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# Liquidation Equilibrium with Seniority and Hidden CDO

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Liquidation Equilibrium with Seniority and Hidden CDO

Abstract

The aim of our paper is to price credit derivatives written on a single name when this name is a bank. Indeed, due to the special structure of the balance sheet of a bank and to the interconnections with other institutions of the financial system, the standard pricing formulas do not apply and their use can imply severe mispricing. The pricing of credit derivatives written on a single bank name requires a joint analysis of the risks of all banks directly or indirectly interconnected with the bank of interest. Each name cannot be priced in isolation, but the banking system must be treated as a whole. It is necessary to analyze the contagion of losses among banks, especially the equilibrium of joint defaults and recovery rates at liquidation time. We show the existence and uniqueness of such an equilibrium. Then the standard pricing formulas are modified by adding a premium to capture the contagion effects.

**Keywords :** Collateralized Debt Obligation, Contagion, Solvency Risk, Value-of-the Firm Model, Liquidation Equilibrium, Contagion premium, Systemic Risk, Stress-Test.

# 1 Introduction

The aim of our paper is to price credit derivatives written on a single name when this name is a bank. Indeed, due to the special structure of the balance sheet of a bank and to the interconnections with other institutions of the financial system, the standard pricing formulas do not apply and their use can imply severe mispricing.

Two alternative approaches are usually followed to price credit derivatives. In the structural approach, introduced by Merton (1974), and used for instance in Basel 2 (Vasicek (1991)), the default of a firm is assumed to occur when the asset component of the balance sheet becomes smaller than its liability component. Then the probability and the price of default are deduced from the historical and risk-neutral properties of these two underlying variables.

An alternative is the reduced form or intensity approach, in which the balance sheet is not taken into account and the historical (or risk-neutral) default

intensities are directly analyzed [see e.g. Duffie, Garleanu (2001), or Duffie, Singleton (1999)]. We focus on the standard pricing formulas based on the structural approach in a two period setup.
Let us consider a digital Credit Default Swap (CDS) written on a single name $i$. Its payoff is equal to 1, if the asset is below the liability, i.e. if $log(A_i) < log(L_i)$, and equal to 0, otherwise. Up to the discounting, its price is deduced from the risk-neutral probability of this default event. It is usually computed by assuming a deterministic level of liability and a Gaussian random log-asset, with a distribution depending on individual characteristics of the name, such as its rating, the expected return and the volatility of its stock. This single factor model, where the factor is the log-asset $log(A_i)$, is not appropriate when the name is a bank. Indeed, the asset component of the bank's balance sheet includes debts of the other banks, and therefore is also sensitive to the risk situations of these banks. In other words, a CDS written on a bank is a CDS written on a portfolio of debts and has to be considered as a more complex multiple name product, e.g. a Collateralized Debt Obligation (CDO). Let us now recall how such a CDO is usually priced. First the composition of the portfolio of interest is described, that is, the log-asset of bank $i$ as a function of the underlying risks is defined:

$$log(A_i) = g\left(log(A_j), log(L_j), j \in J\right),$$

where $J$ is the set of institutions appearing in the balance sheet of bank $i$. Then, the joint distribution of the underlying factors $log(A_j), j \in J$, is specified taking into account their possible dependence by means of a copula (see e.g. Schönbucher, Schubert (2001) or Burtschell, Gregory, Laurent (2009)) not necessarily Gaussian (see the discussion on Li's Copula [Li, (2000)] by Wired (2009) or MacKenzie, Spears (2012)) or by means of unobservable common factor (see Sections 3.1-3.2).
However, this pricing approach of a CDO is also inappropriate in case of banks. Indeed, this approach implicitly assumes an exogenous dependence between the underlying risk factors $log(A_j), j \in J$ (and $log(A_i)$). This does not take into account the fact that the asset component of the balance sheet of another bank $j$ can also depend on $log(A_i)$ for instance. The pricing of credit derivatives written on a single bank name requires a joint analysis of the risks of all banks directly or indirectly interconnected with the bank of interest. We cannot price each name considered in isolation, but we have to treat the banking system as a whole. We have to analyze the contagion of losses among banks, especially the equilibrium of joint defaults and recovery

rates at liquidation date.

In Section 2, we describe the balance sheets of the banks, including stocks and debts with two levels of seniority, and define the bilateral exposure matrices, which characterize the interconnections. We consider the possible defaults of institutions in the system. Due to the interconnections, these defaults have to be considered jointly and this leads to the so-called liquidation equilibrium. The liquidation equilibrium provides the state (either defaulted, or non defaulted) of each institution together with its equilibrium firm value and its junior and senior equilibrium debt values. We prove the existence and uniqueness of an equilibrium in the general case. As an illustration, the special case of two institutions is described in detail in Appendix 1. In Section 3, we focus on the pricing of the junior and senior tranches of the debts of the financial institutions. We adopt a progressive approach of this question. We first consider a financing project in the framework of the Value-of-the-Firm model, then a system of unconnected financial institutions exposed to a common exogenous risk factor and finally we analyze a system of banks, which are interconnected by means of their debt crossholdings. We note that junior and senior tranches written on a given institution, that is, on a single name, are in fact tranches written on several names due to the debt holdings. This explains why the pricing of such a tranche is equivalent to the pricing of a (hidden) Collateralized Debt Obligation (CDO). Moreover, the prices of these hidden CDO's have to take into account the liquidation equilibrium. We show how these CDO's prices under equilibrium can be decomposed in order to highlight the components of these prices due to the presence of interconnections, called contagion premium. In Section 4, we produce numerical illustrations. We provide the dynamics of the sequence of liquidation equilibria when the exogenous asset components are driven by a single common factor and we examine how the prices of hidden CDO's, and their contagion components, depend on the design of the exposure matrices. Section 5 concludes. Proofs are gathered in Appendices.

# 2 Liquidation Equilibrium

## 2.1 Two examples

**i) The equilibrium for a single bank**

Let us first present the standard value-of-the-firm model developed in Merton (1974) and Vacisek (1991), here extended to a unique firm that issued both junior and senior debts. As usual, we distinguish the contractual (or face) value of a debt fixed at the initial date $t = 0$ from its actual value at date $t = 1$ of reimbursement. Indeed, at the date of reimbursement the actual value of the debt can be strictly smaller than the face value, when there is default. Three types of stakeholders are involved: the shareholders whose interest is the net value of the assets over the total value of the debt, that is the so-called value-of-the-firm; the senior debtors, who hold the senior debt and are entitled a right over the assets up to the contractual value of senior debt; finally, the junior debtors have a right on the net value of asset over the contractual senior debt up to the contractual junior debt.

More precisely, let us denote by $L^S$ and $L^J$ (respectively $L^{*S}$ and $L^{*J}$) the actual (respectively, contractual) values of senior and junior debts, $Y$ the value of the firm and $A$ the asset value. Following Merton's analysis, the lines of the balance sheet are constrained by the following equations:

$$L^S = min(A; L^{*S}), \tag{2.1}$$

$$L^J = min(A - L^S; L^{*J}), \tag{2.2}$$

$$Y = \Big(A - (L^J + L^S)\Big)^+. \tag{2.3}$$

This system of equations defines a continuous piecewise linear system in the values of debts and the value of the firm. To invert this system, we have to express the value of the firm $Y$, and the values of the junior and senior debts $L^J$ and $L^S$, as functions of asset value $A$. The invertibility of this system can be analyzed recursively: equation (2.1) provides the value of the senior debt $L^S$; this value can be plugged in equation (2.2) to get the value of the junior debt $L^J$ and finally equation (2.3) gives the value of the firm $Y$ using the two debt values already computed. Therefore this piecewise linear system (2.1) − (2.3) is invertible. The explicit solution is:

$$L^S = \begin{cases} A, & \text{if } A \leq L^{*S}, \\ L^{*S}, & \text{if } L^{*S} < A, \end{cases} \tag{2.4}$$

$$L^J = \begin{cases} 0, & \text{if } A \le L^{*S}, \\ A - L^{*S}, & \text{if } L^{*S} < A \le L^{*S} + L^{*J}, \\ L^{*J}, & \text{if } L^{*S} + L^{*J} < A, \end{cases} \tag{2.5}$$

$$Y = \begin{cases} 0, & \text{if } A \le L^{*S} + L^{*J}, \\ A - L^{*S} - L^{*J}, & \text{if } L^{*S} + L^{*J} < A. \end{cases} \tag{2.6}$$

We get piecewise linear functions of $A$, solutions of the piecewise linear system (2.1) − (2.3). The values $L^S$, $L^J$ and $Y$ are expressed as functions of asset value $A$ (see Figure 1 given for $L^{*J} > L^{*S}$). One can identify three situations depending on the relative position of asset value $A$ with respect of the contractual senior debt value $L^{*S}$ and the total contractual debt value $L^{*S} + L^{*J}$:

- situation 0: the institution does not default, that is, $L^S = L^{*S}$, $L^J = L^{*J}$ and $Y > 0$; since the institution is not in default, debtors get full repayment of the debt and shareholders get a positive value; the situation is called "alive".

- situation 1: the institution defaults only on its junior debt, that is $L^S = L^{*S}$, $L^J < L^{*J}$ and $Y = 0$. Since the institution is in default, the shareholders get zero and the total debt is not fully repaid; but, the default is not very severe since senior debtors are fully repaid, whereas the junior debtors are not. We call this situation "partial default".

- situation 2: the institution defaults on its senior debt, that is, $L^S < L^{*S}$, $L^J = 0$ and $Y = 0$; being in default implies that the value of the firm is zero; in this regime, the situation is serious enough to have erased the junior debt: only the senior debtors have not lost everything; we call this situation "complete default".

**ii) The equilibrium for two unconnected banks**

In the case of two unconnected institutions, we can write system (2.1) − (2.3) for each institution. The balance sheets are constrained by:

$$\begin{cases} L_1^S &= min(A_1; L_1^{*S}), \\ L_1^J &= min(A_1 - L^S; L_1^{*J}), \\ Y_1 &= \Big(A_1 - (L_1^J + L_1^S)\Big)^+, \\ L_2^S &= min(A_2; L_2^{*S}), \\ L_2^J &= min(A_2 - L_2^S; L_2^{*J}), \\ Y_2 &= \Big(A_2 - (L_2^J + L_2^S)\Big)^+. \end{cases} \tag{2.7}$$

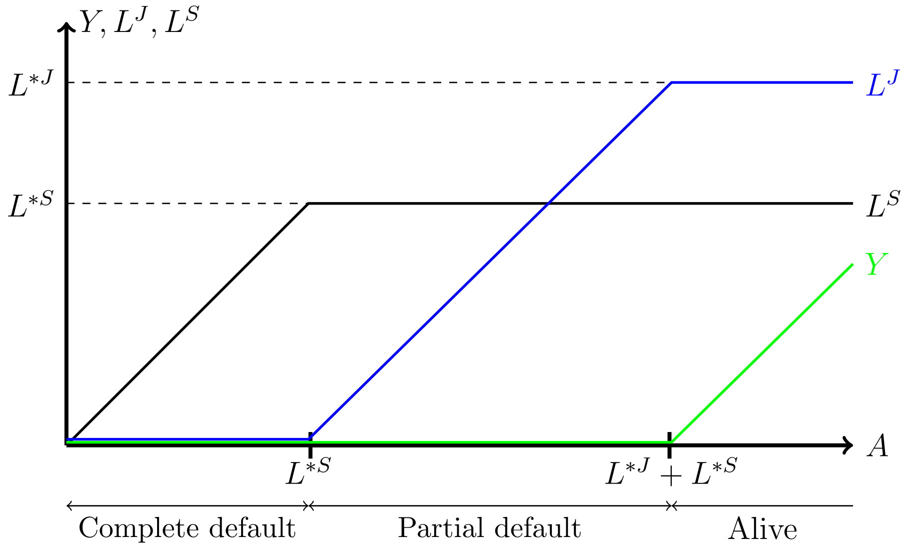



Figure 1: Merton's Model With Two Seniority Levels

The invertibility of this piecewise linear system derives directly from the single institution analysis. Due to the absence of interconnections, the thresholds characterizing the situations of each bank are $L_i^{*S}$ and $L_i^{*S} + L_i^{*J}$ for institution $i = 1, 2$. They are independent of the situation of the other bank. Since there are three situations for each institution defined by the relative position of their specific asset value with respect to their specific thresholds, there are $3^2 = 9$ regimes for the financial system (see Table 1). These regimes are illustrated in the space of asset values $A_1$ and $A_2$ in Figure 2.

| | | Bank 1 | | |
|---|---|---|---|---|
| | | Alive | Partial Default | Complete Default |
| Bank 2 | Alive | Regime $\mathcal{C}_1$ | Regime $\mathcal{C}_4$ | Regime $\mathcal{C}_7$ |
| | Partial Default | Regime $\mathcal{C}_2$ | Regime $\mathcal{C}_5$ | Regime $\mathcal{C}_8$ |
| | Complete Default | Regime $\mathcal{C}_3$ | Regime $\mathcal{C}_6$ | Regime $\mathcal{C}_9$ |

