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ON THE STATIONARITY OF DYNAMIC CONDITIONAL CORRELATION MODELS

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Abstract

We provide conditions for the existence and the unicity of strictly stationary solutions of the usual Dynamic Conditional Correlation GARCH models (DCC-GARCH). The proof is based on Tweedie's (1988) criteria, after having rewritten DCC-GARCH models as nonlinear Markov chains. Moreover, we study the existence of their finite moments.

Key words and phrases: Multivariate dynamic models, conditional correlations, stationarity, DCC.

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1 Introduction

1.1 The problem

In multivariate extensions of GARCH models, modelers are faced with the problem of correlations (between asset returns, in most applications). The simplest idea is to assume that these correlations are constant in time, and constitute only an additional matrix of parameters. This has provided the class of Constant Conditional Correlations models (CCC), first introduced by Bollerslev (1990). Since CCC models can be written as first-order Markov processes, it is relatively easy to prove the existence of strictly stationary and explicit solutions, even if the latter ones are analytically complex: see classical textbooks, for instance Francq and Zakoïan (2010).

It appeared rapidly that the assumption of constant correlations is too strong. It does not correspond to economic intuition and to many empirical features: see the recent paper of Otranto and Bauwens (2013) and the numerous references therein, for instance. Therefore, Engle (2002) and Tse and Tsui (2002) have proposed to extend CCC specifications by adding a particular dynamics on the (conditional) correlation matrices of returns, denoted here by (R_t). To insure modelers are dealing with true correlation matrices, these authors introduced a nonlinear transform: there exists a sequence of variance-covariance matrices (Q_t) such that $R_t = diag(Q_t)^{-1/2}.Q_t.diag(Q_t)^{-1/2}$, and (Q_t)-dynamics are specified instead of (R_t) dynamics directly. This nonlinear transform insures that R_t is always a correlation matrix, i.e. definite positive with ones on its main diagonal. Nonetheless, it complicates a lot the work of stating stationarity conditions of DCC models. Indeed, analytically tractable solutions of such processes do not exist anymore. This explains why the existence of stationarity solutions of DCC models and their unicity have not been established in the literature yet, nor the finiteness of their moments. Particularly, this implies that theoretically sound statistical inference procedures do not exist yet, as noticed in Caporin and McAleer (2013). Nonetheless, despite their theoretical insufficiencies, DCC models have been used intensively among academics and practitioners. Beside numerous applied works, several extensions of the baseline DCC representation have been proposed in the literature: inclusion of asymmetries (Cappiello, Engle, and Sheppard, 2006), of volatility thresholds (Kasch and Caporin, 2013), of macro-variables (Otranto and Bauwens, 2013), of univariate switching regime probabilities (Fermanian and Malongo, 2013), among others. Other authors have revisited the DCC parameterization itself: Billio, Caporin, and Gobbo (2006), Franses and Hafner (2009), etc. Therefore, there is an urgent need for new theoretical results concerning the seminal DCC model itself.

The goal of this paper is to fill this gap, focusing on the stationarity problem. After having introduced some notations, we define DCC models at the beginning of Section 2. They will be rewritten as "almost linear" Markov chains in Subsection 2.2. The existence of strong and weak stationary solutions is stated in Subsection 3.1. Subsection 3.2 exhibits sufficient conditions to get their unicity. The proofs are gathered in the appendices.

1.2 Notations

Consider an (n,m) matrix $M = [m_{ij}]_{1\leq i\leq n, 1\leq j\leq m}$.

- $M \geq 0$ (resp. $M > 0$) means that all elements of M are positive (resp. strictly positive).
- $|M| = [|m_{ij}|]_{1\leq i\leq n, 1\leq j\leq m}$.
- For any vector $\mathbf{x} \in \mathbb{R}^m$, $\|\mathbf{x}\|_2$ denotes the usual euclidian norm of $\mathbf{x}$.
- If $n = m$, let the diagonal matrix $diag(M) = [m_{ij}\mathbf{1}(i = j)]_{1\leq i\leq m, 1\leq j\leq m}$ and the vector $Vecd(M) = [m_{ii}]_{1\leq i\leq m}$ in $\mathbb{R}^m$.
- If $n = m$ and M is symmetrical, $Vech(M)$ denotes the $m(m+1)/2 := m^*$ column vector whose components are read from M column-wise and without redundancy. To formalize, denote $Vech(M) = [\tilde{m}_k]_{1\leq k\leq m^*}$. Actually,

$\tilde{m}_k = m_{ij}$ for the unique couple of indices (i,j) in $\{1,\ldots,m\}^2$, $i \geq j$ such that $[m + (m-1) + \ldots + (m-j+2)]^+ + (i-j+1) = k$. Thus, this defines a one-to-one mapping ϕ between the indices $k \in \{1,\ldots,m^*\}$ and the couples (i,j), $i \geq j$, $1 \leq i,j \leq m$, i.e. $(i,j) = (\phi_1(k), \phi_2(k)) = \phi(k)$.

- $\rho(M)$ denotes the spectral radius of the squared matrix M, i.e. the largest of the modulus of M's eigenvalues. If M is definite and nonnegative, then its smallest eigenvalue is denoted by $\lambda_1(M)$.

- $\otimes$ denotes the usual Kronecker product, and $M^{\otimes p} = M \otimes \ldots \otimes M$ (p times);

- $\odot$ denotes the element-by-element product: if v is a vector in $\mathbb{R}^n$, then $v \odot M = [v_i m_{ij}]_{1 \leq i \leq n, 1 \leq j \leq m}$. If M and N are conformable, then $M \odot N = [m_{ij} n_{ij}]_{1 \leq i \leq n, 1 \leq j \leq m}$.

- An arbitrary matrix norm will be denoted by $\|\cdot\|$. Sometimes, we consider particular norms like $\|M\|_\infty = \max_{1 \leq i \leq n, 1 \leq j \leq m} |m_{ij}|$, $\|M\|_1 := \sum_{i,j} |m_{ij}|$, or the spectral norm defined by

$$\|M\|_s = \max\{\sqrt{\lambda} \mid \lambda \text{ is an eigenvalue of } M'M\} = \max_{\mathbf{x}} \frac{\|M\mathbf{x}\|_2}{\|\mathbf{x}\|_2},$$

  for any squared matrix M. When M is positive, $\|M\|_s = \rho(M)$.

- For any column vector $z \in \mathbb{R}^m$, $\vec{z} := (z_1^2, \ldots, z_m^2)'$.

- e denotes a vector of ones, whose dimension will be implicit and be given by the context.

- 0_m denotes the $m \times m$ matrix of zeros. I_m denotes the $m \times m$ identity matrix.

- If the coefficients of M depend on a vector $\mathbf{x}$, than $\sup_{\mathbf{x} \in A} M(\mathbf{x})$ is the matrix $[\sup_{\mathbf{x} \in A} m_{ij}(\mathbf{x})]$.

2 Dynamic Conditional Correlation models

2.1 The classical DCC specification

Let us recall the standard DCC model, as introduced in Engle (2002) or Tse and Tsui (2002). Consider a stochastic process $(y_t)_{t\in\mathbb{Z}}$ in $\mathbb{R}^m$, typically a vector of m asset returns. The sigma field generated by the past information of this process until (but including) time $t-1$ is denoted by $\mathcal{I}_{t-1}$. Following current practice, we can filter out the mean values of this series. Let $\mu_t(\theta) = E[y_t|\mathcal{I}_{t-1}] \equiv E_{t-1}[y_t]$ be the conditional mean vector of y_t. It depends on a vector of parameters $\theta \in \Theta$. We define a "detrended" series $(z_t)_{t\in\mathbb{Z}}$ by

$$y_t = \mu_t(\theta) + z_t, \;\; E_{t-1}[z_t] = 0.$$

Assume we can write $z_t = H_t^{1/2} e_t$, where

- $(e_t)_{t\in\mathbb{Z}}$ is a standard white noise in $\mathbb{R}^m$: $E[e_t] = 0$, $Var(e_t) = Id_m$ and the vectors e_t are mutually independent.
- $H_t = [h_{ij,t}]_{1\leq i,j\leq m}$ is the "instantaneous" variance-covariance matrix of the t-observations, conditionally on $\mathcal{I}_{t-1}$:

$$Var(y_t|\mathcal{I}_{t-1}) = Var(z_t|\mathcal{I}_{t-1}) = H_t.$$

As usual with DCC-type models, we split the variance-covariance matrix H_t between volatility terms on one side (in D_t), and correlation coefficients on the other side (in R_t):

$$H_t = D_t^{1/2} R_t D_t^{1/2}, \;\; D_t = diag(h_{1,t}, ..., h_{m,t}), \tag{1}$$

where $h_{k,t} := h_{kk,t}$ denotes the "instantaneous variance" of z_k at time t, conditionally on $\mathcal{I}_{t-1}$. Usually in the literature, this conditional volatility of the asset k knowing $\mathcal{I}_{t-1}$ is denoted by $\sigma_{kk,t}$ instead of $h_{k,t}^{1/2}$. We assume GARCH-type models on every margin, but with cross-effects between all these volatilities potentially:

$$Vecd(D_t) = V_0 + \sum_{i=1}^{r} A_i.Vecd(D_{t-i}) + \sum_{j=1}^{s} B_j.\vec{z}_{t-j}, \tag{2}$$

for some deterministic nonnegative matrices $(A_i)_{i=1,\dots,r}$ and $(B_j)_{j=1,\dots,s}$, and for a positive vector V_0 in $\mathbb{R}^m$. We will set $A_i := [a^{(i)}_{k,l}]_{1\leq k,l\leq m}$, $i = 1,\dots,r$, and $B_j := [b^{(j)}_{k,l}]_{1\leq k,l\leq m}$, $j = 1,\dots,s$.

