Centre de Recherche en Économie et Statistique
CREST

Série des Documents de Travail

n° 2014-25

Misspecification of Causal and Noncausal Orders in Autoregressive Processes

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Misspecification of Causal and Noncausal orders in Autoregressive Processes

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September 19, 2014

Abstract

This paper examines the consequences of estimating a past-dependent (causal) AR model from data generated by a stationary noncausal process with a future-dependent component. We show that the outcomes of that estimation depend on the noncausal persistence. When the noncausal persistence is strong, the (pseudo)-ML estimator of the misspecified causal model is consistent, and reveals the presence of a noncausal component as long as the sample is sufficiently non-Normal. When the noncausal persistence is weak, the (pseudo)-ML estimator is inconsistent and leads to a misleading conclusion that the fit of the past-dependent model is correct.
The results are derived theoretically from examining the binding function, and illustrated by simulations of noncausal AR processes with errors that follow mixture distributions with varying proportions of Normal and Cauchy variables.

Keywords: Noncausal Process, Misspecification, Binding Function, Indirect Inference, Speculative Bubble, Stochastic Trend.

JEL number: C 14, G32, G23.

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1 Introduction

Standard practice of estimating causal autoregressive processes may lead to misleading outcomes if the true data generating process (y_t) is stationary and such that:

$$A_0(L)y_t = e_t^*, \tag{1.1}$$

where $A_0(L) = 1 - a_{0,1}L - \cdots - a_{0,p}L^p$, $a_{0,p} \neq 0$, is a lag polynomial of degree p and errors (e_t^*) are i.i.d. variables such that $E(|e_t^*|^s) < \infty$ for a power $s > 0$. This autoregressive model differs from the standard autoregressive model used in the Box, Jenkins methodology in three aspects: 1) the error terms are i.i.d. rather than a weak white noise, 2) the errors do not necessarily have finite second-order moments: for $s \geq 2$, the second-order moment exists, for $s \in [1, 2)$, the errors have infinite variance , but finite first-order moment, for $s \in (0, 1)$ the errors have no first-order moment, 3) the roots of the autoregressive polynomial $A_0(L)$ are not necessarily outside the unit circle.

If the roots of $A_0(L)$ are not on the unit circle, equation (1.1) has a unique strictly stationary solution. This stationary solution is easily derived by distinguishing the roots of A_0 that lie outside and inside the unit circle. Indeed, model (1.1) can also be written as:

$$\Phi_0(L)\Psi_0(L^{-1})y_t = e_t, \tag{1.2}$$

where e_t differ from e_t^* by a scale factor and a lag effect $\forall t$, and $\Phi_0(L)$ (resp. $\Psi_0(L)$) is a lag polynomial of order r (resp. s), with all roots outside the unit circle, where r (resp. s) denotes the number of roots of $A_0(L)$ outside (resp. inside) the unit circle [see Appendix 1 for the expression of $A_0(L)$]. Then, the stationary two-sided moving average representation of y_t in terms of (e_t) is easily written by inverting the causal and noncausal polynomials $\Phi_0(L)$ and $\Psi_0(L^{-1})$, respectively. The stationary solution of (1.1) or (1.2) is known in the literature as a Mixed causal-noncausal Autoregressive process [1], and is called MAR(r,s) henceforth.

Recent literature has revealed the importance of noncausal processes in various economic and financial applications. For example, some structural macro-models lead to mixed processes [see the discussion in Hansen, Sargent (1991), Lippi, Reichlin (1993),

[1] or AR(r,s) in Lanne, Saikkonen (2011).

(1994), Alessi, Barigozzi, Capasso (2011), Gourieroux, Monfort (2014)]. Noncausal models with fat tails can replicate speculative bubbles that characterize the dynamics of commodity prices and exchange rates of electronic currency [see e.g. Gourieroux, Zakoian (2014), Gourieroux, Hencic (2013)]. Moreover, the mixed models have been shown to have better predictive power than the standard ARMA models [see e.g. Lanne, Nyberg, Saarinen (2011), Lanne, Saikkonen (2011), Davis, Song (2012), Chen, Choi, Escanciano (2012), Lanne, Saikkonen (2013)].

The objective of this paper is to study the consequences of misspecification of causal and noncausal orders when the parameters of a noncausal model are estimated from a causal autoregression, which is common practice among econometricians. More precisely, if orders r, s are known and the true error distribution is parametric with scale parameter $e_t \sim 1/\sigma_0 f(e_t/\sigma_0)$, it has been proven that the estimators obtained by maximizing:

$$Max \sum_{t=1}^{T} \left\{ -\log \sigma + \log f \left(\frac{\Phi(L)\Psi(L^{-1})y_t}{\sigma} \right) \right\} \tag{1.3}$$

with respect to σ, Φ, Ψ are consistent of σ_0, Φ_0, Ψ_0 [see, Breidt et al. (1991), Lanne, Saikkonen (2011)]. In contrast, the estimators obtained by maximizing a pseudo-log likelihood:

$$Max \sum_{t=1}^{T} \left\{ -\log \sigma + \log g \left(\frac{A(L)y_t}{\sigma} \right) \right\}, \tag{1.4}$$

with pseudo-density g of the error either equal to or different from f, are expected to be inconsistent.

The pseudo-likelihood (1.4) is based on the causal pseudo-model:

$$A(L)y_t = \epsilon_t, \tag{1.5}$$

where the (ϵ_t) are i.i.d. variables: $\epsilon_t \sim 1/\sigma\, g(\epsilon_t/\sigma)$, such that ϵ_t is independent of $y_{t-1}, y_{t-2}, \ldots.$. The misspecification in causal model (1.5) can be interpreted in three alternative ways:

i) First, the stationary solution of model (1.1) is a two-sided moving average in e_t^*, in general. Therefore, model (1.2) does not satisfy the assumption of independence between the errors and past observations that is incorrectly assumed in pseudo-model (1.5). This implies an endogeneity bias, which explains the inconsistency of the pseudo-maximum likelihood (PML) estimator.

ii) Second, when $A(L)$ has roots inside the unit circle, the solutions of causal model (1.5) are nonstationary.

ii) Third, it is always possible to represent a causal stationary Markov series of order p as a nonlinear autoregression [2] [see e.g. Rosenblatt (2000)]:

$$y_t = H(y_{t-1}, ..., y_{t-p}, \epsilon_t^*), \tag{1.6}$$

where (ϵ_t^*) is a sequence of i.i.d. standard normal variables such that ϵ_t^* is independent of $y_{t-1}, ..., y_{t-p}$. Moreover, this representation is unique if function H is increasing in ϵ_t^*. Pseudo-model (1.5) implicitly assumes a linear function H.

In brief, misspecification in model (1.5) can be seen either as a misspecification concerning the stationarity of the process and/or a misspecification concerning the linearity of the autoregression. It has important consequences for the estimation of the lag polynomial $A(L)$, and the autoregressive coefficients $a_1, ..., a_p$. When the causal misspecified model is Gaussian, i.e. the density g in (1.4) is Gaussian, the PML estimator is inconsistent (see Section 2.1). However, when the misspecified causal model is non-Gaussian, some econometricians may believe that the pseudo-lag polynomial (that is the limit of the PML estimator of $A(L)$, when T tends to infinity) provides valid information on the location of the roots of $A_0(L)$, and reveals the numbers of roots that lie inside and outside the unit circle.

This paper clarifies these issues. We show that for Cauchy mixed processes, the misspecified Cauchy causal AR models yield PML estimators of the lag polynomial that tend to the true $A_0(L)$ for a non-negligeable set of parameter values. For other parameter values, the pseudo-lag polynomial can detect the causal dynamic component of the process, but is unable to reveal the noncausal component of the process. This result is also valid for noncausal processes with mixture distributions of Gaussian and Cauchy.

The paper is organized as follows. Section 2 considers the true noncausal Cauchy AR(1), estimated as a misspecified causal autoregressive process of order 1 and 2, with either Gaussian or Cauchy errors. In Section 3, the analysis is extended to mixed autoregressive process of order 2 that are a MAR(1,1) and MAR(0,2) processes. The misspecifed

[2] Any mixed AR process of order p is a Markov process of order p in calendar time. For pure noncausal processes this is proven in Cambanis, Fakhre-Zakeri (1994) and Gourieroux, Zakoian (2014a). The general result follows easily from results established for pure noncausal processes.

pseudo-model considered in each case is a causal Cauchy AR(2) model. In Sections 2 and 3, we examine the consequences of the misspecification by deriving the binding function in closed form, to show how the pseudo-lag polynomial depends on the true parameter values. In Section 4, we illustrate by simulations the behavior of PMLE in AR(1) models with binding functions that cannot be derived in closed form. Three estimation methods are examined, which are: a Gaussian PMLE, equivalent to the OLS estimator in the causal model, a Cauchy PMLE and a PMLE based on a mixture of Gaussian and Cauchy variables. The behavior of the PMLE is shown to depend on the noncausal persistence, the proportion of Gaussian variables in the mixture and the sample size. Section 5 concludes. The proofs are given in Appendices 1 to 4, and the simulation results are provided in Appendix 5.

2 Misspecified causal analysis of noncausal Cauchy AR(1) process

Let us assume that the true process is a noncausal Cauchy AR(1):

$$y_t = \rho_0 y_{t+1} + \sigma_0 e_t,\ \rho_0 \neq 0,\ |\rho_0| < 1, \tag{2.1}$$

where errors (e_t) are assumed to be i.i.d. standard Cauchy and ρ_0, σ_0 are the true values of the forward autoregressive and scale parameters, respectively. The true autoregressive lag polynomial is $A_0(L) = 1 - (1/\rho_0)L$ (see Appendix 1).

The noncausal stationary processes have been little explored by the practitioners so far, as they generally consider the causal representation:

$$y_t = \rho y_{t-1} + \sigma \epsilon_t, \tag{2.2}$$

where errors (ϵ_t) are i.i.d. and independent of $y_{t-1}, y_{t-2}, \ldots$. Standard practice consists in estimating parameters ρ, σ of model (2.2) by the (pseudo) maximum likelihood, without imposing any restrictions on parameter ρ. This estimation method yields estimates, which for large T, tend to limits $\tilde{\rho} = b_1(\rho_0, \sigma_0)$, $\tilde{\sigma} = b_2(\rho_0, \sigma_0)$, called the pseudo-true values of parameters ρ, σ in model (2.2). Function b that associates the true values to the pseudo-true values is called the binding function [3]. Below, we provide examples of binding

[3] See e.g. Gourieroux, Monfort, Trognon (1984).

functions in closed-form, derived as the limits of PML estimators based on misspecified model (2.2).

2.1 True model: noncausal Cauchy AR(1) model, pseudo-model: causal Gaussian AR(1) model

In this example, the distribution of ϵ_t in (2.2) is assumed standard normal N(0,1). Hence, the PML estimator of ρ is equal to the OLS estimator. Note that the true process (2.1) is such that y_t has a marginal Cauchy distribution up to a scale factor [see e.g. Gourieroux, Zakoian (2014a)]. In particular, variable y_t has no first and second-order moments and ρ cannot be interpreted as an autocorrelation. Nevertheless, the OLS estimator of ρ tends to a limit when T tends to infinity [see Davis, Resnick (1986)].

Proposition 2.1: When the true model is a noncausal Cauchy AR(1) process and the pseudo-model is a causal Gaussian AR(1), we get $b_1(\rho_0, \sigma_0) = \rho_0$.

Then, the pseudo-true value of ρ is equal to the true value ρ_0. Therefore, the pseudo-lag polynomial $A(L) = 1 - \rho_0 L$ is always different from the true lag-polynomial $A_0(L) = 1 - 1/\rho_0 L$ and the PML estimator of $A(L)$ is inconsistent for any admissible true value of the parameter.

2.2 True model: noncausal Cauchy AR(1) model, pseudo-model: causal Cauchy AR(1) model

In this example, both e_t and ϵ_t are assumed i.i.d. standard Cauchy. The proposition below is proven in Appendix 2.

Proposition 2.2: a) When the true model is a noncausal Cauchy AR(1) model and the misspecified model is a causal Cauchy AR(1) model, we have:

$$
\begin{aligned}
b_1(\rho_0, \sigma_0) &= 0, \text{if } |\rho_0| < 0.5, \\
b_1(\rho_0, \sigma_0) &= 1/\rho_0, \text{if } |\rho_0| > 0.5.
\end{aligned}
$$

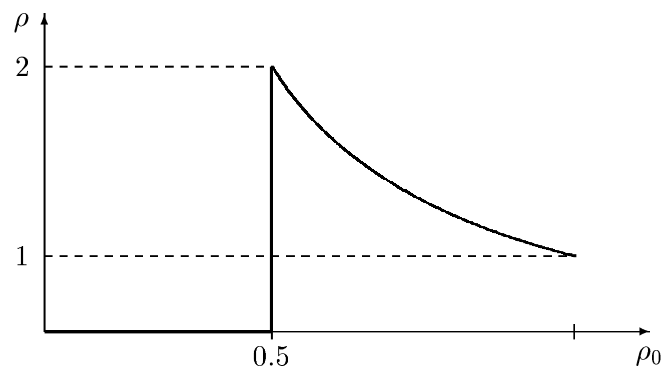
If $\rho_0 = 0.5$, there is an interval of pseudo-true values $b_1(\rho_0, \sigma_0) = (0, 2)$. The pseudo-true value of σ is:

$$
\begin{aligned}
b_2(\rho_0, \sigma_0) &= \sigma_0/(1 - |\rho_0|), \text{if } |\rho_0| \leq 0.5, \\
b_2(\rho_0, \sigma_0) &= \sigma_0/|\rho_0|, \text{if } |\rho_0| \geq 0.5.
\end{aligned}
$$

b) If the PML estimation is performed under a constraint such as $\sigma^2 = 1$, the binding function $b(\rho_0)$ for ρ is equal to $b_1(\rho_0, \sigma_0)$ given above.

