal interest rates has forced them to peg the IOR rate, which is the interest rate that they directly control. In our model, because a permanent IOR-rate peg delivers determinacy, a temporary IOR-rate peg does not give rise to any of the puzzles and paradox.

The (mechanical) connection between indeterminacy under a permanent interest-rate peg and the forward-guidance and fiscal-multiplier puzzles can be found in Carlstrom, Fuerst, and Paustian (2015) and Cochrane (2016a).[6] The basic NK model has a stable eigenvalue under a (permanent or temporary) peg, but no predetermined variable. Under a temporary peg, when we iterate the model forward in time, this eigenvalue magnifies the effects of terminal conditions (at the end of the peg) on initial outcomes (at the start of the peg), so that these effects grow explosively as the duration of the peg goes to infinity, giving rise to the two puzzles. Moreover, as can be easily checked, the indeterminacy property of the basic NK model is also behind the paradox of flexibility: as prices become perfectly flexible, the stable eigenvalue converges towards zero, so that initial outcomes explode even for a peg of given short duration. In our model, by contrast, the stable eigenvalue is matched by a predetermined variable, namely the money stock. This is the feature that delivers determinacy, and it also solves the two puzzles and the paradox.

Most models proposed in the literature to solve either one of the puzzles or the paradox require a discrete departure from the basic NK model.[7] Our model, by contrast, still solves the two

[5]Angeletos and Lian's (2016) qualitative-resolution result is stated in the last sentence of Proposition 3 of their paper. Angeletos and Lian (2016) and Gabaix (2016) also cite work in progress by Farhi and Werning on the forward-guidance puzzle under bounded rationality and incomplete markets, but we do not know whether this work quantitatively attenuates or qualitatively solves the puzzle.

[6]Farhi and Werning (2016) do not explicitly connect the fiscal-multiplier puzzle to indeterminacy under a permanent peg, but this connection is apparent in Equation (4a) and Proposition 2 of their paper.

[7]This requirement is emphasized by Cochrane (2016b) in the context of Gabaix's (2016) model. The only exception we know of comes from García-Schmidt and Woodford (2015), who solve the forward-guidance puzzle for any degree of reflection, in particular for degrees of reflection that are arbitrarily large and hence arbitrarily close to perfect foresight. Our resolution of the forward-guidance puzzle rests on a different mechanism, which

puzzles and the paradox with an *arbitrarily* small departure from the basic NK model, i.e. with arbitrarily small banking costs and convenience yield of bank reserves. In fact, as we show, even a *vanishingly* small departure from the basic NK model is enough to solve the fiscal-multiplier puzzle and the paradox of flexibility, and attenuate the forward-guidance puzzle. This limit result brings the basic NK model at par with Mankiw and Reis's (2002) sticky-information model in terms of their ability to solve or attenuate the puzzles and paradox.[8] It also provides theoretical foundations to Cochrane's (2016a) approach of selecting an equilibrium different from the (puzzling and paradoxical) standard equilibrium in the basic NK model under a temporary interest-rate peg. In particular, it enables us to endogenize his local-to-frictionless equilibrium-selection criterion.[9]

To make our point, we develop a benchmark model of a cashless economy in which bank reserves have a convenience yield. To represent the convenience yield, we assume that holding reserves reduces the costs of banking.[10] Our assumptions about banking costs are quite similar to those of Cúrdia and Woodford (2011), except for our assumption − which we will defend later − that there is no finite satiation level of reserves.[11] In our benchmark model, banks cannot change the aggregate nominal quantity of reserves outstanding. In reality, what the central bank controls is the monetary base, and cash held outside banks makes the quantity of bank reserves endogenous. We will show later that our results are essentially robust to the introduction of household cash into the model.[12]

We first study the global perfect-foresight equilibria of our benchmark model under flexible prices when the central bank permanently pegs the IOR rate and the growth rate of reserves. We find that for a suitably restricted range of IOR-rate values, the model has a unique time-invariant equilibrium, including a uniquely determined initial price level. The other equilibria are deflationary bubbles that involve implosive price paths and converge to a steady state with constant consumption and growing real money balances.[13] We argue that these equilibria are

preserves the basic NK model's analytical tractability and which we show also enables us to solve the fiscal-multiplier puzzle and the paradox of flexibility.

[8]The ability of Mankiw and Reis's (2002) model to solve or attenuate the puzzles and paradox is studied by Carlstrom, Fuerst, and Paustian (2015) and Kiley (2016).

[9]This criterion requires that equilibrium outcomes converge towards flexible-price equilibrium outcomes as prices become perfectly flexible. It does not select a unique equilibrium, but rules out some equilibria.

[10]Our cashless model, as it stands, does not imply a zero lower bound (ZLB) for the IOR rate. Since reserves are useful for reducing banking costs, our banks will hold reserves even at negative IOR rates. It is straightforward to introduce a ZLB into our model by assuming that reserves and vault cash are perfect substitutes in terms of reducing banking costs.

[11]Two other differences between the two models are that (i) we link banking costs to time spent on banking activities, in order to make our global analysis tractable, while Cúrdia and Woodford (2011) link them to goods consumed in banking activities, and (ii) the borrowers in our model are firms (borrowing the wage bill), while they are impatient households in Cúrdia and Woodford (2011).

[12]We think our results would also hold up if we added other realistic features (associated with central-bank operating procedures) that lead to limited endogenous variation in reserves. For example, the Federal Reserve's reverse-repo facility allows financial entities with no access to the IOR rate to affect the monetary base. But the resulting endogeneity is limited because the Federal Reserve sets the reverse-repo rate below the IOR rate.

[13]Our model rules out inflationary bubbles involving explosive price paths because money is "essential" in the sense that marginal banking costs go to infinity as bank reserves fall to zero. The non-existence of inflationary

unlikely to be of much relevance when we address policy questions, and we ignore them in the rest of the paper.

We then introduce price stickiness à la Calvo (1983) and log-linearize the model around its unique steady state (corresponding to the unique time-invariant equilibrium under flexible prices). We show that setting exogenously the IOR rate and the growth rate of reserves leads to local-equilibrium determinacy for all functional forms of the utility and production functions and all values of the structural and policy parameters. As we point out, this policy amounts to following a "shadow Wicksellian rule" for the interest rate on a bond that serves only as a store of value: it is as if the central bank directly controlled this interest rate and set it as an increasing function of output and the price level. Such a rule is well known to ensure determinacy in the basic NK model (as shown in Woodford, 2003, Chapter 4). We show that this rule, given its implied coefficients, also ensures determinacy in our extended NK model with banking costs. This rule is a "shadow rule" in our setup in the sense that what it actually describes is the private sector's behavior, not the central bank's. Since our central bank pegs its policy instruments, it does not react to deviations from equilibrium; we do not need, therefore, to worry about the feasibility of its off-equilibrium reaction (Bassetto, 2005; Loisel, 2016).[14]

Because it delivers determinacy under a permanent (IOR-rate) peg, our model solves the two puzzles and the paradox: the effects of a temporary (IOR-rate) peg do not grow explosively as its duration becomes infinite or as prices become perfectly flexible, but instead converge towards the finite effects of a permanent peg or the finite flexible-price effects; and fiscal interventions in the vanishingly distant future have vanishingly small effects, instead of unboundedly large effects, on current outcomes.

We do not think that our model's ability to solve the puzzles and paradox in a liquidity trap comes at the cost of any controversial implication during "normal times." We check three of these implications. First, we show that if the central bank maintains a fixed spread between the IOR rate and the interbank rate (as in a typical corridor system), then the reduced form of the model becomes isomorphic to the reduced form of the basic NK model for any given interest-rate rule. Therefore, our model then inherits all the standard implications of the basic NK model for equilibrium determinacy and dynamics away from the effective lower bound. Second, we study the effects of monetary-policy shocks in our model and find that they are consistent with standard Keynesian views. In particular, under sufficiently sticky prices, unexpected (temporary or permanent) IOR-rate hikes are contractionary in the short term. Indeed, a higher IOR rate reduces the opportunity cost of holding reserves and thus increases real money demand for any output level; given the existing nominal money stock and the short-term price rigidity, the output level must then fall to clear the money market. Third, we (trivially) show that our

bubbles in models in which fiat money is "essential" to the economy in some restrictive sense is well known (see, for example, Kingston, 1982, and Obstfeld and Rogoff, 1983).

[14] The credibility of its off-equilibrium-behavior threat may, however, still be an issue (Cochrane, 2011), even though this off-equilibrium behavior is passive.

model is Fisherian in the long term, i.e. implies a one-to-one long-term relationship between the interbank rate and the inflation rate. By contrast, as we also show, models that qualitatively solve the forward-guidance puzzle by "discounting" the IS equation and the Phillips curve (such as Angeletos and Lian, 2016, and Gabaix, 2016) make the inflation rate respond negatively to the interest rate in the long term.[15]

In addition, we show that our model can be Fisherian or not in the short term, i.e. can make the inflation rate respond positively or negatively in the short term to a permanent increase in the interbank rate. We thus provide some conditional support for the "neo-Fisherian" view about the short-term inflationary effects of "normalizing" interest rates, which has been at the center of recent work and debate.[16] Whether or not our model is Fisherian in the short term depends not on the equilibrium considered (there is only one, unlike in the NK models used by Cochrane, 2016c), but instead on how the two monetary-policy instruments are used to generate a permanent increase in the interbank rate. We find in particular that the specific combination of stepwise changes in these instruments that generates a stepwise increase in the interbank rate does produce a neo-Fisherian effect.

In our benchmark model, the demand for reserves cannot be satiated because of our assumption of no finite satiation point. We will relax this assumption and show that satiation of the demand for reserves would raise the spectre of indeterminacy issues highlighted in Sargent and Wallace (1985): it would make the marginal convenience yield of bank reserves equal to zero, and the demand for real money balances indeterminate. This would undo the main mechanism that delivers determinacy and solves the puzzles and paradox in our model. Several observers of the U.S. situation (e.g., Cochrane, 2014) assert that the demand for bank reserves is currently satiated. We do not think that the mere fact that the current level of bank reserves is large makes a persuasive case that the convenience yield (transactions services or liquidity value) of reserves has dropped to zero. The stock of U.S. Treasury debt is much larger and its convenience yield − reported in empirical studies like Krishnamurthy and Vissing-Jorgensen (2012) − seems to remain sizeable, and inversely related to the stock of debt.

In the text, we will discuss some model-based criteria for gauging the satiation point of demand for reserves. In the end, however, we cannot make a persuasive argument either way. It seems hard to discriminate between the view that the marginal convenience yield of reserves is exactly zero and our preferred view that it may be small and fairly flat, but still positive and inversely related to the amount of reserves. The latter view substitutes a narrative in which small shocks

[15]Cochrane (2016b) points out that Gabaix's (2016) benchmark model may make inflation respond negatively to the interest rate in the long term. We show that it necessarily does − and that so does Angeletos and Lian's (2016) model − when it delivers determinacy under a permanent interest-rate peg, i.e. when it solves the forward-guidance puzzle. Gabaix (2016) proposes an extension of his benchmark model that can both make inflation respond positively to the interest rate in the long term and deliver determinacy under a permanent interest-rate peg.

[16]Cochrane (2016c) and Schmitt-Grohé and Uribe (2014, 2016) provide examples of models that do produce neo-Fisherian effects, while Cochrane (2016d), García-Schmidt and Woodford (2015), and Kocherlakota (2016) provide examples of models that do not.

may lead to large changes in the demand for reserves for a narrative in which banks are truly indifferent across a range of values for their reserve balances; or, equivalently, a narrative in which quantitative easing has, on the margin, some (possibly very small) effects on the economy, for a narrative in which it has no effects at all.

The rest of the paper is organized as follows. Section 2 presents the benchmark model that is used in the next four sections. Section 3 studies the global perfect-foresight equilibria of this model under flexible prices and a permanent IOR-rate peg. Section 4 log-linearizes the model under sticky prices around its unique steady state, and shows that a permanent IOR-rate peg delivers local-equilibrium determinacy. Section 5 shows that this model solves the forward-guidance puzzle, the fiscal-multiplier puzzle, and the paradox of flexibility. Section 6 studies some other implications of this model, from the effects of corridor systems and monetary-policy shocks to Fisherian and neo-Fisherian effects. Section 7 introduces household cash into our benchmark model and shows that the main results are essentially unaffected. Section 8 introduces a finite satiation point in the demand for reserves, shows that the main results are unaffected if and only if the demand for reserves is not satiated in equilibrium, and discusses whether or not this has been the case in the U.S. over the past few years. We then conclude and provide a technical appendix.

2 Benchmark Model

In our benchmark model, monopolistic firms use labor to produce goods. They need to pay wages before they can produce and sell their output. They borrow the wage bill from banks. Banks use labor and reserves to make loans. The central bank sets the interest rate on bank reserves, and can also change the quantity of reserves through open-market operations or helicopter drops. The model is essentially non-parametric, as we do not specify any functional form for the utility and production functions, in order to broaden the scope of our results. For simplicity, we assume that households do not hold cash and that there is no finite satiation level in the demand for reserves; these assumptions will be relaxed in Sections 7 and 8 respectively.

2.1 Households

Each household consists of workers and bankers. Households get utility from consumption (c) and disutility from labor (h for workers, h^b for bankers). Their intertemporal utility function is

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) - v^b\left(h^b_{t+k}\right) \right] \right\}$$

where $\beta \in (0,1)$. The consumption-utility function u, defined over the set of positive real numbers $\mathbb{R}_{>0}$, is twice differentiable, strictly increasing ($u' > 0$), strictly concave ($u'' < 0$), and satisfies the usual Inada conditions

$$\lim_{c \to 0} u'(c) = +\infty, \tag{1}$$

$$\lim_{c \to +\infty} u'(c) = 0. \tag{2}$$

The labor-disutility functions v and v^b, defined over the set of non-negative real numbers $\mathbb{R}_{\geq 0}$, are twice differentiable, strictly increasing ($v' > 0$ and $v^{b\prime} > 0$), and weakly convex ($v'' \geq 0$ and $v^{b\prime\prime} \geq 0$).

Bankers use their own labor h_t^b and (real) reserves at the central bank m_t to produce (real) loans ℓ_t according to the following technology:

$$\ell_t = f^b\left(h_t^b, m_t\right).$$

The production function f^b, defined over $(\mathbb{R}_{\geq 0})^2$, is twice differentiable, strictly increasing ($f_h^b > 0$ and $f_m^b > 0$), homogeneous of degree $d \in (0, 1]$, and such that $f_{hh}^b < 0$, $f_{mm}^b < 0$, $f_{hm}^b \geq 0$,

$$\forall h_t^b \in \mathbb{R}_{\geq 0}, \lim_{m_t \to +\infty} f_m^b\left(h_t^b, m_t\right) = 0, \tag{3}$$

$$\forall h_t^b \in \mathbb{R}_{\geq 0}, \lim_{m_t \to 0} f_h^b\left(h_t^b, m_t\right) = 0. \tag{4}$$

Assumption (3) is a standard Inada condition, while assumption (4) articulates a sense in which holding reserves is essential for banking. The assumption of decreasing or constant returns to scale ($d \leq 1$) is not necessary for our results, but it simplifies our general analysis.[17] As we show in Appendix A.1, it implies that f^b is concave ($f_{hh}^b f_{mm}^b - \left(f_{hm}^b\right)^2 \geq 0$). Similarly, the assumption that labor and reserves are complements ($f_{hm}^b \geq 0$) could be relaxed to some extent without affecting our results. The set of functions f^b satisfying all these assumptions is broad enough to include, for instance, any constant-elasticity-of-substitution (CES) function, as well as any CES function raised to a power d such that $(s-1)/s \leq d < 1$, where s denotes the elasticity of substitution.

The function f^b is, of course, a convenient short cut to capture the role of bank reserves − which in reality may come, for example, from a maturity mismatch between banks' assets and liabilities. Our results, in particular our resolution of NK puzzles and paradoxes, will not depend on the quantitative importance of this role: the elasticity of loans to reserves, $m_t f_m^b(h_t^b, m_t)/f^b(h_t^b, m_t)$, may be arbitrarily small for any (h_t^b, m_t) in $(\mathbb{R}_{\geq 0})^2$. What we need for our results, however, is that this elasticity is not zero in equilibrium. This condition is necessarily met in our benchmark model, because we assume that there is no finite satiation level in the demand for reserves. We will relax this assumption in Section 8.

[17] We allow for increasing returns to scale ($d > 1$) in Section 7, in the context of a parametric model with cash. It is easy to check that our results do not depend on whether returns to scale are decreasing, constant, or increasing in the cashless version of that parametric model, provided that they are not increasing too much.

Given the properties of f^b, we can invert it and get

$$h_t^b = g^b(\ell_t, m_t),$$

where the function g^b is implicitly and uniquely defined over $(\mathbb{R}_{\geq 0})^2$ by

$$\ell_t = f^b[g^b(\ell_t, m_t), m_t].$$

The utility cost of banking, as a function of loans and reserves, is therefore defined over $(\mathbb{R}_{\geq 0})^2$ by

$$\Gamma(\ell_t, m_t) \equiv v^b\left[g^b(\ell_t, m_t)\right].$$

We derive some properties of the functions g^b and Γ in Appendices A.2, A.3, and A.4. In particular, we establish the following lemma in Appendix A.3:

Lemma 1 (Properties of Function Γ): *The banking-cost function Γ is strictly increasing in loans ($\Gamma_\ell > 0$); strictly decreasing in reserves ($\Gamma_m < 0$); convex ($\Gamma_{\ell\ell} > 0$, $\Gamma_{mm} > 0$, $\Gamma_{\ell\ell}\Gamma_{mm} - (\Gamma_{\ell m})^2 \geq 0$); and such that $\Gamma_{\ell m} < 0$,*

$$\forall \ell_t \in \mathbb{R}_{>0}, \lim_{m_t \to +\infty} \Gamma_m(\ell_t, m_t) = 0, \tag{5}$$

$$\forall \ell_t \in \mathbb{R}_{>0}, \lim_{m_t \to 0} \Gamma_\ell(\ell_t, m_t) = +\infty. \tag{6}$$

The property that Γ_m is not zero (except asymptotically, as $m_t \to +\infty$) reflects our assumption that there is no finite satiation point in the demand for reserves. The negative-cross-derivative property ($\Gamma_{\ell m} < 0$) says that a marginal increase in reserves decreases costs by more the larger are loans, while property (6) reflects our assumption that holding reserves is essential for banking.

In addition to making loans ℓ_t and holding reserve balances m_t at the central bank, households hold bonds b_t (or issue bonds when $b_t < 0$), which serve only as stores of value.[18] Loans, reserves, and bonds are one-period non-contingent assets. We let I_t^ℓ, I_t^m, and I_t^b denote the corresponding gross nominal interest rates. We let P_t denote the price level, and $\Pi_t \equiv P_t/P_{t-1}$ the gross inflation rate. The household budget constraint, expressed in real terms, is then

$$c_t + b_t + \ell_t + m_t \leq \frac{I_{t-1}^b}{\Pi_t} b_{t-1} + \frac{I_{t-1}^\ell}{\Pi_t} \ell_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \omega_t, \tag{7}$$

where w_t represents the real wage and ω_t captures firm profits and the government's lump-sum taxes or transfers.

Households choose b_t, c_t, h_t, ℓ_t, and m_t to maximize their utility function, rewritten as

$$U_t = \mathbb{E}_t\left\{\sum_{k=0}^{\infty} \beta^k \left[u(c_{t+k}) - v(h_{t+k}) - \Gamma(\ell_{t+k}, m_{t+k})\right]\right\},$$

[18] Bonds issued by households can also be thought of as deposits issued by bankers.

subject to their budget constraint (7), taking all prices (I_t^b, I_t^ℓ, I_t^m, P_t, and w_t) as given. Letting λ_t denote the Lagrange multiplier on the period-t budget constraint, the first-order conditions of households' optimization problem are

$$\lambda_t = u'(c_t), \tag{8}$$

$$\frac{1}{I_t^b} = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t \Pi_{t+1}} \right\}, \tag{9}$$

$$\lambda_t w_t = v'(h_t), \tag{10}$$

$$\Gamma_\ell(\ell_t, m_t) + \lambda_t = \beta I_t^\ell \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\},$$

$$\Gamma_m(\ell_t, m_t) + \lambda_t = \beta I_t^m \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}.$$

Using (9), we can rewrite the last two conditions as

$$\frac{I_t^\ell}{I_t^b} = 1 + \frac{\Gamma_\ell(\ell_t, m_t)}{\lambda_t}, \tag{11}$$

$$\frac{I_t^m}{I_t^b} = 1 + \frac{\Gamma_m(\ell_t, m_t)}{\lambda_t}. \tag{12}$$

Condition (11) implies that loans pay more interest than bonds, because the marginal banking cost is positive ($\Gamma_\ell > 0$). Condition (12) implies that reserves pay less interest than bonds, because they serve to reduce banking costs ($\Gamma_m < 0$). Assuming that households' optimization problem is also subject to a standard no-Ponzi-game condition, the transversality condition is

$$\lim_{k \to +\infty} \mathbb{E}_t \left\{ \beta^{t+k} \lambda_{t+k} a_{t+k} \right\} = 0, \tag{13}$$

where $a_t \equiv b_t + \ell_t + m_t$ denotes households' total assets. The second-order conditions of households' optimization problem are met because of the convexity of the banking-cost function Γ.

2.2 Firms

There is a continuum of monopolistically competitive firms owned by households and indexed by $i \in [0, 1]$.[19] Each firm i uses $h_t(i)$ units of labor to produce

$$y_t(i) = f[h_t(i)] \tag{14}$$

units of output. The production function f, defined over $\mathbb{R}_{\geq 0}$, is twice differentiable, strictly increasing ($f' > 0$), weakly concave ($f'' \leq 0$), and such that $f(0) = 0$. To generate a demand for bank loans, we assume that firm i has to borrow its nominal wage bill $W_t h_t(i)$, at the gross nominal interest rate I_t^ℓ, before it can produce and sell its output.[20]

We will consider two main alternative assumptions for price setting: prices may be either flexible, or sticky à la Calvo (1983).[21] These assumptions will be used respectively in Section

[19] The departure from perfect competition plays no particular role in our model with flexible prices.

[20] Our results would be qualitatively unchanged if firms had to borrow a (constant) fraction of their wage bill, instead of their entire wage bill.

[21] In Appendix D.2, we will also consider the case in which prices are set one period in advance.

3 and in Sections 4, 5, and 6. We focus here on the flexible-price assumption, and postpone the consideration of the sticky-price assumption to Section 4. When prices are flexible, firm i chooses its price $P_t(i)$ at date t to maximize

$$\mathbb{E}_t\left\{P_t(i)y_t(i)-\frac{\beta\lambda_{t+1}I_t^{\ell}L_t(i)}{\lambda_t\Pi_{t+1}}\right\}$$

subject to the production function (14), the demand schedule

$$y_t(i)=\left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon}c_t, \tag{15}$$

and the borrowing constraint

$$W_t h_t(i)\leq L_t(i), \tag{16}$$

where $L_t(i)$ denotes the nominal value of firm i's loan, and $\varepsilon>0$ the elasticity of substitution between differentiated goods. The first-order condition of this optimization problem implies

$$P_t(i)=\frac{\varepsilon}{\varepsilon-1}\mathbb{E}_t\left\{\frac{\beta\lambda_{t+1}I_t^{\ell}W_t}{\lambda_t\Pi_{t+1}f'[h_t(i)]}\right\}.$$

Using the Euler equation (9), we can rewrite this pricing equation as

$$P_t(i)=\frac{\varepsilon}{\varepsilon-1}\frac{I_t^{\ell}W_t}{I_t^b f'[h_t(i)]}.$$

In a symmetric equilibrium, all firms set the same price:

$$P_t=\frac{\varepsilon}{\varepsilon-1}\frac{I_t^{\ell}W_t}{I_t^b f'(h_t)}. \tag{17}$$

2.3 Government

The government consists of a monetary authority and a fiscal authority. The monetary authority has two independent instruments: the (gross) nominal interest rate on reserves $I_t^m\geq 0$, and the monetary base, which in our benchmark model is made only of nominal reserves $M_t>0$.[22] Changes in reserve balances are matched by changes in the monetary authority's holdings of bonds issued by households or the fiscal authority. The fiscal authority sets lump-sum transfers T_t (or taxes when $T_t<0$) to households, and satisfies its present-value budget constraint at any prevailing price path (making fiscal policy Ricardian).

The consolidated budget constraint of the government is

$$M_t+B_t=I_{t-1}^m M_{t-1}+I_{t-1}^b B_{t-1}+P_tT_t.$$

We will consider two alternative ways of injecting reserve balances: open-market operations increase M_t holding M_t+B_t constant, while helicopter drops (monetized fiscal transfers) increase M_t holding B_t constant.

[22] In Section 7, the monetary base will be made of bank reserves and household cash.

2.4 Market Clearing

The bond-market-clearing condition is

$$b_t = \frac{B_t}{P_t},$$

the money-market-clearing condition is

$$m_t = \frac{M_t}{P_t}, \tag{18}$$

and the goods-market-clearing condition is

$$c_t = y_t. \tag{19}$$

3 Global Analysis Under Flexible Prices

In this section, we consider the flexible-price version of our benchmark model. We first show that global equilibrium dynamics can be summarized by a single equation relating h_t to h_{t+1}. We then consider the case in which the (gross) nominal interest rate on reserves I_t^m and the (gross) growth rate of nominal reserves $\mu_t \equiv M_t/M_{t-1}$ are permanently pegged to some constant exogenous values $I^m \geq 0$ and $\mu > 0$. We show the existence and uniqueness of a time-invariant equilibrium in this case, for a suitably restricted range of values I^m and μ, and we characterize the other global perfect-foresight equilibria.[23]

3.1 Dynamic Equation in Employment

To derive the key equation summarizing global equilibrium dynamics, we first use (8), (10), (14), (16) holding with equality, and (19), to express loans ℓ_t as a function of employment h_t:

$$\ell_t = \mathcal{L}(h_t) \equiv \frac{h_t v'(h_t)}{u'[f(h_t)]}, \tag{20}$$

where the function $\mathcal{L}$, defined over $\mathbb{R}_{>0}$, is strictly increasing ($\mathcal{L}' > 0$) with

$$\lim_{h_t \to 0} \mathcal{L}(h_t) = 0, \tag{21}$$

$$\lim_{h_t \to +\infty} \mathcal{L}(h_t) = +\infty. \tag{22}$$

Now, under flexible prices, the pricing equation (17) gives the real wage

$$w_t = \frac{\varepsilon - 1}{\varepsilon} f'(h_t) \frac{I_t^b}{I_t^\ell}. \tag{23}$$

We then use households' first-order condition (11), together with (8), (10), (14), (19), (20), and (23), to get a relationship between reserves m_t and employment h_t:

$$\Gamma_\ell [\mathcal{L}(h_t), m_t] = \mathcal{A}(h_t) \equiv u'[f(h_t)] \left\{ \frac{\varepsilon - 1}{\varepsilon} \frac{u'[f(h_t)] f'(h_t)}{v'(h_t)} - 1 \right\}. \tag{24}$$

[23] Some of the results obtained in this section will be used in the next sections to show that the sticky-price version of our benchmark model has a unique steady state and solves the paradox of flexibility.

Because $\Gamma_\ell > 0$, we restrict the domain of definition of $\mathcal{A}$ to $(0, h^\star)$, where, given the Inada conditions (1) and (2), $h^\star > 0$ is implicitly and uniquely defined by

$$\frac{u'\left[f\left(h^\star\right)\right] f'\left(h^\star\right)}{v'\left(h^\star\right)} = \frac{\varepsilon}{\varepsilon - 1}.$$

The function $\mathcal{A}$ is strictly decreasing ($\mathcal{A}' < 0$) with

$$\lim_{h_t \to 0} \mathcal{A}\left(h_t\right) = +\infty, \tag{25}$$

$$\lim_{h_t \to h^\star} \mathcal{A}\left(h_t\right) = 0. \tag{26}$$

Note that $h^\star$ represents the value that h_t would take in the absence of financial frictions, i.e. if the marginal banking cost Γ_ℓ were zero.

Since $\Gamma_{\ell\ell} > 0$, $\Gamma_{\ell m} < 0$, $\mathcal{L}' > 0$, and $\mathcal{A}' < 0$, Equation (24) implicitly and uniquely defines a function $\mathcal{M}$ such that

$$m_t = \mathcal{M}\left(h_t\right). \tag{27}$$

The function $\mathcal{M}$ is strictly increasing ($\mathcal{M}' > 0$). The reason is that under flexible prices, firms' profit maximization makes their real marginal cost equal to the inverse of their mark-up $((\varepsilon - 1)/\varepsilon)$, which is constant over time; since real marginal cost depends positively on employment and negatively on reserves (through borrowing costs), real reserves need to react positively to employment to keep real marginal cost constant. Moreover, given (6), $\mathcal{M}$ is defined over $(0, \overline{h})$, where $\overline{h} \in (0, h^\star]$ is implicitly and uniquely defined by

$$\lim_{m_t \to +\infty} \Gamma_\ell\left[\mathcal{L}\left(\overline{h}\right), m_t\right] = \mathcal{A}\left(\overline{h}\right). \tag{28}$$

The uniqueness of $\overline{h}$ follows from $\mathcal{A}' < 0$, $\mathcal{L}' > 0$, and $\Gamma_{\ell\ell} > 0$, while its existence is ensured by (25) and (26). Finally, given (6) and (28), we have

$$\lim_{h_t \to 0} \mathcal{M}\left(h_t\right) = 0, \tag{29}$$

$$\lim_{h_t \to \overline{h}} \mathcal{M}\left(h_t\right) = +\infty. \tag{30}$$

Thus, in our benchmark model with no satiation point in the demand for reserves, real money balances grow without bound as employment rises towards its upper bound $\overline{h}$. This upper bound coincides with the frictionless employment level $h^\star$ in the case where the marginal banking cost Γ_ℓ converges to zero as real reserves tend to infinity. In general, however, we allow the marginal banking cost to converge to a positive value $-$ in which case we have $\overline{h} < h^\star$, and our economy with the financial friction cannot attain the employment level of the frictionless economy.

Finally, we use households' first-order conditions (9) and (12), together with (8), (14), (18), (19), (20), and (27), to get the dynamic equation in employment:

$$1 + \frac{\Gamma_m\left[\mathcal{L}\left(h_t\right), \mathcal{M}\left(h_t\right)\right]}{u'\left[f\left(h_t\right)\right]} = \beta I_t^m \mathbb{E}_t\left\{\frac{u'\left[f\left(h_{t+1}\right)\right]\mathcal{M}\left(h_{t+1}\right)}{\mu_{t+1} u'\left[f\left(h_t\right)\right]\mathcal{M}\left(h_t\right)}\right\}. \tag{31}$$

In the rest of the section, we use this dynamic equation to characterize the set of perfect-foresight equilibria under permanent pegs.

3.2 Time-Invariant Equilibrium Under Permanent Pegs

We now consider some permanent pegs $I_t^m = I^m \geq 0$ and $\mu_t = \mu > 0$, and study the existence and uniqueness of a time-invariant equilibrium under these pegs $-$ i.e., an equilibrium in which all real variables (and the inflation rate) are constant over time. Given the strict monotonicity of $\mathcal{L}$ and $\mathcal{M}$, the set of time-invariant equilibria coincides with the set of equilibria in which h_t is constant over time. Now, when h_t is constant over time, the dynamic equation (31) boils down to the static equation

$$\mathcal{F}(h_t) \equiv \frac{\Gamma_m \left[\mathcal{L}(h_t), \mathcal{M}(h_t)\right]}{u'\left[f(h_t)\right]} = \frac{\beta I^m}{\mu} - 1, \tag{32}$$

where the function $\mathcal{F}$ is defined over $(0, \overline{h})$. We prove the following lemma in Appendix B.1:

Lemma 2 (Properties of Function $\mathcal{F}$): *The function $\mathcal{F}$ is strictly increasing ($\mathcal{F}' > 0$), with*

$$\begin{aligned} \lim_{h_t \to 0} \mathcal{F}(h_t) &= -\infty, \\ \lim_{h_t \to \overline{h}} \mathcal{F}(h_t) &= 0. \end{aligned} \tag{33}$$

This lemma directly implies that the static equation (32) has a unique solution in h_t if

$$0 \leq \frac{I^m}{\mu} < \frac{1}{\beta} \tag{34}$$

and no solution otherwise, and that this solution is

$$h \equiv \mathcal{F}^{-1}\left(\frac{\beta I^m}{\mu} - 1\right). \tag{35}$$

Therefore, we get the following proposition:

Proposition 1 (Time-Invariant Equilibrium Under Flexible Prices): *In the benchmark model with flexible prices, under the permanent pegs $I_t^m = I^m$ and $\mu_t = \mu$,*

(i) when $I^m/\mu \geq \beta^{-1}$, there is no time-invariant equilibrium;

(ii) when $0 \leq I^m/\mu < \beta^{-1}$, there is a unique time-invariant equilibrium; in this equilibrium, the employment level is strictly increasing in I^m/μ.

