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Income Tax and Retirement Schemes

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Income tax and retirement schemes*

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Abstract

This article aims at understanding the interplay between pension schemes and tax instruments. The model features extensive labor supply in a stationary environment with overlapping generations and perfect financial markets.

Compared with the reference case of a pure taxation economy, we find that taxes become more redistributive when the pension instrument is available, while pensions provide incentives to work.

1 Introduction

In the past ten years, following Prescott (2004)'s claim that the differences in work habits in the US and in Europe were largely due to the differences in the tax systems, a number or researchers have estimated labor supply elasticities both at the microeconomic and macroeconomic levels. It seems that an important, perhaps previously neglected, element is the extensive margin and its reaction to financial incentives at the beginning and at the end of the working life. This has led to a number of models with endogenous retirement dates in a life cycle setup, e.g. Prescott, Rogerson, and Wallenius (2009), Rogerson and Wallenius (2009) and Ljunqvist and Sargent (2014). However there is still little work on the interaction between nonlinear taxes and pension schemes. Indeed Diamond (2009) states

> Apart from some simulation studies, theoretical studies of optimal tax design typically contain neither a mandatory pension system nor the

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behavioral dimensions that lie behind justifications commonly offered for mandatory pensions. Conversely, optimizing models of pension design typically do not include annual taxation of labor and capital incomes. Recognizing the presence of two sets of policy institutions raises the issue of whether normative analysis should be done separately or as a single overarching optimization.

To make progress in this direction one has to choose among the many possible formulations of the problem. Diamond and Mirrlees (1978) focus on the health shocks associated with aging and the social insurance features of pension schemes. Gorry and Oberfield (2012) and Michau (2014) introduce both an intensive and an extensive margins, but keep the fixed cost of going to work constant over the lifetime. Cremer, Lozachmeur, and Pestieau (2004), Lozachmeur (2006) and Cremer, Lozachmeur, and Pestieau (2008) study a two period of life overlapping generations model which allows them to discuss political economy aspects of the problem. Also in an overlapping generations with two period lives and intensive labor supply à la Mirrlees, Brett (2012) describes the comparative statics effects of a change of the trend in population growth on the steady state.

Here we consider a deterministic overlapping generations model in continuous time, where all agents have the same length of life. At each date labor supply is extensive, either 0 or 1. There are a finite number of dynasties, that differ by their (deterministic) profiles of productivity and pecuniary cost of going to work, as well as by their instantaneous utility for consumption.[1] The government wants to redistribute lifetime welfare across dynasties. The policy instruments are an age independent nonlinear tax schedule that depends on the current productivity of the workers (the tax system cannot be based on the privately known cost of going to work nor on the unobserved productivity of the non workers) and a pension scheme, with the level of pension depending on various life-time statistics of the worker activity.[2]

We start by studying the fiscal instrument in a setup without institutional pensions. Optimality of the tax system can be characterized as a situation where a redistributive force, holding labor supply constant, and an efficiency force, changing labor supply and production, compensate. This generalizes Laroque (2011). We are able to describe the shape of the optimal after tax income schedule, which often is piecewise constant. We exhibit circumstances under which

[1] Our setup is similar in a number of respects with that used by Rogerson (2011), Shourideh and Troshkin (2012) or Weinzierl (2011): overlapping generations, deterministic trajectories. Here labor supply is extensive rather than intensive à la Mirrlees while saving and/or borrowing are unrestricted.

[2] It should be noted that the framework of this paper is in the Ramsey tradition rather than in the Mirrlees one. We put a priori restrictions on the shape of the government instruments, and do not derive them from assumptions on asymmetric information, contrary to Diamond and Mirrlees (1978), Grochulski and Kocherlakota (2010) or Golosov, Troshkin, and Tsyvinski (2011).

some agents have their labor supply distorted upwards.

Second, as regards pensions, we study simple schemes where pension transfers depend on a single aggregate statistics of the work life: total time spent working over the life cycle, before or after tax lifetime earnings. We relate distortions to the pattern of the agents social weights. When society favors agents with low skills or high costs of work ("redistributive economy"), the most skilled agent has her labor supply undistorted and all other agents have their labor supply distorted downwards.

Next we assess the extent to which taxation and pension regimes are complementary instruments for redistribution. We explain how the redistributive power of a tax schedule is modified in the presence of a pension regime. The tax schedule serves directly to redistribute income towards the socially favored agents, but also indirectly to facilitate the redistribution operated through the pension transfers. We show that by comparison with the case where there are no pensions, the redistributitve force is larger: the tax instrument tends to specialize into redistribution while pensions are used to provide the appropriate incentives to work. Concentrating our attention on the period from middle age to the end of life, we show that an optimal combination of a tax schedule with a pension policy eliminates any rent coming from productivity differences across agents, suppresses all upward labor supply distortions and reduces downward distortions. In the case where there are only two types of agents, at the optimum the two instruments fully specialize: pensions provide the incentives to work, while taxes do all the redistribution.

The article is organized as follows. Section 2 presents the model. Then Section 3 deals with optimal taxation in the absence of pension policy, introducing the efficiency and redistributive forces. Finally Section 4 analyses the interactions between the tax and pension instruments.

2 Model

We consider an economy in continuous time. All agents have the same life length, normalized to one. There are I types of agents. Agent of type i, if she works at age a, produces at most $w_i(a)$ units of a single homogenous good but suffers a pecuniary cost $\delta_i(a)$, measured in units of good. She has a lifetime utility function of the form

$$\int_0^1 u_i[c_i(a)]\,\mathrm{d}a,$$

where u_i is an increasing concave function and $c_i(a)$ denotes consumption at age a.

The type of an agent is thus characterized by a couple of exogenous, non-negative functions $(w_i(\cdot), \delta_i(\cdot))$ defined on $[0, 1]$ and by the instantaneous utility index $u_i(\cdot)$. The pair $(w_i(a), \delta_i(a))$ as the age a varies determines a curve in the

(w, δ)-space, that we call a *trajectory*. We assume that the functions w_i, δ_i, and u_i are differentiable.

At each date t, for each $i = 1, \ldots, I$, the economy contains a continuum of agents of type i of all ages a in $[0, 1]$; overtime the older agents die and are replaced by newborn of the same type. All cohorts are of the same size, with one agent of each type, and the economy is stationary. An *allocation* specifies the nonnegative consumption $c_i(a)$ and the labor supply $\ell_i(a)$ in $\{0, 1\}$ of all types i along their lives.

Furthermore we assume that there are perfect markets for transferring wealth across time, with a zero interest rate. The agents use these markets to smooth their consumption overtime, $c_i(a) = c_i$ independent of age. From now on we restrict our attention to allocations where consumption is constant and equal to its aggregate value over the lifetime c_i.

Feasibility An allocation is *feasible* if and only if total consumption does not exceed total output net of production cost:

$$\sum_{i=1}^{I} c_i \leq \sum_{i=1}^{I} \int_0^1 [w_i(a) - \delta_i(a)]\ell_i(a)\,\mathrm{d}a. \tag{1}$$

An allocation is *efficient* whenever output net of production costs is maximized, i.e. any agent works whenever her opportunity cost of work is lower than or equal to her productivity, $\ell_i(a) = 1$ if $\delta_i(a) < w_i(a)$ and $\ell_i(a) = 0$ if $\delta_i(a) > w_i(a)$.

Utilitarian optimum (First-Best) The utilitarian optimum is the allocation that maximizes $\sum_i u_i(c_i)$ subject to the feasibility constraint (1). It is the feasible efficient allocation such that marginal utilities are equal:

$$u_i'(c_i) = \lambda,$$

for $i = 1, \ldots, I$.

Laissez-faire The agents maximize their lifetime consumption

$$c_i = \int_0^1 \max(0, w_i(a) - \delta_i(a))\,\mathrm{d}a.$$

They decide to work whenever their productivity is larger than their opportunity cost of work, so the laissez-faire equilibrium is efficient. In general, laissez-faire yields an allocation that differs from the utilitarian optimum.[3]

[3]Suppose that $\delta_i(a)$ is a disutility cost instead of a pecuniary one, i.e. agent i, when working, produces $w_i(a)$ and has instantaneous utility $u(c_i(a)) - \delta_i(a)$, while she has instantaneous utility $u(c_i(a))$ when not working. Then agent i works at age a under laissez-faire if and only if

In all the paper we suppose that the utilitarian government observes the employment status of the agents and, when they work, their productivity w. It never observes the pecuniary cost δ, which is private information. The government has access to two policy instruments, an income tax and a retirement scheme.

The government instruments The first policy instrument is a time invariant income tax schedule. The tax schedule is made of a function $R(w)$, the age-independent after-tax income of a worker with before tax wage w, and of a scalar s equal to the subsistence income of the non-workers.

The second instrument is a pension scheme that relates a lifetime statistics Z, to a (possibly negative) government transfer $P(Z)$, which is equal to the present value of all contributions and benefits associated with the retirement plan. In practice, the pension transfer may depend on individual labor histories through many different channels. We consider here three stripped down legislations bearing on retirement, regimes L, W and N, with $Z = L, W$ or N, where the agent is entitled to get $P(Z)$ provided that his life time performance is at least equal to Z

- length of the agent working life, L,

$$\int_{a=0}^{1} \ell_i(a)\,\mathrm{d}a \geq L_i$$

- lifetime gross earnings collected by the agent, W,

$$\int_{0}^{1} w_i(a)\ell_i(a)\,\mathrm{d}a \geq W_i$$

- lifetime after-tax (or net) earnings, N,

$$\int_{0}^{1} R(w_i(a))\ell_i(a)\,\mathrm{d}a \geq N_i.$$

The three regimes that we analyze are far from exhausting the kinds of legislations found in practice. Note in particular that we do not allow the tax schedule to be age dependent, contrary to Weinzierl (2011), nor do we have the financial market imperfections that underlie some of the pension regimes in practice. In a previous version of the paper, we had studied a situation where the tax schedule differed before and after the retirement age (in fact it was fully confiscatory after the retirement age). Our pension regimes can be seen as a restricted way of introducing age-dependent transfers.

$u_i'(c_i)w_i(a) > \delta_i(a)$, where c_i is her constant, instantaneous consumption level. Hence, this specification entails an *income effect* in labor supply: participation decreases with c_i. Using Pareto-optimality conditions, it can be checked that laissez-faire is efficient. The pecuniary model adopted in this paper avoids these complications.

Second Best Program Facing the tax schedule $(R(\cdot), s)$ and a pension regime associated with transfers $P(\cdot)$, the consumer chooses her labor supply $\ell(a)$, and pension level Z so as to maximize her lifetime utility, i.e.

$$c_i = \max_{\ell, Z} s + \int_0^1 [R(w(a)) - \delta_i(a) - s]\ell(a)\,\mathrm{d}a + P(Z)$$

where $\ell(a)$ belongs to $\{0, 1\}$, and Z is linked to the agent work history depending on the regime.

Feasibility then can be written in two equivalent ways, either as a balanced government budget:

$$\sum_{i=1}^{I} \left\{ \int_0^1 [w_i(a) - R(w_i(a)) + s]\ell_i(a)\,\mathrm{d}a - s - P(Z_i) \right\} \geq 0, \tag{2}$$

or as the equality of aggregate production and aggregate consumption:

$$\sum_{i=1}^{I} \left\{ \int_0^1 [w_i(a) - \delta_i(a)]\ell_i(a)\,\mathrm{d}a - c_i \right\} \geq 0. \tag{3}$$

The second best program, maximizing the sum of utilities under the above constraints, looks formidable. In the next section, we first study the optimal income tax in the absence of retirement schemes, before considering the interaction between taxes and pensions in the remainder of the paper.

3 Only income taxes

When an agent has productivity w at some date, her financial incentive to work is equal to $R(w) - s$, which is to be compared with the opportunity cost of working δ. It is useful to represent the financial incentive to work in the same plan as the individual trajectories $(w(a), \delta(a))$. Hereafter the *incentive schedule* is the curve $(w, R(w) - s)$ as productivity varies. An agent works in regions where her trajectory is located below the incentive schedule, i.e. her opportunity cost of work δ is smaller than the financial incentive to work $R(w) - s$. Her work status changes at points where her trajectory crosses the incentive schedule.

