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Love and Death : A Freund Model with Frailty

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Abstract

We introduce new models for analyzing the mortality dependence between individuals in a couple. The mortality risk dependence is usually taken into account in the actuarial literature by introducing an Archimedean copula. This practice implies symmetric effects on the remaining lifetime of the surviving spouse. The new model allows for both asymmetric reactions by means of a Freund model, and risk dependence by means of an unobservable common risk factor (or frailty). These models allow for distinguishing in the lifetime dependence the component due to common lifetime (frailty) from the broken-heart syndrome (Freund model). The model is applied to insurance products such as joint life policy, last survivor insurance, or contracts with reversionary annuities.

Keywords : Life Insurance, Coupled Lives, Frailty, Freund Model, Broken-Heart, Copula, Last Survivor Insurance, Reversionary Annuities.

1 Introduction

This paper introduces new models for analyzing the mortality dependence between individuals in a couple. This type of model is needed for risk management and pricing of life insurance products written on two heads, such as joint life policy, last survivor insurance policy, or contract with reversionary annuities.

The basic actuarial literature usually assumed the independence between the spouses' mortality risks. Recently the mortality risk dependence has been introduced by means of Archimedean copulas [see e.g. Frees, Carriere, Valdez (1996), Carriere (2000), Youn, Shemyakin (2001), Denuit et alii (2001), Shemyakin, Youn (2006), Luciano, Spreeuw, Vigna (2008), (2010)], and the effect of this dependence on the risk premia starts to be measured. However, copula models imply symmetric reactions of the mortality of a member of the couple when the other dies. An alternative consists in introducing jumps in mortality intensity (the Freund model) at the time of death of the a spouse, to capture the broken-heart syndrome [see e.g. Spreeuw, Wang (2008), Ji, Hardy, Li (2001), Spreeuw, Owadally (2012)]. Our paper extends this literature by mixing the Freund's model, which allows for asymmetric reactions of the mortality intensities at a death event, with unobservable common factor (or frailty), which underlies the Archimedean copulas.

The basic Freund model and its properties in terms of conditional intensities are presented in Section 2. This model allows for jump in the mortality intensity of a given spouse when the other spouse dies. The magnitude of this jump and its variation with respect to the age of the couple is the basis for constructing a convenient association measure, useful to analyse the broken-heart syndrome. The Freund model is extended in Section 3 to include common unobserved static frailty. In particular we discuss the properties of Freund models with latent intensities which are exponential affine functions of the frailty. These models are used in Section 4 to derive the prices of various contracts written on two heads. We consider these prices at the contract issuing as well as during the life of the contract (resp. couple). We emphasize the effect of the dependence between the mortality risks of the two spouses on these prices. Section 5 concludes. Proofs are gathered in appendices.

2 The basic Freund model

This type of model has been introduced by Freund (1961) to construct bivariate survival models for dependent duration variables, while still featuring the lack of memory property. It has been noted by Tosch, Holmes (1980) that such models have an interpretation in terms of latent variables. We follow this interpretation. The model is written for a given couple, without specifying the index of the couple and possibly its observed characteristics such as the birth dates of the spouses, the difference between their ages [Youn, Shemyakin (1999)], or their age at the day of their marriage or common law. In the application, such static couple characteristics will be introduced to capture the generation effects . The analysis is in continuous time and the lifetime variables are continuous variables.

2.1 The latent model

Let us consider a given couple with two spouses 1 and 2. The potential lifetimes of individuals 1 and 2, when both are alive, are denoted by X_1 and X_2, respectively. To get a unique time origin for the two members of the couple, these latent lifetimes are measured since the beginning of the couple. A first individual in the couple dies at date $\min(X_1, X_2)$. He/she is individual 1 (resp. individual 2), if $\min(X_1, X_2) = X_1$ [resp. $\min(X_1, X_2) = X_2$]. After this event, there can be a change in the potential residual lifetime distribution of the surviving individual. The potential residual lifetime of individual 1 (resp. individual 2) after the death of individual 2 (resp. individual 1) is denoted by X_3 (resp. X_4).

The joint distribution of the four latent variables is characterized by

i) the joint survival function of (X_1, X_2) :

$$S_{12}(x_1, x_2) = P[X_1 > x_1, X_2 > x_2]; \tag{2.1}$$

ii) the survival function of X_3 given $X_2 = \min(X_1, X_2) = z$:

$$S_3(x_3; z) = P[X_3 > x_3 | X_2 = \min(X_1, X_2) = z]. \tag{2.2}$$

iii) The survival function of X_4 given $X_1 = \min(X_1, X_2) = z$:

$$S_4(x_4; z) = P[X_4 > x_4 | X_1 = \min(X_1, X_2) = z]. \tag{2.3}$$

These three joint and conditional survival functions, defined on $(0, \infty)$ characterize the latent model for the analysis of the mortality in the couple. In this model there exist at least three generation effects corresponding to the generations of each spouse, and to the generation of the couple, respectively.

2.2 Individual lifetimes

2.2.1 Link between the individual lifetimes and the latent variables

The lifetimes of individuals 1 and 2 (since the beginning of the couple) are denoted by Y_1 and Y_2. They can be expressed in terms of the latent variables as :

$$\begin{cases} Y_1 = X_1 \mathbb{1}_{X_1<X_2} + (X_2+X_3)\mathbb{1}_{X_2<X_1} = \min(X_1,X_2) + X_3 \mathbb{1}_{X_2<X_1}, , \\ \\ Y_2 = X_2 \mathbb{1}_{X_2<X_1} + (X_1+X_4)\mathbb{1}_{X_1<X_2} = \min(X_1,X_2) + X_4 \mathbb{1}_{X_1<X_2}. \end{cases} \tag{2.4}$$

This system can be partially solved. First, the X_1, X_2 variables are related to variables (Y_1, Y_2) since :

$$\min(Y_1,Y_2) = \min(X_1,X_2), \text{ and since } Y_1 > Y_2, \text{ if and only if } X_1 > X_2.$$

Then the variables X_3 and X_4 can be deduced in some regimes since :

$$X_3 \mathbb{1}_{Y_2<Y_1} = Y_1 - \min(Y_1,Y_2) \text{ and } X_4 \mathbb{1}_{Y_1<Y_2} = Y_2 - \min(Y_1,Y_2).$$

As noted in Norberg (1989), the observed model can be interpreted in terms of a chain with four possible states[3], that are:

- state 1: both spouses are alive,
- state 2: husband dead, wife alive,
- state 3: husband alive, wife dead,
- state 4: both spouses are dead,

and transitions can only arise between states 1 and 2, 1 and 3, 2 and 4, and 3 and 4. Since the mortality intensity of a spouse can depend not only on the current state, but potentially on the time elapsed since the death of the other spouse, we get an exemple of semi-Markov chain.

2.2.2 The joint density function and its decomposition

The joint probability density function (pdf) of (Y_1, Y_2) is easily derived from the distribution of the latent variables. We have (see Appendix 1) :

$$\begin{aligned} f(y_1,y_2) &= \left[-\frac{\partial S_{12}}{\partial x_1}(y_1,y_1)\right]\left[-\frac{\partial S_4}{\partial x_4}(y_2-y_1;y_1)\right], \text{ if } y_2 > y_1, \\ &= \left[-\frac{\partial S_{12}}{\partial x_2}(y_2,y_2)\right]\left[-\frac{\partial S_3}{\partial x_3}(y_1-y_2;y_2)\right], \text{ if } y_1 > y_2. \end{aligned} \tag{2.5}$$

[3]In their analysis Ji, Hardy, Li (2001) consider also the possibility of a direct transition from state 1 to state 4 to account for catastrophic events (car accidents, plane crash) implying simultaneous deaths. They use a 5 days cutoff to account for a possible lag in reporting.

Therefore, the joint density function can feature a discontinuity when $y_1 = y_2$.
Let us consider the case $y_2 > y_1$. The density can also be written as :

$$f(y_1, y_2) = -\frac{\partial S^*}{\partial y}(y_1) \left[\frac{\partial S_{12}}{\partial x_1}(y_1, y_1) / \frac{\partial S^*}{\partial y}(y_1)\right] \left[-\frac{\partial S_4}{\partial x_4}(y_2 - y_1; y_1)\right], \tag{2.6}$$

where $S^*(y) = S_{12}(y, y)$ is the survival function of $\min(X_1, X_2)$ and $\frac{\partial S^*}{\partial y}(y) = \frac{\partial S_{12}}{\partial x_1}(y, y) + \frac{\partial S_{12}}{\partial x_2}(y, y)$. Thus, the decomposition of the bivariate density involves three components :

i) $\left[-\frac{\partial S^*}{\partial y}(y_1)\right]$ is the density of the first death event;

ii) the ratio $\left[\frac{\partial S_{12}}{\partial x_1}(y_1, y_1) / \frac{\partial S^*}{\partial y}(y_1)\right]$ is the probability that individual 1 dies at this first death event. It is equal to :

$$P[Y_1 < Y_2 | \min(Y_1, Y_2) = y_1],$$

iii) $\left[-\frac{\partial S_4}{\partial x_4}(y_2 - y_1; y_1)\right]$ is the density of the residual lifetime after this event.