The form of the binding function for ρ is given in Figure 1:

Figure 1: Binding function for ρ ($\rho_0 > 0$)



This is an example of a non-invertible binding function. It shows that, when $\rho_0 < 0.5$, the causal Cauchy AR(1) is not sufficiently informative to reveal the true value of ρ_0.

One may wish to rewrite equation (2.1) as:

$$y_{t+1} = \frac{1}{\rho_0} y_t - \frac{\sigma_0}{\rho_0} e_t \iff y_t = \frac{1}{\rho_0} y_{t-1} + \tilde{\sigma}_0 \epsilon_t$$

and hope to find $b_1(\rho_0, \sigma_0) = 1/\rho_0$. This reasoning is not valid because $(\frac{\sigma_0}{\rho_0} e_t)$ is i.i.d., but not independent of y_t, which can create an endogeneity bias. Surprisingly, for the Cauchy process and $\rho_0 \geq 0.5$, there is no endogeneity bias and we get

$$b_1(\rho_0, \sigma_0) = 1/\rho_0.$$

The reason could be the occurrence of bubbles in the path of a noncausal Cauchy process that are local explosions followed by a crash, whereas a causal AR(1) process with a root greater than 1 represents a stochastic trend that is a global explosion. When $0 < \rho_0 \leq 0.5$, the rate of growth of the bubble is high and equal to $1/\rho_0 \geq 2$, which makes the crash noticeable, so that the mispecified model distinguishes that effect from a stochastic trend. In contrast, when $\rho_0 \geq 0.5$, the rate of growth of the bubble is small and equal to $1 \leq 1/\rho_0 \leq 2$, which makes the crash less noticeable. Therefore, the misspecified model is unable to distinguish this effect from a stochastic trend.

Let us now consider the binding function, when the misspecified model includes a scale parameter. The bivariate binding function:

$$(\rho_0, \sigma_0) \to [b_1(\rho_0, \sigma_0), b_2(\rho_0, \sigma_0)].$$

is still not invertible. More precisely:
if $\rho_0 \geq 0.5$, we identify both ρ_0 and σ_0
if $\rho_0 \leq 0.5$, we identify $\sigma_0/(1 - |\rho_0|)$ only.

If $\rho_0 \geq 0.5$, the PML estimator of $A(L)$ is consistent, but the PML estimator of σ_0 tends to the marginal scale of the true Cauchy process instead of the conditional scale.

2.3 True model: noncausal Cauchy AR(1) model, pseudo-model: causal Cauchy AR(2) model

Let us now examine how the results derived above change when the autoregressive order of the misspecified model is increased to 2, as follows:

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + c\epsilon_t, \tag{2.3}$$

where errors (ϵ_t) are i.i.d. standard Cauchy with ϵ_t assumed to be independent of past observations. The following proposition is proven in Appendix 2, iii).

Proposition 2.3: If the true model is a noncausal Cauchy AR(1) model and the pseudo-model is a causal Cauchy AR(2) model, the binding function for a_1, a_2 is:
$b_1(\rho_0, \sigma_0) = 0,\ b_2(\rho_0, \sigma_0) = 0$, if $|\rho_0| < 0.5$,
$b_1(\rho_0, \sigma_0) = 1/\rho_0,\ b_2(\rho_0, \sigma_0) = 0$, if $|\rho_0| > 0.5$.

These above binding functions do not depend on the value of the true scale parameter and the outcomes are again determined by comparing ρ_0 with 0.5.

We observe that the pseudo-lag polynomial is the same as the pseudo-lag polynomial of the causal AR(1) pseudo-model for $|\rho_0| > 0.5$ (see Proposition 2.3a). Hence, an increase of the autoregressive order of the misspecified model from 1 to 2 does not help detect the noncausal component.

3 Misspecified Mixed AR(2) Cauchy process

Let us now extend the analysis to the true noncausal Cauchy autoregressive process of order 2. Such process can be mixed and have one causal and one noncausal root, and are called the MAR(1,1) model. Alternatively a Cauchy autoregressive process of order 2 can be pure noncausal with two noncausal roots, called the MAR(0,2). In this Section, we consider both MAR(1,1) and MAR(2,0) processes and assume that the misspecified model is a causal Cauchy AR(2) process in each case.

3.1 Misspecified Cauchy MAR(1,1) model

The true process is:

$$(1-\phi_0 L)(1-\psi_0 L^{-1})y_t = e_t, \tag{3.1}$$

where $|\phi_0| < 1, |\psi_0| < 1$ and the (e_t) are i.i.d. standard Cauchy.
The causal pseudo-model is:

$$(1-a_1 L - a_2 L^2)y_t = c\epsilon_t, \tag{3.2}$$

where errors (ϵ_t) are independent standard Cauchy distributed and assumed to be independent of the past values $y_{t-1}, y_{t-2}...$. Parameter c is a scale parameter and is positive. The following proposition is proven in Appendix 3:

Proposition 3.1: If the true model is a MAR(1,1) Cauchy model and the pseudo-model is a causal Cauchy AR(2) model, the binding function for a_1, a_2 is:
$b_1(\phi_0, \psi_0) = \phi_0,\ \ b_2(\phi_0, \psi_0) = 0$, if $|\psi_0| < 0.5$,
$b_1(\phi_0, \psi_0) = \phi_0 + 1/\psi_0,\ \ b_2(\phi_0, \psi_0) = -\phi_0/\psi_0$, if $|\psi_0| > 0.5$.
The associated pseudo lag polynomial is:
$A(L) = 1 - \phi_0 L$, if $|\psi_0| < 0.5$,
$A(L) = 1 - (\phi_0 + \frac{1}{\psi_0})L + \frac{\phi_0}{\psi_0}L^2 = A_0(L)$, if $|\psi_0| > 0.5$. (see Appendix 1)

The results depend on whether the noncausal autoregressive parameter is less or greater than 0.5 in absolute value. The causal component of $A_0(L)$ is always found, but the noncausal component is detected only for $|\psi_0| > 0.5$. Proposition 3.1 implies Proposition 2.3 by setting $\phi_0 = 0$.

3.2 Misspecified Cauchy MAR(0,2) model

In this example, the true process is a pure noncausal process MAR(0,2):

$$(1-\psi_{0,1}L^{-1})(1-\psi_{0,2}L^{-1})y_t = e_t, \tag{3.3}$$

where $1 > \psi_{0,1} > \psi_{0,2} > -1$ and the errors are i.i.d. Cauchy variables. The optimization of the asymptotic pseudo-log likelihood based on a pure causal AR(2) Cauchy process:

$$(1-a_1L-a_2L^2)y_t = \epsilon_t, \tag{3.4}$$

with i.i.d. Cauchy errors (ϵ_t) that are independent of lagged values y_t, is equivalent to the minimization of the following objective function (see Appendix 4):

$$\begin{aligned} d(a_1,a_2) &= (\psi_{0,1}-\psi_{0,2})|a_2| + (\psi_{0,1}-\psi_{0,2})|a_1+a_2(\psi_{0,1}+\psi_{0,2})| \\ &\quad + \sum_{j=0}^{\infty} |\psi_{0,1}^{j+1}A(\psi_{0,1}) - \psi_{0,2}^{j+1}A(\psi_{0,2})|, \end{aligned} \tag{3.5}$$

which has to be optimized numerically. However, some extreme points of the simplex in this linear programming are easily characterized. At the intersection of the two lines representing the first two terms, we get $a_1 = a_2 = 0$. At the intersection of the two lines corresponding to two terms in the infinite sum, we get $A(\psi_{0,1}) = A(\psi_{0,2}) = 0$, that is $A(L) = A_0(L)$ and the PML estimator is consistent. At the intersection of the first term with the j^{th} term in the infinite sum, we get : $a_1 = (\psi_{0,1}^{j+1} - \psi_{0,2}^{j+1})/(\psi_{0,1}^{j+2} - \psi_{0,2}^{j+2})$, $a_2 = 0$.

At the intersection of the second term with the j^{th} term of the infinite sum, we get : $a_1 = \frac{1}{\psi_{0,1}} + \frac{1}{\psi_{0,2}}$, $a_2 = -1/(\psi_{0,1}\psi_{0,2})$. This solution leads to $A(\psi_{0,1}) = A(\psi_{0,2})$ and the PML estimator is still consistent of A_0.

Proposition 3.2: When the true process is a pure noncausal Cauchy process MAR(0,2) and the pseudo-model is a pure causal AR(2) Cauchy process, the admissible lag-polynomials are:

i) $\tilde{A}(L) = 0$,

ii) $\tilde{A}_j(L) = 1 - \left(\frac{\psi_{0,1}^{j+1}-\psi_{0,2}^{j+1}}{\psi_{0,1}^{j+2}-\psi_{0,2}^{j+2}}\right) L \;\;,\;\; j = 0,1,2,...$

iii) the true lag polynomial $A_0(L)$.

Hence, from lag-polynomial ii), we may obtain, a pseudo-autoregressive model of order 1, with autoregressive coefficient:

$$\tilde{a}_{1,j} = \frac{\psi_{0,1}^{j+1} - \psi_{0,2}^{j+1}}{\psi_{0,1}^{j+2} - \psi_{0,2}^{j+2}}.$$

We know from Section 2.3 that if $\psi_{0,2} = 0$, then $\tilde{a}_{1,j} = 1 / \psi_{0,1}$ is independent of j and larger than 1 in absolute value. Let us discuss in more detail the value of $\tilde{a}_{1,j}$ when $\psi_{0,2} \neq 0$. For example, for $j = 0$, we get:

$$\tilde{a}_{1,0} = \frac{\psi_{0,1} - \psi_{0,2}}{\psi_{0,1}^2 - \psi_{0,2}^2} = \frac{1}{\psi_{0,1} + \psi_{0,2}}.$$

The absolute value $|\psi_{0,1}+\psi_{0,2}|$ determines if $|\tilde{a}_{1,0}|$ is greater or less than 1. If $|\psi_{0,1}+\psi_{0,2}| < 1$, the pseudo-model leads to inconsistent estimation of the autoregressive order, but the noncausal autoregressive order is consistently estimated. If $|\psi_{0,1} + \psi_{0,2}| > 1$, neither the autoregressive order, nor the noncausal autoregressive order are consistently estimated.

To display the autoregressive orders implied by various subsets of values $1 > \psi_{0,1} > \psi_{0,2} > -1$, for $\psi_{0,1} \neq 0, \psi_{0,2} \neq 0$ [4], we compute numerically the solutions that minimize $d(a_1, a_2)$ and the associated noncausal pseudo-autoregressive orders, which can be either 0, 1 or 2. The results are displayed in Figure 2 below.

[Insert Figure 2: Noncausal Pseudo-Autoregressive Order]

We observe a symmetric pattern with respect to the line $\psi_{0,1}+\psi_{0,2} = 0$. The estimated noncausal autoregressive order is zero when the process is close to a white noise over a small region of parameter values. The estimated noncausal autoregressive order is 1 when both coefficients are large and of the same sign. It is equal to 2, i.e. to the true noncausal autoregressive order, when both ψ coefficients are large, but of different orders.

4 Misspecification in noncausal AR(1): From Cauchy to Gaussian errors

In this section, we perform simulation experiments to illustrate the behavior of various PMLE based on misspecified causal models with error distributions that are different from Cauchy. The reason is that one may argue that the results derived in Sections 2

[4] to avoid autoregressive orders strictly less than 2 in the true model

and 3 are specific to the assumption of Cauchy error distribution in the true and/or the misspecified autoregressive model and explained by the form of the asymptotic pseudo-maximum likelihood, whose optimization is equivalent to the optimization of a piecewise linear function, with corner solutions. The fat tails of the Cauchy distribution may lead us to missinterpreting the bubbles in the trajectory as a stochastic trend. To clarify this issue, we complete the analysis by bridging the Cauchy and Gaussian autoregressive processes in a simulation study below.

4.1 The true and pseudo-models

The experiment consists in estimating noncausal AR(1) processes from causal AR(1) models with errors that follow various mixtures of distributions.

i) The true process

The true process is a noncausal AR(1) process:

$$y_t = \rho_0 y_{t+1} + e_t, \quad \rho_0 \neq 0, |\rho_0| < 1, \tag{4.1}$$

where the errors are i.i.d. and follow a mixture of Gaussian and Cauchy distributions. More precisely, we get:

$$e_t = z_t \sigma_0 v_t + (1 - z_t)\gamma_0 w_t, \tag{4.2}$$

where z_t is a Bernoulli variable $\mathcal{B}\ (1, \lambda_0)$, v_t is a standard Normal variable and w_t is a standard Cauchy variable. Parameters $\sigma_0, \gamma_0, \sigma_0 > 0, \gamma_0 > 0$ are the scale parameters. The density of the error is:

$$f(e_t) = \lambda_0 \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left(-\frac{e_t^2}{2\sigma_0^2}\right) + (1 - \lambda_0)\frac{\gamma_0}{\pi}\frac{1}{\gamma_0^2 + e_t^2}. \tag{4.3}$$

This mixture of Gaussian and Cauchy distributions is convenient for the analysis, as variable e_t is easy to simulate via (4.2) and the density has a closed form.

ii) The pseudo-models

We consider three different pseudo-models:

$$y_t = \rho y_{t-1} + \epsilon_t, . \tag{4.4}$$

that are the misspecified causal models. They differ with respect to the distribution of the error term, which can be:

1) the same mixture of Cauchy and Gaussian distributions as in the true model;

2) a Gaussian distribution;

3) a Cauchy distribution.