In the unique time-invariant equilibrium, real money balances are constant over time, so that the price level grows at the same rate (μ) as nominal reserves. The Euler equation (9) then implies that I_t^b, the interest rate on a bond that serves as a pure store of value, takes the value $I^b \equiv \mu/\beta$ in equilibrium. Condition (34) requires that the IOR rate be set strictly below I^b. When $I^m \geq I^b$, there is no time-invariant equilibrium because banks would be tempted to issue infinite amounts of debt and deposit the proceeds at the central bank. When $I^m < I^b$, the first-order condition (12) implies that the convenience yield of bank reserves is positive (i.e., we have $\Gamma_m < 0$ in equilibrium), and this basically pins down the demand for real reserves. Since the nominal stock of reserves is exogenous, pinning down the real demand also pins down the path of the price level.

Proposition 1 thus implies that the type of indeterminacy discussed in Sargent and Wallace (1975) does not arise in our model. This type of indeterminacy associates any value in a continuum of initial price levels with the same time-invariant path for real variables (and inflation). In our setup, the initial price level is uniquely pinned down in the time-invariant equilibrium.

In the unique time-invariant equilibrium, the employment level h is strictly increasing in I^m/μ, the equilibrium real IOR rate. This is because an increase in I^m/μ reduces the opportunity cost of holding reserves $I^b/I^m = \mu/(\beta I^m)$. The lower opportunity cost, in turn, decreases the banking cost Γ and the banking spread I^ℓ/I^b. The lower spread (borrowing cost) increases the real wage, which stimulates employment and output. So our model exhibits a departure from superneutrality: money growth affects output in the time-invariant equilibrium.

Our cashless model, as it stands, does not imply a zero lower bound (ZLB) for the net nominal IOR rate. Since banks cannot use vault cash instead of deposits at the central bank, values of the gross nominal IOR rate I^m below one are also consistent with a time-invariant equilibrium. It is easy to modify our model to introduce a ZLB by assuming that vault cash (with no interest payments) is a perfect substitute for deposits at the central bank. More realistically, an expanded model in which vault cash is substitutable to some extent for deposits at the central bank could imply a positive effective lower bound for I^m, although there is no particular reason to think that this lower bound would be unity.

3.3 Other Perfect-Foresight Equilibria Under Permanent Pegs

We now turn to the characterization of the other (i.e., time-varying) perfect-foresight equilibria under permanent pegs. We start by rewriting the dynamic equation (31), when $I_t^m = I^m$ and $\mu_t = \mu$, as

$$1 + \mathcal{F}(h_t) = \frac{\beta I^m}{\mu} \mathbb{E}_t \left\{ \frac{\mathcal{G}(h_{t+1})}{\mathcal{G}(h_t)} \right\}, \tag{36}$$

or equivalently, when $0 < I^m/\mu < \beta^{-1}$, as

$$\mathcal{F}(h_t) - \mathcal{F}(h) = \frac{\beta I^m}{\mu} \mathbb{E}_t \left\{ \frac{\mathcal{G}(h_{t+1})}{\mathcal{G}(h_t)} - 1 \right\}, \tag{37}$$

where the function $\mathcal{G}$ is defined over $\left(0, \overline{h}\right)$ by

$$\mathcal{G}(h_t) \equiv u' \left[f(h_t) \right] \mathcal{M}(h_t).$$

We then prove the following lemma in Appendix B.2:

Lemma 3 (Properties of Function $\mathcal{G}$): *The function $\mathcal{G}$ is strictly increasing ($\mathcal{G}' > 0$), with*

$$\lim_{h_t \to 0} \mathcal{G}(h_t) = 0, \tag{38}$$

$$\lim_{h_t \to \overline{h}} \mathcal{G}(h_t) = +\infty. \tag{39}$$

Using Lemmas 2 and 3, we can easily show that all time-varying perfect-foresight equilibria move away from the time-invariant equilibrium over time: if $h_t > h$, then we sequentially get $\mathcal{F}(h_t) > \mathcal{F}(h)$ (using $\mathcal{F}' > 0$), $\mathcal{G}(h_{t+1}) > \mathcal{G}(h_t)$ (using the dynamic equation (37)), and $h_{t+1} > h_t$ (using $\mathcal{G}' > 0$). More specifically, we establish the following proposition in Appendix B.3:

Proposition 2 (Time-Varying Perfect-Foresight Equilibria Under Flexible Prices): *In the benchmark model with flexible prices, under the permanent pegs $I_t^m = I^m$ and $\mu_t = \mu$,*

(i) when $I^m/\mu \geq \beta^{-1}$, there is no time-varying perfect-foresight equilibrium;

(ii) when $1 < I^m/\mu < \beta^{-1}$, there is an infinity of time-varying perfect-foresight equilibria; these equilibria are indexed by $h_0 \in (h, \overline{h})$ and involve a sequence $\{h_t\}_{t \in \mathbb{N}}$ that is strictly increasing and converges towards $\overline{h}$;

(iii) when $0 \leq I^m/\mu \leq 1$ and under helicopter drops, there is no time-varying perfect-foresight equilibrium;

(iv) when $0 \leq I^m/\mu \leq 1$ and under open-market operations, there is an infinity of time-varying perfect-foresight equilibria; these equilibria are of the same type as those in (ii).

Proposition 2 implies that all time-varying perfect-foresight equilibria involve "deflationary bubbles" that increase real money balances, reduce banking costs, and raise employment over time.[24] These equilibria are unusual in that they make real money balances m_t grow asymptotically at the gross rate $\mu/(\beta I^m) > 1$, so that the price level grows at a rate permanently lower than the nominal money stock, while the consumption level converges towards a finite value. They seem unlikely to be of practical relevance in the context of the policy questions we want to address in this paper. In the rest of the paper, following standard practice, we will ignore them and focus instead on the "determinate" time-invariant equilibrium, which seems a natural "focal point" on which private agents can coordinate.

Proposition 2 also implies that our model rules out "inflationary bubbles" that erode real money balances, raise banking costs, and reduce employment over time. The absence of inflationary

[24] We follow the literature and qualify these equilibria as deflationary, although they do not necessarily make the price level decrease over time; rather, they make the (positive or negative) growth rate of the price level lower than the growth rate of the nominal money stock. Note also that these equilibria deliver higher welfare than the time-invariant equilibrium, which has too little employment.

bubbles mainly rides on the way our model makes bank loans necessary for production, and our assumption (4) makes holding reserves essential for banking. We know from earlier work (e.g., Kingston, 1982, Obstfeld and Rogoff, 1983) that models with fiat money typically have equilibria in which money becomes worthless asymptotically, unless money is "essential" in some sense.[25] We could modify our model (e.g., take out (4) or allow for some production that does not require bank loans) and make inflationary bubbles converging to barter possible. But we will not pursue this point because it is neither theoretically novel nor likely to be of practical relevance in the context of the policy questions we want to address in this paper.

Finally, Proposition 2 implies that when $0 \leq I^m/\mu \leq 1$, deflationary bubbles can arise only under open-market operations, not under helicopter drops. Money injections via helicopter drops involve bond-financed fiscal transfers to households, with the central bank issuing reserves to purchase the bonds. In this case, the net real assets of households (a_t) increase asymptotically at the same rate as real reserves (m_t), i.e. at the gross rate $\mu/(\beta I^m) > 1$. When $I^m/\mu \leq 1$, this violates the transversality condition (13). Alternatively, under open-market operations, i.e. when the central bank injects money by acquiring bonds issued (or previously held) by the private sector, a_t can be constant while m_t grows, so that the transversality condition is met.

4 Local Analysis Under Sticky Prices

We now turn to the sticky-price version of our benchmark model. Because the study of global dynamic perfect-foresight equilibria is too complex under Calvo's (1983) price-setting assumption, we study local rational-expectations equilibria. More specifically, we assume that I_t^m can vary exogenously around a given value $I^m \in (0, \beta^{-1})$, and μ_t around the value $\mu = 1$. Whether prices are flexible or sticky à la Calvo (1983) does not matter for existence and uniqueness of a steady-state equilibrium when $\mu = 1$. Therefore, Proposition 1 still holds when "flexible prices" is replaced by "sticky prices and constant nominal reserves." Thus, the model has a unique steady state (corresponding to the unique time-invariant equilibrium under flexible prices), and this steady state has zero inflation. We log-linearize the model in the neighborhood of this steady state, show that there is a unique local rational-expectations equilibrium, and interpret this determinacy result with the help of a "shadow Wicksellian rule."

4.1 Determinacy Under Exogenous Monetary Policy

Following Calvo (1983), we assume that each firm, whatever its history, has the probability $\theta \in (0, 1)$ not to be allowed to reset its price in any period. In Appendix C.1, we show that

[25] In an endowment economy with separable utility $v(m)$ from holding real money balances, money is essential if the "super Inada condition" $\lim_{m \to 0} mv'(m) > 0$ is satisfied. Kingston (1982) and Obstfeld and Rogoff (1983) summarize earlier contributions suggesting that this condition is quite restrictive, and show that inflationary bubbles can arise in equilibrium if this condition is not satisfied.

the aggregate price-setting behavior of firms is then described by the following log-linearized Phillips curve:

$$\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa_y \widehat{y}_t - \kappa_m \widehat{m}_t, \tag{40}$$

where variables with hats denote log deviations from steady-state values, $\pi_t \equiv \log(\Pi_t)$, and $\kappa_y > 0$ and $\kappa_m > 0$ are two reduced-form parameters defined in Appendix C.1. This Phillips curve departs from the standard NK Phillips curve in two respects. First, the parameter κ_y now depends (positively) on $\Gamma_{\ell\ell}$, as an increase in output raises firms' marginal cost of production also through the resulting increase in banking costs. Second, and more importantly, a new term appears on the right-hand side, $-\kappa_m \widehat{m}_t$, which reflects a cost channel of monetary policy: an increase in real reserve balances reduces firms' marginal cost of production through the resulting decrease in banking costs. The parameter κ_m thus depends (positively) on $|\Gamma_{\ell m}|$.

Log-linearizing the Euler equation (9), and using the goods-market-clearing condition (19), gives the standard IS equation

$$\widehat{y}_t = \mathbb{E}_t \{\widehat{y}_{t+1}\} - \frac{1}{\sigma} \mathbb{E}_t \left\{ i_t^b - \pi_{t+1} \right\}, \tag{41}$$

where $i_t^b \equiv \widehat{I}_t^b$ and $\sigma \equiv -u''(c)c/u'(c) > 0$ (c denoting the steady-state consumption level). This IS equation involves the interest rate on bonds, i_t^b, which is not directly controlled by the central bank. To relate this interest rate to the monetary-policy instruments, we log-linearize the first-order condition (12) in Appendix C.2 and get

$$i_t^b - i_t^m = \sigma \delta_y \widehat{y}_t - \sigma \delta_m \widehat{m}_t, \tag{42}$$

where $i_t^m \equiv \widehat{I}_t^m$ and $\delta_y > 0$ and $\delta_m > 0$ are two reduced-form parameters defined in Appendix C.2. Thus, the spread between the interest rates on bonds and on reserves depends positively on output and negatively on real money balances. The reason is that this spread represents the marginal opportunity cost of holding reserves (rather than bonds serving only as stores of value). It has to be equal to the marginal benefit of holding reserves, i.e. the marginal effect of reserves on banking costs, which depends positively on loans and hence output, and negatively on reserves. The parameter δ_y thus depends (positively) on $|\Gamma_{\ell m}|$, and the parameter δ_m (positively) on Γ_{mm}.

Using the Phillips curve (40), the IS equation (41), the spread equation (42), and the (first difference of the) log-linearized money-market-clearing condition

$$\pi_t = -\left(\widehat{m}_t - \widehat{m}_{t-1}\right) + \widehat{\mu}_t, \tag{43}$$

we then get the following dynamic equation in $\widehat{m}_t$:

$$\mathbb{E}_t \left\{ L\mathcal{P}\left(L^{-1}\right) \widehat{m}_t \right\} = \frac{\kappa_y}{\beta\sigma} i_t^m + \mathbb{E}_t \left\{ \mathcal{Q}\left(L^{-1}\right) \widehat{\mu}_t \right\}, \tag{44}$$

where L denotes the lag operator and

$$\begin{aligned}\mathcal{P}(X) &\equiv X^3 - \left[(1+\delta_y) + \frac{1+\beta-\kappa_m}{\beta} + \frac{\kappa_y}{\beta\sigma}\right] X^2 + ... \\ &\quad \left[(1+\delta_y)\frac{1+\beta-\kappa_m}{\beta} + \frac{1}{\beta} + \left(\frac{1}{\sigma}+\delta_m\right)\frac{\kappa_y}{\beta}\right] X - \left(\frac{1+\delta_y}{\beta}\right), \\ \mathcal{Q}(X) &\equiv X^2 - \left[(1+\delta_y) + \frac{1}{\beta} + \frac{\kappa_y}{\beta\sigma}\right] X + \left(\frac{1+\delta_y}{\beta}\right).\end{aligned}$$

The right-hand side of the dynamic equation (44) involves only monetary-policy instruments and is therefore exogenous, so that $\mathcal{P}(X)$ is the (monic) characteristic polynomial of this dynamic equation. In Appendix C.3, we establish the following lemma $-$ which holds whatever the functional forms of the (dis)utility and production functions u, v, v^b, f, and f^b, the values of the structural parameters $\beta \in (0,1)$, $\varepsilon > 0$, and $\theta \in (0,1)$, and the steady-state value of the IOR rate $I^m \in (0, \beta^{-1})$:

Lemma 4 (Roots of Polynomial $\mathcal{P}$): *The roots of $\mathcal{P}(X)$ are three real numbers ρ, ω_1, and ω_2 such that*

$$0 < \rho < 1 < \omega_1 < \omega_2.$$

This lemma straightforwardly implies that Blanchard and Kahn's (1980) conditions are met, so that we get the following proposition:

Proposition 3 (Local-Equilibrium Determinacy Under Sticky Prices): *In the benchmark model with sticky prices, when I_t^m and μ_t vary exogenously around the values $I^m \in (0, \beta^{-1})$ and $\mu = 1$, there is a unique rational-expectations equilibrium in the neighborhood of the unique steady state.*

This proposition can be viewed as the local counterpart, under sticky prices, of Propositions 1 and 2. On the one hand, it is more restrictive than these propositions, as it deals with local equilibria, not global ones. On the other hand, it extends these propositions along three dimensions: (i) rational-expectations equilibria, not just perfect-foresight equilibria; (ii) exogenous monetary-policy instruments, not just time-invariant ones; and, more importantly, (iii) any degree of price stickiness, not just the limit case of perfect price flexibility (corresponding to $\theta = 0$).

Proposition 3 overturns a well known result of the NK literature. In the basic NK model, there is only one interest rate, namely the interest rate on bonds, and setting it exogenously leads to local-equilibrium *indeterminacy* for all structural-parameter values, as shown in Woodford (2003, Chapter 4) and Galí (2015, Chapter 4). In most extensions of the basic NK model, there is also only one interest rate, and setting it exogenously typically leads to local-equilibrium indeterminacy as well.[26] In our model, by contrast, there are two interest rates, one on bonds and the other on reserves, and setting exogenously the interest rate on reserves (together with the growth rate of nominal reserves) leads to local-equilibrium *determinacy* for all functional forms of the (dis)utility and production functions u, v, v^b, f, and f^b, and all values of the structural and policy parameters β, ε, θ, and I^m.

4.2 A Shadow Wicksellian Rule

The key element at the source of this determinacy result is the equilibrium relationship (42) between the interest rate on bonds, the interest rate on reserves, output, and real money balances. This relationship can be viewed as a "shadow rule" for the interest rate on bonds i_t^b, as if the central bank directly controlled this interest rate. Since the IOR rate and nominal money balances are set exogenously, this shadow rule is "Wicksellian" in the terminology of Woodford (2003): it makes i_t^b react positively to output, the price level (through $\widehat{m}_t$), and no other endogenous variable. It is well known that Wicksellian rules always ensure local-equilibrium determinacy in the basic NK model $-$ as Woodford (2003, Chapter 4) shows. Our model, however, differs from the basic NK model in that it features a cost channel of monetary policy (i.e. $\kappa_m > 0$). We would not always get local-equilibrium determinacy if the parameters κ_y and κ_m of the Phillips curve and the coefficients $\sigma\delta_y$ and $\sigma\delta_m$ of the shadow Wicksellian rule for i_t^b were allowed to take any independent positive values. We always get local-equilibrium determinacy only because these reduced-form parameters and coefficients are related to each other through inequality constraints, as they come from the same primitive functions (most notably the production function f^b and the disutility function v^b).

More specifically, the two key inequality constraints that are proved and used in Appendix C.3 to establish local-equilibrium determinacy are

$$\delta_y \kappa_m < \delta_m \kappa_y, \tag{45}$$

$$\sigma \delta_m < \delta_y. \tag{46}$$

The inequality (45), in particular, corresponds to the well known "Taylor principle" discussed by Woodford (2003, Chapter 4). This principle, which is a necessary condition for local-equilibrium determinacy in the basic NK model under a variety of interest-rate rules, states that the nominal interest rate should react more than one-to-one to the inflation rate in the long run. In our model, the relationship between the nominal interest rate on bonds and the inflation rate in the long run can be easily derived from (40), (42), and (43) as $\Delta i^b = (\delta_m \kappa_y - \delta_y \kappa_m)\sigma\pi/\kappa_y$, where Δi^b and π denote the long-run values of $i_t^b - i_{t-1}^b$ and π_t respectively. Thus, the inequality (45) is both necessary and sufficient for the shadow rule for i_t^b implied by the exogenous setting of i_t^m and $\widehat{\mu}_t$ to satisfy the Taylor principle.

[26] Woodford (2003, Chapter 8) calls the latter result the "Sargent-Wallace property" of NK models.

Another way to see the key role played by this shadow rule in Proposition 3's determinacy result is to consider for a moment the alternative case in which the central bank sets $\widehat{\mu}_t$ exogenously, adopts an exogenous target i_t^{b*} for the interest rate on bonds i_t^b, and sets endogenously the interest rate on reserves i_t^m according to the feedback rule

$$i_t^m = i_t^{b*} - \sigma\delta_y\widehat{y}_t + \sigma\delta_m\widehat{m}_t, \tag{47}$$

which corresponds to (42) in which i_t^b is replaced by i_t^{b*}. In this case, the central bank hits its target i_t^{b*} both in and out of equilibrium, as (42) and (47) imply $i_t^b = i_t^{b*}$. Then, the dynamics of $\widehat{y}_t$, π_t, and $\widehat{m}_t$ are governed by the three-equation system made of the Phillips curve (40), the IS equation (41) in which i_t^b is replaced by i_t^{b*}, and the money-market-clearing condition (43), while i_t^m is residually determined by the feedback rule (47). Thus, the relationship (42) plays no role in local-equilibrium (in)determinacy in this case. Using (40), (41) with $i_t^b = i_t^{b*}$, and (43), we then get the following dynamic equation in π_t:

$$\beta\mathbb{E}_t\left\{\pi_{t+2}\right\} - \left[1 + \beta + \left(\frac{\kappa_y}{\sigma} - \kappa_m\right)\right]\mathbb{E}_t\left\{\pi_{t+1}\right\} + \pi_t = -\frac{\kappa_y}{\sigma}i_t^{b*} + \kappa_m\mathbb{E}_t\left\{\widehat{\mu}_{t+1}\right\}.$$

Now, the inequalities (45) and (46) together imply the inequality $\kappa_y/\sigma - \kappa_m > 0$. Using the last inequality, we easily show that the characteristic polynomial of this dynamic equation has one root inside the unit circle and one root outside.[27] Thus, when the central bank "sets" exogenously i_t^b and $\widehat{\mu}_t$, we always get local-equilibrium *indeterminacy*. What matters for Proposition 3's determinacy result is not the exogeneity of either interest rate and the money-growth rate: it is the exogeneity of the IOR rate and the money-growth rate.

5 Resolution of the Puzzles and Paradox

In the basic NK model and its usual extensions, pegging the interest rate temporarily at its zero lower bound has three puzzling and paradoxical implications: the forward-guidance puzzle, the fiscal-multiplier puzzle, and the paradox of flexibility. In this section, we show that our model does not share any of these implications. The reason is that these implications are mechanically connected to NK models' property of exhibiting indeterminacy under a permanent interest-rate peg. In our model, once interest rates get close to some effective lower bound, the central bank has to suspend whatever rule it would pursue under more normal circumstances and instead peg temporarily the IOR rate, which is the interest rate that it directly controls.[28] Because a permanent peg of the IOR rate (and the money-growth rate) delivers determinacy, a temporary peg of the IOR rate (and the money-growth rate) does not give rise to any of the puzzles and paradox.[29]

[27] This characteristic polynomial is isomorphic to its counterpart in the basic NK model under an interest-rate peg, as one moves from the latter to the former simply by replacing $\kappa/\sigma > 0$ by $\kappa_y/\sigma - \kappa_m > 0$, where κ denotes the slope of the standard NK Phillips curve.

[28] To introduce a zero lower bound for the net IOR rate (i.e. $I_t^m \geq 1$) into our model, we only need to assume that vault cash (with no interest payments) is a perfect substitute for reserves in reducing banking costs.

[29] For simplicity, we maintain in this section the assumption that the money-growth rate is exogenous. In our view, this assumption is not necessarily a bad approximation of reality. The way central banks have conducted

5.1 The Forward-Guidance Puzzle

The first puzzle that we consider is the so-called "forward-guidance puzzle," which can be summarized as follows: in the basic NK model, the effects of a temporary interest-rate peg on current inflation and output become unboundedly large as the duration of this peg goes to infinity, if the central bank is expected to revert, at the end of the peg, to an interest-rate rule ensuring local-equilibrium determinacy. This result, shown in Werning (2012), Carlstrom, Fuerst, and Paustian (2015), and Cochrane (2016a), is a puzzle because a *permanent* interest-rate peg has *finite* effects on inflation and output.[30] There is, thus, a stark discontinuity in the limit as the duration of the peg goes to infinity.

The temporary interest-rate peg in question may be due to a situation in which (i) a liquidity trap compels the central bank to keep the interest rate at its lower bound during T_1 periods, and (ii) the central bank promises to keep the interest rate at its lower bound during T_2 periods after the end of the trap (hence the name of "forward-guidance puzzle"). The puzzle in this case has two manifestations: the log-deviations of inflation and output from their steady-state values, at the start of the trap, become infinitely negative as $T_1 \to +\infty$ for a given T_2, and infinitely positive as $T_2 \to +\infty$ for a given T_1. These two manifestations are, of course, mirror images of each other.[31]

To see how our model solves this puzzle, consider a temporary peg of the IOR rate: assume that the economy is at the steady state at date 0 (so that $\widehat{m}_0 = 0$), and that, unexpectedly, i_t^m (the log deviation of the IOR rate from its steady-state value) takes the value i^* from date 1 to date T, and the value zero from date $T+1$ onwards. As in Werning (2012), Carlstrom, Fuerst, and Paustian (2015), and Cochrane (2016a), we assume for simplicity that T is deterministic and known at date 1. In the NK literature, liquidity traps are typically obtained as the result of a large negative discount-factor shock. For simplicity again, following Carlstrom, Fuerst, and Paustian (2015) and Cochrane (2016a), we do not explicitly introduce such a shock into our model. Thus, in our setup, a positive value of i^* can represent a liquidity-trap situation in which setting the interest rate at its lower bound is not enough to offset the negative discount-factor shock; and a negative value of i^* can represent a situation in which, in accordance with some earlier forward guidance, the central bank keeps the interest rate at its lower bound even though the negative discount-factor shock has ceased to affect the economy.

In the absence of money-growth-rate shocks (i.e., when $\widehat{\mu}_t = 0$), the dynamic equation (44) can

balance-sheet policies over the past few years seems to us more like discretionary changes in policy than rules: we cannot detect any systematic pattern of balance-sheet policies reacting to, say, quarterly data, and the Federal Reserve's current policy is to keep the size of its balance sheet constant (replacing assets as they mature).

[30] A permanent interest-rate peg generates multiple local equilibria in the basic NK model. What we mean is that inflation and output take finite values in each of these equilibria.

[31] Yet another way to see the same puzzle is to note, as Cochrane (2016a) puts it, that "a constant small probability of a trap at any date in the future produces an *infinitely* negative inflation and output today" (his emphasis).

be rewritten as

$$\mathbb{E}_t \left\{ (1-\omega_1 L)(1-\omega_2 L)\, q_{t+2} \right\} = \frac{\kappa_y}{\beta\sigma} i_t^m, \tag{48}$$

where $q_t \equiv \widehat{m}_t - \rho \widehat{m}_{t-1}$. The unique stationary solution to this dynamic equation from date $T+1$ onwards is $q_t = 0$ for $t \geq T+1$. This dynamic equation (48), taken from date 1 to date T, also implies that there exists $(q^*, a_1, a_2) \in \mathbb{R}^3$ such that

$$q_t = \left(1 - a_1 \omega_1^{t-T-1} + a_2 \omega_2^{t-T-1}\right) q^*$$

for $1 \leq t \leq T+2$. We easily get

$$q^* = \frac{\kappa_y i^*}{\beta\sigma (\omega_1 - 1)(\omega_2 - 1)}$$

and, using the terminal conditions $q_{T+1} = q_{T+2} = 0$,

$$a_1 = \frac{\omega_2 - 1}{\omega_2 - \omega_1} > 0 \quad \text{and} \quad a_2 = \frac{\omega_1 - 1}{\omega_2 - \omega_1} > 0.$$

Using the initial condition $\widehat{m}_0 = 0$, the Phillips curve (40), and the money-market-clearing condition (43), we then get

$$\pi_1 = -\left(1 - a_1 \omega_1^{-T} + a_2 \omega_2^{-T}\right) q^*, \tag{49}$$

$$\begin{aligned} \widehat{y}_1 = & \left\{ -(1-\beta\rho-\kappa_m) - a_1 \omega_1^{-T} \left[\beta(\omega_1+\rho-1) + \kappa_m - 1\right] + ... \right. \\ & \left. a_2 \omega_2^{-T} \left[\beta(\omega_2+\rho-1) + \kappa_m - 1\right] \right\} \frac{q^*}{\kappa_y}. \end{aligned} \tag{50}$$

As the duration of the peg goes to infinity, π_1 and $\widehat{y}_1$ converge towards some finite values:

$$\lim_{T\to+\infty} \pi_1 = -q^* \quad \text{and} \quad \lim_{T\to+\infty} \widehat{y}_1 = \frac{-(1-\beta\rho-\kappa_m)\, q^*}{\kappa_y}.$$

These finite limit values coincide with the values that π_1 and $\widehat{y}_1$ would take under a permanent peg. Indeed, if i_t^m took the value i^* at all dates $t \geq 1$, then the unique stationary solution of the dynamic equation (48) would be $q_t = q^*$ for $t \geq 1$. Using $\widehat{m}_0 = 0$, the Phillips curve (40), and the money-market-clearing condition (43), we would then get $\pi_1 = -q^*$ and $\widehat{y}_1 = -(1-\beta\rho-\kappa_m)\, q^*/\kappa_y$. As a consequence, we obtain the following result:

Proposition 4 (Resolution of the Forward-Guidance Puzzle): *In the benchmark model with sticky prices, the responses of π_1 and $\widehat{y}_1$ to a temporary IOR-rate peg of expected duration T ($i_t^m = i^*$ for $1 \leq t \leq T$, $i_t^m = 0$ for $t \geq T+1$) converge, as T goes to $+\infty$, towards their (finite) responses to the corresponding permanent IOR-rate peg ($i_t^m = i^*$ for $t \geq 1$).*

In the basic NK model, as pointed out by Carlstrom, Fuerst, and Paustian (2015) and Cochrane (2016a), the source of the forward-guidance puzzle lies in the model's property of exhibiting indeterminacy under a permanent interest-rate peg. Indeed, this property implies that, under a temporary interest-rate peg, the dynamic system has a "stable eigenvalue" (i.e. an eigenvalue whose modulus is lower than one) that is not matched by any predetermined variable. As a consequence, starting from some terminal condition (at the end of the peg), the economy explodes backward in time. In our model, by contrast, the stable eigenvalue (ρ) is matched by a predetermined variable, namely the lagged money stock ($\widehat{m}_{t-1}$). This is the feature that delivers determinacy under a permanent IOR-rate peg and, therefore, also solves the forward-guidance puzzle.

5.2 The Fiscal-Multiplier Puzzle

We now turn to what we call the "fiscal-multiplier puzzle." Consider an interest-rate peg of known duration, and assume that the government credibly announces that it will increase fiscal expenditures by a given amount at the end of the peg. In the basic NK model, as shown in Farhi and Werning (2016) and Cochrane (2016a), the effect of this expected future fiscal expansion on inflation and output at the start of the peg grows exponentially with the duration of the peg, if the central bank is expected to revert, at the end of the peg, to an interest-rate rule ensuring local-equilibrium determinacy.[32] In fact, this effect still grows exponentially even when the amount of fiscal expenditures declines towards zero as the duration of the peg goes to infinity, provided that it does not decline too fast. Thus, news about fiscal expenditures that are both vanishingly distant and vanishingly small can have exploding effects today.

As pointed out by Cochrane (2016a), the culprit is, again, the basic NK model's property of exhibiting indeterminacy under a permanent interest-rate peg. Indeed, this property implies that, under a temporary interest-rate peg, the dynamic system has a stable eigenvalue but no predetermined variable. As a consequence, when we iterate the model forward in time, this eigenvalue magnifies the effects of the fiscal expansion (at the end of the peg) on initial outcomes (at the start of the peg), so that these effects grow explosively as the duration of the peg goes to infinity, giving rise to the fiscal-multiplier puzzle. Our model, by contrast, delivers determinacy under a permanent IOR-rate peg, as we have shown in the previous section. As a consequence, it solves the fiscal-multiplier puzzle, as we now show.