Let us introduce the so-called "standardized residuals" $\varepsilon_t \equiv D_t^{-1/2} z_t$. The dynamics of correlations are given by the traditional Dynamic Conditional Correlation specification:

$$R_t = diag(Q_t)^{-\frac{1}{2}} Q_t diag(Q_t)^{-\frac{1}{2}}, \tag{3}$$

where the sequence of matrices $(Q_t)_{t\in\mathbb{Z}}$ satisfies

$$Q_t = W_0 + \sum_{k=1}^{\nu} M_k Q_{t-k} M_k' + \sum_{l=1}^{\mu} N_l \varepsilon_{t-l}\varepsilon_{t-l}' N_l', \tag{4}$$

for some deterministic matrices $(M_k)_{k=1,\dots,\nu}$ and $(N_l)_{l=1,\dots,\mu}$, and for a positive definite constant matrix W_0. We will set $M_k := [m^{(k)}_{p,q}]_{1\leq p,q\leq m}$, $k = 1,\dots,\nu$, and $N_l := [n^{(l)}_{p,q}]_{1\leq p,q\leq m}$, $l = 1,\dots,\mu$. In practice, the positive matrix W_0 (or the constant vector $Vech(W_0)$ in $\mathbb{R}^{m^*}$ equivalently) is a parameter to be estimated.

Since $E_{t-1}[\varepsilon_t\varepsilon_t'] = R_t$, there exists a sequence of independent random vectors $(\eta_t)_{t\in\mathbb{Z}}$ in $\mathbb{R}^m$ such that

$$\varepsilon_t = R_t^{1/2}\eta_t, \tag{5}$$

with $E_{t-1}[\eta_t] = 0$ and $E_{t-1}[\eta_t\eta_t'] = I_m$. It can be imposed that $R_t^{1/2}$ is symmetrical and positive definite. In this case, the square root of R_t is uniquely defined: see Serre (2010), Theorem 6.1. This will be our convention throughout the article.

Note that the processes (e_t), (ε_t) or (η_t) can be considered equivalently as the sequences of innovations of our DCC model, because R_t, Q_t and D_t are

$\mathcal{I}_{t-1}$-measurable. In other words, for every t,

$$\mathcal{I}_t = \sigma(e_j, j \leq t) = \sigma(\varepsilon_j, j \leq t) = \sigma(\eta_j, j \leq t).$$

Aielli (2013) has noticed that the estimation of the unknown matrix W_0 is not straightforward, because it cannot be deduced trivially from the unconditional correlation between the standardized residuals ε_t. Therefore, he introduced a new variety of DCC-GARCH models (called cDCC), where (4) is replaced by

$$Q_t = W_0 + \sum_{k=1}^{\nu} M_k Q_{t-k} M_k' + \sum_{l=1}^{\mu} N_l diag(Q_t)^{-1/2} \varepsilon_{t-l} \varepsilon_{t-l}' diag(Q_t)^{-1/2} N_l'. \quad (6)$$

Under this new assumption, cDCC can be seen as a particular BEKK model (Engle and Kroner, 1995). Therefore, Aielli obtained the existence of strictly and/or weakly stationary solutions, applying the conditions of Boussama, Fuchs, and Stelzer (2011) on BEKK processes. Unfortunately, under the usual specification given by (4), DCC models cannot be rewritten as BEKK models anymore and other techniques have to be found. In this paper, we obtain the same type of results as Aielli (2013), but by keeping the original specification of DCC models and without relying on another encompassing family of processes.

2.2 DCC as Markov chains

Actually, it is possible to rewrite the previous DCC model as a Markov chain, that looks like an AR(1) process. This rewriting will become a crucial tool for the study of stationary solutions hereafter. Set

$$X_t := (X_t^{(1)}, X_t^{(2)}, X_t^{(3)}, X_t^{(4)})', \quad (7)$$

where

$$X_t^{(1)} := (Vecd(D_t), \dots, Vecd(D_{t-r+1}))',$$

$$X_t^{(2)} := (\vec{z}_t, \dots, \vec{z}_{t-s+1})',$$

$$X_t^{(3)} := (Vech(Q_t), \ldots, Vech(Q_{t-\nu+1}))',$$

$$X_t^{(4)} := (Vech(\varepsilon_t \varepsilon_t'), \ldots, Vech(\varepsilon_{t-\mu+1}\varepsilon_{t-\mu+1}'))'.$$

The dimensions of the four previous random vectors are rm, sm, νm^* and μm^* respectively. Their sum, the dimension of X_t, is denoted by d. With simple block matrix calculations, there exist random matrices (T_t) and a vector process (ζ_t) such that the dynamics of X_t, any solution of the DCC model, may be rewritten as

$$X_t = T_t.X_{t-1} + \zeta_t, \tag{8}$$

for any t. We will write the block matrix $T_t := [T_{ij,t}]_{1\leq i,j\leq 4}$ with convenient random matrices $T_{ij,t}$.

Knowing (8), the underlying process (X_t) can be seen as a vectorial autoregressive of order one, but with random matrix-coefficients (T_t). Actually, T_t and ξ_t will be stochastic only through ε_t, i.e. through the t-innovation η_t and the $\mathcal{I}_{t-1}$-measurable matrix R_t. This creates a major difficulty to prove the existence of stationary solutions. In particular, this means that T_t depends on some components of X_t. Therefore, it will be difficult to find explicit expressions like $X_t = f_t(\eta_t, \eta_{t-1}, \ldots)$ for some deterministic function f_t, because the link between T_t and the past innovations or observations is highly nonlinear.

Let us detail the AR(1) form of (8):

- set $T_{1k,t} = 0$ when $k = 3, 4$,

$$T_{11,t} := \begin{bmatrix} A_1 & A_2 & \cdots & \cdots & A_r \\ I_m & 0_m & \cdots & \cdots & 0_m \\ 0_m & I_m & 0_m & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_m & \cdots & 0_m & I_m & 0_m \end{bmatrix}, \text{ and } T_{12,t} := \begin{bmatrix} B_1 & B_2 & \cdots & \cdots & B_s \\ 0_m & \cdots & \cdots & \cdots & 0_m \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0_m & \cdots & \cdots & \cdots & 0_m \end{bmatrix}.$$

- We deduce from Equation (2) that

$$D_t\vec{\varepsilon}_t = \vec{\varepsilon}_t \odot Vecd(D_t) = \vec{z}_t = \vec{\varepsilon}_t \odot V_0 + \sum_{i=1}^{r} \vec{\varepsilon}_t \odot A_i.Vecd(D_{t-i}) + \sum_{j=1}^{s} \vec{\varepsilon}_t \odot B_j.\vec{z}_{t-j}. \tag{9}$$

Let us set $T_{23,t} = T_{24,t} = 0$,

$$T_{21,t} := \begin{bmatrix} \vec{\varepsilon}_t \odot A_1 & \vec{\varepsilon}_t \odot A_2 & \cdots & \cdots & \vec{\varepsilon}_t \odot A_r \\ 0_m & \cdots & \cdots & \cdots & 0_m \\ \vdots & & & & \vdots \\ 0_m & \cdots & \cdots & \cdots & 0_m \end{bmatrix}, \text{ and}$$

$$T_{22,t} := \begin{bmatrix} \vec{\varepsilon}_t \odot B_1 & \vec{\varepsilon}_t \odot B_2 & \cdots & \cdots & \vec{\varepsilon}_t \odot B_s \\ I_m & 0_m & \cdots & \cdots & 0_m \\ 0_m & I_m & 0_m & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_m & \cdots & 0_m & I_m & 0_m \end{bmatrix}$$

- Clearly, there exist matrices $\tilde{M}_k$, $k = 1, \ldots, \nu$, such that

$$Vech(M_k Q_{t-k} M_k') = \tilde{M}_k.Vech(Q_{t-k}).$$

Similarly, there exists matrices $\tilde{N}_l$, $l = 1, \ldots, \mu$, such that

$$Vech(N_l \varepsilon_{t-l} \varepsilon_{t-l}' N_l') = \tilde{N}_l.Vech(\varepsilon_{t-l}.\varepsilon_{t-l}').$$

Then, set $T_{31,t} = T_{32,t} = 0$,

$$T_{33,t} := \begin{bmatrix} \tilde{M}_1 & \tilde{M}_2 & \cdots & \cdots & \tilde{M}_\nu \\ I_{m^*} & 0_{m^*} & \cdots & \cdots & 0_{m^*} \\ 0_{m^*} & I_{m^*} & 0_{m^*} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{m^*} & \cdots & 0_{m^*} & I_{m^*} & 0_{m^*} \end{bmatrix}, \text{ and } T_{34,t} := \begin{bmatrix} \tilde{N}_1 & \tilde{N}_2 & \cdots & \cdots & \tilde{N}_\mu \\ 0_{m^*} & \cdots & \cdots & \cdots & 0_{m^*} \\ \vdots & & & & \vdots \\ 0_{m^*} & \cdots & \cdots & \cdots & 0_{m^*} \end{bmatrix}.$$

- $T_{4k,t} = 0$, $k = 1, 2, 3$, and define the $\mu m^* \times \mu m^*$ matrix

$$T_{44,t} := \begin{bmatrix} 0_{m^*} & 0_{m^*} & \cdots & \cdots & 0_{m^*} \\ I_{m^*} & 0_{m^*} & \cdots & \cdots & 0_{m^*} \\ 0_{m^*} & I_{m^*} & 0_{m^*} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{m^*} & \cdots & 0_{m^*} & I_{m^*} & 0_{m^*} \end{bmatrix}.$$

Moreover, rewrite

$$\zeta_t = (\zeta_t^{(1)}, \zeta_t^{(2)}, \zeta_t^{(3)}, \zeta_t^{(4)}),$$

where, with obvious sizes, there vectors are

$$\zeta_t^{(1)} = (V_0, 0_m, \dots, 0_m)', \;\; \zeta_t^{(2)} = (\vec{\varepsilon}_t \odot V_0, 0_m, \dots, 0_m)',$$

$$\zeta_t^{(3)} = (Vech(W_0), 0_{m^*}, \dots, 0_{m^*})', \;\; \zeta_t^{(4)} = (Vech(\varepsilon_t \varepsilon_t'), 0_{m^*}, \dots, 0_{m^*})'.$$

Intuitively, the model (X_t) is $\mathcal{I}$-markovian because it is the case for the process (ζ_t) and (T_t) themselves. Indeed, ε_t (or $\vec{\varepsilon}_t$, or even $Vech(\varepsilon_t \varepsilon_t'))$ is a function of the couple (R_t, η_t) only. Due to (3) and (4), R_t is a deterministic function of X_{t-1}. Since η_t is independent of $\mathcal{I}_{t-1}$, the law of ε_t knowing $\mathcal{I}_{t-1}$ is just the law of ε_t knowing X_{t-1}. The same assertion applies with ζ_t, or with X_t itself, instead of ε_t.