2.2.3 Individual mortality intensities

Let us now derive the individual mortality intensities given the current information concerning the couple. Their expressions depend on the state either alive, or dead, of the other spouse.

i) Let us first consider a date y at which both individuals are still alive, that is, such that $Y_1 \geq y, Y_2 \geq y$. The mortality intensity of individual 1 is defined by :

$$\begin{aligned} \lambda_1(y | Y_1 \geq y, Y_2 \geq y) &= \lim_{dy \to 0^+} \left\{ \frac{1}{dy} P[y \leq Y_1 \leq y + dy | Y_1 \geq y, Y_2 \geq y] \right\} \\ &= \int_y^{\infty} f(y, y_2) dy_2 / S^*(y). \end{aligned} \tag{2.7}$$

After replacing the bivariate density by its expression (2.5) for $y_2 > y_1$ and computing the integral, we get :

$$\lambda_1(y | Y_1 \geq y, Y_2 \geq y) = \left[-\frac{\partial S_{12}}{\partial x_1}(y, y)\right] / S^*(y). \tag{2.8}$$

This is the crude intensity function of individual 1 involved in the decomposition of the joint density function.

Similarly, we have :

$$\begin{aligned}
\lambda_2(y|Y_1 \geq y, Y_2 \geq y) &= \lim_{dy \to 0^+} (\frac{1}{dy} P[y \leq Y_2 \leq y + dy | Y_1 \geq y, Y_2 \geq y]) \\
&= \int_y^{\infty} f(y_1, y) dy_1 / S^*(y). \qquad (2.9) \\
&= \left[-\frac{\partial S_{12}}{\partial x_2}(y, y) \right] / S^*(y).
\end{aligned}$$

ii) The expression of the mortality intensities can change if one of the individual dies exactly at date y. The mortality intensity of individual 1 at date y, if individual 2 dies at date y, becomes :

$$\begin{aligned}
& \lambda_{1|2}(y|Y_1 \geq y, Y_2 = y) \\
= & \lim_{dy \to 0^+} \left[\frac{1}{dy} P(y < Y_1 \leq y + dy | Y_1 \geq y, Y_2 = y) \right] \\
= & [f(y, y)] / \left[-\frac{\partial S_{12}}{\partial x_2}(y, y) \right] \\
= & -\frac{\partial S_3}{\partial x_3}(0, y), \qquad (2.10)
\end{aligned}$$

by applying the expression of the joint density (2.5) with $y_1 = y_2 = y$.

Similarly, we get :

$$\begin{aligned}
& \lambda_{2|1}(y|Y_1 = y, Y_2 \geq y) \\
= & \lim_{dy \to 0^+} \left\{ \frac{1}{dy} P[y \leq Y_2 \leq y + dy | Y_1 = y, Y_2 \geq y] \right\} \\
= & -\frac{\partial S_4}{\partial x_4}(0, y). \qquad (2.11)
\end{aligned}$$

Note that $S_3(0, y) = S_4(0, y) = 1$. Therefore we also have :

$$\begin{aligned}
\lambda_{1|2}(y|Y_1 \geq y, Y_2 > y) &= -\frac{\partial \log S_3}{\partial x_3}(0, y), \\
\text{and } \lambda_{2|1}(y|Y_1 = y, Y_2 \geq y) &= \frac{-\partial \log S_4}{\partial x_4}(0, y),
\end{aligned}$$

which are the expected expressions of the intensities in terms of survival functions.

iii) Finally, we can also consider the mortality intensity of spouse 1, when the other spouse is dead since a given time. We have, for $y > y^*$:

$$
\begin{aligned}
& \lambda_{1|2}(y|Y_1 \geq y, Y_2 = y^*) \\
= \; & \lim_{dy \to 0^+} \frac{1}{dy} P[y < Y_1 < y + dy | Y_1 \geq y, Y_2 = y^*] \\
= \; & f(y, y^*) / \int_y^\infty f(u, y^*) du \\
= \; & -\frac{\partial \log S_3}{\partial x_3}(y - y^*, y^*),
\end{aligned}
$$

which is just the intensity of the residual lifetime X_3 given the date of the first death.

2.2.4 Dependence and Jump in Intensities

It has been suggested in Clayton (1978) to measure the dependence between duration variables by considering the jump in intensities following the news of a death. We get a functional measure of dependence function of the age y of the couple, which is especially appropriate for following the dependence phenomenon during the couple life. These per-cent jumps are the following ones :

When individual 2 dies at date y, the jump at this date of the mortality intensity of individual 1 is :

$$
\begin{aligned}
\gamma_{1|2}(y) &= \lambda_{1|2}(y|Y_1 \geq y, Y_2 = y) / \lambda_1(y|Y_1 \geq y, Y_2 \geq y) \\
&= \left\{ \left[-\frac{\partial S_3}{\partial x_3}(0; y) \right] S^*(y) \right\} / \left[-\frac{\partial S_{12}}{\partial x_1}(y, y) \right]. \qquad (2.12)
\end{aligned}
$$

Symmetrically, we get :

$$
\begin{aligned}
\gamma_{2|1}(y) &= \lambda_{2|1}(y|Y_1 = y, Y_2 \geq y) / \lambda_2(y|Y_1 \geq y, Y_2 \geq y) \\
&= \left\{ \left[-\frac{\partial S_4}{\partial x_4}(0; y) \right] S^*(y) \right\} / \left[-\frac{\partial S_{12}}{\partial x_2}(y, y) \right]. \qquad (2.13)
\end{aligned}
$$

In the standard literature on bivariate survival models, the bivariate density function is continuous at $y_1 = y_2 = y$. Then, the two measures $\gamma_{1|2}(y)$ and $\gamma_{2|1}(y)$

coincide for any age y [see the discussion in Sections 3.2, 3.2.2]. This regularity assumption is not necessarily satisfied in a Freund model. We can observe different reactions of a spouse at the death of the other spouse in the couple.

Definition 1 : We have the broken-heart syndrome for spouse 1 (resp. 2) at date y, if $\gamma_{1|2}(y) > 1$ [resp.$\gamma_{2|1}(y) > 1$].

We can have the broken-heart syndrome (or the reverse broken-heart syndrome when the directional measure of association is strictly smaller than 1), with different magnitude according to the age and spouse. We can even observe reactions in different directions. This arises when the wife is devastated by the death of her husband, with an increase of her mortality intensity, whereas the death of the wife may provide more freedom to her husband and possibly a decrease of his mortality rate. This is the "love and death" phenomenon with the fact that love is not always shared and can be age-dependent.

There exists a few studies trying to measure the effect and showing a positive estimated broken-heart syndrome [see e.g. Jagger, Sutton (1991), Ji, Hardy, Li (2011)]. Moreover it is shown that the broken-heart syndrome affects widowers more than widows [see Spreeuw, Owadally (2012)]. However these studies are based on rather old sets of data, such as a set of contracts of an insurance company over an observation period from 1988 to 1993, specific to the clients of this company. Moreover, by neglecting the frailty effect discussed in Section 3, the estimates may suffer from an omitted heterogeneity biais.

2.3 Observed and latent intensities

Let us now link the distributions of the observed and latent variables. Since the variables (X_1, X_2) and (Y_1, Y_2) are only related by the conditions $\min(X_1, X_2) = \min(Y_1, Y_2)$ and $\mathbb{1}_{X_1>X_2} = \mathbb{1}_{Y_1>Y_2}$, there exist several joint distributions of the pair (X_1, X_2) leading to a given joint distribution of $(\min(X_1, X_2), \mathbb{1}_{X_1>X_2})$. Loosely speaking, under weak regularity conditions, we can choose arbitrarily the form of the copula between X_1 and X_2 [see e.g. Zheng, Klein (1995)]. To solve a part of this identification problem, we assume below that the latent variables X_1 and X_2 are independent. Then the distribution of the latent variables is characterized by the following latent intensities :

i) the latent intensity of X_1 denoted by $a_1(x_1)$;
ii) the latent intensity of X_2 denoted by $a_2(x_2)$;
iii) the latent intensity of X_3 given $X_2 = \min(X_1, X_2) = z$, denoted by $a_3(x_3; z)$;
iv) the latent intensity of X_4 given $X_1 = \min(X_1, X_2) = z$, denoted by $a_4(x_4; z)$.

