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Pricing Default Events : Surprise, Exogeneity and Contagion

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Pricing Default Events : Surprise, Exogeneity and Contagion*

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Abstract: In order to derive closed-form expressions of the prices of credit derivatives, the standard models for credit risk usually price the default intensities but not the default events themselves. The default indicator is replaced by an appropriate prediction and the prediction error, that is the default-event surprise, is neglected. Our paper develops an approach to get closed-form expressions for the prices of credit derivatives written on multiple names without neglecting default-event surprises. The approach differs from the standard one, since the default counts cause the factor process under the risk-neutral probability Q, even if this is not the case under the historical probability. This implies that the default intensities under Q do not exist. A numerical illustration shows the potential magnitude of the mispricing when the surprise on credit events is neglected. We also illustrate the effect of the propagation of defaults on the prices of credit derivatives.

Keywords: Credit Derivative, Default Event, Default Intensity, Frailty, Contagion, Mispricing.

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1 Introduction

Two alternative approaches are usually followed to price credit derivatives such as Credit Default Swaps (CDS). In the structural approach introduced by Merton the default of a corporation occurs when the asset side of the balance sheet becomes smaller than its liability side. Then the probability and price of default are deduced from the historical and risk-neutral properties of these two underlying variables. Another approach is the reduced-form or intensity approach, in which the underlying phenomena are not explicitly modelled and the historical default intensity, assumed to exist, is directly analyzed [see e.g. Duffie, Singleton (1999)]. This latter approach is easily implemented in the framework of factor models, when the default intensity and the stochastic discount factor are exponential affine functions of these factors and these factors feature an affine dynamics [see Duffie, Filipovic, Schachermayer (2003), Duffie (2005) in continuous time, Gourieroux, Monfort, Polimenis (2006) in discrete time]. Indeed the term structure of riskfree as well as risky interest rates admit closed-form expressions and are affine functions of these factors.

However in order to derive these closed-form expressions of interest rates and prices, the reduced-form approach usually prices the default intensity, but not the default indicator itself. In other words, the default indicator is replaced by an appropriate prediction. Thus the prediction error, that is the surprise on default event, is neglected.

There exist a few papers mentioning this approximation and trying to adjust for this practice [see e.g. Jarrow, Yu (2001), Bai, Collin-Dufresne, Goldstein, Helwege (2012) a, b]. However, pricing formulas have no longer closed forms and it seems much more difficult to account for default correlations. This explains why some of these analyses have considered the joint pricing of default for a small number of names, for instance two names in Jarrow, Yu (2001). This literature focuses on the type of default, either simultaneous, or recursive defaults, also called double default.

Our paper develops an approach that results in closed-form formula to price credit derivatives written on any number of names, without neglecting default-event surprises. In Section 2 we review the standard reduced form approach and its limitations. In particular we carefully discuss the link between the assumption that the default count process does not cause the factor process and the existence of a default intensity, both under the historical and risk-neutral probabilities. In Section 3, we consider a homogenous pool of credits and introduce a pricing model, with a joint Compound AutoRegressive (CaR) dynamics for the factor and the default count. When the s.d.f. m is exponential affine in both factor F and default count n, we get linear affine formulas for the term structures of riskfree and risky interest rates. The results are extended in Section 4 to account for a possible heterogeneity of the initial pool of credits. We consider that this pool can be partitioned into J homogenous segments. The model allows for a common systematic factor (e.g. dynamic frailty), and also for contagion phenomena, where a default-event surprise of segment j may have an impact on the prices of credit derivatives written on another segment. Section 5 provides numerical illustrations of our approach. It shows the potential magnitude of the mispricing when the surprise on credit events is neglected. We also illustrate the effect of the propagation of defaults on the prices of credit derivatives. Section 6 concludes. Proofs are gathered in appendices.

2 The standard reduced-form approach and its limitation

2.1 Basic assumptions

We consider a pool of I entities, indexed by $i = 1, \ldots, I$; these entities can be firms or credit contracts. We denote by $d_{i,t}$ the indicator of default of entity i, that is $d_{i,t} = 1$, if entity i is in default at time t (or before), and $d_{i,t} = 0$, otherwise.

We introduce the notations: $d_t = (d_{1,t}, \ldots, d_{I,t})'$, $\underline{d_t} = (d'_1, \ldots, d'_t)'$, $PaR_t = \{i | d_{i,t} =$

$0\}$, $n_t = \Sigma_{i \in PaR_t} d_{i,t}$, $N_t = \Sigma_{\tau=1}^{t} n_\tau$.

PaR_t is the Population-at-Risk that is the set of individuals still alive at date t, n_t is the number of defaults occurring at date t and N_t is the number of defaults at date t or before.

