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The Frontier of Indeterminacy in a Neo-Keynesian Model with Staggered Prices and Wages

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The frontier of indeterminacy in a neo-Keynesian model with staggered prices and wages**

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Abstract

We consider a neo-Keynesian model with staggered prices and wages. When both contracts exhibit sluggish adjustment to market conditions, the policy maker faces a trade-off between stabilizing three welfare relevant variables: output, price inflation and wage inflation. We consider a monetary policy rule designed accordingly: the Central Banker can react to both inflations and the output gap. We generalize the Taylor principle in this case: it embeds the frontier of determinacy derived with staggered prices only, it is also symmetric in price and wage inflations. It follows that when staggered labour contracts are considered, wage inflation is also an illegible and efficient target for the Central Banker.

Keywords: Dynamic Stochastic General Equilibrium model, Monetary Policy Rule, Sun Spot Equilibria, Taylor Principle
JEL : C62, C68, E12, E58, E61

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Introduction

In (Taylor, 1993), John Taylor advocates the use of monetary policy rules where the Central Banker reacts to both price inflation and output as a benchmark to be used judgementally. His design of Wicksellian rule has been extensively studied since then in the context of neo-Keynesian models. In such models, two normative questions arise:[1]

- What kind of policy rule can achieve a social welfare optimum?
- How can one rule out sun-spot fluctuations (as described by (Woodford, 1987))?

In both respects, it has been shown that the Taylor rule has appealing properties (Woodford, 2001): in the simplest neo-Keynesian model, the Taylor rule can be proved optimal in terms of welfare under some assumptions (Rotemberg and Woodford, 1999). It is also key in enforcing solution determinacy: the Taylor principle states that the Central Banker's reaction to inflation must be large enough to ensure the uniqueness of the solution under rational expectations.[2] These results hold under staggered prices and flexible wages. When considering both staggered prices and wages, some of the appealing properties of the standard neo-Keynesian model are weakened. (Blanchard and Gali, 2007) show that allowing for both rigidities generates a trade-off between stabilizing inflation and output even in the absence of cost-push shocks:[3] the social optimum could be achieved when only staggered prices were considered, it is no longer the case with both staggered contracts. (Erceg et al., 2000) study the welfare implications of the addition of staggered wages. They show that is not possible for the monetary policy to fully stabilize more than one of the three objectives: price inflation, wage inflation or output, but the variance of each is detrimental to welfare. Using numerical simulations, they also show that sole price or wage inflation targeting is suboptimal in this context, but a policy rule such as suggested by Taylor or with reactions to both price and wage inflations performs nearly as well as the optimal rule.

In this paper, we consider the same model as (Galí, 2008, chapter 6) or (Erceg et al., 2000) but are mainly concerned with the problem of sun spot fluctuations instead of welfare optimization. We consider a monetary policy rule in line with Erceg et al.'s results: the Central Banker can react to both inflations and the output gap. With straightforward notations, the monetary policy rule takes the following form:

$$i_t = \Phi_p \pi_t^p + \Phi_w \pi_t^w + \Phi_y y_t$$

We find that the necessary and sufficient condition to rule out sun-spot equilibria is symmetric in inflations:

$$\Phi_p + \Phi_w + \frac{1-\beta}{\tilde{\kappa}} \Phi_y > 1$$

with β households' discount factor and $\tilde{\kappa}$ a coefficient depending symmetrically on both slopes of the prices and wages Phillips curves.

The frontier of the Taylor principle with staggered prices only is $\Phi_p + \frac{1-\beta}{\kappa}\Phi_y > 1$ with κ the slope of the Phillips curve on prices (Woodford, 2001). Our results thus generalizes the frontier derived is this simpler case. Though the model's symmetry may not appear straightforward, similar symmetry arises when studying the optimal monetary policy (see the functional form of the welfare criterion derived both by Galí and Erceg et al.). The intuition for this symmetry is given by Blanchard and Gali's comment on (Erceg et al., 2000). In the simple model with staggered prices only, the Phillips curve implies that stabilizing price inflation is equivalent to stabilizing the output gap, a result they present as a *divine coincidence* because it allows the Central Banker to enforce the social optimum. But, as aforementioned, they show that with the addition of staggered wages, this result no longer holds. In Erceg et al.'s model, they note a weaker form of this coincidence: combining the two Phillips curves yields that stabilizing the output gap is equivalent to stabilizing a weighted average of price and wage inflation (with the weight on each inflation being the slope of the others Phillips curve).

In the remainder of this paper, the first section recalls the model. We expose some general mathematical properties of this model in section 2 when the Central Banker can only react to prices and wages inflation ($\Phi_y = 0$). We then

[1] These questions are independent of one another: optimal rules do not necessarily avoid sun-spot fluctuations (Clarida et al., 1999)
[2] (Bullard and Mitra, 2002) shows that the properties of this principle are also key in a model with adaptive learning
[3] In presence of cost-push shocks there is a short run trade-off between the two objectives (Clarida et al., 1999)

study the uniqueness of its solution in this case ($\Phi_y = 0$) (sections 3, 4 and 5). We first consider the limit subcase $\Phi_p + \Phi_w = 1$ (section 3). In section 4, we study the deviations from this subcase ($\Phi_p + \Phi_w \gtrless 1$). In section 5 we derive the frontier of the Taylor principle when $\Phi_y = 0$. Finally we expand this result to the case where the Central Banker can also react to the output gap ($\Phi_y \neq 0$) in section 6. Readers not familiar with this literature can find in appendix some general elements on neo-Keynesian models for monetary policy solved under rational expectations in which we expose the general set-up of this problem.

1 A monetary model with sticky wages and prices

We study the model exposed in (Galí, 2008, chap 6) and (Erceg et al., 2000). This model extends the standard neo-Keynesian model for monetary policy analysis which consist of an *IS* curve relating the output gap to the expected real interest rate, a Phillips curve relating inflation, expected inflation and output gap and a monetary policy rule describing how the interest rate is set by the Central Banker. The present extension of the model considers wage rigidities under the form of Calvo contracts. It follows from this rigidity that real wages may deviate from their flexible equivalent due to exogenous disturbances.

The model takes the following linear form:[4]

$$\pi_t^p = \beta E(\pi_{t+1}^p|t) + \kappa_p y_t + \lambda_p \omega_t \tag{1}$$

$$\pi_t^w = \beta E(\pi_{t+1}^w|t) + \kappa_w y_t - \lambda_w \omega_t \tag{2}$$

$$\omega_{t-1} = \omega_t - \pi_t^w + \pi_t^p + \Delta\omega_t^n \tag{3}$$

$$y_t = E(y_{t+1}|t) - \frac{1}{\sigma}(i_t - E(\pi_{t+1}^p|t) - r_t^n) \tag{4}$$

$$i_t = \Phi_p \pi_t^p + \Phi_w \pi_t^w + \Phi_y y_t + v_t \tag{5}$$

In this system, at each date t, a set of variables ($\pi^p,\ \pi^w,\ \omega,\ y,\ i$) are determined by their current and past value and their expected value at the following date ($E(.|t)$, is the rational expectations operator at date t, i.e. the expectation conditional on the values of every variables up to date t and the model itself). Equations (1) and (2) are the Phillips curves on price inflation (π^p) and wage inflation (π^w). They describe the progressive adjustment of prices and wages to market conditions. Prices may increase with expected inflation or the marginal cost of production. This cost depends positively on the output gap (y_t, defined as the deviation of output from its fully flexible equivalent) and the real wage gap (ω_t, defined as the deviation of real wage from its fully flexible equivalent). Wages may increase with expected wage inflation or decrease with the wage mark-up (taken in deviation from the flexible contracts case). This mark-up depends positively on the real wage gap and negatively on the output gap. Equation (3) describes the fact that because of nominal rigidities, real wages depart from their fully flexible counterpart. Exogenous shocks to the economy affecting the real wage ($\Delta\omega^n$) are not instantaneously transmitted to the actual real wage but only to its flexible counterpart, hence driving a wedge between inflations and the dynamic of the real wage gap. Equation (4) describes the evolution of the output gap (y) as a function of interest rate (i) and expected inflation. The implicit assumption here is that output is driven, in the short run, by private demand. r_t^n is the natural rate of interest, that is the real interest rate which would prevail under fully flexible contracts. Equation (5) describes the interest rate decision of the Central Banker. It is a Taylor rule modified to account for the fact that the Central Banker may react to wages inflation as well as prices inflation. The higher inflations or output are, the higher the Central Banker will set the interest rate in order to temper the economic growth. Moreover, the Central Banker may depart from this rule for exogenous reasons (v).

The parameters of this model are:

- $0 < \beta < 1$, is the discount factor of households.
- $\sigma \geq 0$, is the inverse intertemporal elasticity of substitution of consumption.
- $\Phi_p > 0$, is the Central Banker's reaction to price inflation.(Taylor, 1993) considers $\Phi_p = 1.5$

[4]The complete derivation of the model is exposed in full details in (Galí, 2008, chap 6) with the same notations

- $\Phi_w \geq 0$ is the Central Banker's reaction to wage inflation. In the standard cases the Central Banker only react to price inflation ($\Phi_w = 0$)
- $\Phi_y \geq 0$ is the Central Banker's reaction to the output gap. (Taylor, 1993) considers $\Phi_y = 0.5$.
- $\lambda_p = \frac{(1-\theta_p)(1-\beta\theta_p)}{\theta_p}\frac{1-\alpha}{1-\alpha+\alpha\varepsilon_p}$, where
  - $0 < \theta_p < 1$, is the Calvo parameter on prices, in other words the stickiness of prices (if 0, prices are fully flexible)
  - $0 < \alpha < 1$, with $1 - \alpha$ the elasticity of output with respect to labour
  - $0 < \varepsilon_p \leq 1$, is the elasticity of substitution among goods

$\Rightarrow$ $0 < \lambda_p$

- $\lambda_w = \frac{(1-\theta_w)(1-\beta\theta_w)}{\theta_w(1+\varphi\varepsilon_w)}$, where
  - $0 < \theta_w < 1$, is the Calvo parameter on wages, in other words the stickiness of wages (if 0, wages are fully flexible)
  - $0 < \varphi$, is the Frisch elasticity, in other words the convexity of the cost of labour in terms of welfare.
  - $0 < \varepsilon_w \leq 1$, is the elasticity of substitution among labour types

$\Rightarrow$ $0 < \lambda_w$

- $\kappa_p = \frac{\alpha\lambda_p}{1-\alpha}$, we will also denote later $\lambda_p n_p = \kappa_p$ with $n_p > 0$
- $\kappa_w = \lambda_w(\sigma + \frac{\varphi}{1-\alpha})$ which implies $\kappa_w \geq \lambda_w\sigma$. We will also denote later $\lambda_w n_w = \kappa_w$ with $n_w > 0$ or $\kappa_w = \lambda_w(\sigma+\nu)$ with $\nu > 0$.