We introduce exogenous government expenditures into our model, assuming that they enter households' utility function in a separable way. At each date t, the government consumes g_t goods. The log-linearized goods-market-clearing condition becomes

$$\widetilde{c}_t + \widetilde{g}_t = \widehat{y}_t \tag{51}$$

with $\widetilde{c}_t \equiv (c/y)\widehat{c}_t$ and $\widetilde{g}_t \equiv (g/y)\widehat{g}_t$, where variables without time subscript denote steady-state

[32] The analyses of Christiano, Eichenbaum, and Rebelo (2011), Eggertsson (2011), and Woodford (2011) are related but distinct in two main ways. First, they assume that the length of the liquidity-trap period is stochastic, not deterministic. Second, they consider a fiscal expansion that lasts over the whole liquidity-trap period. Woodford (2011) shows that the resulting fiscal multiplier becomes unboundedly large as the probability to remain in the liquidity trap approaches a certain value (strictly lower than one).

values. Using this condition, we can rewrite the log-linearized Euler equation as the IS equation

$$\widehat{y}_t = \mathbb{E}_t\left\{\widehat{y}_{t+1}\right\} - \frac{1}{\widetilde{\sigma}}\mathbb{E}_t\left\{i_t^b - \pi_{t+1}\right\} + \widetilde{g}_t - \mathbb{E}_t\left\{\widetilde{g}_{t+1}\right\}, \tag{52}$$

where $\widetilde{\sigma} \equiv (y/c)\sigma = -u''(c)y/u'(c)$. In Appendix C.4, we show that the log-linearized Phillips curve and spread equation become respectively

$$\pi_t = \beta\mathbb{E}_t\left\{\pi_{t+1}\right\} + \widetilde{\kappa}_y\widehat{y}_t - \widetilde{\kappa}_m\widehat{m}_t - \kappa_g\widetilde{g}_t, \tag{53}$$

$$i_t^b - i_t^m = \widetilde{\sigma}\widetilde{\delta}_y\widehat{y}_t - \widetilde{\sigma}\widetilde{\delta}_m\widehat{m}_t - \widetilde{\sigma}\delta_g\widetilde{g}_t, \tag{54}$$

where $\widetilde{\kappa}_y$, $\widetilde{\kappa}_m$, κ_g, $\widetilde{\delta}_y$, $\widetilde{\delta}_m$, and δ_g are six positive reduced-form parameters defined in Appendix C.4. Using the money-market-clearing condition (43), the IS equation (52), the Phillips curve (53), and the spread equation (54), we then get the following dynamic equation in $\widehat{m}_t$ in the absence of monetary-policy shocks (i.e., when $i_t^m = \widehat{\mu}_t = 0$):

$$\beta\mathbb{E}_t\left\{L\widetilde{\mathcal{P}}\left(L^{-1}\right)\widehat{m}_t\right\} = \left[\left(1+\widetilde{\delta}_y\right)\kappa_g - \left(1+\delta_g\right)\widetilde{\kappa}_y\right]\widetilde{g}_t + \left(\widetilde{\kappa}_y - \kappa_g\right)\mathbb{E}_t\left\{\widetilde{g}_{t+1}\right\}, \tag{55}$$

where $\widetilde{\mathcal{P}}(X)$ denotes the polynomial obtained by replacing $(\sigma, \kappa_y, \kappa_m, \delta_y, \delta_m)$ by $(\widetilde{\sigma}, \widetilde{\kappa}_y, \widetilde{\kappa}_m, \widetilde{\delta}_y, \widetilde{\delta}_m)$ in $\mathcal{P}(X)$. We also show in Appendix C.4 that $(\widetilde{\sigma}, \widetilde{\kappa}_y, \widetilde{\kappa}_m, \widetilde{\delta}_y, \widetilde{\delta}_m)$ satisfies the same key inequalities as $(\sigma, \kappa_y, \kappa_m, \delta_y, \delta_m)$, so that the roots of $\widetilde{\mathcal{P}}(X)$ are, like those of $\mathcal{P}(X)$, one real number strictly between zero and one (noted $\widetilde{\rho}$) and two real numbers strictly higher than one (noted $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$, with $\widetilde{\omega}_1 < \widetilde{\omega}_2$). The dynamic equation (55) can thus be rewritten as

$$\beta\mathbb{E}_t\left\{\left(1-\widetilde{\omega}_1L\right)\left(1-\widetilde{\omega}_2L\right)\widetilde{q}_{t+2}\right\} = \left[\left(1+\widetilde{\delta}_y\right)\kappa_g - \left(1+\delta_g\right)\widetilde{\kappa}_y\right]\widetilde{g}_t + \left(\widetilde{\kappa}_y - \kappa_g\right)\mathbb{E}_t\left\{\widetilde{g}_{t+1}\right\}, \tag{56}$$

where $\widetilde{q}_t \equiv \widehat{m}_t - \widetilde{\rho}\widehat{m}_{t-1}$.

Now assume that the economy is at the steady state at date 0 (so that $\widehat{m}_0 = 0$), and that the government unexpectedly announces at date 1 that (i) $\widetilde{g}_T = \widetilde{g}^* \neq 0$ for some date $T \geq 2$, and (ii) $\widetilde{g}_t = 0$ for all dates $t \geq 1$ such that $t \neq T$.[33] Using (56) from date $T+1$ onwards, we get $\widetilde{q}_t = 0$ for $t \geq T+1$. Then, using (56) at date T and $\widetilde{q}_{T+1} = \widetilde{q}_{T+2} = 0$, we get

$$\widetilde{q}_T = \left[\left(1+\widetilde{\delta}_y\right)\kappa_g - \left(1+\delta_g\right)\widetilde{\kappa}_y\right]\frac{\widetilde{g}^*}{\beta\widetilde{\omega}_1\widetilde{\omega}_2}.$$

Next, using (56) at date $T-1$, $\widetilde{q}_{T+1} = 0$, and the above expression for $\widetilde{q}_T$, we get

$$\widetilde{q}_{T-1} = \left[\widetilde{\kappa}_y - \kappa_g + \frac{\widetilde{\omega}_1 + \widetilde{\omega}_2}{\widetilde{\omega}_1\widetilde{\omega}_2}\left[\left(1+\widetilde{\delta}_y\right)\kappa_g - \left(1+\delta_g\right)\widetilde{\kappa}_y\right]\right]\frac{\widetilde{g}^*}{\beta\widetilde{\omega}_1\widetilde{\omega}_2}.$$

Finally, the dynamic equation (56) taken from date 1 to date $T-2$ implies that there exists $(\widetilde{a}_1, \widetilde{a}_2) \in \mathbb{R}^2$ such that

$$\widetilde{q}_t = \left(\widetilde{a}_1\widetilde{\omega}_1^{t-T-1} - \widetilde{a}_2\widetilde{\omega}_2^{t-T-1}\right)\frac{\widetilde{g}^*}{\beta\left(\widetilde{\omega}_2 - \widetilde{\omega}_1\right)}$$

[33]For simplicity, we assume that the IOR rate is pegged at its steady-state value not only from date $T+1$ onwards, but also between dates 1 and T (i.e., $i^* = 0$). This assumption does not affect our results since the effect of $\widetilde{g}^*$ is independent of the effect of i^* in our log-linearized setup.

for $1 \leq t \leq T$. The reduced-form parameters $\widetilde{a}_1$ and $\widetilde{a}_2$ are easily obtained, from the above terminal conditions on $\widetilde{q}_{T-1}$ and $\widetilde{q}_T$, as

$$\widetilde{a}_j = \widetilde{\omega}_j \left(\widetilde{\kappa}_y - \kappa_g\right) + \left[\left(1 + \widetilde{\delta}_y\right) \kappa_g - \left(1 + \delta_g\right) \widetilde{\kappa}_y\right] \quad \text{for} \quad j \in \{1, 2\}.$$

Using the initial condition $\widehat{m}_0 = 0$, the money-market-clearing condition (43), and the Phillips curve (53), we then get

$$\pi_1 = \frac{\widetilde{a}_2 \widetilde{\omega}_2^{-T} - \widetilde{a}_1 \widetilde{\omega}_1^{-T}}{\beta \left(\widetilde{\omega}_2 - \widetilde{\omega}_1\right)} \widetilde{g}^*, \tag{57}$$

$$\widehat{y}_1 = \frac{\left[\beta \left(\widetilde{\omega}_1 + \widetilde{\rho} - 1\right) + \widetilde{\kappa}_m - 1\right] \widetilde{a}_1 \widetilde{\omega}_1^{-T} - \left[\beta \left(\widetilde{\omega}_2 + \widetilde{\rho} - 1\right) + \widetilde{\kappa}_m - 1\right] \widetilde{a}_2 \widetilde{\omega}_2^{-T}}{\beta \widetilde{\kappa}_y \left(\widetilde{\omega}_2 - \widetilde{\omega}_1\right)} \widetilde{g}^*, \tag{58}$$

and therefore

$$\lim_{T \to +\infty} \pi_1 = 0 \quad \text{and} \quad \lim_{T \to +\infty} \widehat{y}_1 = 0.$$

This result can be stated in the following way:

Proposition 5 (Resolution of the Fiscal-Multiplier Puzzle): *In the benchmark model with sticky prices, the responses of π_1 and $\widehat{y}_1$ to a given expected fiscal expansion at date T converge towards zero as T goes to $+\infty$.*

Again, the resolution of the puzzle reflects the fact that our model economy does not explode as we go backward in time under a temporary interest-rate peg. And this is because the stable eigenvalue (ρ) is matched by a predetermined variable ($\widehat{m}_{t-1}$).

5.3 The Paradox of Flexibility

The last puzzle that we consider is the so-called "paradox of flexibility." In the basic NK model, the effects of an interest-rate peg of given finite duration on inflation and output become unboundedly large as prices become perfectly flexible, if the central bank is expected to revert, at the end of the peg, to an interest-rate rule ensuring local-equilibrium determinacy.[34] Similarly, the effect on inflation and output of a given fiscal expansion at the end of the peg also grows explosively as prices become perfectly flexible. These results, shown in Werning (2012), Farhi and Werning (2016), and Cochrane (2016a), are puzzling because output and fiscal multipliers take finite values in the limit case of perfectly flexible prices (while inflation is indeterminate).[35] There is, thus, a stark discontinuity in the limit as the degree of price stickiness goes to zero.

Unlike the previous puzzles, the paradox of flexibility is about a discontinuity in the limit not as the duration of the peg goes to infinity, but instead as prices become perfectly flexible. Like

[34] Again, the sign of these unboundedly large effects depend on whether the temporary interest-rate peg in question is due to a liquidity trap ($i^* > 0$) or a forward-guidance policy ($i^* < 0$).

[35] Christiano, Eichenbaum, and Rebelo (2011) find numerically that inflation and output decrease and fiscal multipliers increase with price flexibility in a liquidity trap, but do not obtain limit results.

them, however, it is related to the basic NK model's property of exhibiting indeterminacy under a permanent interest-rate peg. Indeed, under a temporary peg, the stable eigenvalue of the dynamic system, which is not matched by any predetermined variable, converges towards zero as price stickiness vanishes, making endogenous variables explode. In our model, by contrast, endogenous variables converge towards their finite flexible-price values as price stickiness vanishes, because there is no such excess stable eigenvalue.

To prove that our model does indeed solve the paradox of flexibility, we first establish the following lemma in Appendix C.5:

Lemma 5 (Limits of Some Parameters as $\theta \to 0$): *As $\theta \to 0$, we have*

$$\rho \to 0, \qquad \omega_1 \to \omega_1^n \equiv 1 + \frac{\sigma(\delta_m - \delta_y \psi)}{1 - \sigma\psi}, \qquad \omega_2 \to +\infty,$$
$$\kappa_m \rho \to \frac{\sigma\psi(1+\delta_y)}{(1-\sigma\psi) + \sigma(\delta_m - \delta_y\psi)}, \qquad and \qquad \frac{\omega_2}{\kappa_y} \to \frac{1-\sigma\psi}{\beta\sigma},$$

where $\psi \equiv \kappa_m/\kappa_y$ is independent of θ and such that $0 < \psi < \min(\sigma^{-1}, \delta_m/\delta_y)$. These results still hold when κ_y, κ_m, σ, δ_y, δ_m, ψ, ρ, ω_1, ω_2, and ω_1^n are respectively replaced by $\widetilde{\kappa}_y$, $\widetilde{\kappa}_m$, $\widetilde{\sigma}$, $\widetilde{\delta}_y$, $\widetilde{\delta}_m$, $\widetilde{\psi}$, $\widetilde{\rho}$, $\widetilde{\omega}_1$, $\widetilde{\omega}_2$, and $\widetilde{\omega}_1^n$.

Using this lemma, we can easily determine the limits of π_1 in (49) and $\widehat{y}_1$ in (50) as $\theta \to 0$:

$$\lim_{\theta\to 0} \pi_1 = -\left[\frac{1-(\omega_1^n)^{-T}}{\delta_m - \delta_y\psi}\right]\frac{i^*}{\sigma} \quad \text{and} \quad \lim_{\theta\to 0} \widehat{y}_1 = \left[\frac{1-(\omega_1^n)^{-T}}{\delta_m - \delta_y\psi}\right]\frac{\psi i^*}{\sigma}, \tag{59}$$

as well as the limits of π_1 in (57) and $\widehat{y}_1$ in (58) as $\theta \to 0$ (for $T \geq 2$):

$$\lim_{\theta\to 0} \pi_1 = -\left[(1-\vartheta)\,\widetilde{\omega}_1^n + \left(1+\widetilde{\delta}_y\right)\vartheta - (1+\delta_g)\right]\left[\frac{\widetilde{\sigma}\,(\widetilde{\omega}_1^n)^{-T}}{1-\widetilde{\sigma}\widetilde{\psi}}\right]\widetilde{g}^*, \tag{60}$$

$$\lim_{\theta\to 0} \widehat{y}_1 = \left[(1-\vartheta)\,\widetilde{\omega}_1^n + \left(1+\widetilde{\delta}_y\right)\vartheta - (1+\delta_g)\right]\left[\frac{\widetilde{\sigma}\widetilde{\psi}\,(\widetilde{\omega}_1^n)^{-T}}{1-\widetilde{\sigma}\widetilde{\psi}}\right]\widetilde{g}^*, \tag{61}$$

where $\vartheta \equiv \kappa_g/\widetilde{\kappa}_y$ is independent of θ. These limits are finite, unlike their counterparts in the basic NK model. In Appendix B.4, we log-linearize the flexible-price version of our benchmark model (studied in Section 3), with and without fiscal expenditures, and show that the values taken by π_1 and $\widehat{y}_1$ under flexible prices coincide with the above limits. We summarize these results as follows:

Proposition 6 (Resolution of the Paradox of Flexibility): *In the benchmark model with sticky prices, the responses of π_1 and $\widehat{y}_1$ to a temporary IOR-rate peg of expected duration T and to a given expected fiscal expansion at date T converge, as θ goes to 0, towards the (finite) corresponding responses under flexible prices.*

Again, the resolution of the paradox reflects the fact that the stable eigenvalue (ρ) under an interest-rate peg is matched by a predetermined variable ($\widehat{m}_{t-1}$) in our model. So the convergence of this eigenvalue towards zero as prices become perfectly flexible does not translate into explosive equilibrium outcomes under a temporary peg, unlike in the basic NK model.

5.4 The Basic-NK-Model Limit

We have so far shown that our model qualitatively solves the two puzzles and the paradox for any given (dis)utility and production functions u, v, v^b, f, and f^b, any given values of the structural parameters $\beta \in (0,1)$, $\varepsilon > 0$, and $\theta \in (0,1)$, and any given steady-state value of the IOR rate $I^m \in (0, \beta^{-1})$. In this subsection, we first show that the disutility function v^b and the steady-state value I^m can be chosen so as to make our model arbitrarily close, in terms of steady state and reduced form, to the basic NK model characterized by the same (dis)utility and production functions u, v, and f, and the same values of the structural parameters β, ε, and θ. Thus, even an *arbitrarily* small departure from the basic NK model is enough to qualitatively solve the two puzzles and the paradox. We then show that a *vanishingly* small departure from the basic NK model still solves the fiscal-multiplier puzzle and the paradox of flexibility, and attenuates the forward-guidance puzzle.

To show that our model can involve an arbitrarily small departure from the basic NK model in a quantitatively measurable sense, we replace the disutility function v^b by γv^b (and hence the banking-cost function Γ by $\gamma\Gamma$), where $\gamma > 0$ is a scale parameter, and we establish the following proposition in Appendix C.6:

Proposition 7 (Convergence Towards the Steady State and Reduced Form of the Basic NK Model): *As* $(I^m, \gamma) \to (\beta^{-1}, 0)$ *with* $(\beta^{-1} - I^m)/\gamma$ *bounded away from zero and infinity, the steady state and reduced form of the benchmark model with sticky prices converge towards the steady state and reduced form of the basic NK model, i.e.* $h \to h^\star$ *and* $(\kappa_y, \kappa_m, \delta_y, \delta_m) \to (\kappa, 0, 0, 0)$*, where* κ *denotes the slope of the standard NK Phillips curve.*

Making the steady-state IOR rate I^m go to the steady-state interest rate on bonds $I^b = \beta^{-1}$ asymptotically removes the steady-state opportunity cost of holding reserves. Making the banking-cost-scale parameter γ go to zero asymptotically removes: (i) the steady-state marginal banking cost Γ_ℓ, provided that the steady-state value of real reserve balances m is bounded away from zero, and (ii) the steady-state marginal benefit of holding reserves Γ_m, even when m is bounded from above. Imposing that $(\beta^{-1} - I^m)/\gamma$ be bounded away from zero and infinity ensures that the steady-state opportunity cost and marginal benefit of holding reserves go hand in hand to zero, so that m is itself bounded away from zero and infinity. Asymptotically, given that all steady-state costs related to banking and reserve holding are removed, the steady-state employment level takes its frictionless value ($h = h^\star$), the marginal cost of production becomes insensitive to the volume of loans ($\kappa_y = \kappa$), the cost channel of monetary policy is shut down ($\kappa_m = 0$), and the interest-rate spread becomes insensitive to output and reserves ($\delta_y = \delta_m = 0$).

Proposition 7 enables us to consider a sequence of models converging towards the basic NK model, each of them solving qualitatively the puzzles and paradox. In Appendix C.7, we determine the limit of equilibrium outcomes under a temporary IOR-rate peg along this sequence of models, for any given duration of the peg T and any given degree of price stickiness θ. We obtain the following results:

Proposition 8 (Attenuation of the Forward-Guidance Puzzle and Resolution of the Fiscal-Multiplier Puzzle and the Paradox of Flexibility in the Basic-NK-Model Limit): *In the benchmark model with sticky prices, as $(I^m, \gamma) \to (\beta^{-1}, 0)$ with $(\beta^{-1} - I^m)/\gamma$ bounded away from zero and infinity,*

(i) the responses of π_1 and $\widehat{y}_1$ to a temporary IOR-rate peg of expected duration T grow linearly in T as $T \to +\infty$;

(ii) the responses of π_1 and $\widehat{y}_1$ to a given expected fiscal expansion at date T converge towards zero as $T \to +\infty$;

(iii) the responses of π_1 to a temporary IOR-rate peg of expected duration T and to a given expected fiscal expansion at date T converge towards some finite values as $\theta \to 0$;

(iv) the responses of $\widehat{y}_1$ to a temporary IOR-rate peg of expected duration T and to a given expected fiscal expansion at date T converge, as $\theta \to 0$, towards the (finite) corresponding responses in the flexible-price version of the basic NK model.

Proposition 8 implies that, in the limit as $(I^m, \gamma) \to (\beta^{-1}, 0)$, the equilibrium outcomes of our model do not behave like the corresponding equilibrium outcomes of the basic NK model when the central bank is expected to revert, at the end of the peg, to an interest-rate rule ensuring local-equilibrium determinacy. First, these limit equilibrium outcomes are immune from the fiscal-multiplier puzzle: their reaction to a given fiscal expansion at the end of the peg does not grow explosively as the duration of the peg goes to infinity, but instead converges towards zero. Second, they are also immune from the paradox of flexibility: output and inflation do not explode as prices become perfectly flexible; instead, output converges towards its (finite) value in the flexible-price version of the basic NK model, and inflation towards some finite value.[36] And third, these limit equilibrium outcomes suffer from a weaker form of the forward-guidance puzzle: they grow (asymptotically) linearly with the duration of the peg, not exponentially. In all three respects, they behave like the corresponding equilibrium outcomes of Mankiw and Reis's (2002) sticky-information model, studied by Carlstrom, Fuerst, and Paustian (2015) and

[36]The limit value of inflation cannot be related to its value in the flexible-price version of the basic NK model, since the latter is indeterminate.

Kiley (2016).[37] Thus, a vanishingly small departure from the basic NK model is enough to solve the fiscal-multiplier puzzle and the paradox of flexibility, and attenuate the forward-guidance puzzle; and this limit result brings the canonical sticky-price model at par with its sticky-information cousin in terms of their ability to solve or attenuate the puzzles and paradox.[38]

The limit equilibrium outcomes studied in Proposition 8 coincide with the outcomes of a particular equilibrium of the basic NK model − out of an infinity of equilibria − when the central bank temporarily pegs the interest rate at a certain value before permanently pegging it at its steady-state value (i.e. when $i_t = i^*$ for $1 \leq t \leq T$ and $i_t = 0$ for $t \geq T+1$, where i_t denotes the log-deviation of the interest rate from its steady-state value). This particular equilibrium differs from what Cochrane (2016a) calls the "standard equilibrium," which corresponds to $\pi_{T+1} = 0$ and suffers from the two puzzles and the paradox. Therefore, Proposition 8 provides theoretical foundations to Cochrane's (2016a) approach of selecting an equilibrium different from the standard equilibrium in the basic NK model under a temporary interest-rate peg. In particular, since our particular equilibrium does not suffer from the paradox of flexibility, Proposition 8 enables us to *endogenize* Cochrane's (2016a) local-to-frictionless equilibrium-selection criterion, which requires that equilibrium outcomes converge towards flexible-price outcomes as prices become perfectly flexible.[39] It does not, however, enable us to endogenize his backward-stability equilibrium-selection criterion, which requires that equilibrium outcomes do not grow unboundedly backwards in time, since our particular equilibrium still exhibits (a weak form of) the forward-guidance puzzle.

6 Other Implications

In this section, we derive some other implications of our benchmark model with sticky prices. We show in particular that, in this model, (i) a corridor system has standard implications for equilibrium determinacy and dynamics; (ii) the effects of (temporary or permanent) monetary-policy shocks have a familiar Keynesian flavor, provided that prices are sticky enough;[40] and (iii) the standard Fisherian effect holds in the long term, i.e. there is a one-to-one relationship between the inflation rate and the interest rate on bonds in the long term. Thus, the ability of our model to solve the puzzles and paradox in a liquidity trap does not seem to come at the cost of any controversial implication during "normal times." Finally, we derive the implications of our model for the neo-Fisherian view about the short-term inflationary effects of "normalizing"

[37] Of course, the paradox of flexibility solved by Mankiw and Reis's (2002) model is about the effects of information flexibility, not price flexibility.

[38] To our knowledge, the only other paper that qualitatively solves at least one of the puzzles or the paradox without requiring a discrete departure from the basic NK model is García-Schmidt and Woodford (2015). They focus on the forward-guidance puzzle and show that, in the limit as the distance between their model and the basic NK model goes to zero, this puzzle reemerges in the same strong form as in the basic NK model.

[39] This criterion does not select a unique equilibrium, but rules out some equilibria.

[40] In Appendix D, we study the global effects of these shocks under flexible prices and under prices set one period in advance.

interest rates, which has been at the center of recent work and debate.

6.1 Corridor System

In a typical corridor system, the central bank maintains a fixed spread between the IOR rate and the interbank rate. The latter rate, in our model, coincides with the interest rate on bonds. So, in such a system, the spread equation (42) becomes

$$\widehat{m}_t = \frac{\delta_y}{\delta_m}\widehat{y}_t,$$

so that the Phillips curve can be rewritten as

$$\pi_t = \beta\mathbb{E}_t\left\{\pi_{t+1}\right\} + \frac{\delta_m\kappa_y - \delta_y\kappa_m}{\delta_m}\widehat{y}_t.$$

The inequality (45) ensures that this Phillips curve has a positive slope. Therefore, the reduced form of our model, made of the IS equation (41) and this Phillips curve, is then isomorphic to the basic NK model's reduced form for any given rule for the interest rate on bonds.[41] As a consequence, our model then inherits all the standard implications of the basic NK model for equilibrium determinacy and dynamics away from the effective lower bound.

6.2 Temporary Monetary-Policy Shocks

In the presence of temporary (i.i.d.) shocks to the exogenous monetary-policy instruments i_t^m and $\widehat{\mu}_t$, the dynamic equation (44) can be rewritten as

$$\mathbb{E}_t\left\{(1-\omega_1 L)(1-\omega_2 L)\,q_{t+2}\right\} = \frac{\kappa_y}{\beta\sigma}i_t^m + \frac{1+\delta_y}{\beta}\widehat{\mu}_t.$$

Using $\rho\omega_1\omega_2 = -\mathcal{P}(0) = (1+\delta_y)/\beta$, we easily get that the unique stationary solution of this dynamic equation is

$$q_t = \frac{\rho\kappa_y}{\sigma(1+\delta_y)}i_t^m + \rho\widehat{\mu}_t.$$

Using the resulting AR(1) process for $\widehat{m}_t$, the Phillips curve (40), and the money-market-clearing condition (43), we then get the following ARMA(1,1) processes (in i_t^m) and AR(1) processes (in $\widehat{\mu}_t$) for inflation and output:

$$\begin{aligned}
\pi_t &= \rho\pi_{t-1} - \frac{\rho\kappa_y}{\sigma(1+\delta_y)}i_t^m + \frac{\rho\kappa_y}{\sigma(1+\delta_y)}i_{t-1}^m + (1-\rho)\,\widehat{\mu}_t, \\
\widehat{y}_t &= \rho\widehat{y}_{t-1} - \frac{\rho\left[1+\beta(1-\rho)-\kappa_m\right]}{\sigma(1+\delta_y)}i_t^m + \frac{\rho}{\sigma(1+\delta_y)}i_{t-1}^m + \frac{(1-\rho)(1-\beta\rho)+\kappa_m\rho}{\kappa_y}\widehat{\mu}_t.
\end{aligned}$$

Assume that the economy is at the steady state at date 0, and consider the effects of one-off shocks $\widehat{\mu}_1 > 0$ and $i_1^m > 0$ in turn. Following a shock $\widehat{\mu}_1 > 0$, we get, for $t \geq 1$,

$$\pi_t = (1-\rho)\,\rho^{t-1}\widehat{\mu}_1 > 0 \quad \text{and} \quad \widehat{y}_t = \frac{(1-\rho)(1-\beta\rho)+\kappa_m\rho}{\kappa_y}\rho^{t-1}\widehat{\mu}_1 > 0,$$

[41] Woodford (2003, Chapters 2 and 4) obtains a similar isomorphism result in the context of a model in which money is household cash on which interest can be paid, when the central bank maintains a fixed spread between the interest rate on cash and the interest rate on bonds.

so that an unexpected temporary money-growth-rate rise is both inflationary and expansionary on impact. Price stickiness prevents the price level from rising as much as the nominal money stock, thus increasing the real money stock, reducing banking costs, and raising output. Following a shock $i_1^m > 0$, we get

$$\pi_1 = -\frac{\rho \kappa_y}{\sigma(1+\delta_y)} i_1^m < 0 \quad \text{and} \quad \widehat{y}_1 = -\frac{\rho\left[1+\beta(1-\rho)-\kappa_m\right]}{\sigma(1+\delta_y)} i_1^m,$$

and, for $t \geq 2$,

$$\pi_t = \frac{(1-\rho)\kappa_y}{\sigma(1+\delta_y)} \rho^{t-1} i_1^m > 0 \quad \text{and} \quad \widehat{y}_t = \frac{(1-\rho)(1-\beta\rho)+\kappa_m\rho}{\sigma(1+\delta_y)} \rho^{t-1} i_1^m > 0.$$

Thus, an unexpected temporary IOR-rate hike is always disinflationary on impact.[42] By reducing the opportunity cost of holding reserves, the hike raises the demand for real reserves and hence, given the supply of nominal reserves, reduces the price level. The contemporaneous effect on output is ambiguous. On the one hand, the increase in real reserve balances reduces banking costs and thus tends to raise output. On the other hand, price stickiness limits this increase in real reserves and thus exerts downward pressure on output to reduce the demand for real reserves and clear the market for reserves. Since $\lim_{\theta\to 0}\kappa_m = +\infty$ and $\lim_{\theta\to 1}\kappa_m = 0$, the overall effect of the hike is contractionary when the degree of price stickiness θ is sufficiently high for $\kappa_m < 1+\beta(1-\rho)$, and expansionary when θ is sufficiently low for $\kappa_m > 1+\beta(1-\rho)$.

6.3 Permanent Monetary-Policy Shocks

We now assume that the economy is at the steady state at date 0 and that, from date 1 onwards, $(i_t^m, \widehat{\mu}_t)$ is set to $(i^{m*}, \widehat{\mu}^*)$. The dynamic equation (44) can then be rewritten as

$$\mathbb{E}_t\left\{(1-\omega_1 L)(1-\omega_2 L) q_{t+2}\right\} = \frac{\kappa_y}{\beta\sigma} i^{m*} - \left[\frac{\kappa_y}{\beta\sigma} - \frac{\delta_y(1-\beta)}{\beta}\right]\widehat{\mu}^*$$

for $t \geq 1$. Using $(1-\rho)(\omega_1 - 1)(\omega_2 - 1) = \mathcal{P}(1) = (\delta_m\kappa_y - \delta_y\kappa_m)/\beta$, we easily get that the unique stationary solution of this dynamic equation is

$$q_t = q^* \equiv \frac{(1-\rho)\left\{\frac{\kappa_y}{\sigma} i^{m*} - \left[\frac{\kappa_y}{\sigma} - \delta_y(1-\beta)\right]\widehat{\mu}^*\right\}}{\delta_m\kappa_y - \delta_y\kappa_m}$$

for $t \geq 1$. Using $\widehat{m}_0 = 0$, we then get, for $t \geq 1$,

$$\widehat{m}_t = \left(\frac{1-\rho^t}{1-\rho}\right) q^*.$$

Finally, using the Phillips curve (40) and the money-market-clearing condition (43), we obtain, for $t \geq 1$,

$$\pi_t = \widehat{\mu}^* - q^*\rho^{t-1}, \tag{62}$$

$$\widehat{y}_t = \frac{(1-\beta)(1-\rho)\widehat{\mu}^* + \kappa_m q^*}{(1-\rho)\kappa_y} - \left[\frac{(1-\rho)(1-\beta\rho)+\kappa_m\rho}{(1-\rho)\kappa_y}\right] q^*\rho^{t-1}. \tag{63}$$

[42] The responses of π_1 and $\widehat{y}_1$ could also be obtained by setting $T = 1$ and $i^* = i_1^m$ in (49) and (50). The positive response of inflation from date 2 onwards reflects the fact that the price level is stationary following a temporary interest-rate shock.

Consider first a permanent IOR-rate hike alone ($i^{m*} > 0$ and $\widehat{\mu}^* = 0$). On impact, this hike is disinflationary ($\pi_1 < 0$), contractionary when prices are sufficiently sticky ($\widehat{y}_1 < 0$ when $\kappa_m < 1-\beta\rho$), and expansionary when prices are sufficiently flexible ($\widehat{y}_1 > 0$ when $\kappa_m > 1-\beta\rho$), for essentially the same reasons as in the previous subsection. In the long term, the hike has no effect on inflation ($\lim_{t\to+\infty}\pi_t = 0$) and a positive effect on output ($\lim_{t\to+\infty}\widehat{y}_t > 0$): as prices have completed their adjustment, the only remaining effect is the reduction in the opportunity cost of holding reserves, which increases real reserve balances, reduces banking costs, and raises output.

Now consider a permanent rise in nominal-money growth alone ($i^{m*} = 0$ and $\widehat{\mu}^* > 0$). This shock is, of course, inflationary in the long term, given the one-to-one long-term relationship between nominal-money growth and inflation. We show in Appendix C.8 that it is inflationary also in the short term ($\pi_1 > 0$), and therefore at all dates ($\pi_t > 0$ for all $t \geq 1$). We also show in this appendix that the shock is, in the short and long terms, expansionary when prices are sufficiently sticky ($\lim_{\theta\to 1}\widehat{y}_1 > 0$ and $\lim_{\theta\to 1}\lim_{t\to+\infty}\widehat{y}_t > 0$), and contractionary when they are sufficiently flexible ($\lim_{\theta\to 0}\widehat{y}_1 < 0$ and $\lim_{\theta\to 0}\lim_{t\to+\infty}\widehat{y}_t < 0$).

In the long term, the shock has two opposite effects on output. On the one hand, the higher inflation rate raises the opportunity cost of holding reserves, by reducing the real interest rate on reserves without affecting the real interest rate on bonds. This effect tends to decrease real reserve balances, increase banking costs, and lower output. On the other hand, the higher inflation rate tends to raise output because of the non-verticality of the long-term Phillips curve. The stickier prices, the flatter the long-term Phillips curve; in the limit as prices become perfectly sticky (respectively flexible), the Phillips curve becomes horizontal (respectively vertical), the expansionary effect becomes infinite (respectively zero) and dominates (respectively is dominated by) the contractionary effect.[43]

The more flexible prices, the closer the short-term response of output to its long-term response. Therefore, output responds negatively to the shock in the short term if prices are sufficiently flexible. On the contrary, if they are sufficiently sticky, the dominant effect in the short term is the one already mentioned in the previous subsection: price stickiness prevents the price level from rising as much as the nominal money stock, thus increasing the real money stock, reducing banking costs, and raising output.