Table 1: The Nine Regimes in Case of Two Institutions

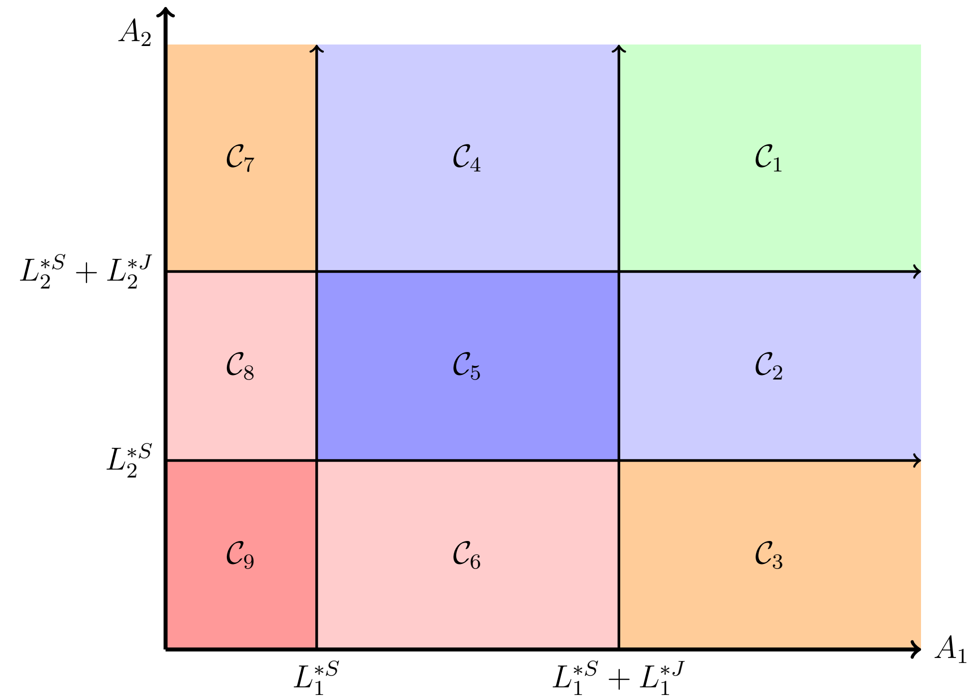



Figure 2: The Regimes for Two Unconnected Institutions

## 2.2 The balance sheets

Let us now extend the analysis above to any set of interconnected institutions. The structure of the balance sheet of bank $i$ is given in Table 2, the upper index either $S$, or $J$, denoting the seniority of the debt, that is, senior and junior, respectively. We denote by $Y_i$, $L_i^S$, $L_i^J$, $i = 1, ..., n$, the value of bank $i$ and its amount of senior and junior debt, respectively. The proportion of shares (resp. senior, junior debt) of bank $j$ held by bank $i$ is denoted by $\pi_{i,j}$ (resp. $\gamma_{i,j}^S$, $\gamma_{i,j}^J$). These proportions are expressed in number of shares (resp. volume), not in value. The exposures $\pi_{i,j}$, $\gamma_{i,j}^S$, $\gamma_{i,j}^J$, $i = 1, ..., n$, are gathered in exposure matrices $\Pi$, $\Gamma^S$, $\Gamma^J$, respectively (see Gourieroux, Heam, Monfort (2012) for examples of exposures matrices for the French banking system). Thus, we implicitly assume that the value of the debt of bank $j$ is allocated in proportion to its initial exposure in case of default. These exposures are nonnegative and the sums of exposures on the financial institutions may be smaller than 1. Thus fractions of the stocks, junior and senior debts can be hold outside the financial system. For instance, a large fraction of the senior

| | | Asset | Liability | | |
|---|---|---|---|---|---|
| interbank cross shareholdings | $\leftrightarrow$ | $\pi_{i,1}Y_1$, $\vdots$, $\pi_{i,n}Y_n$ | $L_i^S$ | $\leftrightarrow$ | senior debt |
| interbank senior lendings | $\leftrightarrow$ | $\gamma_{i,1}^S L_1^S$, $\vdots$, $\gamma_{i,n}^S L_n^S$ | $L_i^J$ | $\leftrightarrow$ | junior debt |
| interbank junior lendings | $\leftrightarrow$ | $\gamma_{i,1}^J L_1^J$, $\vdots$, $\gamma_{i,n}^J L_n^J$ | $Y_i$ | $\leftrightarrow$ | value of the firm |
| external assets | $\leftrightarrow$ | $Ax_i$ | | | |

Table 2: Balance Sheet of Bank $i$

debt can correspond to the deposits on the bank accounts. We consider a unique maturity of the debt, which means in particular that we focus on solvency risk and are not concerned with liquidity features, including market and funding liquidity risks.

The exogenous variables are the contractual debt levels $L_i^{*S}, L_i^{*J}, i = 1, \dots, n$, for both senior and junior debts, the matrices of bilateral exposures $\Pi, \Gamma^S, \Gamma^J$ (with nonnegative elements) and the external asset components $Ax_i, i = 1, \dots, n$. They define the state of the system $\mathcal{S} = \{L^{*S}, L^{*J}, \Pi, \Gamma^S, \Gamma^J, Ax\}$.

## 2.3 The equilibrium conditions

Let us now focus on solvency risk by considering that all assets are perfectly liquid and that the value of a given asset is the same when the bank $i$ is alive, or is under liquidation. We follow the standard Merton's model [see Merton (1974), Vasicek (1991)].

The equilibrium conditions have to account for the recovery of the senior debt before the recovery of the junior debt in case of default. For each bank, the values of the lines of the balance sheet are constrained by the three following equations:

$$\begin{aligned} L_i^S &= min(A_i; L_i^{*S}), \qquad i = 1, \dots, n, & (2.8)\\ L_i^J &= min(A_i - L_i^S; L_i^{*J}), \qquad i = 1, \dots, n, & (2.9)\\ Y_i &= (A_i - L_i)^+, \qquad i = 1, \dots, n, & (2.10) \end{aligned}$$

with $L_i = L_i^S + L_i^J$ and $A_i$ is the total asset of bank $i$ for $i = 1, ..., n$.
The third equation shows the value of the firm as the payment of a call written on asset $A_i$ with strike liability $L_i$. The other equations provide the endogenous recovery rates for the senior and junior debts, equal to $min(1, A_i/L_i^{*S})$ and $min(1, (A_i - L_i^S)/L_i^{*J})$, respectively.
For a given bank $i$ and a given $A_i$, these equations could be applied in a sequential order. The senior debt is paid first, if possible [equation (2.8)]; then, the junior debt is considered based on the remaining amount after payment of the senior debt [equation (2.9)]. Finally, the value of the firm is computed [equation (2.10)]. However, this recursive approach cannot be used to solve system (2.8) − (2.10), since the total asset $A_i$ depends on the values $Y_j$, $L_j^S$ and $L_j^J$ of the other banks.
For each bank, we have three possible regimes:

- regime 0: no default of the bank, that is, $Y_i > 0, L_i^S = L_i^{*S}, L_i^J = L_i^{*J}$.
- regime 1: partial default of the bank : $Y_i = 0, L_i^S = L_i^{*S}, L_i^J < L_i^{*J}$.
- regime 2: complete default of the bank : $Y_i = 0, L_i^S < L_i^{*S}, L_i^J = 0$.

There is a partial default if the bank defaults on its junior debt, but does not default on the senior debt. There is a complete default, when the default occurs for both types of debts (with different recovery rates). Thus, there exist $3^n$ possible joint regimes for the banking system.
By introducing the expressions of the total assets $A_i$, $i = 1, ..., n$ in system (2.8) − (2.10), we get the equilibrium conditions:

$$L_i^S = min\Big(\sum_{j=1}^{n} \pi_{i,j} Y_j + \sum_{j=1}^{n} \gamma_{i,j}^J L_j^J + \sum_{j=1}^{n} \gamma_{i,j}^S L_j^S + Ax_i; L_i^{*S}\Big), \qquad i = 1, \dots, n, \tag{2.11}$$

$$L_i^J = min\Big(\sum_{j=1}^{n} \pi_{i,j} Y_j + \sum_{j=1}^{n} \gamma_{i,j}^J L_j^J + \sum_{j=1}^{n} \gamma_{i,j}^S L_j^S + Ax_i - L_i^S; L_i^{*J}\Big), \qquad i = 1, \dots, n. \tag{2.12}$$

$$Y_i = \Big( \sum_{j=1}^{n} \pi_{i,j} Y_j + \sum_{j=1}^{n} \gamma_{i,j}^J L_j^J + \sum_{j=1}^{n} \gamma_{i,j}^S L_j^S + Ax_i - L_i^J - L_i^S \Big)^+, \quad i = 1, \dots, n, \tag{2.13}$$

The framework above extends the existing structural literature by increasing the number of asset categories through which the financial institutions are interconnected. The major part of the literature, following Furfine (1999), Eisenberg, Noe (2001) and Upper, Worms (2004)[4] consider connections through debts only with a single seniority level and debts totally held within the system. Recently, Gourieroux, Heam, Monfort (2012) considered the case of stocks and debts, but with a single seniority level.

## 2.4 Equilibrium for two interconnected banks

The existence and uniqueness of the equilibrium is equivalent to the invertibility of the piecewise linear system (2.11) $-$ (2.13). We first consider the case of two interconnected banks in order to compare with the case of two unconnected banks discussed in subsection 2.1.*ii*). The analysis is detailed in Appendix 1. We derive the conditions for invertibility (see Proposition $A.1$) and characterize the regimes in the space of the external asset components $Ax_1$, $Ax_2$. The regimes are provided in Figure 3. Compared with Figure 2, we note that the regimes are still defined by means of linear affine boundaries. However, these affine boundaries are no longer parallel to either the $x-$axis, or the $y-$axis. Their slopes depend on the exposure matrices $\Pi$, $\Gamma^S$ and $\Gamma^J$ (see Appendix 1).

Let us discuss some situations on Figure 3. Point $\mathcal{A}$ represents the following situation. Bank 1 suffered a loss for its non-banking assets, and on its own it should (partially) default, since $Ax_1 < L_1^{*S} + L_1^{*J}$. But, at the same time, the non-banking assets of bank 2 is such that $Ax_2 > L_2^{*S} + L_2^{*J}$, so that the shareholders of bank 2 get a surplus. Since bank 1 is a shareholder of bank 2, this large surplus does more than compensate its own loss. Consequently, bank 2 is alive since it has made a surplus and despite its bad result bank 1 is alive too due to its participation in bank 2. In some sense, the cross-participation enables the financial system to share risk.

Let us now consider point $\mathcal{B}$. The non-banking assets of bank 1 have performed (since $Ax_1 > L_1^{*S} + L_1^{*J}$), while bank 2 is in complete default due to an

[4] See also Sheldon, Maurer (1998), Blavak, Nimander (2002), Wells (2002), Degryse, Nguyen (2004), Lubloy (2005).

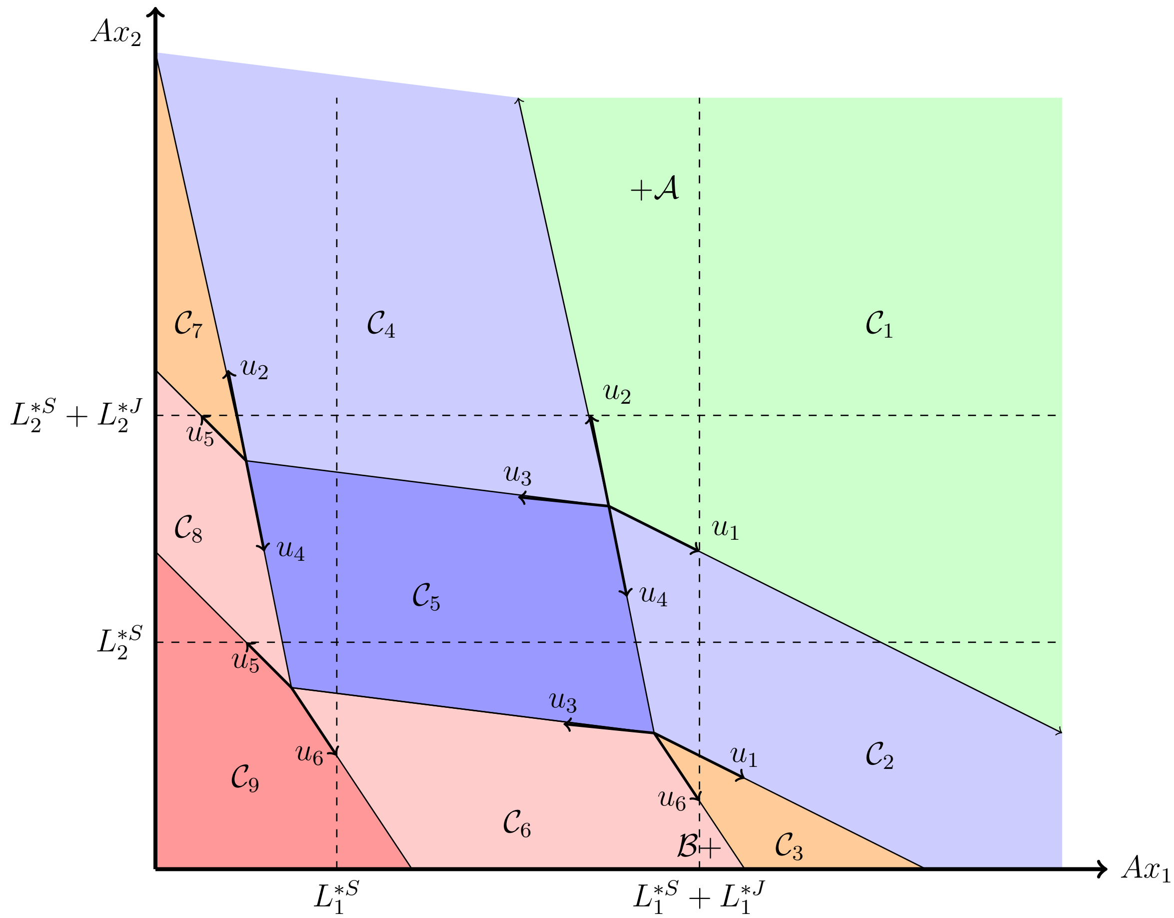



Figure 3: The Regimes for Two Connected Institutions

important loss of its non-banking asset. The situation of (complete) default of bank 2 hits bank 1 in three ways, that is a loss on stock, a loss on junior debt and a loss on senior debt. The interbank asset of bank 1 is sufficiently reduced to lead bank 1 to (partial) default. In this case, cross-participation and cross-lending enhance risk.