3.1 General framework

Assuming that the agents can choose occupations requiring skills below their own ability, no one would choose an occupation whose required productivity belongs to a decreasing part of the function R, preferring to produce less and to earn a higher after-tax income. Formally, we can replace any function R with $\tilde{R}(w) = \max_{w' \leq w} R(w')$. It follows that, without loss of generality, we limit our

attention to functions R that are nondecreasing and assume that workers work at full productivity. Lifetime consumption of agent i is therefore given by

$$c_i^o = s + \int_0^1 [R(w_i(a)) - \delta_i(a) - s]\ell_i^o(a)\ \mathrm{d}a, \tag{4}$$

with

$$\ell_i^o(a) = \mathbb{1}_{R(w_i(a))-\delta_i(a)-s\geq 0}.$$

The superscript o, for 'only tax', indicates the absence of a pension scheme. For notational simplicity, we do not mention the policy instruments R and s in the arguments of the labor supply functions ℓ_i^o. The Lagrangian of the problem in the absence of a pension scheme reduces to

$$\mathcal{L}^o = \sum_{i=1}^{I} u_i(c_i^o) + \lambda \sum_{i=1}^{I} \left[Y_i(\ell_i^o) - c_i^o\right],$$

where λ is the multiplier of the government budget and $Y_i(\ell_i^o)$ is agent i's lifetime net output:

$$Y_i(\ell_i^o) = \int_{a=0}^{1} [w_i(a) - \delta_i(a)]\ell_i^o(a)\,\mathrm{d}a.$$

The problem is to find the tax instruments $(R(\cdot), s)$ which maximizes the Lagrangian $\mathcal{L}^o$ subject to the constraint that $R(\cdot)$ be nondecreasing. An equal translation of $R(\cdot)$ and of the subsistence income s, which does not alter labor supply, yields the first order necessary condition: $\sum_i u_i'(c_i^o) = I\lambda$.

The Lagrangian depends on the tax schedule through two channels: consumption levels c_i and labor supplies ℓ_i. Hereafter, we label "redistribution force" and "efficiency force" the effect of R through these respective channels. The first force is present at all productivity levels while the second is active only at points w where an agent is indifferent between working and not working. Formally we compute the Frechet-derivatives of the Lagrangian of the government problem, seen as a functional that maps the set of functions R into $\mathbb{R}$. To this aim, we evaluate the Lagrangian at a slightly perturbed function $R + \varepsilon h$, compute the ratio $[\mathcal{L}^o(R + \varepsilon h) - \mathcal{L}^o(R)]/\varepsilon$, and let ε tend to zero. A mathematical derivation of the limit can be found in Appendix A. Here we present a heuristic approach of the differentiation.

The redistribution force This force comes from the dependence of lifetime consumptions on the after-tax schedule. Suppose we replace the after-tax income R with $R + \mathrm{d}R$ on the interval $[w, w + \mathrm{d}w]$, with $\mathrm{d}w > 0$. This change in after-tax income translates into a change in consumption for the agents who work at productivity levels in $[w, w + \mathrm{d}w]$. The change in agent i's lifetime consumption is given by

$$\mathrm{d}c_i^o = \mathrm{d}T_i(w; \ell_i^o)\,\mathrm{d}R, \tag{5}$$

where $T_i(w;\ell_i)$ denotes the time spent by agent i with worktime profile ℓ_i working in a productivity lower than or equal to w

$$T_i(w;\ell_i) = \int_0^1 \ell_i(a)\, \mathbb{1}_{w_i(a)\leq w}\, \mathrm{d}a, \tag{6}$$

and, accordingly, its derivative $\mathrm{d}T_i(w;\ell_i)$ represents the time spent by agent i working in a productivity between w and $w + \mathrm{d}w$.

By construction, $T_i(w;\ell_i)$ is a nondecreasing function of w. The limit of $T_i(w;\ell_i)$ as w goes to infinity is the total time agent i works over her life cycle, hereafter denoted L_i.

The derivative of $T_i(w;\ell_i)$ with respect to w, $\mathrm{d}T_i(w;\ell_i)$, is a positive measure which is almost everywhere continuous, possibly having mass points at productivity levels where agent i spends non-infinitesimal periods of time. If we think of agent i's productivity when she works as a random variable, the probability measure $\mathrm{d}T_i(w;\ell_i)/L_i$ can be thought of as the distribution of that random variable. Suppose agent i's trajectory crosses the incentive schedule from below at w_0, i.e. the agent works for $w \leq w_0$ and does not work for $w \geq w_0$ along the trajectory. Then $\mathrm{d}T_i$ has a downward discontinuity at w_0, T_i has a concave kink at w_0. If the trajectory crosses the schedule from above, then the kink of T_i is convex.

By the chain rule, the variation of the Lagrangian coming from the changes in lifetime consumptions is given by $\mathrm{d}\mathcal{L}^o = \mathrm{d}\Phi^o(w;\ell^o)$, where $\Phi^o(w)$ is the social marginal utility of income (net of the cost of public funds) for workers with productivity below w:

$$\Phi^o(w;\ell^o) = \sum_{i=1}^{I} [u_i'(c_i^o) - \lambda]\, T_i(w;\ell_i^o). \tag{7}$$

The term $\mathrm{d}\Phi^o(w;\ell^o)$ reflects the redistributive force. Redistribution induces the government to raise (lower) after-tax income in regions where $\mathrm{d}\Phi^o(w;\ell^o) > 0$ ($\mathrm{d}\Phi^o(w;\ell^o) < 0$). The observation that λ is the average of marginal utilities yields the following result.

Lemma 1. *The net social marginal utility of income of workers with productivity below w, $\Phi^o(w;\ell^o)$, has the same sign as the correlation between marginal utilities $u_i'(c_i^o)$ and working times $T_i(w;\ell_i^o)$.*

Labor supply elasticity A change in the tax schedule may also affect labor supply. We say that there is *indifference* at w if there exists an agent i, having productivity w at some age a_i, $w = w_i(a_i)$, who is indifferent between working and not working at this age, i.e. $R(w) - \delta_i(a_i) = s$. A *switch point* is an indifference point such that the work status of the indifferent agent changes in a neighborhood of w, i.e. the trajectory of agent i crosses the incentive schedule at w. When the

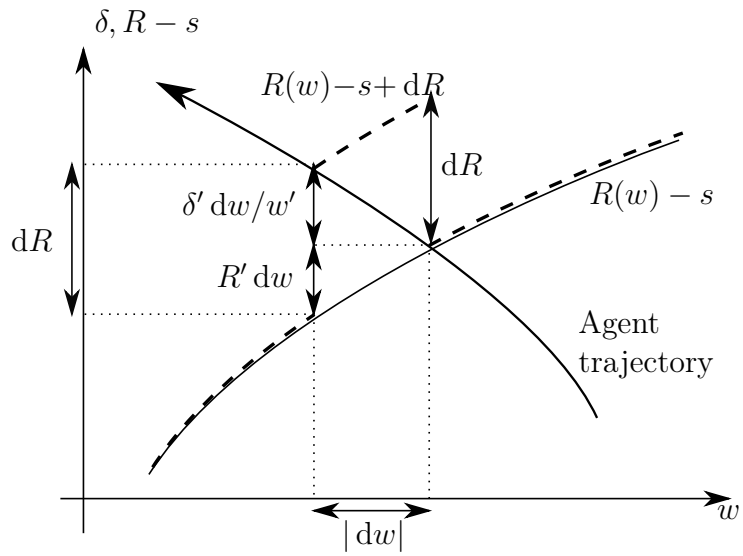


Figure 1: Labor supply elasticity. Original (perturbed) schedule: solid (dashed) line.

slopes of the tax schedule and of the trajectory are different, $R'(w) \neq \delta_i'/w_i'$, the quantity

$$\eta_i^o(w; R) = \frac{1}{|\delta_i'(a_i) - R'(w) w_i'(a_i)|} \tag{8}$$

is positive and finite.

Consider a switch point w and replace R with $R + \mathrm{d}R$ on the interval $[w, w + \mathrm{d}w]$, with $\mathrm{d}R = (\delta_i'/w_i' - R') \, \mathrm{d}w$, as shown on Figure 1. (In the represented example, the trajectory is decreasing in the (w, δ)-space; specifically the agent's productivity and cost of work respectively decline and rise with age.) The perturbation changes the status of the agent on the interval from working to non working. The time spent in the interval is

$$\mathrm{d}a = \mathrm{d}w/|w_i'| = \eta_i^o \, \mathrm{d}R,$$

hence η_i^o is the derivative of labor supply with respect to the tax schedule R. When $R - s$ increases by one percent, the time agent i spends working at a productivity below w is increased by $\varepsilon_i^o(w; R)$ percent, where $\varepsilon_i^o(w; R)$ denotes the elasticity of agent i's labor supply, $T_i(w; \ell_i^o)$, with respect to financial incentives to work:

$$\varepsilon_i^o(w; R) = \frac{R(w) - s}{T_i(w; \ell_i^o)} \, \eta_i^o(w; R). \tag{9}$$

The labor supply elasticity depends on both the gradient of the trajectory and the slope of the after-tax schedule at the switch point. In particular, the steeper

the tax schedule at w, the lower the elasticity, because the agent spends less time in the region affected by the perturbation.

The above formula is readily adapted if agent i's trajectory crosses the tax schedule more than once. Formally, the Frechet-derivative of $T_i(w;\ell_i^o)$ with respect to the tax schedule R is a positive measure made of mass points at agent i's switch points below w, see equation (43) in Appendix A.2. Similarly, the elasticity of the aggregate labor supply, $T(w;\ell^o) = \sum_{i=1}^{I} T_i(w;\ell_i^o)$, is given by

$$\varepsilon^o(w;R) = \frac{R(w)-s}{T(w;\ell^o)} \sum_{i\in\mathcal{I}(w)} \eta_i^o(w;R), \tag{10}$$

where $\mathcal{I}(w)$ is the set of agents who switch at w.

The efficiency force A marginal change of the incentives to work, $\mathrm{d}(R-s)$, on a small interval around a switch point w of agent i has only a second-order effect on her permanent income because she is indifferent between working and not working at this point. Such a change, however, affects the net output she produces over her life cycle:

$$\mathrm{d}Y_i(\ell_i^o) = [w-\delta]\,\eta_i^o(w;R)\,\mathrm{d}R, \tag{11}$$

where $\delta = R(w) - s$ is agent i's cost of work at the switch point. At the same time, the change affects the government revenue. For instance, if $\mathrm{d}R > 0$ and $w > R - s$, the variation of the tax schedule induces the agent to switch from not working to working on a short period of her life, which raises the government revenue. We define the efficiency force as

$$\mathrm{d}\Psi^o(w;\ell^o) = \lambda \sum_i \mathrm{d}Y_i(\ell_i^o).$$

We show formally in Appendix A that this force is a discrete measure concentrated on the set of all switch points

$$\Psi^o(w;\ell^o) = \lambda \sum_{\sigma\in\mathcal{S}} [w_\sigma - R(w_\sigma) + s]\,\eta_i^o(w_\sigma;R)\mathbb{1}_{w_\sigma\le w}, \tag{12}$$

where $\mathcal{S}$ is the set of all agents' switch points, w_σ is the productivity level at σ, and $\varepsilon^o(w;R)$ is the total labor supply elasticity given by (10). The previous analysis is summarized in the following proposition.

Proposition 1. *In the absence of a pension scheme, the Lagrangian $\mathcal{L}^o$ is differentiable at any point $(w, R(w)-s)$ where no trajectory is tangent to the incentive schedule. Its derivative can be written as the sum*

$$\mathrm{d}\mathcal{L}^o = \mathrm{d}\Phi^o(w;\ell^o) + \mathrm{d}\Psi^o(w;\ell^o), \tag{13}$$

where the almost everywhere continuous measure $\mathrm{d}\Phi^o(w;\ell^o)$ given by (7) and the discrete measure $\mathrm{d}\Psi^o(w;\ell^o)$ given by (12) represent the redistribution and efficiency forces.

Raising R at an indifference point increases labor supply, which alleviates the government budget constraint if $w > R - s$ and makes it more stringent if $w < R - s$. Hence, income maximizing pushes the government to raise (lower) after-tax income in regions where $w > R - s$ ($w < R - s$). This force translates into mass points in the derivative of the Lagrangian or even into discontinuity points in the Lagrangian function.

On the other hand, the redistributive force, expressed in the term (7), is absolutely continuous (except at productivity levels where some workers spend a finite time): the redistributive effect of an increase in the after-tax income on an interval of productivities is the integral on the interval of the net social marginal utility of income $\mathrm{d}\Phi^o$.

The above analysis allows to concentrate attention on a particular class of tax schedules (see Appendix B for a formal proof).

Proposition 2. *The second-best optimum may be achieved with an incentive schedule that is piecewise either constant or coincident with an increasing trajectory.*

When the tax schedule coincides with an increasing trajectory, the government faces a particularly strong efficiency force.[4] Otherwise the monotonicity constraint binds. Putting the signs of $\mathrm{d}\Phi^o(w; l^o)$ and $\mathrm{d}\Psi^o(w; l^o)$ on the diagram of trajectories allows to qualitatively separate intervals of productivities where the redistribution and efficiency forces tend to push R up from those where these forces are downwards.

Since we expect bunching to be the norm, it is worthwhile to spell out the form of first order conditions under bunching. Consider a bunching interval $[w_0, w_1]$. We can raise or lower R on the whole bunching interval, raise it on right subintervals $[w, w_1]$, and lower it on left subintervals $[w_0, w]$. None of these variations should increase the Lagrangian, which yields the first-order conditions:

$$\mathrm{d}\mathcal{L}^o([w, w_1]) \leq 0, \tag{14}$$

for all w in the interval, with equality for $w = w_0$. This implies in particular that $\mathrm{d}\mathcal{L}^o$ is nonnegative at w_0 and non-positive at w_1.