Denoting $x_t = [y_t,\ \pi_t^p,\ \pi_t^w,\ \omega_{t-1}]^T$, the endogenous variables, and $z_t = [r_t^n - v_t,\ \Delta\omega_t^n]^T$, the exogenous variables, the equations (1) to (5) can be written in the form:

$$x_t = A^{-1}\left(\ E(x_{t+1}|t) + B\ z_t\right) \tag{6}$$

In the equation (6), the matrix of interest A is:

$$A = \begin{bmatrix} 1 + \dfrac{\kappa_p}{\sigma\beta} + \dfrac{\Phi_y}{\sigma} & \dfrac{\beta\Phi_p - 1 - \lambda_p}{\sigma\beta} & \dfrac{\beta\Phi_w + \lambda_p}{\sigma\beta} & \dfrac{\lambda_p}{\sigma\beta} \\ \dfrac{-\kappa_p}{\beta} & \dfrac{1+\lambda_p}{\beta} & \dfrac{-\lambda_p}{\beta} & \dfrac{-\lambda_p}{\beta} \\ \dfrac{-\kappa_w}{\beta} & \dfrac{-\lambda_w}{\beta} & \dfrac{1+\lambda_w}{\beta} & \dfrac{\lambda_w}{\beta} \\ 0 & -1 & 1 & 1 \end{bmatrix} \tag{7}$$

There are three forward looking variables in this models: $([y_t,\ \pi_t^p,\ \pi_t^w])$.

Lemma 1 *According to (Blanchard and Kahn, 1980), the system* (6) *has a unique solution if and only if the matrix* A *defined by* (7) *has 3 eigenvalues strictly larger than one in modulus and one eigenvalue strictly smaller than one in modulus.*

In this case, there is numerical evidence that the sum $\Phi_p + \Phi_w$ should be larger than 1 when $\Phi_y = 0$ to meet this condition. When $\Phi_y \neq 0$, the condition on $\Phi_p + \Phi_w$ is decreasing with Φ_y (Galí, 2008). Nevertheless, a formal proof to these properties has not been given yet, it is the main objective of this paper.

Main results In the remainder of this paper we show that any monetary policy rule satisfying

$$\Phi_p + \Phi_w + \Phi_y \frac{(1-\beta)}{(n_w+n_p)} \left(\frac{1}{\lambda_p} + \frac{1}{\lambda_w}\right) > 1 \tag{8}$$

rules out sunspot equilibria.

The admissibility of a policy rule symmetrically depends on wage inflation and prices inflation: when the central bank does not respond to changes in output, the condition for monetary policy comes down to $\Phi_p + \Phi_w > 1$ in line with Galí's numerical investigations.

Also in line with Galí's numerical investigations, when the central bank reacts to changes in output, doing so relaxes the constraint above, proportionally to Φ_y with a factor $\frac{(1-\beta)}{(n_w+n_p)} \left(\frac{1}{\lambda_p} + \frac{1}{\lambda_w}\right)$. This coefficient crucially and symmetrically depends on the Phillips curves of prices and wages: more impatient agents (smaller β) or flatter Phillips curves (smaller λ or n), facilitate the task of the Central Banker to prevent sun spot fluctuations.

In this model, a permanent shift in price inflation ($\tilde{\pi}$) implies an identical permanent shift in wage inflation (equation (3)). The Phillips curves (equations (1) and (2)) imply a proportional shift in output gap $\tilde{y} = \frac{(1-\beta)}{(n_w+n_p)} \left(\frac{1}{\lambda_p} + \frac{1}{\lambda_w}\right) \tilde{\pi}$. In turn, the Taylor rule (5) implies that the reaction of the Central Banker is to raise the nominal interest rate by $\tilde{i} = \left[\Phi_p + \Phi_w + \Phi_y \frac{(1-\beta)}{(n_w+n_p)} \left(\frac{1}{\lambda_p} + \frac{1}{\lambda_w}\right)\right] \tilde{\pi}$. Thus, as in the standard neo-Keynesian model without wage rigidities (Woodford, 2011, chapter 4), our frontier of indeterminacy can also be interpreted in terms of the Taylor principle: the *Central Banker reacting more than one for one to permanent changes in inflation.*

Using Dynare (Adjemian et al., 2011), it is possible to verify numerically frontier (8).[5] (Galí, 2008, chapter 6) and (Erceg et al., 2000) show that wage inflation targeting compares with price inflation targeting in terms of welfare. Using Dynare it is possible to confirm their result of symmetry by computing the optimal coefficients for the monetary policy rule considered here.[6] When considering a Central Banker reacting to both inflations and the output gap, we find $\Phi_p = 47.1$, $\Phi_w = 67.8$, $\Phi_y = 231.9$. This optimal rule implies a very sensitive interest rate which is standard when the benefits of a smoothed monetary policy are not considered. More interestingly, the reactions of the optimal interest rate to both inflations are comparable.

Wage inflation and price inflation play similar roles for the design of the optimal monetary policy, we show that they also play symmetric roles for eliminating sun-spot fluctuations. This extended conclusion remains *"at odds with the practice of most central banks, which seem to attach little weight to wage inflation as a target variable"*(Galí, 2008).

Outline of the proof Defining the frontier of indeterminacy is based on the study of the roots of the characteristic polynomial of matrix A, a fourth degree polynomial. Though it is not complex mathematics, it is rather cumbersome. We are particularly grateful to Yvon Maday and other mathematicians at Laboratoire Jacques-Louis Lions for proof-reading and comments.

In sections 2 to 5 we develop the proof in the case $\Phi_y = 0$. In section 2 we study the general properties of this polynomial and its coefficients. We use the intuition that in this case the frontier of determinacy is $\Phi_p + \Phi_w = 1$ and decompose the polynomial as a fourth degree polynomial corresponding to this case plus deviations from this case in both directions (Φ_p, Φ_w). In section 3 we study the polynomial in the case $\Phi_p + \Phi_w = 1$ to show that: 1 is a root of this polynomial; its real roots are non-negative; its complex roots have a modulus strictly greater than one; and at most one real root is in $]0, 1[$. In section 4 we study the deviations from $\Phi_p + \Phi_w = 1$; we show that these deviations are second degree polynomials with positive real roots, one strictly greater than one the other strictly smaller than one. In section 5 we study how the deviations from $\Phi_p + \Phi_w = 1$ modifies the roots of the characteristic polynomial. The complex roots cannot enter the unit circle. The real roots strictly greater or lower

[5] Code available upon request

[6] The welfare criterion to be optimized is derived in Galí, we use his benchmark calibration and in line with his methodology consider technology shocks only.

than one are kept away from 1. The root 1 moves in the direction ensuring the uniqueness of the model's solution (depending on the existence of another root smaller than one) if and only if the deviation from $\Phi_p + \Phi_w = 1$ is positive.

In section 6 we show that the case $\Phi_y \neq 0$ can be treated identically to the case $\Phi_y = 0$. We consider the frontier of indeterminacy under the form $\Phi_p + \Phi_w = 1 - \theta$ and show that setting $\theta = \Phi_y \frac{(1-\beta)}{(n_w + n_p)} \left(\frac{1}{\lambda_p} + \frac{1}{\lambda_w} \right)$ allows a decomposition of the characteristic polynomial which has the same properties as in the case $\Phi_y = 0$. We can conclude that equation (8) generalizes the frontier of indeterminacy.

2 Preliminary properties of the model's characteristic polynomial

According to what has been explained above, the uniqueness result probably holds if and only if $\Phi_w + \Phi_p > 1$ when $\Phi_y = 0$. We begin with the study of this limit case $\Phi_w + \Phi_p = 1$. For that purpose, we introduce a new parameter ϕ_p and use the following parametrization:

$$\Phi_p = \phi_p + \xi \qquad \Phi_w = 1 - \phi_p + \gamma \tag{9}$$

$$0 < \phi_p < 1 \qquad \xi \text{ s.t } \quad \Phi_p > 0 \qquad \gamma \text{ s.t. } \quad \Phi_w > 0 \tag{10}$$

Such values of γ and ξ are called *admissible* throughout the paper. The domain of interest, $D_{p,w}$, is coloured in blue on Figure 1. This parametrization of $D_{p,w}$ is not injective, as three parameters (ϕ_p, ξ, γ) describe a two dimensional domain, but this choice makes the study easier.

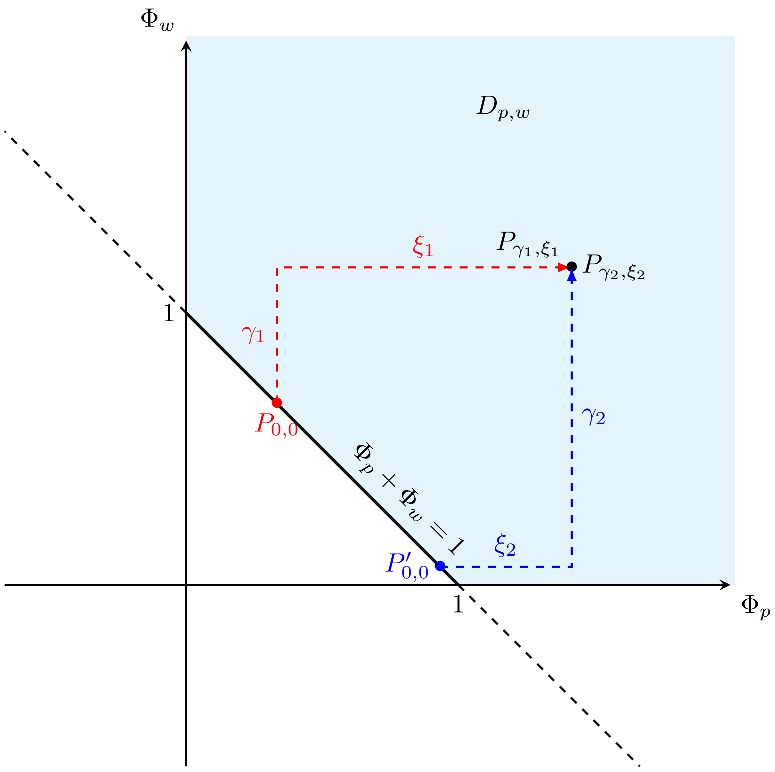

Figure 1: Domain of interest (in blue) and parametrization

Let X denote the vector of the new parameters:

$$X = [\beta, \phi_p, \kappa_p, \lambda_p, \lambda_w, \sigma, \nu] \quad \in]0,1[\times]0,1[\times (\mathbb{R}_+^*)^5 = D_X. \tag{11}$$

it excludes γ and ξ, treated as special parameters.