Accordingly, we examine the "mix" causal pseudo-maximum likelihood estimator, the causal OLS estimator and the causal Cauchy pseudo-maximum likelihood estimator.

In the experiment, only parameters λ_0 and ρ_0 are allowed to vary. By changing the weight in the mixture, we bridge the two pure models i.e. a Cauchy AR(1) model ($\lambda_0 = 0$) and a Gaussian AR(1) model ($\lambda_0 = 1$). These extreme cases have already been discussed. The noncausal Cauchy AR(1) has been examined theoretically in Section 2. When the errors are Gaussian, we know from the literature that the process has also a linear causal AR(1) representation and we cannot identify whether the model is causal or noncausal.

4.2 Estimation results

Parameters σ, γ and λ are assumed to be known. We examine the behavior of the autoregressive coefficient estimator $\hat{\rho}(\lambda_0, \rho_0)$ as a function of λ_0, and the persistence parameter ρ_0 (See Appendix 5).

We use 1000 replications of noncausal AR(1) trajectories of length $T = 400$ and $T = 100$, which are referred to as the large and the small sample, respectively. The true parameter ρ_0 takes the values 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 and 0.9. Parameter λ_0 is set equal to the following values: 0.0, 0.01, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.85, 0.9, 0.95 and 1.

Tables 1.0 to 1.9 display the behavior of the PML estimators of $\hat{\rho}(\lambda_0, \rho_0)$, called the Mix PMLE and Cauchy PMLE and given in columns 1 and 3, respectively, and compare them to the OLS-based first-order autocorrelation estimator, denoted by OLS and given in column 2.

For each estimator, we report the median and the 5th and 95th percentiles of its sampling distribution. We do not provide the sampling mean and sample variance as their theoretical counterparts do not necessarily exist.

We observe that the outcomes depend on the noncausal persistence, the degree of normality of the sample and its size.

As expected, the OLS estimates are close to parameter value ρ_0, for any ρ_0, λ_0. Therefore, we observe the convergence to ρ_0 even if the observations have infinite variance. The OLS estimator leads to a misleading conclusion that the model is causal while it is not.

The results for the Mix and Cauchy PMLE are close, and are commented below. For ease of interpretation, let us recall from Section 3 that for the true noncausal Cauchy model the causal PMLE can converge to 0, if $\rho_0 < 0.5$, to a value from the interval $[0, 2]$, if $\rho_0 = 0.5$ and to $1/\rho_0$, if $\rho_0 > 0.5$. The empirical results are in line with these findings, even for data from a mixture of Gaussian and Cauchy distributions.

a) weak persistence $\rho_0 < 0.5$.

Both the Mix and Cauchy PMLE indicate the absence of causal persistence provided that the proportion of Normal variables in the sample is not too high. More precisely, we observe that $\hat{\rho}$ converges to zero up to the second decimal for $\rho_0 = 0.1$, if the mixture contains less than 80% of Normal variables, for $\rho_0 = 0.2$ if there is less than 70% of Normal variables in the mixture, for $\rho_0 = 0.3$ if there is less than 60 % of Normal variables in the mixture, for $\rho_0 = 0.4$ if there is less than 50 % of Normals in the mixture. These results concern the sample of $T = 400$. In the smaller sample, these proportions can be considerably lower and depend on ρ_0. For example, a zero value of $\hat{\rho}$ up to the second decimal is found for $\rho_0 = 0.4$, only if there is less than 30 % of Normal variables in the mixture.

b) strong persistence $\rho_0 > 0.5$

When ρ_0 lies between 0.6 and 0.9, the Mix and Cauchy PMLE can converge to $1/\rho_0$ when the mixture is sufficiently different from Normal. The values of the reciprocal $1/\rho_0$ are: 1.6 for $\rho_0 = 0.6$, 1.4 for $\rho_0 = 0.7$, 1.2 for $\rho_0 = 0.8$ and 1.1 for $\rho_0 = 0.9$. The PMLE can detect not only the existence of noncausal persistence, but also its magnitude. This effect is observed when the proportions of Normal variables in the mixture are: less than 40% for $\rho_0 = 0.6$, less than 50% for $\rho_0 = 0.7$, less than 80% for $\rho_0 = 0.8$ and less than 90% for $\rho_0 = 0.9$, in the large sample. In the small sample, the PMLE do not always converge to the reciprocal. This is the case of $\rho_0 = 0.6$. For $\rho_0 = 0.7$ and higher, the proportions of Normal variables in the sample that allow for convergence to the reciprocal are lower than in the large sample by about 10%.

The PMLE may provide misleading results and suggest the presence of a unit root in the model when there is a relatively high proportion of Normal variables in the sample. The dependence on the proportion of Normal variables in the mixture as follows: for $\rho_0 = 0.6$ the PMLE is close to 1 if the proportion of Normal variables in the sample is about 60-70 % in the large sample and about 60% in the small sample, for $\rho_0 = 0.7$ if the proportion of Normal variables is about 80-85% in either sample, for $\rho = 0.8$ and $\rho = 0.9$ if the sample is almost entirely Normal.

c) intermediate persistence $\rho_0 = 0.5$

To clarify the behavior of the PMLE estimator in this special case, we illustrate in Figure 3, the distribution of the Mix PMLE computed from a sample of T=400 and replicated 100 times for different values of λ_0. From Section 3, we know that for $\lambda_0 = 0$, the true model is a pure noncausal Cauchy model, the Mix PMLE is equal to the Cauchy PMLE and they both converge to the $[0, 2]$ interval. The top left panel of Figure 3 shows how the limiting value is distributed on $[0, 2]$. The kernel smoothed density estimate of the PMLE features two modes. One of them is close to 2, which is the reciprocal of $1/2$ and the upper bound of the interval , while the other is close to 0.5. This bimodal feature of the distribution, that is the probability of estimating either the causal model with $\rho_0 = 0.5$ or a noncausal model with $\rho_0 = 0.5$ is hard to see from Table 1.5 that shows only the median and sample quantiles. In particular, Figure 3 indicates that, when the proportion of Normal variables in the sample increases, the mode at $2 = 1/0.5$ is given gradually less weight and disappears for $\lambda_0 \geq 0.5$.

[Insert Figure 3: Density of Mix PMLE]

5 Concluding Remarks

The paper shows that the standard practice of estimating the past-dependent causal autoregressive models may lead to misspecification concerning the direction of temporal dependence and the value of the autoregressive coefficients. While the OLS estimator of a misspecified causal model is equivalent to the Gaussian causal PMLE and can never detect the presence of noncausal components, the non-Gaussian causal PMLE is capable of detecting noncausal components provided that the noncausal persistence is sufficiently strong and the sample is sufficiently non-Gaussian. Otherwise, the non-Gaussian PMLE may suggest the presence of a unit root, if the noncausal persistence is strong, but the sample is mostly Normal, or the absence of any persistence, when the true noncausal persistence is weak.

The binding function was used as the instrument for analyzing the consequences of model misspecification. In particular, we examined the patterns of the binding function when a causal autoregressive model is estimated instead of a noncausal or mixed causal-noncausal model. One would expect that the estimated causal pseudo-lag polynomial would have ill located roots, indicating that a mixed autoregressive specification is required. In this respect, we showed that the pseudo-lag polynomial is equal to the true one, up to a scale factor, over a significant set of parameter values.

The binding function can be used in further research as follows: The causal PML estimators can be adjusted for asymptotic bias by applying an indirect inference approach [see e.g. Gourieroux, Monfort, Renault (1993), Gallant, Tauchen (1996)]. To apply the indirect inference method, we would assume that the data generated by the true mixed causal-noncausal model are easy to simulate and the binding function is locally invertible. The simulation of mixed autoregressive models is easy indeed, from the causal and noncausal components of the process [see e.g. Gourieroux, Jasiak (2014)]. However, as shown in this paper,the binding function is not invertible for all the possible true values of the parameter.

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APPENDIX 1

The representations of a mixed MAR process

Let us express $A_0(L)$ and e_t^* in terms of Φ_0, Ψ_0, e_t. We have:

$$\begin{aligned}
&\Phi_0(L)\Psi_0(L^{-1})y_t = e_t \\
&\iff \quad \Phi_0(L)[1 - \psi_{0,1}L^{-1} - \cdots - \psi_{0,s}L^{-s}]y_t = e_t \\
&\iff \quad \Phi_0(L)[L^s - \psi_{0,1}L^{s-1} - \cdots - \psi_{0,s}]y_{t+s} = e_t.
\end{aligned}$$

Let us denote by $\tilde{\Psi}_0$ the polynomial with roots equal to the inverse of the roots of Ψ_0 and such that $\tilde{\Psi}_0(0) = 1$. We get:

$$\begin{aligned}
&\Phi_0(L)\Psi_0(L^{-1})y_t = e_t \\
&\iff \quad \Phi_0(L)\tilde{\Psi}_0(L)y_t = -\frac{e_{t-s}}{\psi_{0,s}},
\end{aligned}$$

whenever $\psi_{0,1} \neq 0$. We find that $A_0(L) = \Phi_0(L)\tilde{\Psi}_0(L)$ and $e_t^* = -\frac{e_{t-s}}{\psi_{0,s}}$. In particular, e_t^* differs from e_t by lag s and a scale factor.

For example, for a MAR(1,1) model:

$$(1 - \phi_0 L)(1 - \psi_0 L^{-1})y_t = e_t, \;\; |\phi_0| < 1, |\psi_0| < 1, \; \psi_0 \neq 0,$$

we get:

$$\begin{aligned}
A_0(L) &= (1 - \phi_0 L)(1 - 1/\psi_0 L) \\
&= 1 - (\phi_0 + 1/\psi_0)L + \phi_0/\psi_0 L^2.
\end{aligned}$$

APPENDIX 2

The binding function of a noncausal Cauchy AR(1) process

i) Pseudo-model: Cauchy error without scale parameter

Let us assume that the true model is a noncausal autoregressive model:

$$y_t = \rho_0 y_{t+1} + e_t, \tag{A.1}$$

where errors (e_t) are i.i.d. Cauchy distributed and the value of the forward autoregressive parameter $\rho_0 \in (-1, 1)$. The process (y_t) is strictly stationary and such that $(1 - |\rho_0|)y_t$ follows a standard Cauchy distribution.

Let us now assume that the econometrician estimates a causal autoregressive process:

$$y_t = \rho y_{t-1} + \epsilon_t, \tag{A.2}$$

where errors (ϵ_t) are i.i.d. Cauchy distributed and independent of $y_{t-1}, y_{t-2}, ...$ Thus, the econometrician makes an error concerning the direction of temporal dependence, but makes no other mistakes concerning the main characteristics of the DGP. The misspecified log-likelihood function is:

$$L_T = \sum_{t=1}^{T} log \left(\frac{1}{\pi} \frac{1}{1 + (y_t - \rho y_{t-1})^2} \right),$$

and its asymptotic counterpart is:

$$\begin{aligned}
\tilde{L}_\infty(\rho) &= lim_{T\to\infty} \frac{1}{T} L_T \\
&= E_0 log \left(\frac{1}{\pi} \frac{1}{1 + (y_t - \rho y_{t-1})^2} \right) \\
&= -log\pi - E_0\{\log[1 + (y_t - \rho y_{t-1})^2]\} \\
&= -log\pi - E_0\{log[1 + [(1 - \rho\rho_0)y_t - \rho e_{t-1}]^2]\}.
\end{aligned} \tag{A.3}$$

Variable y_t is a function of $e_t, e_{t+1}, ..., e_{t+h}, ..$ and is independent of e_{t-1}. From this independence property and the Cauchy marginal distribution of y_t, it follows that:

$$\tilde{L}_\infty(\rho) = -log\pi - E_0 log \left[1 + \left(\frac{1 - \rho\rho_0}{1 - |\rho_0|} e_1 - \rho e_2 \right)^2 \right], \tag{A.4}$$

where e_1 and e_2 are independent standard Cauchy variables. Equivalently, since the combination $\alpha_1 e_1 + \alpha_2 e_2$ of independent standard Cauchy variables is a scaled Cauchy variable $(|\alpha_1| + |\alpha_2|)e$, say, we get:

$$\tilde{L}_\infty(\rho) = -log\pi - E_0 log \left[1 + e^2 \left(\frac{|1 - \rho\rho_0|}{1 - |\rho_0|} + |\rho| \right)^2 \right]. \tag{A.5}$$

It follows that the pseudo-true value of ρ that maximizes $\tilde{L}_\infty(\rho)$ is equal to the minimizer of the following expression:

$$\frac{|1-\rho\rho_0|}{1-|\rho_0|}+|\rho|=d(\rho), \text{ say.}$$

Let us now focus on the optimization of the above objective function. Function d is a piecewise linear function of ρ. More precisely, we have the following three cases (written for $\rho_0 > 0$ for expository purpose):

$$\begin{cases} d(\rho) &= \frac{\rho\rho_0-1}{1-\rho_0}+\rho &= \frac{\rho-1}{1-\rho_0} &, \text{ if } \rho \geq 1/\rho_0, \\ d(\rho) &= \frac{1-\rho\rho_0}{1-\rho_0}+\rho &= \frac{1+\rho(1-2\rho_0)}{1-\rho_0} &, \text{ if } 0 \leq \rho \leq 1/\rho_0, \\ d(\rho) &= \frac{1-\rho\rho_0}{1-\rho_0}-\rho &= \frac{1-\rho}{1-\rho_0} &, \text{ if } \rho \leq 0. \end{cases}$$

We get three patterns of function d depending whether ρ_0 is less or greater then 0.5. They are plotted in Figure 4 below. We deduce the following proposition:

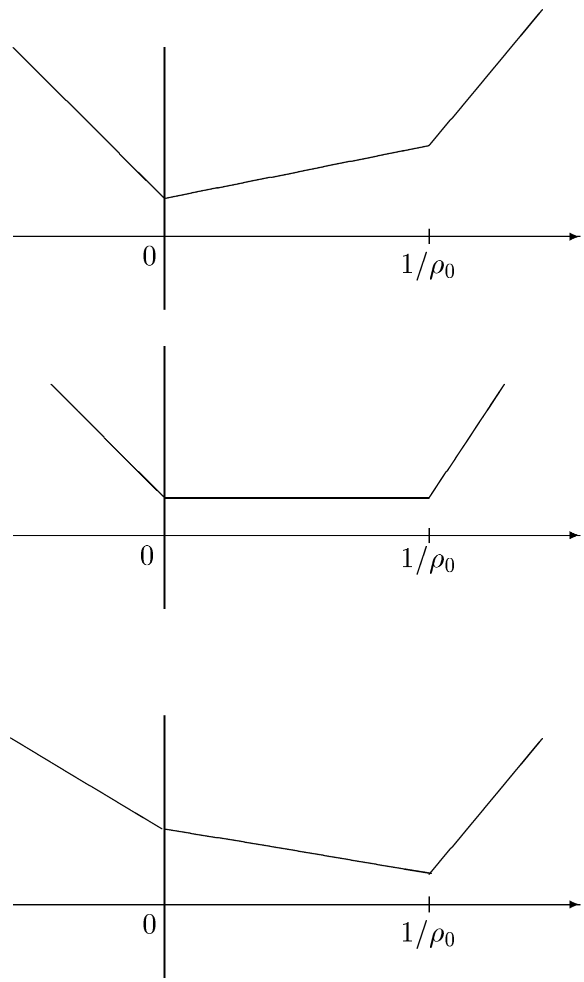
Proposition 2.2a: The pseudo-true value of ρ_0 for $\rho_0 > 0$ is:

$$b_1(\rho_0)=0 \quad , \text{if } \rho_0 < 0.5,$$
$$b_1(\rho_0)=1/\rho_0 \quad , \text{if } \rho_0 > 0.5.$$

There is an interval of pseudo-true values $b(\rho_0)=[0,2]$, if $\rho_0 = 0.5$.

Remark: An alternative proof based on linear programming is used later in the text. The value maximizing the piecewise linear function $d(\rho)$ is necessarily an extreme point of a simplex, either $\rho = 1/\rho_0$, or $\rho = 0$. The associated values of the objective function are: $d(1/\rho_0) = 1/|\rho_0|$ and $d(0) = 1/(1-|\rho_0|)$.
Therefore $1/\rho_0$ is the optimum when $d(1/\rho_0) < d(0)$, that is if $|\rho_0| > 0.5$.

Figure 4: Patterns of the objective function



$\rho_0 < 0.5$

$\rho_0 = 0.5$

$\rho_0 > 0.5$

ii) Pseudo-model: Cauchy error with scale parameter

Let us now examine how the previously established results change when a scale parameter of the Cauchy errors is introduced in both the true and pseudo models. The true noncausal model becomes:

$$y_t = \rho_0 y_{t+1} + \sigma_0 e_t, \tag{A.6}$$

where errors (e_t) are i.i.d standard Cauchy distributed.

The misspecified causal model used by the econometrician is:

$$y_t = \rho y_{t-1} + \sigma \epsilon_t, \tag{A.7}$$

where errors (ϵ_t) are i.i.d. standard Cauchy distributed and depends on two parameters (ρ, σ^2). The asymptotic counterpart of the log-likelihood is:

$$\tilde{L}_\infty(\rho, \sigma^2) = -log\pi - E_0 log\left\{1 + \frac{\sigma_0^2}{\sigma^2} e^2 \left[\frac{|1-\rho\rho_0|}{1-|\rho_0|} + |\rho|\right]^2\right\}, \tag{A.8}$$

where e follows a standard Cauchy distribution.

As before, maximizing the log-likelihood with respect to ρ given σ^2 is equivalent to minimizing the expression:

$$d(\rho) = \frac{|1-\rho\rho_0|}{1-|\rho_0|} + |\rho|.$$

Thus, the component of the binding function associated with ρ is the same as in Proposition (2.2a). In particular, it does not depend on the true scale parameter σ_0.

Let us now concentrate the asymptotic log-likelihood $\tilde{L}_\infty(\rho, \sigma^2)$ with respect to ρ in order to derive the second component of the binding function involving σ^2. The expression of that component depends on whether ρ_0 is less or grater than 0.5. We have:

$$\begin{aligned}
\text{If } \rho_0 > 0.5,\ \ \tilde{L}^c_\infty(\sigma^2) &= -log\pi - log\sigma - E_0 log\left\{1 + \frac{\sigma_0^2}{\sigma^2\rho_0^2} e^2\right\}, \\
\text{If } \rho_0 < 0.5,\ \ \tilde{L}^c_\infty(\sigma^2) &= -log\pi - log\sigma - E_0 log\left\{1 + \frac{\sigma_0^2}{\sigma^2(1-|\rho_0|)^2} e^2\right\}.
\end{aligned}$$

Thus, we can use either equation by replacing ρ_0 with $1-|\rho_0|$. The two expressions are equal for $\rho_0 = 0.5$.

To find the maximizing value of σ^2, we consider the derivative of the concentrated asymptotic pseudo-log-likelihood $\tilde{L}^c_\infty(\sigma^2)$ with respect to σ^2. For instance, for the first expression given above, we have :

$$\begin{aligned}
\frac{d\tilde{L}^c_\infty(\sigma^2)}{d\sigma^2} &= -\frac{1}{2\sigma^2} + E_0\left[\frac{\sigma_0^2 e^2 (\sigma^4\rho_0^2)^{-1}}{1 + \sigma_0^2 e^2 (\sigma^2\rho_0^2)^{-1}}\right] \\
&= -\frac{1}{2\sigma^2} + \frac{1}{\sigma^2} E_0\left[\frac{\sigma_0^2 e^2/\rho_0^2}{\sigma^2 + \sigma_0^2 e^2/\rho_0^2}\right] \\
&= \frac{1}{2\sigma^2} - E_0\left[\frac{1}{\sigma^2 + \sigma_0^2 e^2/\rho_0^2}\right].
\end{aligned}$$

The expectation on the right hand side of the equation has a closed form expression:

$$\begin{aligned}
& E_0\left[\frac{1}{\sigma^2+\sigma_0^2 e^2/\rho_0^2}\right] \\
= \; & \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\sigma^2+\sigma_0^2 x^2/\rho_0^2}\frac{1}{1+x^2}dx \\
= \; & \frac{1}{\pi}\frac{1}{\sigma^2-\sigma_0^2/\rho_0^2}\int_{-\infty}^{\infty}\left[-\frac{\sigma_0^2}{\rho_0^2\sigma^2}\frac{1}{1+\sigma_0^2x^2/(\sigma^2\rho_0^2)}+\frac{1}{1+x^2}\right]dx \\
= \; & \frac{1}{\pi}\frac{1}{\sigma^2-\sigma_0^2/\rho_0^2}\left[-\frac{\sigma_0}{|\rho_0|\sigma}\pi+\pi\right] \\
= \; & (\sigma^2-\sigma_0^2/\rho_0^2)^{-1}(1-\frac{\sigma_0}{|\rho_0|\sigma}) \\
= \; & \frac{1}{\sigma^2}\left(1-\frac{\sigma_0}{|\rho_0|\sigma}\right)\left(1-\frac{\sigma_0^2}{\rho_0^2\sigma^2}\right)^{-1} \\
= \; & \frac{1}{\sigma^2}\left(1+\frac{\sigma_0}{|\rho_0|\sigma}\right)^{-1}.
\end{aligned}$$

Thus, the first-order condition is:

$$\begin{aligned}
\frac{d\tilde{L}^c_\infty(\sigma^2)}{d\sigma^2}=0 \quad & \Longleftrightarrow \quad \frac{1}{2\sigma^2}-\frac{1}{\sigma^2}\left(1+\frac{\sigma_0}{|\rho_0|\sigma}\right)^{-1}=0 \\
& \Longleftrightarrow \quad 1+\frac{\sigma_0}{|\rho_0|\sigma}=2 \\
& \Longleftrightarrow \quad \sigma=\sigma_0/|\rho_0|.
\end{aligned}$$

It provides the second component of the binding function, i.e.

$$\begin{aligned}
b_2(\rho_0,\sigma_0) \;&=\; \sigma_0/|\rho_0|, \text{ if } |\rho_0|\geq 0.5, \\
&=\; \sigma_0/(1-|\rho_0|), \text{ if } |\rho_0|\leq 0.5.
\end{aligned}$$

iii) Pseudo-model: Causal AR(2) Cauchy model

Let us now derive the binding function when the misspecified model is:

$$y_t=a_1y_{t-1}+a_2y_{t-2}+\epsilon_t,$$

where (ϵ_t) are i.i.d. standard Cauchy variables. The asymptotic log-likelihood becomes:

$$\tilde{L}_\infty(a_1,a_2)=E_0 log\left\{\frac{1}{\pi}\frac{1}{1+(y_t-a_1y_{t-1}-a_2y_{t-2})^2}\right\}.$$

By applying the true autoregression with $\sigma_0 = 1$, we get:

$$\begin{aligned}
& y_t - a_1 y_{t-1} - a_2 y_{t-2} \\
&= y_t - a_1 y_{t-1} - a_2(\rho_0 y_{t-1} + e_{t-2}) \\
&= y_t - (a_1 + a_2\rho_0) y_{t-1} - a_2 e_{t-2} \\
&= [1 - \rho_0(a_1 + a_2\rho_0)] y_t - (a_1 + a_2\rho_0) e_{t-1} - a_2 e_{t-2} \\
&= e \left\{ \frac{|1 - \rho_0(a_1 + a_2\rho_0)|}{1 - |\rho_0|} + |a_1 + a_2\rho_0| + |a_2| \right\},
\end{aligned}$$

where e is a standard Cauchy variable. We find that:

$$\begin{aligned}
\tilde{L}_\infty(a_1, a_2) &= -log\pi - E_0 log[1 + (y_t - a_1 y_{t-1} - a_2 y_{t-2})^2] \\
&= -log\pi - E_0 log \left[1 + e^2 \left\{ \frac{|1 - \rho_0(a_1 + a_2\rho_0)|}{1 - |\rho_0|} + |a_1 + a_2\rho_0| + |a_2| \right\} \right],
\end{aligned}$$

is maximized when

$$d(a_1, a_2) = \frac{|1 - a_1\rho_0 - a_2\rho_0^2|}{1 - |\rho_0|} + |a_1 + a_2\rho_0| + |a_2|,$$

is minimized. The objective function is a piecewise linear function in a_1, a_2, which is non-differentiable on the lines $1 - \rho_0(a_1 + a_2\rho_0) = 0$, $a_1 + a_2\rho_0 = 0$, $a_2 = 0$.

Arguments that minimize the objective function are the extreme points located at the intersections of the three lines given above. As the lines $1 - \rho_0(a_1 + a_2\rho_0) = 0$ and $a_1 + a_2\rho_0 = 0$ are parallel, there are only two extreme points:

a) $a_1 = a_2 = 0$, at the intersection of $a_1 + a_2\rho_0 = 0$ and $a_2 = 0$;

b) $a_1 = 1/\rho_0, a_2 = 0$ at the intersection of $1 - \rho_0(a_1 + a_2\rho_0) = 0$ and $a_2 = 0$.

When $a_1 = a_2 = 0$, the value of the objective function is $d_1(\rho_0) = 1/(1 - |\rho_0|)$.
When $a_1 = 1/\rho_0, a_2 = 0$, the value of the objective function is $d_2(\rho_0) = 1/|\rho_0|$.

We find the following binding function for a_1, a_2 depending on the value of ρ_0:
if $|\rho_0| < 0.5,\ b_1(\rho_0) = 0,\ b_2(\rho_0) = 0$,
if $|\rho_0| > 0.5,\ b_1(\rho_0) = 1/(\rho_0),\ b_2(\rho_0) = 0$.

Despite of increasing the number of lags in the misspecified causal Cauchy autoregressive model, the binding function remains non-invertible.

APPENDIX 3

Mixed causal-noncausal MAR(1,1) Cauchy process: Proof of Proposition 3.1

The true process is a MAR(1,1) Cauchy process:

$$(1-\phi_0 L)(1-\psi_0 L^{-1})y_t = e_t, \tag{A.9}$$

where $|\phi_0| < 1, |\psi_0| < 1$ and the errors are independent and standard Cauchy distributed.