3.2 An example: two types and decreasing trajectories

To put in practice the previous analysis, we consider a simple environment with two types of agents, a high type H and a low type L, endowed with the same utility function u, with $w_H(a) > w_L(a)$ and $\delta_H(a) < \delta_L(a)$ for all a in $[0, 1]$. Economically we focus on the second parts of lives, limiting our attention to decreasing trajectories generated by δ functions which are increasing with age,

[4] See the last paragraph of Appendix B.

while the wage functions are decreasing with age. We also suppose that there is a natural retirement age: the trajectories intersect the 45 degree line.

This set of assumptions is consistent with many different patterns. If the agents' productivities are very close while their opportunity costs of work are very different, agent L's trajectory is above agent H's in the (w, δ) plan, see Figure 3. In the opposite case, agent H trajectory lies at the right of that of agent L, see Figure 4. The trajectories may very well cross, possibly many times, meaning that the same characteristics (productivity, cost) are reached by the two agents at different ages. Formally, the following properties hold.

Assumption 3.1 (Decreasing trajectories)**.** *The two agents have the same utility functions u. Their productivities, $w_H(a) > w_L(a)$, decrease with age and their pecuniary costs of work, $\delta_H(a) < \delta_L(a)$, increase with age. There exist ages a_L^* and a_H^* in $(0, 1)$ such that $w_L(a_L^*) = \delta_L(a_L^*)$ and $w_H(a_H^*) = \delta_H(a_H^*)$.*

The fact that type H dominates pointwise type L implies that its consumption and welfare are at least as large, whatever the tax schedule, $c_H \geq c_L$. Any non-decreasing tax schedule crosses each trajectory only once, respectively at ages a_H and a_L, $a_H \geq a_L$, with associated wages $w_H(a_H)$ and $w_L(a_L)$ and opportunity costs of work $\delta_H(a_H)$ and $\delta_L(a_L)$. The wages $w_H(a_H)$ and $w_L(a_L)$ represent the lowest productivities at which the agents work. The following proposition provides the list of all possible configurations at the second-best optimum. Then we present two examples that illustrate how unobserved heterogeneity affects the labor supply distortions.

Proposition 3. *Under Assumption 3.1, the following properties hold:*

(i) There exists an optimal tax schedule with at most two values;

(ii) Agent H has her labor supply distorted downwards;

(iii) Agent L labor supply can be distorted in any direction or undistorted;

(iv) Agent H retires later and enjoys higher lifetime consumption than agent L:

$$c_H - c_L = \int_0^{a_L} [\delta_L(a) - \delta_H(a)] \, \mathrm{d}a + \int_{a_L}^{a_H} [R(w_H(a)) - s - \delta_H(a)] \, \mathrm{d}a > 0. \tag{15}$$

Proof. The inequality $a_H > a_L$ follows from the ordering and the monotonicity of the functions $R(w_i(a)) - s - \delta_i(a)$, $i = H, L$. This inequality, in turn, implies $c_H > c_L$ and $u'(c_H) < u'(c_L)$.

Consider any productivity threshold w greater than $\bar{w} = \max(w_H(a_H), w_L(a_L))$. Only the redistribution force is present above w and its integral over the set $[w, w_H(0)]$,

$$(u'(c_H) - \lambda)(T_H(w; l^o) - T_H(w_H(0); l^o) + (u'(c_L) - \lambda)(T_L(w; l^o) - T_L(w_L(0); l^o),$$

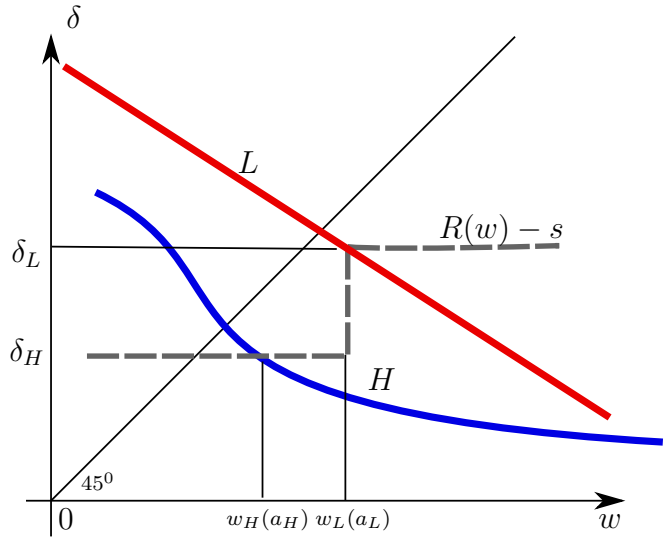


Figure 2: Agent H works at low productivities (case (a) of Proposition 3)

is strictly negative because $u'(c_H) < u'(c_L)$ and agent H spends strictly more time working in that region than agent L, $w_H^{-1}(w) > w_L^{-1}(w)$. The function $r(w) = \min(0, R(\bar{w}) - R(w))$ is non-increasing, and is an admissible variation of the tax schedule as $R + \varepsilon r$ is nondecreasing for small values of ε. We must therefore have:

$$\begin{aligned} < \mathrm{d}\mathcal{L}^o, r > &= \int_{\bar{w}}^{w_H(0)} r(w)\, \mathrm{d}\Phi^o(w; l^o) \\ &= \int_{\bar{w}}^{w_H(0)} r'(w) \left[\Phi^o(w_H(0); l^o) - \Phi^o(w; l^o)\right] \mathrm{d}w \leq 0. \end{aligned}$$

Since the bracketed term in the above inequality is negative and $r' \leq 0$, we get $r' = 0$ and thus $R(\bar{w}) = R(w)$. The after-tax schedule, therefore, is flat above $\max(w_H(a_H), w_L(a_L))$.

We now consider three cases in turn:

(a) Agent H is the only one working at low productivities, $w_H(a_H) < w_L(a_L)$;

(b) Agent L is the only one working at low productivities, $w_L(a_L) < w_H(a_H)$;

(c) Both agents work at low productivities, $w_H(a_H) = w_L(a_L)$.

In case (a), only agent H works at productivities lying between $w_H(a_H)$ and $w_L(a_L)$. Since $u'(c_L) > u'(c_H)$, the redistribution force in that interval pushes downwards, so the financial incentive to work $R - s$ equals $\delta_H(a_H)$ in that interval. From the bunching conditions, it must be the case that the efficiency force is

active and upwards at $w_H(a_H)$, hence $w_H(a_H) > \delta_H(a_H)$: agent H's labor supply is distorted downwards.

To examine agent L's labor supply, we suppose first the tax schedule is continuous at $w_L(a_L)$, i.e. $\delta_L(a_L) = \delta_H(a_H)$. In this case, the after-tax schedule takes only one value for $w \geq w_H(a_H)$, namely the common value of $\delta_L(a_L) = \delta_H(a_H)$, and agent L's labor supply is distorted downwards, because $w_L(a_L) - \delta_L(a_L)$ is larger than $w_H(a_H) - \delta_H(a_H) > 0$. We consider now the case where the schedule is discontinuous at $w_L(a_L)$. We know that $R - s$ is flat and equal to $\delta_L(a_L)$ above that point. We can therefore consider a transformation that pushes $R - s$ down from $\delta_L(a_L)$ to $\delta_H(a_H)$ just above $w(a_L)$. The redistribution and efficiency effects of the transformation respectively bear on type H and type L. The former is positive and of the sign of $(u'_H - \lambda)(\delta_L - \delta_H)$ since it takes $\delta_L - \delta_H$ from type H while leaving L's lifetime consumption level unaffected. The latter is of the sign of $-(w(a_L) - \delta_L)$. For them to sum up to zero, we must have $w(a_L) > \delta_L(a_L)$: agent L's labor supply, again, is distorted downwards.

The analysis of cases (b) and (c) proceeds in the same way and is relegated to Appendix C. Collecting the results obtained in the three cases, we directly get parts *(i)*, *(ii)*, and *(iii)* of the proposition. We have also seen that $a_H > a_L$ and can therefore compute

$$c_H - c_L = \int_0^{a_H} [R(w_H(a)) - s - \delta_H(a)] \, \mathrm{d}a - \int_0^{a_L} [R(w_L(a)) - s - \delta_L(a)] \, \mathrm{d}a.$$

In each of three cases studied above, the tax schedule is flat over $[w_L(a_L), w_H(0)]$, and hence $R(w_L(a)) = R(w_H(a))$ for all $a \leq a_L$, which yields (15). □

We will see in the next section how pension regimes manage to reduce the difference $c_H - c_L$, thus helping to redistribute. Before turning to this central issue, we illustrate the impact of the heterogeneity on labor supply distortions, see point (ii) and (iii) of Proposition 3. We use two examples where the agents differ only in one dimension, either productivity or opportunity cost of work. These examples, therefore, are at the limit of what is permitted by Assumption 3.1. Again, we focus on the second part of the agents' lives where productivity decreases and opportunity cost of work increases, as in the solid lines of the left panels of Figures 3 and 4.

Example 1 (Same productivities, different opportunity costs of work)**.** *In addition to Assumption 3.1, suppose that the agents are equally productive, $w_H(a) = w_L(a)$ for all a, while agent H has a lower pecuniary cost of work: $\delta_H(a) < \delta_L(a)$ for all a. Then at the optimum, both agents have their labor supply distorted downwards.*

In Example 1, we must be in case (a) of the proof of Proposition 3. Moreover the configuration with $\delta_L(a_L) = \delta_H(a_L)$ is not possible here. Indeed, as the two

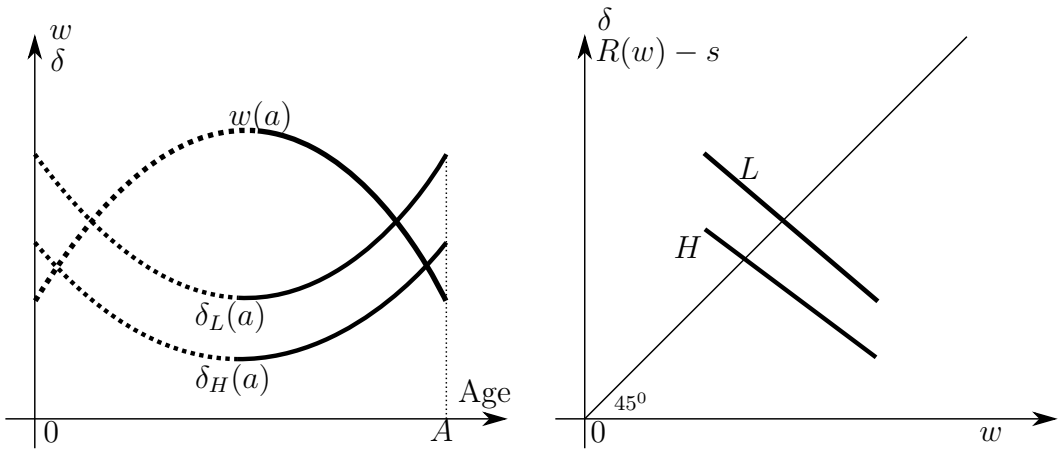


Figure 3: Two types, same productivities

agents would work exactly the same time at productivities above any threshold w, an increase of the tax schedule above $w(a_L) - \varepsilon$ for a small $\varepsilon > 0$ would have no redistributive effect and a positive efficiency effect at $w(a_L)$ –a contradiction. The optimal schedule is therefore discontinuous at $w(a_L)$ as shown on the left panel of Figure 5.