Remark 1 *Please note that we use indistinctly the following notations: $\kappa_p = \lambda_p n_p$ and $\kappa_w = \lambda_w n_w = \lambda_w(\sigma + \nu)$.*

The characteristic polynomial of the matrix defined in (7) above can be expressed as follows, with implicit dependency on X:

$$P_{\gamma,\xi}(t) = at^4 - bt^3 + c_{\gamma,\xi}t^2 - d_{\gamma,\xi}t + e_{\gamma,\xi}. \tag{12}$$

with the coefficients:

$$\begin{aligned}
a &= \sigma\beta^2\\
b &= \beta[\kappa_p + \sigma(2+2\beta+\lambda_p+\lambda_w)]\\
c_{\gamma,\xi} &= \kappa_p[1+\lambda_w+\beta(1+\phi_p+\xi)] \,+\, \lambda_w\nu[\lambda_p+\beta(1-\phi_p)+\beta\gamma]\\
&\quad +\, \sigma[1+4\beta+\beta^2+\lambda_p(1+\beta)+\lambda_w(1+\lambda_p+(2-\phi_p)\beta+\beta\gamma)]\\
d_{\gamma,\xi} &= \kappa_p[1+\lambda_w+\phi_p(1+\beta)+\lambda_w\gamma+(1+\beta+\lambda_w)\xi] \,+\, \lambda_w\nu[\lambda_p+(1-\phi_p)(1+\beta)+(1+\beta+\lambda_p)\gamma+\lambda_p\xi]\\
&\quad +\, \sigma[2\beta+2+\lambda_p+\lambda_w(1+\lambda_p+(1+\beta)(1-\phi_p)+(1+\beta+\lambda_p)\gamma+\lambda_p\xi)]\\
e_{\gamma,\xi} &= \sigma+\kappa_p\phi_p+\kappa_p\xi+\lambda_w(\sigma+\nu)(1-\phi_p)+\lambda_w(\sigma+\nu)\gamma.
\end{aligned}$$

We denote complex numbers $z=\rho(\cos\theta+i\sin\theta)$ and

$$\mathbb{D}=\{z\in\mathbb{C}\quad \text{s.t.}\quad |z|\le 1\}. \tag{13}$$

z is strictly in (resp. out of) $\mathbb{D}$ if $|z|<1$ (resp. $|z|>1$).

We adopt three conventions: a fourth degree polynomial P is said to satisfy the property

(i) if P has one root strictly in $\mathbb{D}$ and three roots strictly out of $\mathbb{D}$,

(ii) if P has two real roots and two complex roots,

(iii) if P has four real roots.

According to the explanation above, the uniqueness of the solution is equivalent to the fact that $P_{\gamma,\xi}$ satisfies (i).

The polynomial $P_{\gamma,\xi}$ satisfies the following properties

Property 1 *For every vector of parameters $X\in D_X$, $(a,b)\in(\mathbb{R}_+^*)^2$ and $\forall\gamma\ge 0$, $\forall\xi\ge 0$, $(c_{\gamma,\xi},d_{\gamma,\xi},e_{\gamma,\xi})\in(\mathbb{R}_+^*)^3$. This implies that $\forall t\le 0$, $P_{\gamma,\xi}(t)>0$.*

Proof: The sign of the coefficients derives from their definition and the definition (11) of D_X. The sign of the polynomial $P_{\gamma,\xi}(t)$ derives from (12). ■

Property 2 *For every vector of parameters $X\in D_X$, for every $\gamma>0$ and for every $\xi>0$, $P_{\gamma,\xi}$ has a root $\lambda_1\in]0,1[$ and a root $\lambda_\infty\in]1,+\infty[$*

Proof:

$$P_{\gamma,\xi}(1) = -\lambda_w\lambda_p(\gamma+\xi)(n_p+n_w) \qquad \text{and} \qquad \lim_{+\infty} P_{\gamma,\xi}=+\infty$$

and as $P_{\gamma,\xi}(0)>0$, the property is proved. ■

Property 3 *The following inequalities hold:*

$$\forall X\in D_X \quad \forall\gamma\ge 0 \quad \forall\xi\ge 0 \qquad b>4a \qquad c_{\gamma,\xi}>2a+b>6a \qquad e_{\gamma,\xi}>a$$

Property 4 *The discriminant of the second order derivative P''_γ is $\Delta(P''_{\gamma,\xi})=12\,[3b^2-8ac_{\gamma,\xi}]$. Moreover, $\forall\gamma\ge 0$, $\forall\xi\ge 0$ if $3b^2<8ac_{\gamma,\xi}$ then $P_{\gamma,\xi}$ has two conjugate complex roots*

Proof: If the discriminant of $P''_{\gamma,\xi}$ is negative, this polynomial is positive for every t. In this case, $P'_{\gamma,\xi}$ is strictly increasing and has only one real root, and $P_{\gamma,\xi}$ is strictly decreasing and then strictly increasing. As we already know that it has two real roots, the two others are complex. ■

From now on, we use the simplifying notations $P_0 = P_{0,0}$, $c = c_{0,0}$, $d = d_{0,0}$ and $e = e_{0,0}$ and the following decomposition:

$$P_{\gamma,\xi}(t) = P_0(t) + \gamma\,\lambda_w\,Q(t) + \xi\,\lambda_p\,S(t) \qquad \text{where} \tag{14}$$
$$Q(t) = \beta n_w\,t^2 - [\kappa_p + n_w(1+\beta+\lambda_p)]t + n_w \tag{15}$$
$$S(t) = \beta n_p\,t^2 - [\kappa_w + n_p(1+\beta+\lambda_w)]t + n_p \tag{16}$$

Property 5 *For all $X \in D_X$, $P_0'(1) = 4a - 3b + 2c - d$ and $6a - 3b + c$ cannot be both negative.*

Proof: We proceed by contradiction

Assumption 1 *Exists $X \in D_X$ such that $P_0'(1) = 4a - 3b + 2c - d < 0$ and $6a - 3b + c < 0$.*

The first inequality rewrites:

$$\begin{aligned} 4a - 3b + 2c - d = \kappa_p[\lambda_w + (1-\beta)(1-\phi_p)] \,&+\, \lambda_w\nu[\lambda_p - (1-\beta)(1-\phi_p)] \\ &+\, \sigma[(1-\beta)(\lambda_p + \lambda_w\phi_p) + \lambda_w\lambda_p] \,<\, 0. \end{aligned}$$

The only way to do so is:

$$\lambda_p \,<\, (1-\beta)(1-\phi_p) \tag{17}$$

Reintroducing this into the last inequality, we obtain:

$$\lambda_w\nu \,>\, \frac{\kappa_p[\lambda_w + (1-\beta)(1-\phi_p)] + \sigma[(1-\beta)(\lambda_p + \lambda_w\phi_p) + \lambda_w\lambda_p]}{(1-\beta)(1-\phi_p) - \lambda_p} = m. \tag{18}$$

The second inequality can be expressed as:

$$\begin{aligned} 6a - 3b + c = \kappa_p[\lambda_w + 1 - \beta - \beta(1-\phi_p)] \,&+\, \lambda_w\nu[\lambda_p + \beta(1-\phi_p)] \\ &+\, \sigma[(1-\beta)^2 + \lambda_p(1-\beta) + \lambda_w(1+\lambda_p) - \lambda_p\beta - \lambda_w\beta(1+\phi_p)] \,<\, 0 \end{aligned}$$

and this implies another bound for $\lambda_w\nu$:

$$\lambda_w\nu \,<\, \frac{-\kappa_p[\lambda_w + 1 - \beta - \beta(1-\phi_p)] - \sigma[(1-\beta)^2 + \lambda_p(1-\beta) + \lambda_w(1+\lambda_p) - \lambda_p\beta - \lambda_w\beta(1+\phi_p)]}{\lambda_p + \beta(1-\phi_p)} = M. \tag{19}$$

Now let us denote by D_m and D_M the denominators of m and M respectively, and compute: $\Delta \equiv (m - M)D_m D_M$

$$\begin{aligned} \Delta = \kappa_p[\lambda_w + (1-\beta)(1-\phi_p)][\lambda_p + \beta(1-\phi_p)] &+ \sigma[(1-\beta)(\lambda_p + \lambda_w\phi_p) + \lambda_w\lambda_p][\lambda_p + \beta(1-\phi_p)] \\ &+ \kappa_p[\lambda_w + 1 - \beta - \beta(1-\phi_p)][(1-\beta)(1-\phi_p) - \lambda_p] \\ + \sigma[(1-\beta)^2 + \lambda_p(1-\beta) &+ \lambda_w(1+\lambda_p) - \lambda_p\beta - \lambda_w\beta(1+\phi_p)][(1-\beta)(1-\phi_p) - \lambda_p] \\ &= \sigma\Delta_\sigma + \kappa_p\Delta_{\kappa_p} \end{aligned}$$

where, after simplification:

$$\begin{aligned} \Delta_\sigma &= (1-\beta)(\lambda_p + 1 - \beta)[(1-\beta)(1-\phi_p) - \lambda_p] + \lambda_w(1-\beta)^2(1-\phi_p) + \lambda_p\lambda_w\beta + \lambda_p^2 \\ \Delta_{\kappa_p} &= (1-\beta)[(1-\beta)(1-\phi_p) - \lambda_p] + (\lambda_p + \lambda_w)(1-\phi_p). \end{aligned}$$

It is not difficult to check that the two cofactors Δ_σ and Δ_{κ_p} are strictly positive. Hence, we obtain that $m - M > 0$ and this is not possible, considering (18) and (19). ■

3 The eigenvalues in the limit case $\Phi_p + \Phi_w = 1$: P_0 study

P_0 satisfies the property 1, but not the property 2 as explained below.

Property 6 *$P_0(1) = 0$ and we can write $P_0(t) = (t-1)R_0(t)$ with*

$$R_0(t) = at^3 - (b-a)t^2 + (a-b+c)t + a - b + c - d.$$

Moreover, P_0 has at least one root strictly greater than one.

We are going to prove that R_0 has at most one root in the unit disk $\mathbb{D}$. We first note that $a - b + c - d = -e$, and the property 3 enables us to tell that the product of the roots of R_0 is greater than one.

3.1 The real roots

Let us study the case in which P_0 satisfies (iii), namely has four real roots. Thanks to property 4, we know that this implies $3b^2 > 8ac$.

Lemma 2 *If P_0 has four real roots, at most one belongs to $]0, 1[$.*

Proof: Property 4, implies $3b^2 > 8ac$, and P_0'' has two real roots, denoted t_- and t_+. As $P_0''(t) = 12at^2 - 6bt + 2c$, we easily obtain that:

$$t_\pm = \frac{b}{4a} \pm \left(\left(\frac{b}{4a}\right)^2 - \frac{c}{6a}\right)^{\frac{1}{2}}.$$

Moreover

$$t_- + t_+ = \frac{b}{2a} > 2 \qquad t_- t_+ = \frac{c}{6a} > 1.$$

We rule out the case in which the three roots of R_0 are smaller than one, because the product of these roots must be larger than one, and proceed by contradiction:

Assumption 2 *P_0 has two roots $(\lambda_i, \lambda_j) \in]0, 1[\times]0, 1[$.*

Figure 2 shows how P_0 could look like under this assumption.