The associated cumulated intensities, that are their primitives with respect to the x argument, are denoted by $A_1(x_1), A_2(x_2), A_3(x_3; z), A_4(x_4; z)$, respectively. We deduce that :

$$S_{12}(x_1, x_2) = \exp\{-[A_1(x_1) + A_2(x_2)]\}, S_3(x_3; z) = \exp[-A_3(x_3; z)],$$

$$S_4(x_4; z) = \exp[-A_4(x_4; z)]$$

Then, the expression (2.5) of the bivariate probability density function becomes :

$$\begin{aligned} f(y_1, y_2) &= a_1(y_1)\exp\{-[A_1(y_1) + A_2(y_2)]\}a_4(y_2 - y_1; y_1)\exp[-A_4(y_2 - y_1; y_1)], \text{ if } y_2 > y_1, \\ &= a_2(y_2)\exp[-(A_1(y_1) + A_2(y_2))]a_3(y_1 - y_2; y_2)\exp[-A_3(y_1 - y_2; y_2)], \text{ if } y_1 > y_2. \end{aligned} \tag{2.14}$$

Similarly the directional measures of association can be written in terms of the latent intensities by using the expressions (2.12)-(2.13).

Property 1 :

The directional measures of association are :

$$\gamma_{1|2}(y) = a_3(0; y)/a_1(y), \gamma_{2|1}(y) = a_4(0; y)/a_2(y). \tag{2.15}$$

3 Freund model with static frailty

The notion of (shared) frailty has been introduced by Vaupel, Manton, Stallard (1979). The idea is to introduce unobserved heterogeneity (or frailty) in bivariate duration models in order to create an additional dependence between lifetimes. In the basic specification, this frailty is static, since it depends on the couple only, neither on time, nor age. It represents the effect of common lifestyle, or common disasters encountered by the couple. In the extended model, the dependence between the lifetimes are due to either the exogenous shock (the frailty),or to the so-called contagion effects, that are the jumps in the intensities at the time of default. This type of specification allows to disentangle these two effects. We first extend the Freund model of Section 2.4 to include unobserved frailty. Then, we discuss special cases.

3.1 The model

Let us denote by F the frailty variable, possibly multivariate. We consider a Freund model with the structure introduced in Section 2.4, where X_1 and X_2 are independent conditional on F, with latent intensities conditional on F given by : $a_1(x_1; F), a_2(x_2; F),\ a_3(x_3; z; F), a_4(x_4; z, F)$. Let us now derive the latent[4] survival functions $S_{12}(x_1, x_2), S_3(x_3; z), S_4(x; z)$, when frailty F has been integrated out. We have :

$$\begin{aligned} S_{12}(x_1, x_2) &= EP[X_1 \geq x_1, X_2 \geq x_2 | F] \\ &= E\{\exp -[A_1(x_1; F) + A_2(x_2; F)]\}, \end{aligned}$$

where the expectation is taken with respect to the distribution of F.

Similarly we get :

$$\begin{aligned} S_3(x_3; z) &= P[X_3 > x_3 | X_2 = \min(X_1, X_2) = z] \\ &= P[X_3 > x_3 | X_2 = z, X_1 > z] \\ &= \frac{E[a_2(z, F) \exp(-[A_1(z, F) + A_2(z; F) + A_3(x_3; z; F)])]}{E[a_2(z; F) \exp(-[A_1(z; F) + A_2(z; F)])]}. \end{aligned}$$

These formulas can be used as inputs to derive the bivariate observed density (2.5) and the directional measures of association (2.12)-(2.13). For instance, we have by (2.12) :

$$\gamma_{1|2}(y) = \frac{E\{a_3(0; y; F) a_2(y, F) \exp(-[A_1(y; F) + A_2(y; F)]\} E[\exp(-[A_1(y; F) + A_2(y; F)])]}{E\{a_2(y; F) \exp(-[A_1(y; F) + A_2(y; F)])\} E\{a_1(y; F) \exp[-A_1(y; F) + A_2(y; F)]\}}$$

We deduce the property below.

Property 2 :

$$\gamma_{1|2}(y) = \frac{\overset{Q_y}{E}[a_3(0; y; F) a_2(y; F)]}{\overset{Q_y}{E}[a_1(y; F)]\ \overset{Q_y}{E}[a_2(y; F)]}, \tag{3.1}$$

where Q_y denotes the probability distribution with density :

$$q_y(F) = \exp\{-[A_1(y) + A_2(y)]F\} / E[\exp(-(A_1(y) + A_2(y))F],$$

[4]Note that the model has two layers of latent variables, first F, second X_1, X_2, X_3, X_4.

with respect to the distribution of F.

The change of probability is due to the aging of the heterogeneity structure in the population of surviving couples, called Population-at-Risk (PaR) at age y [see e.g. Vaupel et alii (1979), eq. (5)].

Since the conditional directional measure of association is [see (2.15)] :

$$\gamma_{1|2}(y;F) = a_3(0,y;F)/a_1(y,F),$$

we can also write the corresponding unconditional measure as :

$$\begin{aligned}
\gamma_{1|2}(y) &= \frac{\overset{Q_y}{E}\left[\gamma_{1|2}(y;F)a_1(y;F)a_2(y;F)\right]}{\overset{Q_y}{E}\left[a_1(y;F)\right]\overset{Q_y}{E}\left(a_2(y;F)\right]} \\
&= \overset{\tilde{Q}_y}{E}\left[\gamma_{1|2}(y;F)\right]\frac{\overset{Q_y}{E}\left[a_1(y;F)a_2(y;F)\right]}{\overset{Q_y}{E}\left[a_1(y;F)\right]\overset{Q_y}{E}\left[a_2(y;F)\right]}, \\
\text{where : } d\tilde{Q}^y &= \frac{a_1(y;F)a_2(y;F)}{\overset{Q_y}{E}\left[a_1(y;F)a_2(y;F)\right]}dQ^y.
\end{aligned}$$

Thus the unconditional directional measure of association $\gamma_{1|2}(y)$ is an average of the conditional directional measures of association with respect to a modified probability distribution, and adjusted for the dependence between $a_1(y;F)$ and $a_2(y;F)$, since the adjustment term equals 1, when these variable are not correlated under Q^y.

3.2 Single proportional frailty

Following Vaupel, Manton, Stallard (1979), it is usual to consider a single positive frailty with the same effect on all latent intensities. This implies an Archimedean copula for the bivariate latent variables X_1 and X_2 [see Oakes (1989)], but not for the observed variables Y_1, Y_2, due to the changes in intensities after the first death event. More precisely, if :

$$a_1(x_1;F) = a_1(x_1)F, a_2(x_2;F) = a_2(x_2)F, a_3(x_3;z;F) = a_3(x_3;z)F; a_4(x_4;z;F) = a_4(x_4;z)F,$$

we deduce from Property 2 eq.(3.1) that :

$$\gamma_{1|2}(y) = \frac{a_3(0;y)}{a_1(y)} \frac{\overset{Q_y}{E}(F^2)}{[\overset{Q_y}{E}(F)]^2}, \gamma_{2|1}(y) = \frac{a_4(0;y)}{a_2(y)} \frac{\overset{Q_y}{E}(F^2)}{[\overset{Q_y}{E}(F)]^2}. \tag{3.2}$$

In this simple case, the directional measures of association given F are [see (2.15)] :

$$\gamma_{1|2}(y;F) = \frac{a_3(0;y)F}{a_1(y)F} = \frac{a_3(0;y)}{a_1(y)}, \gamma_{2|1}(y;F) = \frac{a_4(0;y)}{a_2(y)}.$$

They are independent of the frailty F, but not necessarily equal, which allows for asymmetric reactions.