Example 2 (Different productivities, same opportunity costs of work)**.** *In addition to Assumption 3.1, suppose that the agents have the same pecuniary costs of work, $\delta_H(a) = \delta_L(a)$ for all a, while agent H is more productive: $w_L(a) < w_H(a)$. Then at the optimum, agent L has her labor supply distorted upwards.*

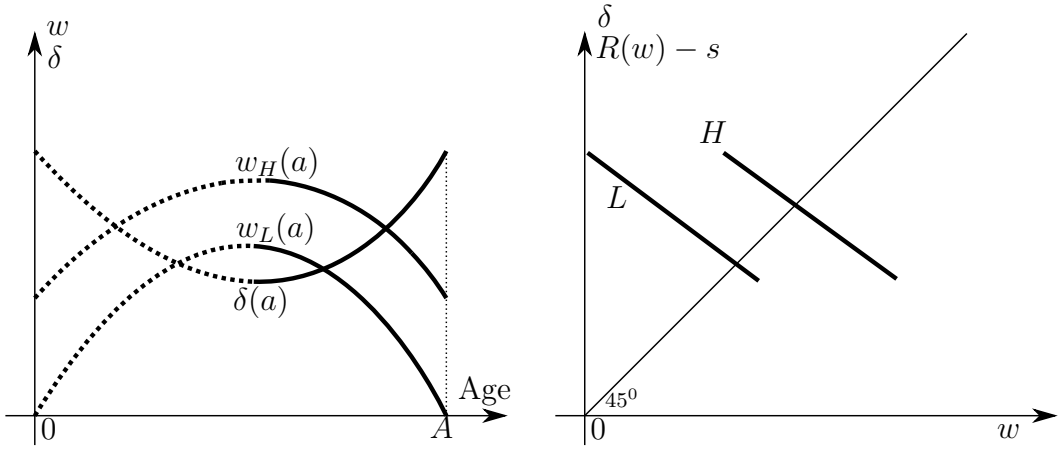


Figure 4: Two types, same opportunity costs of work

In Example 2, we must be in case (b) of the proof of Proposition 3. Moreover the configuration where the tax schedule is flat is not possible here. Indeed, the equality $\delta_L(a_L) = \delta_H(a_L)$ would imply $a_L = a_H$, meaning that the two agents

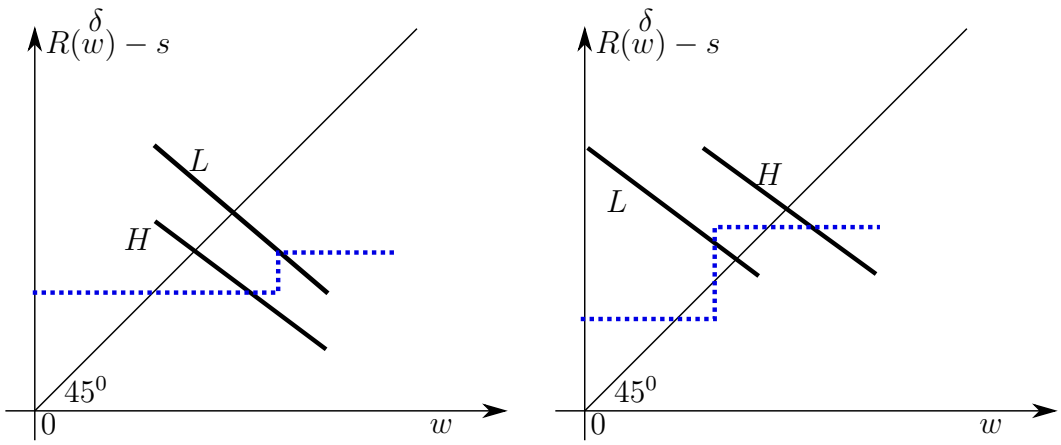


Figure 5: The optima with same productivities (left), same opportunity costs of work (right)

would have the same total working time: a uniform increase of $R - s$ would thus have no redistributive effect and a positive efficiency effect at $w_H(a_H)$ – a contradiction. The optimal schedule is therefore discontinuous at $w_L(a_L)$ as shown on the right panel of Figure 5.

In Example 1, the heterogeneity primarily comes from the opportunity cost of work while in Example 2 it comes from the productivity. In both cases, the government cannot implement the first-best in these two-type economies. The direction of the distortions, however, is sensitive to the source of the heterogeneity.

4 Retirement schemes

To redistribute welfare across types, we now allow the government to use a retirement scheme on top of income taxation. We shall limit ourselves to one of the three retirement schemes Z in $\{L, N, W\}$ presented page 5. The presence of such a scheme makes the subsistence income s superfluous: increasing pension transfers by s, $P(Z)$ becoming $P(Z) + s$, and simultaneously increasing taxes by s, $R(w)$ becoming $R(w) - s$, leaves the second best program of page 6 unchanged. Hereafter we set $s = 0$. Agent i lifetime consumption is the sum of lifetime earnings and pension wealth

$$c_i = \int_0^1 [R(w_i(a)) - \delta_i(a)]\ell_i(a)\,\mathrm{d}a + P(Z_i).$$

It is useful to introduce the optimal lifetime earnings, the function $\gamma_i(Z_i; R)$ which is the maximum of

$$\int_0^1 [R(w_i(a)) - \delta_i(a)]\ell_i(a)\,\mathrm{d}a \tag{16}$$

over $\ell_i(.)$, subject to the relevant pension constraint, namely

- under regime L: $L_i \leq \int_0^1 \ell_i(a)\,\mathrm{d}a$;
- under regime W: $W_i \leq \int_0^1 w_i(a)\ell_i(a)\,\mathrm{d}a$;
- under regime N: $N_i \leq \int_0^1 R(w_i(a))\ell_i(a)\,\mathrm{d}a$.

Agent i lifetime consumption is then

$$c_i = \gamma_i(Z_i; R) + P(Z_i). \tag{17}$$

To make sure that the level of pension Z_i is chosen by agent i, it is standard to replace the maximization with respect to Z with a set of incentive constraints. Let $(\mathrm{IC})_{i,j}$ denote the constraint that agent i does not strictly prefer agent j's allocation

$$P(Z_i) - P(Z_j) \geq \gamma_i(Z_j; R) - \gamma_i(Z_i; R) \Longleftrightarrow c_i \geq c_j - \gamma_j(Z_j; R) + \gamma_i(Z_j; R). \tag{18}$$

Under a given pension regime, the government selects the retirement scheme $P(Z)$ and the tax schedule $R(.)$ that maximize its utilitarian objective

$$\max \sum_i u_i(c_i) = \sum_i u_i(\gamma_i(Z_i; R) + P(Z_i))$$

subject to the feasibility constraint (2) or (3) and the family of incentive constraints (18).

Our aim is to understand how the retirement schemes interact with income tax. We proceed as follows. In Section 4.1, we explain how the agents' lifetime consumption and labor supply depend on the tax schedule R and the pension requirement Z. Section 4.2 exhibits conditions under which the single crossing property holds and attention can thus be restricted to local incentive constraints. In Section 4.3, we link the pattern of active binding incentive constraints to the shape of social weights. In Section 4.4, we look for the optimal pension requirements Z_i and derive a necessary condition on the agents' labor supply at the optimum. Section 4.5 presents necessary conditions for the optimal tax schedule. Section 4.6 shows that the three considered pension schemes are equivalent when the agent's trajectories are decreasing and spell out how pensions allow to improve upon the second best optimum of Section 3 where the only available instrument is income tax.

4.1 Properties of labor supply and consumption demand

The presence of a retirement scheme modifies the agents' behaviors. For a pension requirement Z, letting $z(w)$ respectively equal to 1, w or $R(w)$ in regimes L, W and N, the constraint associated with the retirement scheme takes the form[5]

$$\int_0^1 z(w(a))\ell^z(a)\,\mathrm{d}a \geq Z. \tag{19}$$

The behavioral impact of the retirement scheme goes through this constraint, and the associated nonnegative Lagrange multiplier π. Agent i's labor supply depends directly on the tax schedule R and indirectly, through the multiplier π_i, on the pension requirement Z_i. More precisely, when making her labor supply decision, agent i takes into consideration the adjusted tax schedule $R(w)+\pi_i z(w)$. The first component $R(w)$ represents the instantaneous after-tax income while the second term $\pi_i z(w)$ represents the (deferred) pension benefit associated with before-tax earning w. The multiplier π_i can therefore be thought of as an implicit conversion rate between after-tax earnings and pension benefits. Formally labor supply is given by[6]

$$\ell_i^z(a) = \mathbb{1}_{R(w_i(a))+\pi_i z(w_i(a))-\delta_i(a)\geq 0}. \tag{20}$$

We analyse how it depends on the after-tax schedule and on the pension requirements. A change of the after-tax schedule around a switch point affects labor supply through two channels.

First, it affects the time spent working around the switch point in the same way as in Section 3. To get the expression of this direct effect, we replace the after-tax schedule $R(w)$ in (8) with the adjusted schedule $R(w) + \pi_i z(w)$:

$$\eta_i^z(w; R, \pi_i) = \frac{1}{|\delta_i' - R'(w)w_i' - \pi_i z'(w)w_i'|}, \tag{21}$$

which yields the labor supply elasticity: $\varepsilon_i^z = \eta_i^z R/T_i$. The static elasticity under regime L is the same as in the absence of pension because the derivative z' is identically zero in that case. For decreasing trajectories, we have

$$\eta_i^z(w; R, \pi_i) \geq \eta_i^o(w; R)$$

because the adjusted after-tax schedule $R(w)+\pi_i z(w)$ is steeper than the original schedule $R(w)$ (recall the qualitative analysis below Figure 1).

Second, when the pension constraint (19) is binding, the change in the after-tax schedule affects the multipliers π_i. Increasing R around a switch point translates into a decrease of π_i, i.e. less pressure placed by the pension scheme on

[5] We use the superscript z to differentiate the behavioral functions according to the regimes, ℓ, w or n.

[6] For notational simplicity, we do not mention R and Z_i in the arguments of ℓ_i^z.

the agent's labor supply. This feedback effect of the tax schedule on the pension multipliers, in turn, affects labor supply in a nonlocal way through (20). By "nonlocal", we mean that a change of R around a particular switch point alters labor supply around *all* switch points of agent i.

The following lemma describes the effect of the tax schedule R, but also of the pension requirements Z_i, on labor supply behavior.

Lemma 2 (Labor supply)**.** *If agent i switches work status at w, a marginal increase $\mathrm{d}R$ of after-tax income $R(w)$ on $[w, w + \mathrm{d}w]$*

- *directly increases her labor supply around w by $\eta_i^z \,\mathrm{d}R$, where η_i^z is given by (21);*
- *indirectly decreases, if the pension constraint (19) is binding, the time spent working in the neighborhood of any switch point w'*

$$-\frac{\eta_i^z(w)\eta_i^z(w')z(w)z(w')}{\sum_{\sigma\in\mathcal{S}_i} z^2(w_\sigma)\eta_i^z(w_\sigma)}\,\mathrm{d}R, \tag{22}$$

  where $\mathcal{S}_i$ is the set of all switch points of agent i.

When w is not a switch point of agent i and the pension constraint is binding, the same perturbation has no effect on her labor supply in regimes L and W and has a second-order, indirect effect on labor supply around all switch points w'

$$-\frac{\eta_i^z(w')R(w')}{\sum_{\sigma\in\mathcal{S}_i} R^2(w_\sigma)\eta_i^z(w_\sigma)}\,\mathrm{d}T_i(w;\ell_i^z)\,\mathrm{d}R \tag{23}$$

in regime N.

Finally, when the pension constraint is binding, a marginal increase $\mathrm{d}Z_i$ of the pension requirement increases the time agent i spends working in the neighborhood of all switch points w' by

$$\frac{\eta_i^z(w')z(w')}{\sum_{\sigma\in\mathcal{S}_i} z^2(w_\sigma)\eta_i^z(w_\sigma)}\,\mathrm{d}Z. \tag{24}$$

Proof. In appendix D, we derive the expression for the feedback effect of the after-tax schedule on the pension multipliers. Specifically, we show that when the pension constraint (19) is binding, a marginal increase $\mathrm{d}R$ of after-tax income $R(w)$ around a switch point w decreases π_i by

$$\mathrm{d}\pi_i = -\frac{z(w)\eta_i(w)}{\sum_{\sigma\in\mathcal{S}_i} z^2(w_\sigma)\eta_i^z(w_\sigma)}\,\mathrm{d}R. \tag{25}$$

Similarly, we show that a marginal increase $\mathrm{d}Z$ of the pension requirement Z increases π_i by

$$\mathrm{d}\pi_i = \frac{1}{\sum_{\sigma\in\mathcal{S}_i} z^2(w_\sigma)\eta_i^z(w_\sigma)}\,\mathrm{d}Z. \tag{26}$$

The nonlocal effects at any switch point w_σ of agent i, (22) and (24), follow by the chain rule.

Finally we also prove in the appendix that in regime N, if w is not a switch point, a marginal increase $\mathrm{d}R$ of after-tax income $R(w)$ around w decreases π_i by

$$\mathrm{d}\pi_i = -\frac{1}{\sum_{\sigma\in\mathcal{S}_i} R^2(w_\sigma)\eta_i^z(w_\sigma)}\,\mathrm{d}T_i(w;\ell_i^z)\,\mathrm{d}R, \tag{27}$$

which yields (23). □

As regards the perturbation of the tax schedule at a switch point w (first part of Lemma 2), it is worth noticing that, at the considered switch point $w' = w$, the direct effect $\eta_i^z(w;R,\pi_i)\,\mathrm{d}R$ weakly dominates the indirect effect given by (22). That is, labor supply weakly increases around w following a local increase in R ($\mathrm{d}R > 0$), while it decreases around the other switch points. If there are no other switch points than w and the agent labor supply is constrained by the pension system ($\pi_i > 0$), both effects cancel out exactly. This is because when w is the only switch point on the agent trajectory, labor supply is entirely determined by the pension requirement (19), so changing R around the switch point has no effect on labor supply.

The properties of consumption, from (17), follow from those of the function $\gamma_i(Z;R)$:

$$\gamma_i(Z;R) = \max\left\{\int_0^1 [R(w_i(a)) - \delta_i(a)]\,\ell_i^z(a)\,\mathrm{d}a + \pi_i\left(\int_0^1 z(w_i(a))\ell_i^z(a)\,\mathrm{d}a - Z\right)\right\}.$$

Lemma 3 (Consumption demand)**.** *Under regimes L, N and W, the functions $\gamma_i(Z;R)$, $i = 1,\dots,I$ are non-increasing and concave in Z with their derivative being given by*

$$\frac{\partial\gamma_i(Z;R)}{\partial Z} = -\pi_i.$$

The Frechet-derivative of consumption with respect to the tax schedule is given by

$$\frac{\partial\gamma_i(Z;R)}{\partial R(w)} = \mathrm{d}T_i(w;\ell_i^z) \tag{28}$$

under regimes L and W and by

$$\frac{\partial\gamma_i(Z;R)}{\partial R(w)} = (1+\pi_i)\,\mathrm{d}T_i(w;\ell_i^z) \tag{29}$$

under regime N, where $T_i(w;\ell_i^z)$ is the time spent working below productivity w.