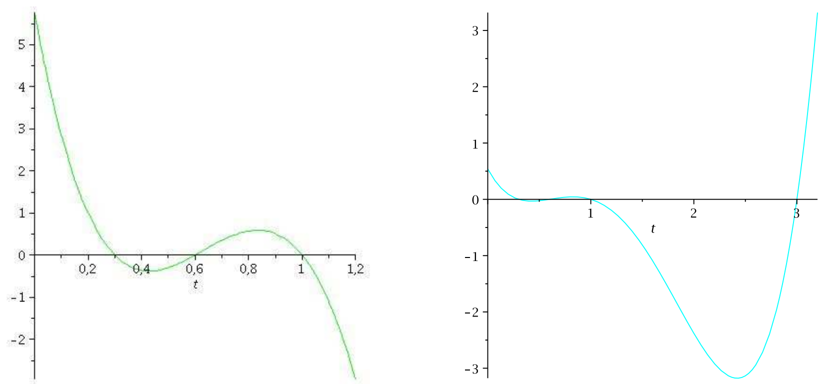

Figure 2: Example of a polynomial satisfying the assumption 2

A straightforward study of the variations of P_0 and P_0' show that our assumption implies that $P_0'(1) < 0$ and $t_- < 1$. This last inequality gives:

$$\frac{b}{4a} - 1 < \left(\left(\frac{b}{4a}\right)^2 - \frac{c}{6a}\right)^{\frac{1}{2}} \iff$$

$$1 - \frac{b}{2a} < -\frac{c}{6a} \iff 6a - 3b + c < 0.$$

Combining this with $P_0'(1) < 0$, we obtain that necessarily:

$$4a - 3b + 2c - d < 0 \qquad 6a - 3b + c < 0 \tag{20}$$

and this is in contradiction with property 5. ■

3.2 The complex roots

Now, we are interested in the possible complex roots of P_0.

Lemma 3 *If P_0 has two complex roots z and $\bar{z}$, they are conjugate and outside the unit disk: $|z| > 1$.*

Proof: We proceed by contradiction.

Assumption 3 *R_0 has two complex roots $z = \rho(\cos\theta + i\sin\theta)$ and $\bar{z}$ such that $\delta = 1 - \rho^2 \geq 0$.*

We have

$$R_0(z) = a\rho^3(\cos^3\theta - 3\cos\theta\sin^2\theta) + (a-b)\rho^2(\cos^2\theta - \sin^2\theta) + (a-b+c)\rho\cos\theta + \\ a - b + c - d + i[a\rho^3(3\sin\theta\cos^2\theta - \sin^3\theta) + 2(a-b)\rho^2\sin\theta\cos\theta + \rho(a-b+c)\sin\theta]$$

If $R_0(z) = 0$, both its real and imaginary parts are equal to zero. Let us study the imaginary part, in which $\rho\sin\theta$ can be put in factor. Hence, we focus on $\Im^*(R_0(z))$ in which $\Im(R_0(z)) = \rho\sin\theta\,\Im^*(R_0(z))$ and

$$\Im^*(R_0(z)) = 4a(\rho\cos\theta)^2 - 2(b-a)\rho\cos\theta + a(1-\rho^2) - b + c = F(\rho\cos\theta)$$
$$F(t) = 4a\left(t^2 - \frac{2(b-a)}{4a}t + \frac{\delta a - b + c}{4a}\right). \tag{21}$$

Let us denote by r_F^- and r_F^+ the two roots of F. It easily follows that

$$r_F^{\pm} = \frac{b-a}{4a} \pm \left(\frac{b-c-\delta a}{4a} + \left(\frac{b-a}{4a}\right)^2\right)^{\frac{1}{2}}.$$

Thanks to the assumption 3, we know that $\rho \leq 1$, and as $\Im^*(R_0(z)) = F(\rho\cos\theta)$, F has at least one real root in $]-1;1[$. The product of the roots is greater than $\frac{3}{2}$, the sum is greater than $\frac{1}{4}$ (due to property 3). Let us express a necessary condition to ensure that $r_F^- < 1$:

$$\frac{b-a}{4a} - \left(\frac{b-c-\delta a}{4a} + \left(\frac{b-a}{4a}\right)^2\right)^{\frac{1}{2}} < 1 \iff$$
$$1 - 2\frac{(b-a)}{4a} < \frac{b-c-\delta a}{4a} \iff 4a - 2(b-a) < b - c - \delta a$$
$$\iff 6a - 3b + c < -\delta a < 0. \tag{22}$$

Next, let us remark that the product of the four roots of P_0 being greater than one and P_0 vanishing in 1, the assumption 3 ensures that the other real root is strictly greater than one. This implies that $P_0'(1) < 0$, and property 5 gives us the contradiction with (22). ■

Property 7 *For all $\theta \in]0,\pi[$, denoting by $z = \cos\theta + i\sin\theta$, we proved that*

$$\Im(R_0(z)) = \sin\theta\left[4a(\cos\theta)^2 - 2(b-a)\cos\theta - b + c\right] > 0.$$

Proof: Obviously, $\forall\,\theta \in]0,\pi[$, $\sin\theta > 0$. We showed in the last proof that the roots of the polynomial F defined by (21) are strictly outside $[0,1]$, for all $\delta \in [0,1]$. For $\delta = 0$, we obtain that

$$F(t) = 4at^2 - 2(b-a)t - b + c$$

Evaluating in $t = 0$, it follows that $F(0) = c - b > 0$ due to property 3. ■

Conclusion to P_0 study We have showed that

(I) 1 is a root of P_0,

(II) P_0 has no negative root,

(III) if P_0 has complex roots, their modulus is strictly larger than one,

(IV) P_0 has at most one root in $]0, 1[$.

Figure 3 illustrates the four configurations of roots for P_0:

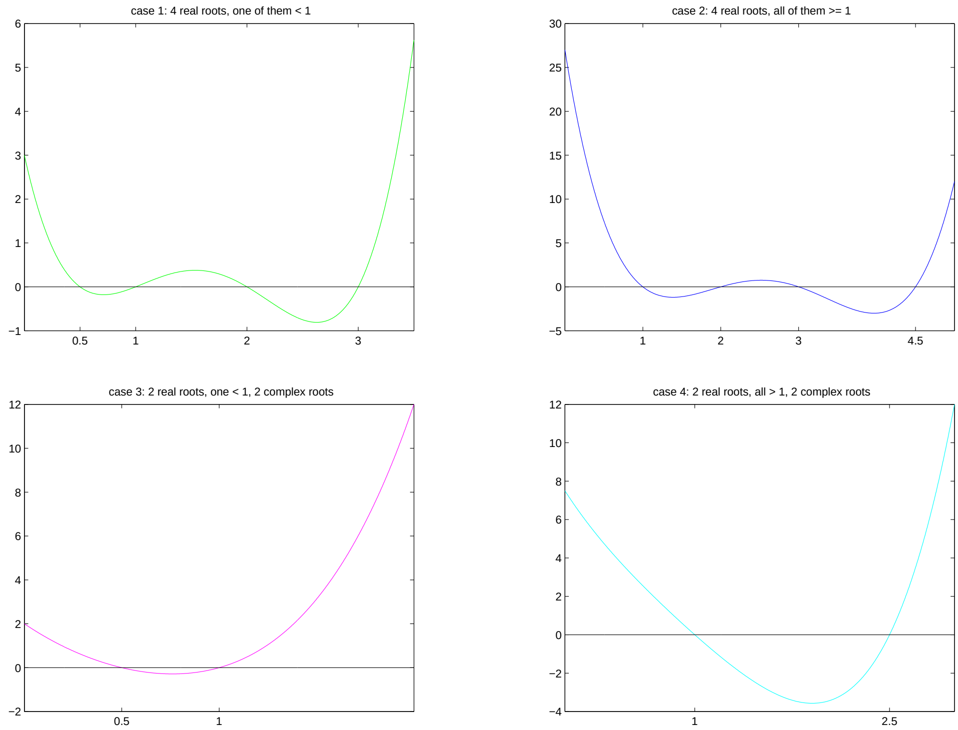

Figure 3: The four possible configurations of P_0 roots: cases 1 to 4

4 Deviations from the limit case : Q and S study

In this section, we study the polynomials Q and S defined, respectively, by (15) and (16). We give the expression of their real roots, and study them for complex values, to ease the proof of the final theorem.

4.1 On the real axis

Let r_Q^- and r_Q^+ denote the roots of Q.

$$Q(t) = \beta n_w t^2 - [\kappa_p + n_w(1+\beta+\lambda_p)]t + n_w. \tag{23}$$

The sum and the product of the roots are non-negative, and we easily obtain that

$$\frac{\Delta_Q}{4} = -\frac{1}{\beta} + \frac{1}{4}\left(1 + \frac{1+\lambda_p}{\beta} + \frac{\kappa_p}{\beta n_w}\right)^2 > \frac{1}{4}\left(1 - \frac{1}{\beta} + \frac{\lambda_p}{\beta} + \frac{\kappa_p}{\beta n_w}\right)^2.$$

Property 8 *Q has two positive roots*

$$r_Q^{\pm} = \frac{1}{2}\left(1 + \frac{1+\lambda_p}{\beta} + \frac{\kappa_p}{\beta n_w}\right) \pm \frac{1}{2}\left(-\frac{4}{\beta} + \left(1 + \frac{1+\lambda_p}{\beta} + \frac{\kappa_p}{\beta n_w}\right)^2\right)^{\frac{1}{2}}$$

$r_Q^- \in]0,1[$ *and* $r_Q^+ \in \left]\frac{1}{\beta}, +\infty\right[$.

Proof:

$$Q(0) = n_w > 0 \qquad Q(1) = -\lambda_p(n_p + n_w) < 0$$

We denote

$$B = 1 + \frac{1+\lambda_p}{\beta} + \frac{\kappa_p}{\beta n_w} \qquad \Delta = B^2 - \frac{4}{\beta}$$

Let us assume that $r_Q^+ = \frac{1}{2}(B + \sqrt{\Delta}) \leq \frac{1}{\beta}$, then

$$\sqrt{\Delta} \leq \frac{2}{\beta} - B \quad \Rightarrow \quad -\frac{1}{\beta} \leq \frac{1}{\beta^2} - \frac{B}{\beta} \quad \Rightarrow \quad B \leq 1 + \frac{1}{\beta}$$

The last inequality does not hold. Hence, $r_Q^+ > \frac{1}{\beta}$. ■

Let $r_S^{\pm}$ denote the two roots of S.

$$S(t) = \beta n_p t^2 - [\kappa_w + n_p(1+\beta+\lambda_w)]t + n_p$$

the discriminant of S satisfies:

$$\frac{\Delta_S}{4} = -\frac{1}{\beta} + \frac{1}{4}\left(1 + \frac{1+\lambda_w}{\beta} + \frac{\kappa_w}{\beta n_p}\right)^2 > \frac{1}{4}\left(1 - \frac{1}{\beta} + \frac{\lambda_w}{\beta} + \frac{\kappa_w}{\beta n_p}\right)^2.$$

Property 9 *S has two positive roots*

$$r_S^{\pm} = \frac{1}{2}\left(1 + \frac{1+\lambda_w}{\beta} + \frac{\kappa_w}{\beta n_p}\right) \pm \frac{1}{2}\left(-\frac{4}{\beta} + \left(1 + \frac{1+\lambda_w}{\beta} + \frac{\kappa_w}{\beta n_p}\right)^2\right)^{\frac{1}{2}}$$

$r_S^- \in]0,1[$ *and* $r_S^+ \in \left]\frac{1}{\beta}, +\infty\right[$.