Let us first assume that the pool is homogenous and that the default dependence is driven by an exogenous (multivariate) factor F_t. Let us denote by $\Omega_t = (\underline{F_t}, \underline{d_t})$, $\Omega_t^* = (\underline{F_{t+1}}, \underline{d_t}) = (F_{t+1}, \Omega_t)$ the information sets, where $\underline{F_t} = \{F_\tau, \tau \leq t\}$. Thus we have nested filtrations satisfying $\Omega_t \subset \Omega_t^* \subset \Omega_{t+1}$. The assumptions, which will be relaxed in Subsection 2.5 and in Section 4, can be formalized in the following way:

Assumption A_0:

i) The variables $d_{i,t+1}, i = 1, \ldots, I$, are independent conditional on $\Omega_t^* = (F_{t+1}, \Omega_t)$, state 1 is an absorbing state, the variables $\{d_{i,t+1}, i \in PaR_t\}$ are identically distributed, conditional on[4] Ω_t^*, and this conditional distribution is only function of F_{t+1}.

ii) The conditional distribution of F_{t+1} given Ω_t is equal to the distribution of F_{t+1} given F_t.

Assumption A_0 ii) means that the process (d_t) does not cause the process (F_t), or equivalently that process (F_t) is exogenous, and that process (F_t) is Markov of order 1. The usual definition of noncausality introduced by Granger is characterized by the conditions:

$$f(F_{t+1}|\underline{F_t}, \underline{d_t}) = f(F_{t+1}|\underline{F_t}), \forall t, \tag{2.1}$$

where $f(.|.)$ denotes a conditional probability density function (p.d.f) [Granger (1980)]. It has been shown [see e.g. Florens, Mouchart (1982), and Gourieroux, Monfort (1989)

[4] Note that the population-at-risk PaR_t is measurable with respect to Ω_t.

Property 1.2 for a simpler proof] that these conditions are also equivalent to:

$$f(d_{t+1}|\underline{d_t}, \underline{F_T}) = f(d_{t+1}|\underline{d_t}, \underline{F_{t+1}}) = f(d_{t+1}|\Omega_t^*), \forall t, T, T \geq t. \tag{2.2}$$

This is the Sims' definition of noncausality [Sims (1972)].

Moreover, it is easily shown by projecting the noncausality condition (2.1) on the information $\underline{F_t}, \underline{n_t}$ that we have also:

$$f(F_{t+1}|\underline{F_t}, \underline{n_t}) = f(F_{t+1}|\underline{F_t}), \forall t,$$

that is, the count process (n_t) does not cause the process (F_t).

2.2 The standard pricing approach

Under the assumption of no arbitrage opportunity the credit derivatives can be priced by introducing stochastic discount factors (s.d.f.) from the issuing date t_0, say. The standard pricing approach assumes that the short term s.d.f. is specified as a function of the current factor value only, that is, the s.d.f. for period $(t, t+1)$ is of the type: $\tilde{m}_{t,t+1} = \tilde{m}(F_{t+1})$, say. Then the price at t_0 of the derivative written on the total number of default and paying $g(N_{t_0+h})$ at date $t_0 + h$ is:

$$\tilde{\Pi}(g, h) = E_{t_0}\left[\Pi_{k=1}^h \tilde{m}_{t_0+k-1,t_0+k} g(N_{t_0+h})\right] = E_{t_0}\left[\Pi_{k=1}^h \tilde{m}(F_{t_0+k}) g(N_{t_0+h})\right], \tag{2.3}$$

where E_{t_0} is the conditional expectation given $\Omega_{t_0} = (\underline{F_{t_0}}, d_{t_0} = 0)$, since all the credits are alive at the issuing of the pool.

By applying the iterated expectation theorem, we get:

$$\tilde{\Pi}(g, h) = E_{t_0}\left[\Pi_{k=1}^h \tilde{m}(F_{t_0+k}) E(g(N_{t_0+h})|\underline{F_{t_0+h}})\right] = \tilde{\Pi}(\tilde{g}, h), \tag{2.4}$$

where $\tilde{g}(\underline{F_{t_0+h}}) = E[g(N_{t_0+h})|\underline{F_{t_0+h}}]$.

In other words it is equivalent to price the derivative proposed on the market with payoff $g(N_{t_0+h})$ written on the cumulated number of defaults , or to price the derivative with payoff $\tilde{g}(\underline{F_{t_0+h}})$ written on the factor history. Thus the choice of a s.d.f. that is function of the latent factor only greatly simplifies the derivation of closed-form formulas for the prices of credit derivatives [see e.g. Lando (1998), Duffie, Singleton (1999), Duffie, Garleanu (2001) for pricing in continuous time, Gourieroux, Monfort, Polimenis (2006) for pricing in discrete time].