This model is consistent with the previous literature on interconnectedness: a high level of interconnectedness helps protecting against the effect of small systematic shocks, but can jeopardize the whole system for extreme shocks. This is also compatible with the theoretical literature saying that a more symmetric structure of interconnections can make the system less vulnerable [see e.g. Allen, Gale (2000)].

### 2.5 General case

The previous analysis, namely the detailed case of two interconnected banks, is extended to a general framework and sufficient conditions for existence and uniqueness of the liquidation equilibrium are given below.

**Proposition 1 :** For two seniority levels, the liquidation equilibrium exists and is unique if :

$$\sum_{i=1}^{n} \pi_{i,j} < 1, \sum_{i=1}^{n} \gamma_{i,j}^{J} < 1, \sum_{i=1}^{n} \gamma_{i,j}^{S} < 1, \quad j = 1, \ldots, n.$$

**Proof :** See Appendix 2.

This result completes the result in Gourieroux, Heam, Monfort (2012), where there exist only stocks and one type of debt, but requires a specific proof to treat the seniority feature. Elsinger [(2009) Theorem 8] has derived the existence and uniqueness of the liquidation equilibrium under a set of similar conditions on the exposure matrices by a different approach. The similar results obtained in Eisenberg, Noe (2001), or in Demange (2012) are special cases of Proposition 1. First, a common feature of these two papers is the absence of interconnections through stocks, that is : $\Pi = 0$. Second, their model have a single seniority level. In this respect let us discuss more carefully Demange (2012): in her paper, seniority is present under the term of "absolute priority", that is, "creditors outside the banking system have priority over those inside". But this seniority aspect matches perfectly the difference between external creditors and internal creditors. In our framework, this means that interbank loans are exclusively junior ($\Gamma^S = 0$), but there exists a senior debt ($L^S > 0$). Moreover, the "junior" debts are only composed of loans from the banks in the network : in our setting, this would be written as : $\sum_{i} \gamma_{i,j}^{J} = 1$ for $j = 1, ..., n$. In a different perspective, Elliott, Golub and Jackson (2012) propose to consider only cross-holdings and study integration and diversification effects across institutions.

# 3 Pricing bank debts in a CDO perspective

The liability side of the balance sheet of a financial institution can be analyzed in terms of Collateralized Debt Obligation (CDO). The junior and senior debts of each bank are tranches written on its total debt. Moreover due to the interconnections between banks, these tranches are implicitly written on a portfolio of debts of the other connected banks. We first review the standard pricing formulas of these tranches for unconnected banks and explain how a common risk factor affecting the external asset components can be introduced to capture the exogenous component of default dependence. However, the defaults of the banks are also related by means of the interconnections existing in the balance sheets. When we consider the more general framework of connected banks, the derivation of the prices of the tranches written on a single name requires to solve the liquidation equilibrium discussed in Section 2. Then we can disentangle in the default dependence and its effect on prices the component due to the exogenous shocks and the component due to the interconnections.
For expository purpose and CDO interpretation, we consider interconnections through debts only. Indeed, the standard CDO are defined on portfolio of debts, not on portfolio of debts and stocks. However, it is easily seen that the general pricing formula is still valid if stocks are also included. The pricing formulas are also derived under the assumption of a deterministic short term riskfree rate[5].

## 3.1 Pricing tranches in the extended Merton's model

### 3.1.1 CDO interpretation

Let us consider a financing project funded by a nominal debt $L^*$. The debt is divided into a junior debt and a senior debt of nominal values $L^{*J}$ and $L^{*S}$, respectively, which have to be reimbursed at the predetermined date $t = 1$. We have $L^* = L^{*J} + L^{*S}$. The project is represented by an asset of value $Ax$. The corresponding balance sheet at date $t = 1$ is given in Table 3 where the liability components depend on the final state of the project.

The values $L^S$ and $L^J$ of the debts at $t = 1$ are derived from Merton's

[5]Or at least the assumption of a short term rate independent of the other variables under the risk-neutral dynamics.

| Asset | Liability |
|---|---|
| $Ax$ | $L^J$ |
| | $L^S$ |
| | $Y$ |

Table 3: Balance Sheet of a Financing Project with Two Seniority Levels

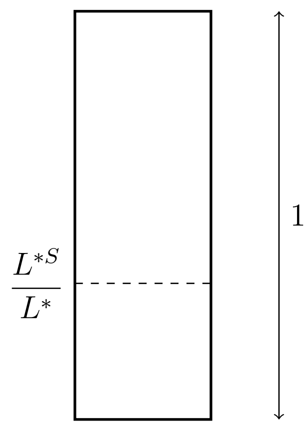



Table 4: Normalized Debt Side of a Financing Project with Two Seniority Levels

model :

$$
\begin{aligned}
L^S &= min(L^{*S}; Ax), && (3.1)\\
L^J &= min(L^{*J}; Ax - L^S). && (3.2)
\end{aligned}
$$

In a CDO pricing perspective, let us normalize the debt side as in Table 4. The ratio $L^{*S}/L^*$ defines the detachment point. From this CDO perspective, we can define the payoff of the zero-coupon of junior debt (respectively, senior debt), denoted $ZC^J$ (respectively, $ZC^S$), in the following way:

$$
ZC^S = \begin{cases} 1, & \text{if } L^{*S} < Ax, \\ \dfrac{Ax}{L^{*S}}, & \text{otherwise.} \end{cases} \tag{3.3}
$$

$$
ZC^J = \begin{cases} 1, & \text{if } L^{*J} + L^{*S} < Ax, \\ \dfrac{Ax - L^{*S}}{L^{*J}}, & \text{if } L^{*S} < Ax < L^{*J} + L^{*S}, \\ 0, & \text{if } Ax < L^{*S}. \end{cases} \tag{3.4}
$$

Thus, the associated normalized losses, denoted by $\mathcal{L}^J$ and $\mathcal{L}^S$, are :

$$\mathcal{L}^S = 1 - ZC^S = \left(1 - \frac{Ax}{L^{*S}}\right)^+,$$

$$\mathcal{L}^J = 1 - ZC^J = 1 - \left(\frac{Ax}{L^{*J}} - \frac{L^{*S}}{L^{*J}}\right)^+ + \left(\frac{Ax}{L^{*J}} - \frac{L^{*S} + L^{*J}}{L^{*J}}\right)^+.$$

### 3.1.2 Decomposition of the prices of the tranches in PD and ELGD

The prices of the tranches are deduced from the expected value of the loss under the risk-neutral probability $\mathbb{Q}$. The riskfree short term rate is assumed deterministic and the riskfree zero-coupon bond is denoted by $B_{rf}$.

**i) Price of the senior tranche**
The difference between the price $B_{rf}$ of the riskfree bond and the price $B^S$ of the senior zero-coupon is :

$$B_{rf} - B^S \quad = \quad B_{rf} \times PD^S \times ELGD^S, \tag{3.5}$$

where $PD^S = \mathbb{Q}(Ax < L^{*S})$ is the risk-neutral probability of default on the senior debt and $ELGD^S = \mathbb{E}^{\mathbb{Q}}\Big(1 - \frac{Ax}{L^{*S}}\Big|Ax < L^{*S}\Big)$ is the expected loss given default on the senior debt.

**ii) Price of the junior tranche**
Let us now consider the price, $B^J$, of the junior zero-coupon. We have :

$$B_{rf} - B^J \quad = \quad B_{rf} \times \Big(PD^S + PD^{J,S} ELGD^{J,S}\Big), \tag{3.6}$$

where $PD^{J,S} = \mathbb{Q}(L^{*S} < Ax < L^{*S} + L^{*J})$ is the risk-neutral probability of the junior tranche with no default on the senior tranche, and $ELGD^{J,S} = \mathbb{E}^{\mathbb{Q}}\Big(\frac{Ax - L^{*S}}{L^{*J}}|L^{*S} < Ax < L^{*S} + L^{*J}\Big)$ is the associated expected loss given default.

In the Basel 2 regulation terminology, the price of the junior (defaultable) zero-coupon $ZC^J$ would be written as:

$$B_{rf} - B^J = B_{rf} \times PD^J \times ELGD^J, \tag{3.7}$$

where $PD^J$ denotes the probability of default on the junior debt and $ELGD^J$ the associated expected loss given default. Comparing equations (3.6) and (3.7), we see how to switch from a tranche approach to the Basel 2 approach. We get :

$$PD^J = PD^S + PD^{J,S} \quad \text{and} \quad ELGD^J = \frac{PD^S}{PD^J} + \frac{PD^{J,S}}{PD^J} ELGD^{J,S}. \tag{3.8}$$

The probability of default on the junior debt, $PD^J$, and the expected loss given default, $ELGD^J$, are decomposed in two terms, depending on the default on the senior debt. The decompositions directly derive from the fact that a default on the senior debt implies a total default on the junior debt.

**iii) Price of a portfolio of tranches**

Let us finally consider a mixed investment with a share $\gamma$ of senior payoff $ZC^S$ and a share $(1-\gamma)$ of junior payoff $ZC^J$. The price of this portfolio is :

$$B(\gamma) = \gamma B^S + (1-\gamma) B^J. \tag{3.9}$$

Therefore we have :

$$B_{rf} - B(\gamma) = B_{rf} \times \Big( \gamma \times PD^S \times ELGD^S + (1-\gamma) \times PD^J \times ELGD^J \Big). \tag{3.10}$$

In the Basel 2 terminology, it would be written as :

$$B_{rf} - B(\gamma) = B_{rf} \times PD(\gamma) \times ELGD(\gamma). \tag{3.11}$$

For $\gamma < 1$, the portfolio has a strictly positive allocation in junior tranche. So it suffers loss as soon as the junior tranche does. Therefore, $PD(\gamma) = PD^J$. The probability of default of the portfolio is the probability of default on the junior debt. Moreover, we have :

$$ELGD(\gamma) = \gamma \frac{PD^S}{PD^J} ELGD^S + (1-\gamma) ELGD^J. \tag{3.12}$$

The expected loss given default of the portfolio is a linear combination of the expected loss given default on the junior debt and on the senior debt. The weights are the weights of portfolio allocation combined with the ratio of the probabilities of default on the junior and senior debts.

If $\gamma = 1$, the portfolio is a pure investment in senior tranche. So $PD(1) = PD^S$, and $ELGD(1) = ELGD^S$.

## 3.2 Pricing tranches for unconnected banks

Let us now consider a portfolio of junior and senior tranches of $n$ banks. We have $n$ balance sheets similar to the balance sheet displayed in Table 3. For each bank, there are two tranches on its debt, junior and senior, respectively. Therefore, there is a market of $2n$ tranches. Each tranche is either a junior, or a senior tranche on a bank debt. The standard pricing approaches will assume that the banks are unconnected, but can feature default risk dependence by means of a common exogenous factor.

To price this set of tranches, we consider the following standard factor model[6], written under the risk-neutral probability :

$$log Ax_i = \beta_i F + \varepsilon_i, \quad i = 1, ..., n,$$

where $F$ is a common factor (or a vector of common factors), $\beta_i$ is the sensitivity of bank $i$ to the common factor, and $\varepsilon_i$ a shock specific to bank $i$. We assume that $F$ and $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)'$ are independent and that $\varepsilon \sim \mathcal{N}(0, diag(\sigma_i^2))$ where $\sigma_i$ is the volatility specific to bank $i$. We denote by $\Phi$ (respectively, $\varphi$) the c.d.f. (the p.d.f.) of the standard Gaussian distribution.

The prices of two tranches written on two different banks are linked through the common factor $F$, while the prices of the two tranches of a same bank are linked through both the common factor $F$ and the idiosyncratic shock $\varepsilon_i$.

### 3.2.1 Pricing with an observed factor

The prices of the tranches are easily derived if we assume that the specific shocks are unobserved by the investor, whereas the common factor is. In this case, the computations are performed conditional on the factor values (see Appendix 3).

Let us introduce the variables:

$$\mathcal{A}_i = \frac{log(L_i^{*S}) - \beta_i F}{\sigma_i} \quad \text{and} \quad \mathcal{B}_i = \frac{log(L_i^{*S} + L_i^{*J}) - \beta_i F}{\sigma_i}, \tag{3.13}$$

which depend on factor $F$.

[6]The latent factor is introduced in order to capture the dependence between the external asset components. When $F$ is non Gaussian, this is a convenient way to get non Gaussian copulas.