The omitted heterogeneity introduces a positive bias on these measures. Indeed, we have $\overset{Q_y}{E}(F^2)/[\overset{Q_y}{E}(F)]^2 \geq 1$, by Cauchy-Schwartz inequality and more generally the property below :

Property 3 : In a Freund model with single proportional frailty the unconditional directional measures of association are larger than the conditional ones. They are equal if and only if frailty F is constant, that is, if there is no omitted heterogeneity :

$$\gamma_{1|2}(y) \geq \gamma_{1|2}(y;F), \gamma_{2|1}(y) \geq \gamma_{2|1}(y;F), \forall F.$$

However the per-cent adjustment for omitted heterogeneity is independent of age y and of the direction, which is considered. In particular the symmetry condition between spouses is preserved since :

$$\gamma_{1|2}(y;F) = \gamma_{2|1}(y;F) \Longleftrightarrow \gamma_{1|2}(y) = \gamma_{2|1}(y).$$

3.3 The actuarial literature

The models with mortality dependence considered in the actuarial literature are often special cases of the single proportional frailty model of Section 3.2.1, assuming moreover the continuity of the latent intensities :

Continuity assumption of the latent intensities

$$\begin{aligned} a_3(x_3;z) &= a_1(x_3+z), \forall x_3, z, \\ a_4(x_4;z) &= a_2(x_4+z), \forall x_4, z. \end{aligned}$$

Under the continuity assumption, the lifetimes Y_1, Y_2 are independent given the shared frailty F, with joint conditional survivor function :

$$S_{12}(y_1, y_2|F) = \exp[-[A_1(y_1) + A_2(y_2)]F].$$

To ensure the positivity of the intensity, the frailty F has to be positive. Let us denote by ψ its Laplace transform defined for positive arguments u by :

$$\psi(u) = E[\exp(-uF)]. \tag{3.3}$$

By integrating out the frailty, we deduce the joint survivor function :

$$S_{12}(y_1, y_2) = \psi[A_1(y_1) + A_2(y_2)]. \tag{3.4}$$

A similar computation can be performed to derive the marginal survivor functions. We get :

$$S_1(y_1) = \psi[A_1(y_1)], S_2(y_2) = \psi[A_2(y_2)]. \tag{3.5}$$

Since the Laplace transform of F is continuous and strictly increasing, it is invertible. We deduce the expression of S_{12} in terms of S_1, S_2 and ψ:

$$S_{12}(y_1, y_2) = \psi[\psi^{-1}[S_1(y_1)] + \psi^{-1}[S_2(y_2)]] \tag{3.6}$$

This is the standard definition of a copula [Sklar (1959)]:

$$S_{12}(y_1, y_2) = C[S_1(y_1), S_2(y_2)], \tag{3.7}$$

with a survivor Archimedean copula [Genest, McKay (1986)]:

$$C(u_1, u_2) = \psi[\psi^{-1}(u_1) + \psi^{-1}(u_2)], \tag{3.8}$$

Property 4 : Let us consider a Freund model with single proportional frailty. Under the continuity assumption, the dependence between the lifetime variables Y_1, Y_2 is summarized by an Archimedean copula with the Laplace transform of the frailty as generator.

Therefore, any Archimedean copula admits an interpretation in term of common shock and there is no reason to distinguish the two approaches [see e.g. Das (2004) for such a distinction].

The actuarial literature has considered this special case with different choices of the marginal distributions of the lifetimes and of the copulas [see Tables 3.1 and 3.2, for the actuarial literature, and Nelsen (1999) for a rather extensive list of copulas].

Table 3.1

Selected Marginal Distributions

Gompertz	Frees et alii, (1996), Youn, Shemyakin (2001), Luciano et alii (2008), (2010)
Weibull	Frees et alii (1996), Youn, Shemyakin (1999), (2001), Shemyakin, Youn (2006), Luciano et alii (2010)

Table 3.2

Selected Copula

Frank	Frees et alii, (1996), Youn, Shemyakin (2001), Luciano et alii (2008), (2010)
Gumbell-Hougaard	Youn, Shemyakin (1999), (2001), Shemyakin, Youn (2006), Spreeuw (2006) Luciano et alii (2010)
Linear mixing frailty	Frees et alii (1996)
Clayton	Luciano et alii (2008), (2010), Spreeuw (2006)
4.2.20 Nelsen copula[5]	Luciano et alii (2008), (2010)

A more recent literature [see e.g. Denuit, Cornet (1999), Spreeuw, Wang (2008), Ji, Hardy, Li (2011), Spreeuw, Owadally (2012)] focus on the broken-heart syndrome, but without introducing frailty in the specification of the intensities.

3.4 Affine intensity model

A simple extension of the bivariate survival model discussed in Sectin 3.2 is obtained by introducing an intercept in the basic proportional frailty model [the so-called Generalized Shared Frailty model developed in Iachine (2004) in a special case]. The specification becomes :

$$\begin{aligned} a_1(x_1;F) &= a_1(x_1)F + b_1(x_1), a_2(x_2;F) = a_2(x_2)F + b_2(x_2), \\ a_3(x_3;z;F) &= a_3(x_3;z)F + b_3(x_3;z), a_4(x_4;z;F) = a_4(x_4;z)F + b_4(x_4;z). \end{aligned}$$

[5]The numbers 4.2.20 indicate the copula in the list provided by Nelsen (1999).

This extended version allows for conditional directional measures of association $\gamma_{1|2}(y;F)$ and $\gamma_{2|1}(y;F)$ depending on frailty F, and leads to non Archimedean copulas, when considering the joint distribution of latent lifetimes X_1 and X_2.

The affine specification is likely the most appropriate one for representing the effect of common lifestyle F and especially the memory features. After the death of a spouse, we expect that the effect of common lifestyle will diminish and asymptotically vanish. Thus, we expect that the latent intensity $a_3(x_3;z)$ [resp. $a_4(x_4;z)$] is a decreasing function of x_3 (resp. x_4) tending to zero at infinity. Then functions b_3 and b_4 provide the limiting mortality intensity a long time after the death of the other spouse.

Finally, note that this affine intensity models assumes implicitly no remarriage or new common law of the surviving spouse. This assumption is rather realistic for our purpose, since the insurance policies of interest are generally taken by rather old couples to profit of estate tax reductions, or to provide a rent to the surviving spouse.

4 Pricing bivariate contracts

We will now derive the pricing formulas for insurance contracts written on two heads such as joint life policies, last survivor policies and policies with reversionary annuities. By considering extended Freund models (under the risk-neutral probability), we analyze the effect of jumps in intensity on prices at the contract issuing as well as on the premium updating during the life of the contract.

4.1 Prices at the signature of the contracts

The premium computations for the joint policies are based on the joint remaining lifetimes risk-neutral distribution conditional on the ages of the spouses at the beginning of their couple y_{10}^*, y_{20}^*, say, and on the fact that both spouses are still alive with an age of the couple equal to z_0, say, at the signature of the contract. Thus, the joint risk-neutral density of the remaining lifetimes $\tilde{y}_j = Y_j - z_0, j = 1,2$ at the signature of the contract is[6] :

[6]The link between the historical and risk-neutral bivariate distributions of the lifetimes is discussed in Appendix 2. Note that the insurance literature often price the insurance contracts by means of the historical distributions to get the so called fair premium [see e.g. Ji, Hardy, Li (2001), Section 5.6].

$$
\begin{aligned}
& \tilde{f}_0(\tilde{y}_1, \tilde{y}_2 | z_0) \\
= & \lim_{dy_1, dy_2 \to 0} \left\{ \frac{1}{dy_1 dy_2} P[Y_1 \in (\tilde{y}_1 + z_0, \tilde{y}_1 + z_0 + dy_1), Y_2 \in (\tilde{y}_2 + z_0, \tilde{y}_2 + z_0 + dy_2) \right. \\
& |Y_1 > z_0, Y_2 \geq z_0, y^*_{10}, y^*_{20}] \\
= & f_0(\tilde{y}_1 + z_0, \tilde{y}_2 + z_0) / S_0(z_0), \qquad (4.1)
\end{aligned}
$$

where the index 0 means that the distribution characteristics of Section 3 can now depend on the initial ages y^*_{10}, y^*_{20}.

Let us now illustrate the premium computation in a continuous time framework with instantaneous constant interest rate r. For each insurance product, we have to analyze the risk-neutral distribution of the discounted cash-flows.

i) Joint life policy

Let us denote by a the premium rate and consider a unitary insurance payoff at the first death of a spouse. The discounted sequence of cash-flows measured at the signature of the contract is :

$$
\begin{aligned}
C_0^{(1)}(a, r, z_0; Y_1, Y_2) &= a \int_0^{\min(Y_1, Y_2) - z_0} \exp(-rh) dh - \exp[-r(\min(Y_1, Y_2) - z_0)] \\
&= \frac{a}{r} \{1 - \exp[-r(\min(Y_1, Y_2) - z_0)]\} - \exp[-r(\min(Y_1, Y_2) - z_0)]\}.
\end{aligned}
\qquad (4.2)
$$

There exist different ways for balancing the stochastic positive and negative cash-flows. In particular the premium rate [7] can be defined by fixing equal expectations to these sequences. We get :

$$
a_0^{*(1)}(r) = r \frac{E_0\{\exp[-r(\min(Y_1, Y_2) - z_0)] | Y_1 \geq z_0, Y_2 \geq z_0]\}}{1 - E_0\{\exp[-r(\min(Y_1, Y_2) - z_0]) | Y_1 \geq z_0, Y_2 \geq z_0)\}}. \qquad (4.3)
$$

ii) Last survivor policy

Let us now assume that the death event written in the policy is the second death of a spouse. The formulas are the same as for the joint life policy above after substituting $\max(Y_1, Y_2)$ to $\min(Y_1, Y_2)$. For instance, the fair premium becomes :