6.4 Fisherian Effect

Because it involves the standard IS equation (41), our model trivially implies a standard Fisherian effect in the long term, i.e. a *one-to-one* long-term relationship between the inflation rate

[43]When $i^{m*} = \widehat{\mu}^* > 0$, the opportunity cost of holding reserves is left unchanged, as neither the real interest rate on reserves nor the real interest rate on bonds is affected. Therefore, the contractionary effect of $\widehat{\mu}^* > 0$ is exactly offset by the expansionary effect of $i^{m*} > 0$, and what remains is the expansionary effect of $\widehat{\mu}^* > 0$: output rises.

and the interest rate on bonds. Thus, a permanent rise in the nominal-money-growth rate will raise the inflation rate and the interest rate on bonds by the same amount in the long term.

By contrast, as we now show, models that qualitatively solve the forward-guidance puzzle by "discounting" the IS equation and the Phillips curve, such as Angeletos and Lian's (2016) model and Gabaix's (2016) benchmark model, necessarily make the inflation rate respond *negatively* to the interest rate in the long term. To see this, consider the class of reduced forms made of an IS equation and a Phillips curve of type

$$\widehat{y}_t = \nu_1 \mathbb{E}_t \left\{ \widehat{y}_{t+1} \right\} - \frac{1}{\sigma} \mathbb{E}_t \left\{ i_t^b - \nu_2 \pi_{t+1} \right\}, \tag{64}$$

$$\pi_t = \beta \nu_3 \mathbb{E}_t \left\{ \pi_{t+1} \right\} + \kappa \left[\widehat{y}_t - (1 - \nu_4) \mathbb{E}_t \left\{ \widehat{y}_{t+1} \right\} \right], \tag{65}$$

where $\beta \in (0,1)$, $\kappa > 0$, $\sigma > 0$, and $(\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{R}^4$ are such that $\varphi \equiv \beta\sigma\nu_1\nu_3 + \kappa\nu_2 (1 - \nu_4) \neq 0$. This class nests the reduced form of the basic NK model as the special case $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$, and generalizes this special case by allowing the coefficients of the expectation terms to take any real-number values consistent with $\varphi \neq 0$.[44] It also encompasses the reduced form of Gabaix's (2016) benchmark model, with $(\nu_1, \nu_3) \in (0,1)^2$ and $\nu_2 = \nu_4 = 1$, and the reduced form of Angeletos and Lian's (2016) model, with $(\nu_1, \nu_2, \nu_3, \nu_4) \in (0,1)^4$. We establish the following proposition in Appendix C.9:

Proposition 9 (No Fisherian Effect in "Discounting Models"): *In models whose reduced form is made of an IS equation of type (64) and a Phillips curve of type (65) with $\nu_4 > 0$, if a permanent peg of i_t^b ensures local-equilibrium determinacy, then π_t responds negatively to i_t^b in the long term.*

Thus, modifications brought to the basic NK model that affect its reduced form only through the coefficients of the expectation terms, such as the introduction of bounded rationality in Gabaix (2016) and lack of common knowledge in Angeletos and Lian (2016), cannot imply both local-equilibrium determinacy under a permanent interest-rate peg and a *positive* (let alone one-to-one) response of the inflation rate to the interest rate in the long term, as long as they imply $\nu_4 > 0$ (as is the case in Gabaix, 2016, and Angeletos and Lian, 2016).[45] Therefore, they cannot both qualitatively solve the forward-guidance puzzle and imply a long-term relationship consistent in sign (let alone in size) with the standard Fisherian effect. Our model manages to both qualitatively solve the two puzzles and the paradox and generate the standard Fisherian effect only by bringing other changes to the basic NK model's reduced form − most notably, by introducing price-level terms in this reduced form, through real-money-balance terms.

[44] The restriction $\varphi \neq 0$ rules out a zero-measure degenerate case in which the system made of the IS equation (64) and the Phillips curve (65) has zero or only one non-predetermined variable.

[45] Cochrane (2016b) has already made this point under a particular calibration of Gabaix's (2016) benchmark model. Our contribution is to show that Cochrane's point applies more generally to any calibration of any model of the class we consider (provided that $\nu_4 > 0$).

One simple way to interpret Proposition 9 involves, again, a shadow interest-rate rule and the Taylor principle. The question (negatively) answered by Proposition 9 is whether the system made of the modified IS equation (64), the modified Phillips curve (65), and the permanent peg $i_t^b = i^{b*}$ can have a unique stationary solution and make inflation, in this unique stationary solution, depend positively on i^{b*}. This question will receive exactly the same answer if that system is replaced by the system made of the standard IS equation (41), the modified Phillips curve (65), and the shadow interest-rate rule

$$i_t^b = i^{b*} + \sigma (1 - \nu_1) \mathbb{E}_t \{\widehat{y}_{t+1}\} + (1 - \nu_2) \mathbb{E}_t \{\pi_{t+1}\}. \tag{66}$$

Indeed, the two systems have exactly the same implications for local-equilibrium determinacy and the dynamics of inflation and output (they differ only in terms of the implied dynamics for i_t^b). So consider the latter system. The Taylor principle, shown in Appendix C.9 to be valid in this context (when $\nu_4 > 0$), states that a necessary condition for local-equilibrium determinacy is that the modified Phillips curve (65) and the shadow interest-rate rule (66) should make the interest rate react more than one-to-one to the inflation rate in the long term, that is to say

$$\zeta \equiv \sigma (1 - \nu_1) \frac{1 - \beta \nu_3}{\kappa \nu_4} + (1 - \nu_2) > 1. \tag{67}$$

In the unique local equilibrium, the (constant) interest rate i^b and the (constant) inflation rate π are therefore linked to each other by the relationship $i^b = i^{b*} + \zeta \pi$, where $\zeta > 1$. Now, the standard IS equation (41) implies that they should be equal to each other: $i^b = \pi$. As a consequence, we get

$$\pi = \frac{-i^{b*}}{\zeta - 1}.$$

Thus, the necessary condition for local-equilibrium determinacy (67) imposed by the Taylor principle requires that π be *negatively* related to i^{b*}.[46]

In our model, the IS equation (41) is standard, not discounted, so that we necessarily get the long-term Fisherian effect − i.e., the one-to-one relationship between the inflation rate and the interest rate on bonds in the long term. Despite the standard nature of its IS equation, however, our model delivers local-equilibrium determinacy under a permanent interest-rate peg because the interest rate pegged is the IOR rate, not the interest rate on bonds. Under a permanent IOR-rate peg, the interest rate on bonds evolves according to the shadow Wicksellian rule (42), which ensures local-equilibrium determinacy.

6.5 Neo-Fisherian Effect

One topic linked to indeterminacy problems of NK models has been at the center of recent work and debate. This topic is the "neo-Fisherian" view about the short-term inflationary effects

[46] Similarly, it requires that i^b be negatively related to i^{b*}. García-Schmidt and Woodford (2015) also consider, in a related context, a model in which the equilibrium interest rate may be negatively related to the exogenous intercept of the interest-rate rule.

of "normalizing" interest rates. Will the next interest-rate hike be inflationary? Cochrane's (2016c) analysis highlights how mixed the answer is if we ask the question in the context of the basic NK model, which exhibits indeterminacy under a permanent interest-rate peg: whether the permanent interest-rate hike is inflationary in the short term or not depends on the equilibrium considered. We revisit this question in the context of our model, which delivers determinacy under a permanent IOR-rate peg.

In his benchmark thought experiment, Cochrane (2016c) considers an exogenous step-function rise in the interest rate on bonds. In our model, the interest rate on bonds is endogenous: the two monetary-policy instruments are the interest rate on reserves and the money-growth rate. For consistency of comparison, we consider all the exogenous step-function changes in these two instruments that make the interest rate on bonds strictly higher at all dates, as well as in the long-term limit, than initially. Some specific combinations of such policy-instrument changes will result in a step-function rise in the interest rate on bonds, but we also allow for the other combinations, which result in richer dynamics for the rise in the interest rate on bonds. For simplicity, we focus on the case in which these policy changes are unexpected.

So, as in Subsection 6.3, we assume that the economy is at the steady state at date 0 and that $(i_t^m, \widehat{\mu}_t)$ is set to $(i^{m*}, \widehat{\mu}^*)$ from date 1 onwards. Using (42), (62), and (63), we obtain, after some simple algebra,

$$i_t^b = \widehat{\mu}^* - \xi q^* \rho^{t-1}, \tag{68}$$

where

$$\xi \equiv \frac{\sigma}{\kappa_y} \left[\delta_y (1 - \beta\rho) - \frac{(\delta_m \kappa_y - \delta_y \kappa_m)\, \rho}{1 - \rho} \right].$$

Therefore, $i_t^b > 0$ for all $t \geq 1$ and for $t \to +\infty$ if and only if $\widehat{\mu}^* > 0$ and $\widehat{\mu}^* - \xi q^* > 0$. In Appendix C.10, we establish the following lemma:

Lemma 6 (Properties of Parameter ξ): *$0 < \xi < \rho$ and $0 < \eta(\xi) < \eta(1)$, where, for $x \in \{\xi, 1\}$,*

$$\eta(x) \equiv \left\{ 1 + \frac{\sigma}{\kappa_y} \left[\frac{\delta_m \kappa_y - \delta_y \kappa_m}{(1-\rho)\, x} - (1 - \beta)\, \delta_y \right] \right\}^{-1}.$$

This lemma implies that $\widehat{\mu}^* - \xi q^* > 0$ if and only if $\widehat{\mu}^* > \eta(\xi)\, i^{m*}$. As a consequence, $i_t^b > 0$ for all $t \geq 1$ and for $t \to +\infty$ if and only if $\widehat{\mu}^* > \max[0, \eta(\xi)\, i^{m*}]$. Moreover, (62), (68), and Lemma 6 together imply that the following three statements are equivalent to each other: (i) $\pi_1 > 0$, (ii) $i_1^b > (1 - \xi) \lim_{t \to +\infty} i_t^b$, and (iii) $\widehat{\mu}^* > \eta(1)\, i^{m*}$. We summarize these findings as follows:

Proposition 10 (Neo-Fisherian Effects): *In the benchmark model with sticky prices, following an unexpected shock at date 1 moving $(i_t^m, \widehat{\mu}_t)$ permanently to $(i^{m*}, \widehat{\mu}^*)$,* *(i)* $i_t^b > 0$ *for all* $t \geq 1$ *and for* $t \to +\infty$ *if and only if* $\widehat{\mu}^* > \max[0, \eta(\xi)\, i^{m*}]$*;*

(ii) if $\max[0, \eta(\xi)\, i^{m*}] < \widehat{\mu}^* < \eta(1)\, i^{m*}$*, then* $i_1^b < (1-\xi)\lim_{t\to+\infty} i_t^b$ *and* $\pi_1 < 0$*;*

(iii) if $\widehat{\mu}^* > \max[0, \eta(1)\, i^{m*}]$*, then* $i_1^b > (1-\xi)\lim_{t\to+\infty} i_t^b$ *and* $\pi_1 > 0$*.*

Thus, we get a neo-Fisherian effect ($\pi_1 > 0$) if and only if the short-term increase in i_t^b is sufficiently large relatively to its long-term increase, more specifically if and only if $i_1^b > (1-\xi)\lim_{t\to+\infty} i_t^b$. In particular, we get a neo-Fisherian effect for all the values of $(i^{m*}, \widehat{\mu}^*)$ that generate a step-function rise in i_t^b (implying $i_1^b = \lim_{t\to+\infty} i_t^b$), which is arguably the closest case to the benchmark thought experiment of Cochrane (2016c). Given (68), this specific case requires that $q^* = 0$. Given (62) and (63), $q^* = 0$ implies in turn that not only the interest rate on bonds, but also inflation and output (and more generally all endogenous variables) jump to their long-term values at date 1 and keep these values thereafter $-$ in particular, inflation jumps from 0 to $\widehat{\mu}^* > 0$.

If prices are sufficiently flexible for $\kappa_y/\sigma > (1-\beta)\delta_y$, then this specific case of stepwise changes in all endogenous variables requires an IOR-rate hike ($i^{m*} > 0$). Indeed, under sufficiently flexible prices, following a shock $\widehat{\mu}^* > 0$ alone, real reserve balances decrease in the long term, as the negative effect stemming from the higher opportunity cost of holding reserves then dominates the positive effect stemming from the non-verticality of the long-term Phillips curve. To erode real reserve balances, inflation overshoots its long-term value in the short term ($\pi_1 > \lim_{t\to+\infty} \pi_t = \widehat{\mu}^* > 0$) and converges towards this value from above. To generate a stepwise increase in inflation, the shock $\widehat{\mu}^* > 0$ has therefore to be complemented by a shock $i^{m*} > 0$, which is disinflationary in the short term.

Alternatively, if prices are sufficiently sticky for $\kappa_y/\sigma < (1-\beta)\delta_y$, then this specific case requires an IOR-rate cut ($i^{m*} < 0$). Indeed, under sufficiently sticky prices, following a shock $\widehat{\mu}^* > 0$ alone, real reserve balances increase in the long term, so that inflation undershoots its long-term value in the short term ($\lim_{t\to+\infty} \pi_t = \widehat{\mu}^* > \pi_1 > 0$). To generate a stepwise increase in inflation, the shock $\widehat{\mu}^* > 0$ has then to be complemented by a shock $i^{m*} < 0$, which is inflationary in the short term.

7 Model With Cash

Our benchmark model is specific in that households hold money only in the form of reserves, in their capacity as bankers. This makes our point stark because banks cannot collectively change the nominal stock of reserves. In reality, bank reserves can fall if households demand more cash. We now illustrate that our results do not unravel when we allow for such leakages out of reserve balances. We introduce household cash into our benchmark model, and we study the consequences of pegging or setting exogenously the IOR rate I_t^m and the growth rate μ_t of the monetary base (made of bank reserves and household cash).

We first show, in a non-parametric setup, that our global-determinacy results under flexible prices are unchanged when the elasticity of intertemporal substitution is higher than or equal to one. We then show, in the context of a parametric example, that these results are also unchanged when this elasticity is strictly less than one, except for implausible calibrations. Finally, we show that our local-determinacy results under sticky prices are also unchanged except for the same implausible calibrations.

7.1 Non-Parametric Global Analysis Under Flexible Prices

We introduce cash in advance, held by households, into the non-parametric benchmark model presented in Section 2. More specifically, we assume that a fraction $\phi \in (0,1]$ of consumption has to be bought with cash. Thus, households choose bonds b_t, consumption c_t, work hours h_t, loans ℓ_t, reserves m_t, and now cash m_t^c to maximize the same intertemporal utility function as previously, rewritten as

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) - \Gamma\left(\ell_{t+k}, m_{t+k}\right) \right] \right\}$$

subject to the budget constraint

$$(1-\phi)\, c_t + m_t^c + b_t + \ell_t + m_t \leq \frac{m_{t-1}^c - \phi c_{t-1}}{\Pi_t} + \frac{I_{t-1}^b}{\Pi_t} b_{t-1} + \frac{I_{t-1}^\ell}{\Pi_t} \ell_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \omega_t$$

and the cash-in-advance constraint

$$m_t^c \geq \phi c_t, \tag{69}$$

taking all prices (I_t^b, I_t^ℓ, I_t^m, P_t, and w_t) as given. Letting λ_t and λ_t^c denote the Lagrange multipliers on these two constraints respectively, the first-order conditions of households' optimization problem are again (8), (9), (10), (11), (12), and now

$$\frac{\lambda_t^c}{\phi} + \frac{\beta \lambda_{t+1}}{\Pi_{t+1}} - \lambda_t = 0.$$

None of the other equilibrium conditions is changed, except the money-market-clearing condition, which becomes

$$m_t + m_t^c = \frac{M_t}{P_t}, \tag{70}$$

as the monetary base controlled by the central bank is now made not only of bank reserves, but also of household cash. Therefore, the relationships (20) and – under flexible prices – (27) still hold. Using households' first-order conditions (9) and (12), together with (8), (14), (19), (20), (27), (69) holding with equality, and (70), we obtain the following dynamic equation in h_t, under flexible prices and under policies that peg $\mu_t = \mu > 0$ and $I_t^m = I^m \geq 0$:

$$1 + \mathcal{F}\left(h_t\right) = \frac{\beta I^m}{\mu} \mathbb{E}_t \left\{ \frac{\widetilde{\mathcal{G}}\left(h_{t+1}\right)}{\widetilde{\mathcal{G}}\left(h_t\right)} \right\},$$

where

$$\widetilde{\mathcal{G}}\left(h_t\right) \equiv u'\left[f\left(h_t\right)\right]\left[\mathcal{M}\left(h_t\right)+\phi f\left(h_t\right)\right]=\mathcal{G}\left(h_t\right)+\phi u'\left[f\left(h_t\right)\right] f\left(h_t\right).$$

Since the left-hand side of this dynamic equation is the same as previously, the necessary and sufficient condition on I^m and μ for existence and uniqueness of a time-invariant equilibrium is again (34), so that Proposition 1 still holds when "benchmark model" is replaced by "model with cash."

Moreover, if the elasticity of intertemporal substitution is always higher than or equal to one ($\forall c>0$, $-u''(c)c/u'(c) \leq 1$), then $u'\left[f\left(h_t\right)\right] f\left(h_t\right)$ is weakly increasing in h_t, so that $\widetilde{\mathcal{G}}$ is strictly increasing. Therefore, using

$$\lim_{h_t \rightarrow 0} \widetilde{\mathcal{G}}\left(h_t\right) \geq 0 \quad \text{and} \quad \lim_{h_t \rightarrow \overline{h}} \widetilde{\mathcal{G}}\left(h_t\right)=+\infty,$$

we obtain, in the same way as previously, that the set of time-varying perfect-foresight equilibria under permanent pegs is of the same type as in the benchmark model, i.e. that Proposition 2 still holds when "benchmark model" is replaced by "model with cash, when the elasticity of intertemporal substitution is higher than or equal to one." In particular, the unique time-invariant equilibrium, when the pegs fall in the suitable range, is the globally unique "determinate" perfect-foresight equilibrium.

In the alternative case where the elasticity of intertemporal substitution can be lower than one ($\exists c>0$, $-u''(c)c/u'(c)>1$), the function $\widetilde{\mathcal{G}}$ may not be always strictly increasing, and the analysis becomes too complex to be carried out in our non-parametric setup. In the next subsection, we address this case in the context of a parametric example.

7.2 Parametric Global Analysis Under Flexible Prices

For simplicity, we now assume that all utility and production functions are isoelastic:

$$\begin{aligned}
u\left(c_t\right) & \equiv (1-\sigma)^{-1}\left(c_t\right)^{1-\sigma}, \\
v\left(h_t\right) & \equiv \delta(1+\chi)^{-1}\left(h_t\right)^{1+\chi}, \\
v^b\left(h_t^b\right) & \equiv \delta_b\left(1+\chi_b\right)^{-1}\left(h_t^b\right)^{1+\chi_b}, \\
f^b\left(h_t^b, m_t\right) & \equiv A_b\left(h_t^b\right)^{\alpha_b}\left(m_t\right)^{\gamma_b}, \\
f\left(h_t\right) & \equiv A\left(h_t\right)^\alpha,
\end{aligned}$$

where $\sigma>1$, $\delta>0$, $\delta_b>0$, $\chi \geq 0$, $\chi_b \geq 0$, $A>0$, $A_b>0$, $0<\alpha<1$, $\alpha_b>0$, $0<\gamma_b<1$, and $\left(1-\gamma_b\right)\left(1+\chi_b\right)>\alpha_b$. Notice that (i) we focus on the case in which σ (the inverse of the elasticity of intertemporal substitution) is higher than one, following the previous discussion, and (ii) we allow the degree of homogeneity of f^b to be higher than one, provided it is lower than $1+\left(1-\gamma_b\right) \chi_b$.

These specifications imply that

$$
\begin{aligned}
g^{b}\left(\ell_{t}, m_{t}\right) &= A_{b}^{\frac{-1}{\alpha_{b}}}\left(\ell_{t}\right)^{\frac{1}{\alpha_{b}}}\left(m_{t}\right)^{\frac{-\gamma_{b}}{\alpha_{b}}}, \\
\Gamma\left(\ell_{t}, m_{t}\right) &= \delta_{b}\left(1+\chi_{b}\right)^{-1} A_{b}^{\frac{-\left(1+\chi_{b}\right)}{\alpha_{b}}}\left(\ell_{t}\right)^{\frac{1+\chi_{b}}{\alpha_{b}}}\left(m_{t}\right)^{\frac{-\gamma_{b}\left(1+\chi_{b}\right)}{\alpha_{b}}}, \\
\mathcal{L}\left(h_{t}\right) &= \delta A^{\sigma}\left(h_{t}\right)^{1+\chi+\alpha\sigma}, \\
\mathcal{A}\left(h_{t}\right) &= A^{-\sigma}\left(h_{t}\right)^{-\alpha\sigma}\left[\alpha\delta^{-1}(\varepsilon-1)\varepsilon^{-1}A^{1-\sigma}\left(h_{t}\right)^{-\alpha(\sigma-1)-(1+\chi)}-1\right], \\
\mathcal{M}\left(h_{t}\right) &= A_{b}^{\frac{-1}{\gamma_{b}}}\left\{\alpha_{b}^{-1}\delta_{b}\left[\mathcal{L}\left(h_{t}\right)\right]^{\frac{1+\chi_{b}}{\alpha_{b}}-1}\left[\mathcal{A}\left(h_{t}\right)\right]^{-1}\right\}^{\frac{\alpha_{b}}{\gamma_{b}\left(1+\chi_{b}\right)}}, \\
\mathcal{F}\left(h_{t}\right) &= -\alpha_{b}^{-1}\gamma_{b}\delta_{b}A^{\sigma}A_{b}^{\frac{-\left(1+\chi_{b}\right)}{\alpha_{b}}}\left(h_{t}\right)^{\alpha\sigma}\left[\mathcal{L}\left(h_{t}\right)\right]^{\frac{1+\chi_{b}}{\alpha_{b}}}\left[\mathcal{M}\left(h_{t}\right)\right]^{\frac{-\gamma_{b}\left(1+\chi_{b}\right)}{\alpha_{b}}-1}, \\
\widetilde{\mathcal{G}}\left(h_{t}\right) &= A^{-\sigma}\left(h_{t}\right)^{-\alpha\sigma}\left[\mathcal{M}\left(h_{t}\right)+\phi A\left(h_{t}\right)^{\alpha}\right], \\
\overline{h}=h^{\star} &= \left[\alpha\delta^{-1}(\varepsilon-1)\varepsilon^{-1}A^{1-\sigma}\right]^{\frac{1}{1+\chi+\alpha(\sigma-1)}}.
\end{aligned}
$$

Let again h denote the unique steady-state value of h_t, defined by (35). We establish the following lemma in Appendix E.1:

Lemma 7 (Properties of Function $\widetilde{\mathcal{G}}$): *In the parametric model with cash, the function $\widetilde{\mathcal{G}}$ is U-shaped, with*

$$\lim_{h_t\to 0}\widetilde{\mathcal{G}}\left(h_t\right)=\lim_{h_t\to\overline{h}}\widetilde{\mathcal{G}}\left(h_t\right)=+\infty,$$

and a sufficient condition on the parameters for $\widetilde{\mathcal{G}}'(h)>0$ is

$$(1+\chi)+(\sigma-1)\left[\alpha-\left(1-\frac{\beta I^{m}}{\mu}\right)\frac{1+\chi_{b}}{\alpha_{b}}\frac{\varepsilon}{\varepsilon-1}\phi\right]>0. \tag{71}$$

Lemma 7 implies that Condition (71) is sufficient for the unique time-invariant equilibrium to be (at least locally) "determinate." Now, we view this condition as likely to be met.[47] Consider indeed the standard value 0.66 for the elasticity of output to labor α. Assume conservatively that the elasticity of substitution between goods is $\varepsilon = 6$ (implying a 20% markup), that the inverse of the Frisch elasticity of bankers' labor supply is $\chi_b = 5$, and that the fraction of consumption that has to be bought with cash is $\phi = 0.3$.[48] For an elasticity of loans to labor

[47]The necessary and sufficient condition for $\widetilde{\mathcal{G}}'(h) > 0$ involves the steady-state value h, which cannot be expressed analytically as a function of parameters. We focus on the sufficient condition (71) because it is more transparent and, as we argue, likely to be met anyway.

[48]These assumptions are conservative in the sense that a higher value for ε and lower values for χ_b and ϕ seem arguably more likely, and considering such values would only strengthen our point. In particular, the value 5 for the inverse of a Frisch elasticity of labor supply lies at the upper end of the range of microeconomic estimates, and is much higher than values commonly considered in macroeconomics. And, for the U.S., the value 0.3 for the fraction of consumption that has to be bought with cash lies at the upper end of the range of values reported by the Survey of Consumer Payments Choice for the ratio of the *number* of cash transactions to the total number of transactions between 2008 and 2014 (Greene, Schuh, and Stavins, 2016), and is much higher than the values reported by the Diary of Consumer Payment Choice for the ratio of the *value* of cash transactions to the total value of transactions in 2012 and 2015 (Matheny, O'Brien, and Wang, 2016).

α_b equal to 0.66, as long as the spread between the interest rate on bonds and the interest rate on reserves does not exceed 20 percentage points (i.e. $1 - (\beta I^m)/\mu \leq 0.20$), the expression in square brackets on the left-hand side of (71) will be positive, and hence Condition (71) will be met for any value of χ and σ. Even if the elasticity of loans to labor were half as large ($\alpha_b = 0.33$), the threshold value for the interest-rate spread above which there exist values of χ and σ such that Condition (71) is not met would still be a comfortable 10%. We conclude that, in the context of this parametric model, the unique time-invariant equilibrium remains (at least locally) "determinate" when the elasticity of intertemporal substitution is lower than one, except for implausible calibrations.

7.3 Local Analysis Under Sticky Prices

We now turn to the sticky-price version of our model with cash, in which prices are set à la Calvo (1983). We assume that I_t^m can vary exogenously around a given value $I^m \in (0, \beta^{-1})$, and μ_t around the value $\mu = 1$. The previous analysis implies that the model has a unique steady state, and that this steady state has zero inflation. We log-linearize the model around this unique steady state, and study local rational-expectations equilibria.

The log-linearized Phillips curve (40), IS equation (41), and spread equation (42) are unaffected by the introduction of cash, as the money-market-clearing condition plays no role in their derivation. Using the goods-market-clearing condition (19) and the binding cash-in-advance constraint (69), we can log-linearize the new money-market-clearing condition (70) as

$$\widehat{\left(\frac{M_t}{P_t}\right)} = (1 - \alpha_c)\, \widehat{m}_t + \alpha_c \widehat{y}_t, \tag{72}$$

or equivalently, in first difference,

$$\pi_t = -\,(1 - \alpha_c)\,(\widehat{m}_t - \widehat{m}_{t-1}) - \alpha_c\,(\widehat{y}_t - \widehat{y}_{t-1}) + \widehat{\mu}_t, \tag{73}$$

where

$$\alpha_c \equiv \frac{\phi f\,(h)}{\phi f\,(h) + \mathcal{M}\,(h)} \in (0, 1)$$

denotes the steady-state share of household cash in the monetary base. Equation (73) is the counterpart, in the model with cash, of Equation (43) in the benchmark model, and one can move from the former to the latter by (arbitrarily) replacing α_c by zero.

Using the Phillips curve (40), the IS equation (41), the spread equation (42), and the money-market-clearing condition (73), we easily get the following dynamic equation in $\widehat{m}_t$ and $\widehat{y}_t$:

$$\mathbb{E}_t \left\{ \begin{bmatrix} \widehat{m}_{t+1} \\ \widehat{m}_t \\ \widehat{y}_{t+1} \\ \widehat{y}_t \end{bmatrix} \right\} = \mathbf{A} \begin{bmatrix} \widehat{m}_t \\ \widehat{m}_{t-1} \\ \widehat{y}_t \\ \widehat{y}_{t-1} \end{bmatrix} + \mathbb{E}_t \left\{ \mathbf{B} \begin{bmatrix} i_t^m \\ \widehat{\mu}_{t+1} \\ \widehat{\mu}_t \end{bmatrix} \right\}, \tag{74}$$

where the 4×4 matrix $\mathbf{A}$ and 4×3 matrix $\mathbf{B}$ are defined in Appendix E.2. In Appendix E.3, we establish the following lemma:

Lemma 8 (Eigenvalues of Matrix A): *In the non-parametric (respectively parametric) model with cash and sticky prices, the matrix* $\mathbf{A}$ *has two eigenvalues inside the unit circle and two eigenvalues outside if* $\sigma \leq 1$ *(respectively if* $\sigma > 1$ *and Condition (71), with* $\mu = 1$*, is met).*

This lemma straightforwardly implies, through Blanchard and Kahn's (1980) conditions, the following proposition:

Proposition 11 (Local-Equilibrium Determinacy in the Model With Cash Under Sticky Prices): *In the non-parametric (respectively parametric) model with cash and sticky prices, when* I_t^m *and* μ_t *vary exogenously around the values* $I^m \in (0, \beta^{-1})$ *and* $\mu = 1$*, there is a unique rational-expectations equilibrium in the neighborhood of the unique steady state if* $\sigma \leq 1$ *(respectively if* $\sigma > 1$ *and Condition (71), with* $\mu = 1$*, is met).*

This proposition is the counterpart of Proposition 3 in the model with cash. It can be interpreted in the same way as Proposition 3, with the help of a "shadow rule" for the interest rate on bonds. This shadow rule is still Wicksellian, i.e. it still makes the interest rate (on bonds) react positively to output, the price level, and no other endogenous variable. Indeed, the spread equation (42) makes i_t^b depend positively on $\widehat{y}_t$ and negatively on $\widehat{m}_t$; in turn, $\widehat{m}_t$ now depends, through the new money-market-clearing condition (72), negatively on output and the price level.

Again, it is well known that Wicksellian rules always ensure local-equilibrium determinacy in the basic NK model. Because of its cost channel of monetary policy (i.e. $\kappa_m > 0$), however, our model differs from the basic NK model, and the shadow Wicksellian rule for i_t^b does not always ensure local-equilibrium determinacy. When $\sigma \leq 1$, this rule ensures determinacy in the non-parametric model. When $\sigma > 1$, a sufficient condition for this rule to ensure determinacy in the parametric model is (71) with $\mu = 1$. This condition, as we have argued in Subsection 7.2, seems likely to be met. We conclude that, at least in the context of this parametric model, setting exogenously the IOR rate and the growth rate of the monetary base delivers local-equilibrium determinacy, except for implausible calibrations.[49]

[49]To solve the forward-guidance and fiscal-multiplier puzzles (as in Subsections 5.1 and 5.2), we need local-equilibrium determinacy for all empirically relevant calibrations. To solve the paradox of flexibility (as in Subsection 5.3), and to solve or attenuate the puzzles and paradox in the basic-NK-model limit (as in Subsection 5.4), we also need local-equilibrium determinacy when the price-stickiness parameter θ and the scale parameter δ_b (which plays the same role as parameter γ in Subsection 5.4) are small enough and when I^m is close enough to β^{-1}. This additional need is satisfied because Condition (71) involves neither θ nor δ_b, and is necessarily met when I^m is close enough to β^{-1}.

8 Model With a Satiation Level

Our benchmark model assumes that there is no finite satiation level of demand for reserves − i.e., that the marginal convenience yield of reserves remains positive at any finite level of reserve balances. This assumption is analytically convenient but cannot be literally true. For example, if reserves reduce banking costs only by helping banks manage the liquidity risk associated with short-term deposits, then the marginal convenience yield must be zero once reserves are as large as deposits.

In this section, we first summarize (relegating formal statements and details to Appendix F) the consequences of introducing a finite satiation level of demand for reserves into our benchmark model, as in Cúrdia and Woodford (2011). Not surprisingly, we find that our main results remain essentially intact if and only if reserves are below the satiation level in equilibrium. We then discuss arguments that seem relevant for gauging whether or not the demand for reserves is currently satiated in the United States.