In other words, the non-linearity of the DCC model is coming from $\vec{\varepsilon}_t$ in T_t.

But there exists constants matrices (of zeros and ones) F and G such that (8) can be rewritten

$$X_t = (\vec{\varepsilon}_t \otimes F) \odot T_o X_{t-1} + (\vec{\varepsilon}_t \otimes G) \odot \zeta_o, \tag{10}$$

where T_o (resp. ζ_o) is the T_t matrix (resp. ζ_t vector) when $\vec{\varepsilon}_t = 1$. Since $\varepsilon_t = R_t^{1/2}\eta_t$ and since R_t is a measurable function of X_{t-1}, then X_t is clearly a function of X_{t-1} and of the innovation η_t only. These arguments prove the markovian structure of the (X_t) process.

3 Stationarity of DCC models

3.1 Existence of stationary DCC solutions

To obtain the existence of stationary solutions of the previous DCC model, we will invoke Tweedie's (1988) criterion. The latter result will provide the existence of an invariant probability measure for the Markov chain defined by (8). This technique has already been used in several papers in econometrics, notably Ling and McAleer (2003) or Ling (1999).

To get the stationarity conditions of (z_t), the "size" of the matrix T_t does not have to explode. This matrix is random because it depends on the random variables ε_{kt}^2, $k = 1, \ldots, m$. The latter variables have a variance one, but they are not independent. This is in contrast with Ling and McAleer (2003). Moreover, unfortunately, the joint law of $\vec{\varepsilon}_t$ is a function of R_t, i.e. a function of X_{t-1}. That is why we need the following condition.

Assumption E1: For some $p \geq 1$, $E[\|\eta_t\|^{2p}] < \infty$ and $\rho\,(T^*) < 1$, where

$$T^* := \sup_{\mathbf{x} \in \mathbb{R}^d} E[|T_t^{\otimes p}| \mid X_{t-1} = \mathbf{x}].$$

Recall that T_t depends on $\vec{\varepsilon}_t$, that $\varepsilon_t = R_t^{1/2}\eta_t$, and that the components of η_t are uncorrelated. Then, the supremum above exists and is finite because all the coefficients of R_t are less than one (in absolute values).

Theorem. 1 *Under Assumption E1, the process (z_t, D_t, R_t) as defined by Equations (1), (2), (3) and (4) possesses a strictly stationary solution. This solution is measurable w.r.t. the σ-field $\mathcal{I}$ induced by the white noise (η_t). Moreover, (z_t) is second-order stationary and the $2p$-th moments of z_t are finite.*

To apply the previous theorem, it may be hard to check the condition on the spectral radius of T^*, due to the analytical complexity of $T_t^{\otimes p}$. In the next theorem, we provide simpler and more explicit conditions.

Theorem. 2 *If*

$$\sum_{i=1}^{r} \sup_{l=1,\ldots,m} |\sum_{k=1}^{m} a_{k,l}^{(i)}| + \sum_{j=1}^{s} \sup_{l=1,\ldots,m} |\sum_{k=1}^{m} b_{k,l}^{(i)}| < 1, \tag{11}$$

and

$$\sum_{k=1}^{\nu} \sup_{p,q} \sum_{i,j=1;i\geq j}^{m} |m_{i,p}^{(k)} m_{j,q}^{(k)}| < 1, \tag{12}$$

then the process (z_t, D_t, R_t) given by Equations (1)-(5) possesses a strictly stationary solution. This solution is measurable w.r.t. the σ-field $\mathcal{F}$ induced by the white noise (η_t).

Note that the previous result applies whatever the sequence of matrices (N_l), $l = 1, \ldots, \mu$.

Obviously, when a stationary solution exists, it is nonanticipative and ergodic, because the process X_t can be generated as $f(\eta_t, \eta_{t-1}, \ldots)$, for some measurable function f from $\mathbb{R}^\infty$ to $\mathbb{R}^d$.

Example 1: Consider a diagonal-type DCC model, where all the matrices of parameters are diagonal, assuming no "cross-effects" in terms of volatilities and/or correlations. Here, there exists real numbers $a_u^{(i)}$, $b_u^{(j)}$, $m_u^{(k)}$ and $n_u^{(l)}$, $u = 1, \ldots, m$, such that

$$A_i = diag(a_1^{(i)}, \ldots, a_m^{(i)}),\ i = 1, \ldots, r,\ B_j = diag(b_1^{(j)}, \ldots, b_m^{(j)}),\ j = 1, \ldots, s,$$

$$M_k = diag(m_1^{(k)}, \dots, m_m^{(k)}),\ k = 1, \dots, \nu,\ N_l = diag(n_1^{(l)}, \dots, n_m^{(l)}),\ l = 1, \dots, \mu.$$

In this case, Condition (11) becomes

$$\sum_{i=1}^{r} \sup_{l=1,\dots,m} |a_l^{(i)}| + \sum_{j=1}^{s} \sup_{l=1,\dots,m} |b_l^{(i)}| < 1,$$

and Condition (12) is $\sum_{k=1}^{\nu} \sup_{p=1,\dots,m} |m_p^{(k)}|^2 < 1$. To reduce even more the number of free parameters, scalar-DCC models are often assumed. In this case, all the unknown matrices are simply the product of a scalar and an identity matrix:

$$A_i = a^{(i)} I_m,\ i = 1, \dots, r,\ B_j = b^{(j)} I_m,\ j = 1, \dots, s,$$

$$M_k = m^{(k)} I_m,\ k = 1, \dots, \nu,\ N_l = n^{(l)} I_m,\ l = 1, \dots, \mu.$$

Such models are very popular, because they allow a dramatic reduction of the number of free parameters. With obvious notations, we have to satisfy

$$\sum_{i=1}^{r} |a^{(i)}| + \sum_{j=1}^{s} |b^{(j)}| < 1, \text{ and } \sum_{k=1}^{\nu} |m_p^{(k)}|^2 < 1.$$

Actually, to insure that the process generate positive variances, we have assumed that the previous scalars are nonnegative. Then, we recover the usual condition of stationarity of GARCH-type models:

$$0 \leq a^{(i)}, b^{(j)} \leq 1,\ \sum_{i=1}^{r} a^{(i)} + \sum_{j=1}^{s} b^{(j)} < 1.$$

3.2 Unicity of stationary DCC solutions

Even if there exist stationary solutions of the DCC model, we are not insured *a priori* that they are unique. Unfortunately, such a result is not given "for free" by Tweedie's Lemma 6. Moreover, the usual arguments concerning the unicity of stationary GARCH-type solutions do not apply here. Indeed, under the Markov-chain form given by Equation (8), the matrix T_t is itself a function of the random vector X_t through the $\vec{\varepsilon}_t$ factors. It is a major difference with the CCC case, notably. That is why we need to find another strategy. Now, we provide some unicity results under some more or less restrictive assumptions. The major one is the boundedness of the innovations (η_t) almost surely.

Assumption U1: there exists a constant C_η such that $\|\eta_t\|_\infty \leq C_\eta$ a.e. for every t.

Even questionable, this condition of boundedness is necessary to avoid some "pathological" trajectories of (Q_t). Indeed, without any constraint on the noises (η_t), it is always possible to generate (Q_t) paths that induce correlation matrices R_t whose determinants are arbitrarily close to zero, i.e. "almost" non-invertible. Such a behavior would create instabilities. Moreover, the previous upper bound C_η will be multiplied by the norms of the parameter matrices N_l hereafter, that are typically small in practice (less than 0.1 for asset returns, typically). Then, we feel Assumption U1 is not dramatically strong.

Assumption U2: $\|T_{33}\|_s < 1$.

The matrix T_{33} has been introduced in Subsection 2.2, under the name $T_{33,t}$. Since $T_{33,t}$ does not depend on time, we have removed the index t here.

Assumption U3: The underlying DCC model is "partially" scalar, i.e. there exist scalars $m^{(k)}$ such that $M_k = m^{(k)} I_m$ for all $k = 1, \ldots, \nu$. Moreover, $\rho(M^*) < 1$ by setting

$$M^* := \begin{bmatrix} (m^{(1)})^2 & (m^{(2)})^2 & \cdots & \cdots & (m^{(\nu)})^2 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Actually, U3 will not be mandatory to get our unicity result, even if allows a weakening of the other technical conditions. In every case, this "partially" scalar case is in line with the common practice of scalar DCC (or scalar multivariate GARCH) models.