**i) Conditional price of the senior tranche**

To price the senior zero-coupon of bank $i$, we first have to compute $PD_i^S$ and $ELGD_i^S$. We get :

$$PD_i^S = \Phi\big(\mathcal{A}_i\big), \tag{3.14}$$

$$ELGD_i^S = 1 - \frac{exp\left(-\sigma_i \mathcal{A}_i + \frac{\sigma_i^2}{2}\right)}{\Phi(\mathcal{A}_i)} \Phi(\mathcal{A}_i - \sigma_i). \tag{3.15}$$

The price of the senior zero-coupon of bank $i$ is such that :

$$B_{rf} - B_i^S = B_{rf} \times \Big(\Phi(\mathcal{A}_i) - exp\left(-\sigma_i \mathcal{A}_i + \frac{\sigma_i^2}{2}\right) \Phi(\mathcal{A}_i - \sigma_i)\Big). \tag{3.16}$$

The pricing formulas are similar to those obtain through the Black-Scholes model. We can write the payoff of a senior zero-coupon as $Ax_i/L_i^{*S} - (Ax_i/L_i^{*S} - 1)^+$ and apply the Black-Scholes formula to the second component which is the payoff of a European call option. Then, equation (3.16) becomes:

$$B_i^S = \underbrace{B_{rf} exp\left(-\sigma_i \mathcal{A}_i + \frac{\sigma_i^2}{2}\right)}_{\text{prices of } Ax_i/L_i^{*S}} \underbrace{- B_{rf} exp\left(-\sigma_i \mathcal{A}_i + \frac{\sigma_i^2}{2}\right) \Phi\big(-\mathcal{A}_i + \sigma_i\big) + B_{rf}\Phi\big(-\mathcal{A}_i\big)}_{\text{call option}}$$

**ii) Conditional price of the junior tranche**

To price the junior zero-coupon of bank $i$, we first have to compute $PD_i^{J,S}$ and $ELGD_i^{J,S}$. We get :

$$PD_i^{J,S} = \Phi\big(\mathcal{B}_i\big) - \Phi\big(\mathcal{A}_i\big), \tag{3.17}$$

$$ELGD_i^{J,S} = 1 + \frac{L_i^{*S}}{L_i^{*J}} - \frac{L_i^{*S}}{L_i^{*J}} exp\left(-\sigma_i \mathcal{A}_i + \frac{\sigma_i^2}{2}\right) \frac{\Phi\big(\mathcal{B}_i - \sigma_i\big) - \Phi\big(\mathcal{A}_i - \sigma_i\big)}{\Phi\big(\mathcal{B}_i\big) - \Phi\big(\mathcal{A}_i\big)}. \tag{3.18}$$

We deduce the price of the junior zero-coupon of bank $i$ as:

$$\begin{aligned} B_{rf} - B_i^J = B_{rf} \times \Big[\Phi\big(\mathcal{A}_i\big) + \Big(\Phi\big(\mathcal{B}_i\big) - \Phi\big(\mathcal{A}_i\big)\Big) \times \\ \left(1 + \frac{L_i^{*S}}{L_i^{*J}} - \frac{L_i^{*S}}{L_i^{*J}} exp\left(-\sigma_i \mathcal{A}_i + \frac{\sigma_i^2}{2}\right) \frac{\Phi\big(\mathcal{B}_i - \sigma_i\big) - \Phi\big(\mathcal{A}_i - \sigma_i\big)}{\Phi\big(\mathcal{B}_i\big) - \Phi\big(\mathcal{A}_i\big)}\right)\Big]. \end{aligned} \tag{3.19}$$

### 3.2.2 Pricing with unobserved factor

Pricing formulas of subsection 3.2.1 have been derived conditional on factor $F$. They depend on $F$ through $\mathcal{A}_i(F)$ and $\mathcal{B}_i(F)$. When the factor is not observed by the investor, the conditional pricing formulas have to be integrated out with respect to $F$ under the risk-neutral probability. This marginalizing step is needed to take into account the default dependence through the dependence of the asset components of the balance sheet. In other words, this dependence is introduced through common shocks external to the system.

## 3.3 Connected banks and hidden CDO

The standard pricing formulas of subsection 3.2 are derived under the assumption of unconnected banks. Let us now consider a network of $n$ banks linked through their junior and senior debts. We will see that the standard pricing formulas are modified by adding a contagion premium.

### 3.3.1 The prices of the tranches

The payoffs of junior and senior defaultable zero-coupons must now be jointly determined. These payoffs are solutions of the following system, which is a standardized version of the equilibrium system (2.12) – (2.13):

$$ZC_i^S = \frac{L_i^S}{L_i^{*S}} = min\Big[\frac{Ax_i}{L_i^{*S}} + \sum_{j=1}^{n} \gamma_{i,j}^J \frac{L_j^J}{L_i^{*S}} + \sum_{j=1}^{n} \gamma_{i,j}^S \frac{L_j^S}{L_i^{*S}}; 1\Big], \tag{3.20}$$

$$ZC_i^J = \frac{L_i^J}{L_i^{*J}} = min\Big[\frac{Ax_i}{L_i^{*J}} + \sum_{j=1}^{n} \gamma_{i,j}^J \frac{L_j^J}{L_i^{*J}} + \sum_{j=1}^{n} \gamma_{i,j}^S \frac{L_j^S}{L_i^{*J}} - \frac{L_i^S}{L_i^{*J}}; 1\Big], \tag{3.21}$$

for $i = 1, ..., n$ and $j = 1, ..., n$.
Equivalently, they are solutions of:

$$ZC_i^S = min\Big[\frac{Ax_i}{L_i^{*S}} + \sum_{j=1}^{n} \gamma_{i,j}^J \frac{L_j^{*J}}{L_i^{*S}} ZC_j^J + \sum_{j=1}^{n} \gamma_{i,j}^S \frac{L_j^{*S}}{L_i^{*S}} ZC_j^S; 1\Big], \tag{3.22}$$

$$ZC_i^J = min\Big[\frac{Ax_i}{L_i^{*J}} + \sum_{j=1}^{n} \gamma_{i,j}^J \frac{L_j^{*J}}{L_i^{*J}} ZC_j^J + \sum_{j=1}^{n} \gamma_{i,j}^S \frac{L_j^{*S}}{L_i^{*J}} ZC_j^S - \frac{L_i^{*S}}{L_i^{*J}} ZC_i^S; 1\Big], \tag{3.23}$$

for $i = 1, ..., n$ and $j = 1, ..., n$.
The prices of the tranches are deduced by:

i) first solving system (3.22)$-$(3.23) to get the payoffs in terms of external assets, the nominal debts values and the bilateral exposures. This requires an appropriate algorithm to find numerically this equilibrium (see Section 4.1).

ii) Then computing their discounted risk-neutral expectations with respect to both the common factor $F$ and the specific errors $\varepsilon_i$, $i = 1, ..., n$. We will not get closed form expressions of derivative prices[7], but these prices are easily computed by simulations.

The computation of the prices of the tranches is much more complicated than in the unconnected framework. In fact, the asset side of the bank includes different debts of the other banks. Thus, by buying (or selling) a tranche written on a single bank, that is, on a single name, we buy (or sell) a portfolio of tranches written on the other banks, that is, a CDO written on $n$ names if all the banks are connected.
Formulas (3.22) and (3.23) clearly show that a junior or senior zero-coupon written on bank $i$ involves several names. The involved names are not only the names directly obtained with the strictly positive exposures of bank $i$, which are all the banks $j$ such that $\gamma_{i,j}^S$, or $\gamma_{i,j}^J$ are not zero. They are also the banks connected to these banks through a chain of debt holdings. The hidden CDO is not a simple pooling of names with exogenous weights. Its design involves the structure of the network, that is the exposure matrices $\Gamma^J$ and $\Gamma^S$ and the structures of the balance sheets, that are the ratios $Ax_j/L_j^{*S}$ and $Ax_j/L_j^{*J}$. Moreover, the pricing of these tranches, that are hidden CDO's, have to take into account the simultaneity of the liquidation process, that is, for a contagion effect (see subsection 3.4).

### 3.3.2 Seniority and rating

The seniority, $S$ or $J$, of the debt is not a perfect signal of overall risk quality. It should only be interpreted as an ordering of the debts for each given bank: the junior debt of a bank is riskier than the senior debt of the same bank. In other words, the rating of the junior debt of a bank is smaller than the rating of its senior debt. But, this does not mean that all senior debts (or all junior debts) have a same rating. The rating of junior debt of bank $i$ might be greater than the rating of the senior debt of bank $j \neq i$, since the quality

[7] Even conditional on $F$.

| Bank 1 | | Bank 2 | | Bank 3 | |
|---|---|---|---|---|---|
| Asset | Liability | Asset | Liability | Asset | Liability |
| $Ax_1$ | $L_1^J$ | $Ax_2$ | $L_2^J$ | $\gamma_{3,1}^J L_1^J$ | $L_3^J$ |
| | $L_1^S$ | | $L_2^S$ | $\gamma_{3,1}^S L_1^S$ | $L_3^S$ |
| | $Y_1$ | | $Y_2$ | $\gamma_{3,2}^J L_2^J$ | $Y_3$ |
| | | | | $\gamma_{3,2}^S L_2^S$ | |

Table 5: Balance Sheet of Two Retail Banks and One Lender of Last Resort

of the tranches depend on the asset-liability ratios $Ax_i/L_i^{*S}$ and $Ax_i/L_i^{*J}$ and on the debt exposures. For instance, the senior debt of a bank with small asset/liability ratio can be riskier than the junior debt of a bank with a large asset/liability ratio. In particular, it might be misleading to consider a basket of senior tranches as weakly risky.
The seniority should only be interpreted as a priority rule for allocating the asset in case of liquidation. In this framework, defining the priority rule for each bank is enough to define endogenously the recovery rates on all debts, that is, to fix the equilibrium conversion rates between the prices of the different junior and senior tranches (see Section 2.5).
As an illustration of a junior debt which may be less risky than a senior debt, let us consider a basic network composed of three banks with two purely retail banks and one lender of last resort. Bank 1 and bank 2 have no interbanking exposures. Bank 3 plays the role of lender of last resort and has only debts of bank 1 and 2 in its asset side. The balance sheets of the banks are presented in Table 5. In this extreme case, where there is no external asset component for bank 3, a tranche on this lender of last resort is a pure CDO, written in fact on two names and mixing junior and senior debts.

The payoffs of the senior and junior zero-coupons issued by bank 3 are:

$$ZC_3^S = min\left[\gamma_{3,1}^J \frac{L_1^J}{L_3^{*S}} + \gamma_{3,2}^J \frac{L_2^J}{L_3^{*S}} + \gamma_{3,1}^S \frac{L_1^S}{L_3^{*S}} + \gamma_{3,2}^S \frac{L_2^S}{L_3^{*S}}; 1\right], \tag{3.24}$$

$$ZC_3^J = min\left[\gamma_{3,1}^J \frac{L_1^J}{L_3^{*J}} + \gamma_{3,2}^J \frac{L_2^J}{L_3^{*J}} + \gamma_{3,1}^S \frac{L_1^S}{L_3^{*J}} + \gamma_{3,2}^S \frac{L_2^S}{L_3^{*J}} - \frac{L_3^S}{L_3^{*J}}; 1\right], \tag{3.25}$$

These equations can be rewritten to show the impacts of the situations of bank 1 and bank 2 on the senior zero-coupon issued by bank 3 :

$$ZC_3^S = min\left[\gamma_{3,1}^J \frac{min\left(\left(Ax_1 - L_1^{*S}\right)^+; L_1^{*J}\right)}{L_3^{*S}} + \gamma_{3,2}^J \frac{min\left(\left(Ax_2 - L_2^{*S}\right)^+; L_2^{*J}\right)}{L_3^{*S}}\right.$$
$$\left. + \gamma_{3,1}^S \frac{min\left(Ax_1; L_1^{*S}\right)}{L_3^{*S}} + \gamma_{3,2}^S \frac{min\left(Ax_2; L_2^{*S}\right)}{L_3^{*S}}; 1\right], \quad (3.26)$$

Let us assume that bank 2 is alive: $Ax_2 > L_2^{*J} + L_2^{*S}$, and let us focus on the senior zero-coupon of bank 3. If the external asset of bank 1 decreases, it may happen that the senior zero-coupon issued by bank 3 is riskier than the junior zero-coupon issued by bank 2. This arises if

$$Ax_1 < min\left(\frac{L_3^{*S} - \gamma_{3,2}^J L_2^{*J} - \gamma_{3,2}^S L_2^{*S}}{\gamma_{3,1}^S}; L_1^{*S}\right),$$

where we get:

$$ZC_3^S = \gamma_{3,2}^J \frac{L_2^{*J}}{L_3^{*S}} + \gamma_{3,1}^S \frac{min\left(Ax_1; L_1^{*S}\right)}{L_3^{*S}} + \gamma_{3,2}^S \frac{L_2^{*S}}{L_3^{*S}} < 1, \quad (3.27)$$

Therefore there is a loss for the senior zero-coupon of bank 3 while the junior zero-coupon issued by bank 2 has a plain payoff.