The misspecified model is:

$$(1-a_1 L - a_2 L^2)y_t = c\epsilon_t, \tag{A.10}$$

where (ϵ_t) are independent standard Cauchy variables and c is the scale parameter, $c > 0$. The asymptotic pseudo-log-likelihood is:

$$\tilde{L}_\infty(a_1, a_2, c) = E_0 \left\{ -\log c - \log \pi - \log \left[1 + \frac{(y_t - a_1 y_{t-1} - a_2 y_{t-2})^2}{c^2} \right] \right\}. \tag{A.11}$$

As in Appendix 2, we rewrite $y_t - a_1 y_{t-1} - a_2 y_{t-2}$ in terms of scaled Cauchy variables. For this purpose, we rewrite process y_t in terms of its causal and noncausal components [see, Gourieroux, Jasiak (2014)] as:

$$y_t = \frac{1}{1-\phi_0\psi_0}(\phi_0 v_{t-1} + u_t),$$

where the causal component v_t is defined by:

$v_t = \frac{1}{1-\phi_0 L} e_t \iff v_t = \phi_0 v_{t-1} + e_t,$

and the noncausal component u_t is:

$u_t = \frac{1}{1-\psi_0 L^{-1}} e_t \iff u_t = \psi_0 u_{t+1} + e_t.$

Thus v_{t-1} is a function of all past components of (e_t), and u_t is a function of the current and future components of (e_t). We have:

$$\begin{aligned} & y_t - a_1 y_{t-1} - a_2 y_{t-2} \\ = \ & \frac{1}{1-\phi_0\psi_0}[\phi_0 v_{t-1} + u_t - a_1(\phi_0 v_{t-2} + u_{t-1}) - a_2(\phi_0 v_{t-3} + u_{t-2})] \\ = \ & \frac{1}{1-\phi_0\psi_0}\{\phi_0[\phi_0^2 v_{t-3} + \phi_0 e_{t-2} + e_{t-1}] + u_t - a_1[\phi_0(\phi_0 v_{t-3} + e_{t-2}) + \psi_0 u_t + e_{t-1}] \end{aligned}$$

$$
\begin{aligned}
& -a_2[\phi_0 v_{t-3} + \psi_0^2 u_t + \psi_0 e_{t-1} + e_{t-2}]\} \\
= \ & \frac{1}{1-\phi_0\psi_0}\{v_{t-3}[\phi_0^3 - a_1\phi_0^2 - a_2\phi_0] + e_{t-2}[\phi_0^2 - a_1\phi_0 - a_2] \\
& + e_{t-1}[\phi_0 - a_1 - a_2\psi_0] + u_t[1 - a_1\psi_0 - a_2\psi_0^2]\}.
\end{aligned}
$$

The variables $v_{t-3}, e_{t-2}, e_{t-1}, u_t$ are independent Cauchy variables with scales $1/(1-|\phi_0|), 1, 1, 1/(1-|\psi_0|)$, respectively. Therefore, variable $y_t - a_1 y_{t-1} - a_2 y_{t-2}$ is a Cauchy variable with scale:

$$
y_t - a_1 y_{t-1} - a_2 y_{t-2} = \frac{e}{1-\phi_0\psi_0}\left\{\frac{|\phi_0^2 - a_1\phi_0 - a_2|}{1-|\phi_0|} + |\phi_0 - a_1 - a_2\psi_0| + \frac{|1 - a_1\psi_0 - a_2\psi_0^2|}{1-|\psi_0|}\right\},
$$

and

$$
\begin{aligned}
& \tilde{L}_\infty(a_1, a_2, c) \\
& = \ E_0\left\{-logc - \log\pi - \log\left[\frac{e^2}{c^2(1-\phi_0\psi_0)^2}\left\{\frac{|\phi_0^2 - a_1\phi_0 - a_2|}{1-|\phi_0|}\right.\right.\right. \\
& \quad + \ \left.\left.\left.|\phi_0 - a_1 - a_2\psi_0| + \frac{|1 - a_1\psi_0 - a_2\psi_0^2|}{1-|\psi_0|}\right\}\right]^2\right\}.
\end{aligned}
$$

The maximization of $\tilde{L}_\infty$ with respect to a_1, a_2 is equivalent to the minimization of the piecewise linear function:

$$
d(a_1, a_2) = \frac{|\phi_0^2 - a_1\phi_0 - a_2|}{1-|\phi_0|} + |\phi_0 - a_1 - a_2\psi_0| + \frac{|1 - a_1\psi_0 - a_2\psi_0^2|}{1-|\psi_0|}.
$$

The optimization of a piecewise linear function can be performed by linear programming. The solution is one of the extreme points of the simplex. Among these extreme points is point (a_1, a_2) which is the solution of the following system:

$$
\begin{cases}
\phi_0^2 - a_1\phi_0 - a_2 & = 0, \\
1 - a_1\psi_0 - a_2\psi_0^2 & = 0,
\end{cases}
$$

that is the lag-polynomial $A_0(L) = 1 - a_{1,0}L - a_{2,0}L^2$, whose roots are $1/\phi_0$ and ψ_0. When this extreme point minimizes the objective function, the causal PML estimator is consistent of $A_0(L)$. The corresponding solutions are: $a_1 = \phi_0 + 1/\psi_0$, $a_2 = -\phi_0/\psi_0$.
The other extreme points can be:

i) the solutions of the system:

$$\begin{cases} \phi_0^2 - a_1\phi_0 - a_2 & = 0, \\ \phi_0 - a_1 - a_2\psi_0 & = 0, \end{cases}$$

that are $a_1 = \phi_0, a_2 = 0$, or equivalently $A(L) = 1 - \phi_0 L$.

ii) the solutions of the system:

$$\begin{cases} \phi_0 - a_1 - a_2\psi_0 & = 0, \\ 1 - a_1\psi_0 - a_2\psi_0^2 & = 0. \end{cases}$$

By multiplying the first equation by $-\psi_0$ and adding it to the second equation, we get $1 - \phi_0\psi_0 = 0$, which is impossible.

To summarize, the optimization leads only to the admissible solutions $A(L) = A_0(L)$ or $A(L) = 1 - \phi_0 L$.

The values of the objective function at those two extreme points are $|\frac{1}{\psi_0} - \phi_0|$ and $\frac{1-\phi_0\psi_0}{1-|\psi_0|}$, respectively. By comparing these values of the objective function, we find the minimizing solution.

a) Let us first consider the case $0 < \psi_0 < 1$.

The PML estimator is consistent iff

$$\begin{aligned} & \frac{1-\phi_0\psi_0}{1-\psi_0} > \frac{1}{\psi_0} - \phi_0 \\ \iff & \quad 1 - \phi_0\psi_0 > \frac{1}{\psi_0} - 1 - \phi_0 + \phi_0\psi_0 \\ \iff & \quad 2 - \frac{1}{\psi_0} > \phi_0(2\psi_0 - 1) \\ \iff & \quad \frac{2\psi_0 - 1}{\psi_0} > \phi_0(2\psi_0 - 1) \end{aligned}$$

Thus, if $2\psi_0 - 1 > 0 \iff \psi_0 > 0.5$, the inequality is always satisfied. If $2\psi_0 - 1 < 0$, the inequality becomes $\phi_0 > 1/\psi_0$ and is never satisfied.

b) Let us now consider the case $-1 < \psi_0 < 0$.

The PML estimator is consistent iff:

$$\frac{1-\phi_0\psi_0}{1+\psi_0} > \phi_0 - \frac{1}{\psi_0}$$

$$
\begin{aligned}
&\Longleftrightarrow \quad 2(1-\phi_0\psi_0) > (1-\phi_0\psi_0)(-\frac{1}{\psi_0}) \\
&\Longleftrightarrow \quad 2 > -\frac{1}{\psi_0} = \frac{1}{|\psi_0|} \\
&\Longleftrightarrow \quad |\psi_0| > 0.5.
\end{aligned}
$$

To summarize, the choice between the two extreme points depends on whether $|\psi_0|$ is less or greater than 0.5, and is independent of the value of ϕ_0.

APPENDIX 4

Pure noncausal AR(2) Cauchy process: Proof of Proposition 3.2

The process is a noncausal AR(2) Cauchy process:

$$(1-\psi_{0,1}L^{-1})(1-\psi_{0,2}L^{-1})y_t = e_t, \tag{A.12}$$

where $1 > \psi_{0,1} > \psi_{0,2} > -1$ and e_t errors are independent Cauchy variables. We get:

$$
\begin{aligned}
y_t &= \frac{1}{(1-\psi_{0,1}L^{-1})(1-\psi_{0,2}L^{-1})} e_t \\
&= \frac{1}{\psi_{0,1}-\psi_{0,2}} \left(\frac{\psi_{0,1}}{1-\psi_{0,1}L^{-1}} - \frac{\psi_{0,2}}{1-\psi_{0,2}L^{-1}} \right) e_t \\
&= \frac{1}{\psi_{0,1}-\psi_{0,2}} \sum_{j=0}^{\infty} [(\psi_{0,1}^{j+1} - \psi_{0,2}^{j+1}) e_{t+j}].
\end{aligned}
$$

The pseudo-model is a pure causal AR(2) Cauchy process:

$$(1-a_1L-a_2L^2)y_t = \epsilon_t,$$

where errors (ϵ_t) are i.i.d. Cauchy variables. The variable involved in the pseudo-log-likelihood is:

$$
\begin{aligned}
&(1-a_1L-a_2L^2)y_t \\
&\quad = \frac{1}{\psi_{0,1}-\psi_{0,2}} \{ \sum_{j=0}^{\infty} [(\psi_{0,1}^{j+1} - \psi_{0,2}^{j+1}) e_{t+j} \\
&\qquad - a_1 \sum_{j=0}^{\infty} [(\psi_{0,1}^{j+1} - \psi_{0,2}^{j+1}) e_{t+j-1} - a_2 \sum_{j=0}^{\infty} [(\psi_{0,1}^{j+1} - \psi_{0,2}^{j+1}) e_{t+j-2} \}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\psi_{0,1}-\psi_{0,2}}\{e_{t-2}[-a_2(\psi_{0,1}-\psi_{0,2})]+e_{t-1}[-a_1(\psi_{0,1} \\
&\quad -\psi_{0,2})-a_2(\psi_{0,1}^2-\psi_{0,2}^2)]+\sum_{j=0}^{\infty}e_{t+j}\{\psi_{0,1}^{j+1} \\
&\quad -\psi_{0,2}^{j+1}-a_1(\psi_{0,1}^{j+2}-\psi_{0,2}^{j+2})-a_2(\psi_{0,1}^{j+3}-\psi_{0,2}^{j+3})\} \\
&\overset{d}{\equiv} e\{(\psi_{0,1}-\psi_{0,2})|a_2|+(\psi_{0,1}-\psi_{0,2})|a_1+a_2(\psi_{0,1}+\psi_{0,2})| \\
&\quad +\sum_{j=0}^{\infty}|\psi_{0,1}^{j+1}A(\psi_{0,1})-\psi_{0,2}^{j+1}A(\psi_{0,2})|\},
\end{aligned}
$$

where e follows a standard Cauchy distribution. The minimization of the quantity inside the brackets with respect to a_1, a_2 involves a much larger number of extreme points, including some that have already been evaluated before (see Section 3.2).

APPENDIX 5

Table 1.0: Estimates of $\rho = 0.0$

lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=400								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0	0.000	-0.0093	0.0108	-0.0024	-0.0548	0.0444	0.0000	-0.0093	0.0108
0.0100	0.0001	-0.0094	0.0091	-0.0024	-0.0541	0.0431	0.0000	-0.0097	0.0103
0.1000	0.0001	-0.0109	0.0100	-0.0024	-0.0451	0.0415	0.0001	-0.0104	0.0098
0.2000	0.0000	-0.0115	0.0114	-0.0024	-0.0486	0.0408	0.0000	-0.0105	0.0113
0.3000	0.0001	-0.0113	0.0107	-0.0023	-0.0475	0.0437	0.0000	-0.0104	0.0124
0.4000	0.0000	-0.0127	0.0113	-0.0024	-0.0489	0.0399	0.0000	-0.0130	0.0148
0.5000	0.0000	-0.0150	0.0118	-0.0024	-0.0509	0.0448	0.0001	-0.0135	0.0126
0.6000	0.0001	-0.0179	0.0174	-0.0024	-0.0493	0.0367	0.0002	-0.0167	0.0188
0.7000	0.0000	-0.0206	0.0175	-0.0027	-0.0510	0.0421	0.0000	-0.0192	0.0238
0.8000	-0.0002	-0.0266	0.0272	-0.0020	-0.0532	0.0405	0.0000	-0.0288	0.0277
0.8500	0.0000	-0.0330	0.0276	-0.0026	-0.0574	0.0470	0.0001	-0.0373	0.0413
0.9000	0.0000	-0.0406	0.0422	-0.0033	-0.0606	0.0558	0.0003	-0.0507	0.0486
0.9500	-0.0001	-0.0638	0.0543	-0.0028	-0.0742	0.0609	0.0002	-0.0647	0.0690
1.0000	-0.0081	-0.0821	0.0749	-0.0081	-0.0821	0.0749	-0.0006	-0.0970	0.1007
lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=100								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	0.0003	-0.0334	0.0382	-0.0104	-0.1246	0.1105	0.0003	-0.0334	0.0382
0.0100	0.0000	-0.0347	0.0395	-0.0103	-0.1219	0.1171	0.0003	-0.0364	0.0423
0.1000	0.0001	-0.0395	0.0443	-0.0102	-0.1191	0.0970	0.0000	-0.0439	0.0441
0.2000	0.0000	-0.0483	0.0452	-0.0108	-0.1335	0.1008	0.0001	-0.0441	0.0401
0.3000	-0.0001	-0.0437	0.0485	-0.0094	-0.1161	0.1067	0.0002	-0.0432	0.0517
0.4000	0.0001	-0.0454	0.0536	-0.0107	-0.1149	0.1020	-0.0003	-0.0456	0.0518
0.5000	0.0004	-0.0474	0.0558	-0.0109	-0.1177	0.1171	0.0001	-0.0583	0.0598
0.6000	0.0002	-0.0585	0.0617	-0.0105	-0.1256	0.0903	0.0000	-0.0715	0.0729
0.7000	-0.0006	-0.0748	0.0755	-0.0113	-0.1275	0.0947	-0.0001	-0.0875	0.0780
0.8000	-0.0001	-0.0898	0.0840	-0.0115	-0.1328	0.1239	-0.0005	-0.1097	0.1014
0.8500	-0.0023	-0.1102	0.0969	-0.0124	-0.1394	0.1176	0.0002	-0.1171	0.1100
0.9000	-0.0004	-0.1317	0.1097	-0.0110	-0.1535	0.1357	-0.0013	-0.1364	0.1281
0.9500	-0.0028	-0.1481	0.1388	-0.0113	-0.1645	0.1391	-0.0045	-0.1750	0.1559
1.0000	-0.0105	-0.1700	0.1622	-0.0105	-0.1700	0.1622	-0.0050	-0.1964	0.1879