Thanks to the latter assumptions, we will be able to bound $\|Q_t\|_\infty$ from above and below, and $\lambda_1(Q_t)$ from below. There tools will be crucial to prove the unicity of stationary DCC solutions.

Lemma. 3 *Under Assumption U1 and U2, for almost every trajectory of a solution (Q_t) of the DCC model, we have*

$$\|Q_t\|_\infty \le C_Q := \frac{\|Vech(W_0)\|_s + \sum_{l=1}^{\mu} \|\tilde{N}_l\|_s C_\varepsilon}{1 - \|T_{33}\|_s},$$

$$C_\varepsilon := \sqrt{\frac{m(m+1)}{2}} m^2 C_\eta^2.$$

Lemma. 4 *Under Assumption U1 and U2, for almost every trajectory of a solution (Q_t) of the DCC model, we have*

$$\lambda_1(Q_t) \ge C_\lambda, \text{ and } \min_{i=1,\ldots,m} q_{ii,t} \ge C_q,$$

where $C_\lambda = \lambda_1(W_0)$ and $C_q := \min_{i=1,\ldots,m}(W_0)_{ii}$. In addition, if we assume U3, then we can set

$$C_\lambda = \frac{\lambda_1(W_0)}{1 - \sum_{k=1}^{\nu}(m^{(k)})^2} \text{ and } C_q = \frac{\min_{i=1,\ldots,m}(W_0)_{ii}}{1 - \sum_{k=1}^{\nu}(m^{(k)})^2}.$$

The proofs of these lemmas are postponed to the end of the appendix.

Let $\kappa = \max(\nu, \mu)$ and, for every $j = 1, \ldots, \kappa$, set

$$\beta_j := \mathbf{1}(j \le \nu)\|M_j\|_s^2 + \mathbf{1}(j \le \mu)\|N_j\|_s^2 \frac{m^{3/2}\sqrt{C_Q}}{C_q\sqrt{C_\lambda}} \left[1 + \frac{mC_Q}{C_q}\right].$$

Let N be the (κ, κ) squared matrix

$$N^* := \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \cdots & \beta_\kappa \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Assumption U4: $\rho(N^*) < 1$.

Such conditions on spectral radius are standard in the GARCH literature (see Francq and Zakoian, 2010, e.g.). Actually, the technical assumptions U1-U4 above will insure the unicity of (ε_t), (Q_t) and (R_t) only. To get the unicity of (D_t) and then of (z_t) itself, we need a last assumption: with the notations of Subsection 2.2, set

$$\bar{T}_t := \begin{bmatrix} T_{11,t} & T_{12,t} \\ T_{21,t} & T_{22,t} \end{bmatrix}, \text{ and } \bar{T}^* = E[\bar{T}_t].$$

Note that $\bar{T}^*$ does not depend on any particular sequence (ε_t) nor t, because $E[\varepsilon_{kt}^2] = 1$ for every k.

Assumption U5: the spectral norm of $\bar{T}^*$ is strictly smaller than one.

Theorem. 5 *Under the assumptions of Lemmas 3 and 4, and under U4-U5, the strictly stationary solution of the DCC model is unique, given the sequence* (η_t).

Example 1 (Continued): In the case of scalar DCC models of order one, it is easy to specify the conditions above. Here, $r = s = \nu = \mu = 1$,

$$A_1 = a^{(1)} I_m,\ B_1 = b^{(1)} I_m,\ M_1 = m^{(1)} I_m,\ N_1 = n^{(1)} I_m.$$

Assumptions U2 and U3 are equivalent and mean $|m^{(1)}| < 1$. Assumption U4 is written

$$\beta_1 := \left(m^{(1)}\right)^2 + \left(n^{(1)}\right)^2 \frac{m^{3/2}\sqrt{C_Q}}{C_q\sqrt{C_\lambda}} \left[1 + \frac{mC_Q}{C_q}\right].$$

Finally,

$$\bar{T}^* = \begin{bmatrix} a^{(1)} & b^{(1)} \\ a^{(1)} & b^{(1)} \end{bmatrix} \otimes I_m.$$

Through elementary algebra, it can checked that the characteristic function of $\bar{T}^*$ is the function $x \mapsto (-x)^m(a^{(1)} + b^{(1)} - x)^m$. Then Assumption U5 means $a^{(1)} + b^{(1)} < 1$.

A Technical lemmas

We recall Tweedie's criterion, a key tool to prove the existence of an invariant probability measure for a Markov chain. His noteworthy advantage w.r.t. other techniques is to avoid any irreducibility conditions. Let $(X_t)_{t=1,2,\dots}$ be a temporally homogeneous Markov chain with a locally compact completely separable metric state space $(S, \mathcal{B})$. The transition probability is $P(x, A) = P(X_t \in A|\, X_{t-1} = x)$, where $x \in S$ and $A \in \mathcal{B}$. Theorem 2 of Tweedie (1988) provides:

Lemma. 6 *Suppose that (X_t) is a Feller chain, i.e. for each bounded continuous fonction h on S, the function of $\mathbf{x}$ given by $E[h(X_{t-1}) \,|\, X_{t-1} = \mathbf{x}]$ is also continuous.*

1. *If there exists, for some compact set $A \in \mathcal{B}$, a nonnegative function g and $\varepsilon > 0$ satisfying*

$$\int_{A^c} P(x, dy) g(y) \leq g(x) - \varepsilon, \quad x \in A^c, \tag{13}$$

   then there exists a $\sigma-$finite invariant measure μ for P with $0 < \mu(A) < \infty$.

2. Furthermore, if

$$\int_A \mu(dx)\left[\int_{A^c} P(x,dy)g(y)\right] < \infty, \tag{14}$$

then μ is finite and hence $\pi = \mu/\mu(S)$ is an invariant probability measure.

3. Furthermore, if

$$\int_{A^c} P(x,dy)g(y) \leq g(x) - f(x), \;\; x \in A^c, \tag{15}$$

then μ admits a finite f-moment, that is $\int_S \mu(dy)f(y) < \infty$.

Lemma. 7 *For a given squared matrix T, if $\rho(|T|) < 1$, then there exists a vector $M > 0$ such that $(Id - |T|')M > 0$.*

Proof of Lemma 7: Due to the condition on the spectral radius, the squared matrix $Id - |T|'$ is invertible, and its inverse is given by

$$(Id - |T|')^{-1} = Id + \sum_{j=1}^{\infty} (|T|')^j \,.$$

Because every element of $(|T|)^j$ is nonnegative, for any vector $L > 0$, $(Id - |T|')^{-1} L > 0$. Then, set $M = (Id - |T|')^{-1} L$. This completes the proof. ■

B Proof of Theorem 1:

First, let us check that (X_t) is a Feller chain. Let h be a bounded and continuous function on $\mathbb{R}^d$. Clearly,

$$\begin{aligned} E[h(X_t) \,|\, X_{t-1} = \mathbf{x}] &= E[h(T_t\mathbf{x} + \zeta_t) \,|\, X_{t-1} = \mathbf{x}] \\ &= E[h(\psi_1(\varepsilon_t\varepsilon_t')\mathbf{x} + \psi_2(\varepsilon_t\varepsilon_t')) \,|\, X_{t-1} = \mathbf{x}], \end{aligned}$$

for some continuous transforms ψ_1 and ψ_2. Note that $\varepsilon_t = R_t^{1/2}\eta_t$ and that $R_t^{1/2}$ is a continuous function of X_{t-1}. Indeed, $R_t \mapsto R_t^{1/2}$ is continuous (see

Proposition 6.3 in Serre (2010), e.g.) and invoke $X_{t-1} \mapsto R_t$ is continuous too by construction. Then,

$$E[h(X_t)\,|\,X_{t-1} = \mathbf{x}] = E[h(\tilde{\psi}_1(\mathbf{x})\eta_t.\eta_t' + \tilde{\psi}_2(\mathbf{x}))\,|\,X_{t-1} = \mathbf{x}]$$
$$= \quad E[h(\tilde{\psi}_1(\mathbf{x})\eta_t.\eta_t' + \tilde{\psi}_2(\mathbf{x}))],$$

for some continuous transforms $\tilde{\psi}_1$ and $\tilde{\psi}_2$. Moreover, the values taken by $R_t^{1/2}$ belongs to some compact subset when X_{t-1} describes a compact subset in $\mathbb{R}^d$. Therefore, applying the dominated convergence theorem, $\mathbf{x} \mapsto E[h(X_t)\,|\,X_{t-1} = \mathbf{x}]$ is continuous and (X_t) is Feller.

Second, set $g(\mathbf{x}) = 1 + |\mathbf{x}^{\otimes p}|' M$, for an arbitrary positive vector M, that will be chosen after. Clearly,

$$E[g(X_t)\,|\,X_{t-1} = \mathbf{x}] = 1 + E\left[|(T_t\mathbf{x} + \zeta_t)^{\otimes p}|'\,|\,X_{t-1} = \mathbf{x}\right] M.$$

By expanding the Kronecker products, we can check that

$$(T_t\mathbf{x} + \zeta_t)^{\otimes p} = (T_t\mathbf{x})^{\otimes p} + R(\mathbf{x}),$$

with

$$\|R(\mathbf{x})\| \leq \alpha_0 \left(\|\zeta_t\|.\|(T_t\mathbf{x})^{\otimes(p-1)}\| + \ldots + \|\zeta_t\|^{p-1}.\|(T_t\mathbf{x})\| + \|\zeta_t\|^p \right),$$

for some positive constant α_0 and any multiplicative matrix norm $\|\cdot\|$.