## 3.4 Contagion premium

The prices of the tranches can be decomposed into a part that accounts for contagion phenomena and a part without contagion phenomena. The last component corresponds to the standard price of a tranche (see Section 3.2) Let us consider an initial financial system defined by the bilateral exposure matrices, the values of external assets and the nominal values of junior and senior debts: $\mathcal{S}^0 = \{\Pi, \Gamma^J, \Gamma^S, L^{*J}, L^{*S}, Ax^0\}$. We assume that initially all banks are alive : $Y_i^0 > 0$, $i = 1, ..., n$. The prices of the junior and senior zero-coupons can be written as functions of this initial financial system $\mathcal{S}^0$ :

$$ZC_i^J(\mathcal{S}^0) \text{ and } ZC_i^S(\mathcal{S}^0), \qquad i = 1, ..., n.$$

Let us now consider an alternative initial financial system, where all the banks sell their cross holdings of stocks, junior and senior debts. These sells transform these assets into external assets. The after sell initial financial system is $\mathcal{S}^1 = \{0, 0, 0, L^{*J}, L^{*S}, Ax^1\}$, where $Ax^1 = \Pi Y^0 + \Gamma^J L^{*J} + \Gamma^S L^{*S} + Ax^0 = Y^0 + L^{*J} + L^{*S}$. Of course the banks have the same values at date $t = 0$ in both systems $\mathcal{S}^0$ and $\mathcal{S}^1$.
Then let us apply a shock on the external asset components $Ax^0$, and thus also on $Ax^1$. This leads to new systems $\mathcal{S}^0 + \delta\mathcal{S}^0$ and $\mathcal{S}^1 + \delta\mathcal{S}^1$, respectively, depending on the initial system, which is considered. The prices of the junior and senior zero-coupons after the shock are denoted by $ZC_i^J(\mathcal{S}^0 + \delta\mathcal{S}^0)$ and $ZC_i^S(\mathcal{S}^0 + \delta\mathcal{S}^0)$, and by $ZC_i^J(\mathcal{S}^1 + \delta\mathcal{S}^1)$ and $ZC_i^S(\mathcal{S}^1 + \delta\mathcal{S}^1)$, respectively. Since there are no interconnections, the prices $ZC_i^J(\mathcal{S}^1 + \delta\mathcal{S}^1)$ and $ZC_i^S(\mathcal{S}^1 + \delta\mathcal{S}^1)$ are immune to contagion and derived by the standard formulas of Section 3.2. By comparing the two types of prices, we define a contagion premium for the junior and senior zero-coupons :

$$ZC_i^J(\mathcal{S}^0 + \delta\mathcal{S}^0) = \underbrace{ZC_i^J(\mathcal{S}^1 + \delta\mathcal{S}^1)}_{\text{standard price}} + \underbrace{\Big(ZC_i^J(\mathcal{S}^0 + \delta\mathcal{S}^0) - ZC_i^J(\mathcal{S}^1 + \delta\mathcal{S}^1)\Big)}_{\text{contagion premium}}, \tag{3.28}$$

$$ZC_i^S(\mathcal{S}^0 + \delta\mathcal{S}^0) = \underbrace{ZC_i^S(\mathcal{S}^1 + \delta\mathcal{S}^1)}_{\text{standard price}} + \underbrace{\Big(ZC_i^S(\mathcal{S}^0 + \delta\mathcal{S}^0) - ZC_i^S(\mathcal{S}^1 + \delta\mathcal{S}^1)\Big)}_{\text{contagion premium}}, \tag{3.29}$$

for $i = 1, ..., n$.
The effect of interconnexions can increase or decrease the value of a junior, or senior zero-coupon, that is, the contagion premium in equations (3.28) and (3.29) can be either positive, or negative. Moreover, the sign of this premium may vary across banks and across seniority levels for a given bank. If the contagion premium is positive, the interconnections make the defaultable zero-coupon safer whereas a negative contagion premium makes it riskier.
Finally note that both the standard price and contagion premium depend on the dependence due to the common shock $F$ exogenous to the system. The standard price (resp. the contagion premium) may themselves be decomposed to measure the effect of the common shock (resp. the joint effect of exogenous shock and contagion).

# 4 Applications

For illustration purpose, we consider a system of 3 banks, which are interconnected through their debt holdings only, and we assume that no bank has self-holding of its debt. We give an example of a liquidation equilibrium path when the external asset components of the balance sheets are driven by a single common dynamic factor. Then, we provide the prices of hidden CDO's and their contagion components for different junior and senior exposures.

## 4.1 The liquidation program

Let us first explain how to solve numerically the piecewise linear system defining the equilibrium. The equilibrium conditions for the junior and senior debts in a model without interconnections by stocks are given by:

$$L_i^S = min(L_i^{*S}; \sum_{j \neq i} \gamma_{i,j}^S L_j^S + \sum_{j \neq i} \gamma_{i,j}^J L_j^J + Ax_i), \qquad i = 1, \ldots, n, \tag{4.1}$$

$$L_i^J = min(L_i^{*J}; \sum_{j \neq i} \gamma_{i,j}^S L_j^S + \sum_{j \neq i} \gamma_{i,j}^J L_j^J + Ax_i - L_i^S), \qquad i = 1, \ldots, n. \tag{4.2}$$

How to solve this $2n$ dimensional piecewise linear system in practice? If $n$ is large, it is computationally unfeasible to consider the $3^n$ regimes and select the one corresponding to the equilibrium. Proposition 2 provides an approach, which involves a number of computations of order $n$ instead of $3^n$ to find the liquidation equilibrium. This extends Lemma 4 in Eisenberg, Noe (2001). The idea is that the liquidation equilibrium is the solution of a linear program.

**Proposition 2 :** The solution of the linear program :

$$\begin{array}{lll}
\max\limits_{L_i^S, L_i^J} & \left( \sum\limits_{i=1}^n L_i^S + \theta \sum\limits_{i=1}^n L_i^J \right) & \\
s.t. & L_i^S \leq L_i^{*S}, & i = 1, ..., n, \\
& L_i^S \leq \sum\limits_{j \neq i} \gamma_{i,j}^S L_j^S + \sum\limits_{j \neq i} \gamma_{i,j}^J L_j^J + Ax_i, & i = 1, ..., n, \\
& L_i^J \leq L_i^{*J}, & i = 1, ..., n, \\
& L_i^J \leq \sum\limits_{j \neq i} \gamma_{i,j}^S L_j^S + \sum\limits_{j \neq i} \gamma_{i,j}^J L_j^J + Ax_i - L_i^S, & i = 1, ..., n,
\end{array} \tag{4.3}$$

satisfies the equilibrium conditions (4.1)-(4.2), if the positive penalty coefficient $\theta$ is sufficiently small.

**Proof :** See Appendix 4.

Therefore the liquidation equilibrium can be derived by applying a simplex method to the linear program in Proposition 2. This optimization problem can be seen as describing the behavior of an authority, which has to define the liquidation process. The criterion of the authority considers the debt recoveries, while ensuring the priority of the senior debts with respect to the junior debts by the appropriate choice of the weighting scalar[8] $\theta$. Then, by maximizing the objective function under the contractual and limited liabilities restrictions, this optimizing authority will reach the liquidation equilibrium.
As noted in Elsinger(2009), the result of Proposition 2 cannot be extended when there are also stock crossholdings. It is still possible to use a fixed point algorithm converging to the liquidation equilibrium, but the limiting equilibrium is not necessarily reached in a finite number of steps

## 4.2 Liquidation equilibrium path

For illustrative purpose, let us now consider a basic network of three banks. The exposure matrices are set to :

$$\Pi = \begin{pmatrix} 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \end{pmatrix}, \quad \Gamma^J = \begin{pmatrix} 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 \\ 0.35 & 0.15 & 0.00 \end{pmatrix} \text{ and } \quad \Gamma^S = \begin{pmatrix} 0.00 & 0.00 & 0.00 \\ 0.20 & 0.00 & 0.00 \\ 0.05 & 0.05 & 0.00 \end{pmatrix}.$$

The liability components of the balance sheets are :

$$L^{*J} = \begin{pmatrix} 0.75 \\ 0.75 \\ 0.10 \end{pmatrix} \text{ and } L^{*S} = \begin{pmatrix} 1.75 \\ 1.25 \\ 0.90 \end{pmatrix}.$$

The external assets are driven by the following historical dynamic equation :

$$log(Ax_{i,t}) = \alpha_i + \rho \; log(Ax_{i,t-1}) + \beta_i F_t + \sigma_i \varepsilon_{i,t}, \quad \text{for } t = 2, ..., 5, \text{ and } i = 1, 2, 3, \tag{4.4}$$

[8]The discussion in Appendix 4 shows that scalar $\theta$ has to be chosen smaller than 1.

with initial conditions:

$$\begin{pmatrix} Ax_{1,1}, & Ax_{2,1}, & Ax_{3,1} \end{pmatrix}' = \begin{pmatrix} 2.6, & 2.1, & 0.6 \end{pmatrix}', \tag{4.5}$$

parameters set to:

$$\begin{aligned} \begin{pmatrix} \sigma_1, & \sigma_2, & \sigma_3 \end{pmatrix}' &= \begin{pmatrix} 0.16, & 0.06, & 0.01 \end{pmatrix}', \\ \rho &= 0.95, \\ \beta &= (1.2, 0.8, 0.4)', \end{aligned} \tag{4.6}$$

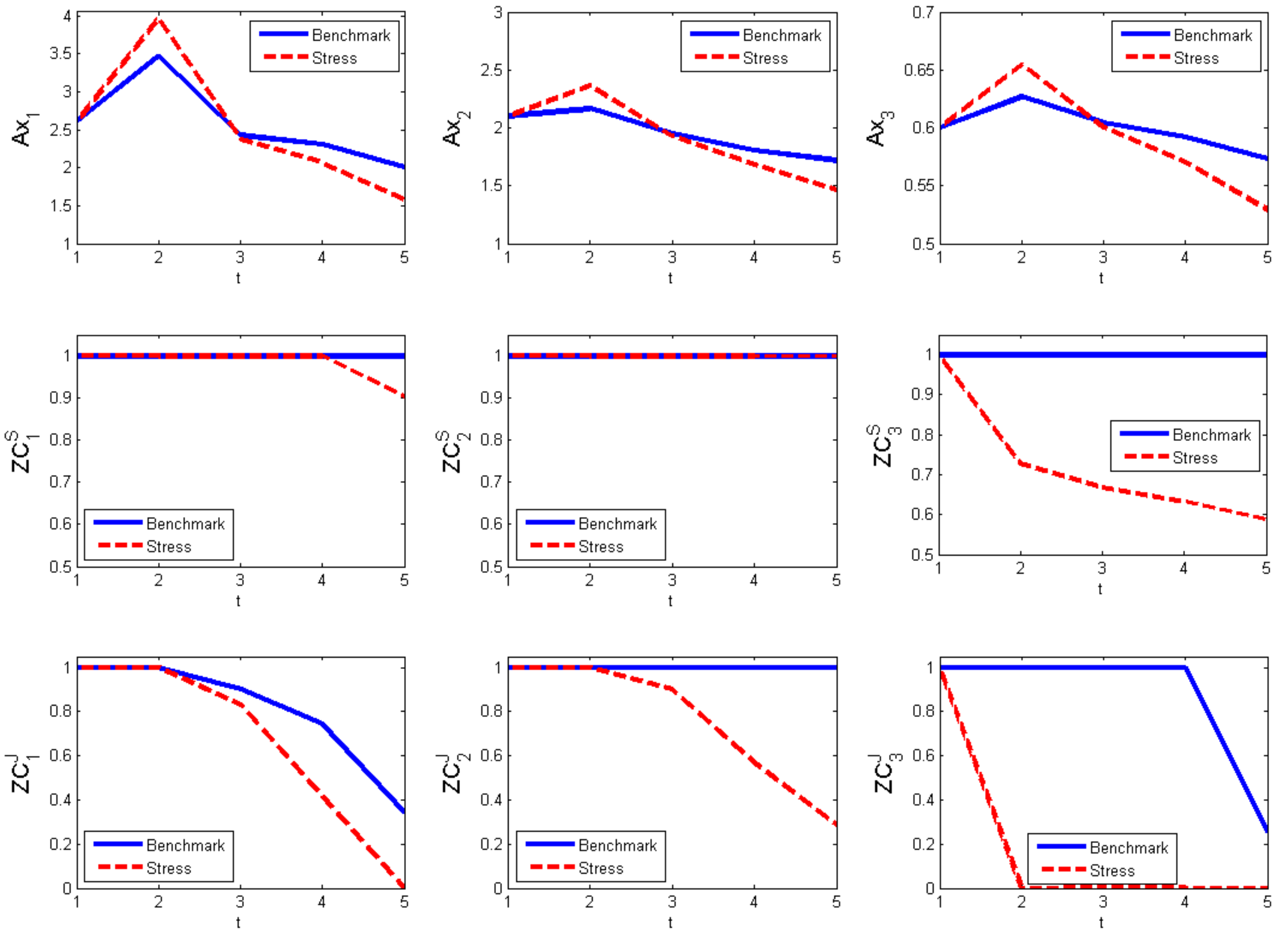
Gaussian common factor $F_t \sim II\mathcal{N}(0; 0.001)$, and independent standard normal specific error terms $\varepsilon_{i,t} \sim II\mathcal{N}(0; 1)$. The $\alpha_i$'s are set such that $\mathbb{E}(Ax_{i,2}|Ax_{i,1}) = Ax_{i,1}$.

Intuitively, bank 1 is a pure retail bank, which is very sensitive to the systematic factor. Bank 3 plays the role of lender of last resort : it lends to the two other banks and has very few assets on its own. The asset component of bank 2 includes interbank lending (towards bank 1) and moderately risky external assets.

The dynamic $(4.4)-(4.6)$ defines the benchmark scenario. The external asset components at initial date 1 have been chosen to ensure that the three banks are alive at this date. Then at the next date, these external asset components receive specific shocks through the new drawing of the $\varepsilon_{i,t}$ and new common shocks through the new drawing of the common factor. At some dates these shocks can be adverse shocks implying the default of one or several banks.

The dynamic model above can also serve for stress-tests. A stress-test compares the outcomes of benchmark and stress scenarios. We build a stress scenario where the common factor $F$ is twice bigger than in the benchmark scenario. Figure 4 displays the payoffs of the junior and senior tranches in the benchmark and stress scenarios for given systematic and idiosyncratic trajectories.

Under the benchmark scenario, the senior zero-coupons are not risky for the three banks, but we observe default of bank 1 at date 3 (respectively, of bank 3 at date 5) on the junior debt. Under the stress scenario, the payoffs are smaller than under the benchmark scenario. The default can now reach the senior debts, for instance for banks 3 and 1. Moreover, we observe first a default of bank 3 at date 2, then a joint default of banks 1 and 2 at date 3. Finally, note that the payoff of the junior debt of bank 3 in the benchmark scenario is higher than the payoff of its senior debt under the stress scenario. This means that the rating of a junior (resp. senior) debt can vary over time.