Table 1.1: Estimates of $\rho = 0.1$

lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=400								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	0.0002	-0.0085	0.0158	0.0976	0.0551	0.1466	0.0002	-0.0085	0.0158
0.0100	0.0001	-0.0082	0.0141	0.0976	0.0549	0.1445	0.0001	-0.0082	0.0136
0.1000	0.0002	-0.0080	0.0138	0.0976	0.0595	0.1454	0.0002	-0.0081	0.0147
0.2000	0.0002	-0.0087	0.0173	0.0978	0.0617	0.1384	0.0002	-0.0078	0.0149
0.3000	0.0003	-0.0084	0.0161	0.0974	0.0549	0.1415	0.0003	-0.0080	0.0181
0.4000	0.0002	-0.0103	0.0199	0.0976	0.0608	0.1402	0.0004	-0.0089	0.0195
0.5000	0.0006	-0.0112	0.0227	0.0975	0.0585	0.1406	0.0006	-0.0104	0.0198
0.6000	0.0010	-0.0132	0.0295	0.0973	0.0582	0.1426	0.0008	-0.0116	0.0318
0.7000	0.0013	-0.0129	0.0385	0.0969	0.0504	0.1381	0.0013	-0.0133	0.0438
0.8000	0.0037	-0.0119	0.0582	0.0973	0.0513	0.1390	0.0033	-0.0145	0.0618
0.8500	0.0078	-0.0110	0.0793	0.0973	0.0447	0.1496	0.0059	-0.0130	0.0812
0.9000	0.0155	-0.0109	0.1076	0.0967	0.0356	0.1537	0.0107	-0.0166	0.1094
0.9500	0.0406	-0.0103	0.1400	0.0968	0.0252	0.1665	0.0334	-0.0168	0.1420
1.0000	0.0981	0.0190	0.1856	0.0981	0.0190	0.1856	0.0944	0.0129	0.1941
lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=100								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	0.0014	-0.0396	0.0607	0.0887	-0.0214	0.1790	0.0014	-0.0396	0.0607
0.0100	0.0014	-0.0346	0.0582	0.0888	-0.0213	0.1816	0.0005	-0.0391	0.0513
0.1000	0.0016	-0.0354	0.0614	0.0886	-0.0196	0.1812	0.0008	-0.0402	0.0532
0.2000	0.0016	-0.0359	0.0660	0.0889	-0.0254	0.1928	0.0009	-0.0406	0.0512
0.3000	0.0026	-0.0368	0.0632	0.0889	-0.0148	0.1936	0.0009	-0.0376	0.0671
0.4000	0.0014	-0.0384	0.0777	0.0896	-0.0245	0.1910	0.0010	-0.0364	0.0800
0.5000	0.0022	-0.0452	0.0751	0.0894	-0.0183	0.1922	0.0020	-0.0482	0.0778
0.6000	0.0035	-0.0435	0.0937	0.0884	-0.0235	0.2131	0.0034	-0.0512	0.1065
0.7000	0.0066	-0.0437	0.1229	0.0889	-0.0391	0.2069	0.0048	-0.0523	0.1247
0.8000	0.0132	-0.0511	0.1570	0.0886	-0.0444	0.2157	0.0104	-0.0601	0.1720
0.8500	0.0222	-0.0530	0.1845	0.0871	-0.0694	0.2201	0.0172	-0.0659	0.1975
0.9000	0.0297	-0.0627	0.2051	0.0886	-0.0459	0.2358	0.0290	-0.0728	0.2243
0.9500	0.0571	-0.0667	0.2495	0.0887	-0.0640	0.2268	0.0612	-0.0794	0.2648
1.0000	0.0922	-0.0741	0.2729	0.0922	-0.0741	0.2729	0.1068	-0.0906	0.2892

Table 1.2: Estimates of $\rho = 0.2$

lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=400								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	0.0006	-0.0076	0.0255	0.1968	0.1548	0.2422	0.0006	-0.0076	0.0255
0.0100	0.0007	-0.0073	0.0253	0.1970	0.1553	0.2422	0.0007	-0.0073	0.0207
0.1000	0.0010	-0.0070	0.0283	0.1969	0.1520	0.2369	0.0008	-0.0084	0.0218
0.2000	0.0009	-0.0068	0.0266	0.1969	0.1559	0.2448	0.0007	-0.0078	0.0248
0.3000	0.0012	-0.0070	0.0323	0.1972	0.1560	0.2367	0.0012	-0.0072	0.0309
0.4000	0.0013	-0.0081	0.0384	0.1970	0.1517	0.2485	0.0012	-0.0079	0.0332
0.5000	0.0023	-0.0067	0.0399	0.1969	0.1567	0.2430	0.0019	-0.0080	0.0414
0.6000	0.0035	-0.0069	0.0485	0.1968	0.1506	0.2358	0.0027	-0.0069	0.0558
0.7000	0.0053	-0.0063	0.0770	0.1969	0.1599	0.2412	0.0048	-0.0072	0.0779
0.8000	0.0168	-0.0051	0.1272	0.1974	0.1538	0.2543	0.0112	-0.0083	0.1237
0.8500	0.0332	-0.0033	0.1623	0.1970	0.1519	0.2512	0.0210	-0.0086	0.1651
0.9000	0.0615	-0.0013	0.1935	0.1980	0.1382	0.2575	0.0646	-0.0042	0.2106
0.9500	0.1363	0.0069	0.2405	0.1975	0.1351	0.2669	0.1367	-0.0001	0.2449
1.0000	0.2011	0.1189	0.2803	0.2011	0.1189	0.2803	0.2037	0.1118	0.2878
lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=100								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	0.0035	-0.0273	0.0992	0.1884	0.0581	0.3035	0.0035	-0.0273	0.0992
0.0100	0.0026	-0.0253	0.0989	0.1884	0.0596	0.3022	0.0034	-0.0267	0.0863
0.1000	0.0031	-0.0230	0.1030	0.1873	0.0616	0.3059	0.0029	-0.0285	0.1037
0.2000	0.0036	-0.0274	0.1112	0.1875	0.0770	0.3177	0.0046	-0.0262	0.1006
0.3000	0.0054	-0.0252	0.1190	0.1877	0.0666	0.3108	0.0038	-0.0266	0.1088
0.4000	0.0072	-0.0241	0.1338	0.1879	0.0751	0.3104	0.0068	-0.0251	0.1225
0.5000	0.0078	-0.0305	0.1397	0.1869	0.0692	0.3166	0.0074	-0.0289	0.1365
0.6000	0.0138	-0.0280	0.1654	0.1877	0.0666	0.2944	0.0091	-0.0306	0.1498
0.7000	0.0182	-0.0219	0.2196	0.1884	0.0650	0.3217	0.0140	-0.0325	0.2170
0.8000	0.0367	-0.0205	0.2508	0.1869	0.0624	0.3105	0.0298	-0.0286	0.2688
0.8500	0.0625	-0.0226	0.2737	0.1880	0.0516	0.3036	0.0528	-0.0318	0.2826
0.9000	0.0961	-0.0141	0.2825	0.1869	0.0340	0.3287	0.0904	-0.0291	0.3136
0.9500	0.1374	-0.0128	0.3126	0.1885	0.0181	0.3378	0.1379	-0.0238	0.3412
1.0000	0.1943	0.0388	0.3539	0.1943	0.0388	0.3539	0.1947	0.0071	0.3803

Table 1.3: Estimates of $\rho = 0.3$

lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=400								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	0.0022	-0.0048	0.0460	0.2970	0.2471	0.3428	0.0022	-0.0048	0.0460
0.0100	0.0020	-0.0053	0.0474	0.2968	0.2482	0.3419	0.0023	-0.0048	0.0437
0.1000	0.0018	-0.0060	0.0521	0.2968	0.2474	0.3448	0.0023	-0.0052	0.0510
0.2000	0.0027	-0.0051	0.0531	0.2968	0.2494	0.3398	0.0023	-0.0052	0.0526
0.3000	0.0036	-0.0052	0.0596	0.2971	0.2488	0.3426	0.0027	-0.0044	0.0657
0.4000	0.0047	-0.0043	0.0724	0.2967	0.2548	0.3423	0.0030	-0.0049	0.0733
0.5000	0.0062	-0.0049	0.0867	0.2966	0.2433	0.3425	0.0051	-0.0054	0.0882
0.6000	0.0097	-0.0034	0.1183	0.2962	0.2468	0.3393	0.0073	-0.0049	0.1161
0.7000	0.0213	-0.0026	0.1801	0.2968	0.2464	0.3400	0.0165	-0.0045	0.1854
0.8000	0.0703	-0.0007	0.2409	0.2966	0.2487	0.3381	0.0551	-0.0029	0.2446
0.8500	0.1101	0.0000	0.2738	0.2969	0.2476	0.3456	0.1297	-0.0005	0.2859
0.9000	0.1705	0.0043	0.3116	0.2970	0.2363	0.3512	0.1989	0.0006	0.3206
0.9500	0.2376	0.0659	0.3462	0.2963	0.2323	0.3581	0.2519	0.0876	0.3589
1.0000	0.2999	0.2156	0.3746	0.2999	0.2156	0.3746	0.3009	0.2057	0.3879
lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=100								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	0.0075	-0.0277	0.2065	0.2858	0.1564	0.4038	0.0075	-0.0277	0.2065
0.0100	0.0071	-0.0287	0.1928	0.2857	0.1628	0.4035	0.0075	-0.0206	0.1940
0.1000	0.0059	-0.0296	0.2132	0.2857	0.1563	0.4077	0.0084	-0.0208	0.2002
0.2000	0.0095	-0.0244	0.1930	0.2859	0.1774	0.3971	0.0091	-0.0253	0.2075
0.3000	0.0124	-0.0224	0.2205	0.2852	0.1554	0.4001	0.0104	-0.0216	0.1904
0.4000	0.0151	-0.0222	0.2304	0.2864	0.1679	0.4009	0.0120	-0.0229	0.2206
0.5000	0.0229	-0.0192	0.2591	0.2849	0.1473	0.4054	0.0151	-0.0241	0.2570
0.6000	0.0342	-0.0184	0.2932	0.2855	0.1583	0.4089	0.0238	-0.0185	0.3185
0.7000	0.0582	-0.0133	0.3258	0.2858	0.1595	0.4087	0.0380	-0.0184	0.3365
0.8000	0.0994	-0.0112	0.3770	0.2865	0.1581	0.4084	0.1067	-0.0141	0.3833
0.8500	0.1531	-0.0050	0.3891	0.2860	0.1296	0.4215	0.1387	-0.0114	0.3853
0.9000	0.2039	-0.0001	0.4168	0.2848	0.1408	0.4263	0.1930	-0.0047	0.4150
0.9500	0.2574	0.0200	0.4261	0.2867	0.1302	0.4268	0.2481	-0.0033	0.4470
1.0000	0.3011	0.1268	0.4319	0.3011	0.1268	0.4319	0.3028	0.0925	0.4598

Table 1.4: Estimates of $\rho = 0.4$

lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=400								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	0.0103	-0.0034	0.1960	0.3964	0.3546	0.4440	0.0103	-0.0034	0.1960
0.0100	0.0076	-0.0037	0.1753	0.3964	0.3545	0.4430	0.0101	-0.0029	0.1991
0.1000	0.0079	-0.0037	0.1814	0.3965	0.3530	0.4505	0.0098	-0.0031	0.2012
0.2000	0.0109	-0.0032	0.1972	0.3963	0.3503	0.4420	0.0129	-0.0031	0.2167
0.3000	0.0124	-0.0026	0.2159	0.3967	0.3540	0.4463	0.0127	-0.0030	0.2356
0.4000	0.0177	-0.0024	0.2431	0.3961	0.3508	0.4404	0.0187	-0.0020	0.2505
0.5000	0.0308	-0.0022	0.3206	0.3964	0.3558	0.4400	0.0329	-0.0025	0.2863
0.6000	0.0686	-0.0016	0.3426	0.3965	0.3517	0.4411	0.0732	-0.0014	0.3306
0.7000	0.1582	0.0004	0.4011	0.3965	0.3490	0.4443	0.1658	-0.0005	0.3818
0.8000	0.2631	0.0060	0.4225	0.3961	0.3491	0.4377	0.2666	0.0054	0.4331
0.8500	0.3041	0.0418	0.4400	0.3959	0.3346	0.4452	0.3118	0.0361	0.4429
0.9000	0.3349	0.1404	0.4555	0.3959	0.3408	0.4530	0.3422	0.1727	0.4517
0.9500	0.3724	0.2357	0.4656	0.3965	0.3311	0.4576	0.3708	0.2582	0.4719
1.0000	0.4000	0.3233	0.4712	0.4000	0.3233	0.4712	0.3974	0.3066	0.4841
lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=100								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	0.0547	-0.0166	0.6310	0.3848	0.2676	0.4982	0.0547	-0.0166	0.6310
0.0100	0.0508	-0.0166	0.6466	0.3850	0.2708	0.4999	0.0387	-0.0157	0.4214
0.1000	0.0528	-0.0153	0.6347	0.3853	0.2783	0.4991	0.0408	-0.0149	0.4053
0.2000	0.0478	-0.0141	0.6429	0.3857	0.2620	0.5029	0.0419	-0.0150	0.4122
0.3000	0.0599	-0.0121	0.5970	0.3851	0.2615	0.4903	0.0542	-0.0144	0.4334
0.4000	0.0764	-0.0104	0.5871	0.3843	0.2781	0.5000	0.0582	-0.0103	0.4653
0.5000	0.0978	-0.0073	0.6013	0.3861	0.2707	0.4949	0.0922	-0.0105	0.5139
0.6000	0.1334	-0.0078	0.5571	0.3854	0.2557	0.4982	0.1094	-0.0088	0.5104
0.7000	0.2051	-0.0030	0.5859	0.3861	0.2623	0.5169	0.1890	-0.0048	0.5480
0.8000	0.2865	0.0006	0.5541	0.3838	0.2522	0.4953	0.2877	-0.0018	0.5569
0.8500	0.3107	0.0268	0.5376	0.3830	0.2517	0.5147	0.3060	0.0005	0.5579
0.9000	0.3354	0.0458	0.5473	0.3846	0.2400	0.5098	0.3422	0.0248	0.5574
0.9500	0.3778	0.1333	0.5465	0.3835	0.2283	0.5222	0.3695	0.0925	0.5634
1.0000	0.4056	0.2384	0.5370	0.4056	0.2384	0.5370	0.3940	0.2172	0.5686