Note that that $(T_t\mathbf{x})^{\otimes k} = T_t^{\otimes k}.\mathbf{x}^{\otimes k}$. Recall that T_t is a function of $\vec{\varepsilon}_t$, i.e. of ε_t. Then, its conditional law depends on R_t, i.e. it is a function of X_{t-1}. We deduce

$$E[|(T_t\mathbf{x})^{\otimes p}|\,|\,X_{t-1} = \mathbf{x}]' M \leq |\mathbf{x}^{\otimes p}|' E[|T_t^{\otimes p}|'\,|\,X_{t-1} = \mathbf{x}] M$$
$$\leq \quad |\mathbf{x}^{\otimes p}|' \left(\sup_{\mathbf{x} \in \mathbb{R}^d} E[|T_t^{\otimes p}|'\,|\,X_{t-1} = \mathbf{x}] \right) M$$
$$\leq \quad |\mathbf{x}^{\otimes p}|' (T^*)' M.$$

Now, choose M as given by Lemma 7, when T is replaced by the non negative matrix T^*.

Moreover, $\varepsilon_t = R_t^{1/2}\eta_t$, and the (positive definite) matrix $R_t^{1/2}$ can be chosen so that all its coefficients are less than $m^{1/2}$ (diagonalize this matrix in an orthonormal basis and invoke Cauchy-Schwartz inequality). This implies there exist constants α_k such that $\|Vech(\varepsilon_t\varepsilon_t')^{\otimes k}\| \leq \alpha_k \|Vech(\eta_t\eta_t')^{\otimes k}\|$ when $k \leq p$. Since $E[\|\eta\|^{2p}] < \infty$ by assumption, there exist some constants $c_{k,l}$ such that $E_{t-1}[\|\zeta_t\|^k.\|\vec{\varepsilon}_t\|^l] < c_{k,l}$ for any couple (k,l), $k+l \leq p$. We deduce the boundedness of $T_t \otimes k$, $k \leq p$, and

$$E[\|R(\mathbf{x})\| \,|\, X_{t-1} = \mathbf{x}] \leq \alpha_1 \left(\|\mathbf{x}^{\otimes(p-1)}\| + \ldots + \|\mathbf{x}\| + 1 \right),$$

for some positive constant α_1. We have obtained

$$\begin{aligned} E[g(X_t) \,|\, X_{t-1} = \mathbf{x}] &\leq 1 + |\mathbf{x}^{\otimes p}|'(T^*)'M + O\left(\sum_{k=0}^{p-1} \|\mathbf{x}^{\otimes k}\| \right) \\ &\leq \quad g(\mathbf{x}) - |\mathbf{x}^{\otimes p}|' \left(Id - (T^*)' \right) M + O\left(\sum_{k=0}^{p-1} \|\mathbf{x}^{\otimes k}\| \right). \end{aligned} \tag{16}$$

By Lemma 7, $(Id - (T^*)')M$ is strictly positive. Then, there exists a positive constant c_0 such that

$$|\mathbf{x}^{\otimes p}|' \left(Id - (T^*)' \right) M \geq c_0 \sum_{j=1}^{d} |x_j|^p,$$

for every d-dimensional vector $\mathbf{x}$. Set $N(\mathbf{x}) := \sum_{j=1}^d |x_j|^p$. By a similar reasoning, there exists a positive constant c_1 such that $g(\mathbf{x}) \geq c_1 N(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^d$. Moreover, by applying Hölder's inequality,

$$\sum_{i_1,\ldots,i_k} |x_{i_1} \cdots x_{i_k}| = \left(\sum_{i=1}^{d} |x_i| \right)^k \leq \left(\sum_{i=1}^{d} |x_i|^p \right)^{k/p} d^k,$$

for every $k \leq p$. Then there exists a positive constant c_2 such that

- $g(\mathbf{x}) \leq 1 + \|M\| \sum_{i_1,\ldots,i_p} |x_{i_1} \cdots x_{i_p}| \leq 1 + c_2 N(\mathbf{x})$, and
- every "residual" term $\|\mathbf{x}^{\otimes k}\|$ is bounded above by (a scalar times) $N(\mathbf{x})^{k/p}$, when $k < p$.

Therefore, this provides

$$
\begin{aligned}
E[g(X_t) \,|\, X_{t-1} = \mathbf{x}] &\leq g(\mathbf{x}) \left[1 - c_0 \frac{N(\mathbf{x})}{g(\mathbf{x})} + O\left(\sup_{k=0,\ldots,p-1} \frac{N(\mathbf{x})^{k/p}}{g(\mathbf{x})} \right) \right] \\
&\leq \quad g(\mathbf{x}) \left[1 - \frac{c_0 N(\mathbf{x})}{1 + c_2 N(\mathbf{x})} + O\left(\sup_{k=0,\ldots,p-1} \frac{N(\mathbf{x})^{k/p}}{c_1 N(\mathbf{x})} \right) \right].
\end{aligned}
$$

Let us define the set $A := \{\mathbf{x} \in \mathbb{R}^d \,|\, N(\mathbf{x}) \leq \Delta\}$, for some $\Delta > 1$. When Δ is sufficiently large and for any $\mathbf{x} \notin A$,

$$
0 \leq E[g(X_t) \,|\, X_{t-1} = \mathbf{x}] \leq g(\mathbf{x}) \left[1 - \frac{c_0}{2c_2} + O\left(\frac{\Delta^{-1/p}}{c_1} \right) \right] < g(\mathbf{x}) \left[1 - \frac{c_0}{3c_2} \right]. \tag{17}
$$

Since $g(\mathbf{x}) \geq 1$, it follows that $E[g(X_t) \,|\, X_{t-1} = \mathbf{x}] \leq g(\mathbf{x}) - \varepsilon$ for some $\varepsilon > 0$. This proves Equation (13) in Lemma 6. Therefore, there exists a σ-finite invariant measure μ for the Markov chain (X_t), and $0 < \mu(A) < \infty$.

For any $\mathbf{x} \in A$, Equation (16) provides

$$
E[g(X_t) \,|\, X_{t-1} = \mathbf{x}] \leq g(\mathbf{x}) + O\left(\sum_{k=0}^{p-1} \|\mathbf{x}^{\otimes k}\| \right) \leq C\Delta^p
$$

for some constant C that does not depend on $\mathbf{x}$. Then,

$$
\int_A \mu(dx) \left[\int_{A^c} P(x, dy) g(y) \right] \leq \int_A \mu(dx) E[g(X_t) \,|\, X_{t-1} = \mathbf{x}] \leq C\Delta^p \mu(A) < \infty.
$$

We deduce that μ is finite and hence $\pi = \mu/\mu(\mathbb{R}^d)$ is an invariant probability measure of (X_t). This implies there exists a strictly stationary solution satisfying (8), still denoted by X_t.

Third, by invoking Equation (17), we get (15) in Lemma 6 with $f(\mathbf{x}) =$

$\beta g(\mathbf{x})$, for some $\beta \in (0,1)$. Since $g(\mathbf{x}) \geq c_1 N(\mathbf{x})$, we obtain

$$E_\pi[N(X_t)] < \infty. \tag{18}$$

In particular and by Hölder's inequality, this implies that $E_\pi[z_{it}^{2k}] < \infty$, for every $i = 1, \ldots, m$ and every $k \leq p$. ■

Remark. 8 *Equation (18) provides a lot more than only the finiteness of z's moments. Globally, this means that*

$$E_\pi\left[\sum_{i=1}^m h_{it}^p\right] < \infty, \quad E_\pi\left[\sum_{i=1}^m z_{it}^{2p}\right] < \infty,$$

$$E_\pi\left[\sum_{i,j=1}^m |Q_{ij,t}|^p\right] < \infty, \quad E_\pi\left[\sum_{i=1}^m |\varepsilon_{it}|^{2p}\right] < \infty.$$

C Proof of Theorem 2:

Let us use the same technique as in Theorem 1, but now with the function

$$g(\mathbf{x}) := 1 + |\mathbf{x}|'\mathbf{v},$$

where $\mathbf{v} = (\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \mathbf{v}^{(4)})'$ will be a positive vector. Obviously, the dimensions of the constant subvectors $\mathbf{v}^{(k)}$, $k = 1, \ldots, 4$ are consistent with those of X_t in (7). With obvious notations, let us rewrite

$$\begin{aligned}
E[g(X_t) \,|\, X_{t-1} = \mathbf{x}] &= 1 + E\left[|T_t X_{t-1} + \zeta_t|' \,|\, X_{t-1} = \mathbf{x}\right]\mathbf{v} \\
&= 1 + \sum_{i=1}^4 E\left[|\sum_{j=1}^4 T_{ij,t} X_{t-1}^{(j)} + \zeta_t^{(i)}|' \,|\, X_{t-1} = \mathbf{x}\right]\mathbf{v}^{(i)} \\
&\leq g(\mathbf{x}) + \sum_{i=1}^4 \left\{ E\left[|\sum_{j=1}^4 T_{ij,t} X_{t-1}^{(j)} + \zeta_t^{(i)}|' \,|\, X_{t-1} = \mathbf{x}\right]\mathbf{v}^{(i)} - |\mathbf{x}^{(i)}|'\mathbf{v}^{(i)} \right\} \\
&:= g(\mathbf{x}) + \sum_{i=1}^4 r_i(\mathbf{x}).
\end{aligned}$$

We will bound from above every term r_k, $k = 1, 2, 3, 4$, and we will choose $\mathbf{v}$ so that the sum of them becomes negative. Set $\mathbf{v}^{(1)} := (\alpha_0 e, \alpha_1 e, \ldots, \alpha_{r-1} e)'$, with convenient vector sizes and positive real coefficients α_i, $i = 1, \ldots, r-1$. Knowing $X_{t-1} = \mathbf{x}$, we have

$$\begin{aligned} r_1(\mathbf{x}) \leq \alpha_0 |V_0 + \sum_{i=1}^{r} A_i.Vecd(D_{t-i}) + \sum_{j=1}^{s} B_j.\vec{z}_{t-j}|'e + (\alpha_1 - \alpha_0)|Vecd(D_{t-1})|'e \\ + \quad \ldots + (\alpha_{r-1} - \alpha_{r-2})|Vecd(D_{t-r+1})|'e - \alpha_{r-1}|Vecd(D_{t-r})|'e. \quad (19) \end{aligned}$$