Proof: The proof is symmetric to that of property 8. ■

Property 10 *Exists two real , t_1 and t_2 such that $t_2 > \frac{1}{\beta}$, $t_1 < 1$ and $\forall\, t \in [t_1, t_2]$,*

$$Q(t) < 0 \qquad \textit{and} \qquad S(t) < 0.$$

4.2 In the complex plane $\mathbb{C}$

We consider here $z = \rho(\cos\theta + i\sin\theta)$ and as we know that $P_0(1) = 0$, we focus on

$$\Im\left(\frac{Q(z)}{z-1}\right) = \frac{1}{|z-1|^2}\Im(Q(z)(\bar{z}-1)) \quad \text{and} \quad \frac{1}{|z-1|^2}\Im(S(z)(\bar{z}-1))$$

and we have

$$Q(z) = 2\beta n_w \rho^2 \cos^2\theta - [\kappa_p + n_w(1+\beta+\lambda_p)]\rho\cos\theta + (1-\beta)n_w \\ + i\rho\sin\theta[2\beta n_w \rho\cos\theta - \kappa_p - n_w(1+\beta+\lambda_p)]$$

$$S(z) = 2\beta n_p \rho^2 \cos^2\theta - [\kappa_w + n_p(1+\beta+\lambda_w)]\rho\cos\theta + (1-\beta)n_p \\ + i\rho\sin\theta[2\beta n_p \rho\cos\theta - \kappa_w - n_p(1+\beta+\lambda_w)].$$

All simplifications being made:

$$\Im(Q(z)(\bar{z}-1)) = \rho\sin\theta\,[\beta n_w(\rho-\cos\theta)^2 + \kappa_p + n_w(1+\beta\,(1-\cos\theta)]$$
$$\Im(S(z)(\bar{z}-1)) = \rho\sin\theta\,[\beta n_p(\rho-\cos\theta)^2 + \kappa_w + n_p(1+\beta\,(1-\cos\theta)]$$

from which we deduce the following property:

Property 11 *For every $\theta \in \,]0,\pi[$, denoting $z = \rho(\cos\theta + i\sin\theta)$ we have that:*

$$\Im(Q(z)(\bar{z}-1)) > 0 \qquad \textit{and} \qquad \Im(S(z)(\bar{z}-1)) > 0.$$

5 The rank condition when $\Phi_y = 0$

5.1 Preliminary property

Lemma 4 *Let P be a polynomial (and more generally, a $\mathcal{C}^1$ function), and λ a simple root of this polynomial: $P(\lambda) = 0$ but $P'(\lambda) \neq 0$. If we add a real quantity q sufficiently small to the polynomial, it translates the root λ in the direction defined by the sign of:*

$$-P'(\lambda)\,q$$

+ defining a translation to the right, - to the left.

Proof Let us choose $\delta > 0$, arbitrarily small. P' being smooth, there exists w such that $\forall\, h \in [-w,w]\ |P'(\lambda+h)| > \delta$. This ensures that P is a bijection from $I = [\lambda - w, \lambda + w]$ on $P(I)$. Now, we denote by

$$q_0 = \min\{|P(\lambda - w)|, |P(\lambda + w)|\}.$$

Assume that $P'(\lambda) > 0$. We define, for every real q such that $|q| < q_0$

$$P_q : [-w,w] \to [P(\lambda-w)+q, P(\lambda+w)+q]$$
$$t \mapsto P(\lambda+t)+q$$

P_q is a bijection, and since $P_q(0) = P(\lambda) + q = q > 0$ and $(P(\lambda-w)+q)(P(\lambda+w)+q) < 0$ by choice of q, if $q > 0$ (resp. < 0), the root is shifted to the left (resp. to the right). The result is the same when $P'(\lambda) < 0$. ∎

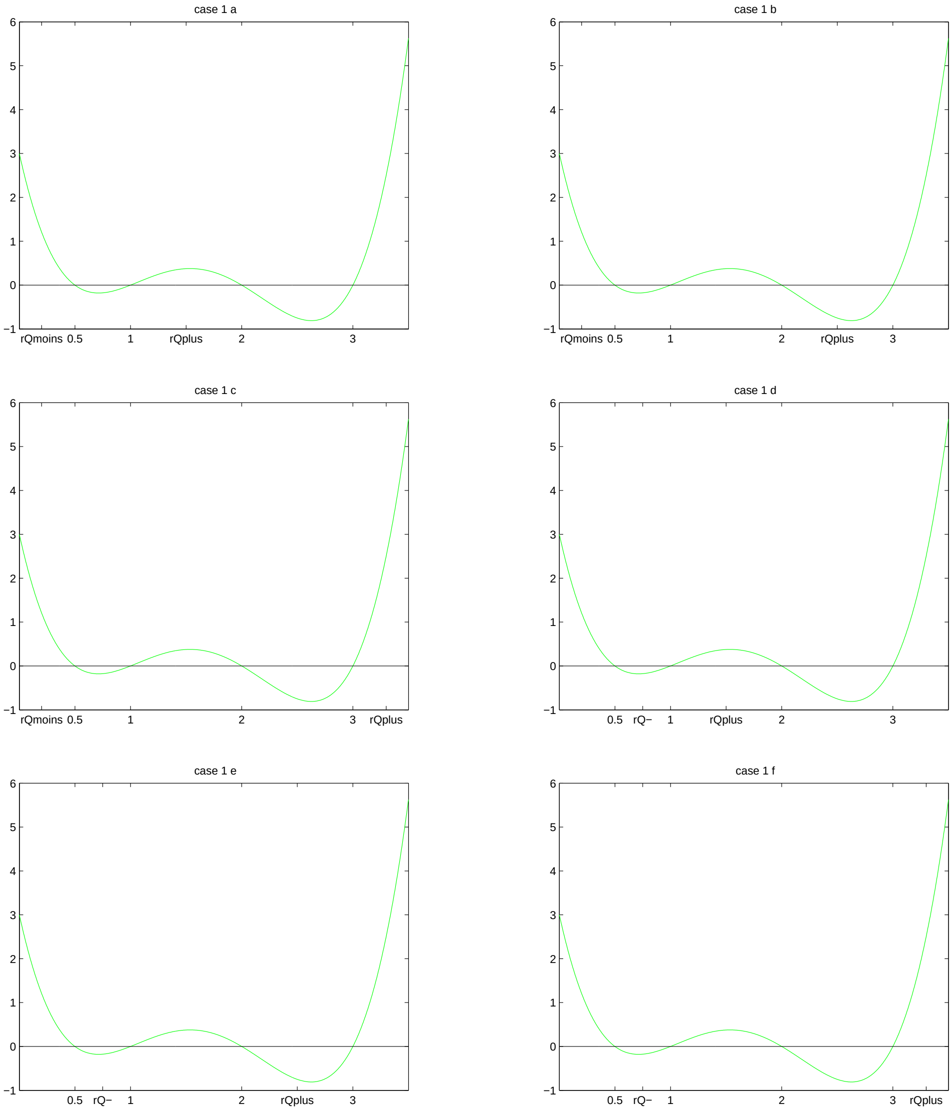

Figure 4: Case 1

5.2 General values of γ, $\xi = 0$

For the sake of clarity, we begin with the study of $P_{\gamma,0}$, that we denote by P_γ in this section. Before giving the main theorem of this part and its proof, let us locate the roots of P_0 with respect to the ones of Q, denoted by $r_Q^{\pm}$. There are exactly 14 possible configurations: 4 configurations just for the roots of P_0, each of this configuration leading to several possibilities, depending on the location of $r_Q^{\pm}$.

Theorem 1 *For every vector $X \in D_X$ and for every $\gamma > 0$, P_γ satisfies property (i): three of its roots are strictly out of $\mathbb{D}$ and the other one is strictly in $\mathbb{D}$.*

Proof

Real roots of P_γ The figures (4), (5) and (6) show all the possible configurations for P_0. Remembering that $P_\gamma = P_0 + \gamma\lambda_w Q$, we consider the four following facts:

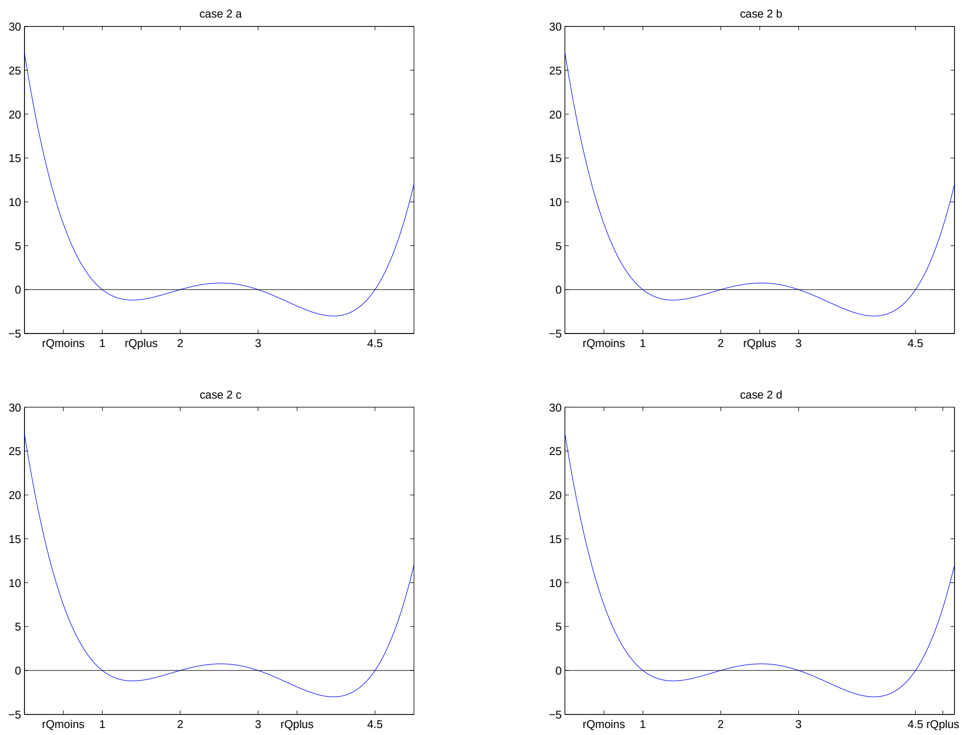

Figure 5: Case 2

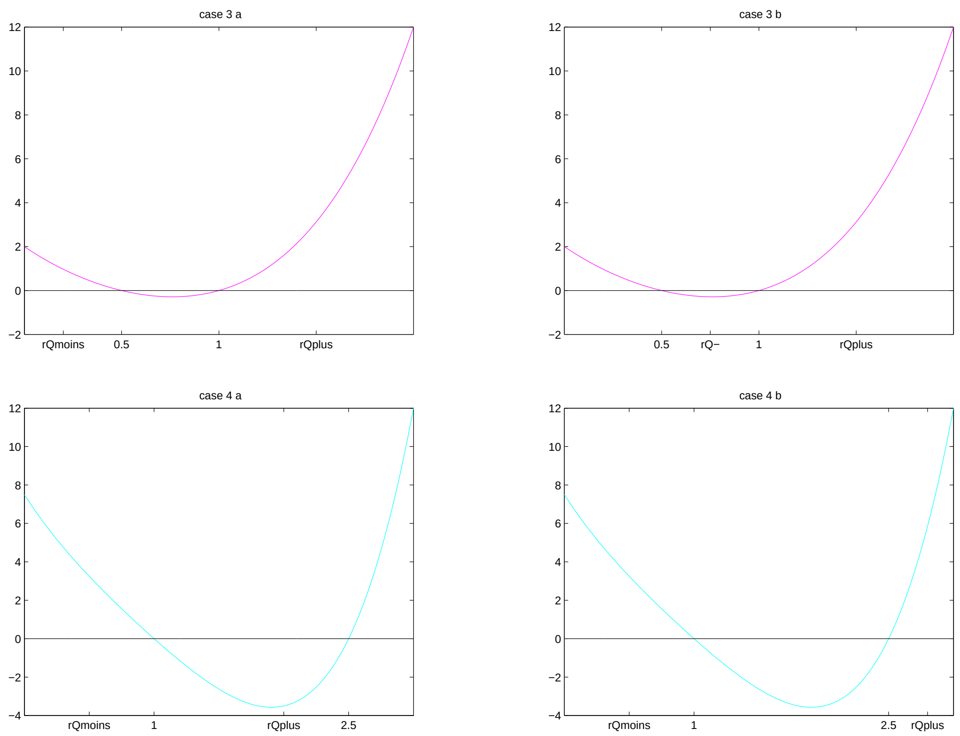

Figure 6: Cases 3 and 4

- with γ increasing, a root of P_γ can never *cross* a root of Q: otherwise it stays *on* the root of Q (we add a null quantity with γ increasing),
- Q is strictly negative between its two roots, namely on $\left]r_Q^-;r_Q^+\right[$, positive outside its roots,
- the roots of P_0, and more generally of P_γ are moving according to lemma (4),
- starting from the cases 1 or 2, and increasing γ, X being kept constant, two of the real roots melt and disappear to create two complex roots for a given value of γ.