Table 1.5: Estimates of $\rho = 0.5$

lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=400								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	0.2895	0.0031	1.9889	0.4961	0.4462	0.5354	0.2895	0.0031	1.9889
0.0100	0.3943	0.0123	1.9881	0.4961	0.4450	0.5349	0.2865	0.0029	1.9380
0.1000	0.3773	0.0105	1.9615	0.4963	0.4409	0.5357	0.2966	0.0015	1.8270
0.2000	0.5018	0.0256	1.9547	0.4961	0.4507	0.5392	0.3123	0.0020	1.5902
0.3000	0.5017	0.0235	1.6148	0.4961	0.4390	0.5336	0.3485	0.0049	1.4594
0.4000	0.4994	0.0288	1.2169	0.4961	0.4372	0.5390	0.3037	0.0100	1.1022
0.5000	0.5125	0.0771	0.9732	0.4963	0.4390	0.5435	0.4464	0.0204	0.9453
0.6000	0.5026	0.3095	0.6560	0.4961	0.4407	0.5477	0.4836	0.0979	0.8061
0.7000	0.4989	0.3359	0.6320	0.4962	0.4392	0.5388	0.4998	0.2413	0.7395
0.8000	0.5058	0.3676	0.6166	0.4957	0.4419	0.5428	0.5035	0.3269	0.6476
0.8500	0.4979	0.3929	0.5914	0.4958	0.4413	0.5456	0.4994	0.3579	0.6284
0.9000	0.4965	0.4190	0.5665	0.4961	0.4386	0.5455	0.5023	0.3777	0.6058
0.9500	0.5026	0.3975	0.5884	0.4957	0.4328	0.5554	0.4985	0.4050	0.5969
1.0000	0.4985	0.4236	0.5670	0.4985	0.4236	0.5670	0.4975	0.4109	0.5775
lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=100								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	0.2956	-0.0003	1.9983	0.4839	0.3609	0.6034	0.2956	-0.003	1.9983
0.0100	0.3059	0.0015	1.9960	0.4839	0.3578	0.6034	0.2954	-0.0003	1.8226
0.1000	0.4208	0.0032	1.9918	0.4841	0.3698	0.6082	0.3000	0.0001	1.7127
0.2000	0.5146	0.0058	1.9966	0.4837	0.3746	0.6125	0.3265	0.0019	1.7050
0.3000	0.5117	0.0038	1.9790	0.4831	0.3535	0.5936	0.3284	-0.0009	1.6713
0.4000	0.5154	0.0067	1.8983	0.4849	0.3693	0.6127	0.3709	0.0011	1.7174
0.5000	0.5093	0.0092	1.7158	0.4833	0.3396	0.6074	0.4296	0.0062	1.4924
0.6000	0.4966	0.1335	0.8097	0.4824	0.3506	0.6004	0.4704	0.0284	1.1954
0.7000	0.5032	0.1477	0.7998	0.4833	0.3576	0.5934	0.4918	0.0178	0.9573
0.8000	0.4964	0.2305	0.7158	0.4826	0.3521	0.6043	0.5030	0.0526	0.8762
0.8500	0.4947	0.2656	0.6844	0.4832	0.3484	0.6081	0.5079	0.1290	0.8049
0.9000	0.4947	0.3360	0.6214	0.4821	0.3392	0.6142	0.5002	0.2263	0.7051
0.9500	0.4949	0.2980	0.6731	0.4874	0.3451	0.6141	0.4999	0.2872	0.6692
1.0000	0.4948	0.3425	0.6283	0.4948	0.3425	0.6283	0.4977	0.3300	0.6393

Table 1.6: Estimates of $\rho = 0.6$

lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=400								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	1.6499	1.2725	1.6683	0.5960	0.5518	0.6357	1.6499	1.2725	1.6683
0.0100	1.6492	1.2520	1.6683	0.5960	0.5526	0.6358	1.6446	1.2117	1.6684
0.1000	1.6448	1.2097	1.6683	0.5960	0.5515	0.6355	1.6381	1.1755	1.6680
0.2000	1.6341	1.1402	1.6678	0.5961	0.5483	0.6359	1.6237	1.0886	1.6680
0.3000	1.6135	1.0352	1.6678	0.5957	0.5512	0.6303	1.6043	1.0146	1.6679
0.4000	1.5343	0.9012	1.6673	0.5960	0.5507	0.6384	1.5509	0.8870	1.6672
0.5000	1.3090	0.7904	1.6668	0.5959	0.5552	0.6373	1.3377	0.7894	1.6670
0.6000	1.0809	0.7255	1.6658	0.5956	0.5504	0.6420	1.0954	0.7127	1.6664
0.7000	0.8901	0.6443	1.6310	0.5959	0.5480	0.6411	0.8851	0.6562	1.6571
0.8000	0.7601	0.5996	1.0608	0.5960	0.5473	0.6414	0.7550	0.6057	1.0778
0.8500	0.7177	0.5829	0.9301	0.5954	0.5421	0.6371	0.7098	0.5740	0.8752
0.9000	0.6741	0.5598	0.8284	0.5957	0.5425	0.6415	0.6705	0.5521	0.7874
0.9500	0.6383	0.5474	0.7403	0.5956	0.5392	0.6529	0.6323	0.5377	0.7310
1.0000	0.6002	0.5294	0.6610	0.6002	0.5294	0.6610	0.6024	0.5232	0.6728
lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=100								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	1.5358	0.5173	1.6721	0.5845	0.4660	0.7089	1.5358	0.5173	1.6721
0.0100	1.5338	0.5163	1.6719	0.5845	0.4652	0.7103	1.4576	0.2092	1.6719
0.1000	1.5283	0.5670	1.6720	0.5842	0.4715	0.7089	1.4421	0.2129	1.6700
0.2000	1.4749	0.5172	1.6714	0.5836	0.4681	0.7130	1.4200	0.2360	1.6709
0.3000	1.4432	0.4926	1.6710	0.5831	0.4697	0.7076	1.3262	0.2240	1.6703
0.4000	1.3350	0.5239	1.6703	0.5835	0.4554	0.7003	1.2683	0.2952	1.6697
0.5000	1.1840	0.4969	1.6674	0.5829	0.4668	0.6930	1.1276	0.3024	1.6674
0.6000	0.9855	0.4891	1.6663	0.5827	0.4650	0.6910	0.9789	0.2868	1.6673
0.7000	0.8526	0.4625	1.6600	0.5842	0.4305	0.7072	0.8419	0.4115	1.6661
0.8000	0.7469	0.4491	1.5501	0.5836	0.4642	0.6948	0.7293	0.4243	1.6441
0.8500	0.6982	0.4521	1.2485	0.5824	0.4459	0.7060	0.6930	0.4206	1.4897
0.9000	0.6620	0.4248	1.0383	0.5832	0.4358	0.6978	0.6539	0.4254	1.0218
0.9500	0.6169	0.4298	0.8352	0.5818	0.4353	0.6957	0.6195	0.4219	0.8564
1.0000	0.5886	0.4427	0.7097	0.5886	0.4427	0.7097	0.5960	0.4156	0.7365

Table 1.7: Estimates of $\rho = 0.7$

lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=400								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	1.4233	1.3542	1.4302	0.6957	0.6550	0.7410	1.4233	1.3542	1.4302
0.0100	1.4231	1.3497	1.4304	0.6957	0.6555	0.7408	1.4234	1.3558	1.4301
0.1000	1.4224	1.3423	1.4305	0.6957	0.6523	0.7393	1.4229	1.3380	1.4301
0.2000	1.4213	1.3304	1.4300	0.6951	0.6455	0.7395	1.4208	1.3042	1.4301
0.3000	1.4181	1.2844	1.4299	0.6955	0.6529	0.7443	1.4193	1.2841	1.4297
0.4000	1.4160	1.2637	1.4298	0.6957	0.6512	0.7395	1.4158	1.2529	1.4299
0.5000	1.4094	1.1809	1.4293	0.6956	0.6538	0.7403	1.4091	1.1730	1.4297
0.6000	1.3936	1.0723	1.4291	0.6961	0.6536	0.7351	1.3909	1.0567	1.4297
0.7000	1.2750	0.9456	1.4287	0.6956	0.6496	0.7330	1.2943	0.9115	1.4290
0.8000	1.0490	0.8218	1.4278	0.6955	0.6499	0.7423	1.0480	0.7997	1.4284
0.8500	0.9433	0.7636	1.4214	0.6964	0.6538	0.7429	0.9244	0.7560	1.4276
0.9000	0.8667	0.7250	1.2530	0.6953	0.6493	0.7406	0.8385	0.7063	1.4056
0.9500	0.7760	0.6710	0.9571	0.6960	0.6383	0.7438	0.7612	0.6657	0.8926
1.0000	0.7001	0.6358	0.7541	0.7001	0.6358	0.7541	0.6980	0.6276	0.7573
lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=100								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	1.4098	1.1366	1.4359	0.6816	0.5488	0.7832	1.4098	1.1366	1.4359
0.0100	1.4090	1.1301	1.4356	0.6816	0.5488	0.7784	1.4056	1.0637	1.4353
0.1000	1.4043	1.1076	1.4347	0.6815	0.5444	0.7798	1.4009	1.0190	1.4354
0.2000	1.4025	1.0919	1.4344	0.6820	0.5496	0.7869	1.3983	1.0160	1.4341
0.3000	1.3943	1.0294	1.4335	0.6825	0.5467	0.7810	1.3887	0.9625	1.4344
0.4000	1.3805	0.9393	1.4332	0.6805	0.5480	0.7935	1.3794	0.8982	1.4330
0.5000	1.3560	0.8344	1.4339	0.6821	0.5582	0.7875	1.3410	0.8646	1.4338
0.6000	1.2999	0.7858	1.4322	0.6800	0.5473	0.7782	1.3200	0.7778	1.4324
0.7000	1.2138	0.7373	1.4299	0.6806	0.5679	0.7822	1.1867	0.7308	1.4304
0.8000	1.0039	0.6565	1.4270	0.6800	0.5522	0.7747	0.9878	0.6319	1.4284
0.8500	0.9271	0.6318	1.4261	0.6826	0.5460	0.7872	0.8796	0.6274	1.4283
0.9000	0.8217	0.6028	1.4054	0.6797	0.5430	0.7840	0.8075	0.5801	1.4192
0.9500	0.7646	0.5738	1.3286	0.6779	0.5451	0.7852	0.7445	0.5696	1.3498
1.0000	0.6925	0.5421	0.8025	0.6767	0.5350	0.7850	0.6915	0.5353	0.8138

Table 1.8: Estimates of $\rho = 0.8$

lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=400								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	1.2477	1.2249	1.2515	0.7954	0.7506	0.8347	1.2477	1.2249	1.2515
0.0100	1.2480	1.2234	1.2518	0.7955	0.7479	0.8347	1.2477	1.2245	1.2515
0.1000	1.2475	1.2201	1.2517	0.7953	0.7524	0.8302	1.2474	1.2183	1.2516
0.2000	1.2468	1.2166	1.2514	0.7953	0.7450	0.8292	1.2472	1.2184	1.2516
0.3000	1.2461	1.2079	1.2512	0.7955	0.7475	0.8293	1.2466	1.2061	1.2518
0.4000	1.2458	1.2007	1.2515	0.7953	0.7573	0.8363	1.2450	1.1935	1.2514
0.5000	1.2437	1.1810	1.2515	0.7950	0.7548	0.8317	1.2435	1.1796	1.2513
0.6000	1.2390	1.1477	1.2509	0.7954	0.7537	0.8301	1.2395	1.1501	1.2509
0.7000	1.2326	1.0805	1.2506	0.7957	0.7536	0.8334	1.2300	1.0757	1.2511
0.8000	1.2064	0.9841	1.2504	0.7952	0.7554	0.8252	1.1986	0.9690	1.2503
0.8500	1.1523	0.9112	1.2500	0.7951	0.7484	0.8365	1.1177	0.9021	1.2500
0.9000	1.0421	0.8548	1.2496	0.7951	0.7463	0.8357	1.0211	0.8394	1.2493
0.9500	0.9156	0.7925	1.2441	0.7947	0.7475	0.8390	0.8860	0.7829	1.2444
1.0000	0.7987	0.7447	0.8435	0.7940	0.7369	0.8414	0.7968	0.7351	0.8515
lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=100								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	1.2423	1.1491	1.2571	0.7791	0.6641	0.8876	1.2423	1.1491	1.2571
0.0100	1.2422	1.1449	1.2571	0.7798	0.6652	0.8874	1.2422	1.1518	1.2582
0.1000	1.2410	1.1279	1.2562	0.7803	0.6601	0.8870	1.2415	1.1277	1.2577
0.2000	1.2387	1.1249	1.2570	0.7791	0.6484	0.8795	1.2394	1.1266	1.2571
0.3000	1.2346	1.1024	1.2561	0.7778	0.6589	0.8999	1.2372	1.0825	1.2568
0.4000	1.2303	1.0847	1.2550	0.7795	0.6597	0.8810	1.2325	1.0566	1.2561
0.5000	1.2217	1.0247	1.2543	0.7803	0.6709	0.8861	1.2267	1.0068	1.2574
0.6000	1.2120	0.9652	1.2532	0.7807	0.6646	0.9081	1.2115	0.9659	1.2552
0.7000	1.1775	0.8908	1.2527	0.7801	0.6733	0.9007	1.1798	0.8747	1.2545
0.8000	1.1162	0.7959	1.2510	0.7765	0.6443	0.8710	1.0914	0.8040	1.2513
0.8500	1.0443	0.7630	1.2504	0.7784	0.6501	0.8650	1.0282	0.7716	1.2513
0.9000	0.9537	0.7272	1.2471	0.7799	0.6530	0.8694	0.9514	0.7289	1.2500
0.9500	0.8500	0.6985	1.2256	0.7733	0.6519	0.8617	0.8526	0.7015	1.2295
1.0000	0.7919	0.6649	0.8802	0.7719	0.6412	0.8569	0.7918	0.6595	0.8864