[7] The fair premium rate is obtained by replacing the risk-neutral distribution by the historical distribution in formula (4.3). Otherwise the premium rate accounts for a risk premium.

$$a_0^{*(2)}(r) = r\frac{E_0(\exp[-r(\max(Y_1,Y_2)-z_0)]|Y_1 \geq z_0, Y_2 \geq z_0)}{1-E_0\{\exp[-r(\max(Y_1,Y_2)-z_0)]|Y_1 \geq z_0, Y_2 \geq z_0\}} \tag{4.4}$$

iii) Reversionary annuities

Finally, let us consider a product in which the premium is paid when both spouses are alive and a unitary annuity is paid to the surviving spouse up to his/her death. The discounted sequence of cash-flows becomes :

$$\begin{aligned} C^{(3)}(a,r,z_0;Y_1,Y_2) &= a\int_0^{\min(Y_1,Y_2)-z_0} \exp(-rh)dh - \int_{\min(Y_1,Y_2)-z_0}^{\max(Y_1,Y_2)-z_0} \exp(-rh)dh \\ &= \frac{a}{r}\{1-\exp(-r[\min(Y_1,Y_2)-z_0])\} \\ &\quad -\frac{1}{r}\{\exp[-r(\min(Y_1,Y_2)-z_0)] \\ &\quad -\exp[-r(\max(Y_1,Y_2)-z_0)]\}. \end{aligned} \tag{4.5}$$

The associated premium rate is :

$$a_0^{*(3)}(r) = \frac{E_0\{\exp(-r[\min(Y_1,Y_2)-z_0]) - \exp(-r[\max(Y_1,Y_2)-z_0])|Y_1 \geq z_0, Y_2 \geq z_0\}}{1-E_0\{\exp(-r[\min(Y_1,Y_2)-z_0])|Y_1 \geq z_0, Y_2 \geq z_0\}}. \tag{4.6}$$

iv) Individual products

The premia for joint products have naturally to be compared with the premia of the individual insurance products written on a single head $j=1,2$. The associated fair premium is :

$$a_{j,0}^{*}(r) = r\frac{E_0(\exp[-r(Y_j-z_0)]|Y_j \geq z_0)}{1-E_0(\exp[-r(Y_j-z_0)]|Y_j \geq z_0]}, \tag{4.7}$$

if only information on spouse j is taken into account and

$$a_{j,0}^{**}(r) = \frac{rE_0(\exp[-r(Y_j-z_0)]|Y_1 \geq z_0, Y_2 \geq z_0)}{1-E_0[\exp[-r(Y_j-z_0))|Y_1 \geq z_0, Y_2 \geq z_0]}, \tag{4.8}$$

if the information on the couple is taken into account.

In the limiting case of a zero risk-free rate $r=0$, the expressions of the premia are obtained by a Taylor expansion. We get :

$$a_0^{*(1)}(0) = \frac{1}{E_0\{[\min(Y_1,Y_2)-z_0]|Y_1 \geq z_0, Y_2 \geq z_0)\}},$$

$$a_0^{*(2)}(0) = \frac{1}{E_0\{[\max(Y_1, Y_2) - z_0]|Y_1 \geq z_0, Y_2 \geq z_0)\}},$$

$$a_0^{*(3)}(0) = \frac{E_0\{\max(Y_1, Y_2) - \min(Y_1, Y_2)|Y_1 \geq z_0, Y_2 \geq z_0)\}}{E_0\{\min(Y_1, Y_2)|Y_1 \geq z_0, Y_2 \geq z_0)\}},$$

$$a_{j,0}^{*}(0) = \frac{1}{E_0\{Y_j - z_0|Y_j \geq z_0)\}},$$

$$a_{j,0}^{**}(0) = \frac{1}{E_0\{Y_j - z_0|Y_1 \geq z_0, Y_2 \geq z_0)\}}.$$

Note that the pricing of the individual contracts of two spouses cannot be done seperately. The price of the contract of a widow has to account for the time elapsed since the death of her husband.

4.2 Effect of risk dependence on prices

Let us now illustrate the effect on policy prices of risk dependencies: due to the frailty and to the asymmetric jump in intensities existing in a Freund model.

We consider a model with single proportional frailty (see Section 3.2). The population of couples is such that the two spouses have the same age 30. The distribution of the heterogeneity F at age 30 is assumed to be a gamma distribution. Note that when there is no jump in latent intensities, the joint distribution of the lifetimes is associated to a Clayton copula. Due to the mover-stayer phenomenon, as the population ages, the distribution given that both spouses survive up to age $z_0 > 30$, that is, the heterogeneity distribution that the insurance company applies to price a contract for a couple with an underwriting age $z_0 > 30$, will depend on age z_0. Intensities of the latent duration variables X_1 (female), X_2 (male) are of the following form:

$$a_1(x_1) = \exp(\alpha_1 x_1 + \beta_1), \qquad \forall x_1 > 0,$$

and

$$a_2(x_2) = \exp(\alpha_2 x_2 + \beta_2), \qquad \forall x_2 > 0.$$

We assume that the death of the spouse has a constant multiple effect γ on the mortality intensity of the survivor. Thus, given $z = \min(X_1, X_2)$, the conditional intensities of X_3, X_4 are of the form:

$$a_3(x_3, z) = \gamma \exp\Big(\alpha_1(z + x_3) + \beta_1\Big), \qquad \forall x_3 > 0,$$

and

$$a_4(x_4, z) = \gamma \exp\Big(\alpha_2(z + x_4) + \beta_2\Big), \qquad \forall x_4 > 0,$$

where the constant $\gamma = \frac{a_3(0,z)}{a_1(z)} = \frac{a_4(0,z)}{a_2(z)}$ is non smaller than 1 to reflect the broken-heart syndrome. For numerical illustrations, parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ are chosen to fit the marginal intensities of American females and males at ages $31, 32, ..., 110$, provided by the Human Mortality Database[8]. Their values are reported below:

$$\alpha_1 = 0.089, \beta_1 = -7.613, \alpha_2 = 0.081, \beta_2 = -6.934.$$

The measure of association γ is the same in both directions with values $\gamma \in \{1, 3, 5\}$. $\gamma = 5$ corresponds to a very huge impact of the death of the spouse on the survivor lifetime and $\gamma = 1$ corresponds to the case of no impact (at the individual level, indeed, even in this case there is still jump of intensity when the heterogeneity is integrated out, see e.q.(3.2)). The gamma distribution of the heterogeneity at age 30 is set to have a shape parameter k and a scale parameter $1/k$. Therefore, the average mortality intensity at age 30 is the same for each value of k, since $\mathbb{E}(F) = 1/k \cdot k = 1$ does not depend on k. The heterogeneity parameter k will be set to $k \in \{2, 5, 10\}$. $k = 10$ corresponds to a low heterogeneity level and $k = 2$ corresponds to a high one. This specification of the duration distribution is the risk-neutral distribution, which can be used to price the different life insurance contracts described in Section 4.1. The risk-free interest rate is set to $r = 1\%$. We provide in Figure 1 the evolution of the premium rates as a function of the underwriting age $z_0 \in 31, 32, ..., 80$, for different contracts and for $\gamma = 5, k = 2$. The contracts include a joint life policy, a last survivor policy, a contract with reversionary annuities, and the individual insurance products for female with, or without, the information on the survival of the husband up to z_0.

These premia are not directly comparable, since the premia paid by the insured peoples (resp. the payments by the insurance company) do not correspond to a same period. Nevertheless for each product, the premium rate is increasing with the age of underwriting of the couple, which is in conformity with the usual premium structure without heterogeneity.

In general, in a model with heterogeneity, the average intensity (as well as of the premium) can be or not increasing in z_0. Indeed, the aging of the population has a positive impact on the premium when z_0 increases, while the mover-stayer phenomenon has a negative impact on the premium since couples with higher risks die out more quickly, hence the average heterogeneity is improving in time. In this example, the first effect is more important, which results in an increasing premium.

Besides, the premium rate of an individual insurance contract for a female is always lower when the insurance company know that her spouse is still alive,

[8]The Human Mortality Database (HMD) was created to provide detailed mortality and population data to researchers, students, journalists, policy analysts, and others interested in the history of human longevity. It is maintained by the University of California, Berkeley, and the Max Planck Institute for Demographic Research in Rostock, Germany; its official website is http://www.mortality.org

as shown in the lower right panel. The difference is negligible at low ages, but increases significantly with respect to z_0. We also observe that the curves of the premia are convex, except for reversionary annuities, where the trend is almost linear.