With those observations, one can check easily that, starting from P_0:

- in the cases 1 and 3, the root smaller than one is trapped in $]0,1[$, while the root 1 moves toward the right to become one of the root outside $\mathbb{D}$,
- in the case 2 and 4, the root 1 moves toward the left to become the root in $\mathbb{D}$,
- all the real roots of P_0 strictly greater than one cannot enter the interval $]0,1[$.

Those observations remain true when increasing γ without any limit.

Complex roots of P_γ The complex roots of P_0 lie outside the unit disk $\mathbb{D}$ (see Lemma 3). With $\gamma > 0$ we show that the complex roots of P_γ can not *cross* the unit circle. We proceed by contradiction and assume that for a given γ, the root z is on the unit disk.

Assumption 4 *There exist $\gamma > 0$ and $z \in \mathbb{D}$ such that $P_\gamma(z) = 0$.*

Under this assumption, let us study:

$$\begin{aligned}
\frac{P_\gamma(z)}{z-1} &= R_0(z) + \gamma\lambda_w \frac{Q(z)}{z-1} = R_0(z) + \gamma\lambda_w \frac{Q(z)(\bar{z}-1)}{|z-1|^2} \\
&= \frac{1}{|z-1|^2}\left[R_0(z)|z-1|^2 + \gamma\lambda_w Q(z)(\bar{z}-1)\right] \\
\Im\left(\frac{P_\gamma(z)}{z-1}\right) &= \frac{1}{|z-1|^2}\left[\Im(R_0(z))|z-1|^2 + \gamma\lambda_w \Im(Q(z)(\bar{z}-1))\right]
\end{aligned}$$

Taking z belonging to the unit circle $z = \cos\theta + i\sin\theta$, we obtain:

$$\Im(R_0(z))|z-1|^2 + \gamma\lambda_w\Im(Q(z)(\bar{z}-1)) = 2(1-\cos\theta)\Im(R_0(z)) + \gamma\lambda_w\Im(Q(z)(\bar{z}-1)). \tag{24}$$

Thanks to properties 7 and 11, we know that the two quantities on the right-hand side are strictly positive on the unit disk and that their sum cannot be equal to zero. Hence, assumption 4 is false. ■

5.3 General values of ξ, $\gamma = 0$

We study in this section $P_{0,\xi}$ denoted in this section P_ξ.

$$P_\xi(t) = P_0(t) + \xi\,\lambda_p\,S(t)$$

The following result holds

Theorem 2 *For all given $X \in D_X$, $\forall \xi > 0$, P_ξ satisfies property (i): three of its roots are strictly out of the unit disk $\mathbb{D}$ and the other one is strictly inside.*

Proof The proof is symmetric to that of theorem 1. ■

5.4 Rank condition

Lemma 5 *$\forall \gamma \geq 0$, $\forall \xi \geq 0$, if $\gamma \neq 0$ or $\xi \neq 0$ then $P_{\gamma,\xi}$ satisfies property (i).*

Proof

Complex roots If P_0 satisfies (ii), the complex roots of $P_{\gamma,\xi}$ stay outside $\mathbb{D}$. To prove it, we can use the exact same proof as for lemma 3, by adding the contribution of S to the equation (24). We obtain that on the disk:

$$\Im(R_0(z))|z-1|^2 + \gamma\,\lambda_w\,\Im(Q(z)(\bar{z}-1)) + \xi\,\lambda_p\,\Im(S(z)(\bar{z}-1)) = \\ 2(1-\cos\theta)\Im(R_0(z)) + \gamma\lambda_w\,\Im(Q(z)(\bar{z}-1)) + \xi\,\lambda_p\,\Im(S(z)(\bar{z}-1)) \quad (25)$$

and due to property 11 we know that the quantity we added is strictly positive and the same conclusion holds.

Real roots To get the polynomial $P_{\gamma,\xi}$, one starts from P_0, goes to P_γ, and finally obtains $P_{\gamma,\xi}$. We already proved that P_γ satisfies the required property. Now, adding $\xi\lambda_w S$ to this polynomial, we study the evolution of the roots. As the roots of S can not be *crossed*, the situation is the exact same as previously: denoting by $r_\gamma^\pm$ the two real roots of P_γ, we can see that the four following situations:

- $r_\gamma^- < r_S^- < 1 < r_S^+ < r_\gamma^+$
- $r_\gamma^- < r_S^- < 1 < r_\gamma^+ < r_S^+$
- $r_S^- < r_\gamma^- < 1 < r_S^+ < r_\gamma^+$
- $r_S^- < r_\gamma^- < 1 < r_\gamma^+ < r_S^+$

lead to a stable configuration, namely one root in $]0,1[$, three out of $\mathbb{D}$. ■

Theorem 3 *For every $X \in D_X$, for all admissible values of γ and ξ (namely such that $\phi_p+\xi > 0$ and $1-\phi_p+\gamma > 0$), $\gamma+\xi > 0$ **if and only if** $P_{\gamma,\xi}$ satisfies property (i).*

Proof Any point of the domain $D_{p,w}$ can be reached starting from a point on the axis $\phi_p + \phi_w = 1$ and adding positive values of ξ and γ (Figure 1). Hence, the ending point of any path using a negative value of ξ or γ, but such that $\gamma+\xi > 0$ can be reached with a path such that both γ and ξ remain positive. In which case Lemma 5 applies and $\phi_p + \phi_w > 1$ is a sufficient condition to ensure that $P_{\gamma,\xi}$ satisfies the required property for the uniqueness of the equilibrium.

By similar reasoning, it is a necessary one: any polynomial $P_{\gamma,\xi}$ satisfying property (i) has been reached starting from a polynomial P_0. If we come back to the study of the dynamics of its roots, the crucial point is the movement of the root 1 of P_0. According to Lemma **??** and independently from the fact that it is the first or second real root of P_0, root 1 moves in the proper direction if and only if:

$$\xi\lambda_p S(1) + \gamma\lambda_w Q(1) < 0$$

and computing $S(1)$ and $Q(1)$, we find that it requires that

$$(\xi+\gamma)\lambda_p\lambda_w(n_w+n_p) > 0 \quad (26)$$

namely that $\xi+\gamma > 0$. ■

6 The rank condition in the case $\Phi_y \neq 0$

Even when the mandate of the Central Banker is just to control the inflation, as the European central bank does, in practice, one can find that the reaction of this Central Banker to the output gap Φ_y is strictly positive. In such countries, this reaction is nevertheless smaller than the one of the Federal Reserve bank for instance, whose mandate is to react also to the output gap. Thus, in order for our uniqueness result to be useful as a benchmark for monetary policy, it should be a real improvement of the standard Taylor rule, *i.e.* being studied in the case $\Phi_y > 0$.
We are going to see that this case is just a simple generalization of the case $\Phi_y = 0$.

A priori the frontier of indeterminacy can be written $\phi_p + \phi_w = 1 - \theta$, with θ positive and to be determined (Galí, 2008). The characteristic polynomial P of A is still defined by (12):

$$P_{\gamma,\xi}(t) = at^4 - bt^3 + c_{\gamma,\xi,\theta}t^2 - d_{\gamma,\xi,\theta}t + e_{\gamma,\xi,\theta}.$$

with new coefficients:

$$\begin{aligned}
a &= \sigma\beta^2 \\
b &= \beta[\kappa_p + \Phi_y\beta + \sigma(2 + 2\beta + \lambda_p + \lambda_w)] \\
c_{\gamma,\xi,\theta} &= \kappa_p[1 + \lambda_w + \beta(1 + \phi_p + \xi)] + \lambda_w\nu[\lambda_p + \beta(1 - \theta - \phi_p) + \beta\gamma] + \Phi_y\beta[\lambda_p + \lambda_w + 2 + \beta] \\
&\quad + \sigma[1 + 4\beta + \beta^2 + \lambda_p(1 + \beta) + \lambda_w(1 + \lambda_p + (2 - \phi_p - \theta)\beta + \beta\gamma)] \\
d_{\gamma,\xi,\theta} &= \kappa_p[1 + \lambda_w(1 - \theta) + \phi_p(1 + \beta) + \lambda_w\gamma + (1 + \beta + \lambda_w)\xi] + \\
&\quad \lambda_w\nu[\lambda_p(1 - \theta) + (1 - \theta - \phi_p)(1 + \beta) + (1 + \beta + \lambda_p)\gamma + \lambda_p\xi] + \Phi_y[1 + \lambda_w + \lambda_p + 2\beta] \\
&\quad + \sigma[2\beta + 2 + \lambda_p + \lambda_w(1 + \lambda_p(1 - \theta) + (1 + \beta)(1 - \theta - \phi_p) + (1 + \beta + \lambda_p)\gamma + \lambda_p\xi)] \\
e_{\gamma,\xi,\theta} &= \sigma[1 + \lambda_w(1 - \theta - \phi_p)] + \kappa_p\phi_p + \kappa_p\xi + \lambda_w\nu(1 - \theta - \phi_p) + \lambda_w(\sigma + \nu)\gamma + \Phi_y.
\end{aligned}$$

There is now a third dimension (Φ_y) to the domain defined by Figure 1. We recover the decomposition (14) with strictly identical polynomials Q and S by choosing the baseline polynomial P_0 accordingly. θ is set such that $P_0(1) = 0$, which implies that θ is proportional to Φ_y:

$$\theta = \Phi_y \frac{(1-\beta)(\lambda_p + \lambda_w)}{\lambda_w\lambda_p[\sigma + \nu + n_p]}. \tag{27}$$

Hence the limit condition $\phi_p + \phi_w = 1 - \theta$ defines a tetrahedron in the space (ϕ_p, ϕ_w, Φ_y):

$$\phi_p + \phi_w + \Phi_y \frac{(1-\beta)(\lambda_p + \lambda_w)}{\lambda_w\lambda_p[\sigma + \nu + n_p]} = 1 \qquad \phi_p \geq 0, \quad \phi_w \geq 0, \quad \Phi_y \geq 0. \tag{28}$$

This tetrahedron is the frontier of indeterminacy when $\Phi_y \neq 0$.