Table 1.9: Estimates of $\rho = 0.9$

lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=400								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	1.1104	1.1018	1.1124	0.8948	0.8564	0.9242	1.1104	1.1018	1.1124
0.0100	1.1104	1.1010	1.1123	0.8947	0.8552	0.9247	1.1105	1.1010	1.1126
0.1000	1.1102	1.1003	1.1124	0.8950	0.8582	0.9254	1.1103	1.1002	1.1123
0.2000	1.1100	1.0979	1.1125	0.8952	0.8580	0.9261	1.1100	1.0991	1.1125
0.3000	1.1097	1.0970	1.1123	0.8951	0.8572	0.9237	1.1098	1.0954	1.1127
0.4000	1.1093	1.0950	1.1122	0.8950	0.8588	0.9278	1.1095	1.0906	1.1125
0.5000	1.1086	1.0862	1.1123	0.8950	0.8588	0.9253	1.1090	1.0883	1.1125
0.6000	1.1075	1.0784	1.1121	0.8952	0.8613	0.9306	1.1080	1.0756	1.1126
0.7000	1.1042	1.0574	1.1119	0.8950	0.8527	0.9233	1.1049	1.0581	1.1122
0.8000	1.0963	1.0252	1.1114	0.8952	0.8564	0.9245	1.0987	1.0192	1.1120
0.8500	1.0892	0.9894	1.1115	0.8944	0.8540	0.9327	1.0881	0.9902	1.1116
0.9000	1.0699	0.9445	1.1110	0.8944	0.8506	0.9255	1.0580	0.9351	1.1112
0.9500	0.9961	0.8915	1.1093	0.8927	0.8499	0.9242	0.9913	0.8908	1.1098
1.0000	0.8977	0.8580	0.9283	0.8905	0.8441	0.9241	0.8984	0.8478	0.9320
lambda	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)	med	q(0.05)	q(0.95)
	T=100								
	Mix PMLE			**OLS**			**Cauchy PMLE**		
0.0000	1.1087	1.0732	1.1164	0.8776	0.7758	0.9997	1.1087	1.0732	1.1164
0.0100	1.1086	1.0728	1.1165	0.8776	0.7724	0.9945	1.1086	1.0753	1.1163
0.1000	1.1083	1.0664	1.1169	0.8783	0.7713	1.0117	1.1080	1.0720	1.1160
0.2000	1.1076	1.0656	1.1165	0.8769	0.7637	1.0161	1.1075	1.0683	1.1158
0.3000	1.1064	1.0615	1.1159	0.8772	0.7767	0.9962	1.1069	1.0544	1.1180
0.4000	1.1047	1.0495	1.1161	0.8774	0.7630	1.0175	1.1052	1.0421	1.1169
0.5000	1.1018	1.0274	1.1149	0.8774	0.7787	1.0139	1.1021	1.0239	1.1178
0.6000	1.0984	1.0127	1.1148	0.8770	0.7678	1.0001	1.0993	1.0001	1.1164
0.7000	1.0929	0.9621	1.1150	0.8769	0.7642	0.9818	1.0884	0.9624	1.1159
0.8000	1.0708	0.9147	1.1148	0.8758	0.7592	0.9602	1.0680	0.9060	1.1147
0.8500	1.0477	0.8818	1.1127	0.8771	0.7697	0.9756	1.0514	0.8785	1.1135
0.9000	0.9923	0.8453	1.1117	0.8765	0.7722	0.9655	1.0054	0.8493	1.1118
0.9500	0.9334	0.8219	1.1068	0.8744	0.7540	0.9481	0.9429	0.8148	1.1073
1.0000	0.8898	0.7848	0.9451	0.8708	0.7678	0.9356	0.8915	0.7887	0.9551

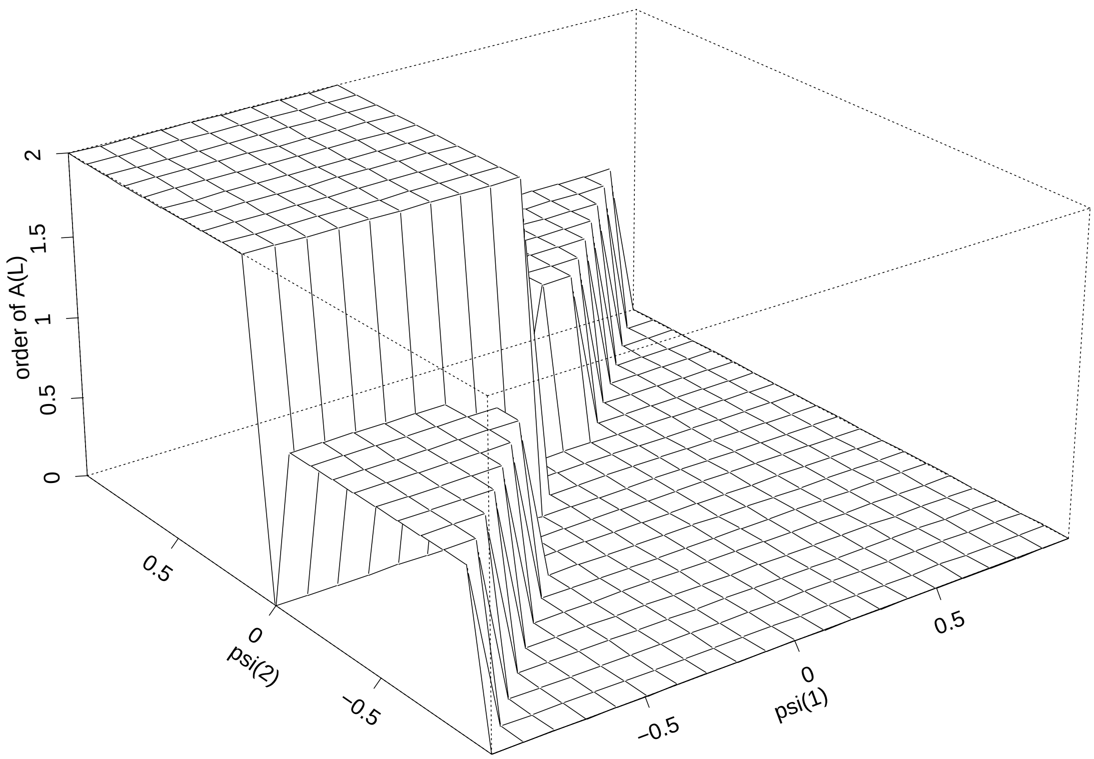


Fig 2: Noncausal Pseudo-Autoregressive Order

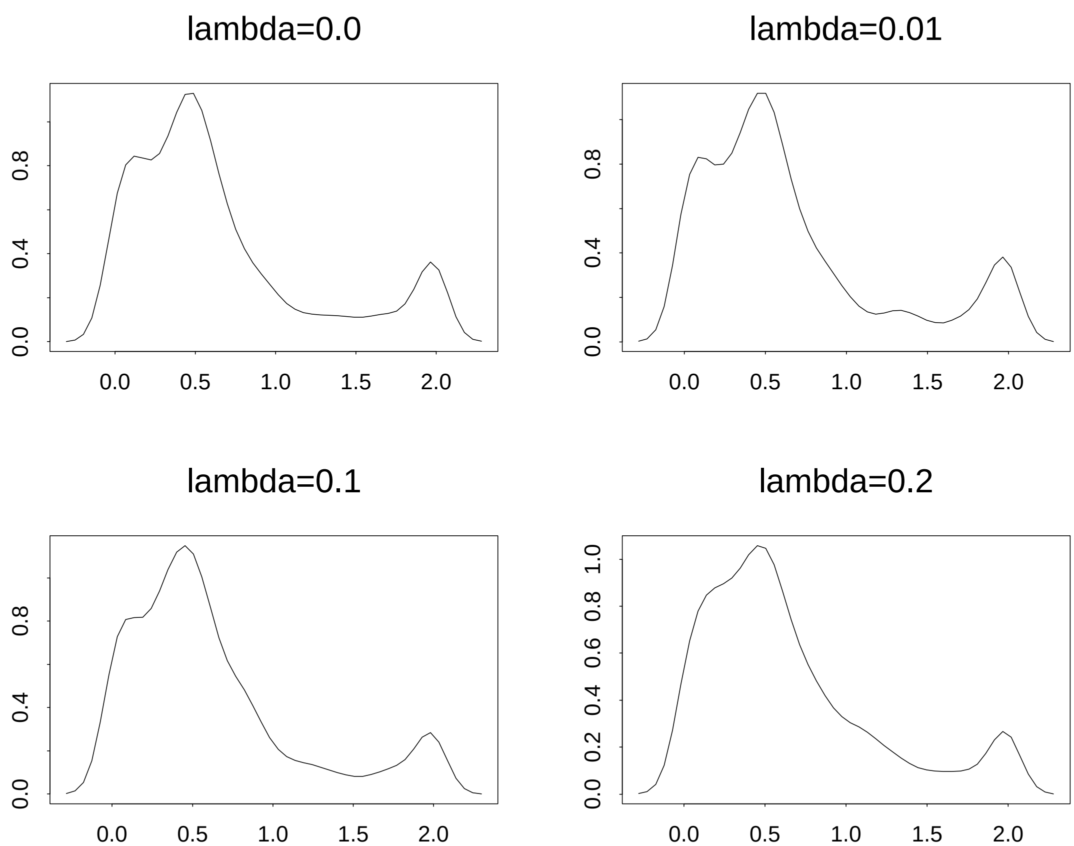


Fig 3: Density of Mix PMLE

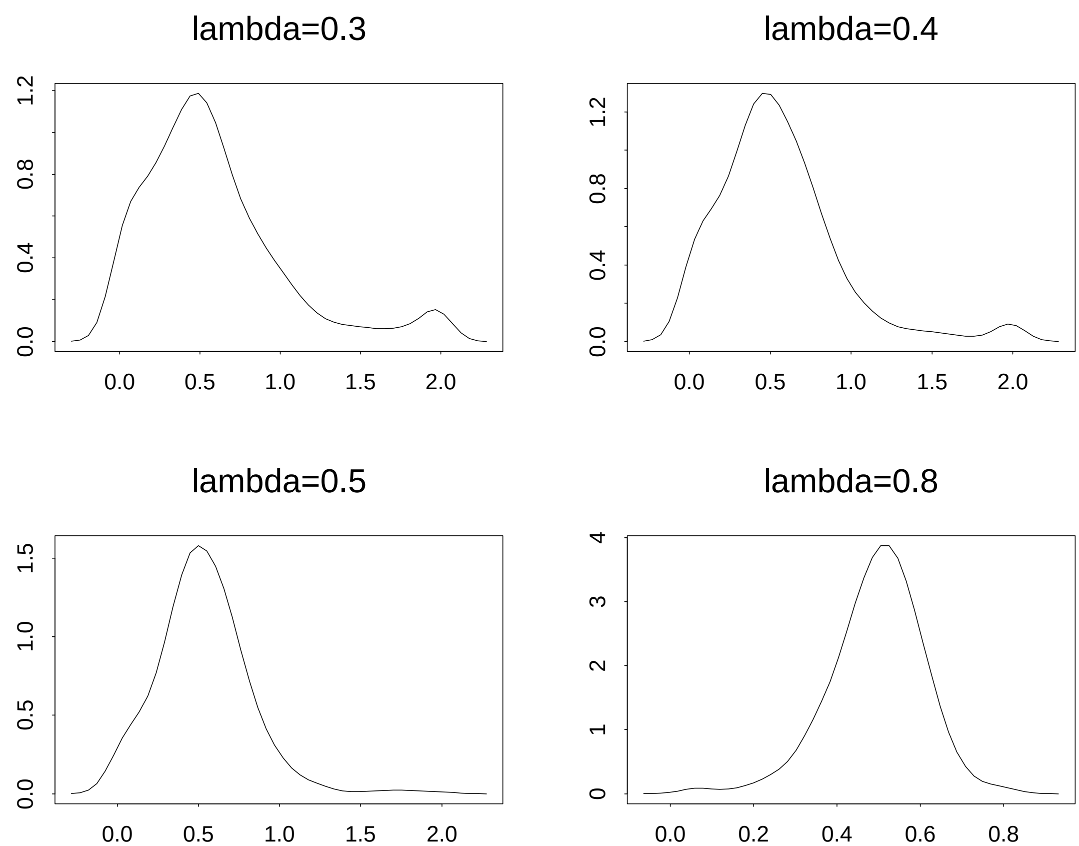


Fig 3 cont: Density of Mix PMLE