8.1 Summary of the Results

To introduce a finite satiation level of demand for reserves into our benchmark model, we remove the assumption that $f_m^b > 0$ and $f_{mm}^b < 0$ for all $\left(h_t^b, m_t\right) \in (\mathbb{R}_{\geq 0})^2$, and replace it by the following assumption: for any $h_t^b \in \mathbb{R}_{>0}$, there exists $\underline{m}^f\left(h_t^b\right) \in \mathbb{R}_{>0}$ such that (i) $f_m^b = f_{mm}^b = 0$ if $m_t \geq \underline{m}^f\left(h_t^b\right)$, and (ii) $f_m^b > 0$ and $f_{mm}^b < 0$ if $m_t < \underline{m}^f\left(h_t^b\right)$. This change affects the properties of the banking-cost function Γ stated in Lemma 1. We no longer have $\Gamma_m < 0$, $\Gamma_{mm} > 0$, and $\Gamma_{\ell m} < 0$ for all $(\ell_t, m_t) \in (\mathbb{R}_{\geq 0})^2$. Instead, we get that for any $\ell_t \in \mathbb{R}_{>0}$, there exists $\underline{m}(\ell_t) \in \mathbb{R}_{>0}$ such that (i) $\Gamma_m = \Gamma_{mm} = \Gamma_{\ell m} = 0$ if $m_t \geq \underline{m}(\ell_t)$, and (ii) $\Gamma_m < 0$, $\Gamma_{mm} > 0$, and $\Gamma_{\ell m} < 0$ if $m_t < \underline{m}(\ell_t)$.

As we show in Appendix F, our results for perfect-foresight equilibria under flexible prices, when the central bank permanently pegs its monetary-policy instruments ($I_t^m = I^m$ and $\mu_t = \mu$), do not change dramatically. The model has no equilibria under policies that peg the instruments in the range with $I^m/\mu > \beta^{-1}$. Any peg policy with $0 \leq I^m/\mu < \beta^{-1}$ is associated with a unique time-invariant equilibrium outside the satiation region. As in our benchmark model, such a policy is also consistent with a continuum of time-varying equilibria. The new element here is that these equilibria now involve real money balances that converge to the satiation range (while in the benchmark model, real money balances would grow without bound). The other novelty is that peg policies with $I^m/\mu = \beta^{-1}$ are now consistent with existence of equilibrium − and the resulting equilibria exhibit the kind of indeterminacy discussed in Sargent and Wallace (1985). More precisely, because we have a representative-agent setup (in contrast to Sargent and Wallace's overlapping-generations setup), the indeterminacy in our model implies that any level of real money balances in the satiation range can be an equilibrium, but this indeterminacy does not permeate to equilibrium values of other real variables. These policies with the ratio of instruments (I^m/μ) set exactly at a critical value (β^{-1}), however, represent a knife-edge case in our model with permanently pegged instruments.

Turning to our local analysis under sticky prices, we obtain exactly the same results as the benchmark model if we log-linearize the model around a steady state outside the satiation region. By contrast, a log-linear approximation around a steady state inside the satiation region leads to a reduced form isomorphic to the reduced form of the basic NK model. That is, compared to our benchmark model, we have $\kappa_m = \delta_y = \delta_m = 0$.[50] Thus, to apply our proposed resolution of the NK puzzles and paradox to the current U.S. situation, we must assume that the demand for reserves is not currently satiated in the U.S.

8.2 Gauging the Satiation Threshold

The fact that the stock of bank reserves is large (currently about $2 trillion) does not necessarily mean that their marginal convenience yield is zero. The daily flow of transactions on Fedwire is currently about the same size as the stock of reserves. For some small banks, the marginal convenience yield of reserves may still be linked to the risk of being forced to borrow from the Discount Window, or to borrow federal funds at rates exceeding the IOR rate, in order to satisfy reserve requirements. For large banks, the risk of hitting the required reserve threshold is negligible, but the risk of hitting the lower bound on a capital requirement may be an important consideration. Reserves have emerged as their preferred short-term asset in their liquidity portfolios; the convenience yield of reserves is thus linked to the need for liquidity.[51]

If we could associate an observable interest rate with the rate I_t^b in our model, then satiation would correspond to $I_t^m = I_t^b$, and our benchmark model would involve $I_t^m < I_t^b$. The problem with this approach is that we have not developed a structural model of how banking costs arise and how holding reserves reduces these costs. If we associate these costs with liquidity management, then other money-market instruments, like Treasury bills, also seem likely to have a convenience yield and therefore cannot serve as our proxy for I_t^b.

We do not view the federal-funds rate as an observable counterpart of I_t^b in our model. The effective federal-funds rate has remained below the IOR rate in the aftermath of the crisis, contradicting both versions of our model (with and without satiation threshold); but the reasons presumably have to do with institutional details not captured in our model. One likely reason (noted, for example, in Williamson, 2015) is that government-sponsored enterprises (GSEs) are not eligible to receive interest on deposits at the Federal Reserve. The GSEs have become large

[50] We also have $\kappa_y \geq \kappa$, where again κ denotes the slope of the basic NK model's Phillips curve. If marginal banking costs Γ_ℓ are zero under satiation, then $\kappa_y = \kappa$ and the reduced form of our model becomes exactly identical to the reduced form of the basic NK model.

[51] Osborne and Sim (2015) also emphasize the relevance of new liquidity standards in their commentary on demand for reserves in the U.K.

lenders in the federal-funds market; and money-market mutual funds are large lenders in the Eurodollar market. Banks borrowing these funds pay the federal-funds rate and receive the IOR rate on their deposits at the Federal Reserve. If the marginal bank is a large domestic institution, it incurs balance-sheet costs such as deposit-insurance fees (which depend on total assets) and capital costs from increased leverage. These costs can create a wedge, keeping the effective federal-funds rate below the IOR rate. Foreign banking organizations (FBOs) have a cost advantage in these transactions because they do not pay the Federal Deposit Insurance Corporation insurance premium (and, more recently, also because of cross-country differences in implementation of the Basel III liquidity-coverage regulations). And FBOs have emerged as disproportionately large borrowers in the federal-funds and Eurodollar markets.[52]

Against this background, it does not seem farfetched to us to assume that the marginal dollar of reserves provides some liquidity services to a bank − or, stretching beyond our model, generates transaction services indirectly for an individual whose money-market fund lends overnight funds to a bank. It seems easier to envision an equilibrium in which FBOs have a higher effective pecuniary return on reserves and hold disproportionately large reserve balances, in the context of our benchmark model; the higher non-pecuniary return associated with the smaller reserve balances of domestic banks would compensate for the difference in effective pecuniary returns. We find it harder to envision an equilibrium under satiation in which FBOs face a higher effective IOR rate and yet do not eliminate the spread with the federal-funds rate.

Our interpretation of some commentary about satiation of demand for reserves is that many policymakers and economists ask whether other liquid assets like Treasury bills are now very close substitutes for bank reserves.[53] We do not think that the case for our benchmark model necessarily rests on the answer to this question. We could, for example, assume that T-Bills and reserves are perfect substitutes in our specification of banking costs, as long as they *both* have a convenience yield and their supply is essentially determined by policy. A number of empirical contributions (e.g., Friedman and Kuttner, 1998, Greenwood and Vayanos, 2014, and Krishnamurthy and Vissing-Jorgensen, 2012) support the view that government debt has a convenience yield that is inversely related to its outstanding stock.

Nor does our benchmark model necessarily contradict the view that the demand for reserves may be very flat, and that small shocks may lead to large changes in the demand for reserves. Our model only assumes that banks are not truly indifferent across a range of values for their reserve balances. As long as this is granted, we can allow for a convenience yield that is very

[52]Ennis and Wolman (2015), for example, note that in 2011 the assets of FBOs were about 10 to 20 percent of the assets of domestic institutions, but the two groups had roughly the same amount of reserve balances.

[53]For example, Osborne and Sim (2015) state that reserves lost some of their "specialness" after the Bank of England started remunerating them at the policy rate, and under these circumstances reserves "become much more like part of the broader spectrum of liquid assets, assessed against their liquidity value and expected return." And Ennis and Wolman (2015) argue that reserves were still special in their data set. They report a negative correlation between the T-Bill-IOR spread and the reserve-to-deposit ratio in weekly data ending in 2011; they interpret the negative correlation as evidence in favor of Ireland's (2014) model in which reserves play a special role, and the demand for reserves is determinate even when the IOR rate is equal to a market rate.

small and very flat.

9 Conclusion

In this paper, we have proposed a resolution of two puzzles and one paradox that arise in the basic NK model under a temporary interest-rate peg (e.g., in the context of a liquidity trap): the forward-guidance puzzle, the fiscal-multiplier puzzle, and the paradox of flexibility. This resolution rests on the ability of our model to deliver equilibrium determinacy under a permanent interest-rate peg. In turn, this ability comes from the fact that our central bank can set the interest rate on bank reserves and the supply of bank reserves independently, because these reserves reduce the costs of banking (i.e., have a convenience yield). The introduction of household cash alongside bank reserves into the monetary base leaves all these results essentially unaffected.

Our model's ability to solve the puzzles and paradox in a liquidity trap does not seem to come at the cost of any controversial implication during "normal times." In particular, unlike other models proposed in the literature to solve the forward-guidance puzzle, our model preserves the long-term Fisherian relationship; and, with a corridor system, it inherits all the standard implications of the basic NK model for equilibrium determinacy and dynamics away from the effective lower bound.

Moreover, while these other models require a discrete departure from the basic NK model to solve the forward-guidance puzzle, our model still solves the two puzzles and the paradox with an *arbitrarily* small departure from the basic NK model, i.e. with arbitrarily small banking costs and convenience yield of bank reserves. In fact, even a *vanishingly* small departure from the basic NK model is enough to solve the fiscal-multiplier puzzle and the paradox of flexibility, and attenuate the forward-guidance puzzle. This limit result brings the basic NK model at par with Mankiw and Reis's (2002) sticky-information model in terms of their ability to solve or attenuate the puzzles and paradox. It also provides theoretical foundations to Cochrane's (2016a) approach of selecting an equilibrium different from the (puzzling and paradoxical) standard equilibrium in the basic NK model under a temporary interest-rate peg. In particular, it enables us to endogenize his local-to-frictionless equilibrium-selection criterion.

We identify two main avenues for future research. First, we plan to investigate the implications of our model for other interesting issues raised by NK liquidity traps. In particular, in the basic NK model, positive supply shocks − such as downward shifts in the labor-disutility function, labor-tax cuts, technology improvements, and reductions in market power − are expansionary under a standard interest-rate rule, but contractionary under a temporary interest-rate peg (the so-called "paradox of toil").[54] This implication of the basic NK model is not puzzling

[54]These results can be found in, e.g., Eggertsson (2010, 2011, 2012), Eggertsson and Krugman (2012), Eggertsson, Ferrero, and Raffo (2014), Kiley (2016), and Wieland (2016). The phrase "paradox of toil" was coined by

in the same way as the three implications that we have addressed in this paper: it is about sign reversals under a temporary interest-rate peg, not about (stark) discontinuities in the limit as the duration of the peg goes to infinity, or as the degree of price stickiness goes to zero. Preliminary work suggests nonetheless that our model may not share this implication of the basic NK model either.[55]

Second, we plan to explore further the implications of our model for central banks' exit strategies. We have already shown, in this paper, that neo-Fisherian effects can arise in our model, i.e. that a permanent increase in the interbank rate can be inflationary in the short term. Unlike Cochrane's (2016c) analysis of such effects in the basic NK model, our analysis does not hinge on any equilibrium-selection argument, since a permanent IOR-rate peg conveniently delivers local-equilibrium determinacy in our model. But whether neo-Fisherian effects arise or not in our model depends on how the two monetary-policy instruments (the IOR rate and the growth rate of reserves) are used to generate the permanent increase in the interbank rate. This issue deserves further scrutiny in future work because central banks' exit strategies can involve several interesting combinations of when and how interest rates are normalized and balance-sheet adjustments occur.

Appendix A: Benchmark Model

In this appendix and the following ones, we omit time subscripts and function arguments whenever no ambiguity results.

A.1 Concavity of Function f^b

Since f^b is homogeneous of degree d, we have $\forall x \in \mathbb{R}_{\geq 0}$, $f^b\left(xh_t^b, xm_t\right) = x^d f^b\left(h_t^b, m_t\right)$. Computing the first derivative of the left- and right-hand sides of this equation with respect to x at $x = 1$ leads to

$$df^b = h^b f_h^b + m f_m^b. \tag{A.1}$$

In turn, computing the first derivative of the left- and right-hand sides of the last equation with respect to h^b and m leads to

$$\begin{aligned} -(1-d) f_h^b &= h^b f_{hh}^b + m f_{hm}^b, \\ -(1-d) f_m^b &= h^b f_{hm}^b + m f_{mm}^b, \end{aligned}$$

Eggertsson (2010).

[55] Other puzzling implications of NK models under a temporary interest-rate peg are that capital-tax cuts can reduce investment (the so-called "paradox of thrift," obtained by Eggertsson, 2011) and that capital-destruction shocks and negative oil-supply shocks can be expansionary (as shown by Wieland, 2016). Considering these implications would require to introduce capital and oil into our model.

which can be rewritten as

$$
\begin{aligned}
f_{hh}^{b} &= -\frac{(1-d)\, f_{h}^{b} + m f_{hm}^{b}}{h^{b}}, \\
f_{mm}^{b} &= -\frac{(1-d)\, f_{m}^{b} + h^{b} f_{hm}^{b}}{m}.
\end{aligned}
$$

Using these expression for f_{hh}^{b} and f_{hm}^{b}, as well as (A.1), we get

$$
\begin{aligned}
f_{hh}^{b} f_{mm}^{b} - \left(f_{hm}^{b}\right)^{2} &= \frac{1-d}{h^{b} m} \left[(1-d)\, f_{h}^{b} f_{m}^{b} + f_{hm}^{b} \left(h^{b} f_{h}^{b} + m f_{m}^{b}\right)\right] \\
&= \frac{1-d}{h^{b} m} \left[(1-d)\, f_{h}^{b} f_{m}^{b} + d f^{b} f_{hm}^{b}\right] \\
&\geq 0,
\end{aligned}
$$

so that the function f^{b} is (weakly) concave.

A.2 Properties of Function g^{b}

Computing the first and second derivatives of the left- and right-hand sides of $\ell_t = f^{b}[g^{b}(\ell_t, m_t), m_t]$ with respect to ℓ_t and m_t gives

$$
\begin{aligned}
1 &= f_{h}^{b} g_{\ell}^{b}, \\
0 &= f_{h}^{b} g_{m}^{b} + f_{m}^{b}, \\
0 &= f_{hh}^{b} \left(g_{\ell}^{b}\right)^{2} + f_{h}^{b} g_{\ell\ell}^{b}, \\
0 &= f_{hh}^{b} g_{\ell}^{b} g_{m}^{b} + f_{hm}^{b} g_{\ell}^{b} + f_{h}^{b} g_{\ell m}^{b}, \\
0 &= f_{hh}^{b} \left(g_{m}^{b}\right)^{2} + 2 f_{hm}^{b} g_{m}^{b} + f_{h}^{b} g_{mm}^{b} + f_{mm}^{b}.
\end{aligned}
$$

Using these equations and $f_{h}^{b} > 0$, $f_{m}^{b} > 0$, $f_{hh}^{b} < 0$, $f_{hm}^{b} \geq 0$, and $f_{mm}^{b} < 0$, we sequentially get

$$
\begin{aligned}
g_{\ell}^{b} &= \frac{1}{f_{h}^{b}} > 0, \\
g_{m}^{b} &= \frac{-f_{m}^{b}}{f_{h}^{b}} < 0, \\
g_{\ell\ell}^{b} &= \frac{-f_{hh}^{b}}{\left(f_{h}^{b}\right)^{3}} > 0, \\
g_{\ell m}^{b} &= \frac{f_{m}^{b} f_{hh}^{b}}{\left(f_{h}^{b}\right)^{3}} - \frac{f_{hm}^{b}}{\left(f_{h}^{b}\right)^{2}} < 0, \\
g_{mm}^{b} &= \frac{-f_{hh}^{b} \left(f_{m}^{b}\right)^{2}}{\left(f_{h}^{b}\right)^{3}} + 2 \frac{f_{m}^{b} f_{hm}^{b}}{\left(f_{h}^{b}\right)^{2}} - \frac{f_{mm}^{b}}{f_{h}^{b}} > 0.
\end{aligned}
$$

Then, using these expressions for $g_{\ell\ell}^{b}$, g_{mm}^{b}, $g_{\ell m}^{b}$, and the concavity of f^{b} established in Appendix A.1, we easily get that

$$
g_{\ell\ell}^{b} g_{mm}^{b} - \left(g_{\ell m}^{b}\right)^{2} = \frac{f_{hh}^{b} f_{mm}^{b} - \left(f_{hm}^{b}\right)^{2}}{\left(f_{h}^{b}\right)^{4}} \geq 0,
$$

so that the function g^b is (weakly) convex.

Moreover, since f^b is homogeneous of degree d, we have $\forall x \in \mathbb{R}_{\geq 0}$, $g^b\left(x^d \ell_t, x m_t\right) = x g^b\left(\ell_t, m_t\right)$. Computing the first derivative of the left- and right-hand sides of this equation with respect to x at $x = 1$ leads to

$$g^b = d\ell g^b_\ell + m g^b_m. \tag{A.2}$$

In turn, computing the first derivative of the left- and right-hand sides of the last equation with respect to ℓ and m leads to

$$\begin{aligned}(1-d)\, g^b_\ell &= d\ell g^b_{\ell\ell} + m g^b_{\ell m}, && \text{(A.3)}\\ 0 &= d\ell g^b_{\ell m} + m g^b_{mm}, && \text{(A.4)}\end{aligned}$$

which can be rewritten as

$$\begin{aligned}g^b_{\ell\ell} &= \frac{(1-d)\, g^b_\ell - m g^b_{\ell m}}{d\ell}, && \text{(A.5)}\\ g^b_{mm} &= \frac{-d\ell g^b_{\ell m}}{m}. && \text{(A.6)}\end{aligned}$$

Finally, as a consequence of (3) and (4), we have

$$\begin{aligned}\forall \ell_t &\in \mathbb{R}_{\geq 0}, \lim_{m_t \to +\infty} g^b_m(\ell_t, m_t) = 0, && \text{(A.7)}\\ \forall \ell_t &\in \mathbb{R}_{\geq 0}, \lim_{m_t \to 0} g^b_\ell(\ell_t, m_t) = +\infty. && \text{(A.8)}\end{aligned}$$

A.3 Proof of Lemma 1

Computing the first and second derivatives of the left- and right-hand sides of $\Gamma(\ell_t, m_t) \equiv v^b[g^b(\ell_t, m_t)]$ with respect to ℓ_t and m_t gives

$$\begin{aligned}\Gamma_\ell &= v^{b\prime} g^b_\ell > 0,\\ \Gamma_m &= v^{b\prime} g^b_m < 0,\\ \Gamma_{\ell\ell} &= v^{b\prime\prime}\left(g^b_\ell\right)^2 + v^{b\prime} g^b_{\ell\ell} > 0,\\ \Gamma_{mm} &= v^{b\prime\prime}\left(g^b_m\right)^2 + v^{b\prime} g^b_{mm} > 0,\\ \Gamma_{\ell m} &= v^{b\prime\prime} g^b_\ell g^b_m + v^{b\prime} g^b_{\ell m} < 0,\end{aligned}$$

where the inequalities follow from $v^{b\prime} > 0$, $v^{b\prime\prime} \geq 0$, $g^b_\ell > 0$, $g^b_m < 0$, $g^b_{\ell\ell} > 0$, $g^b_{mm} > 0$, and $g^b_{\ell m} < 0$. In addition, using first (A.5) and (A.6) and then (A.2), we easily get that

$$\begin{aligned}
\Gamma_{\ell\ell}\Gamma_{mm} - (\Gamma_{\ell m})^2 &= \left(v^{b\prime}\right)^2 \left[g^b_{\ell\ell}g^b_{mm} - \left(g^b_{\ell m}\right)^2\right] + ... \\
& \quad v^{b\prime}v^{b\prime\prime}\left[\left(g^b_\ell\right)^2 g^b_{mm} + \left(g^b_m\right)^2 g^b_{\ell\ell} - 2g^b_\ell g^b_m g^b_{\ell m}\right] \\
&= \frac{-\left(1-d\right)\left(v^{b\prime}\right)^2 g^b_\ell g^b_{\ell m}}{m} + ... \\
& \quad \frac{v^{b\prime}v^{b\prime\prime}}{d\ell m}\left[-g^b_{\ell m}\left(d\ell g^b_\ell + m g^b_m\right)^2 + \left(1-d\right) m g^b_\ell \left(g^b_m\right)^2\right] \\
&= \frac{-\left(1-d\right)\left(v^{b\prime}\right)^2 g^b_\ell g^b_{\ell m}}{m} + ... \\
& \quad \frac{v^{b\prime}v^{b\prime\prime}}{d\ell m}\left[-\left(g^b\right)^2 g^b_{\ell m} + \left(1-d\right) m g^b_\ell \left(g^b_m\right)^2\right] \\
&\geq 0,
\end{aligned} \tag{A.9}$$

so that the function Γ is (weakly) convex. Finally, using (A.7) and (A.8), we straightforwardly get (5) and (6).

A.4 Other Properties of Function Γ

Using (A.2) and (A.5), we get

$$\begin{aligned}
\ell\Gamma_{\ell\ell} + m\Gamma_{\ell m} &= \left(1-d\right)\ell\Gamma_{\ell\ell} + d\ell\Gamma_{\ell\ell} + m\Gamma_{\ell m} \\
&= \left(1-d\right)\ell\Gamma_{\ell\ell} + d\ell\left[v^{b\prime\prime}\left(g^b_\ell\right)^2 + v^{b\prime}g^b_{\ell\ell}\right] + m\left(v^{b\prime\prime}g^b_\ell g^b_m + v^{b\prime}g^b_{\ell m}\right) \\
&= \left(1-d\right)\ell\Gamma_{\ell\ell} + v^{b\prime\prime}g^b_\ell\left(d\ell g^b_\ell + m g^b_m\right) + v^{b\prime}\left(d\ell g^b_{\ell\ell} + m g^b_{\ell m}\right) \\
&= \left(1-d\right)\ell\Gamma_{\ell\ell} + v^{b\prime\prime}g^b g^b_\ell + \left(1-d\right)v^{b\prime}g^b_\ell \\
&\geq 0.
\end{aligned} \tag{A.10}$$

Similarly, using (A.2) and (A.4), we also get

$$\begin{aligned}
\ell\Gamma_{\ell m} + m\Gamma_{mm} &= \left(1-d\right)\ell\Gamma_{\ell m} + d\ell\Gamma_{\ell m} + m\Gamma_{mm} \\
&= \left(1-d\right)\ell\Gamma_{\ell m} + d\ell\left[v^{b\prime\prime}g^b_\ell g^b_m + v^{b\prime}g^b_{\ell m}\right] + m\left[v^{b\prime\prime}\left(g^b_m\right)^2 + v^{b\prime}g^b_{mm}\right] \\
&= \left(1-d\right)\ell\Gamma_{\ell m} + v^{b\prime\prime}g^b_m\left(d\ell g^b_\ell + m g^b_m\right) + v^{b\prime}\left(d\ell g^b_{\ell m} + m g^b_{mm}\right) \\
&= \left(1-d\right)\ell\Gamma_{\ell m} + v^{b\prime\prime}g^b g^b_m \\
&\leq 0.
\end{aligned} \tag{A.11}$$

Finally, we have

$$\frac{\Gamma_{\ell m}}{\Gamma_\ell} - \frac{\Gamma_{mm}}{\Gamma_m} = \frac{\left(v^{b\prime}\right)^2\left(g^b_m g^b_{\ell m} - g^b_\ell g^b_{mm}\right)}{\Gamma_\ell\Gamma_m} = \frac{\left(v^{b\prime}\right)^2\left(f^b_h f^b_{mm} - f^b_m f^b_{hm}\right)}{\Gamma_\ell\Gamma_m\left(f^b_h\right)^3} > 0. \tag{A.12}$$

Appendix B: Benchmark Model Under Flexible Prices

B.1 Proof of Lemma 2

Using (24), we can rewrite $\mathcal{F}(h_t)$ as

$$\mathcal{F}(h_t) = \mathcal{F}_1(h_t)\,\mathcal{F}_2(h_t),$$

where the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined over $(0, \overline{h})$ by

$$\begin{aligned}
\mathcal{F}_1(h_t) &\equiv \frac{\Gamma_m[\mathcal{L}(h_t), \mathcal{M}(h_t)]}{\Gamma_\ell[\mathcal{L}(h_t), \mathcal{M}(h_t)]}, \\
\mathcal{F}_2(h_t) &\equiv \frac{\varepsilon - 1}{\varepsilon}\frac{u'[f(h_t)]\,f'(h_t)}{v'(h_t)} - 1.
\end{aligned}$$

Since

$$\mathcal{F}_1(h_t) \equiv \frac{\Gamma_m[\mathcal{L}(h_t), \mathcal{M}(h_t)]}{\Gamma_\ell[\mathcal{L}(h_t), \mathcal{M}(h_t)]} = \frac{g_m^b[\mathcal{L}(h_t), \mathcal{M}(h_t)]}{g_\ell^b[\mathcal{L}(h_t), \mathcal{M}(h_t)]},$$

we have

$$\begin{aligned}
\left(g_\ell^b\right)^2 \mathcal{F}_1' &= g_\ell^b\left(g_{\ell m}^b\mathcal{L}' + g_{mm}^b\mathcal{M}'\right) - g_m^b\left(g_{\ell\ell}^b\mathcal{L}' + g_{\ell m}^b\mathcal{M}'\right) \\
&= -g_{\ell m}^b\left(d\mathcal{L}g_\ell^b + \mathcal{M}g_m^b\right)\left(\frac{\mathcal{M}'}{\mathcal{M}} - \frac{\mathcal{L}'}{d\mathcal{L}}\right) - (1-d)\,g_\ell^b g_m^b\frac{\mathcal{L}'}{d\mathcal{L}} \\
&= -g^b g_{\ell m}^b\left(\frac{\mathcal{M}'}{\mathcal{M}} - \frac{\mathcal{L}'}{d\mathcal{L}}\right) - (1-d)\,g_\ell^b g_m^b\frac{\mathcal{L}'}{d\mathcal{L}},
\end{aligned}$$

where the second equality is obtained by using (A.5) and (A.6), and the third equality by using (A.2). Now, deriving the left- and right-hand sides of (24) with respect to h_t gives

$$\Gamma_{\ell m}\mathcal{M}'(h_t) + \Gamma_{\ell\ell}\mathcal{L}'(h_t) = \mathcal{A}'(h_t) < 0.$$

This inequality, together with (A.10), implies

$$\frac{\mathcal{M}'(h_t)}{\mathcal{M}(h_t)} > \frac{\mathcal{L}'(h_t)}{d\mathcal{L}(h_t)}, \tag{B.1}$$

from which we conclude that $\mathcal{F}_1' > 0$. Then, using $\mathcal{F}_1' > 0$, $\mathcal{F}_1 < 0$, $\mathcal{F}_2' < 0$, and $\mathcal{F}_2 > 0$, we get that the function $\mathcal{F}$ is strictly increasing ($\mathcal{F}' > 0$). Moreover, $\mathcal{F}_1' > 0$ and $\mathcal{F}_1 < 0$ imply that $\lim_{h_t \to 0}\mathcal{F}_1(h_t) < 0$, while the Inada condition (1) implies that $\lim_{h_t \to 0}\mathcal{F}_2(h_t) = -\infty$, so that

$$\lim_{h_t \to 0}\mathcal{F}(h_t) = -\infty.$$

Finally, the Inada condition (5) implies

$$\lim_{h_t \to \overline{h}}\mathcal{F}(h_t) = 0.$$

B.2 Proof of Lemma 3

Using (20), we can rewrite $\mathcal{G}(h_t)$ as

$$\mathcal{G}(h_t) = h_t v'(h_t) \frac{\mathcal{M}(h_t)}{\mathcal{L}(h_t)}.$$

Given that $0 < d \leq 1$, $\mathcal{L} > 0$, and $\mathcal{L}' > 0$, (B.1) implies

$$\frac{\mathcal{M}'(h_t)}{\mathcal{M}(h_t)} > \frac{\mathcal{L}'(h_t)}{\mathcal{L}(h_t)},$$

so that $\mathcal{M}(h_t)[\mathcal{L}(h_t)]^{-1}$ is strictly increasing in h_t. Together with $v'' \geq 0$, $v' > 0$, $\mathcal{L} > 0$, and $\mathcal{M} > 0$, this implies in turn that $\mathcal{G}$ is strictly increasing ($\mathcal{G}' > 0$). Moreover, since $\mathcal{M}(h_t)[\mathcal{L}(h_t)]^{-1}$ is strictly increasing in h_t and positive, it converges towards a finite limit as h_t goes to zero, so that

$$\lim_{h_t \to 0} \mathcal{G}(h_t) = 0.$$

Finally, $\mathcal{L}' > 0$, (21), and (22) together imply that $\mathcal{L}(h_t)$ converges towards a finite limit as h_t goes to $\overline{h}$. Given (30), we therefore get that

$$\lim_{h_t \to \overline{h}} \mathcal{G}(h_t) = +\infty.$$

B.3 Proof of Proposition 2

We first consider the case with $I^m/\mu \geq \beta^{-1}$. We show by contradiction that there is no perfect-foresight equilibrium in this case. Indeed, if there were one, then (36) and $\mathcal{F} < 0$ would imply that

$$\frac{\mathcal{G}(h_{t+1})}{\mathcal{G}(h_t)} < \frac{\mu}{\beta I^m} \leq 1$$

in this equilibrium. Given that $\mathcal{G}' > 0$, this would in turn imply that the sequence $(h_t)_{t\in\mathbb{N}}$ is strictly decreasing. Using (36) and $\mathcal{F}' > 0$, we would then get

$$\frac{\mathcal{G}(h_{t+1})}{\mathcal{G}(h_t)} = \frac{\mu}{\beta I^m}[1 + \mathcal{F}(h_t)] < \frac{\mu}{\beta I^m}[1 + \mathcal{F}(h_0)] < 1,$$

so that the sequence $[\mathcal{G}(h_t)]_{t\in\mathbb{N}}$ would converge towards zero. Given (38), the sequence $(h_t)_{t\in\mathbb{N}}$ would then converge towards zero too. However, (33) and (36) would then imply that the ratio $\mathcal{G}(h_{t+1})/\mathcal{G}(h_t)$ turns negative for h_t sufficiently close to zero, which is impossible.

We then consider the alternative case with $I^m/\mu < \beta^{-1}$. We show by contradiction that there is no perfect-foresight equilibrium with $0 < h_0 < h$. Indeed, if there were one, then, using (37), $\mathcal{F}' > 0$, and $\mathcal{G}' > 0$, we would get by recurrence that the sequence $(h_t)_{t\in\mathbb{N}}$ is strictly decreasing. This would imply that

$$\frac{\mathcal{G}(h_{t+1})}{\mathcal{G}(h_t)} = 1 - \frac{\mu}{\beta I^m}[\mathcal{F}(h) - \mathcal{F}(h_t)] < 1 - \frac{\mu}{\beta I^m}[\mathcal{F}(h) - \mathcal{F}(h_0)] < 1,$$

so that the sequence $[\mathcal{G}(h_t)]_{t\in\mathbb{N}}$ would converge towards zero. Given (38), the sequence $(h_t)_{t\in\mathbb{N}}$ would then converge towards zero too. However, (33) and (36) would then imply that the ratio $\mathcal{G}(h_{t+1})/\mathcal{G}(h_t)$ turns negative for h_t sufficiently close to zero, which is impossible.

Now assume that $h < h_0 < \overline{h}$. Then, using (37), $\mathcal{F}' > 0$, and $\mathcal{G}' > 0$, we get by recurrence that the sequence $(h_t)_{t\in\mathbb{N}}$ is strictly increasing. This implies that

$$\frac{\mathcal{G}(h_{t+1})}{\mathcal{G}(h_t)} = 1 + \frac{\mu}{\beta I^m}\left[\mathcal{F}(h_t) - \mathcal{F}(h)\right] > 1 + \frac{\mu}{\beta I^m}\left[\mathcal{F}(h_0) - \mathcal{F}(h)\right] > 1,$$

so that the sequence $[\mathcal{G}(h_t)]_{t\in\mathbb{N}}$ goes to infinity. Given (39), the sequence $(h_t)_{t\in\mathbb{N}}$ then converges towards $\overline{h}$. Such a sequence is an equilibrium path if and only it satisfies the transversality condition (13). If money injections are done by helicopter drops, then the net real assets of households (a_t) increase asymptotically at the same rate as real reserves (m_t), i.e. at the gross rate $\mu/(\beta I^m) > 1$. As long as we have $I^m/\mu > 1$, this does not violate the transversality condition (13) and such sequences are equilibrium paths. Alternatively, if the central bank injects money by acquiring bonds issued (or previously held) by the private sector, then a_t can be constant while m_t grows, so that all sequences of this type satisfy the transversality condition and are equilibrium paths.