Similarly, set the positive vectors

$$\mathbf{v}^{(2)} := (\beta_0 e, \beta_1 e, \ldots, \beta_{s-1} e)', \ \ \mathbf{v}^{(3)} := (\gamma_0 e, \gamma_1 e, \ldots, \gamma_{\nu-1} e)',$$

$$\mathbf{v}^{(4)} := (\delta_0 e, \delta_1 e, \ldots, \delta_{\mu-1} e)'.$$

Since $E[\vec{\varepsilon}_t \,|\, X_{t-1} = \mathbf{x}] = e$, we get

$$\begin{aligned} r_2(\mathbf{x}) \leq \beta_0 |V_0 + \sum_{i=1}^{r} A_i.Vecd(D_{t-i}) + \sum_{j=1}^{s} B_j.\vec{z}_{t-j}|'e + (\beta_1 - \beta_0)|\vec{z}_{t-1}|'e \\ + \quad \ldots + (\beta_{s-1} - \beta_{s-2})|\vec{z}_{t-s+1}|'e - \beta_{s-1}|\vec{z}_{t-s}|'e. \quad (20) \end{aligned}$$

Moreover,

$$\begin{aligned} r_3(\mathbf{x}) \leq \gamma_0 |Vech(W_0) + \sum_{k=1}^{\nu} \tilde{M}_k.Vech(Q_{t-k}) + \sum_{l=1}^{\mu} \tilde{N}_l.Vech(\varepsilon_{t-l}.\varepsilon'_{t-l})|'e \\ + \quad (\gamma_1 - \gamma_0)|Vech(Q_{t-1})|'e + \ldots + (\gamma_{\nu-1} - \gamma_{\nu-2})|Vech(Q_{t-\nu+1})|'e - \gamma_{\nu-1}|Vech(Q_{t-\nu})|'e, \end{aligned}$$

and

$$\begin{aligned} r_4(\mathbf{x}) \leq \delta_0 E[|Vech(\varepsilon_t \varepsilon'_t)| \,|\, X_{t-1} = \mathbf{x}]'e + (\delta_1 - \delta_0)|Vech(\varepsilon_{t-1}\varepsilon'_{t-1})|'e \\ + \ldots + (\delta_{\mu-1} - \delta_{\mu-2})|Vech(\varepsilon_{t-\mu+1}\varepsilon'_{t-\mu+1})|'e - \delta_{\mu-1}|Vech(\varepsilon_{t-\mu}\varepsilon'_{t-\mu})|'e. \end{aligned}$$

Note that $E[|Vech(\varepsilon_t \varepsilon'_t)|' \,|\, X_{t-1} = \mathbf{x}]e \leq m^*$, because $E[|\varepsilon_{kt}\varepsilon_{lt}| \,|\, X_{t-1} = \mathbf{x}] \leq 1$ for every couple (k, l) (Cauchy-Schwartz inequality).

Now, let us choose the vector $\mathbf{v}$ so that there exist some constants $\varepsilon > 0$ and C_0 such that, for every $\mathbf{x} \in \mathbb{R}^d$,

$$E[g(X_t) \mid X_{t-1} = \mathbf{x}] \leq g(\mathbf{x}) - \varepsilon |\mathbf{x}|' e + C_0. \tag{21}$$

For instance, consider the coefficient of the l-th component of D_{t-i}, i.e. $h_{t-i,l}$, for $i = 1, \ldots, r$ and $l = 1, \ldots, m$, that appeared in the r.h.s. of (19) and (20). This coefficient is

$$(\alpha_0 + \beta_0) \sum_{k=1}^{m} a_{k,l}^{(i)} + \alpha_i - \alpha_{i-1}$$

where $\alpha_r = 0$ by convention. Such a coefficient should be negative for every l. Therefore, by setting

$$a_i^* := \sup_{l=1,\ldots,m} \sum_{k=1}^{m} a_{k,l}^{(i)},$$

we would like to satisfy

$$(\alpha_0 + \beta_0) a_i^* + \alpha_i - \alpha_{i-1} < 0, \;\; \forall i = 1, \ldots, r. \tag{22}$$

Similarly, dealing with $\mathbf{v}^{(2)}$, we have to satisfy the following condition:

$$(\alpha_0 + \beta_0) b_j^* + \beta_j - \beta_{j-1} < 0, \;\; \forall j = 1, \ldots, s, \tag{23}$$

where $\beta_s = 0$ and $b_j^* := \sup_{l=1,\ldots,m} \sum_{k=1}^m b_{k,l}^{(i)}$. To deal with $\mathbf{v}^{(3)}$, note that, knowing $X_{t-1} = \mathbf{x}$,

$$\begin{aligned} r_3(\mathbf{x}) &\leq \gamma_0 \{ |Vech(W_0)|' e + |\sum_{k=1}^{\nu} \tilde{M}_k . Vech(Q_{t-k})|' e + |\sum_{l=1}^{\mu} \tilde{N}_l . Vech(\varepsilon_{t-l} . \varepsilon_{t-l}')|' e \} \\ &+ \quad (\gamma_1 - \gamma_0) |Vech(Q_{t-1})|' e + \ldots + (\gamma_{\nu-1} - \gamma_{\nu-2}) |Vech(Q_{t-\nu+1})|' e \\ &- \quad \gamma_{\nu-1} |Vech(Q_{t-\nu})|' e. \end{aligned} \tag{24}$$

The coefficient of the (p,q) element of Q_{t-k} is denoted by $q_{p,q,k}$. Recall that

$$\tilde{M}_k.Vech(Q_{t-k}) = Vech(M_k Q_{t-k} M_k') = Vech\left(\left[\sum_{p,q=1}^{m} m_{i,p}^{(k)} m_{j,q}^{(k)} q_{p,q,k}\right]_{i,j}\right).$$

Then, the coefficient of $q_{p,q,k}$ in the r.h.s. of (24) is (less than) then

$$\gamma_0 \sum_{i,j=1;i\geq j}^{m} |m_{i,p}^{(k)} m_{j,q}^{(k)}| + (\gamma_k - \gamma_{k-1}).$$

Setting

$$m_k^* := \sup_{p,q} \sum_{i,j=1;i\geq j}^{m} |m_{i,p}^{(k)} m_{j,q}^{(k)}|,\ k = 1,\dots,\nu,$$

we would like to satisfy

$$\gamma_0 m_k^* + \gamma_k - \gamma_{k-1} < 0,\ \ \forall k = 1,\dots,\nu, \tag{25}$$

with $\gamma_\nu := 0$.

Let us do the same analysis with the coefficients of $Vech(\varepsilon_{t-l}\varepsilon_{r-l}')$, $l = 1,\dots,\mu$. For instance, the coefficient corresponding to the cell (p,q) is given by

$$\gamma_0 \sum_{i,j=1;i\geq j}^{m} |n_{i,p}^{(l)} n_{j,q}^{(l)}| + (\delta_l - \delta_{l-1}),$$

with $\delta_\mu := 0$. Let us define

$$n_l^* := \sup_{p,q} \sum_{i,j=1;i\geq j}^{m} |n_{i,p}^{(l)} n_{j,q}^{(l)}|,\ l = 1,\dots,\mu.$$

Then, setting $\delta_\nu = 0$, we have to satisfy

$$\gamma_0 n_l^* + \delta_l - \delta_{l-1} < 0,\ \ \forall l = 1,\dots,\mu. \tag{26}$$

We argue it is possible to find $\mathbf{v}$ that satisfies the constraints (22), (23), (25) and (26) simultaneously. Define

$$a^* = \sum_{i=1}^{r} a_i^*, \quad b^* = \sum_{j=1}^{s} b_j^*, \quad m^* = \sum_{k=1}^{\nu} m_k^*.$$

A solution of the latter problem is given by

$$\alpha_0 = \frac{a^*(r+s)}{1-a^*-b^*} + r,\ \beta_0 = \frac{b^*(r+s)}{1-a^*-b^*} + s,\ \gamma_0 = \frac{\nu}{1-m^*},$$

$$\alpha_i = (r-i) + (\alpha_0+\beta_0)[a_r^* + a_{r-1}^* + \ldots + a_{i+1}^*], \quad i = 1, \ldots, r-1,$$

$$\beta_j = (s-j) + (\alpha_0+\beta_0)[b_s^* + b_{s-1}^* + \ldots + b_{j+1}^*], \quad j = 1, \ldots, s-1,$$

$$\gamma_k = (\nu-k) + \gamma_0[m_\nu^* + m_{\nu-1}^* + \ldots + m_{k+1}^*], \quad k = 1, \ldots, \nu-1, \text{ and}$$

$$\delta_l = (\mu-l) + \gamma_0[n_\nu^* + n_{\nu-1}^* + \ldots + n_{l+1}^*], \quad l = 0, \ldots, \mu-1.$$

Therefore, we have proved the inequality (21), with the constant

$$C_0 = (\alpha_0+\beta_0)|V_0|'e + \gamma_0|Vech(W_0)|'e + \delta_0 m^*.$$

Then, it is easy to conclude, following the same arguments as in Theorem 1. Particularly, to apply Tweedie's criterion (Lemma 6), define a set $A := \{\mathbf{x} \in \mathbb{R}^d \,|\, |\mathbf{x}|'e \leq \Delta\}$, with $\Delta > 1$. For a sufficiently large Δ, and when $\mathbf{x}$ does not belong to A, we obtain

$$E[g(X_t) \,|\, X_{t-1} = \mathbf{x}] \leq g(\mathbf{x}) - \varepsilon/2, \tag{27}$$

providing Equation (13). This concludes the proof. ■

D Proof of Theorem 5:

Imagine there exist two strongly stationary solutions (X_t) and $(\tilde{X}_t)$. Since both of them satisfy Equation (8), with obvious notations, we can write for every t

$$X_t = T_t.X_{t-1} + \zeta_t, \text{ and } \tilde{X}_t = \tilde{T}_t.\tilde{X}_{t-1} + \tilde{\zeta}_t.$$

Note that the difference between T_t and $\tilde{T}_t$ is only due to the (a priori different) factors ε_t and $\tilde{\varepsilon}_t$. We want to prove that, for every t, we have in fact $X_t = \tilde{X}_t$ almost surely.