The x-axis corresponds to time with $t = 1$ as initial date. The y-axis represents the values of external assets (top row), senior zero-coupons (middle row) and junior zero-coupons (bottom row). The left column of the three plots refers to bank 1, the middle column of the three plots refers to bank 2 and the right column of the three plots refers to bank 3. The plain lines correspond to values under the benchmark scenario while the dashed lines correspond to values under the stress scenario.

Figure 4: Payoffs in the Benchmark and Stress Scenarios

## 4.3 Decomposition of hidden CDO prices

Let us now focus on the decomposition of the junior and senior hidden CDO prices into the standard prices and the contagion premiums. We assume that the risk-neutral dynamics is the autoregressive system with systematic

| | Standard Price | | | Contagion Premium | | |
|---|---|---|---|---|---|---|
| | $ZC_1^S$ | $ZC_2^S$ | $ZC_3^S$ | $ZC_1^S$ | $ZC_2^S$ | $ZC_3^S$ |
| $h=0$ | 100.00 | 100.00 | 100.00 | 0.00 | 0.00 | 0.00 |
| $h=1$ | 99.79 | 100.00 | 100.00 | 0.00 | 0.00 | -0.41 |
| $h=2$ | 98.90 | 100.00 | 100.00 | 0.00 | 0.00 | -1.20 |
| $h=3$ | 97.68 | 100.00 | 100.00 | 0.00 | 0.00 | -1.97 |
| $h=4$ | 96.54 | 100.00 | 100.00 | 0.00 | 0.00 | -2.56 |
| | $ZC_1^J$ | $ZC_2^J$ | $ZC_3^J$ | $ZC_1^J$ | $ZC_2^J$ | $ZC_3^J$ |
| $h=0$ | 100.00 | 100.00 | 100.00 | 0.00 | 0.00 | 0.00 |
| $h=1$ | 79.62 | 99.80 | 99.98 | 0.00 | 0.13 | -13.67 |
| $h=2$ | 71.07 | 98.81 | 99.62 | 0.00 | 0.54 | -23.38 |
| $h=3$ | 66.25 | 97.36 | 98.84 | 0.00 | 0.93 | -28.19 |
| $h=4$ | 63.42 | 96.16 | 97.86 | 0.00 | 1.11 | -30.55 |

Table 6: Prices of Tranches and Contagion Premiums

factor given in the benchmark scenario (4.4). The prices are computed by simulation with 10 000 iterations. The results are given in Table 6 (in %) for date $t = 1$ at horizon $h = 0, ..., 4$. There is no default at current date $t = 1$ and the initial conditions are given in (4.5). For the financial system without contagion, we consider that banks sell their interbank asset and invest them in their external assets, whose dynamics is given by (4.4). Since bank 1 does not lend to other banks, there is no contagion premium on the tranches of bank 1. For the senior tranches, the tranches of bank 3 are affected by a negative contagion premium. For the junior tranches, the contagion premium is negative for bank 3 whereas it is positive for bank 2. The interconnections have a positive (respectively, negative) impact on bank 2 (respectively, on bank 3). In this special case, the contagion premium for bank 1 is null, is positively increasing with the horizon for bank 2 and negatively decreasing with the horizon for bank 3. The negative risk premium for bank 3 was expected. Indeed the bank has an interpretation of lender in last resort and has to support the increased common risk introduced in the stochastic scenario. When the horizon $h$ increases, it has to support both the direct and the indirect effects due to the interconnections.

## 5 Concluding remarks

An adverse exogenous shock on the banks can create joint defaults and a joint determination of the recovery rates of the debts. We have shown in this paper that the new situation of the financial system depends on the structure of the interconnections between the banks through the different types of assets included in their balance sheets, in particular through the matrices of exposures for stocks, junior and senior debts. We have proved the existence and uniqueness of the liquidation equilibrium under conditions on the exposures, which are generally fulfilled in practice.

These interconnections and the associated liquidation equilibrium have significant consequences on the pricing of the junior and senior debts written on a single name, when this name is a bank. Those contractual single name assets are in fact CDO's written on the names of all interconnected banks. Moreover to account for the interconnection between banks the standard CDO's pricing formulas have to be modified by adding a premium to capture the contagion effects.

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## Appendix 1

### The equilibrium for two banks

We consider system (2.11) – (2.13) with $n = 2$. There are nine possible joint default regimes for the two banks. The idea is first to consider the solution of the system in each regime, second to characterize each regime in the space of external assets $Ax_1, Ax_2$, and finally to write conditions to insure that the regions defining the regimes form a partition of this space. For each given regime, there exists a unique solution. These solutions are given below.

**Regime 1 :** No default. We get : $(Id - \Pi)Y = \Delta Ax$, and this regime occurs iff :

$$\Delta Ax \in \begin{pmatrix} 1-\pi_{1,1} & -\pi_{1,2} \\ -\pi_{2,1} & 1-\pi_{2,2} \end{pmatrix} I\!R^{+2} \equiv \mathcal{C}_1.$$

**Regime 2 :** bank 1 is alive ; bank 2 is in partial default. We get :

$$(Id - \Pi)\begin{bmatrix} Y_1 \\ 0 \end{bmatrix} + (\Gamma^J - Id)\begin{bmatrix} 0 \\ \Delta L_2^J \end{bmatrix} = \Delta Ax,$$

and this regime occurs iff :

$$\Delta Ax \in \begin{pmatrix} 1-\pi_{1,1} & \gamma_{1,2}^J \\ -\pi_{2,1} & \gamma_{2,2}^J - 1 \end{pmatrix} I\!R^{+} \times [0; L_2^{*J}] \equiv \mathcal{C}_2.$$

**Regime 3 :** bank 1 is alive ; bank 2 is in complete default. We get :

$$(Id - \Pi)\begin{bmatrix} Y_1 \\ 0 \end{bmatrix} + (\Gamma^J - Id)\begin{bmatrix} 0 \\ L_2^{*J} \end{bmatrix} + (\Gamma^S - Id)\begin{bmatrix} 0 \\ \Delta L_2^S \end{bmatrix} = \Delta Ax,$$

and this regime occurs iff :

$$\Delta Ax - \begin{pmatrix} \gamma_{1,2}^J L_2^{*J} \\ (\gamma_{2,2}^J - 1)L_2^{*J} \end{pmatrix} \in \begin{pmatrix} 1-\pi_{1,1} & \gamma_{1,2}^S \\ -\pi_{2,1} & \gamma_{2,2}^S - 1 \end{pmatrix} I\!R^{+} \times [0; L_2^{*S}] \equiv \mathcal{C}_3.$$

**Regime 4 :** bank 1 is in partial default ; bank 2 is alive. We get :

$$(Id - \Pi)\begin{bmatrix} 0 \\ Y_2 \end{bmatrix} + (\Gamma^J - Id)\begin{bmatrix} \Delta L_1^J \\ 0 \end{bmatrix} = \Delta Ax,$$

and this regime occurs iff :

$$\Delta Ax \in \begin{pmatrix} \gamma^J_{1,1}-1 & -\pi_{1,2} \\ \gamma^J_{1,2} & 1-\pi_{2,2} \end{pmatrix} [0; L_1^{*J}] \times I\!R^+ \equiv \mathcal{C}_4.$$

**Regime 5 :** bank 1 is in partial default ; bank 2 is in partial default. We get : $(\Gamma^J - Id)\Delta L^J = \Delta Ax$, and this regime occurs iff :

$$\Delta Ax \in \begin{pmatrix} \gamma^J_{1,1}-1 & \gamma^J_{1,2} \\ \gamma^J_{2,1} & \gamma^J_{2,2}-1 \end{pmatrix} [0; L_1^{*J}] \times [0; L_2^{*J}] \equiv \mathcal{C}_5.$$

**Regime 6 :** bank 1 is in partial default ; bank 2 is in complete default. We get :

$$(\Gamma^S - Id) \begin{bmatrix} 0 \\ \Delta L_2^S \end{bmatrix} + (\Gamma^J - Id) \begin{bmatrix} \Delta L_1^J \\ 0 \end{bmatrix} + (\Gamma^J - Id) \begin{bmatrix} 0 \\ L_2^{*J} \end{bmatrix} = \Delta Ax,$$

and this regime occurs iff :

$$\Delta Ax - \begin{pmatrix} \gamma^J_{1,2} L_2^{*J} \\ (\gamma^J_{2,2}-1) L_2^{*J} \end{pmatrix} \in \begin{pmatrix} \gamma^J_{1,1}-1 & \gamma^S_{1,2} \\ \gamma^J_{2,1} & \gamma^S_{2,2}-1 \end{pmatrix} [0; L_1^{*J}] \times [0; L_2^{*S}] \equiv \mathcal{C}_6.$$

**Regime 7 :** bank 1 is in complete default ; bank 2 is alive. We get :

$$(Id - \Pi) \begin{bmatrix} 0 \\ Y_2 \end{bmatrix} + (\Gamma^S - Id) \begin{bmatrix} \Delta L_1^S \\ 0 \end{bmatrix} + (\Gamma^J - Id) \begin{bmatrix} L_1^{*J} \\ 0 \end{bmatrix} = \Delta Ax,$$

and this regime occurs iff :

$$\Delta Ax - \begin{pmatrix} (\gamma^J_{1,1}-1) L_1^{*J} \\ \gamma^J_{1,2} L_1^{*J} \end{pmatrix} \in \begin{pmatrix} \gamma^S_{1,1}-1 & -\pi_{1,2} \\ \gamma^S_{1,2} & 1-\pi_{2,2} \end{pmatrix} [0; L_1^{*J}] \times I\!R^+ \equiv \mathcal{C}_7.$$

**Regime 8 :** bank 1 is in complete default ; bank 2 is in partial default. We get :

$$(\Gamma^S - Id) \begin{bmatrix} \Delta L_1^S \\ 0 \end{bmatrix} + (\Gamma^J - Id) \begin{bmatrix} L_1^{*J} \\ 0 \end{bmatrix} + (\Gamma^J - Id) \begin{bmatrix} 0 \\ \Delta L_2^J \end{bmatrix} = \Delta Ax,$$

and this regime occurs iff :

$$\Delta Ax - \begin{pmatrix} (\gamma^J_{1,1}-1)L^{*J}_1 \\ \gamma^J_{1,2}L^{*J}_1 \end{pmatrix} \in \begin{pmatrix} \gamma^S_{1,1}-1 & \gamma^J_{1,2} \\ \gamma^S_{1,2} & \gamma^J_{2,2}-1 \end{pmatrix} [0; L^{*S}_1] \times [0; L^{*J}_2] \equiv \mathcal{C}_8.$$

**Regime 9 :** bank 1 is in complete default ; bank 2 is in complete default.

We get :

$$(\Gamma^S - Id) \begin{bmatrix} \Delta L^S_1 \\ \Delta L^S_2 \end{bmatrix} + (\Gamma^J - Id) \begin{bmatrix} L^{*J}_1 \\ L^{*J}_2 \end{bmatrix} = \Delta Ax,$$

and this regime occurs iff :

$$\Delta Ax - \begin{pmatrix} (\gamma^J_{1,1}-1)L^{*J}_1 + \gamma^J_{1,2}L^{*J}_2 \\ \gamma^J_{2,1}L^{*J}_1 + (\gamma^J_{2,2}-1)L^{*J}_2 \end{pmatrix} \in \underbrace{\begin{pmatrix} \gamma^S_{1,1}-1 & \gamma^S_{1,2} \\ \gamma^S_{2,1} & \gamma^S_{2,2}-1 \end{pmatrix} [0; L^{*S}_1] \times [0; L^{*S}_2]}_{\equiv \mathcal{C}_9}.$$

We get a partition of the set of external asset values $Ax = (Ax_1, Ax_2)'$ into 9 bounded or unbounded quadrilaterals defined by means of 4 nodes. These quadrilaterals are generated by the six vectors $u_1$,...,$u_6$, spreading from $Ax^*$, $Ax^* + X_1$, $Ax^* + X_2$, or $Ax^* + X_1 + X_2$, where :

$$u_1 = \begin{pmatrix} 1-\pi_{1,1} \\ -\pi_{2,1} \end{pmatrix}, \quad u_3 = \begin{pmatrix} \gamma^J_{1,1}-1 \\ \gamma^J_{2,1} \end{pmatrix}, \quad u_5 = \begin{pmatrix} \gamma^S_{1,1}-1 \\ \gamma^S_{2,1} \end{pmatrix},$$

$$u_2 = \begin{pmatrix} -\pi_{1,2} \\ 1-\pi_{2,2} \end{pmatrix}, \quad u_4 = \begin{pmatrix} \gamma^J_{1,2} \\ \gamma^J_{2,2}-1 \end{pmatrix}, \quad u_6 = \begin{pmatrix} \gamma^S_{1,2} \\ \gamma^S_{2,2}-1 \end{pmatrix},$$

$$\text{and : } X_1 = \begin{pmatrix} (\gamma^J_{1,1}-1)L^{J*}_1 \\ \gamma^J_{2,1}L^{J*}_1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} \gamma^J_{1,2}L^{J*}_2 \\ (\gamma^J_{2,2}-1)L^{J*}_2 \end{pmatrix}.$$

This partition is described in Figure 3.

In case of two banks, the existence and uniqueness of the liquidation equilibrium is given below.