We keep the notation θ to ease the reading, though θ is given by (27). In the case $\Phi_y = 0$, the crucial condition to ensure the required properties on P_0 is property 5. Property 12 generalizes this property when $\Phi_y \neq 0$.

Property 12 *For all $X \in D_X$, $P_0'(1) = 4a - 3b + 2c - d$ and $6a - 3b + c$ cannot be both negative.*

Proof We proceed again by contradiction and assume that $4a - 3b + 2c - d$ and $6a - 3b + c$ are both negative for a given set of parameters X. After simplification, we obtain:

$$\begin{aligned}
p_1 = 4a - 3b + 2c - d = {} & \kappa_p[\lambda_w(1 + \theta) + (1 - \beta)(1 - \phi_p)] + \lambda_w\nu[\lambda_p(1 + \theta) - (1 - \beta)(1 - \theta - \phi_p)] \\
& + \sigma[(1 - \beta)(\lambda_p + \lambda_w(\phi_p + \theta)) + \lambda_w\lambda_p(1 + \theta)] + \Phi_y[(2\beta - 1)(\lambda_p + \lambda_w) - (1 - \beta)^2] < 0
\end{aligned}$$

and

$$\begin{aligned}
p_2 = 6a - 3b + c = {} & \kappa_p[\lambda_w + 1 - \beta - \beta(1 - \phi_p)] + \lambda_w\nu[\lambda_p + \beta(1 - \theta - \phi_p)] \\
& + \Phi_y\beta[2(1 - \beta) + \lambda_p + \lambda_w] + \sigma[\lambda_p\lambda_w + (1 - \beta)(1 - \beta + \lambda_p + \lambda_w) - \beta(\lambda_p + \lambda_w(\phi_p + \theta))] < 0.
\end{aligned}$$

We denote by $c_i[r]$ the cofactor of the variable r in the expression p_i. Now, let us study the structure of those inequalities

$$p_1 = \kappa_p c_1[\kappa_p] + \sigma c_1[\sigma] + \lambda_w \nu c_1[\lambda_w \nu] + \Phi_y c_1[\Phi_y] < 0 \tag{29}$$
$$p_2 = \kappa_p c_2[\kappa_p] + \sigma c_2[\sigma] + \lambda_w \nu c_2[\lambda_w \nu] + \Phi_y c_2[\Phi_y] < 0 \tag{30}$$

where

$$c_1[\kappa_p] > 0, \quad c_1[\sigma] > 0, \quad c_2[\lambda_w\nu] > 0, \quad c_2[\Phi_y] > 0, \text{ and}$$
$$c_1[\lambda_w\nu],\ c_1[\Phi_y],\ c_2[\kappa_p] \text{ and } c_2[\sigma] \text{ are of unknown sign.}$$

At least one of the cofactor of unknown sign of (29) and one of the cofactor of unknown sign of (30) has to be negative to ensure the negativity of the whole expression p_1 and p_2. We begin with a simple property

Property 13 *If $c_1[\Phi_y] < 0$, then $c_2[\sigma] \geq 0$.*

Indeed, let us assume that $c_1[\Phi_y] < 0$ and $c_2[\sigma] < 0$. This writes:

$$(2\beta - 1)(\lambda_p + \lambda_w) < (1-\beta)^2$$
$$\lambda_p\lambda_w + (1-\beta)^2 + (1-\beta)(\lambda_p + \lambda_w) < \beta(\lambda_p + \lambda_w(\phi_p + \theta))$$

Inserting the first inequality into the second one, we obtain that

$$(2\beta - 1)(\lambda_p + \lambda_w) + \lambda_p\lambda_w + (1-\beta)(\lambda_p + \lambda_w) < \beta(\lambda_p + \lambda_w(\phi_p + \theta)) \quad \Rightarrow$$
$$\lambda_p\lambda_w \quad - \beta(1 - \theta - \phi_p)$$

and this is not possible, considering that λ_w and λ_p are strictly positive. Hence, to ensure that p_1 and p_2 are strictly negative, the possibilities are the following:

(I) $c_1[\lambda_w\nu] < 0$ and $c_1[\Phi_y] < 0$, so $c_2[\sigma] \geq 0$ and $c_2[\kappa_p] < 0$.

(II) $c_1[\lambda_w\nu] < 0$ so $c_2[\sigma] \geq 0$ and $c_2[\kappa_p] < 0$. $c_1[\Phi_y] \geq 0$.

(III) $c_1[\lambda_w\nu] < 0$ and $c_1[\Phi_y] \geq 0$.

Let us rule out those cases in the order.

Assumption 5 *$c_1[\lambda_w\nu] < 0$ and $c_1[\Phi_y] < 0$, so $c_2[\sigma] \geq 0$ and $c_2[\kappa_p] < 0$.*

This implies:

$$[\lambda_w(1+\theta) + (1-\beta)(1-\phi_p)]\kappa_p < \Phi_y[(1-\beta)^2 - (2\beta-1)(\lambda_p+\lambda_w)] +$$
$$\lambda_w\nu[(1-\theta-\phi_p)(1-\beta) - \lambda_p(1+\theta)] - \sigma[(1-\beta)(\lambda_p + \lambda_w(\phi_p+\theta)) + \lambda_w\lambda_p(1+\theta)] = M$$

and

$$[\beta(1-\phi_p) - \lambda_w - (1-\beta)]\kappa_p > s[\lambda_p\lambda_w + (1-\beta)(1-\beta+\lambda_p+\lambda_w) - \beta(\lambda_p + \lambda_w(\phi_p+\theta))] +$$
$$\lambda_w\nu[\lambda_p + \beta(1-\theta-\phi_p)] + \Phi_y\beta[2(1-\beta) + \lambda_p + \lambda_w] = m$$

Now, we compute $[\lambda_w(1+\theta) + (1-\beta)(1-\phi_p)]m - [\beta(1-\phi_p) - \lambda_w - (1-\beta)]M$ and prove that it is positive. For that purpose, let us study the terms in factor of σ, $\lambda_w\nu$ and Φ_y. For σ, we can see directly that it is strictly positive (because of the negative sign of its cofactor in M). For Φ_y, we obtain:

$$\lambda_w(1+\theta)\beta(2+\lambda_p) + \lambda_w^2(1-\beta+\beta\theta) + \lambda_w(1-\beta)(1-\beta+\lambda_p) +$$
$$(1-\beta)[(1-\beta)^2 - (2\beta-1)(\lambda_p+\lambda_w)] + \beta^2(1-\phi_p)(\lambda_p+\lambda_w) > 0$$

and finally, we obtain for $\lambda_w\nu$

$$\lambda_p\lambda_w(1+\theta) + \lambda_p(1-\beta)(1-\phi_p) + \lambda_w\beta(1+\theta)(1-\theta-\phi_p) + \lambda_w(1-\theta-\phi_p)(1-\beta) +$$
$$(1-\beta)^2(1-\theta-\phi_p) + \lambda_p(1+\theta)[\beta(1-\phi_p) - \lambda_w - (1-\beta)] > 0$$

Hence, the assumption 5 cannot hold.

Assumption 6 $c_1[\sigma] < 0$ *so* $c_2[\sigma] \geq 0$ *and* $c_2[\kappa_p] < 0$. $c_1[\lambda_w \nu] \geq 0$

Now we use Φ_y and obtain

$$[(1-\beta)^2 - (2\beta-1)(\lambda_p+\lambda_w)]\Phi_y > \kappa_p[\lambda_w(1+\theta)+(1-\beta)(1-\phi_p)] + \lambda_w\nu[\lambda_p(1+\theta)-(1-\theta-\phi_p)(1-\beta)] + \sigma[(1-\beta)(\lambda_p+\lambda_w(\phi_p+\theta))+\lambda_w\lambda_p(1+\theta)] = m$$

and

$$\Phi_y\beta[2(1-\beta)+\lambda_p+\lambda_w] < -\sigma[\lambda_p\lambda_w+(1-\beta)(1-\beta+\lambda_p+\lambda_w)-\beta(\lambda_p+\lambda_w(\phi_p+\theta))] - \lambda_w\nu[\lambda_p+\beta(1-\theta-\phi_p)] + \kappa_p[\beta(1-\phi_p)-\lambda_w-(1-\beta)] = M.$$

We proceed as previously and compute $\beta[2(1-\beta)+\lambda_p+\lambda_w]m - [(1-\beta)^2-(2\beta-1)(\lambda_p+\lambda_w)]M$. It is straightforward to see that the cofactors of σ and $\lambda_w\nu$ are positive. The cofactor of κ_p is

$$\beta\lambda_w(1+\theta)[2(1-\beta)+\lambda_w+\lambda_p] + \beta(1-\beta)^2(1-\phi_p) + \beta^2(1-\phi_p)(\lambda_p+\lambda_w) + \lambda_w(1-\beta)[(1-\beta)^2-(2\beta-1)(\lambda_p+\lambda_w)] > 0$$

because all the terms involved are positive. As previously, we get a contradiction and assumption 6 cannot hold.

Assumption 7 $c_1[\lambda_w\nu] < 0$ *and* $c_1[\Phi_y] \geq 0$

Now, we have to use $\lambda_w\nu$. We define

$$[(1-\theta-\phi_p)(1-\beta)-\lambda_p(1+\theta)]\lambda_w\nu > \kappa_p[\lambda_w(1+\theta)+(1-\beta)(1-\phi_p)] + \Phi_y[(2\beta-1)(\lambda_p+\lambda_w)-(1-\beta)^2] + \sigma[(1-\beta)(\lambda_p+\lambda_w(\phi_p+\theta))+\lambda_w\lambda_p(1+\theta)] = m$$

and

$$[\lambda_p+\beta(1-\theta-\phi_p)]\lambda_w\nu < \kappa_p[\lambda_w+1-\beta-\beta(1-\phi_p)] - \Phi_y\beta[2(1-\beta)+\lambda_p+\lambda_w] - \sigma[\lambda_p\lambda_w+(1-\beta)(1-\beta+\lambda_p+\lambda_w)-\beta(\lambda_p+\lambda_w(\phi_p+\theta))] = M$$

The cofactor of Φ_y in $[\lambda_p+\beta(1-\theta-\phi_p)]m - [(1-\theta-\phi_p)(1-\beta)-\lambda_p(1+\theta)]M$ is strictly positive, and we just have to compute the cofactors of σ and κ_p. The cofactor of σ is:

$$\begin{aligned}&\lambda_p(1-\beta)+\lambda_p\lambda_w(1-\beta)(\phi_p+\theta)+\lambda_p\lambda_w\beta(1+\theta)(1-\theta-\phi_p)+\lambda_p\lambda_w(1-\beta)(1-\theta-\phi_p)+\\ &\lambda_p^2\beta(1+\theta)+\lambda_p\lambda_w\beta(1+\theta)(\phi_p+\theta)+(1-\beta)^2(1-\theta-\phi_p)(\lambda_p+\lambda_w)+(1-\beta)^3(1-\theta-\phi_p)\\ &-\lambda_p(1+\theta)(1-\beta)^2-\lambda_p(1+\theta)(1-\beta)(\lambda_p+\lambda_w).\end{aligned}$$