Proof. When Z_i is lower than $Z_i^0 = \int_0^1 z(w_i(a))\mathbb{1}_{R(w_i(a))-\delta_i(a)}\,\mathrm{d}a$, the agent is not constrained by the pension requirement (19). In this region, the multiplier π_i is zero and the consumption net of the pension transfer $P(Z_i)$ is flat, as shown

on Figure 6. For positive π_i, the derivative with respect to Z can be computed using the envelope theorem. The variables π_i and $\ell_i^z(a)$ are jointly solutions of equations (19) and (20). Drawing labor supply from (20), $\ell_i^z(a)$ is nondecreasing in π_i. Substituting into (19), π_i is nondecreasing in Z. Hence $\gamma_i(Z;R)$ is concave in Z for all i.

To express the Frechet-derivatives with respect to the tax schedule R, differentiate $\gamma_i(Z;R+\varepsilon h)$ with respect to ε by the envelope theorem and use the change of variables $w = w_i(a)$ as in the pure tax case (see the proof of (5) in Appendix A.1). □

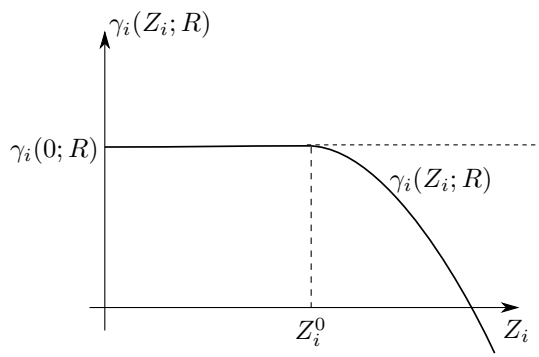


Figure 6: Consumption net of pension transfer

4.2 Incentive constraints

By (17), agent i's lifetime consumption is $c_i = \gamma_i(Z_i;R) + P_i$. The single-crossing property for the iso-consumption curves $c_i(Z_i, P_i)$ is equivalent to ordering the slopes $\gamma_i'(Z;R)$ with respect to the agent type i.

Assumption 4.1 (Type ordering)**.** *The productivities and pecuniary costs are ordered: for all ages a,*

$$w_1(a) \leq w_2(a) \leq \ldots \leq w_I(a) \quad \textit{and} \quad \delta_1(a) > \delta_2(a) > \cdots > \delta_I(a).$$

Lemma 4. *Under Assumption 4.1, the single-crossing property holds:*

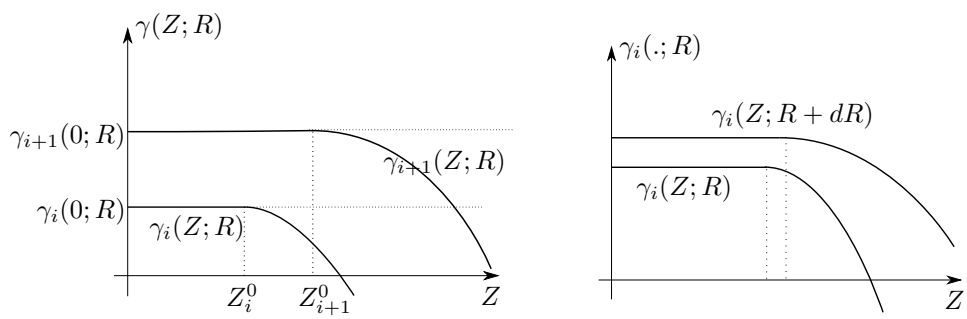
$$\frac{\partial \gamma_1(Z;R)}{\partial Z} \leq \frac{\partial \gamma_2(Z;R)}{\partial Z} \leq \ldots \leq \frac{\partial \gamma_{I-1}(Z;R)}{\partial Z} \leq \frac{\partial \gamma_I(Z;R)}{\partial Z}$$

for all Z. It follows that an allocation (P_i, Z_i), $i = 1, \ldots, I$, is incentive compatible if and only if the local incentive constraints (i.e. those corresponding to adjacent types) are satisfied, together with the monotonicity condition

$$Z_1 \leq Z_2 \leq \ldots \leq Z_{N-1} \leq Z_N.$$

Proof. We already know that $\pi_i = -\partial\gamma_i(Z;R)/\partial Z$, together with $\ell_i(a)$, are solutions of the system made of (19) and (20). Moreover an increase in π_i raises $\ell_i(a)$ by (20), and therefore the left-hand side of (19). Now under Assumption 4.1, note that both $w_i(a)$ and $\ell_i(a)$, the latter since by construction $R(w_i(a)) + \pi_i z(w_i(a))$ is nondecreasing in $w_i(a)$, are nondecreasing in $w_i(a)$. Hence the left-hand side of (19) is also nondecreasing in $w_i(a)$. Furthermore, $\ell_i(a)$ and therefore the left-hand side of (19) decreases with $\delta_i(a)$. It follows that the left-hand side of (19) increases with the type i. Therefore π_i is non-increasing in i under Assumption 4.1, which yields the single-crossing property. The second part of the Lemma is standard. □

The pension multipliers $\pi_i = -\partial\gamma_i/\partial Z$ reflect the pressure placed by the pension system on labor supply behavior. The single-crossing property expresses the fact that a same pension requirement Z places less pressure on more productive agents. Figure 7 shows the shape of the functions $\gamma_i(Z;R)$ as the agent type i and the tax schedule R vary. The functions are non-increasing and concave in Z and their slope $\gamma_i' = -\pi_i$ increases as i and R locally rises. The latter point follows from the observation that π_i decreases with R in all three regimes, see (25) and (27).



(a) The functions $\gamma_i(Z;R)$ are concave and steeper as i rises, in regimes L, N and W.

(b) The function $\gamma_i(Z;R)$ is steeper than $\gamma_i(Z;R+dR)$, $dR > 0$.

Figure 7: Shape of $\gamma_i(Z;R)$ as the agent type and the tax schedule vary

4.3 Taxonomy of economies

Denoting by λ the multiplier associated with the feasibility constraint and $\mu_{i+1,i}$ and $\mu_{i,i+1}$ those associated respectively with the downward and upward incentive

constraints, the Lagrangian of the government problem can be written as

$$\begin{aligned}\mathcal{L}^z &= \sum_{i=1}^{I}\left\{u_i(c_i)+\lambda\left(\int_0^1[w_i(a)-\delta_i(a)]\ell_i^z(a)\,\mathrm{d}a-c_i\right)\right\}\\ &\quad+\sum_{i,j}\mu_{i,j}\left\{c_i-c_j+\gamma_j(Z_j;R)-\gamma_i(Z_j;R)\right\},\end{aligned}$$

where the second sum is taken over adjacent agents i and j.

Holding the tax schedule $R(w)$ and the Z_i's fixed, it is equivalent to differentiate the Lagrangian with respect to consumption levels c_i or with respect to the pension transfers P_i, because $c_i = P_i + \gamma_i(Z_i; R)$. Doing so yields a simple taxonomy of economies associated with different patterns of binding incentive constraints. We say that an economy

- is *redistributive* if $u_1'(c_1) > u_2'(c_2) > \cdots > u_I'(c_I)$, or more generally where the average weight of the agents less productive than any given agent j exceeds the cost of public funds, $\sum_{i=1}^{j} u_i'(c_i) > \lambda j$;
- is *anti-redistributive* if $u_1'(c_1) < u_2'(c_2) < \cdots < u_I'(c_I)$, or more generally if $\sum_{i=1}^{j} u_i'(c_i) < \lambda j$ for all j;
- favors *middle classes* if the social weights of intermediate agents are above average, while those of low and high types are below average, so that $\sum_{i=1}^{j} u_i'(c_i)$ is smaller (larger) than λj for small (large) j.

Lemma 5. *In* (anti-)redistributive *economies, all the (upward) downward incentive constraints are binding. In* middle class *societies, upward (downward) incentive constraints are binding in the low (high) end of the population.*

Proof. The government program is well-behaved in the variables $(c_i)_{i=1,\ldots,I}$, with a concave objective function and linear constraints defining a non-empty set. By construction, $\mu_{i+1,i}$ positive means that agent $i+1$ is indifferent between her allocation and that of agent i, while if she strictly prefers her own, $\mu_{i+1,i}$ is zero.

Setting $\nu_i = \mu_{i,i+1} - \mu_{i+1,i}$ for $i = 1, \ldots, I-1$ and $\nu_0 = \nu_I = 0$, we can write the first order conditions with respect to the c's as:

$$u_i'(c_i) + \nu_i - \nu_{i-1} - \lambda = 0,$$

for $i = 1, \ldots, I$. Summing up these equalities yields $\sum u_i'(c_i) = I\lambda$, which shows that λ is positive. Note that given a consumption vector (c_i), the above conditions allow to compute all the multipliers, first λ, then the ν's. For $i = 1, \ldots, I$, we have

$$\nu_i = \sum_{j=1}^{i}\left[\lambda - u_j'(c_j)\right], \tag{30}$$

with $\lambda = (\sum u_i')/I$. Finally the μ's are given by

$$\mu_{i,i+1} = \max(0, \nu_i) \quad \mu_{i+1,i} = -\min(0, \nu_i). \tag{31}$$

Lemma 5 follows directly. □

Hereafter, we restrict attention to redistributive economies. The Lagrangian takes the form

$$\begin{aligned} \mathcal{L}^z &= \sum_{i=1}^{I} \{u_i(c_i) + \lambda [Y_i(Z_i; R) - c_i]\} \\ &\quad + \sum_{i=1}^{I-1} \mu_{i+1,i} \{c_{i+1} - c_i + \gamma_i(Z_i; R) - \gamma_{i+1}(Z_i; R)\}, \end{aligned} \tag{32}$$

where $Y_i(Z; R)$ denotes agent i's lifetime net output

$$Y_i(Z; R) = \int_0^1 [w_i(a) - \delta_i(a)] \ell_i^z(a) \, \mathrm{d}a$$

and ℓ_i^z is agent i's labor supply, given by (20).

4.4 The optimal choice of the pension requirements

We now differentiate the Lagrangian with respect to the pension requirements Z_i, holding lifetime consumptions c_i and the tax schedule $R(w)$ fixed. When the pension constraint (19) is binding, a change in Z_i alters the multiplier π_i, and in turn agent i's labor supply and lifetime net output.

More precisely, an increase in Z_i increases π_i by $\mathrm{d}\pi_i > 0$ as shown in (26), and hence increases labor supply. For instance, in regime L, the adjusted after-tax schedule is shifted upwards, see Figure 8. The requirement to get the pension transfer P_i is stronger and the agent has to work longer. Assuming that the pension constraint (19) is binding, we compute in Appendix D the derivative of agent i's lifetime net output $Y_i(\ell_i^z)$ with respect to the pension requirement Z_i as

$$\frac{\partial Y_i}{\partial Z_i} = \frac{\sum_{\sigma \in \mathcal{S}_i} (w_\sigma - \delta_\sigma) z(w_\sigma) \eta_i(w_\sigma)}{\sum_{\sigma \in \mathcal{S}_i} z^2(w_\sigma) \eta_i(w_\sigma)}, \tag{33}$$

where $\mathcal{S}_i$ is the set of agent i's switch points. The term $(w_\sigma - \delta_\sigma)/z(w_\sigma)$ is a measure of the local distortion of labor supply around the concerned switch point. At any efficient allocation, this term is zero at all switch points. Labor supply is locally distorted downwards (upwards) when this term is positive (negative). The derivative of an agent lifetime net output thus appears as a weighted average of adjusted distortions $(w - \delta)/z$ over her switch points.

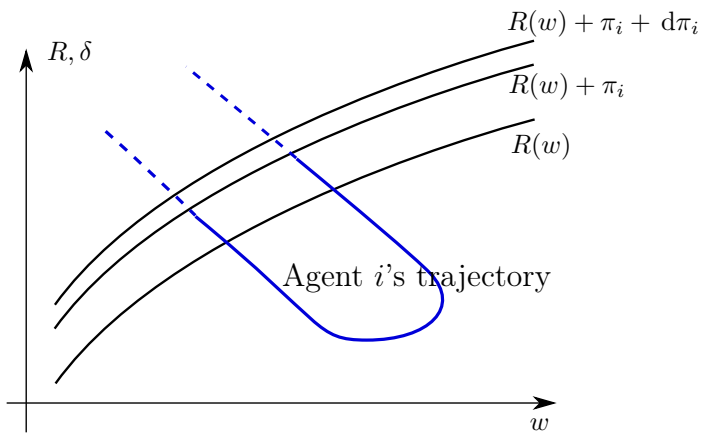


Figure 8: In regime L, a small increase in Z_i raises the adjusted after tax-schedule $R + \pi_i$ by $\mathrm{d}\pi_i > 0$

Differentiating (32) with respect to Z_i, $i = 1, \ldots, I-1$, while keeping c_i and R fixed, we find, in a redistributive economy

$$\lambda \frac{\partial Y_i}{\partial Z_i} = \mu_{i+1,i} \left[\frac{\partial \gamma_{i+1}(Z_i; R)}{\partial Z} - \frac{\partial \gamma_i(Z_i; R)}{\partial Z} \right] \geq 0, \tag{34}$$

where the inequality follows from the ordering assumption 4.1 and Lemma 4. The derivative is zero for the highest type, $i = I$.