**Proposition A.1:** For a system with two banks and two seniority levels, the liquidation equilibrium exists and is unique, iff the determinants :

$$\begin{aligned}
\det(u_1,u_2) &= (1-\pi_{1,1})(1-\pi_{2,2})-\pi_{1,2}\pi_{2,1},\\
\det(u_2,u_3) &= (1-\pi_{2,2})(1-\gamma^J_{1,1})-\pi_{1,2}\gamma^J_{2,1},\\
\det(u_3,u_4) &= (1-\gamma^J_{1,1})(1-\gamma^J_{2,2})-\gamma^J_{1,2}\gamma^J_{2,1},\\
\det(u_4,u_1) &= (1-\pi_{1,1})(1-\gamma^J_{2,2})-\gamma^J_{1,2}\pi_{2,1},\\
\det(u_2,u_5) &= (1-\pi_{2,2})(1-\gamma^S_{1,1})-\pi_{1,2}\gamma^S_{2,1},\\
\det(u_5,u_4) &= (1-\gamma^S_{1,1})(1-\gamma^J_{2,2})-\gamma^J_{1,2}\gamma^S_{2,1},\\
\det(u_5,u_6) &= (1-\gamma^S_{1,1})(1-\gamma^S_{2,2})-\gamma^S_{1,2}\gamma^S_{2,1},\\
\det(u_3,u_6) &= (1-\gamma^J_{1,1})(1-\gamma^S_{2,2})-\gamma^S_{1,2}\gamma^J_{2,1},\\
\det(u_5,u_1) &= (1-\pi_{1,1})(1-\gamma^S_{2,2})-\gamma^S_{1,2}\pi_{2,1},
\end{aligned}$$

have the same sign.

**Proof :** Let us consider two vectors $u$, $v$ of $I\!R^2$ with a same length. Vector $v$ can be deduced from vector $u$ by a rotation of angle $\theta \in (-\pi,\pi)$. The sign of this angle is equal to the sign of the determinant $det(u,v)$. More generally, when the two vectors have different lengths, the sign of the determinant still provides either the positive, or negative direction of the rotation to pass from $\frac{u}{||u||}$ to $\frac{v}{||v||}$.

Let us now consider Figure 3. The equilibrium exists and is unique if and only if the (bounded and unbounded) quadrilaterals form a partition of $I\!R^2$. This happens if and only if the directions of the rotations are the same at the four nodes. This provides the conditions of Proposition A.1.

QED

These determinantal conditions are implied by the conditions :

$$\sum_{i=1}^{2}\pi_{i,j}<1, \sum_{i=1}^{2}\gamma^J_{i,j}<1, \sum_{i=1}^{2}\gamma^S_{i,j}<1, j=1,2,$$

since under these inequalities all determinants of Proposition A.1 are strictly positive. Thus, the equilibrium exists and is unique if nonzero fractions of stocks and junior and senior debts are hold outside the system.

## Appendix 2

### The equilibrium for $n$ interconnected banks: Proof of Proposition 1

The proof of the existence and uniqueness of the equilibrium in the general framework needs more complicated arguments than in the case of two banks. It requires a condition for the invertibility of a piecewise linear function in a multidimensional space like in Gourieroux, Heam, Monfort (2012) and arguments of projective geometry.

Let us consider a set of $n$ interconnected banks. There are three situations for each bank and therefore $3^n$ joint regimes indexed by a sequence $z = (z_1, ..., z_n)$, where $z_i = 0$, if bank $i$ does not default, $z_i = 1$, if bank $i$ partially defaults and $z_i = 2$, otherwise. For each state $z_i$ and for each bank, there is only one variable among $Y$, $L^S$, $L^J$, which is not known.

The equilibrium conditions (2.11) − (2.13) can be written in terms of $Y$, $\Delta L^S = L^{*S} - L^S$ and $\Delta L^J = L^{*J} - L^J$, where $Y_i$, $\Delta L_i^S$, $\Delta L_i^J$ are nonnegative variables. Let us denote $\Delta Ax = Ax - (Id - \Gamma^S)L^{*S} - (Id - \Gamma^J)L^{*J} = Ax - Ax^*$, where $Ax^*$ denotes the out of stock net nominal assets, that is, $Ax_i^*$ is the out of stock nominal profit and loss of bank $i$.

These equations are equivalent to :

$$Y_i = \Big( \sum_{j=1}^n \pi_{i,j} Y_j - \sum_{j=1}^n \gamma_{i,j}^J \Delta L_j^J - \sum_{j=1}^n \gamma_{i,j}^S \Delta L_j^S + \Delta L_i^J + \Delta L_i^S + \Delta Ax_i \Big)^+, \qquad i = 1, \ldots, n, \tag{a.1}$$

$$\Delta L_i^S = \Big( - \sum_{j=1}^n \pi_{i,j} Y_j + \sum_{j=1}^n \gamma_{i,j}^J \Delta L_j^J + \sum_{j=1}^n \gamma_{i,j}^S \Delta L_j^S - L_i^{*J} - \Delta Ax_i \Big)^+, \qquad i = 1, \ldots, n, \tag{a.2}$$

$$\Delta L_i^J = \Big( - \sum_{j=1}^n \pi_{i,j} Y_j + \sum_{j=1}^n \gamma_{i,j}^J \Delta L_j^J + \sum_{j=1}^n \gamma_{i,j}^S \Delta L_j^S - \Delta L_i^S - \Delta Ax_i \Big)^+, \qquad i = 1, \ldots, n. \tag{a.3}$$

Let us now denote $B = Id - \Pi$, $C^J = \Gamma^J - Id$, $C^S = \Gamma^S - Id$, and $b_i$, $c_i^J$ and $c_i^S$ the i$^{th}$ columns of matrices $B$, $C^J$ and $C^S$, respectively. Let us also introduce the dummy variable $z_{i,j}$ equal to 1, if $z_i = j$, and equal to 0, otherwise, and the $(n,n)$ matrix $D(z)$, whose i$^{th}$ column is given by :

$$d_i(z) = z_{i,0}b_i + z_{i,1}c_i^J + z_{i,2}c_i^S. \tag{a.4}$$

Then the equilibrium conditions become :

$$(1 - z_{i,0})Y_i + (1 - z_{i,1})\Delta L_i^J + (1 - z_{i,2})\Delta L_i^S = 0, \quad i = 1, \dots, n. \tag{a.5}$$

$$D(z) \begin{bmatrix} \vdots \\ z_{i,0}Y_i + z_{i,1}\Delta L_i^J + z_{i,2}\Delta L_i^S \\ \vdots \end{bmatrix} + (\Gamma^J - Id) \begin{bmatrix} \vdots \\ z_{i,2}L_i^{*J} \\ \vdots \end{bmatrix} = \Delta Ax. \tag{a.6}$$

Equation $(a.5)$ defines the regimes. For instance, if $z_{i,0} = 1$, we get $z_{i,1} = z_{i,2} = 0$, then $\Delta L_i^J = \Delta L_i^S = 0$, that is $L_i^J = L_i^{*J}$ and $L_i^S = L_i^{*S}$, since the variables $\Delta L_i^J$ and $\Delta L_i^S$ are nonnegative.
The $n-$dimensional system $(a.6)$ provides the closed form expression of the unconstrained variable in each regime. For instance, if $z_{i,0} = 1$ for all $i = 1, ..., n$, it provides the expression of the value of the firm, by solving :

$$BY = (Id - \Pi)Y = \Delta Ax. \tag{a.7}$$

The equilibrium conditions $(a.5) - (a.6)$ can be rewritten in an equivalent way, which is more appropriate to study the existence and uniqueness of the liquidation equilibrium. The aim of the transformation is to consider a vector space in which the boundaries defining the regimes are orthogonal. Let us consider the new variables defined by :

$$X_i^J = -\Delta L_i^J \in [-L_i^{*J}; 0],$$

$$X_i^S = -\Delta L_i^S - L_i^{*J} \in [-L_i^{*J} - L_i^{*S}; -L_i^{*J}].$$

With these new variables, the regimes can be defined by considering the location of $U_i = z_{i,0}Y_i + z_{i,1}X_i^J + z_{i,2}X_i^S$ with respect to the thresholds $-L_i^{*J}$ and 0. We get the description of such regimes in Figure 5 for the case of two banks.

With the new notations, the equilibrium conditions involve new matrices. Let us define the matrix $Q(z)$, whose $i^{th}$ column is the $i^{th}$ column of $\Pi$, if $z_i =$

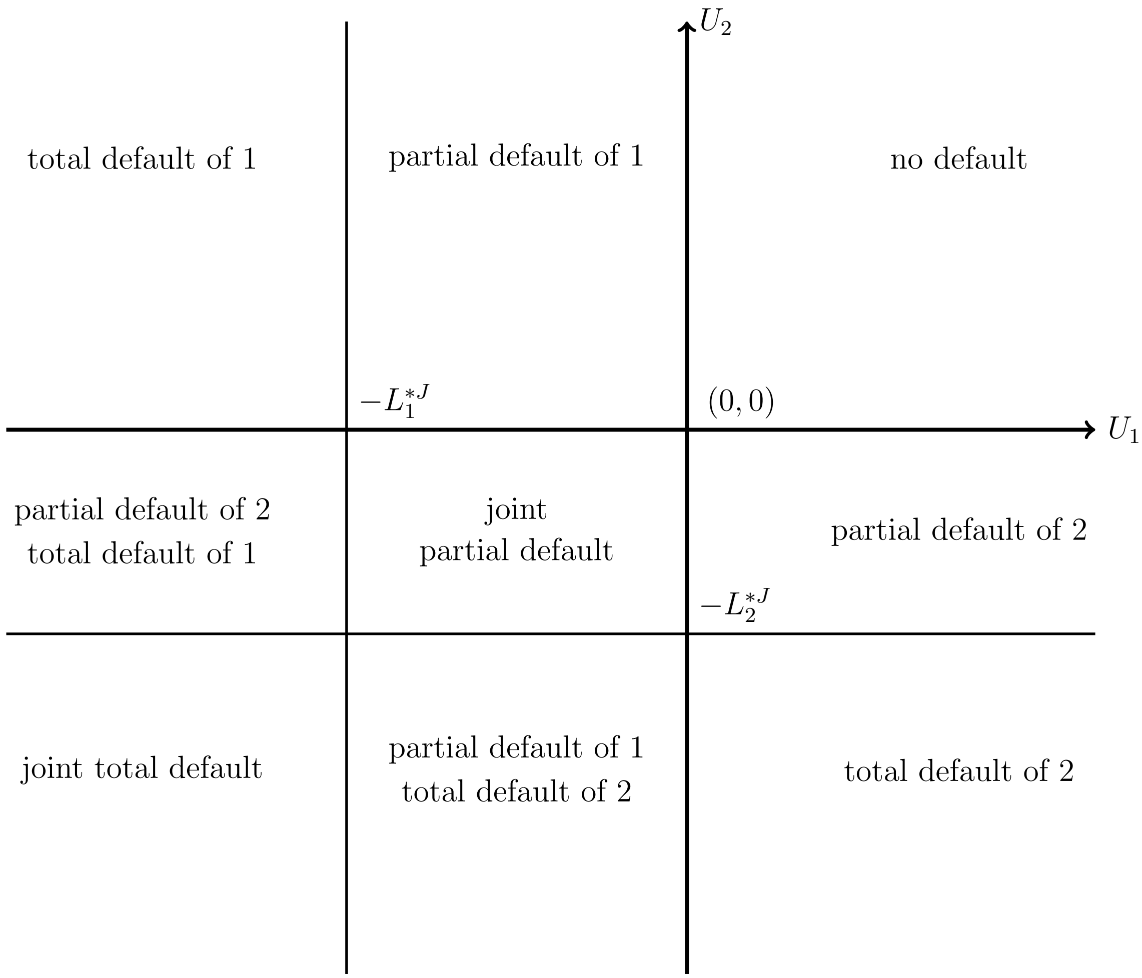


Figure 5: The Transformed Quadrilaterals

0, the $i^{th}$ column of $\Gamma^J$, if $z_i = 1$, and the $i^{th}$ column of $\Gamma^S$, otherwise. The equilibrium conditions on the space of values of $U = (U_1, ..., U_n)'$ become :

$$\Delta Ax = \big(Id - Q(z)\big) \begin{pmatrix} \vdots \\ U_i \\ \vdots \end{pmatrix} + (\Gamma^J - \Gamma^S) \begin{pmatrix} \vdots \\ z_{i2} L_i^{*J} \\ \vdots \end{pmatrix}. \tag{a.8}$$

The function of interest is a continuous piecewise linear function with $3^n$ regimes, corresponding to bounded or unbounded quadrilaterals, characterized by $2^n$ nodes.

Let us now derive sufficient conditions for the piecewise linear function defined in Section 2.4 to be a one-to-one mapping.

i) A necessary condition for the global invertibility of this function is its local invertibility at each node. In a neighborhood of a given node, we have $3^n$ regimes. The condition of local invertibility is the condition of identical sign of the determinants of the corresponding linear components of the application in each of the $3^n$ regimes.

ii) Moreover the border of one regime is exactly the border of the very adjacent regime. If there were an overlapping, there would be several equilibria. To avoid this situation we need the signs of the determinants to be the same for the $2^n$ nodes.

iii) Finally, we have to ensure that the unbounded polygons do not intersect.

Let us now discuss the different conditions.

Let us first establish the following Lemma.

**Lemma A.2:** Under the inequality conditions of Proposition 1, $det[Id - Q(z)] > 0, \forall z$.

Proof : By the assumptions in Proposition 1, the matrices $Q'(z)$ have nonnegative coefficients, which sum up to a value strictly smaller than 1 per row. By applying Perron-Froebenius theorem, we deduce that the eigenvalues of $Q'(z)$, which are also equal to the eigenvalues of $Q(z)$, have a modulus strictly smaller than 1. Therefore the eigenvalues of $Id - Q(z)$ are either complex conjugates, or real positive, and their product equal to $det[Id - Q(z)]$ is strictly positive.