Let us now illustrate the effect of risk dependencies and of the heterogeneity for the different insurance contracts. We first illustrate in Tables 5.1 and 5.2 the effect of the measure of association γ for two different ages 30 and 50. This parameter has no effect on the joint insurance policies: indeed, the contract terminates up to the first death whereas the measure of association impacts only the residual lifetime beyond the first death event. Therefore, premium rates of the joint insurance are not reported in the Tables. The two last columns correspond to the individual insurance contract for a female with and without information on the survival of her spouse. We get premia, which increase with the γ parameter, except for the reversionary annuities. Indeed, unlike other contracts which concern death benefit, a reversionary annuity pays survival benefits; therefore its relationship with the deterioration of mortality is opposite to other products. Then we illustrate in Tables 5.3 and 5.4 the effect of heterogeneity, characterized by parameter k, for two different ages 30 and 50. This effect is less clear than the effect of γ. For instance, for a product with death benefit, in a more heterogeneous population ($k = 2$) there are more couples of extremely high risk, as well as more couples of extremely low risk. The first couples contribute to increase the premium whereas the latter couples contribute to diminish the premium. For the reversionary annuity, a riskier couple is expected to trigger annuity payment earlier, which means less premium income, but the payment is also expected to terminate earlier, which spells less total payment. In our simulation studies, we observe that, for each product, the premium rate is decreasing in the heterogeneity, both for age 30 and 50. Figure 2 plots, for each k, simulated lifetimes distributions for the last survivor, respectively for $z_0 = 30$ and 50.

Special attention should be paid when comparing premium rates at age 50 for different values of parameter k. Indeed, for each value of k, $\gamma(k, 1/k)$ is the heterogeneity distribution at age 30, but the heterogeneity distribution conditional on the survival of both spouses up to age 50 is no longer the same. However, it is still a gamma distribution $\gamma(k, 1/[k + A_1(z_1 - z_0) + A_2(z_1 - z_0)])$, where $z_0 = 30$, $z_1 = 50$ and A_1, A_2 are the cumulative intensities (see Appendix 3). Therefore, the mean of the heterogeneity is $k/[k+A_1(z_1-z_0)+A_2(z_1-z_0)]$, and quotient between the variance at age 50 and that at age 30 is $k^2/[k + A_1(z_1 - z_0) + A_2(z_1 - z_0)]^2$. Both quantities are decreasing functions of k, that is, the mean and the variance of the heterogeneity diminish (in proportion) faster in the population with initially the highest heterogeneity ($k = 2$). Figure 3 plots, for each k, the probability density function of the heterogeneity both at age 30 and at age 50. The gamma distribution parameters at age 50 are reported in Table 5.5.

4.3 Evolution of the price of the contract during the life of the contract

A premium level a_0 is fixed at the signature of each contract (see Section 4.1). However, it is important to evaluate regularly the residual value of this contract during its life, for instance, to include it correctly in the balance sheet, or, if it is securitized, to evaluate the price of the corresponding component of the Insurance Linked Security.

Let us first focus on the joint life policy. The fair value of this contract at a date where both spouses are still alive and the age of the couple is $z_1, z_1 \geq z_0$, is given by :

$$\begin{aligned} & C^{(1)}_{1|0}(a_0, r, z_1; Y_1, Y_2) \\ = \; & E_0[C^{(1)}_0(a_0, r, z_1; Y_1, Y_2)|Y_1 \geq z_1, Y_2 \geq z_1]. \end{aligned} \tag{4.9}$$

a_0 is for instance equal to the fair premium $a_0 = a_0^{*(1)}$ given in (4.3) when $z_1 = z_0$.

The price updating is more complicated for the reversionary annuities product, since we have to distinguish the two possible regimes existing during the life of the contract. In the first regime the two spouses are both alive, with an age of the couple equal to z_1. In the second regime, there is just one surviving spouse, the available information includes the date of the first death and the fact that the surviving spouse is the husband, or the wife. In both regimes, the residual value is systematically negative. First, in the second regime the only cash flows are the payment of the annuity, which are negative. Second, in the first regime, the premium rate of the reversionary annuity is increasing in z_0 (see Figure 1), therefore, couples who entered into the contract at age $z_0 < z_1$ pay, at age z_1, less premium than newly underwritten couples of age z_1, while the two groups have the same heterogeneity distribution, thus the same risk profile.

For illustration, let us calculate the residual value of a reversionary annuity underwritten at the age of 30. At date $t > 30$, the residual value of this contract depends on the survival status of the couple. We use the same model as in the previous section and Figure 4 displays the evolution of the residual value of the contract, first when both spouses are still alive at date t, then when one of the spouse died before t. The parameters are $\gamma = 5, k = 2, z_0 = 30$. As expected we observe that in both case, the value of the contrat is negative. We observe also in the second case, that the value of the contract is smaller for widows than for widowers. Indeed, at the same age and with the same marital status, women have a smaller mortality intensity than men have.

5 Concluding remarks

The standard insurance literature for analyzing and pricing insurance contracts written on two heads are pure models. A first category assumes a continuous bivariate distribution of the spouses' lifetimes with a continuous probability density function. This continuity assumption implies no jump in intensity when a spouse dies. A second category of models apply a pure Freund model to describe the broken-heart syndrome. These two effects impact the price of insurance contracts and of annuity values in different ways, not only the price of contracts written on two heads, but also the prices of individual contracts[9]. By considering appropriate extensions of the Freund model, we have explained how to account for both individual heterogeneity and potential jumps at the time of a spouse's death.

A similar problem arises in the credit risk literature where the death event is replaced by a default event. The standard credit risk literature prices the default intensity, not the default event itself, leading to possible mispricing of credit derivatives. The idea of introducing jumps in intensity to correct such a mispricing has been proposed in Jarrow, Yu (2001) for a credit derivative, written on two corporations[10] [see also the discussions in Benzoni et al. (2012) and Bay, Collin-Dufresne, Goldstein, Helwege (2013)]. Recently Gourieroux, Monfort, Renne (2013) derived the pricing formulas for credit derivatives written on a large pool of corporations and taking into account the jumps arising when corporations in the pool default.

Finally formulas providing the prices of insurance contracts written on two heads depend on parameters explaining how the exogenous variable impact the bivariate lifetime (risk-neutral) distribution. These variables include the individual characteristics of the couple, including the information on their generation. This generation information for each given age allows for taking into account the time dependence of the mortality rate. These parameters have to be calibrated, especially the parameters measuring the magnitude of the jumps (or of the association measures), the parameters capturing the memory effect and how they depend on generation (i.e. time). Data on individual contracts considered in isolation will not be sufficient to identify these parameters and there is clearly a need of data on couples and prices of contracts written on two heads.

[9]For the same reason they can impact the price of health insurance contracts or of long term care contracts, for instance since the risk of entering long term institutional care after the death of a spouse can increase [Nihtila, Martikainen (2008)].

[10]which is equivalent to an insurance product written on two heads.

R E F E R E N C E S

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Appendix 1

Joint density of lifetimes

Let us assume $y_1 < y_2$. We have :

$$\begin{aligned} f(y_1, y_2) &= \lim_{dy_1, dy_2 \to 0} \frac{1}{dy_1 dy_2} P[Y_1 \in (y_1, y_1 + dy_1), Y_2 \in (y_2, y_2 + dy_2)] \\ &= \lim_{dy_1, dy_2 \to 0} \frac{1}{dy_1 dy_2} P[X_1 < X_2, X_1 \in (y_1, y_1 + dy_1), X_1 + X_4 \in (y_2, y_2 + dy_2)] \\ &= \lim_{dy_1, dy_2 \to 0} \left[\frac{1}{dy_1} P[y_1 < X_2, X_1 \in (y_1, y_1 + dy_1)] \right. \\ &\quad \left. \frac{1}{dy_2} P[X_4 \in (y_2 - y_1, y_2 - y_1 + dy_2) | X_1 = \min(X_1, X_2) = y_1] \right] \\ &= \left[-\frac{\partial S_{12}}{\partial x_1}(y_1, y_1) \right] \left[-\frac{\partial S_4}{\partial x_4}(y_2 - y_1; y_1) \right]. \end{aligned}$$

Appendix 2

Link between the historical and risk-neutral distributions

For expository purpose we set the riskfree rate $r = 0$. Then we have to consider jointly the historical (or physical) distribution, with characteristics indexed by P, and the risk-neutral (or adjusted for risk) distribution, with characteristics indexed by Q. Since we are in an incomplete market frameworks, these two distributions can be specified independently. Let us now discuss the possible effects of the change of probability.

i) The stochastic discount factor (sdf) is the ratio between the risk-neutral and historical densities:

$$m(y_1, y_2, F) = \frac{f^Q(y_1, y_2, F)}{f^P(y_1, y_2, F)},$$

for a model with frailty for instance. A discontinuity of the risk-neutral density f^Q on the 45° line $y_1 = y_2$, that is, jumps in the risk-neutral intensities, can result from either jumps in the historical intensities, or jumps in the adjustment for risk (sdf) when a death occurs.