B.4 Log-Linearization

Log-linearizing the pricing equation (23) gives

$$\widehat{w}_t = \frac{ff''}{(f')^2}\widehat{y}_t + i_t^b - i_t^\ell. \tag{B.2}$$

Using (C.5), (C.6), (C.7), and (B.2), we get

$$\widehat{y}_t = \psi\widehat{m}_t, \tag{B.3}$$

where ψ is defined in Lemma 5. Using the IS equation (41), the spread equation (42), the money-market-clearing condition (43), and (B.3), we get

$$\omega_1^n\widehat{y}_t = \mathbb{E}_t\left\{\widehat{y}_{t+1}\right\} + \frac{\psi}{1-\sigma\psi}\left(i_t^m - \mathbb{E}_t\left\{\widehat{\mu}_{t+1}\right\}\right),$$

where ω_1^n is defined in Lemma 5. Since $\omega_1^n > 1$, this dynamic equation meets Blanchard and Kahn's (1980) conditions and therefore has a unique stationary solution. When $i_t^m = i^*$ for $1 \leq t \leq T$, $i_t^m = 0$ for $t \geq T+1$, and $\widehat{\mu}_t = 0$ for $t \geq 1$, this solution, for $t = 1$, is

$$\widehat{y}_1 = \left[\frac{1-(\omega_1^n)^{-T}}{\delta_m - \delta_y\psi}\right]\frac{\psi i^*}{\sigma}. \tag{B.4}$$

When in addition $\widehat{m}_0 = 0$, using (43), (B.3), and (B.4), we get

$$\pi_1 = -\left[\frac{1-(\omega_1^n)^{-T}}{\delta_m - \delta_y\psi}\right]\frac{i^*}{\sigma}.$$

These values of $\widehat{y}_1$ and π_1 coincide with the values reported in (59).

Now introduce fiscal expenditures in the same way as in Subsection 5.2. Using (B.2), (C.15), (C.16), and (C.17), we get

$$\widehat{y}_t = \widetilde{\psi}\widehat{m}_t + \vartheta\widetilde{g}_t, \tag{B.5}$$

where ϑ is defined in Subsection 5.3. Using the money-market-clearing condition (43), the IS equation (52), the spread equation (54), and (B.5), we get

$$\begin{aligned}\widetilde{\omega}_1^n\widehat{y}_t &= \mathbb{E}_t\left\{\widehat{y}_{t+1}\right\} + \frac{\widetilde{\psi}}{1-\widetilde{\sigma}\widetilde{\psi}}\left(i_t^m - \mathbb{E}_t\left\{\widehat{\mu}_{t+1}\right\}\right) + ... \\ &\left[\frac{\left(\vartheta - \widetilde{\sigma}\widetilde{\psi}\right) + \widetilde{\sigma}\left(\widetilde{\delta}_m\vartheta - \delta_g\widetilde{\psi}\right)}{1-\widetilde{\sigma}\widetilde{\psi}}\right]\widetilde{g}_t - \left(\frac{\vartheta - \widetilde{\sigma}\widetilde{\psi}}{1-\widetilde{\sigma}\widetilde{\psi}}\right)\mathbb{E}_t\left\{\widetilde{g}_{t+1}\right\}.\end{aligned}$$

Since $\widetilde{\omega}_1^n > 1$, this dynamic equation meets Blanchard and Kahn's (1980) conditions and therefore has a unique stationary solution. When $i_t^m = \widehat{\mu}_t = 0$ for $t \geq 1$, $\widetilde{g}_T = \widetilde{g}^* \neq 0$ for some date $T \geq 2$, and $\widetilde{g}_t = 0$ for all dates $t \geq 1$ such that $t \neq T$, this solution, for $t = 1$, is

$$\widehat{y}_1 = \left[(1-\vartheta)\widetilde{\omega}_1^n + \left(1+\widetilde{\delta}_y\right)\vartheta - (1+\delta_g)\right]\left[\frac{\widetilde{\sigma}\widetilde{\psi}\left(\widetilde{\omega}_1^n\right)^{-T}}{1-\widetilde{\sigma}\widetilde{\psi}}\right]\widetilde{g}^*. \tag{B.6}$$

When in addition $\widehat{m}_0 = 0$, using (43), (B.5), and (B.6), we get

$$\pi_1 = -\left[(1-\vartheta)\widetilde{\omega}_1^n + \left(1+\widetilde{\delta}_y\right)\vartheta - (1+\delta_g)\right]\left[\frac{\widetilde{\sigma}\left(\widetilde{\omega}_1^n\right)^{-T}}{1-\widetilde{\sigma}\widetilde{\psi}}\right]\widetilde{g}^*.$$

These values of $\widehat{y}_1$ and π_1 coincide with the values reported in (60) and (61).

Appendix C: Benchmark Model Under Sticky Prices

C.1 Phillips Curve

Firm i chooses $\widetilde{P}_t(i)$ to maximize

$$\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\left[\frac{\lambda_{t+k}}{\lambda_t\Pi_{t,t+k}}\widetilde{P}_t(i)\,y_{t+k}(i) - \beta\frac{\lambda_{t+k+1}}{\lambda_t\Pi_{t,t+k+1}}I_{t+k}^{\ell}W_{t+k}f^{-1}\left[y_{t+k}(i)\right]\right]\right\}$$

subject to

$$y_{t+k}(i) = \left[\frac{\widetilde{P}_t(i)}{P_{t+k}}\right]^{-\varepsilon}y_{t+k},$$

where $\Pi_{t,t+k} \equiv P_{t+k}/P_t$ for any $k \in \mathbb{N}$. The first-order condition is

$$\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\left[\frac{\lambda_{t+k}}{\lambda_t\Pi_{t,t+k}}\widetilde{P}_t(i) - \frac{\beta\varepsilon}{\varepsilon-1}\frac{\lambda_{t+k+1}}{\lambda_t\Pi_{t,t+k+1}}\frac{I_{t+k}^{\ell}W_{t+k}}{f'\left[h_{t+k}(i)\right]}\right]y_{t+k}(i)\right\} = 0.$$

Using the law of iterated expectations and the Euler equation (9), we can rewrite this first-order condition as

$$\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\,\frac{\lambda_{t+k}}{\lambda_t\Pi_{t,t+k}}\left[\widetilde{P}_t\left(i\right)-\frac{\varepsilon}{\varepsilon-1}\frac{I^\ell_{t+k}}{I^b_{t+k}}\frac{W_{t+k}}{f'\left[h_{t+k}\left(i\right)\right]}\right]y_{t+k}\left(i\right)\right\}=0,$$

or equivalently

$$\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\,\frac{\lambda_{t+k}}{\lambda_t\Pi_{t,t+k}}\left[\frac{\widetilde{P}_t\left(i\right)}{P_t}-\frac{\varepsilon}{\varepsilon-1}\frac{I^\ell_{t+k}}{I^b_{t+k}}\frac{w_{t+k}\Pi_{t,t+k}}{f'\left[h_{t+k}\left(i\right)\right]}\right]y_{t+k}\left(i\right)\right\}=0.$$

Log-linearizing this equation around the unique zero-inflation steady state leads to

$$\widetilde{p}_t-p_t=(1-\beta\theta)\,\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\left(i^\ell_{t+k}-i^b_{t+k}+\widehat{w}_{t+k}+p_{t+k}-p_t-\widehat{mp}_{t+k|t}\right)\right\},\tag{C.1}$$

where $\widetilde{p}_t\equiv\log(\widetilde{P}_t)$, $p_t\equiv\log(P_t)$, variables with hats denote log deviations from steady-state values, $i^\ell_t\equiv\widehat{I}^\ell_t$, $i^b_t\equiv\widehat{I}^b_t$, and $mp_{t+k|t}$ denotes the marginal productivity in period $t+k$ for a firm whose price was last set in period t. Now, log-linearizing the production function (14) and the goods-market-clearing condition (19) gives

$$\widehat{h}_t = \frac{f}{f'h}\widehat{y}_t,\tag{C.2}$$

$$\widehat{c}_t = \widehat{y}_t,\tag{C.3}$$

so that $\widehat{mp}_{t+k|t}$ can be rewritten as

$$\begin{aligned}\widehat{mp}_{t+k|t} &= \frac{f''h}{f'}\widehat{h}_{t+k|t}=\widehat{mp}_{t+k}+\frac{f''h}{f'}\left(\widehat{h}_{t+k|t}-\widehat{h}_{t+k}\right)=\widehat{mp}_{t+k}+\frac{ff''}{(f')^2}\left(\widehat{y}_{t+k|t}-\widehat{y}_{t+k}\right)\\ &= \widehat{mp}_{t+k}-\frac{\varepsilon ff''}{(f')^2}\left(\widetilde{p}_t-p_{t+k}\right),\end{aligned}$$

where mp_{t+k} denotes the average marginal productivity in period $t+k$. Using this result and

$$\pi_t\equiv\log\left(\Pi_t\right)=(1-\theta)\left(\widetilde{p}_t-p_{t-1}\right),$$

and following the same steps as in, e.g., Galí (2015, Chapter 3), we can rewrite (C.1) as

$$\pi_t=\beta\mathbb{E}_t\left\{\pi_{t+1}\right\}+\frac{(1-\theta)\,(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon ff''}{(f')^2}\right]}\left(i^\ell_t-i^b_t+\widehat{w}_t-\widehat{mp}_t\right).\tag{C.4}$$

Now, log-linearizing the first-order condition (11), and using (C.3), gives

$$i^\ell_t-i^b_t=\alpha_\ell\frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\widehat{\ell}_t+\alpha_\ell\frac{\Gamma_{\ell m}m}{\Gamma_\ell}\widehat{m}_t-\alpha_\ell\frac{u''c}{u'}\widehat{y}_t,\tag{C.5}$$

where

$$\alpha_\ell\equiv\frac{I^\ell-I^b}{I^\ell}\in(0,1)\,.$$

Log-linearizing the first-order condition (10), and using (C.2) and (C.3), gives

$$\widehat{w}_t=\left(-\frac{u''c}{u'}+\frac{v''h}{v'}\frac{f}{f'h}\right)\widehat{y}_t.\tag{C.6}$$

Log-linearizing the constraint (16) holding with equality, and using (C.2) and (C.6), gives

$$\widehat{\ell}_t = \left(-\frac{u''c}{u'} + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\widehat{y}_t. \tag{C.7}$$

Using (C.5), (C.6), (C.7), and

$$\widehat{mp}_t = \frac{ff''}{(f')^2}\widehat{y}_t,$$

we can then rewrite (C.4) as the Phillips curve

$$\pi_t = \beta\mathbb{E}_t\left\{\pi_{t+1}\right\} + \kappa_y\widehat{y}_t - \kappa_m\widehat{m}_t,$$

where

$$\begin{aligned}\kappa_y &\equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon ff''}{(f')^2}\right]}\left[\alpha_\ell\frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\left(-\frac{u''c}{u'} + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right) - \ldots\right. \\ &\quad \left.(1+\alpha_\ell)\frac{u''c}{u'} + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2}\right] > 0, \\ \kappa_m &\equiv -\frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon ff''}{(f')^2}\right]}\alpha_\ell\frac{\Gamma_{\ell m}m}{\Gamma_\ell} > 0.\end{aligned}$$

C.2 Spread Between the Interest Rates on Bonds and on Reserves

Log-linearizing the first-order condition (12), and using (C.3), gives

$$i_t^m - i_t^b = \alpha_m\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\widehat{\ell}_t + \alpha_m\frac{\Gamma_{mm}m}{\Gamma_m}\widehat{m}_t - \alpha_m\frac{u''c}{u'}\widehat{y}_t, \tag{C.8}$$

where $i_t^m \equiv \widehat{I}_t^m$ and

$$\alpha_m \equiv \frac{I^m - I^b}{I^m} < 0.$$

Using (C.7), we can rewrite (C.8) as

$$i_t^b = i_t^m + \sigma\delta_y\widehat{y}_t - \sigma\delta_m\widehat{m}_t,$$

where

$$\begin{aligned}\delta_y &\equiv -\alpha_m + \alpha_m\frac{u'}{u''c}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(-\frac{u''c}{u'} + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right) > 0, \\ \delta_m &\equiv -\alpha_m\frac{u'}{u''c}\frac{\Gamma_{mm}m}{\Gamma_m} > 0.\end{aligned}$$

C.3 Proof of Lemma 4

We can rewrite $\mathcal{P}(X)$ as

$$\begin{aligned}\mathcal{P}(X) &= X^3 - \left(\frac{1+2\beta}{\beta} + K_1 + \frac{K_2}{\beta}\right)X^2 + \ldots \\ &\quad \left(\frac{2+\beta}{\beta} + \frac{1+\beta}{\beta}K_1 + \frac{K_2}{\beta} + \frac{K_3}{\beta}\right)X - \frac{1+K_1}{\beta},\end{aligned}$$

where

$$\begin{aligned}
K_1 &\equiv \delta_y > 0, \\
K_2 &\equiv \frac{\kappa_y}{\sigma} - \kappa_m > 0, && \text{(C.9)} \\
K_3 &\equiv \delta_m \kappa_y - \delta_y \kappa_m > 0. && \text{(C.10)}
\end{aligned}$$

The inequality $K_3 > 0$ follows from

$$\begin{aligned}
\frac{-\theta\left[1-\frac{\varepsilon f f''}{(f')^2}\right]}{\alpha_m(1-\theta)(1-\beta\theta)} K_3 &= \frac{u'}{u''c}\frac{\Gamma_{mm}m}{\Gamma_m}\left[\alpha_\ell \frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\left(-\frac{u''c}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right) - \ldots\right. \\
&\quad \left.(1+\alpha_\ell)\frac{u''c}{u'}+\frac{v''h}{v'}\frac{f}{f'h}-\frac{ff''}{(f')^2}\right] + \ldots \\
&\quad \alpha_\ell \frac{\Gamma_{\ell m}m}{\Gamma_\ell}\left[1-\frac{u'}{u''c}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(-\frac{u''c}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right)\right] \\
&= \frac{\alpha_\ell \ell m}{\Gamma_\ell \Gamma_m}\frac{u'}{u''c}\left(-\frac{u''c}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right)\left[\Gamma_{\ell\ell}\Gamma_{mm}-(\Gamma_{\ell m})^2\right] + \ldots \\
&\quad \frac{u'}{u''c}\frac{\Gamma_{mm}m}{\Gamma_m}\left[-\frac{u''c}{u'}+\frac{v''h}{v'}\frac{f}{f'h}-\frac{ff''}{(f')^2}\right] + \ldots \\
&\quad \alpha_\ell m\left(\frac{\Gamma_{\ell m}}{\Gamma_\ell}-\frac{\Gamma_{mm}}{\Gamma_m}\right) \\
&> 0,
\end{aligned}$$

where the last inequality is obtained using (A.9) and (A.12).

The inequality $K_2 > 0$ follows from $K_3 > 0$ and

$$\begin{aligned}
\sigma\delta_m - \delta_y &= \alpha_m \frac{\Gamma_{mm}m}{\Gamma_m} + \alpha_m - \alpha_m \frac{u'}{u''c}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(-\frac{u''c}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right) \\
&= \frac{\alpha_m}{\Gamma_m}(\ell\Gamma_{\ell m}+m\Gamma_{mm}) + \alpha_m - \alpha_m \frac{u'}{u''c}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right) \\
&< 0, && \text{(C.11)}
\end{aligned}$$

where the last inequality is obtained using (A.11).

We have

$$\begin{aligned}
\mathcal{P}(0) &= -\left(\frac{1+K_1}{\beta}\right) < 0, && \text{(C.12)} \\
\mathcal{P}(1) &= \frac{K_3}{\beta} > 0. && \text{(C.13)}
\end{aligned}$$

By rewriting $\mathcal{P}(X)$ as

$$\mathcal{P}(X) = (X-1-K_1)\left[X^2 - \left(\frac{1+\beta+K_2}{\beta}\right)X + \frac{1}{\beta}\right] - \left(\frac{K_1K_2-K_3}{\beta}\right)X$$

and noting that (C.11) implies

$$K_3 < K_1 K_2, \tag{C.14}$$

we also get

$$\begin{aligned}\mathcal{P}(1+K_1) &= \frac{-(K_1K_2-K_3)(1+K_1)}{\beta}<0,\\ \mathcal{P}(1+K_4) &= \frac{-(K_1K_2-K_3)(1+K_4)}{\beta}<0,\end{aligned}$$

where

$$K_4 \equiv \frac{1+\beta+K_2+\sqrt{(1+\beta+K_2)^2-4\beta}}{2\beta}>0.$$

Therefore, the roots of $\mathcal{P}(X)$ are three real numbers ρ, ω_1, and ω_2 such that

$$0<\rho<1<\omega_1<1+\min(K_1,K_4)\leq 1+\max(K_1,K_4)<\omega_2.$$

C.4 Introduction of Fiscal Expenditures

Log-linearizing the first-order condition (11), and using (51), gives

$$i_t^\ell - i_t^b = \alpha_\ell \frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\widehat{\ell}_t + \alpha_\ell \frac{\Gamma_{\ell m}m}{\Gamma_\ell}\widehat{m}_t - \alpha_\ell \frac{u''y}{u'}\widehat{y}_t + \alpha_\ell \frac{u''y}{u'}\widetilde{g}_t. \tag{C.15}$$

Log-linearizing the first-order condition (10), and using (51) and (C.2), gives

$$\widehat{w}_t = \left(-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h}\right)\widehat{y}_t + \frac{u''y}{u'}\widetilde{g}_t. \tag{C.16}$$

Log-linearizing the constraint (16) holding with equality, and using (C.2) and (C.16), gives

$$\widehat{\ell}_t = \left(-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\widehat{y}_t + \frac{u''y}{u'}\widetilde{g}_t. \tag{C.17}$$

Using (C.15), (C.16), (C.17), and

$$\widehat{mp}_t = \frac{ff''}{(f')^2}\widehat{y}_t,$$

we can then rewrite (C.4) as the Phillips curve

$$\pi_t = \beta \mathbb{E}_t\{\pi_{t+1}\} + \widetilde{\kappa}_y \widehat{y}_t - \widetilde{\kappa}_m \widehat{m}_t - \kappa_g \widetilde{g}_t,$$

where

$$\begin{aligned}\widetilde{\kappa}_y &\equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon ff''}{(f')^2}\right]}\left[\alpha_\ell\frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\left(-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right) - (1+\alpha_\ell)\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2}\right]>0,\\ \widetilde{\kappa}_m &\equiv -\frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon ff''}{(f')^2}\right]}\alpha_\ell\frac{\Gamma_{\ell m}m}{\Gamma_\ell}>0,\\ \kappa_g &\equiv -\frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon ff''}{(f')^2}\right]}\frac{u''y}{u'}\left[1+\alpha_\ell\left(1+\frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\right)\right]>0.\end{aligned}$$

Moreover, log-linearizing the first-order condition (12), and using (51), gives

$$i_t^m - i_t^b = \alpha_m \frac{\Gamma_{\ell m}\ell}{\Gamma_m}\widehat{\ell}_t + \alpha_m \frac{\Gamma_{mm}m}{\Gamma_m}\widehat{m}_t - \alpha_m \frac{u''y}{u'}\widehat{y}_t + \alpha_m \frac{u''y}{u'}\widetilde{g}_t.$$

Using (C.17), we can rewrite this equation as

$$i_t^b - i_t^m = \widetilde{\sigma}\widetilde{\delta}_y\widehat{y}_t - \widetilde{\sigma}\widetilde{\delta}_m\widehat{m}_t - \widetilde{\sigma}\delta_g\widetilde{g}_t,$$

where

$$\begin{aligned}
\widetilde{\delta}_y &\equiv -\alpha_m + \alpha_m \frac{u'}{u''y}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right) > 0, \\
\widetilde{\delta}_m &\equiv -\alpha_m \frac{u'}{u''y}\frac{\Gamma_{mm}m}{\Gamma_m} > 0, \\
\delta_g &\equiv -\alpha_m\left(1 + \frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right) > 0.
\end{aligned}$$

If $g > 0$, then the reduced-form parameters $\widetilde{\kappa}_y$, $\widetilde{\kappa}_m$, $\widetilde{\delta}_y$, and $\widetilde{\delta}_m$ differ from their counterparts κ_y, κ_m, δ_y, and δ_m in the following two ways. First, the elasticity $u''c/u'$ in the latter is replaced by $(y/c)u''c/u'$ in the former, where $y/c > 1$. Second, for any given (dis)utility and production functions u, v, v^b, f, and f^b, any given values of the structural parameters $\beta \in (0,1)$, $\varepsilon > 0$, and $\theta \in (0,1)$, and any given steady-state value of the IOR rate $I^m \in (0, \beta^{-1})$, the steady state changes when one moves from $g = 0$ to $g > 0$, so that the reduced-form parameter α_ℓ and the elasticities $u''c/u'$, $v''h/v'$, $f'h/f$, $f''h/f'$, $\Gamma_{\ell\ell}\ell/\Gamma_\ell$, $\Gamma_{mm}m/\Gamma_m$, $\Gamma_{\ell m}\ell/\Gamma_m$, and $\Gamma_{\ell m}m/\Gamma_\ell$ in $\widetilde{\kappa}_y$, $\widetilde{\kappa}_m$, $\widetilde{\delta}_y$, and $\widetilde{\delta}_m$ are not evaluated at the same point as in κ_y, κ_m, δ_y, and δ_m. These two differences notwithstanding, it is straightforward to show, by following the same steps as in Appendix C.3, that

$$\begin{aligned}
\widetilde{K}_1 &\equiv \widetilde{\delta}_y > 0, \\
\widetilde{K}_2 &\equiv \frac{\widetilde{\kappa}_y}{\widetilde{\sigma}} - \widetilde{\kappa}_m > 0, \\
\widetilde{K}_3 &\equiv \widetilde{\delta}_m\widetilde{\kappa}_y - \widetilde{\delta}_y\widetilde{\kappa}_m \in \left(0, \widetilde{K}_1\widetilde{K}_2\right),
\end{aligned}$$

and to conclude that the roots of $\widetilde{\mathcal{P}}(X)$ are three real numbers $\widetilde{\rho}$, $\widetilde{\omega}_1$, and $\widetilde{\omega}_2$ such that

$$0 < \widetilde{\rho} < 1 < \widetilde{\omega}_1 < 1 + \min\left(\widetilde{K}_1, \widetilde{K}_4\right) \leq 1 + \max\left(\widetilde{K}_1, \widetilde{K}_4\right) < \widetilde{\omega}_2,$$

where

$$\widetilde{K}_4 \equiv \frac{1 + \beta + \widetilde{K}_2 + \sqrt{\left(1 + \beta + \widetilde{K}_2\right)^2 - 4\beta}}{2\beta} > 0.$$

C.5 Proof of Lemma 5

As prices become perfectly flexible ($\theta \to 0$), we have $\kappa_y \to +\infty$ and $\kappa_m \to +\infty$. For any $X \in \mathbb{R}$, we have

$$\lim_{\theta \to 0} \frac{\mathcal{P}(X)}{\kappa_y} = \frac{-(1-\sigma\psi)}{\beta\sigma}X^2 + \frac{(1-\sigma\psi) + \sigma(\delta_m - \delta_y\psi)}{\beta\sigma}X, \tag{C.18}$$

where $\psi \equiv \kappa_m/\kappa_y$ is independent of θ. The two roots of the polynomial on the right-hand side of (C.18) are 0 and $1 + \sigma(\delta_m - \delta_y\psi)/(1-\sigma\psi)$. The latter root is strictly higher than 1, because of (C.9) and (C.10), and strictly lower than $1+\delta_y$, because of (C.14). Since ρ is the unique root of $\mathcal{P}(X)$ inside $(-1,1)$, and ω_1 its unique root inside $(1,1+\delta_y)$, we conclude that

$$\lim_{\theta\to 0}\rho = 0, \tag{C.19}$$

$$\lim_{\theta\to 0}\omega_1 = 1+\frac{\sigma\left(\delta_m-\delta_y\psi\right)}{1-\sigma\psi}. \tag{C.20}$$

Moreover, using

$$\rho\omega_1\omega_2 = -\mathcal{P}\left(0\right) = \frac{1+\delta_y}{\beta}, \tag{C.21}$$

$\rho > 0$, (C.19), and (C.20), we get that

$$\lim_{\theta\to 0}\omega_2 = +\infty. \tag{C.22}$$

Next, using (C.19) and

$$\begin{aligned} 0=\mathcal{P}\left(\rho\right) &= \rho^3-\left[\frac{1+\beta-\kappa_m}{\beta}+\frac{\kappa_y}{\beta\sigma}+\left(1+\delta_y\right)\right]\rho^2+... \\ &\left[\left(1+\delta_y\right)\frac{1+\beta-\kappa_m}{\beta}+\frac{1}{\beta}+\left(\frac{1}{\sigma}+\delta_m\right)\frac{\kappa_y}{\beta}\right]\rho-\left(\frac{1+\delta_y}{\beta}\right), \end{aligned}$$

we get that

$$\lim_{\theta\to 0}\kappa_m\rho = \frac{\sigma\psi\left(1+\delta_y\right)}{\left(1-\sigma\psi\right)+\sigma\left(\delta_m-\delta_y\psi\right)}. \tag{C.23}$$

Finally, using (C.20), (C.21), and (C.23), we get

$$\lim_{\theta\to 0}\frac{\omega_2}{\kappa_y} = \frac{1-\sigma\psi}{\beta\sigma}.$$

C.6 Proof of Proposition 7

The proof of steady-state convergence largely rests on reasonings similar to the ones conducted in Subsections 3.1 and 3.2 and in Appendix B.1. With the introduction of parameter γ, Equation (24) becomes

$$\gamma\Gamma_\ell\left[\mathcal{L}\left(h\right),m\right] = \mathcal{A}\left(h\right). \tag{C.24}$$

This equation implicitly and uniquely defines a function $\widetilde{\mathcal{M}}$ such that

$$m = \widetilde{\mathcal{M}}\left(h,\gamma\right).$$

The function $\widetilde{\mathcal{M}}$ is strictly increasing in each of its two arguments ($\widetilde{\mathcal{M}}_h > 0$ and $\widetilde{\mathcal{M}}_\gamma > 0$). For any $\gamma > 0$, the function $h \mapsto \widetilde{\mathcal{M}}(h,\gamma)$ is defined over $(0,\overline{h})$, where $\overline{h} \in (0,h^\star]$ is implicitly and uniquely defined by

$$\lim_{m_t\to+\infty}\gamma\Gamma_\ell\left[\mathcal{L}\left(\overline{h}\right),m_t\right] = \mathcal{A}\left(\overline{h}\right).$$

Note that $\overline{h}$ depends on γ and satisfies

$$\lim_{\gamma\to 0}\overline{h} = h^\star. \tag{C.25}$$

Now, with the introduction of parameter γ, Equation (32) becomes

$$\widetilde{\mathcal{F}}(h,\gamma) \equiv \frac{\gamma\Gamma_m\left[\mathcal{L}(h), \widetilde{\mathcal{M}}(h,\gamma)\right]}{u'[f(h)]} = \beta I^m - 1. \tag{C.26}$$

Lemma 2 implies that, for any $\gamma > 0$,

$$\widetilde{\mathcal{F}}_h > 0 \quad \text{and} \quad \lim_{h_t \to \overline{h}} \widetilde{\mathcal{F}}(h_t,\gamma) = 0. \tag{C.27}$$

We can rewrite $\widetilde{\mathcal{F}}(h,\gamma)$ as

$$\widetilde{\mathcal{F}}(h,\gamma) = \widetilde{\mathcal{F}}_1(h,\gamma)\,\mathcal{F}_2(h),$$

where, for any $\gamma > 0$, the function $h \mapsto \widetilde{\mathcal{F}}_1(h,\gamma)$ is defined over $(0,\overline{h})$ by

$$\widetilde{\mathcal{F}}_1(h,\gamma) \equiv \frac{\Gamma_m\left[\mathcal{L}(h), \widetilde{\mathcal{M}}(h,\gamma)\right]}{\Gamma_\ell\left[\mathcal{L}(h), \widetilde{\mathcal{M}}(h,\gamma)\right]} = \frac{g^b_m\left[\mathcal{L}(h), \widetilde{\mathcal{M}}(h,\gamma)\right]}{g^b_\ell\left[\mathcal{L}(h), \widetilde{\mathcal{M}}(h,\gamma)\right]},$$

while $\mathcal{F}_2$ is defined in Appendix B.1. We have

$$\left(g^b_\ell\right)^2 \widetilde{\mathcal{F}}_{1,\gamma} = \left(g^b_\ell g^b_{mm} - g^b_m g^b_{\ell m}\right)\widetilde{\mathcal{M}}_\gamma = -g^b_{\ell m}\left(d\mathcal{L}g^b_\ell + \mathcal{M}g^b_m\right)\frac{\widetilde{\mathcal{M}}_\gamma}{\widetilde{\mathcal{M}}} = -g^b g^b_{\ell m}\frac{\widetilde{\mathcal{M}}_\gamma}{\widetilde{\mathcal{M}}} > 0,$$

where the second equality is obtained by using (A.6), and the third equality by using (A.2). Therefore, we get $\widetilde{\mathcal{F}}_{1,\gamma} > 0$ and hence, using $\mathcal{F}_2 > 0$,

$$\widetilde{\mathcal{F}}_\gamma > 0. \tag{C.28}$$

Using (C.25), (C.26), (C.27), and (C.28), we conclude that

$$\lim_{(I^m,\gamma)\to(\beta^{-1},0)} h = h^\star. \tag{C.29}$$

As a consequence, the steady-state values of all endogenous variables converge, as $(I^m,\gamma) \to (\beta^{-1},0)$, towards their counterparts in the corresponding basic NK model $-$ with the exception of the steady-state value of real reserves m, which does not exist in the basic NK model.

We now show that m is bounded away from zero and infinity as $(I^m,\gamma) \to (\beta^{-1},0)$ when $(\beta^{-1} - I^m)/\gamma$ is bounded away from zero and infinity. Rewrite (C.26) as

$$\frac{-\Gamma_m[\mathcal{L}(h), m]}{u'[f(h)]} = \frac{1-\beta I^m}{\gamma}. \tag{C.30}$$

Since the right-hand side of this equation is bounded away from zero, (5) and (C.29) imply that m is bounded from above. Moreover, (C.28) and $\widetilde{\mathcal{F}} < 0$ imply that, for any h_t,

$$\lim_{\gamma\to 0} \frac{-\widetilde{\mathcal{F}}(h_t,\gamma)}{\gamma} = +\infty. \tag{C.31}$$

Now, using (6) and (C.24), we get, for any h_t,

$$\lim_{\gamma\to 0} \widetilde{\mathcal{M}}(h_t,\gamma) = 0$$

and therefore

$$\lim_{\gamma\to 0}\frac{-\widetilde{\mathcal{F}}(h_t,\gamma)}{\gamma}=\lim_{\gamma\to 0}\frac{-\Gamma_m\left[\mathcal{L}(h_t),\widetilde{\mathcal{M}}(h_t,\gamma)\right]}{u'[f(h_t)]}=\lim_{m_t\to 0}\frac{-\Gamma_m[\mathcal{L}(h_t),m_t]}{u'[f(h_t)]}. \tag{C.32}$$

Using (C.31) and (C.32), we then get, for any h_t,

$$\lim_{m_t\to 0}\frac{-\Gamma_m[\mathcal{L}(h_t),m_t]}{u'[f(h_t)]}=+\infty.$$

Using this result, (C.29), (C.30), and the fact that the right-hand side of (C.30) is bounded from above, we conclude that m is bounded away from zero.