Proposition 4. *At the second-best allocation of a redistributive economy, a weighted average of adjusted local distortions $(w_\sigma - \delta_\sigma)/z_\sigma$ over an agent set of switch points is zero for the highest type and positive for the other types.*

When an agent's trajectory has only one switch point, for instance when it is decreasing as in section 4.6 below, we conclude that *labor supply is undistorted for the most productive agent and is distorted downwards for the other agents.*

4.5 Optimal income tax

As in the pure taxation case, it is useful to analyze the effect of a change in income tax through the efficiency and redistribution forces. From (32) the Lagrangian is given by

$$\begin{aligned} \mathcal{L}^z &= \sum_{i=1}^{I} \{u_i(c_i) + \lambda [Y_i(\ell_i^z) - c_i]\} \\ &\quad + \sum_{i=1}^{I-1} \mu_{i+1,i} \{P_{i+1} - P_i + \gamma_{i+1}(Z_{i+1}; R) - \gamma_{i+1}(Z_i; R)\}. \end{aligned} \tag{35}$$

We proceed as in Section 3 to break down the derivative of the Lagrangian as $\mathrm{d}\mathcal{L}^z = \mathrm{d}\Phi^z + \mathrm{d}\Psi^z$, where $\Phi^z(w;\ell_i^z)$ and $\Psi^z(w;\ell_i^z)$ represent the net social marginal utility of income and the efficiency force. To this aim, we perturb the tax schedule respectively in an interval that does not contain any switch point and around a switch point. We implement these perturbations holding the pension scheme (P_i, Z_i) fixed.

We consider first the redistribution force. The following proposition shows that the correction for the presence of the incentive constraints increases the social marginal utility of income relative to the pure taxation case. Thus, as regards the redistribution motive, the presence of a pension scheme pushes after-tax income up.

Proposition 5. *The incentive constraints attached to pension decisions push up the redistribution force. Formally, at the second-best optimum with a pension scheme (P_i, Z_i):*

$$\mathrm{d}\Phi^z(w;\ell^z) \geq \mathrm{d}\Phi^o(w;\ell^z). \tag{36}$$

Proof. A perturbation of the tax schedule outside switch points does not only change the actual consumption levels, $c_i = P_i + \gamma_i(Z_i;R)$, but also the counterfactual consumption levels the agents would enjoy if they deviated to allocations designed for other agents, $P_j + \gamma_i(Z_j;R)$, $j \neq i$. We obtain the net social marginal utility of income by differentiating (35) with respect to the tax schedule, keeping the (Z,P)s constant. Specifically, using the derivatives of the γ_i's given by (28) and (29), we get, in a redistributive economy

$$\begin{aligned}\mathrm{d}\Phi^z(w;\ell^z) \quad = \quad & \sum_{i=1}^{I}(u_i' - \lambda)\,\mathrm{d}T_i(w;\ell_i^z) \\ & + \sum_{i=1}^{I-1}\mu_{i+1,i}\left[\mathrm{d}T_{i+1}(w;\ell_{i+1}^z) - \mathrm{d}T_{i+1}(w;\ell_{i+1}^z(Z_i))\right]\end{aligned} \tag{37}$$

for regimes L and W, where $\ell_{i+1}^z(Z_i)$ is the optimal labor supply of type $i+1$ when she wants to get the pension $P(Z_i)$ of type i, given the tax schedule R. The formula must be corrected by replacing T_i with $(1+\pi_i)T_i$ for all i for the regime N. The first term of (37) corresponds to the social marginal utility of income (net of the cost of public funds) that the government would consider if it wrongly ignored the presence of the incentive constraints, namely $\mathrm{d}\Phi^o(w;\ell^z)$. The second term is nonnegative because $Z_{i+1} \geq Z_i$ for $i < I$ and hence $\mathrm{d}T_{i+1}(w;\ell_{i+1}^z(Z_{i+1})) \geq \mathrm{d}T_{i+1}(w;\ell_{i+1}^z(Z_i))$ for any productivity level w (by Lemma 2 and equation (24) labor supply increases with Z). □

It is worthwhile noticing that the redistribution force in fact involves *only* incentives. Indeed, differentiating (32) with respect to the after-tax income R,

keeping the (Z, c)s rather than (Z, P)s constant, we get an alternative expression for the redistribution force:

$$\Phi^z(w; \ell^z) = \sum_{i=1}^{I-1} \mu_{i+1,i} \left[T_i(w; \ell_i^z(Z_i)) - T_{i+1}(w; \ell_{i+1}^z(Z_i)) \right]. \tag{38}$$

The equivalence of (37) and (38) is easily checked by replacing $u_i'(c_i) - \lambda$ with $\mu_{i+1,i} - \mu_{i,i-1}$ using (30) and (31). Thus, the introduction of pensions changes the channel through which the redistribution force operates. Instead of redistributing directly across agents, the after tax schedule serves as a facilitating device for redistribution through pension transfers. This new channel actually increases the redistribution force, $\mathrm{d}\Phi^z \geq \mathrm{d}\Phi^o$. Other things being equal –in particular the efficiency force being kept fixed– after-tax income should be raised to alleviate the incentive constraints.

Turning to efficiency, we examine how the presence of the pension scheme modifies the expression $\mathrm{d}\Psi^o(w; \ell^o)$ seen in Section 3. Holding the pension requirements fixed and perturbing the tax schedule at a switch point w_σ of agent i, we first compute the effect on the agent lifetime net output for a given level of the pension multiplier π_i. Using (11) and replacing η_i^o with the appropriate elasticity η_i^z, we get

$$\frac{\partial Y_i}{\partial R}_{|\pi_i} = [w_\sigma - \delta_\sigma]\, \eta_i^z(w_\sigma; R), \tag{39}$$

where the cost of work at the switch point δ_σ equals $R(w_\sigma) + \pi z_\sigma$ under the pension system Z. (In contrast, $\delta_\sigma = R(w_\sigma) - s$ in the absence of pension system.) Yet the perturbation of the tax schedule, at fixed pension requirement Z_i, alters the pension multiplier π_i so as to maintain equations (19) and (20).

Proposition 6. *The feedback effect due to the endogeneity of pension multipliers reduces the derivative of net output with respect to after tax income:*

$$\frac{\partial Y_i}{\partial R} \leq \frac{\partial Y_i}{\partial R}_{|\pi_i}. \tag{40}$$

Proof. The efficiency force reflects the *total* effect of tax perturbations around switch points on the net output in the economy:

$$\mathrm{d}\Psi^z(w; \ell^z) = \lambda \sum_i \frac{\partial Y_i}{\partial R}.$$

The direct effect –at given levels of the pension multipliers– follows from (39). When the pension constraint (19) is binding, a tax rise around a switch point increases π_i according to (25). In turn, the agent net output Y_i is modified as follows

$$\frac{\partial Y_i}{\partial \pi_i} \frac{\partial \pi_i}{\partial R} = \left(\sum_{\sigma \in \mathcal{S}_i} (w_\sigma - \delta_\sigma) z(w_\sigma) \eta_i^z(w_\sigma) \right) \left(- \frac{z(w) \eta_i^z(w)}{\sum_{\sigma \in \mathcal{S}_i} z^2(w_\sigma) \eta_i^z(w_\sigma)} \right).$$

The sign of the above term at the second-best optimum is known. Indeed, using (33) and the optimality condition regarding the pension requirements Z_i, equation (34), we find that the feedback effect of the change in pension multipliers on net output is negative:

$$\frac{\partial Y_i}{\partial \pi_i}\frac{\partial \pi_i}{\partial R} = -z(w)\eta_i^z(w)\frac{\partial Y_i}{\partial Z_i} \leq 0.$$

Thus, the effect of a tax decrease on net outptut is reduced by the presence of the pension requirements and the associated changes in the endogenous pension multipliers. □

As is transparent in the proof of Proposition 6, raising taxes around a switch point increases the pressure placed by the pension system on the concerned agent's labor supply, thus attenuating the depressive effect of the tax rise on labor supply. In other words, the presence of the pension system introduces an implicit income effect. To meet the pension requirements, the agents respond less vigorously to tax rises, reducing their net output by less than they would do in the absence of the pension system.

The pension requirements provide the government with new instruments to control labor supply. The presence of these extra instruments, in turn, makes it optimal for the government to place more (less) emphasis on redistributive (efficiency) concerns when setting taxes. In other words, the government can be bolder in redistributing through taxes as another instrument is available to mitigate the negative consequences of taxes on labor supply.

4.6 Decreasing trajectories

We now revisit the above general analysis in the two-type example of Section 3.2 where the agent trajectories are decreasing and the single-crossing property holds, recall Assumption 3.1.

When the trajectories are decreasing, the quantity $R(w_i(a))+\pi_i z(w_i(a))-\delta_i(a)$ decreases with age for $i = H, L$, implying from (20) that the agents work up to a retirement age a where $R(w_i(a)) + \pi_i z(w_i(a)) - \delta_i(a) = 0$ and do not work afterwards. An allocation is therefore given by a quadruple (a_H, a_L, c_H, c_L).

When agent i is constrained by the pension requirement ($\pi_i > 0$), her retirement age is given by the equality in (19) or

$$\int_0^{a_i} z(w_i(a))\,\mathrm{d}a = Z_i.$$

Under this circumstance, labor supply is determined by the pension system only. As it does not depend on the after-tax schedule R, the agent does not contribute to the efficiency force. On the other hand, when $\pi_i = 0$, the efficiency force takes the same form as under the pure taxation case studied in Section 3.

In this section, we first show that the efficiency force is identically zero at the second best optimum: $\Psi^z(w;\ell^z) = 0$ for all w. This property makes it easy to characterize the second best allocation.

Lemma 6. *At the second-best optimum, agent H's labor supply if she picked agent L's pension plan would be unconstrained by the requirement Z_L:*

$$\frac{\partial \gamma_H(Z_L;R)}{\partial Z} = 0.$$

Proof. Suppose by contradiction that agent H's labor supply were constrained under requirement Z_L, i.e. $\partial\gamma_H(Z_L;R)/\partial Z < 0$. Then by single crossing, Lemma 4, agent L would a fortiori be constrained under that requirement: $\partial\gamma_L(Z_L;R)/\partial Z < \partial\gamma_H(Z_L;R)/\partial Z < 0$ or $\pi_L > 0$. In other words, the pension requirement (19) would be binding for both agents with $Z = Z_L$, implying that agent H would retire earlier than L. As furthermore H is everywhere more productive than L, we find that agent H would spend less time working at low productivities than agent L. Using the expression (38) of the redistribution force, we get

$$\Phi^z(w;\ell^z) = \mu_{HL}\left[T_L(w,\ell_L^z(Z_L)) - T_H(w,\ell_H^z(Z_L))\right] > 0$$

for all w in each of the three considered regimes. On the other hand, the efficiency force in this configuration would be zero because agent H never contributes to that force (as her labor supply is known to be undistorted at the optimum, recall Section 4.4) and agent L does not contribute either because $\pi_L > 0$ by assumption. The government would thus have an incentive to raise after-tax income, the desired contradiction. □

Lemma 7. *The efficiency force is inactive at the second-best optimum: $\Psi^z(w;\ell^z) = 0$ for all productivity level w.*

Proof. As already mentioned, agent H does not contribute to the efficiency force at the second-best optimum because her labor supply is undistorted. As regards agent L, we use the first-order conditions for the pension requirements. We rewrite (33) as

$$\frac{\partial Y_L}{\partial Z_L} = \frac{w_L(a_L) - \delta_L(a_L)}{z(w_L(a_L))},$$

and (34) as

$$\lambda\frac{w_L(a_L) - \delta_L(a_L)}{z(w_L(a_L))} = \mu\left[\frac{\partial\gamma_H(Z_L;R)}{\partial Z} - \frac{\partial\gamma_L(Z_L;R)}{\partial Z}\right],$$

and finally, using Lemma 6,

$$\lambda\frac{w_L(a_L) - \delta_L(a_L)}{z(w_L(a_L))} = -\mu\frac{\partial\gamma_L(Z_L;R)}{\partial Z}.$$

The above equation shows that if agent L's labor supply is unconstrained by the pension system ($\partial\gamma_L(Z_L;R)/\partial Z = 0$), then it must be undistorted, implying that the agent does not contribute to the efficiency force. When $\pi_L > 0$, a_L is determined by Z_L and independent of the tax scheme, as already mentioned. □

Proposition 7. *Under Assumption 3.1, the following properties hold at a second best allocation (a_H, a_L, c_H, c_L) with the income tax and pension instruments:*