The standard insurance literature computing the prices from a specification of the historical distribution and the sdf has omitted the second possibility. This is typical of the practice of pricing by Esscher transforms [Esscher (1932), Gerber, Shiu (1994), Yao (2002)] written on factor F, that is choosing $m(y_1, y_2, F) = \exp(\alpha + \beta F)$, where α and β are such that $E^P[\exp(\alpha + \beta F)] = 1$ to get the zero riskfree rate.

Intuitively to reintroduce the effect of death event while using the practice of Esscher transforms, we may introduce the Esscher transforms on the distributions of the latent variables, that is,

$$\text{for the pair } (X_1, X_2) : \exp(\alpha_{12} + \beta_{12} F), \text{ say,}$$

$$\text{for the pair } X_3 : \exp(\alpha_3 + \beta_3 F), \text{ say,}$$

$$\text{for the pair } (X_4) : \exp(\alpha_4 + \beta_4 F), \text{ say.}$$

with parameters linked by the condition of zero riskfree rate.

Appendix 3

Probability distribution function of the heterogeneity given survival up to time t.

We derive the probability density function of the heterogeneity of the set of couples such that both spouses survive up to age $z_0 + x$. It is denoted g_x, We also denote by g_0 the heterogeneity distribution at age $z_0 = 30$, which equals $\gamma(k, 1/k)$, therefore:

$$g_0(f) \propto f^{k-1} \exp[-kf].$$

The unconditional survival probability that both survive up to age $z_0 + x$ is:

$$\begin{aligned} S(x) &= \mathbb{P}(Y_1 > z_0 + x, Y_2 > z_0 + x | Y_1 > z_0, Y_1 > z_0) \\ &= \int \exp[-[A_1(x) + A_2(x)] f] g_0(f) \mathrm{d}f, \end{aligned}$$

where A_1 and A_2 are cumulative intensities. Then the unconditional mortality intensity at age $z_0 + x$ is:

$$\begin{aligned}\lambda(x) &= -\frac{\mathrm{d}}{\mathrm{d}x}\log S(x)\\ &= \frac{\displaystyle\int [a_1(x)+a_2(x)]f\exp[-[A_1(x)+A_2(x)]f]g_0(f)\mathrm{d}f}{\displaystyle\int \exp[-[A_1(x)+A_2(x)]f]g_0(f)\mathrm{d}f}.\end{aligned}$$

Therefore, we deduce that the heterogeneity distribution function is:

$$\begin{aligned}g_x(f) &= \frac{g_0(f)\exp[-[A_1(x)+A_2(x)]f]}{\displaystyle\int g_0(f)\exp[-[A_1(x)+A_2(x)]f]\mathrm{d}f}\\ &\propto f^{k-1}\exp[-[k+A_1(x)+A_2(x)]f],\end{aligned}$$

which is a gamma distribution with shape parameter k and scale parameter $1/(k+A_1(x)+A_2(x))$.

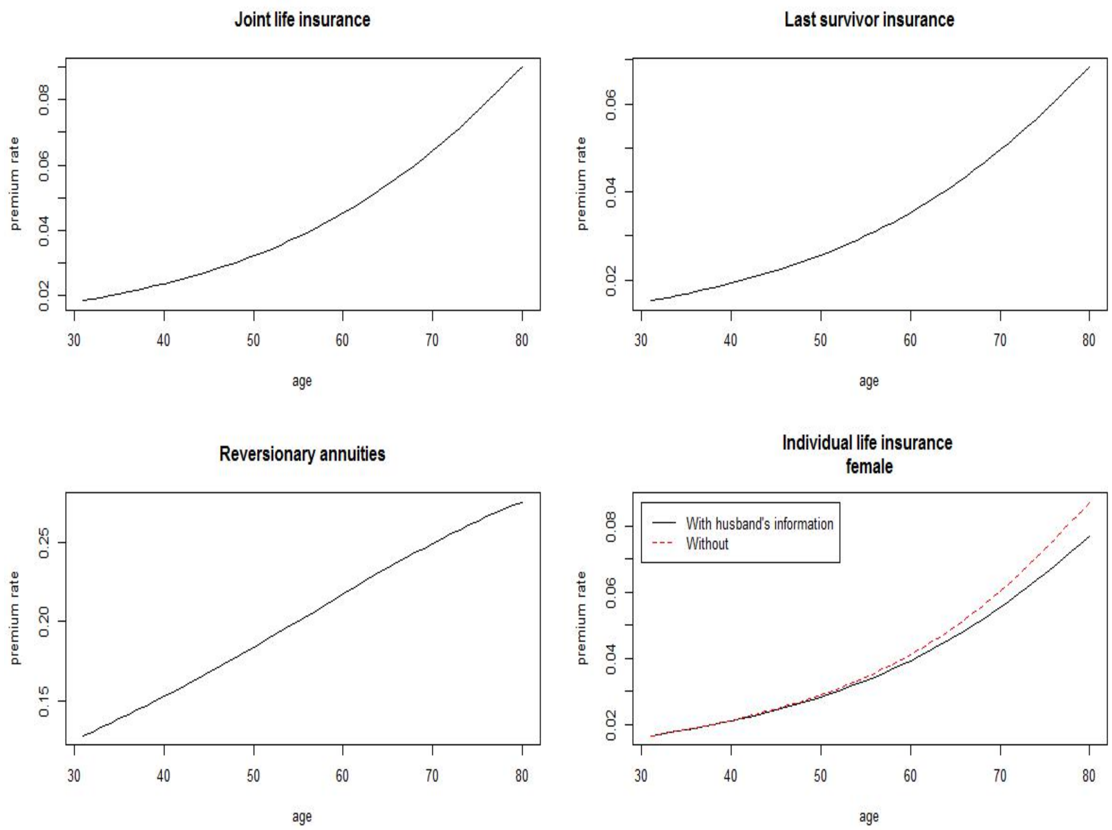

Figure 1: Premium rate as a function of the age of the couple at the time of underwriting. In the lower right panel for individual life insurance policies, the dashed line (respectively solid line) represents the premium rates when the information on the spouse is (respectively is not) taken into account.

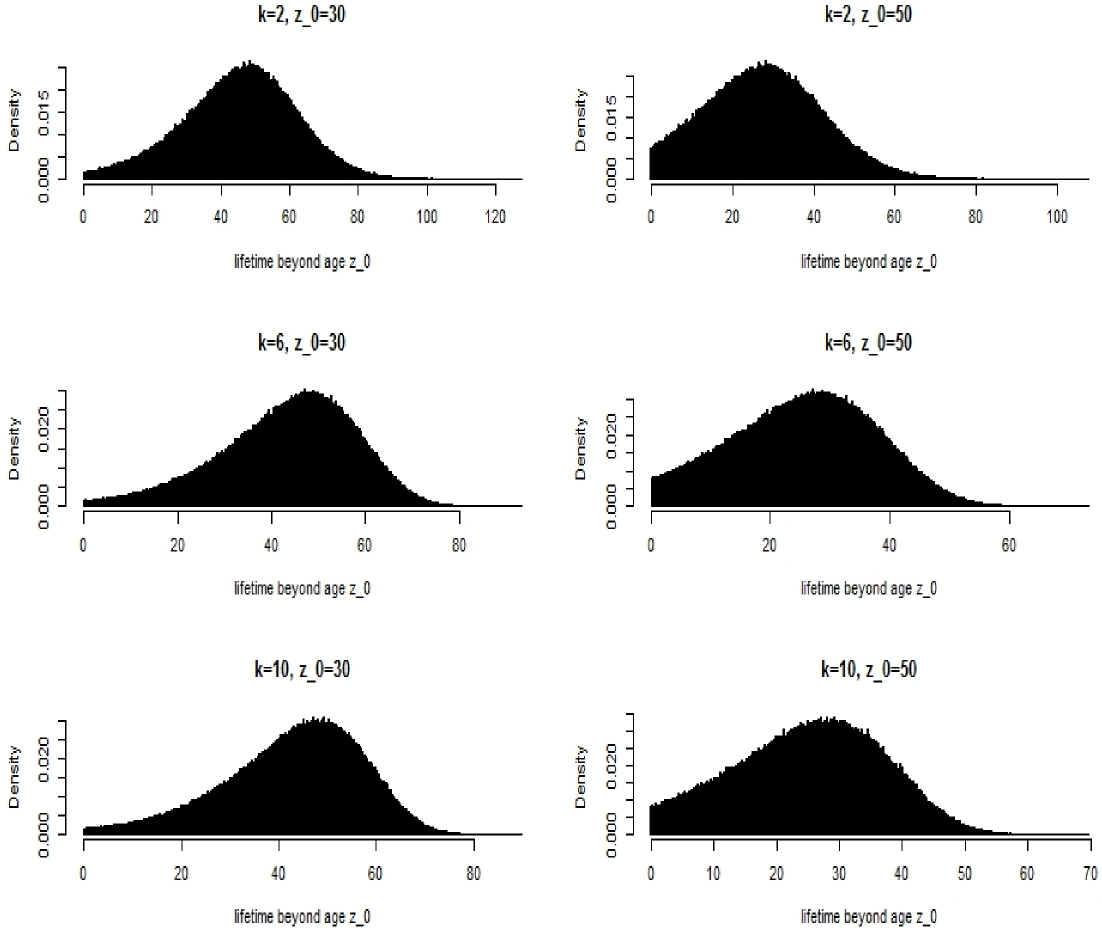

Figure 2: Probability density functions of the last survivor's lifetime upon z_0, for $z_0 = 30, 50$.