Finally, (C.29) and the boundedness of m away from zero and infinity imply that, as $(I^m,\gamma)\to(\beta^{-1},0)$, (i) the elasticities $\Gamma_{\ell\ell}\ell/\Gamma_\ell$, $\Gamma_{mm}m/\Gamma_m$, $\Gamma_{\ell m}\ell/\Gamma_m$, and $\Gamma_{\ell m}m/\Gamma_\ell$ are themselves bounded away from zero and infinity, and (ii) the parameter $\alpha_\ell\equiv(I^\ell-I^b)/I^\ell$, which can be rewritten as

$$\alpha_\ell=\frac{\gamma\Gamma_\ell(\ell,m)}{u'[f(h)]+\gamma\Gamma_\ell(\ell,m)}$$

by using the first-order condition (11) amended to take into account the introduction of parameter γ, converges towards zero. Since $\alpha_m\equiv(I^m-I^b)/I^m$ also converges towards zero as $(I^m,\gamma)\to(\beta^{-1},0)$, using the definitions of κ_y, κ_m, δ_y, and δ_m, we conclude that

$$\kappa_y\to\kappa\equiv\frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon ff''}{(f')^2}\right]}\left[\frac{u''c}{u'}+\frac{v''h}{v'}\frac{f}{f'h}-\frac{ff''}{(f')^2}\right],\quad \kappa_m\to 0,\quad \delta_y\to 0,\quad \text{and}\quad \delta_m\to 0$$

as $(I^m,\gamma)\to(\beta^{-1},0)$, where the elasticities in κ are evaluated at $h=h^\star$ and $c=f(h^\star)$.

C.7 Proof of Proposition 8

In this appendix, for the sake of brevity, we replace "$(I^m,\gamma)\to(\beta^{-1},0)$ with $(\beta^{-1}-I^m)/\gamma$ bounded away from zero and infinity" by "$D\to 0$," where D stands for "Distance between the basic NK model and our model." Using Proposition 7, we easily get that, as $D\to 0$,

$$\mathcal{P}(X)\to\frac{1}{\beta}(X-1)\left[\beta X^2-\left(1+\beta+\frac{\kappa}{\overline{\sigma}}\right)X+1\right]$$

for any $X\in\mathbb{R}$, where $\overline{\sigma}\equiv -u''[f(h^\star)]f(h^\star)/u'[f(h^\star)]>0$, from which we conclude that $\omega_1\to 1$ and

$$\begin{aligned}\rho &\to \overline{\rho}\equiv\frac{1}{2\beta}\left[1+\beta+\frac{\kappa}{\overline{\sigma}}-\sqrt{\left(1+\beta+\frac{\kappa}{\overline{\sigma}}\right)^2-4\beta}\right]\in(0,1),\\ \omega_2 &\to \overline{\omega}_2\equiv\frac{1}{2\beta}\left[1+\beta+\frac{\kappa}{\overline{\sigma}}+\sqrt{\left(1+\beta+\frac{\kappa}{\overline{\sigma}}\right)^2-4\beta}\right]>1.\end{aligned}$$

Using these results and L'Hospital's rule, we can easily determine the limits of π_1 in (49) and $\widehat{y}_1$ in (50) as $D\to 0$:

$$\begin{aligned}\lim_{D\to 0}\pi_1 &= \frac{-\kappa i^*}{\beta\overline{\sigma}(\overline{\omega}_2-1)}\left[T-\frac{1-\overline{\omega}_2^{-T}}{\overline{\omega}_2-1}\right],\\ \lim_{D\to 0}\widehat{y}_1 &= \frac{-i^*}{\beta\overline{\sigma}(\overline{\omega}_2-1)}\left[(1-\beta\overline{\rho})T+\left(\beta-\frac{1-\beta\overline{\rho}}{\overline{\omega}_2-1}\right)\left(1-\overline{\omega}_2^{-T}\right)\right],\end{aligned}$$

from which we get

$$\lim_{T\to+\infty}\lim_{D\to 0}\frac{\pi_1}{T}=\frac{-\kappa i^*}{\beta\overline{\sigma}\left(\overline{\omega}_2-1\right)},\quad \lim_{T\to+\infty}\lim_{D\to 0}\frac{\widehat{y}_1}{T}=\frac{-\left(1-\beta\overline{\rho}\right)i^*}{\beta\overline{\sigma}\left(\overline{\omega}_2-1\right)},$$
$$\lim_{\theta\to 0}\lim_{D\to 0}\pi_1=-i^*T,\quad \lim_{\theta\to 0}\lim_{D\to 0}\widehat{y}_1=0,$$

which proves Point (i) and half of Points (iii) and (iv) of the proposition. We can also easily determine the limits of π_1 in (57) and $\widehat{y}_1$ in (58) as $D\to 0$:

$$\begin{aligned}\lim_{D\to 0}\pi_1 &= \frac{\widetilde{\kappa}-\overline{\kappa}_g}{\beta}\overline{\widetilde{\omega}}_2^{\,-T}\widetilde{g}^*,\\ \lim_{D\to 0}\widehat{y}_1 &= \frac{-\left[\beta\left(\overline{\widetilde{\omega}}_2+\overline{\widetilde{\rho}}-1\right)-1\right]\left(\widetilde{\kappa}-\overline{\kappa}_g\right)\overline{\widetilde{\omega}}_2^{\,-T}}{\beta\overline{\widetilde{\kappa}}}\widetilde{g}^*,\end{aligned}$$

where $(\widetilde{\kappa},\overline{\kappa}_g,\overline{\widetilde{\rho}},\overline{\widetilde{\omega}}_2)\equiv\lim_{D\to 0}(\widetilde{\kappa}_y,\kappa_g,\widetilde{\rho},\widetilde{\omega}_2)$, from which we get

$$\lim_{T\to+\infty}\lim_{D\to 0}\pi_1=\lim_{T\to+\infty}\lim_{D\to 0}\widehat{y}_1=0$$

and, for $T\geq 2$,

$$\lim_{\theta\to 0}\lim_{D\to 0}\pi_1=\lim_{\theta\to 0}\lim_{D\to 0}\widehat{y}_1=0,$$

which proves Point (ii) and the other half of Points (iii) and (iv) of the proposition.

C.8 Effects of a Permanent Increase in the Money-Growth Rate

When $\widehat{\mu}^*>0$ and $i^{m*}=0$, using (62) for $t=1$, we get

$$\begin{aligned}\frac{\pi_1}{\widehat{\mu}^*} &= 1+\frac{1-\rho}{\sigma\left(\delta_m-\delta_y\psi\right)}-\frac{\left(1-\rho\right)\left(1-\beta\right)\delta_y}{\delta_m\kappa_y-\delta_y\kappa_m}\\ &= 1+\frac{1-\rho}{\sigma\left(\delta_m-\delta_y\psi\right)}-\frac{\left(1-\beta\right)\rho}{1-\beta\rho}\left[1+\left(\frac{\xi}{\rho}\right)\frac{1-\rho}{\sigma\left(\delta_m-\delta_y\psi\right)}\right]\\ &> \left[1-\frac{\left(1-\beta\right)\rho}{1-\beta\rho}\right]\left[1+\frac{1-\rho}{\sigma\left(\delta_m-\delta_y\psi\right)}\right]\\ &> 0,\end{aligned}$$

where the second equality comes from the definition (C.40) of ξ, and the first inequality from (C.42). Using (63) for $t\to+\infty$, we also get

$$\frac{1}{\widehat{\mu}^*}\lim_{t\to+\infty}\widehat{y}_t=\frac{\left(1-\beta\right)\delta_m}{\kappa_y\left(\delta_m-\delta_y\psi\right)}-\frac{\psi}{\sigma\left(\delta_m-\delta_y\psi\right)}.$$

Since $\lim_{\theta\to 0}\kappa_y=+\infty$ and $\lim_{\theta\to 1}\kappa_y=0$, we obtain

$$\lim_{\theta\to 0}\lim_{t\to+\infty}\widehat{y}_t=\frac{-\psi\widehat{\mu}^*}{\sigma\left(\delta_m-\delta_y\psi\right)}<0\quad\text{and}\quad\lim_{\theta\to 1}\lim_{t\to+\infty}\widehat{y}_t=+\infty.$$

Finally, using (63) for $t=1$, we get

$$\begin{aligned}\frac{\widehat{y}_1}{\widehat{\mu}^*} &= \frac{\left(1-\beta\right)\delta_m}{\kappa_y\left(\delta_m-\delta_y\psi\right)}-\frac{\psi}{\sigma\left(\delta_m-\delta_y\psi\right)}+...\\ &\quad\left[\frac{\left(1-\rho\right)\left(1-\beta\rho\right)}{\kappa_y}+\psi\rho\right]\left[\frac{1}{\sigma\left(\delta_m-\delta_y\psi\right)}-\frac{\left(1-\beta\right)\delta_y}{\kappa_y\left(\delta_m-\delta_y\psi\right)}\right].\end{aligned}\tag{C.33}$$

Using Lemma 5, we then obtain

$$\lim_{\theta\to 0}\widehat{y}_1=\lim_{\theta\to 0}\lim_{t\to+\infty}\widehat{y}_t=\frac{-\psi\widehat{\mu}^*}{\sigma\left(\delta_m-\delta_y\psi\right)}<0.$$

To determine the limit of $\widehat{y}_1$ as $\theta\to 1$, we first use the definition (C.40) of ξ to rewrite (C.33) as

$$\begin{aligned}\frac{\widehat{y}_1}{\widehat{\mu}^*} &= \frac{(1-\beta)\,\delta_m}{\kappa_y\left(\delta_m-\delta_y\psi\right)}-\frac{\psi}{\sigma\left(\delta_m-\delta_y\psi\right)}+\left[\frac{(1-\rho)\,\xi}{\sigma\delta_y}+\frac{\delta_m\rho}{\delta_y}\right]\left[\frac{1}{\sigma\left(\delta_m-\delta_y\psi\right)}-\frac{(1-\beta)\,\delta_y}{\kappa_y\left(\delta_m-\delta_y\psi\right)}\right]\\ &= \left(\frac{1-\beta}{\delta_m-\delta_y\psi}\right)\left(\delta_m-\frac{\xi}{\sigma}\right)\left(\frac{1-\rho}{\kappa_y}\right)+\frac{1}{\sigma\left(\delta_m-\delta_y\psi\right)}\left[\frac{(1-\rho)\,\xi}{\sigma\delta_y}+\frac{\delta_m\rho}{\delta_y}-\psi\right].\end{aligned}\tag{C.34}$$

We then determine the limits of ρ, $(1-\rho)/\kappa_y$, and ξ as $\theta\to 1$. Using $\lim_{\theta\to 1}\kappa_y=0$ and $\lim_{\theta\to 1}\kappa_m=0$, we get, after some algebra, for any $X\in\mathbb{R}$,

$$\lim_{\theta\to 1}\mathcal{P}\left(X\right)=\left(X-1\right)\left(X-\beta^{-1}\right)\left(X-1-\delta_y\right).\tag{C.35}$$

The polynomial on the right-hand side of (C.35) has a unique root inside $[-1,1]$, which is 1. Since ρ is the unique root of $\mathcal{P}(X)$ inside $[-1,1]$, we conclude that

$$\lim_{\theta\to 1}\rho=1\quad\text{and}\quad\lim_{\theta\to 1}\left\{\omega_1,\omega_2\right\}=\left\{\beta^{-1},1+\delta_y\right\}.\tag{C.36}$$

Moreover, (C.13) implies that

$$\frac{1-\rho}{\kappa_y}=\frac{\delta_m-\delta_y\psi}{\beta\left(\omega_1-1\right)\left(\omega_2-1\right)},$$

so that, using (C.36), we get

$$\lim_{\theta\to 1}\frac{1-\rho}{\kappa_y}=\frac{\delta_m-\delta_y\psi}{\delta_y\left(1-\beta\right)}.\tag{C.37}$$

Finally, using (C.36) and (C.43), we get

$$\lim_{\theta\to 1}\xi=\frac{\delta_y-\sigma\delta_m}{\delta_y}.\tag{C.38}$$

Using (C.34), (C.36), (C.37), and (C.38), we conclude that

$$\lim_{\theta\to 1}\widehat{y}_1=\frac{\left(1+\delta_y\right)\delta_m\widehat{\mu}^*}{\delta_y^2}>0.$$

C.9 Proof of Proposition 9

Under a permanent peg $i_t^b=i^{b*}$, the system made of the IS equation (64) and the Phillips curve (65) can easily be rewritten as

$$\mathbb{E}_t\left\{\begin{bmatrix}\widehat{y}_{t+1}\\ \pi_{t+1}\end{bmatrix}\right\}=\mathbf{C}\begin{bmatrix}\widehat{y}_t\\ \pi_t\end{bmatrix}+\mathbf{D}i^{b*},$$

where

$$\mathbf{C}\equiv\frac{1}{\varphi}\begin{bmatrix}\kappa\nu_2+\beta\sigma\nu_3 & -\nu_2\\ \kappa\sigma\left(1-\nu_1-\nu_4\right) & \sigma\nu_1\end{bmatrix}\quad\text{and}\quad\mathbf{D}\equiv\frac{1}{\varphi}\begin{bmatrix}\beta\nu_3\\ \kappa\left(1-\nu_4\right)\end{bmatrix}.$$

The characteristic polynomial of $\mathbf{C}$ is

$$\mathcal{C}(X) \equiv X^2 - \frac{\sigma\nu_1 + \kappa\nu_2 + \beta\sigma\nu_3}{\varphi} X + \frac{\sigma}{\varphi}.$$

If the permanent peg ensures local-equilibrium determinacy, then $\mathcal{C}(X)$ must have no root inside the unit circle, because the system has two non-predetermined variables. In particular, $\mathcal{C}(X)$ must have no root inside the real-number interval $[0,1]$, which requires that $\mathcal{C}(0)\,\mathcal{C}(1) > 0$, i.e. equivalently

$$\sigma(1-\nu_1)(1-\beta\nu_3) - \kappa\nu_2\nu_4 > 0. \tag{C.39}$$

In the unique local equilibrium, the (constant) inflation rate is easily obtained as

$$\pi_t = \pi^* \equiv \frac{-\kappa\nu_4 i^{b*}}{\sigma(1-\nu_1)(1-\beta\nu_3) - \kappa\nu_2\nu_4}.$$

Given $\nu_4 > 0$ and (C.39), π^* is negatively related to i^{b*}.

C.10 Proof of Lemma 6

We have

$$\begin{aligned}
\xi &\equiv \frac{\sigma}{\kappa_y}\left[\delta_y(1-\beta\rho) - \frac{(\delta_m\kappa_y - \delta_y\kappa_m)\rho}{1-\rho}\right] && \text{(C.40)}\\
&= \frac{\beta\sigma}{(1-\rho)\kappa_y}\left\{\delta_y\rho^2 - \left[\frac{(1+\beta)\delta_y}{\beta} + \frac{\delta_m\kappa_y - \delta_y\kappa_m}{\beta}\right]\rho + \frac{\delta_y}{\beta}\right\}\\
&= \frac{\beta\sigma}{(1-\rho)\kappa_y}\left\{-\mathcal{P}(\rho) + \rho^3 - \left[\frac{1+2\beta}{\beta} + \frac{1}{\beta}\left(\frac{\kappa_y}{\sigma} - \kappa_m\right)\right]\rho^2 + \ldots\right.\\
&\quad \left.\left[\frac{2+\beta}{\beta} + \frac{1}{\beta}\left(\frac{\kappa_y}{\sigma} - \kappa_m\right)\right]\rho - \frac{1}{\beta}\right\}\\
&= \rho - \frac{\sigma}{\kappa_y}\left[\kappa_m\rho + (1-\beta\rho)(1-\rho)\right] && \text{(C.41)}\\
&< \rho, && \text{(C.42)}
\end{aligned}$$

where the last equality is obtained by using $\mathcal{P}(\rho) = 0$. We also have

$$\begin{aligned}
\xi &\equiv \frac{\sigma}{\kappa_y}\left[\delta_y(1-\beta\rho) - \frac{(\delta_m\kappa_y - \delta_y\kappa_m)\rho}{1-\rho}\right]\\
&= \frac{-\delta_y}{1-\rho}\left\{\rho - \frac{\sigma}{\kappa_y}\left[\kappa_m\rho + (1-\beta\rho)(1-\rho)\right]\right\} + \frac{(\delta_y - \sigma\delta_m)\rho}{1-\rho}\\
&= \frac{-\delta_y}{1-\rho}\xi + \frac{(\delta_y - \sigma\delta_m)\rho}{1-\rho},
\end{aligned}$$

where the last equality is obtained by using (C.41), so that we get

$$\xi = \frac{(\delta_y - \sigma\delta_m)\rho}{1-\rho+\delta_y} > 0, \tag{C.43}$$

where the inequality is obtained by using (C.11). Finally, we have

$$\frac{\kappa_y}{\sigma\eta(\xi)} = \frac{\kappa_y}{\sigma} + \frac{\delta_m\kappa_y - \delta_y\kappa_m}{(1-\rho)\xi} - (1-\beta)\delta_y > \frac{\kappa_y}{\sigma} + \frac{\delta_m\kappa_y - \delta_y\kappa_m}{1-\rho} - (1-\beta)\delta_y = \frac{\kappa_y}{\sigma\eta(1)},$$

where the inequality is obtained by using (C.10), and

$$\begin{aligned}
\frac{\kappa_y}{\sigma\eta(1)} &= \frac{\kappa_y}{\sigma} + \frac{\delta_m\kappa_y - \delta_y\kappa_m}{1-\rho} - (1-\beta)\,\delta_y \\
&= \frac{\kappa_y}{\sigma} - \left[\delta_y(1-\beta\rho) - \frac{(\delta_m\kappa_y - \delta_y\kappa_m)\,\rho}{1-\rho}\right] + \beta(1-\rho)\,\delta_y + (\delta_m\kappa_y - \delta_y\kappa_m) \\
&= \frac{\kappa_y}{\sigma}(1-\xi) + \beta(1-\rho)\,\delta_y + (\delta_m\kappa_y - \delta_y\kappa_m) \\
&> 0,
\end{aligned}$$

where the last equality is obtained by using the definition of ξ, and the inequality by using (C.10). Therefore, we get $\eta(1) > \eta(\xi) > 0$.

Appendix D: Global Effects of Monetary-Policy Shocks

In this appendix, we conduct a global analysis of the effects of monetary-policy shocks in our benchmark model under two alternative price-setting assumptions in turn: flexible prices, and prices set one period in advance. For simplicity, we focus on "MIT shocks," which occur unexpectedly at date 1 and cannot occur afterwards. These shocks may affect either the value at which the interest rate is pegged, or the value at which the growth rate of reserves is pegged, and may be either temporary or permanent.

More specifically, until date 0 included, I_t^m and μ_t are pegged to constant exogenous values I^m and μ in the range defined by (34), and the economy is at the corresponding time-invariant equilibrium. At date 1, unexpectedly, the central bank sets I_1^m to I^{m*} and μ_1 to μ^*, where I^{m*} and μ^* are two exogenous values such that either $I^{m*} > I^m$ and $\mu^* = \mu$, or $I^{m*} = I^m$ and $\mu^* < \mu$. The shock, thus, is either an interest-rate hike, or a money-growth-rate cut. In both cases, it "tightens" monetary policy. The central bank also announces at date 1 whether I_t^m and μ_t will be pegged, from date 2 onwards, to I^m and μ (in which case the shock is temporary) or to I^{m*} and μ^* (in which case the shock is permanent). In the latter case, we impose that $I^{m*}/\mu^* < \beta^{-1}$, so that an equilibrium exists. Following the previous discussion, we focus on the "determinate" equilibrium and ignore the deflationary bubbles. To simplify the notations, we make use of the function $\mathcal{H}$ defined by $\mathcal{H}(x) \equiv \mathcal{F}^{-1}(\beta x - 1)$ for any $x \in [0, \beta^{-1})$.

D.1 Flexible Prices

In the case where the shock is permanent, the economy jumps to the new time-invariant equilibrium at date 1, given the absence of state variable. Given that $\mathcal{H}$ is strictly increasing ($\mathcal{H}' > 0$), we therefore have

$$h_1 = \mathcal{H}\left(\frac{I^{m*}}{\mu^*}\right) > \mathcal{H}\left(\frac{I^m}{\mu}\right) = h_0, \tag{D.1}$$

so that the shock is expansionary. As explained in Subsection 6.3, this expansionary effect comes from the lower opportunity cost of holding reserves, which reduces banking costs. Using

(D.1) and $\mathcal{M}' > 0$, we then get

$$\Pi_1 \equiv \frac{P_1}{P_0} = \frac{\mu^* m_0}{m_1} \leq \frac{\mu m_0}{m_1} = \frac{\mu \mathcal{M}(h_0)}{\mathcal{M}(h_1)} < \mu = \Pi_0, \tag{D.2}$$

so that the shock is disinflationary in the short term. The shock increases the demand for real reserves and hence, given the supply of nominal reserves, reduces the price level below the value that it would have taken in the absence of the shock.

In the alternative case where the shock is temporary, the economy is back to the initial time-invariant equilibrium at date 2, again because of the absence of state variable. So, for an interest-rate hike, the dynamic equation (36) implies

$$[1 + \mathcal{F}(h_1)]\,\mathcal{G}(h_1) = \frac{\beta I^{m*}}{\mu} \mathcal{G}\left[\mathcal{H}\left(\frac{I^m}{\mu}\right)\right] > \frac{\beta I^m}{\mu} \mathcal{G}\left[\mathcal{H}\left(\frac{I^m}{\mu}\right)\right] = [1 + \mathcal{F}(h_0)]\,\mathcal{G}(h_0). \tag{D.3}$$

This inequality, together with $1 + \mathcal{F}(h_0) > 0$, $\mathcal{G} > 0$, $\mathcal{F}' > 0$, and $\mathcal{G}' > 0$, implies in turn that $h_1 > h_0$. The shock is thus expansionary and, given (D.2), also disinflationary, for the same reasons as before. For a money-growth-rate cut, the inequality in (D.3) becomes an equality, so that we get $h_1 = h_0$ and

$$\Pi_1 \equiv \frac{P_1}{P_0} = \frac{\mu^* m_0}{m_1} < \frac{\mu m_0}{m_1} = \frac{\mu \mathcal{M}(h_0)}{\mathcal{M}(h_1)} = \mu = \Pi_0.$$

The shock thus has no effect on employment, as prices fall in period 1 to leave real money balances unchanged. We summarize these findings as follows:

Result 1: *In the benchmark model with flexible prices, an unexpected monetary-policy tightening, whether temporary or permanent, is (non-strictly) expansionary and (strictly) disinflationary in the short term: $h_1 \geq h_0$ and $\Pi_1 < \Pi_0$ following a date-1 shock, with $h_1 = h_0$ only for a temporary money-growth-rate cut.*

The short-term expansionary effect of a (temporary or permanent) monetary-policy "tightening" is, of course, an unusual result. In the next appendix (Appendix D.2), we show that this result is overturned under prices set one period in advance.

D.2 Prices Set One Period in Advance

When prices are set one period in advance, firm i chooses its price $P_t(i)$ at date $t-1$ to maximize

$$\mathbb{E}_{t-1}\left\{P_t(i) y_t(i) - \frac{\beta \lambda_{t+1} I_t^\ell L_t(i)}{\lambda_t \Pi_{t+1}}\right\}$$

subject to the production function (14), the demand schedule (15), and the borrowing constraint (16). The first-order condition of this optimization problem implies

$$P_t(i) = \frac{\varepsilon}{\varepsilon - 1} \frac{\mathbb{E}_{t-1}\left\{\frac{\beta \lambda_{t+1} I_t^\ell W_t y_t(i)}{\lambda_t \Pi_{t+1} f'[h_t(i)]}\right\}}{\mathbb{E}_{t-1}\{y_t(i)\}}.$$

Using the Euler equation (9) and the law of iterated expectations, this pricing equation can be rewritten as

$$P_t(i) = \frac{\varepsilon}{\varepsilon - 1} \frac{\mathbb{E}_{t-1}\left\{\frac{I_t^\ell W_t y_t(i)}{I_t^b f'[h_t(i)]}\right\}}{\mathbb{E}_{t-1}\left\{y_t\left(i\right)\right\}}.$$

In a symmetric equilibrium, all firms set the same price:

$$P_t = \frac{\varepsilon}{\varepsilon - 1} \frac{\mathbb{E}_{t-1}\left\{\frac{I_t^\ell W_t y_t}{I_t^b f'(h_t)}\right\}}{\mathbb{E}_{t-1}\left\{y_t\right\}}.$$

Using households' first-order conditions (9) and (12) at date 1, together with (8), (14), (18), (19), and (20), we get

$$1 + \frac{\Gamma_m\left[\mathcal{L}\left(h_1\right), m_1\right]}{u'\left[f\left(h_1\right)\right]} = \beta I^{m*} \frac{u'\left[f\left(h_2\right)\right]}{u'\left[f\left(h_1\right)\right] \Pi_2}. \tag{D.4}$$

Since prices are set one period in advance, it is now from date 2 onwards that the economy is at the time-invariant equilibrium corresponding to I^{m*} and μ^* (if the shock is permanent) or I^m and μ (if the shock is temporary) – focusing again on the "determinate" equilibrium, i.e. ignoring the deflationary bubbles. Therefore, we have $m_1 = (\mu^*/\mu)m_0$ and $m_2 = \mathcal{M}(h_2)$.

In the case where the shock is permanent, we have

$$\Pi_2 \equiv \frac{P_2}{P_1} = \frac{\mu^* m_1}{m_2} = \frac{\mu^* m_1}{\mathcal{M}\left(h_2\right)},$$

so that we can rewrite (D.4) as

$$\mathcal{K}\left(h_1, m_1\right) \equiv m_1 \left\{u'\left[f\left(h_1\right)\right] + \Gamma_m\left[\mathcal{L}\left(h_1\right), m_1\right]\right\} = \frac{\beta I^{m*}}{\mu^*} \mathcal{G}\left(h_2\right), \tag{D.5}$$

where the function $\mathcal{K}$ is strictly decreasing in its first argument ($\mathcal{K}_h < 0$), strictly increasing in its second argument ($\mathcal{K}_m > 0$), and such that

$$\forall m_t \in \mathbb{R}_{>0}, \ \lim_{h_t \to 0} \mathcal{K}\left(h_t, m_t\right) = +\infty. \tag{D.6}$$

Using (D.5), $\mathcal{K}_m > 0$, $\mathcal{G}' > 0$, and

$$h_2 = \mathcal{H}\left(\frac{I^{m*}}{\mu^*}\right) > \mathcal{H}\left(\frac{I^m}{\mu}\right) = h_0, \tag{D.7}$$

we then get

$$\mathcal{K}\left(h_1, m_0\right) = \mathcal{K}\left(h_1, \frac{\mu}{\mu^*} m_1\right) \geq \mathcal{K}\left(h_1, m_1\right) > \frac{\beta I^m}{\mu} \mathcal{G}\left(h_2\right) > \frac{\beta I^m}{\mu} \mathcal{G}\left(h_0\right) = \mathcal{K}\left(h_0, m_0\right).$$

From this inequality, $\mathcal{K}_h < 0$, and (D.6), we conclude that h_1 exists, is unique, and is such that $h_1 < h_0$. Finally, using (D.7) and $\mathcal{M}' > 0$, we get

$$\Pi_2 \equiv \frac{P_2}{P_1} = \frac{\mu^* m_1}{m_2} = \frac{\left(\mu^*\right)^2 m_0}{\mu m_2} = \frac{\left(\mu^*\right)^2 \mathcal{M}\left(h_0\right)}{\mu \mathcal{M}\left(h_2\right)} < \frac{\left(\mu^*\right)^2}{\mu} \leq \mu = \Pi_0 = \Pi_1.$$

In the alternative case where the shock is temporary, we have

$$\Pi_2 \equiv \frac{P_2}{P_1} = \frac{\mu m_1}{m_2} = \frac{\mu m_1}{\mathcal{M}(h_2)},$$

so that we can rewrite (D.4) as

$$\mathcal{K}(h_1, m_1) = \frac{\beta I^{m*}}{\mu} \mathcal{G}(h_2). \tag{D.8}$$

Using this equality, $\mathcal{K}_m > 0$, and

$$h_2 = \mathcal{H}\left(\frac{I^m}{\mu}\right) = h_0, \tag{D.9}$$

we then get

$$\mathcal{K}(h_1, m_0) = \mathcal{K}\left(h_1, \frac{\mu}{\mu^*} m_1\right) \geq \mathcal{K}(h_1, m_1) \geq \frac{\beta I^m}{\mu} \mathcal{G}(h_2) = \frac{\beta I^m}{\mu} \mathcal{G}(h_0) = \mathcal{K}(h_0, m_0),$$

where one of these two inequalities is strict, so that

$$\mathcal{K}(h_1, m_0) > \mathcal{K}(h_0, m_0).$$

From this inequality, $\mathcal{K}_h < 0$, and (D.6), we conclude that h_1 exists, is unique, and is such that $h_1 < h_0$. Finally, using (D.9), we get

$$\Pi_2 \equiv \frac{P_2}{P_1} = \frac{\mu m_1}{m_2} = \frac{\mu^* m_0}{m_2} = \frac{\mu^* \mathcal{M}(h_0)}{\mathcal{M}(h_2)} = \mu^* \leq \mu = \Pi_0 = \Pi_1,$$

where the inequality is an equality for an interest-rate hike and a strict inequality for a money-growth-rate cut.

We summarize these findings as follows:

Result 2: *In the benchmark model with prices set one period in advance, an unexpected monetary-policy tightening, whether temporary or permanent, is (strictly) contractionary in the short term and (non-strictly) disinflationary in the medium term:* $h_1 < h_0$ *and* $\Pi_2 \leq \Pi_0 = \Pi_1$ *following a date-*1 *shock, with* $\Pi_2 = \Pi_0 = \Pi_1$ *only for a temporary interest-rate hike.*

Thus, unlike the flexible-price results, the synchronized-sticky-price results have a familiar Keynesian flavor: unexpected monetary-policy tightening, whether temporary or permanent, is always contractionary in the short term. The reason is that it reduces the opportunity cost of holding reserves, and hence increases the demand for real reserves for any output level; given the existing nominal money stock and the short-term price rigidity, the output level must then fall to clear the money market.

Appendix E: Model With Cash

E.1 Proof of Lemma 7

After some simple algebra, we can rewrite $\mathcal{F}(h_t)$ and $\widetilde{\mathcal{G}}(h_t)$ as

$$\begin{aligned}\mathcal{F}(h_t) &= -Z_1 (h_t)^{-z_1} \left[Z_2 (h_t)^{-z_2} - 1\right]^{z_3},\\ \widetilde{\mathcal{G}}(h_t) &= Z_3 (h_t)^{z_4} \left[Z_2 (h_t)^{-z_2} - 1\right]^{-z_5} + Z_4 (h_t)^{-z_6},\end{aligned}$$

where

$$\begin{aligned}Z_1 &\equiv \alpha_b^{\frac{\alpha_b}{\gamma_b(1+\chi_b)}} \gamma_b \delta^{1-\frac{1}{\gamma_b}+\frac{\alpha_b}{\gamma_b(1+\chi_b)}} \delta_b^{\frac{-\alpha_b}{\gamma_b(1+\chi_b)}} A^{-\sigma\frac{1-\gamma_b}{\gamma_b}} A_b^{\frac{1}{\gamma_b}} > 0,\\ Z_2 &\equiv \alpha\delta^{-1}(\varepsilon - 1)\varepsilon^{-1} A^{1-\sigma} > 0,\\ Z_3 &\equiv \alpha_b^{\frac{-\alpha_b}{\gamma_b(1+\chi_b)}} \delta^{\frac{1}{\gamma_b}-\frac{\alpha_b}{\gamma_b(1+\chi_b)}} \delta_b^{\frac{\alpha_b}{\gamma_b(1+\chi_b)}} A^{\sigma\frac{1-\gamma_b}{\gamma_b}} A_b^{\frac{-1}{\gamma_b}} > 0,\\ Z_4 &\equiv \phi A^{1-\sigma} > 0,\\ z_1 &\equiv \alpha\sigma\frac{1-\gamma_b}{\gamma_b} + (1+\chi)\frac{(1-\gamma_b)(1+\chi_b) - \alpha_b}{\gamma_b(1+\chi_b)} > 0,\\ z_2 &\equiv 1 + \chi + \alpha(\sigma - 1) > 0,\\ z_3 &\equiv 1 + \frac{\alpha_b}{\gamma_b(1+\chi_b)} > 0,\\ z_4 &\equiv \alpha\sigma\frac{1-\gamma_b}{\gamma_b} + (1+\chi)\frac{1+\chi_b-\alpha_b}{\gamma_b(1+\chi_b)} > 0,\\ z_5 &\equiv \frac{\alpha_b}{\gamma_b(1+\chi_b)} > 0,\\ z_6 &\equiv \alpha(\sigma - 1) > 0.\end{aligned}$$

We can also write $\widetilde{\mathcal{G}}'(h_t)$ as

$$\begin{aligned}\widetilde{\mathcal{G}}'(h_t) &= Z_3 (h_t)^{z_4-1}\left\{(z_4 + z_2 z_5)\left[Z_2 (h_t)^{-z_2} - 1\right]^{-z_5} + z_2 z_5 \left[Z_2 (h_t)^{-z_2} - 1\right]^{-z_5-1}\right\} - ...\\ &\quad Z_4 z_6 (h_t)^{-z_6-1}.\end{aligned}$$

Since

$$z_4 - 1 > z_4 - (1+\chi) = z_1 > 0,$$

$\widetilde{\mathcal{G}}'(h_t)$ is strictly increasing in h_t, that is to say that $\widetilde{\mathcal{G}}$ is convex. In turn, the convexity of $\widetilde{\mathcal{G}}$ and its limit properties

$$\lim_{h_t\to 0} \widetilde{\mathcal{G}}(h_t) = \lim_{h_t\to \bar{h}} \widetilde{\mathcal{G}}(h_t) = +\infty$$

imply that $\widetilde{\mathcal{G}}$ is U-shaped.