(i) After tax income can be taken to be constant;

(ii) Agent H has her labor supply undistorted, $a_H = a_H^*$;

(iii) Agent L labor supply is distorted downwards;

(iv) The rent granted to agent H is reduced to the minimum unobservable difference in opportunity costs of work:

$$c_H - c_L = \int_0^{a_L} [\delta_L(a) - \delta_H(a)]\ \mathrm{d}a > 0. \tag{41}$$

Proof. As any constant function is an admissible variation in the government optimization problem, we get that

$$< \mathrm{d}\Phi^z, 1 >= \Phi^z(\infty, \ell^z) = \mu_{HL}[T_L(\infty, \ell_L^z(Z_L)) - T_H(\infty, \ell_H^z(Z_L))] = 0,$$

meaning that agent H, if she picked agent L's pension plan, would retire at age a_L: $T_H(\infty, \ell_H^z(Z_L)) = T_L(\infty, \ell_L^z(Z_L))$. The binding incentive constraint can therefore be rewritten as (see (18) and (16))

$$\begin{aligned} c_H - c_L &= \gamma_H(Z_L;R) - \gamma_L(Z_L;R) \\ &= \int_0^{a_L} [R(w_H(a)) - R(w_L(a)) + \delta_L(a) - \delta_H(a)]\ \mathrm{d}a. \end{aligned} \tag{42}$$

The functions R and $-R$ also constitute admissible variations in the government optimization problem as both $(1+\varepsilon)R$ and $(1-\varepsilon)R$ are nondecreasing functions for $0 < \varepsilon < 1$. It follows that

$$< \mathrm{d}\mathcal{L}^z, R >=< \mathrm{d}\Phi^z, R > + < \mathrm{d}\Psi^z, R >= 0,$$

and hence, by Lemma 7, $< \mathrm{d}\Phi^z, R >= 0$. Now

$$< \mathrm{d}\Phi^z, R >= \mu_{HL} \int_w R(w)[\,\mathrm{d}T_L(w, \ell_L^z(Z_L)) - \mathrm{d}T_H(w, \ell_H^z(Z_L))] = 0.$$

It follows that the terms involving the tax schedule in (42) vanish:

$$\int_0^{a_L} [R(w_H(a)) - R(w_L(a))]\ \mathrm{d}a =< \mathrm{d}\Phi^z, R >= 0,$$

which yields (41). Using the feasibility condition

$$c_H + c_L = \int_0^{a_L} [w_L(a) - \delta_L(a)]\,\mathrm{d}a + \int_0^{a_H} [w_H(a) - \delta_H(a)]\,\mathrm{d}a,$$

we can solve for lifetime consumption levels

$$2c_H = \int_0^{a_L} [w_L(a) - \delta_L(a)]\,\mathrm{d}a + \int_0^{a_H} [w_H(a) - \delta_H(a)]\,\mathrm{d}a + \int_0^{a_L} [\delta_L(a) - \delta_H(a)]\,\mathrm{d}a$$

and

$$2c_L = \int_0^{a_L} [w_L(a) - \delta_L(a)]\,\mathrm{d}a + \int_0^{a_H} [w_H(a) - \delta_H(a)]\,\mathrm{d}a - \int_0^{a_L} [\delta_L(a) - \delta_H(a)]\,\mathrm{d}a,$$

and replace them in the government objective $u(c_H) + u(c_L)$. Differentiating the objective with respect to a_H yields $w_H(a_H) = \delta_H(a_H)$, agent H retires at the efficient age, which we already know from Section 4.4. Differentiating with respect to a_L yields

$$[u'(c_H) + u'(c_L)]\,[w_L(a_L) - \delta_L(a_L)] = [u'(c_L) - u'(c_H)]\,[\delta_L(a_L) - \delta_H(a_L)]\,.$$

The first factor at the right-hand side is positive as $c_L < c_H$ from (41). The second factor is positive by Assumption 3.1. It follows that $w_L(a_L) > \delta_L(a_L)$, agent L retires inefficiently early, point iii of the proposition. □

Propositions 5 and 6 materialize in a particularly strong form in the special circumstances considered here – an economy with two types of agents with decreasing trajectories. Indeed here the actual impact of the tax system on labor supply (the "true" efficiency force) is zero as labor supply is fully controlled by the pension instruments. It is thus optimal for the government to use taxes to redistribute with no consideration for efficiency, hence an extreme form of taxation in this particular instance, 100% marginal rate.

The exact level of the after-tax income is undetermined because the average redistribution force $\Phi^z(\infty, \ell^z)$ is zero: a vertical translation of the after-tax schedule leaves the Lagrangian unchanged. In fact, while the allocation (a_H, a_L, c_H, c_L) is completely determined, the pension transfers P_L and P_H, together with the constant after-tax income R, only satisfy

$$\begin{aligned} P_H + a_H R &= c_H + \int_0^{a_H} \delta_H(a)\,\mathrm{d}a \\ P_L + a_L R &= c_L + \int_0^{a_L} \delta_L(a)\,\mathrm{d}a, \end{aligned}$$

which leaves a degree of freedom when implementing the allocation ($R = 0$ is a possible choice).

Finally, comparing (41) with (15), we see that in an economy with two types of agents and decreasing trajectories, the pension instrument allows to eliminate all the rents that a pure tax system would have to concede to the high productivity types. Also, see Proposition 7, it suppresses all upward labor supply distortions and reduces downward distortions. Moreover the two instruments fully specialize: pensions provide the incentives to work, while taxes do all the redistribution.

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Appendix

A Proof of Proposition 1

A.1 Derivative of lifetime consumption and redistribution force

We first compute the Frechet-derivative of lifetime consumption levels with respect to the tax schedule R. We consider a perturbation $R + \varepsilon h$ of the tax schedule, where h is a nonnegative test function h with compact support. Using the expression of c_i, equation (4), and the change of variables $w = w_i(a)$, we find that the ratio $[c_i(R + \varepsilon h) - c_i(R)]/\varepsilon$ tends to

$$\int_0^1 h(w_i(a))\ell_i(a)\,\mathrm{d}a = \int h(w)\ \mathrm{d}T_i(w;\ell_i)$$

as ε goes to zero, meaning that the positive measure $\mathrm{d}T_i(w;\ell_i)$ is the Frechet-derivative of c_i. This is the formal statement corresponding to equation (5). $T_i(w)/L_i$ is the cumulative distribution function of w_i seen as a random variable. If we think of agent i's productivity when she works as a random variable and denote that variable by $\tilde{W}_i$, the above integral can be seen as the expectation of $h(\tilde{W}_i)$, multiplied by L_i.

The chain rule then yields the redistribution force (7). Keeping labor supply constant, the ratio $[\mathcal{L}(R + \varepsilon h) - \mathcal{L}(R)]/\varepsilon$ tends to

$$\sum_{i=1}^{N}[u_i'(c_i) - \lambda] \int h(w)\ \mathrm{d}T_i(w)$$

as ε goes to zero, which yields (7).

A.2 Labor supply elasticity and efficiency force

Labor supply is changed under the perturbed schedule $R+\varepsilon h$ only if the support of h contains switching points. For ease of exposition, we assume that the support contains only one switching point, that we denote by $\bar{w}$. We denote by i the switching agent and by a_i the age at which agent i switches at $\bar{w}$. We have: $w_i(a_i) = \bar{w}$ and $R(\bar{w}) - s = \delta_i(a_i)$. To fix ideas, we suppose that both $\delta'(a_i)$ and $w'(a_i)$ are positive and that the slope of the indifferent agent's trajectory is larger than the slope of the schedule: $\delta_i'(a_i)/w'(a_i) > R'(\bar{w})$.

The perturbed schedule $R + \varepsilon h$ crosses agent i's trajectory at points w such that there exists a with $w = w(a)$ and $\Theta(a, \varepsilon) = 0$, where

$$\Theta(a, \varepsilon) = \varepsilon h(w(a)) - \delta(a) + R(w(a)) - s.$$

As $\partial\Theta/\partial\varepsilon(a_i,0) = h(\bar{w})$ and $\partial\Theta/\partial a(a_i,0) = -\delta_i'(a_i) + R'(\bar{w})w_i'(a_i)$, the ratio $[T_i(w;R+\varepsilon h) - T_i(w;R)]/\varepsilon$ tends to

$$\begin{cases} \dfrac{h(\bar{w})}{\delta_i'(a_i) - R'(\bar{w})w_i'(a_i)} & \text{for } w > \bar{w} \\ 0 & \text{for } w \leq \bar{w}. \end{cases}$$

as ε goes to zero. If the slope of the tax schedule is larger than that of the trajectory, $\delta_i'(a_i)/w'(a_i) < R'(\bar{w})$, replacing R with $R+\varepsilon h$ changes labor supply on the left of $\bar{w}$ and the ratio $[T_i(w;R+\varepsilon h) - T_i(w;R)]/\varepsilon$ tends to

$$\begin{cases} \dfrac{h(\bar{w})}{|\delta_i'(a_i) - R'(\bar{w})w_i'(a_i)|} & \text{for } w \geq \bar{w} \\ 0 & \text{for } w < \bar{w}. \end{cases}$$

as ε goes to zero. This yields expression (9) for the elasticity of agent i's labor supply. The Frechet-derivative of $T_i(w;R)$ is thus given by

$$\frac{\partial T_i(w;R)}{\partial(R-s)} = \sum_{\sigma\in\mathcal{S}_i(w)} \varepsilon_i(w_\sigma;R)\frac{T_i(w_\sigma;R)}{R(w_\sigma)-s}\,\zeta(w_\sigma), \tag{43}$$

where $\mathcal{S}_i(w)$ is the set of agent i's switch points σ located below w, $w_\sigma \leq w$ is the agent's productivity at σ, and $\zeta(w_\sigma)$ denotes the mass point at w_σ. The Frechet-derivative of the total labor supply T has the same expression as above, replacing $\mathcal{S}_i(w)$ with $\mathcal{S}(w)$, the set of *all* agents' switch points located below w.

We use the same method to compute the Frechet-derivative of the term $\int_0^1 [w_i(a) - \delta_i(a)]\ell_i(a)\,\mathrm{d}a$. The only difference with the above analysis is the presence of the multiplicative term $w_i(a) - \delta_i(a)$, which, at $a = a_i$, is equal to $\bar{w} - R(\bar{w}) + s$, given that $\bar{w}$ is a switch point. This yields (12) and (13).

Discontinuous Lagrangian Consider an indifference point w such that the incentive schedule is locally tangent to the indifferent agent's trajectory. (In other words, we have: $\sigma = R'$.) Then the Lagrangian is discontinuous at w as an infinitesimally small increase in R implies a non-infinitesimal change in the Lagrangian. In other words, the efficiency force is particulary strong, creating a discontinuity in the Lagrangian, whose sign is the same as that of $w - R + s$. This is in particular the case where the tax schedule locally coincides with an agent trajectory.

B From increasing to piecewise constant schedules

Lemma B.1. *Let R be any nondecreasing tax schedule. Let $\underline{w} < \bar{w}$ be such that none of the agents' trajectories $(w_i(a), \delta_i(a))$, $a \in [0,1]$, $i = 1,\ldots,I$, intersects*

the rectangle $[\underline{w}, \bar{w}] \times [R(\underline{w}), R(\bar{w})]$. *Assume that the functions* T_i *have at most finitely many discontinuity points.*

Then there exists a nondecreasing tax schedule $\bar{R}$, *such that* $\bar{R}$ *is piecewise constant, with finitely many pieces, on* $[\underline{w}, \bar{w}]$, $\bar{R}$ *takes its values in* $[R(\underline{w}), R(\bar{w})]$ *on this interval, and*

$$\int_{\underline{w}}^{\bar{w}} \bar{R}(w)\,\mathrm{d}T_i(w) = \int_{\underline{w}}^{\bar{w}} R(w)\,\mathrm{d}T_i(w)$$

for $i = 1, \ldots, I$. *If all the functions* T_i *are continuous on* $[\underline{w}, \bar{w}]$, *then the schedule* $\bar{R}$ *has at most* $I + 1$ *pieces on* $[\underline{w}, \bar{w}]$.