The annoying terms are the one of the last line, because of their negative sign. But using the fact that $c_1[\lambda_w\nu] < 0$, which writes

$$\lambda_p(1+\theta) < (1-\beta)(1-\theta-\phi_p)$$

we get that the fourth line is greater than:

$$-(1-\beta)^3(1-\theta-\phi_p)-(1-\beta)^2(1-\theta-\phi_p)(\lambda_p+\lambda_w)$$

and those terms exactly cancel the one of the third line. Hence the cofactor of σ is strictly positive. Now we compute the cofactor of κ_p. Note that if $\lambda_w+1-\beta-\beta(1-\phi_p) < 0$ we immediately know that this cofactor is positive. Hence, we can assume that this quantity is positive. Hence, $c_2[\sigma] < 0$, and that can be written

$$\lambda_w(1-\beta)+(1-\beta)^2 < \beta(\lambda_p+\lambda_w(\phi_p+\theta))-\lambda_p\lambda_w-\lambda_p(1-\beta)$$

The cofactor of κ_p writes

$$\begin{aligned}&2\lambda_p\lambda_w(1+\theta)+\lambda_w(1+\theta)\beta(1-\theta-\phi_p)+\lambda_p(1-\beta)(1-\phi_p)+\beta(1-\beta)(1-\phi_p)(1-\theta-\phi_p)+\\ &(1-\beta)\lambda_p(1+\theta)+\beta(1-\phi_p)[(1-\theta-\phi_p)(1-\beta)-\lambda_p(1+\theta)]\\ &-\lambda_w(1-\beta)(1-\theta-\phi_p)-(1-\beta)^2(1-\theta-\phi_p)\end{aligned}$$

using the former inequality, the annoying terms on the third line are greater than

$$\lambda_p\lambda_w(1-\theta-\phi_p)+\lambda_p(1-\beta)(1-\theta-\phi_p)-\beta\lambda_p(1-\theta-\phi_p)-\beta\lambda_w(\phi_p+\theta)(1-\theta-\phi_p)$$

we have directly that

$$-\beta\lambda_w(\phi_p+\theta)(1-\theta-\phi_p)+\beta\lambda_w(1+\theta)(1-\theta-\phi_p)\ >\ 0.$$

Now we use

$$\lambda_p(1+\theta)\ <\ (1-\theta-\phi_p)(1-\beta)$$

to obtain that

$$-\beta\lambda_p(1-\theta-\phi_p)+\beta(1-\beta)(1-\phi_p)(1-\theta-\phi_p)\ >\ -\beta(1-\beta)(1-\theta-\phi_p)^2+\beta(1-\beta)(1-\phi_p)(1-\theta-\phi_p)\ >\ 0$$

hence, the cofactor of κ_p is also strictly positive. ■

From property 12, the decomposition of polynomial P, the definition of its coefficients and the definition of θ (27), all the developments derived in the case $\Phi_y = 0$ apply and theorem 3 can be generalized.

Theorem 4 *For every $X \in D_X$, for all admissible values of γ and ξ (namely such that $\phi_p+\xi\ >\ 0$ and $1-\theta-\phi_p+\gamma\ >\ 0$ with $\theta = \Phi_y \frac{(1-\beta)(\lambda_p+\lambda_w)}{\lambda_w\lambda_p[\sigma+\nu+n_p]} > 0$), $\gamma+\xi\ >\ 0$* ***if and only if*** *$P_{\gamma,\xi,\theta}$ satisfies property (i).*

In other words, the neo-Keneysian model with staggered prices and wages (1) to (5) has a unique solution under rational expectations if and only if $\Phi_p + \Phi_w + \Phi_y \frac{(1-\beta)}{(n_w+n_p)}\left(\frac{1}{\lambda_p}+\frac{1}{\lambda_w}\right)\ >\ 1.$

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A General elements on neo-Keynesian models for monetary policy solved under rational expectations

This paper was written as a collaboration between an economist and an applied mathematician. This appendix provides the necessary definitions to set our problem for someone not familiar with neo-Keynesian models for monetary policy solved under rational expectations.

Definition 1 (Dynamic Stochastic General Equilibrium Models (DSGE)) *A DSGE model is a set of equations which describes jointly the short term fluctuations of key macroeconomic variables (growth, inflation, interest rate...). The dynamic of these variables, referred to as endogenous variables, will depend on their past, present and forecast values, but also on exogenous variables (shocks), whose dynamic is not described by the model.*

In this class of models are neo-Keynesian models, particularly useful in the context of monetary analysis and Real Business Cycle models in which nominal variables (in particular prices) are supposed flexible.

These models borrow from microeconomists their modelling of different economic agents (households, firms, governments, banks, monetary authority...) on different markets (goods, assets, labour...). They have triggered an extensive literature at the frontier between economics and applied mathematics.
Economic decisions, such as savings, investments, consumption, wage setting, will then be written as functions of the past and present values of various economic variables and the expectation formed by the deciding agent over the future values of some of these economic variables. These latter variables will be referred to as forward looking variables. For instance, a household may increase its savings and reduce today's consumption if it believes the remuneration of its savings will increase tomorrow.
To solve such a model, one must define how agents form their expectations over the future. One way to define this function is to make the (strong) assumption that every economic agent knows the others' functions of decision and will form expectations compatible with them.[7] As a consequence, all agents will form the same expectations.
To understand better the concept of expectations in this framework, it is important to point out the stochastic dimension of DSGE models, which contains two types of variables[8].
Endogenous variables are the result of the agents' decisions (consumption, prices, production, savings, investments...). Their dynamic is described by the model.
Exogenous ones are perturbations, or shocks on the endogenous variables which are not explained in the model (oil shock, change in the bargaining power of trade unions, fiscal changes, soccer team winning the world cup, hurricanes destroying houses or factories...). These variables will be summarized with random variables (usually iid white noises) impacting the economy in various ways.
Since the realizations of these random variables are not known in advance, agents must try to guess them. From these guesses, they can deduce future values of the endogenous variables.

Definition 2 (Rational expectations) *The expectations mentioned so far are called rational expectations.*
They are the statistical operator expectation (or first moment), over the distribution of possible realizations of the exogenous variables, conditional on the values of the endogenous and exogenous variables up to now and the equations of the model, which we will denote $E(.|t)$

To sum up, if the endogenous variables, *i.e.* the variables of interest, are gathered in the vector $X(t)$, and the exogenous variables in the vector $z(t)$ our model can be written as

$$E(X(t+1)|t) = AX(t) + Bz(t) \tag{31}$$

The vector X can include endogenous variables at the current and the previous period to allow for the representation of the model above, in which case the number of forward looking variables will be strictly lower than the dimension of X. The components of the vector $z(t)$ are simply the different exogenous variables at the current period. The matrix B describes the way in which these shocks (the exogenous variables) impact the endogenous variables.

[7] This idea is attributed to Robert Lucas (Lucas, 1976) and has been a cornerstone of macroeconomics since then.
[8] A variable being a function of time, a sequence indexed by time or a time series, depending on where you come from in science.

A solution to such a model is a trajectory of endogenous variables compatible with the equations of the model (including the expectation mechanism) for any realization of the exogenous variables over time. It is not unique *a priori*. The non-uniqueness of the solution would mean that in such an economy, there can be spontaneous fluctuations, as agents beliefs or coordination "jump" from one solution to another. Such equilibria are called sunspot equilibria and are problematic as they increase the volatility of the economy (Woodford, 1987).

An important characteristic of the model is the number of forward looking variables, i.e. the variables upon which the agents must form some expectations. In a seminal paper of 1980, Blanchard and Kahn have shown that there is a unique solution to model (31) if and only if there are as many forward looking variables as eigenvalues of A which are strictly larger than one in modulus.

Theorem 5 (Blanchard, Kahn) *We consider a linearised DSGE model:*

$$E(X(t+1)|t) = AX(t) + Bz(t) \tag{32}$$

in which X gather endogenous variables and z exogenous variables. If solved under rational expectations, this model has a unique solution if and only if there are as many forward looking variables in X as there are eigenvalues of A which are strictly larger than one in modulus.

DSGE models are widely used in central banks to enlighten the conduct of monetary policy. Indeed, such models provide a normative framework to analyse the effect of the Central Bank on the economy and describe how its behaviour can be optimal, that is maximize a welfare criterion.

Definition 3 (Central Bank and monetary policy) *A central bank is a political institution. Historically, these institutions have been created to print and issue the money needed for economic transactions. Nowadays, they also usually serve control and regulation purposes on the private banking system.*
More important for this paper is the central bank's role in conducting monetary policy : by setting the interest rate at which it lends money to private banks, the central banks can control inflation and activity. The choice of this short term interest rate is the monetary policy instrument through which the central bank monitors (though imperfectly) the economy.
For instance, the European central bank's[9] *mandate is to maintain price stability, understood as an average inflation of consumption prices lower but close to 2%.*

(Taylor, 1993) showed that the Federal Reserve (the USA Central Bank) monetary policy is to set its interest rate following a function of price inflation and the level of production.

Definition 4 (Taylor rule) *The Taylor rule is a function which describes monetary policy decisions, i.e. the choice of the interest rate by the Central Banker, as a linear function of inflation and output:*

$$i_t = i^* + \phi_\pi \pi_t + \phi_y y_t$$

where i, i^*, π, y are the interest rate set, the interest rate target of the Central Banker, price inflation and output gap[10]. *Many variations of this rule have been studied, using expected future inflation, output growth, over four quarters inflation...*

(Taylor, 1993) argues that this rule is a good benchmark for monetary policy committees. Indeed later research have shown that such a rule is optimal in the standard neo-Keynesian model for monetary policy analysis (Woodford, 2001).

Another key issue is the uniqueness of the model's solution: the solution is unique if and only if the model satisfies the condition of theorem (5) also known as the rank condition or Blanchard and Kahn condition. On the standard neo-Keynesian model for monetary policy analysis, it has been shown that this condition solely depends on the Central Banker's policy rule. Numerically on more complex models, we generally find that ensuring the uniqueness of the

[9] http://www.ecb.int/press/govcdec/html/index.en.html
[10] The output gap is the deviation of output (i.e. production) from its potential or optimal value. It captures the extend to which the economy is overheating.

solution relies crucially on the Central Banker. This result is known as the Taylor principle: the Central Banker's reaction to price inflation must be higher than one. More simply, it means that if inflation increases by 1 point, the Central Banker should increase its interest rate by more than 1 point.

In this paper we determine the necessary and sufficient conditions to ensure the uniqueness of the solution to a model less constrained than the standard neo-Keynesian model for monetary policy analysis. This model is borrowed from (Erceg et al., 2000) and (Galí, 2008), it differs from the standard framework by including wage rigidities in addition to price rigidities. In this model the Central Banker aims at stabilizing altogether price and wage inflation and output, so we consider a policy rule reacting to these three endogenous variables. We show that the necessary and sufficient condition to ensure the uniqueness of the solution depends only on the reaction function of the Central Banker, *i.e.* on how the interest rate is set.