The problem will be solved if we prove the unicity of the process $(X_t^{(3)}, X_t^{(4)})$, given by subvectors of (X_t). For the moment, assume it has been proved. Recall that

$$X_t^{(3)} := (Vech(Q_t), \dots, Vech(Q_{t-\nu+1}))', \text{ and}$$

$$X_t^{(4)} := (Vech(\varepsilon_t\varepsilon_t'), \dots, Vech(\varepsilon_{t-\mu+1}\varepsilon_{t-\mu+1}'))'.$$

Then (R_t) is unique, due to (3). Moreover, the sequence of the random matrices (T_t) and of the noises (ζ_t) are unique too, similarly to the CCC case. It remains to prove the unicity of $Y_t := (X_t^{(1)}, X_t^{(2)})$, that is related to the instantaneous volatilities (D_t) and to the returns (z_t) themselves. With our notations, we have

$$Y_t = \bar{T}_t Y_{t-1} + \bar{\zeta}_t, \text{ and } \tilde{Y}_t = \bar{T}_t \tilde{Y}_{t-1} + \bar{\zeta}_t,$$

for every t, by setting $\bar{\zeta}_t = (\zeta_t^{(1)}, \zeta_t^{(2)})$. The arguments are then standard: for instance, see Theorem 2.4's proof in Francq and Zakoian (2010). To get the unicity of (Y_t), it is sufficient to assume that the top Lyapunov exponent γ of the sequence of random matrices $(\bar{T}_t)$ is strictly negative. This is the case under Assumption U5 because, for every sequence (ε_t),

$$E[\ln \|\bar{T}_t\bar{T}_{t-1}\dots\bar{T}_1\|_1] \leq \ln E[\|\bar{T}_t\bar{T}_{t-1}\dots\bar{T}_1\|_1] \leq \ln \| \left(\bar{T}^*\right)^t \|_1,$$

by invoking the matrix norm $\|A\|_1 := \sum_{i,j} |a_{ij}|$. The fist inequality is due to Jensen's inequality. The second one is a consequence of the conditional independence between all the r.v. $\varepsilon_t, \ldots, \varepsilon_1$. Indeed, every term of the random matrix $\bar{T}_t$, say the (i,j)-th, is the product of a random variable $\varepsilon_{k_{ij},t}^{2\alpha_{ij}}$ and a deterministic term b_{ij}, where $\alpha_{ij} \in \{0,1\}$ and k_{ij} is an index between 1 and m. Denote by b_{ij} the (i,j)-th term of the matrix $\bar{T}^*$. Actually, $\|\bar{T}_t\bar{T}_{t-1}\ldots\bar{T}_1\|_1$ is a sum of terms like

$$\varepsilon_{k_{i_1j_1},t}^{2\alpha_{i_1j_1}}\varepsilon_{k_{i_2j_2},t-1}^{2\alpha_{i_2j_2}}\ldots\varepsilon_{k_{i_tj_t},1}^{2\alpha_{i_tj_t}}|b_{i_1j_1}\ldots b_{i_tj_t}|,$$

over some collection of indices $i_1, j_1, \ldots, i_t, j_t$. The expectation of this term is simply $|b_{i_1j_1}\ldots b_{i_tj_t}|$. By collecting all the latter terms, we get $\| \left(\bar{T}^*\right)^t \|_1$. We deduce there exists a constant C s.t.

$$E[\ln\|\bar{T}_t\bar{T}_{t-1}\ldots\bar{T}_1\|_1] \leq \ln\left\{C\| \left(\bar{T}^*\right)^t \|_s\right\} \leq \ln\left(C\|\bar{T}^*\|_s^t\right).$$

Therefore, since $\gamma = \lim_{t\to\infty} t^{-1}E[\ln\|\bar{T}_t\bar{T}_{t-1}\ldots\bar{T}_1\|_1]$, γ is strictly negative under Assumption U5, providing the unicity of the processes (D_t) and (z_t) (once we assume the unicity of the processes (Q_t) and (ε_t)).

Now, let us prove the unicity of $(X_t^{(3)}, X_t^{(4)})$ or, in other terms, of (Q_t, ε_t). This is clearly more tricky, because we will have to deal with the nonlinear feature of the DCC specification. Here, the convenient matrix norm will be the spectral norm $\|\cdot\|_s$. Consider two stationary solutions (Q_t, ε_t) and $(\tilde{Q}_t, \tilde{\varepsilon}_t)$. Since the spectral norm is multiplicative, we deduce from (4) that

$$\begin{aligned} E[\|Q_t - \tilde{Q}_t\|_s] \leq &\sum_{k=1}^{\nu} \|M_k\|_s^2 E[\|Q_{t-k} - \tilde{Q}_{t-k}\|_s] \\ + &\sum_{l=1}^{\mu} \|N_l\|_s^2 E[\|\varepsilon_{t-l}\varepsilon_{t-l}' - \tilde{\varepsilon}_{t-l}\tilde{\varepsilon}_{t-l}'\|_s]. \end{aligned} \tag{28}$$

The key point will be to bound from above the terms $E[\|\varepsilon_{t-l}\varepsilon_{t-l}' - \tilde{\varepsilon}_{t-l}\tilde{\varepsilon}_{t-l}'\|_s]$ by a function of $E[\|Q_{t-l} - \tilde{Q}_{t-l}\|_s]$. To lighten the indices, we assume $l = 0$.

Clearly, we have

$$\begin{aligned}\|\varepsilon_t\varepsilon_t' - \tilde\varepsilon_t\tilde\varepsilon_t'\|_s &= \|R_t^{1/2}\eta_t\eta_t' R_t^{1/2} - \tilde R_t^{1/2}\eta_t\eta_t'\tilde R_t^{1/2}\|_s \\ &\leq \|(R_t^{1/2} - \tilde R_t^{1/2})\eta_t\eta_t' R_t^{1/2}\|_s + \|\tilde R_t^{1/2}\eta_t\eta_t'(R_t^{1/2} - \tilde R_t^{1/2})\|_s \\ &\leq \|R_t^{1/2} - \tilde R_t^{1/2}\|_s\|\eta_t\eta_t'\|_s\|R_t^{1/2}\|_s + \|\tilde R_t^{1/2}\|_s\|\eta_t\eta_t'\|_s\|R_t^{1/2} - \tilde R_t^{1/2}\|_s.\end{aligned}$$

Since the rank of $\eta_t\eta_t'$ is one, then $\|\eta_t\eta_t'\|_s = Tr(\eta_t\eta_t') = \|\eta_t\|_2^2$ and $E_{t-1}[\|\eta_t\eta_t'\|_s] = m$. Moreover,

$$\|R_t^{1/2}\|_s = \rho(R_t)^{1/2} \leq Tr(R_t)^{1/2} = \sqrt{m}.$$

We deduce

$$E_{t-1}[\|\varepsilon_t\varepsilon_t' - \tilde\varepsilon_t\tilde\varepsilon_t'\|_s] \leq 2m^{3/2}\|R_t^{1/2} - \tilde R_t^{1/2}\|_s, \tag{29}$$

because $R_t^{1/2}$ and $\tilde R_t^{1/2}$ are $\mathcal{I}_{t-1}$-measurable. Since the spectral norm is unitarily invariant, Theorem 6.2 in Hingham (2008) provides

$$\|R_t^{1/2} - \tilde R_t^{1/2}\|_s \leq \frac{1}{\lambda_1(R_t)^{1/2} + \lambda_1(\tilde R_t)^{1/2}}\|R_t - \tilde R_t\|_s. \tag{30}$$

Note that, for any t,

$$\begin{aligned}\lambda_1(R_t) = \min_{\mathbf{x}}\frac{\mathbf{x}'R_t\mathbf{x}}{\mathbf{x}'\mathbf{x}} &= \min_{\mathbf{x}}\frac{\mathbf{x}'diag(Q_t)^{-1/2}Q_t diag(Q_t)^{-1/2}\mathbf{x}}{\mathbf{x}'\mathbf{x}} \\ &\geq \min_{\mathbf{y}}\frac{\mathbf{y}'Q_t\mathbf{y}}{\mathbf{y}'\mathbf{y}}\min_{\mathbf{x}}\frac{\|diag(Q_t)^{-1/2}\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2} \\ &\geq \lambda_1(Q_t)\min_i\frac{1}{q_{ii,t}}.\end{aligned}$$

Invoking Lemmas 3 and 4, we deduce

$$\lambda_1(R_t) \geq \lambda_1(Q_t)/C_Q \geq C_\lambda/C_Q, \tag{31}$$

and the same inequality applies with $\lambda_1(\tilde R_t)$. Therefore, we get a.e.

$$\frac{1}{\lambda_1(R_t)^{1/2} + \lambda_1(\tilde R_t)^{1/2}} \leq \frac{\sqrt{C_Q}}{2\sqrt{C_\lambda}}. \tag{32}$$