The unique steady-state value of h_t is defined by (35), that is to say by

$$(h)^{z_1}\left[Z_2 (h)^{-z_2} - 1\right]^{-z_3} = Z_1\left(1 - \frac{\beta I^m}{\mu}\right)^{-1}.$$

Using this relationship and

$$\begin{aligned} z_1 &= z_4 - (1+\chi), \\ z_2 &= z_6 + (1+\chi), \\ z_3 &= z_5 + 1, \end{aligned}$$

we can rewrite $\widetilde{\mathcal{G}}'(h)$ as

$$\begin{aligned} \widetilde{\mathcal{G}}'(h) &= (h)^{\chi}\left\{ Z_1 Z_3 z_4 \left(1-\frac{\beta I^m}{\mu}\right)^{-1}\left[Z_2(h)^{-z_2}-1\right]+... \right. \\ & \left. \left[Z_1 Z_2 Z_3 z_2 z_5 \left(1-\frac{\beta I^m}{\mu}\right)^{-1} - Z_4 z_6\right](h)^{-z_2}\right\}. \end{aligned} \tag{E.1}$$

So a sufficient condition on the parameters for $\widetilde{\mathcal{G}}'(h) > 0$ is

$$Z_1 Z_2 Z_3 z_2 z_5 \left(1-\frac{\beta I^m}{\mu}\right)^{-1} > Z_4 z_6,$$

that is to say equivalently

$$(1+\chi)+(\sigma-1)\left[\alpha-\left(1-\frac{\beta I^m}{\mu}\right)\frac{1+\chi_b}{\alpha_b}\frac{\varepsilon}{\varepsilon-1}\phi\right] > 0.$$

E.2 Matrices A and B

$$\mathbf{A} \equiv \begin{bmatrix} \frac{1+\beta}{\beta}-\frac{\kappa_m}{\beta} & \frac{-1}{\beta} & \frac{\kappa_y}{\beta} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{\beta\sigma}-\delta_m-\frac{\kappa_m}{\beta\sigma} & \frac{-1}{\beta\sigma} & 1+\delta_y+\frac{\kappa_y}{\beta\sigma} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \alpha_c \begin{bmatrix} \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{-1}{\beta\sigma} \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{-1}{\beta\sigma} \\ 0 & 0 & 0 & 0 \end{bmatrix} + ...$$

$$\frac{\alpha_c}{1-\alpha_c}\begin{bmatrix} \delta_m+\frac{(1-\sigma)\kappa_m}{\beta\sigma} & 0 & \frac{-(1-\sigma)}{\beta\sigma}-\delta_y-\frac{(1-\sigma)\kappa_y}{\beta\sigma} & \frac{1-\sigma}{\beta\sigma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{B} \equiv \begin{bmatrix} 0 & 1 & \frac{-1}{\beta} \\ 0 & 0 & 0 \\ \frac{1}{\sigma} & 0 & \frac{-1}{\beta\sigma} \\ 0 & 0 & 0 \end{bmatrix} + \frac{\alpha_c}{1-\alpha_c}\begin{bmatrix} \frac{-1}{\sigma} & 1 & \frac{1-\sigma}{\beta\sigma} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

E.3 Proof of Lemma 8

After some simple algebra, we get

$$\det(\mathbf{A}-X\mathbf{I}_4) = X\left[\mathcal{P}(X)+\frac{\alpha_c}{1-\alpha_c}\mathcal{R}(X)\right],$$

where $\mathbf{I}_4$ denotes the 4×4 identity matrix and

$$\mathcal{R}(X) \equiv -\left[\delta_m+\frac{(1-\sigma)\kappa_m}{\beta\sigma}\right]X^2+\left[\frac{(1+\beta)\delta_m}{\beta}+\frac{(1-\sigma)\kappa_m}{\beta\sigma}+\frac{\delta_m\kappa_y-\delta_y\kappa_m}{\beta}\right]X-\frac{\delta_m}{\beta}.$$

Therefore, the eigenvalues of $\mathbf{A}$ are 0 and the roots of

$$\mathcal{S}(X) \equiv \mathcal{P}(X) + \frac{\alpha_c}{1-\alpha_c}\mathcal{R}(X).$$

We have

$$\mathcal{R}(0) = \frac{-\delta_m}{\beta} < 0, \tag{E.2}$$

$$\mathcal{R}(1) = \frac{\delta_m \kappa_y - \delta_y \kappa_m}{\beta} = \frac{K_3}{\beta} > 0, \tag{E.3}$$

so that, using (C.12) and (C.13), we get

$$\begin{aligned} \mathcal{S}(0) &= \frac{-(1+K_1)}{\beta} - \frac{\alpha_c \delta_m}{(1-\alpha_c)\beta} < -1, \\ \mathcal{S}(1) &= \frac{K_3}{(1-\alpha_c)\beta} > 0. \end{aligned}$$

Therefore, $\mathcal{S}(X)$ has either one root or three roots in the real-number interval $[0,1]$. Now, the product of the three roots of $\mathcal{S}(X)$ is equal to $-\mathcal{S}(0) > 1$, so that $\mathcal{S}(X)$ has at least one root outside the unit circle. As a consequence, $\mathcal{S}(X)$ has exactly one root inside the real-number interval $[0,1]$.

The other roots of $\mathcal{S}(X)$ are either two real numbers outside $[0,1]$, or two conjugate complex numbers. In the latter case, both are outside the unit circle, since $\mathcal{S}(X)$ has at least one root outside it. Therefore, $\mathcal{S}(X)$ has exactly two roots outside the unit circle if and only if it has no root inside the real-number interval $[-1,0)$. Since $\mathcal{S}(0) < 0$, the latter condition is equivalent to $\mathcal{S}(X) < 0$ for all $X \in [-1,0]$.

If $\sigma \leq 1$, then the coefficient of X^2 in $\mathcal{R}(X)$ is negative. Given (E.2) and (E.3), we then have $\mathcal{R}(X) < 0$ for all $X \in [-1,0]$. Since $\mathcal{P}(X) < 0$ for all $X \in [-1,0]$, we get $\mathcal{S}(X) < 0$ for all $X \in [-1,0]$. Therefore, $\mathcal{S}(X)$ has exactly two roots outside the unit circle.

Alternatively, if $\sigma > 1$, then rewrite $\mathcal{P}(X)$ and $\mathcal{R}(X)$ as

$$\begin{aligned} \mathcal{P}(X) &= \frac{\kappa_y}{\beta}\left[\mathcal{P}_1(X) + \mathcal{P}_2(X)\right], \\ \mathcal{R}(X) &= \frac{\kappa_y}{\beta}\left[\mathcal{R}_1(X) + \mathcal{R}_2(X)\right], \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}_1(X) &\equiv \frac{-1}{\kappa_y}(1-X)(1-\beta X)(1+\delta_y - X), \\ \mathcal{P}_2(X) &\equiv \left(\frac{1}{\sigma} - \psi\right) X(1-X) + (\delta_m - \delta_y \psi) X, \\ \mathcal{R}_1(X) &\equiv \frac{-\delta_m}{\kappa_y}(1-X)(1-\beta X), \\ \mathcal{R}_2(X) &\equiv \frac{-(\sigma-1)\psi}{\sigma} X(1-X) + (\delta_m - \delta_y \psi) X. \end{aligned}$$

Whatever $X \in [-1, 0]$, $\mathcal{P}_1(X)$ and $\mathcal{R}_1(X)$ are strictly decreasing functions of θ. Therefore, $\mathcal{S}(X) < 0$ for all $X \in [-1, 0]$ and all $\theta \in (0, 1)$ if and only if $\mathcal{S}(X) < 0$ for all $X \in [-1, 0]$ as $\theta \to 0$. Since $\mathcal{P}_1(X)$ and $\mathcal{R}_1(X)$ converge towards zero as θ goes to zero whatever $X \in [-1, 0]$, the latter condition is equivalent to $\mathcal{S}_2(X) < 0$ for all $X \in [-1, 0)$, where

$$\mathcal{S}_2(X) \equiv \mathcal{P}_2(X) + \frac{\alpha_c}{1-\alpha_c}\mathcal{R}_2(X).$$

Now, we have

$$\mathcal{S}_2(X) = X\left[ZX + \left(\frac{\delta_m - \delta_y\psi}{1-\alpha_c} - Z\right)\right],$$

where

$$Z \equiv -\left(\frac{1}{\sigma} - \psi\right) + \frac{\alpha_c}{1-\alpha_c}\frac{(\sigma-1)\psi}{\sigma},$$

so that $\mathcal{S}_2(X) < 0$ for all $X \in [-1, 0)$ if and only if

$$Z < \frac{\delta_m - \delta_y\psi}{2(1-\alpha_c)}. \tag{E.4}$$

Now, using $I^b = \beta^{-1}$, we get

$$\alpha_m = \frac{-(1-\beta I^m)}{\beta I^m}.$$

Moreover, in the context of our parametric example, after some simple algebra, we get, using (11),

$$\alpha_\ell = \frac{Z_2(h)^{-z_2} - 1}{Z_2(h)^{-z_2}},$$

and, using (35),

$$\frac{\alpha_c}{1-\alpha_c} = \frac{(1-\beta I^m) Z_4(h)^{-z_2}}{Z_1 Z_3\left[Z_2(h)^{-z_2} - 1\right]}.$$

Using these expressions for $\alpha_c/(1-\alpha_c)$, α_ℓ, and α_m, after again some simple algebra, we can then rewrite (E.4) as $\widetilde{\mathcal{G}}'(h) > 0$, where $\widetilde{\mathcal{G}}'(h)$ is given by (E.1) with $\mu = 1$. Therefore, Appendix E.1 implies that a sufficient condition for (E.4) to be met is (71) with $\mu = 1$. As a consequence, (71) with $\mu = 1$ is also a sufficient condition for $\mathcal{S}(X)$ to have exactly two roots outside the unit circle.

Appendix F: Model With a Satiation Level

F.1 Global Analysis Under Flexible Prices

The introduction of a finite satiation level of reserves brings only three changes to Subsection 3.1's analysis. First, Equation (30) is replaced by

$$\lim_{h_t \to \overline{h}} \mathcal{M}(h_t) = \underline{m}\left[\mathcal{L}\left(\overline{h}\right)\right],$$

so that we have

$$h_t < \overline{h} \quad \text{and} \quad m_t = \mathcal{M}(h_t) < \underline{m}\left[\mathcal{L}\left(\overline{h}\right)\right]$$

when the economy is outside the satiation range at date t. Second, the economy can now also be inside the satiation range at date t, in which case we have

$$h_t = \overline{h} \quad \text{and} \quad m_t \geq \underline{m}\left[\mathcal{L}\left(\overline{h}\right)\right].$$

Third, the dynamic equation (31) is replaced by

$$1 + \frac{\Gamma_m\left[\mathcal{L}\left(h_t\right), m_t\right]}{u'\left[f\left(h_t\right)\right]} = \beta I_t^m \mathbb{E}_t\left\{\frac{u'\left[f\left(h_{t+1}\right)\right] m_{t+1}}{\mu_{t+1} u'\left[f\left(h_t\right)\right] m_t}\right\}. \tag{F.1}$$

For convenience, we extend the domain of definition of $\mathcal{M}$ from $\left(0, \overline{h}\right)$ to $\left(0, \overline{h}\right]$, and define $\mathcal{M}(\overline{h}) \equiv \underline{m}[\mathcal{L}(\overline{h})]$.

Under the permanent pegs $I_t^m = I^m$ and $\mu_t = \mu$, the set of time-invariant equilibria is still characterized by the static equation (32), where the function $\mathcal{F}$ still has all the properties stated in Lemma 2. The only novelty is that $\mathcal{F}$ is now also defined at point $\overline{h}$, with $\mathcal{F}(\overline{h}) = 0$. Therefore, we get the following proposition, which replaces Proposition 1:

Proposition 12 (Time-Invariant Equilibria in the Model With a Satiation Level Under Flexible Prices): *In the model with a satiation level and flexible prices, under the permanent pegs $I_t^m = I^m$ and $\mu_t = \mu$,*

(i) when $I^m/\mu > \beta^{-1}$, there is no time-invariant equilibrium;

(ii) when $I^m/\mu = \beta^{-1}$, there is an infinity of time-invariant equilibria; in each of these equilibria, the employment level is equal to $\overline{h}$; these equilibria differ from each other only in terms of the constant value of real reserves and the initial price level;

(iii) when $0 \leq I^m/\mu < \beta^{-1}$, there is a unique time-invariant equilibrium; in this equilibrium, the employment level is lower than $\overline{h}$ and strictly increasing in I^m/μ.

To study time-varying perfect-foresight equilibria under the permanent pegs $I_t^m = I^m$ and $\mu_t = \mu$, we rewrite the dynamic equation (F.1) as

$$1 + \frac{\Gamma_m\left[\mathcal{L}\left(h_t\right), m_t\right]}{u'\left[f\left(h_t\right)\right]} = \frac{\beta I^m}{\mu} \mathbb{E}_t\left\{\frac{u'\left[f\left(h_{t+1}\right)\right] m_{t+1}}{u'\left[f\left(h_t\right)\right] m_t}\right\}, \tag{F.2}$$

and we define the function $\mathcal{G}$ over $\left(0, \overline{h}\right]$ by $\mathcal{G}(h_t) \equiv u'[f(h_t)]\mathcal{M}(h_t)$. The function $\mathcal{G}$ still has all the properties stated in Lemma 3, except the property (39).

We first consider the case in which $I^m/\mu > \beta^{-1}$. If the economy starts outside the satiation range, then it stays outside forever. Indeed, if it were outside at some date t and inside at date $t+1$, then (F.2) would imply

$$\frac{u'\left[f\left(\overline{h}\right)\right] m_{t+1}}{u'\left[f\left(h_t\right)\right] \mathcal{M}\left(h_t\right)} < \frac{\mu}{\beta I^m} < 1,$$

which would contradict the fact that

$$u'\left[f\left(\overline{h}\right)\right]m_{t+1}\geq u'\left[f\left(\overline{h}\right)\right]\underline{m}\left[\mathcal{L}\left(\overline{h}\right)\right]=\mathcal{G}\left(\overline{h}\right)>\mathcal{G}\left(h_t\right)=u'\left[f\left(h_t\right)\right]\mathcal{M}\left(h_t\right). \tag{F.3}$$

Now, if the economy stays outside the satiation range forever, then (F.2) implies

$$\frac{\mathcal{G}\left(h_{t+1}\right)}{\mathcal{G}\left(h_t\right)}<\frac{\mu}{\beta I^m}<1,$$

which implies in turn, through the same reasoning as in Appendix B.3, that the ratio $\mathcal{G}(h_{t+1})/\mathcal{G}(h_t)$ turns negative for t sufficiently large, which is impossible. Thus, there is no time-varying perfect-foresight equilibrium starting outside the satiation range.

Alternatively, if the economy starts inside the satiation range, then, as long as it stays inside, (F.2) boils down to $m_{t+1}/m_t=\mu/(\beta I^m)<1$, so that m_t decreases over time at a constant rate. Therefore, the economy leaves the satiation range at some finite date. Once it leaves this range, it stays outside forever, for the same reason as previously. Then, as previously, we get, through the same reasoning as in Appendix B.3, that the ratio $\mathcal{G}(h_{t+1})/\mathcal{G}(h_t)$ turns negative for t sufficiently large, which is impossible. Thus, there is no time-varying perfect-foresight equilibrium starting inside the satiation range. We conclude that there is no time-varying perfect-foresight equilibrium at all in the case $I^m/\mu>\beta^{-1}$.

We then turn to the case in which $I^m/\mu=\beta^{-1}$. If the economy starts outside the satiation range, then it stays outside forever. Indeed, if it were outside at some date t and inside at date $t+1$, then (F.2) would imply

$$\frac{u'\left[f\left(\overline{h}\right)\right]m_{t+1}}{u'\left[f\left(h_t\right)\right]\mathcal{M}\left(h_t\right)}<\frac{\mu}{\beta I^m}=1,$$

which would contradict (F.3). Now, if the economy stays outside the satiation range forever, then (F.2) implies

$$\frac{\mathcal{G}\left(h_{t+1}\right)}{\mathcal{G}\left(h_t\right)}<\frac{\mu}{\beta I^m}=1,$$

which implies in turn, through the same reasoning as in Appendix B.3, that the ratio $\mathcal{G}(h_{t+1})/\mathcal{G}(h_t)$ turns negative for t sufficiently large, which is impossible. Thus, there is no time-varying perfect-foresight equilibrium starting outside the satiation range.

Alternatively, if the economy starts inside the satiation range, then it stays inside forever. Indeed, if it were inside at some date t and outside at date $t+1$, then (F.2) would imply

$$\frac{u'\left[f\left(h_{t+1}\right)\right]\mathcal{M}\left(h_{t+1}\right)}{u'\left[f\left(\overline{h}\right)\right]m_t}=\frac{\mu}{\beta I^m}=1,$$

which would contradict the fact that

$$u'\left[f\left(h_{t+1}\right)\right]\mathcal{M}\left(h_{t+1}\right)=\mathcal{G}\left(h_{t+1}\right)<\mathcal{G}\left(\overline{h}\right)=u'\left[f\left(\overline{h}\right)\right]\underline{m}\left[\mathcal{L}\left(\overline{h}\right)\right]\leq u'\left[f\left(\overline{h}\right)\right]m_t. \tag{F.4}$$

Now, if the economy stays inside the satiation range forever, then (F.2) boils down to $m_{t+1}=m_t$, so that the equilibrium is not time-varying. Thus, there is no time-varying perfect-foresight equilibrium starting inside the satiation range. We conclude that there is no time-varying perfect-foresight equilibrium at all in the case $I^m/\mu = \beta^{-1}$.

Finally, we consider the case in which $0 \leq I^m/\mu < \beta^{-1}$. If the economy starts inside the satiation range, then it stays inside forever. Indeed, if it were inside at some date t and outside at date $t+1$, then (F.2) would imply

$$\frac{u'\left[f\left(h_{t+1}\right)\right]\mathcal{M}\left(h_{t+1}\right)}{u'\left[f\left(\overline{h}\right)\right]m_t} = \frac{\mu}{\beta I^m} > 1,$$

which would contradict (F.4). Now, if the economy stays inside the satiation range forever, then (F.2) boils down to $m_{t+1}/m_t = \mu/(\beta I^m) > 1$, so that m_t increases over time at a constant rate. As in Appendix B.3, when the central bank injects money by acquiring bonds issued (or previously held) by the private sector, these paths are always equilibrium paths; when money injections are done by helicopter drops, they are equilibrium paths only if $I^m/\mu > 1$.

Alternatively, if the economy starts outside the satiation range from some $h_0 \in (0, h)$, where h denotes the value of h_t at the unique time-invariant equilibrium, then it stays outside forever. Indeed, if it were outside until some date t and inside at date $t+1$, then the sequence $(h_k)_{k\in\mathbb{N}}$ would be decreasing until date t, as shown in Appendix B.3, so that we would have $h_t < h$; therefore, (F.2) would imply

$$\frac{u'\left[f\left(\overline{h}\right)\right]m_{t+1}}{u'\left[f\left(h_t\right)\right]\mathcal{M}\left(h_t\right)} = 1 + \frac{\mu}{\beta I^m}\left[\mathcal{F}\left(h_t\right) - \mathcal{F}\left(h\right)\right] < 1,$$

which would contradict (F.3). Now, if the economy stays outside the satiation range forever, then we get, through the same reasoning as in Appendix B.3, that the ratio $\mathcal{G}(h_{t+1})/\ \mathcal{G}(h_t)$ turns negative for t sufficiently large, which is impossible. Thus, there is no time-varying perfect-foresight equilibrium starting outside the satiation range from some $h_0 \in (0, h)$.

Alternatively, if the economy starts outside the satiation range from some $h_0 \in (h, \overline{h})$, then it enters the satiation range at some finite date. Indeed, if it stayed forever outside this range, then we would get, through the same reasoning as in Appendix B.3, that the sequence $[\mathcal{G}(h_t)]_{t\in\mathbb{N}}$ goes to infinity, which is impossible. Once the economy enters the satiation range, it stays inside forever, for the same reason as previously, and the analysis is then exactly the same as previously.

These results can be summarized by the following proposition, which is the counterpart of Proposition 2 in the model with a satiation level:

Proposition 13 (Time-Varying Perfect-Foresight Equilibria in the Model With a Satiation Level Under Flexible Prices): *In the model with a satiation level and flexible prices, under the permanent pegs $I_t^m = I^m$ and $\mu_t = \mu$,*

(i) when $I^m/\mu \geq \beta^{-1}$, there is no time-varying perfect-foresight equilibrium;

(ii) when $1 < I^m/\mu < \beta^{-1}$, there is an infinity of time-varying perfect-foresight equilibria; these equilibria are indexed by $h_0 \in \left(h, \overline{h}\right]$ and involve a non-decreasing sequence $\{h_t\}_{t\in\mathbb{N}}$ that is constantly equal to $\overline{h}$ from some finite date onwards;

(iii) when $0 \leq I^m/\mu \leq 1$ and under helicopter drops, there is no time-varying perfect-foresight equilibrium;

(iv) when $0 \leq I^m/\mu \leq 1$ and under open-market operations, there is an infinity of time-varying perfect-foresight equilibria; these equilibria are of the same type as those in (ii).

F.2 Local Analysis Under Sticky Prices

We assume that I_t^m can vary exogenously around a given value $I^m \in (0, \beta^{-1}]$, and μ_t around the value $\mu = 1$. Whether prices are flexible or sticky à la Calvo (1983) does not matter for existence and uniqueness of a steady-state equilibrium when $\mu = 1$. Therefore, Proposition 12 still holds when "flexible prices" is replaced by "sticky prices and constant nominal reserves."

Thus, if $I^m < \beta^{-1}$, then the model has a unique steady state; this steady state lies outside the satiation range and has zero inflation. Naturally, log-linearizing the model around this steady state gives exactly the same reduced form as the reduced form of our benchmark model (without a satiation level).

Alternatively, if $I^m = \beta^{-1}$, then the model has an infinity of steady states; all these steady states lie inside the satiation range and have zero inflation. Log-linearizing the model around any of these steady states leads to the reduced form made of the Phillips curve (40) with $\kappa_m = 0$ (because $\Gamma_{\ell m} = 0$), the IS equation (41), the spread equation (42) with $\delta_y = \delta_m = 0$ (because $\Gamma_m = 0$), and the money-market-clearing condition (43). This reduced form is isomorphic to the reduced form of the basic NK model. If $\Gamma_\ell > 0$, then the slope κ_y of our model's Phillips curve (defined in Appendix C.1) is larger than the slope κ of the basic NK model's Phillips curve (defined in Appendix C.6). If $\Gamma_\ell = 0$, then the two slopes are equal to each other, and the reduced form of our model becomes exactly identical to the reduced form of the basic NK model.

References

[1] Adão, B., Correia, I., and Teles, P. (2003), "Gaps and Triangles," *Review of Economic Studies*, 70(4), 699-713.

[2] Angeletos, G.-M., and Lian, C. (2016), "Forward Guidance Without Common Knowledge," NBER Working Paper No. 22785.

[3] Bassetto, M. (2005), "Equilibrium and Government Commitment," *Journal of Economic Theory*, 124(1), 79-105.

[4] Blanchard, O.J., and Kahn, C.M. (1980), "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48(5), 1305-1311.

[5] Calvo, G.A. (1983), "Staggered Prices in a Utility-Maximising Framework," *Journal of Monetary Economics*, 12(3), 383-398.

[6] Canzoneri, M.B., and Diba, B.T. (2005), "Interest-Rate Rules and Price Determinacy: The Role of Transactions Services of Bonds," *Journal of Monetary Economics*, 52(2), 329-343.

[7] Carlstrom, C.T., Fuerst, T.S., and Paustian, M. (2015), "Inflation and Output in New Keynesian Models With a Transient Interest-Rate Peg," *Journal of Monetary Economics*, 76, 230-243.

[8] Christiano, L., Eichenbaum, M., and Rebelo, S. (2011), "When Is the Government Spending Multiplier Large?," *Journal of Political Economy*, 119(1), 78-121.

[9] Cochrane, J.H. (2011), "Determinacy and Identification with Taylor Rules," *Journal of Political Economy*, 119(3), 565-615.

[10] Cochrane, J.H. (2014), "Monetary Policy With Interest on Reserves," *Journal of Economic Dynamics and Control*, 49, 74-108.

[11] Cochrane, J.H. (2016a), "The New-Keynesian Liquidity Trap," mimeo.

[12] Cochrane, J.H. (2016b), "Comments on 'A Behavioral New Keynesian Model' by Xavier Gabaix," mimeo.

[13] Cochrane, J.H. (2016c), "Michelson-Morley, Occam and Fisher: The Radical Implications of Stable Inflation at Near-Zero Interest Rates," mimeo.

[14] Cochrane, J.H. (2016d), "Stepping on a Rake: Replication and Diagnosis," NBER Working Paper No. 22979.

[15] Cúrdia, V., and Woodford, M. (2011), "The Central-Bank Balance Sheet as an Instrument of Monetary Policy," *Journal of Monetary Economics*, 58(1), 54-79.

[16] Del Negro, M., Giannoni, M.P., and Patterson, C. (2015), "The Forward-Guidance Puzzle," mimeo.

[17] Eggertsson, G.B. (2010), "The Paradox of Toil," Federal Reserve Bank of New York Staff Report No. 433.

[18] Eggertsson, G.B. (2011), "What Fiscal Policy is Effective at Zero Interest Rates?," in Acemoglu, D., and Woodford, M. (eds) *NBER Macroeconomics Annual 2010*, University of Chicago Press, 25, 59-112.

[19] Eggertsson, G.B. (2012), "Was the New Deal Contractionary?", *American Economic Review*, 102(1), 524-555.

[20] Eggertsson, G.B., and Krugman, P. (2012), "Debt, Deleveraging, and the Liquidity Trap: A Fisher-Minsky-Koo Approach," *Quarterly Journal of Economics*, 127(3), 1469-1513.

[21] Eggertsson, G.B., Ferrero, A., and Raffo, A. (2014), "Can Structural Reforms Help Europe?," *Journal of Monetary Economics*, 61, 2-22.

[22] Ennis, H.M., and Wolman, A.L. (2015), "Large Excess Reserves in the United States: A View from the Cross-Section of Banks," *International Journal of Central Banking*, 11(1), 251-289.

[23] Farhi, E., and Werning, I. (2016), "Fiscal Multipliers: Liquidity Traps and Currency Unions," forthcoming in Taylor, J.B., and Uhlig, H. (eds) *Handbook of Macroeconomics*, Amsterdam: Elsevier Science, 2.

[24] Friedman, B.M., and Kuttner, K.N. (1998), "Indicator Properties of the Paper-Bill Spread: Lessons From Recent Experience," *Review of Economics and Statistics*, 80(1), 34-44.

[25] Gabaix, X. (2016), "A Behavioral New Keynesian Model," mimeo.

[26] Galí, J. (2015), *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and its Applications*, Princeton: Princeton University Press, second edition.

[27] García-Schmidt, M., and Woodford, M. (2015), "Are Low Interest Rates Deflationary? A Paradox of Perfect-Foresight Analysis," mimeo.

[28] Greene, C., Schuh, S., and Stavins, J. (2016), "The 2014 Survey of Consumer Payment Choice: Summary Results," Federal Reserve Bank of Boston Research Data Report No. 16-3.

[29] Greenwood, R., and Vayanos, D. (2014), "Bond Supply and Excess Bond Returns," *Review of Financial Studies*, 27(3), 663-713.

[30] Ireland, P.N. (2014), "The Macroeconomic Effects of Interest on Reserves," *Macroeconomic Dynamics*, 18(6), 1271-1312.

[31] Kiley, M.T. (2016), "Policy Paradoxes in the New Keynesian Model," *Review of Economic Dynamics*, 21, 1-15.

[32] Kingston, G.H. (1982), "The Semi-Log Portfolio Balance Schedule Is Tenuous," *Journal of Monetary Economics*, 9(3), 389-399.

[33] Kocherlakota, N. (2016), "Note on Neo-Fisherianism," mimeo.

[34] Krishnamurthy, A., and Vissing-Jorgensen, A. (2012), "The Aggregate Demand for Treasury Debt," *Journal of Political Economy*, 120(2), 233-267.

[35] Loisel, O. (2016), "The Implementation of Stabilization Policy," mimeo.

[36] Mankiw, G.N., and Reis, R. (2002), "Sticky Information Versus Sticky Prices: A Proposal to Replace the New Keynesian Phillips Curve," *Quarterly Journal of Economics*, 117(4), 1295-1328.

[37] Matheny, W., O'Brien, S., and Wang, C. (2016), "The State of Cash: Preliminary Findings from the 2015 Diary of Consumer Payment Choice," FedNote, November 3.

[38] McKay, A., Nakamura, E., and Steinsson, J. (2016), "The Power of Forward Guidance Revisited," *American Economic Review*, 106(10), 3133-3158.

[39] Obstfeld, M., and Rogoff, K. (1983), "Speculative Hyperinflations in Maximizing Models: Can We Rule Them Out?," *Journal of Political Economy*, 91(4), 675-687.

[40] Osborne, M., and Sim, M. (2015), "Are Reserves Still Special?," *Bank Underground*, September 28, https://bankunderground.co.uk/2015/09/28/are-reserves-still-special/.

[41] Sargent, T.J., and Wallace, N. (1975), "Rational Expectations, the Optimal Monetary Intrument and the Optimal Money Supply Rule," *Journal of Political Economy*, 83(2), 241-254.

[42] Sargent, T.J., and Wallace, N. (1985), "Interest on Reserves," *Journal of Monetary Economics*, 15(3), 279-290.

[43] Schmitt-Grohé, S., and Uribe, M. (2014), "Liquidity Traps: An Interest-Rate-Based Exit Strategy," *Manchester School*, 82(S1), 1-14.

[44] Schmitt-Grohé, S., and Uribe, M. (2016), "Liquidity Traps and Jobless Recoveries," forthcoming in *American Economic Journal: Macroeconomics*.

[45] Werning, I. (2012), "Managing a Liquidity Trap: Monetary and Fiscal Policy," mimeo.

[46] Wiederholt, M., "Empirical Properties of Inflation Expectations and the Zero Lower Bound," mimeo.

[47] Wieland, J.F. (2016), "Are Negative Supply Shocks Expansionary at the Zero Lower Bound?," mimeo.

[48] Williamson, S.D. (2015), "Interest on Reserves, Interbank Lending, and Monetary Policy," Federal Reserve Bank of St. Louis Working Paper No. 2015-024.

[49] Woodford, M. (2003), *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton: Princeton University Press.

[50] Woodford, M. (2011), "Simple Analytics of the Government Expenditure Multiplier," *American Economic Journal: Macroeconomics*, 3(1), 1-35.