Proof: By assumption, labor supply is not affected as long as the schedule remains between $R(\underline{w})$ and $R(\bar{w})$. We can therefore drop the second argument in the functions T_i, writing $T_i(w)$ rather than $T_i(w, R)$. Let $w_1, \ldots, w_N$ be the discontinuity points of the functions T_i. Let $w_0 = \underline{w}$ and $w_{N+1} = \bar{w}$. We have:

$$\int_{\underline{w}}^{\bar{w}} R(w)\,\mathrm{d}T_i(w) = \sum_j \int_{w_j}^{w_{j+1}} R(w)\,\mathrm{d}T_i(w) + R(w_j)[T_i(w_j^+) - T_i(w_j^-)].$$

It is sufficient to prove the result on each interval $[w_j, w_{j+1}]$. Integrating by parts yields:

$$\int_{w_j}^{w_{j+1}} R(w)\,\mathrm{d}T_i(w) = [T_i(w_{j+1}^-) - T_i(w_j^+)]R(w_{j+1}) - \int_{w_j}^{w_{j+1}} [T_i(w) - T_i(w_j^+)]\,\mathrm{d}R(w).$$

We now apply Lemma A.1 (p. 1260) of Ghosal and Van der Vaart (2001) with the compact set $K = [w_j, w_{j+1}]$, the probability measure $F_0 = \mathrm{d}R(w)/[R(w_{j+1}) - R(w_j)]$ and the functions $\Psi_i(w) = T_i(w) - T_i(w_j^+)$, $i = 1, \ldots, I$, which are continuous on K. The Lemma yields a discrete probability measure ν on K with at most $I + 1$ support points such that

$$\frac{1}{[R(w_{j+1}) - R(w_j)]} \int_{w_j}^{w_{j+1}} [T_i(w) - T_i(w_j^+)]\,\mathrm{d}R(w) = \int_{w_j}^{w_{j+1}} [T_i(w) - T_i(w_j^+)]\,\mathrm{d}\nu(w)$$

for all $i = 1, \ldots, I$. Integrating again by parts yields

$$\begin{aligned}
\int_{w_j}^{w_{j+1}} R(w)\,\mathrm{d}T_i(w) &= [T_i(w_{j+1}^-) - T_i(w_j^+)]R(w_{j+1}) \\
&\quad -[R(w_{j+1}) - R(w_j)] \int_{w_j}^{w_{j+1}} [T_i(w) - T_i(w_j^+)]\,\mathrm{d}\nu(w) \\
&= [T_i(w_{j+1}^-) - T_i(w_j^+)]R(w_{j+1}) \\
&\quad -[R(w_{j+1}) - R(w_j)][T_i(w_{j+1}^-) - T_i(w_j^+)] \\
&\quad +[R(w_{j+1}) - R(w_j)] \int_{w_j}^{w_{j+1}} \nu(w)\,\mathrm{d}T_i(w) \\
&= \int_{w_j}^{w_{j+1}} \bar{R}(w)\,\mathrm{d}T_i(w)
\end{aligned}$$

with $\bar{R}(w) = R(w_j) + [R(w_{j+1}) - R(w_j)]\nu(w)$. The schedule $\bar{R}$ is nondecreasing and piecewise constant, with at most $I + 1$ pieces. It takes its values in $[R(w_j), R(w_{j+1})]$. □

Proof of Proposition 2. Consider an interval where the schedule is increasing. The schedule can locally coincide with an increasing trajectory, in which case efficiency and redistribution play in opposite directions: the schedule is slightly below the trajectory if $\mathrm{d}\Phi^o > 0$ and $w < R - s$, slightly above if $\mathrm{d}\Phi^o < 0$ and $w > R - s$. For instance, in the former case, lowering (raising) R entails an infinitesimal (a non-infinitesimal) fall in the Lagrangian through the redistribution (efficiency) effect.[7]

Now consider an interval $[w, w']$ where the schedule is increasing and does not coincide with an increasing trajectory.[8] By compactness of $[w, w']$, there exists a finite sequence $w = w_1 < \cdots < w_h = w'$ such that no trajectory crosses the rectangles $[w_j, w_{j+1}] \times [R(w_j), R(w_{j+1})]$. On each interval $[w_j, w_{j+1}]$, we apply Lemma B.1 and replace R with a piecewise constant schedule that takes its values in $[R(w_j), R(w_{j+1})]$ and leaves the government revenue and the agents' lifetime consumption and labor supply unchanged. □

C Decreasing trajectories

In this section, we examine case (b) and case (c) in the proof of Proposition 3.

In case (b), we have $\delta_H(a_H) = R(w_H(a_H)) - s \geq R(w_L(a_L)) - s = \delta_L(a_L)$. We deal separately with the situation where the two agents have the same opportunity costs when they stop working, and when that of H is larger than that of L. Suppose first that $\delta_L(a_L) = \delta_H(a_H)$. Then the financial incentive to work $R(w) - s$ is equal to that common opportunity cost for all $w \geq w_L(a_L)$. The efficiency force $w_L(a_L) - \delta_L(a_L)$ cannot be downwards at $w_L(a_L)$ as this would violate the first-order condition on the bunching interval starting at $w_L(a_L)$ (in practice, the government would slightly decrease the after tax income at $w_L(a_L)$), hence $w_H(a_H) > w_L(a_L) \geq \delta_L(a_L) = \delta_H(a_H)$: agent H's labor supply is distorted downwards.

Suppose now that $\delta_L(a_L) < \delta_H(a_H)$. Since $u'(c_L) > u'(c_H)$ and only agent L works at productivities lying between $w_L(a_L)$ and $w_H(a_H)$, the redistribution force pushes upwards and the financial incentive to work $R - s$ equals $\delta_H(a_H)$ in that interval. The tax schedule, therefore, is discontinuous at $w_L(a_L)$ and equal to $\delta_H(a_H)$ above that point. Consider the perturbation that moves the discontinuity point $w_L(a_L)$ in the tax schedule slightly to the left while maintaining $R - s =$

[7] In other words, the Lagrangian is locally discontinuous in productivity regions where the tax schedule is locally tangent to a trajectory, see Appendix A.2.

[8] The first-order conditions imply that the net social marginal utility of income, $\mathrm{d}\Phi^o$, is identically zero on $[w, w']$ and that $R(w) - s = w$ at any switch point in this region.

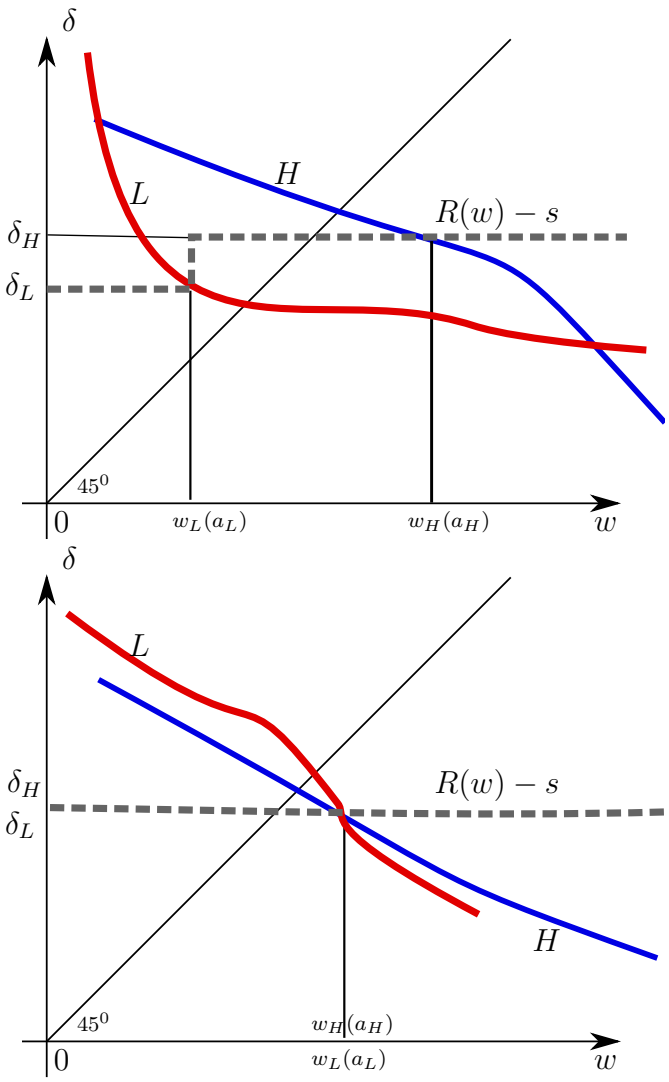

Figure 9: Decreasing trajectories: Case (b) (top) and (3) (bottom) of Proposition 3

$\delta_H(a_H)$. This perturbation, which does not affect agent H, increases agent L's labor supply and consumption. Consumption is increased by a first-order quantity because the agent receives positive extra income $\delta_H(a_H) - \delta_L(a_L) > 0$ during a small time interval, hence a positive redistributive effect. The efficiency part of the perturbation is a change in the Lagrangian of the sign of $w_L(a_L) - \delta(a_L)$. Expressing that the latter must outweigh the former, the first-order condition on R in the bunching interval, yields $w_L(a_L) < \delta(a_L)$, an upward distortion in L's labor supply.

Finally we consider case (c), denoting by $\underline{w}$ the common value of $w_H(a_H)$ and $w_L(a_L)$. We first show that the tax schedule necessarily intersects the two trajectories at the same point: $\delta_H(a_H)$ and $\delta_L(a_L)$ must be equal. Suppose for instance that $\delta_H(a_H) < \delta_L(a_L)$. A small increase $\mathrm{d}R_H$ in after-tax income below $\underline{w}$ would put agent H to work on a small time interval of length $\mathrm{d}T_H = \eta_H^o \,\mathrm{d}R_H$. Similarly a small decrease $-\mathrm{d}R_L$ in after-tax income above $\underline{w}$ would put agent L out of work on a small time interval of length $\mathrm{d}T_L = \eta_L^o \,\mathrm{d}R_L$. These transformations have redistribution effects that are of the second order. Choosing $\mathrm{d}R_H$ and $\mathrm{d}R_L$ such that $\mathrm{d}T_H = \mathrm{d}T_L$, we find by (12) that the associated changes in the Lagrangian would be respectively $\lambda(\underline{w} - \delta_H(a))\,\mathrm{d}T$ and $-\lambda(\underline{w} - \delta_L(a))\,\mathrm{d}T$. The sum of these two quantities would be of the sign of $\delta_L(a) - \delta_H(a)$, therefore positive, implying that one of the above changes would increase the Lagrangian through the efficiency force –a contradiction. A similar contradiction is found if $\delta_H(a_H) > \delta_L(a_L)$, hence the announced equality.

The tax schedule is flat above $\underline{w}$. A slight decrease of its constant level has a positive redistribution effect, and must therefore have a negative efficiency effect, implying that both agents have their labor supply distorted downwards.

D Labor supply elasticity under a pension scheme

The multiplier $\pi_i(Z; R)$ is defined jointly by (19) and (20), which can be rewritten as $K(\pi_i) = Z$ with

$$K(\pi_i) = \int_0^1 z(w_i(a)) \mathbb{1}_{R(w_i(a)) + \pi_i z(w_i(a)) - \delta_i(a) \geq 0} \,\mathrm{d}a.$$

We obtain the derivative of K with respect to π_i by the same method as in Appendix A.2, using the function $\Theta(a, \pi_i) = R(w_i(a)) + \pi_i z(w_i(a)) - \delta_i(a)$:

$$\frac{\partial K}{\partial \pi_i} = \sum_{\sigma \in \mathcal{S}_i} z^2(w_\sigma) \eta_i^z(w_\sigma; R, \pi_i).$$

The inverse function theorem yields (26). Similarly the Frechet-derivative of K with respect to R is given by

$$\frac{\partial K}{\partial R} = \sum_{\sigma \in \mathcal{S}_i} z(w_\sigma)\eta_i^z(w_\sigma; R, \pi_i)\zeta(w_\sigma)$$

in regime L and W and

$$\frac{\partial K}{\partial R} = \mathrm{d}T_i(w; \ell_i^z) + \sum_{\sigma \in \mathcal{S}_i} R(w_\sigma)\eta_i^z(w_\sigma; R, \pi_i)\zeta(w_\sigma)$$

in regime N, where $\zeta(w_\sigma)$ denotes the mass point at w_σ. Applying the implicit function theorem yields the Frechet-derivative of π_i with respect to R:

$$\frac{\partial \pi_i}{\partial R} = -\frac{\sum_{\sigma \in \mathcal{S}_i} z(w_\sigma)\eta_i^z(w_\sigma; R, \pi_i)\zeta(w_\sigma)}{\sum_{\sigma \in \mathcal{S}_i} z^2(w_\sigma)\eta_i^z(w_\sigma; R, \pi_i)} \tag{44}$$

in regime L and W, and by

$$\frac{\partial \pi_i}{\partial R} = -\frac{\mathrm{d}T_i(w; \ell_i^z) + \sum_{\sigma \in \mathcal{S}_i} R(w_\sigma)\eta_i^z(w_\sigma; R, \pi_i)\zeta(w_\sigma)}{\sum_{\sigma \in \mathcal{S}_i} R^2(w_\sigma)\eta_i^z(w_\sigma; R, \pi_i)} \tag{45}$$

in regime N. In this last regime, increasing R outside a switch points increases the after-tax income collected during the lifetime, modifies (19), hence the new term T_i at the numerator in the expression of π_i. Equations (44) and (45) are the formal counterparts of (25) and (27).

Finally, to compute the derivative of an agent's net output Y_i with respect to the pension requirement Z_i, we use the chain rule

$$\frac{\partial Y_i}{\partial Z_i} = \frac{\partial Y_i}{\partial \pi_i}\frac{\partial \pi_i}{\partial Z_i},$$

where the second term, $\partial \pi_i / \partial Z_i$, is given by (26). Computing the first term, $\partial Y_i / \partial \pi_i$ with the same method as above yields (33).