Moreover,

$$\begin{aligned} R_t - \tilde{R}_t &= diag(Q_t)^{-1/2}(Q_t - \tilde{Q}_t)diag(Q_t)^{-1/2} \\ &+ \quad (diag(Q_t)^{-1/2} - diag(\tilde{Q}_t)^{-1/2})\tilde{Q}_t diag(Q_t)^{-1/2} \\ &+ \quad diag(\tilde{Q}_t)^{-1/2}\tilde{Q}_t(diag(Q_t)^{-1/2} - diag(\tilde{Q}_t)^{-1/2}) := \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3. \end{aligned}$$

By Lemma 4, we obtain

$$\begin{aligned} \|\mathcal{R}_1\|_s &= \|diag(Q_t)^{-1/2}(Q_t - \tilde{Q}_t)diag(Q_t)^{-1/2}\|_s \leq \|diag(Q_t)^{-1/2}\|_s^2 \|Q_t - \tilde{Q}_t\|_s \\ &\leq \quad \frac{1}{\min_i q_{ii,t}} \|Q_t - \tilde{Q}_t\|_s \leq \frac{1}{C_q}\|Q_t - \tilde{Q}_t\|_s. \end{aligned}$$

Since $\|A\|_\infty \leq \|A\|_s \leq m\|A\|_\infty$ for any matrix A, we get

$$\begin{aligned} \|\mathcal{R}_2\|_s &\leq \|diag(Q_t)^{-1/2}\|_s \|\tilde{Q}_t\|_s \|diag(Q_t)^{-1/2} - diag(\tilde{Q}_t)^{-1/2}\|_s \\ &\leq \quad m\|Q_t\|_\infty C_q^{-1/2} \|diag\left(\frac{q_{ii,t} - \tilde{q}_{ii,t}}{q_{ii,t}^{1/2}\tilde{q}_{ii,t}^{1/2}(q_{ii,t}^{1/2} + \tilde{q}_{ii,t}^{1/2})}\right)\|_s \\ &\leq \quad \|Q_t - \tilde{Q}_t\|_s \frac{mC_Q}{2C_q^2}. \end{aligned}$$

Similarly,

$$\|\mathcal{R}_3\|_s \leq \|Q_t - \tilde{Q}_t\|_s \frac{mC_Q}{2C_q^2}.$$

Globally, we get

$$\|R_t - \tilde{R}_t\|_s \leq \frac{1}{C_q}\left[1 + \frac{mC_Q}{C_q}\right]\|Q_t - \tilde{Q}_t\|_s \tag{33}$$

everywhere. Recalling (29), (30) (32) and (33), we deduce

$$\|R_t^{1/2} - \tilde{R}_t^{1/2}\|_s \leq \frac{\sqrt{C_Q}}{2C_q\sqrt{C_\lambda}}\left[1 + \frac{mC_Q}{C_q}\right]\|Q_t - \tilde{Q}_t\|_s, \text{ and} \tag{34}$$

$$E_{t-1}[\|\varepsilon_t\varepsilon_t' - \tilde{\varepsilon}_t\tilde{\varepsilon}_t'\|_s] \leq \frac{m^{3/2}\sqrt{C_Q}}{C_q\sqrt{C_\lambda}}\left[1 + \frac{mC_Q}{C_q}\right]\|Q_t - \tilde{Q}_t\|_s. \tag{35}$$

Set $v_t := E[\|Q_t - \tilde{Q}_t\|_s]$. By using the previous inequality and taking successive conditional expectations in (28), we obtain

$$v_t \leq \sum_{k=1}^{\nu} \|M_k\|_s^2 v_{t-k} + \sum_{l=1}^{\mu} \|N_l\|_s^2 \frac{m^{3/2}\sqrt{C_Q}}{C_q\sqrt{C_\lambda}} \left[1 + \frac{mC_Q}{C_q}\right] v_{t-l} := \sum_{j=1}^{\kappa} \beta_j v_{t-j}, \tag{36}$$

for all t and with our notations. Under Assumption U4, $v_t \to 0$ when $t \to \infty$. This implies that $Q_t = \tilde{Q}_t$ a.e. because (v_t) can be initialized arbitrarily far in the past. We deduce that $R_t = \tilde{R}_t$ a.e. and that $\varepsilon_t = \tilde{\varepsilon}_t$ a.e., knowing (η_t). This concludes the proof. ■

Proof of Lemma 3: With the notations of Subsection 2.2, consider the dynamics of the random vector $X_t^{(3)} := (Vech(Q_t), \dots, Vech(Q_{t-\nu+1}))'$. Clearly,

$$X_t^{(3)} = T_{33} X_{t-1}^{(3)} + \pi_t,$$

where

$$\pi_t := Vech(W_0) + \sum_{l=1}^{\mu} \tilde{N}_l Vech(\varepsilon_{t-l}\varepsilon'_{t-l}) := C_\pi.$$

Under U1, (π_t) is bounded from above. Moreover, under Assumption U2, the sum $\sum_{k=0}^{+\infty} T_{33}^k \pi_{t-k}$ is absolutely convergent a.e., and then $X_t^{(3)} = \sum_{k=0}^{+\infty} T_{33}^k \pi_{t-k}$. To be specific, since $\|\cdot\|_s$ is a multiplicative norm, we have for every realization

$$\|\pi_t\|_s \leq \|Vech(W_0)\|_s + \sum_{l=1}^{\mu} \|\tilde{N}_l\|_s C_\varepsilon.$$

Indeed, we check that, for any t,

$$\|Vech(\varepsilon_t\varepsilon'_t)\|_s = \|Vech(\varepsilon_t\varepsilon'_t)\|_2 \leq \sqrt{\frac{m(m+1)}{2}} \|\varepsilon_t\|_\infty^2.$$

Moreover, since $\|\mathbf{x}\|_s = \|\mathbf{x}\|_2$ for any vector $\mathbf{x}$ and since $\|A\|_\infty \leq \|A\|_s$ for any matrix A (Lütkepohl, 1996, p. 111), we get

$$\begin{aligned} \|\varepsilon_t\|_\infty \leq \|\varepsilon_t\|_s &\leq \|R_t^{1/2}\eta_t\|_s \leq \|R_t^{1/2}\|_s \|\eta_t\|_s \\ &\leq \|R_t\|_s^{1/2} \|\eta_t\|_2 \leq Tr(R_t)^{1/2} \sqrt{m} C_\eta \leq m C_\eta. \end{aligned}$$

This provides the inequality

$$\|Vech(\varepsilon_{t-l}\varepsilon'_{t-l})\|_s \leq \sqrt{\frac{m(m+1)}{2}} m^2 C_\eta^2 := C_\varepsilon,$$

for every t and l. We deduce

$$\|X_t^{(3)}\|_s \leq \sum_{k=0}^{+\infty} \|T_{33}\|_s^k C_\pi = \frac{C_\pi}{1 - \|T_{33}\|_s} := C_Q, \text{ and}$$

$$\|Q_t\|_\infty \leq \|X_t^{(3)}\|_\infty \leq \|X_t^{(3)}\|_s \leq C_Q. \blacksquare$$

Proof of Lemma 4: Is it known that, for any two squared definite positive matrices A and B, $\lambda_1(A+B) \geq \lambda_1(A)+\lambda_1(B)$ (Weyl's Theorem. See Lütkepohl, 1996, p. 75). In our case, we deduce obviously that $\lambda_1(Q_t) \geq \lambda_1(W_0)$ everywhere, due to Equation (4).

We can improve this lower bound in the particular case of "partially" scalar DCC models. Indeed, in this case, we have

$$\lambda_1(Q_t) \geq \lambda_1(W_0)+\sum_{k=1}^{\nu} \lambda_1((m^{(k)})^2 Q_{t-k}) \geq \lambda_1(W_0)+\sum_{k=1}^{\nu}(m^{(k)})^2 \lambda_1(Q_{t-k}). \quad (37)$$

Introduce the random vector $\vec{\lambda}_t := (\lambda_1(Q_t), \ldots, \lambda_1(Q_{t-\nu+1}))'$ and $\vec{\lambda}_W := (\lambda_1(W_0), 0, \ldots, 0)'$. Because of (37), we have for every t

$$\vec{\lambda}_t \geq M^* \vec{\lambda}_{t-1} + \vec{\lambda}_W.$$

Under Assumption U3, it is easy to check that $\sum_{k=0}^{+\infty}(M^*)^k$ is absolutely convergent and that

$$\vec{\lambda}_t \geq \sum_{k=0}^{+\infty}(M^*)^k \vec{\lambda}_W := \vec{\lambda}_\infty,$$

for every t. Obviously, $M^*\vec{\lambda}_\infty + \vec{\lambda}_W = \vec{\lambda}_\infty$. Due to the definition of M^*, this implies that all the components of $\vec{\lambda}_\infty$ are the same, i.e. there exists a real number λ_∞ such that $\vec{\lambda}_\infty = \lambda_\infty e$, $e \in \mathbb{R}^\nu$. Taking the first component of the vectorial equation $\lambda_\infty M^* e + \vec{\lambda}_W = \lambda_\infty e$ provides $\lambda_\infty \sum_{k=1}^{\nu} (m^{(k)})^2 + \lambda_1(W_0) = \lambda_\infty$. This proves the lower bound of $\lambda_1(Q_t)$ under U3.

Consider a fixed index $i = 1, \ldots, m$. The reasoning for the sequence $(q_{ii,t})_t$ is exactly similar, because

$$q_{ii,t} \geq (W_0)_{ii} + \sum_{k=1}^{\nu} (m^{(k)})^2 q_{ii,t-k},$$

for all t, this inequality playing the same role as (37). So the result. ■

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