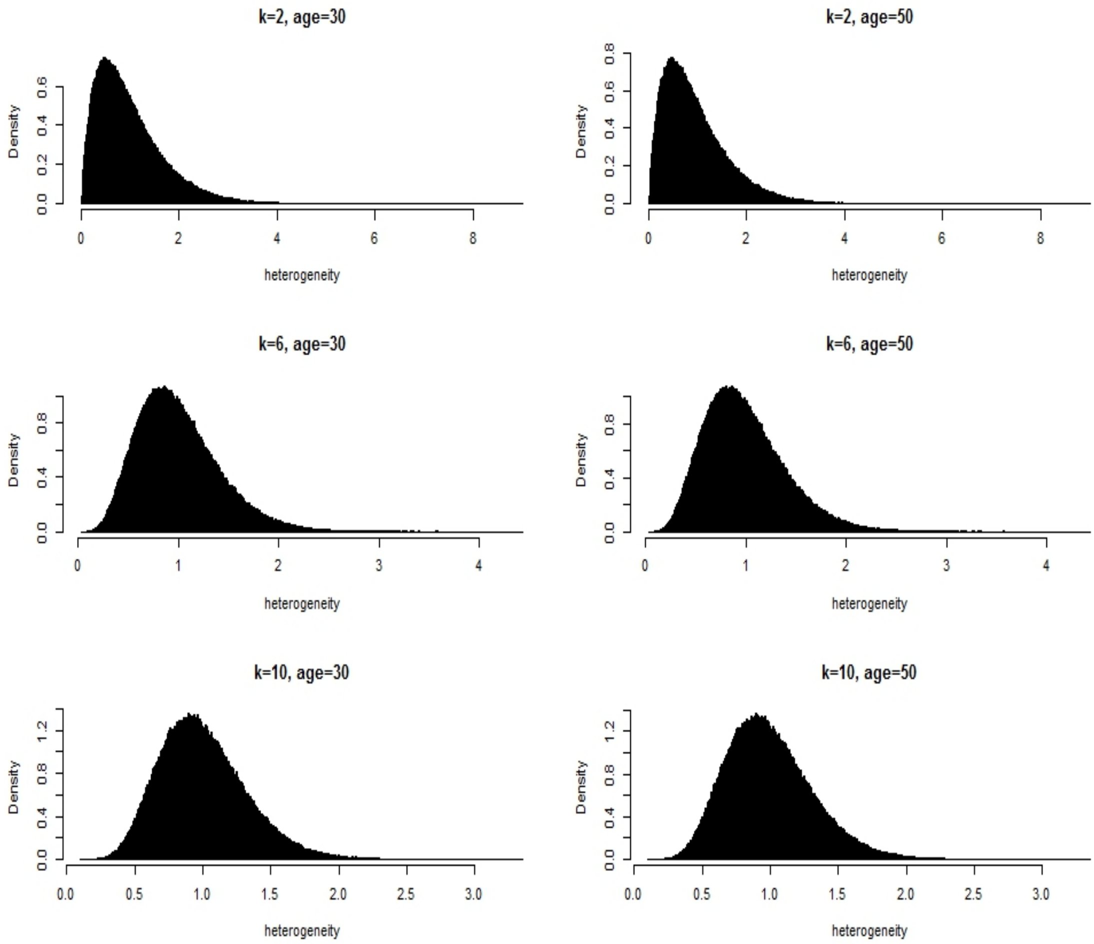

Figure 3: Probability density functions of the heterogeneity, at ages 30 and 50.

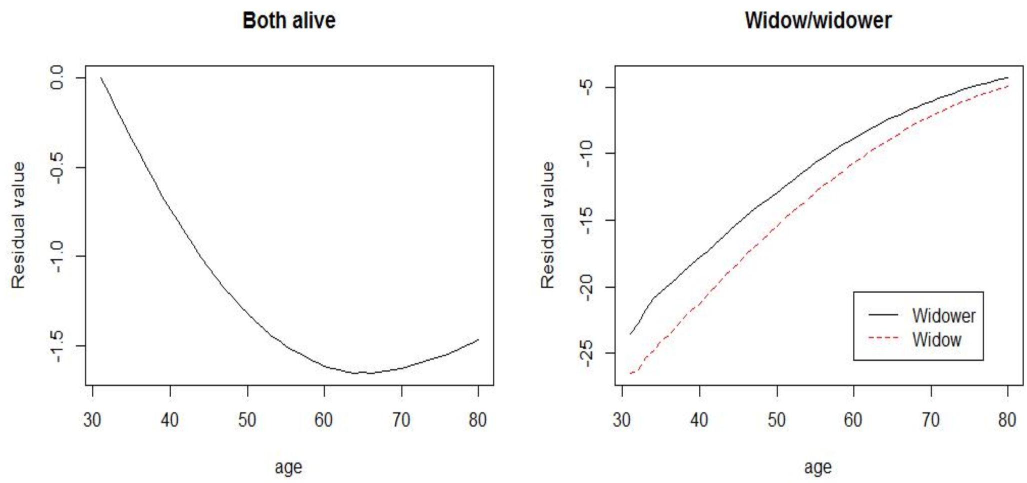

Figure 4: Evolution of the residual value of a reversionary annuity. Left panel: both spouses are still alive. Right panel: one of the spouses died before t.

	Last survivor	Reversion annuity	Individual, female, with husband's information	Individual, female, without husband's information
$\gamma = 5$	0.0194	0.134	0.0212	0.0210
$\gamma = 3$	0.0182	0.181	0.0203	0.0202
$\gamma = 1$	0.0153	0.318	0.0184	0.0183

Table 5.1: Effect of the broken heart syndrome on premium rates with a fixed heterogeneity distribution ($k = 6$), at age 30.

	Last survivor	Reversion annuity	Individual, female, with husband's information	Individual, female, without husband's information
$\gamma = 5$	0.0279	0.166	0.0319	0.0303
$\gamma = 3$	0.0260	0.225	0.0309	0.0290
$\gamma = 1$	0.0214	0.404	0.0275	0.0258

Table 5.2: Effect of the broken heart syndrome on premium rates with a fixed heterogeneity distribution ($k = 6$), at age 50.

	Joint life	Last survivor	Reversion annuity	Individual, female, with husband's information	Individual, female, without husband's information
$k = 2$	0.0186	0.0153	0.129	0.0167	0.0167
$k = 6$	0.0196	0.0161	0.135	0.0176	0.0176
$k = 10$	0.0197	0.0162	0.136	0.0177	0.0177

Table 5.3: Effect of heterogeneity on premium rates with a fixed broken heart syndrome ($\gamma = 5$), at age 30.

	Joint life	Last survivor	Reversion annuity	Individual, female, with husband's information	Individual, female, without husband's information
$k = 2$	0.0334	0.0265	0.188	0.0299	0.0293
$k = 6$	0.0364	0.0287	0.199	0.0324	0.0318
$k = 10$	0.0371	0.0292	0.203	0.0329	0.0323

Table 5.4: Effect of heterogeneity on premium rates with a fixed broken heart syndrome ($\gamma = 5$), at age 50.

	Shape parameter	Scale parameter	$\sqrt{\dfrac{\text{Variance at age 50}}{\text{Variance at age 30}}}$
$k = 2$	0.4816	2	0.9279
$k = 6$	0.1646	6	0.9750
$k = 10$	0.0992	10	0.9849

Table 5.5: Gamma distribution parameters at age 50 for different gamma distributions $\gamma(k, 1/k)$ at age 30. The scale parameter is the same as at age 30. The fourth column gives values of $k/[k + A_1(x) + A_2(x)]$, which equals also the mean of the heterogeneity distribution. It measures the reduction of the heterogeneity due to the mover-stayer phenomenon.