QED

We can now apply Theorem 1 in Gourieroux, Laffont, Monfort (1980) The condition $det[Id - Q(z)] > 0, \forall z$ is exactly the condition ensuring the global invertibility of the piecewise linear function $(a.8)$ in a bounded region including all the nodes.

Thus, to finish the proof we have just to check that the unbounded polygons do not overlap, or equivalently that the piecewise linear mapping is locally invertible at "infinity". This condition can be derived by considering the projective geometry [see e.g. Bennett (1995)], which provides the transformation of the asymptotic directions associated with the piecewise linear function. Let us denote $\Delta Ax^{as}, Y_i^{as}, \Delta L_i^{S,as}$ the asymptotic directions associated with $\Delta Ax, Y_i, \Delta L_i^S$, respectively. (There is no asymptotic direction for the junior debt, which is bounded). In the projective space the piecewise linear transformation becomes :

$$\left\{ \begin{array}{l} (1 - z_i^*)Y_i^{as} + z_i^* \Delta L_i^{S,as} = 0, i = 1, \ldots, n. \\ \Delta Ax^{as} = [Id - Q^{as}(z^*)](z_i^* Y_i^{as} + (1 - z_i^*)\Delta L_i^{S,as}), \end{array} \right. \tag{a.9}$$

where $z_i^* = 0$, if $Y_i^{as} \neq 0, \Delta L_i^{S,as} = 0$, $z_i^* = 1$, if $Y_i^{as} = 0, \Delta L_i^{S,as} \neq 0$, and the matrix $Q^{as}(z^*)$ is such that its $i^{th}$ column is the $i^{th}$ column of $\Pi$, if $z_i^* = 0$, and the $i^{th}$ column of $\Gamma^S$, otherwise.

We get an asymptotic system, which involves only the bilateral exposures in stocks and senior debts. By reapplying Theorem 1 in Gourieroux, Laffont, Monfort (1980), the invertibility of projective system (a.9) is obtained if the $det[Id - Q^{as}(z^*)], \forall z^*$, have the same sign. This condition is satisfied if: $\sum_{i=1}^{n} \pi_{i,j} < 1, \sum_{i=1}^{n} \gamma_{i,j}^S < 1, j = 1, \ldots, n$ ; it is in particular satisfied under the inequality conditions of Proposition 1.

QED

## Appendix 3

### Computations of the $PD$ and $ELGD$ for tranches

We denote $\varepsilon_i = \sigma_i u_i$ where $u_i \sim \mathcal{N}(0,1)$. Let us first consider the senior debt and compute the associated probability of default and the expected loss given default.

$$\begin{aligned} PD_i^S &= \mathbb{Q}(Ax_i < L_i^{*S}) \\ &= \mathbb{Q}\big(log(Ax_i) < log(L_i^{*S})\big) \\ &= \mathbb{Q}\big(\beta_i F + \varepsilon_i < log(L_i^{*S})\big) \\ &= \Phi\Big(\frac{log(L_i^{*S}) - \beta_i F}{\sigma_i}\Big) \\ &= \Phi(\mathcal{A}_i). \end{aligned}$$

$$\begin{aligned} ELGD_i^S &= \mathbb{E}^{\mathbb{Q}}\left(1 - \frac{Ax_i}{L_i^{*S}}\middle| Ax_i < L_i^{*S}\right) \\ &= 1 - \mathbb{E}^{\mathbb{Q}}\left(\frac{Ax_i}{L_i^{*S}}\middle| Ax_i < L_i^{*S}\right) \\ &= 1 - \mathbb{E}^{\mathbb{Q}}\left(exp\left(log\left(Ax_i\right) - log\left(L_i^{*S}\right)\right)\middle| log\left(Ax_i\right) - log\left(L_i^{*S}\right) < 0\right) \\ &= 1 - \mathbb{E}^{\mathbb{Q}}\left(exp\left(\sigma_i u_i - \sigma_i \mathcal{A}_i\right)\middle| \sigma_i u_i - \sigma_i \mathcal{A}_i < 0\right) \\ &= 1 - exp\left(-\sigma_i \mathcal{A}_i\right)\mathbb{E}^{\mathbb{Q}}\left(exp\left(\sigma_i u_i\right)\middle| u_i - \mathcal{A}_i < 0\right) \\ &= 1 - \frac{exp\left(-\sigma_i \mathcal{A}_i\right)}{\Phi(\mathcal{A}_i)}\int_{-\infty}^{\mathcal{A}_i}\frac{1}{\sqrt{2\pi}}exp\left(\sigma_i u_i - \frac{1}{2}u_i^2\right)du_i \\ &= 1 - \frac{exp\left(-\sigma_i \mathcal{A}_i\right)}{\Phi(\mathcal{A}_i)}exp(\frac{1}{2}\sigma_i^2)\int_{-\infty}^{\mathcal{A}_i}\frac{1}{\sqrt{2\pi}}exp\left(-\frac{1}{2}(u_i - \sigma_i)^2\right)du_i \\ &= 1 - \frac{exp\left(-\sigma_i \mathcal{A}_i\right)}{\Phi(\mathcal{A}_i)}exp(\frac{1}{2}\sigma_i^2)\int_{-\infty}^{\mathcal{A}_i - \sigma_i}\frac{1}{\sqrt{2\pi}}exp\left(-\frac{1}{2}(u_i)^2\right)du_i \\ &= 1 - \frac{exp\left(-\sigma_i \mathcal{A}_i + \frac{1}{2}\sigma_i^2\right)}{\Phi\left(\mathcal{A}_i\right)}\Phi\left(\mathcal{A}_i - \sigma_i\right) \end{aligned}$$

Second, we derive the expression of the probability that the asset value are between the boundaries defining the junior tranche, and the corresponding expected loss given default.

$$\begin{aligned}
PD_i^{J,S} &= \mathbb{Q}(L_i^{*S} < Ax_i < L_i^{*S} + L_i^{*J}) \\
&= \mathbb{Q}\Big(log(L_i^{*S}) < log(Ax_i) < log(L_i^{*S} + L_i^{*J})\Big) \\
&= \mathbb{Q}\Big(log(L_i^{*S}) < \beta_i F + \varepsilon_i < log(L_i^{*S} + L_i^{*J})\Big) \\
&= \mathbb{Q}\big(\frac{log(L_i^{*S}) - \beta_i F}{\sigma_i} < u_i < \frac{log(L_i^{*S} + L_i^{*J}) - \beta_i F}{\sigma_i}\big) \\
&= \Phi\Big(\frac{log(L_i^{*S} + L_i^{*J}) - \beta_i F}{\sigma_i}\Big) - \Phi\Big(\frac{log(L_i^{*S}) - \beta_i F}{\sigma_i}\Big) \\
&= \Phi(\mathcal{B}_i) - \Phi(\mathcal{A}_i).
\end{aligned}$$

$$\begin{aligned}
ELGD_i^{J,S} &= \mathbb{E}^{\mathbb{Q}}\Big(1 - \frac{Ax_i - L_i^{*S}}{L_i^{*J}}\Big| L_i^{*S} < Ax_i < L_i^{*S} + L_i^{*J}\Big) \\
&= 1 - \mathbb{E}^{\mathbb{Q}}\left(\frac{exp\,(\beta_i F + \sigma_i u_i) - L_i^{*S}}{L_i^{*J}}\Big| L_i^{*S} < exp\,(\beta_i F + \sigma_i u_i) < L_i^{*S} + L_i^{*J}\right) \\
&= 1 + \frac{L_i^{*S}}{L_i^{*J}} - \frac{exp\,(\beta_i F)}{L_i^{*J}} \frac{\mathbb{E}^{\mathbb{Q}}\left(exp(\sigma_i u_i)\mathbb{1}_{\mathcal{A}_i < u_i < \mathcal{B}_i}\right)}{\Phi(\mathcal{B}_i) - \Phi(\mathcal{A}_i)} \\
&= 1 + \frac{L_i^{*S}}{L_i^{*J}} - \frac{exp\,(\beta_i F)}{L_i^{*J}} \frac{\mathbb{E}^{\mathbb{Q}}\left(exp(\sigma_i u_i)\mathbb{1}_{u_i < \mathcal{B}_i}\right) - \mathbb{E}^{\mathbb{Q}}\left(exp(\sigma_i u_i)\mathbb{1}_{u_i < \mathcal{A}_i}\right)}{\Phi(\mathcal{B}_i) - \Phi(\mathcal{A}_i)} \\
&= 1 + \frac{L_i^{*S}}{L_i^{*J}} - \frac{exp\,(\beta_i F)}{L_i^{*J}} \frac{exp\left(\frac{1}{2}\sigma_i^2\right)\Big(\Phi(\mathcal{B}_i - \sigma_i) - \Phi(\mathcal{A}_i - \sigma_i)\Big)}{\Phi(\mathcal{B}_i) - \Phi(\mathcal{A}_i)} \\
&= 1 + \frac{L_i^{*S}}{L_i^{*J}} - \frac{L_i^{*S}}{L_i^{*J}} exp\left(-\sigma_i \mathcal{A}_i + \frac{1}{2}\sigma_i^2\right) \frac{\Phi(\mathcal{B}_i - \sigma_i) - \Phi(\mathcal{A}_i - \sigma_i)}{\Phi(\mathcal{B}_i) - \Phi(\mathcal{A}_i)}
\end{aligned}$$

## Appendix 4

## Proof of Proposition 2

The result is deduced from Lemma 4 in Eisenberg, Noe (2001) by first concentrating with respect to the junior debts.

**i) Concentration with respect to the junior debts.**
Let us consider fixed values for the senior debts satisfying the constraints and let us optimize with respect to the junior debt only. The maximization problem becomes :

$$\begin{aligned}
\max_{L_i^J} \quad & \Big(\sum_{i=1}^n L_i^J\Big) \\
s.t. \quad & L_i^S \leq \sum_{j\neq i} \gamma_{i,j}^S L_j^S + \sum_{j\neq i} \gamma_{i,j}^J L_j^J + Ax_i, && i=1,...,n \\
& L_i^J \leq L_i^{*J}, && i=1,...,n \\
& L_i^J \leq \sum_{j\neq i} \gamma_{i,j}^S L_j^S + \sum_{j\neq i} \gamma_{i,j}^J L_j^J + Ax_i - L_i^S, && i=1,...,n
\end{aligned}$$

In this maximization problem, only the two last types of inequalities matter since the first type of inequalities is implied by the third type of inequalities. Then, by Lemma 4 in Eisenberg, Noe (2001), we know that the solutions $L_i^J$ satisfy :

$$L_i^J = min\Big(L_i^{*J} \quad ; \quad \sum_{j\neq i} \gamma_{i,j}^S L_j^S + \sum_{j\neq i} \gamma_{i,j}^J L_j^J + Ax_i - L_i^S\Big), \qquad i=1,...,n.$$

This $n-$dimensional system can be solved to express the argmax as piecewise linear functions of the levels of senior debts :

$$L_i^J(L^S, Ax, \Gamma^S, \Gamma^J), \text{ say.}$$

**ii) The concentrated optimization problem**
By concentration, we get :

$$\begin{aligned}
\max_{L_i^S} \quad & \Big(\sum_{i=1}^n L_i^S + \theta \sum_{i=1}^n L_i^J(L^S, Ax, \Gamma^S, \Gamma^J)\Big) \\
s.t. \quad & L_i^S \leq L_i^{*S}, && i=1,...,n \\
& L_i^S \leq \sum_{j\neq i} \gamma_{i,j}^S L_j^S + \sum_{j\neq i} \gamma_{i,j}^J L_j^J(L^S, Ax, \Gamma^S, \Gamma^J) + Ax_i, && i=1,...,n.
\end{aligned}$$

The criterion function is a piecewise linear functions in $L_i^S$, $i = 1, ..., n$. To apply the argument used in the proof of Lemma 4 in Eisenberg, Noe (2001), we have to ensure that this criterion function is increasing. In some regimes, some slopes coefficients of $L_j^S$ in the expression of $L_j^J(L^S, Ax, \Gamma^S, \Gamma^J)$ will be negative. But the presence of $L_j^S$ in the first part of criterion will balance this sign whenever $\theta$ is chosen sufficiently small. Under this condition, we deduce that at the optimum, we have :

$$L_i^S = min\Big(L_i^{*S} \quad ; \quad \sum_{j \neq i} \gamma_{i,j}^S L_j^S + \sum_{j \neq i} \gamma_{i,j}^J L_j^J + Ax_i\Big), \qquad i = 1, ..., n.$$

**iii) A system with a single bank**

To understand the need for the weighting scalar in the criterion function, let us consider the case of a single bank $n = 1$. The equilibrium conditions are :

$$\begin{cases} L^S = min(Ax, L^{*S}), \\ L^J = min(Ax - L^S, L^{*J}). \end{cases}$$

The optimization problem of Proposition 3 is :

$$\begin{array}{rl} \max\limits_{L^J, L^J} & L^S + \theta L^J \\ s.t. & L^S \leq Ax, \\ & L^S \leq L^{*S}, \\ & L^J \leq L^{*J}, \\ & L^J \leq Ax - L^S, \end{array}$$

with $\theta > 0$.

The optimization with respect to $L^J$ provides the solution:

$$L^J(L^S, Ax, L^{*S}, L^{*J}) = min(Ax - L^S, L^{*J}).$$

Then the concentrated optimization problem is:

$$\begin{array}{rl} \max\limits_{L^S} & L^S + \theta \; min(Ax - L^S, L^{*J}) \\ s.t. & L^S \leq Ax, \\ & L^S \leq L^{*S}. \end{array}$$

In this very simple case, we have just to choose a value of $\theta$ strictly smaller than 1 to ensure that the concentrated criterion is a strictly increasing function of $L^S$.