2.3 Risk premia associated with default events.

The standard practice described above may induce mispricing, since the default events themselves have not been included in the s.d.f.. Let us now consider a short term s.d.f. depending on both F_{t+1} and n_{t+1}:

$$m_{t,t+1} = m(F_{t+1}, n_{t+1}). \tag{2.5}$$

The pricing formula becomes:

$$\Pi(g,h) = E_{t_0}\left\{\Pi_{k=1}^{h} m(F_{t_{0+k}}, n_{t_0+k}) g(N_{t_0+h})\right\}. \tag{2.6}$$

Since the pricing operator is linear, we get:

$$\begin{aligned}\Pi(g,h) &= E_{t_0}\left[\Pi_{k=1}^{h} m(F_{t_0+k}, n_{t_0+k})\tilde{g}(\underline{F_{t_0+h}})\right] \\ &+ E_{t_0}\left\{\Pi_{k=1}^{h} m(F_{t_0+k}, n_{t_0+k})[g(N_{t_0+h}) - \tilde{g}(\underline{F_{t_0+h}})]\right\} \\ &= E_{t_0}\left\{E[\Pi_{k=1}^{h} m(F_{t_0+k}, n_{t_0+k})|\underline{F_{t_0+h}}]\tilde{g}(\underline{F_{t_0+h}})\right\} \\ &+ E_{t_0}\left\{\Pi_{k=1}^{h} m(F_{t_0+k}, n_{t_0+k})[g(N_{t_0+h}) - \tilde{g}(\underline{F_{t_0+h}})]\right\}. \end{aligned} \tag{2.7}$$

By the iterated expectation theorem and by using the Sims' version (2.2) of the noncausality assumption A_0 ii), we get (see Appendix 1):

$$E\left[\Pi_{k=1}^{h} m(F_{t_0+k}, n_{t_0+k})|\underline{F_{t_0+h}}\right] = \Pi_{k=1}^{h}\tilde{m}(F_{t_0+k}), \tag{2.8}$$

with:

$$\tilde{m}(F_{t+1}) = E[m(F_{t+1}, n_{t+1})|F_{t+1}]. \tag{2.9}$$

Let us now interpret equations (2.8)-(2.9). The pricer $\tilde{\Pi}$ based on factor values only is obtained by considering the expectation $\tilde{m}(F_{t+1})$ of the s.d.f. of different maturities conditional on factor values. From (2.8)-(2.9), we see that the projection of the s.d.f. for maturity h is the product of the short term projections. This feature is needed for these approximated s.d.f.'s to be compatible with no dynamic arbitrage opportunity for an investor using information $(\underline{F_t})$ in his portfolio updating and interested in pricing derivatives written on the factor process. However, even if the "projected" s.d.f.'s are coherent, they differ from the initial s.d.f., and this implies mispricing for derivatives written on default counts. Let us discuss this mispricing. From (2.7), we get:

$$\Pi(g,h) = \Pi(\tilde{g},h) + \Pi(g-\tilde{g},h),$$

with $g-\tilde{g} = g(N_{t_0+h}) - E[g(N_{t_0+h})|\underline{F_{t_0+h}}]$. Then, by applying (2.8)-(2.9):

$$\Pi(g,h) = \tilde{\Pi}(\tilde{g},h) + \Pi(g-\tilde{g},h) = \tilde{\Pi}(g,h) + \Pi(g-\tilde{g},h). \tag{2.10}$$

The true price is the standard one based on the projected s.d.f. plus an adjustment term. This adjustment term $\Pi(g-\tilde{g},h)$ is the price of the *surprise* on default events: $g(N_{t_0+h}) - E[g(N_{t_0+h})|\underline{F_{t_0+h}}]$. When the s.d.f.:

$$m(F_{t+1}, n_{t+1}) \equiv m^{*}[F_{t+1}, n_{t+1} - E(n_{t+1}|F_{t+1})],$$

does not depend on the default-event surprise of date $t+1$, that is on $n_{t+1} - E(n_{t+1}|F_{t+1})$, this adjustment term vanishes. Otherwise, there is a risk premium for the surprise and a need for price adjustment.

2.4 Risk-neutral dynamics, default intensities and non causality

The individual point processes $d_i = (d_{i,t}), i = 1, \ldots, I$, are independent conditional on the factor process (F_t) (see Appendix 1). Let us now consider the transition of such a point process conditional on (F_t). Using Sims' noncausality condition (2.2), we know that $P(d_{t+1}|\underline{d_t}, (F_t)) = P(d_{t+1}|\Omega_t^*)$. Moreover, we have:

$$P(d_{i,t+1} = 0|d_{i,t} = 1, \Omega_t^*) = 0,$$

since state 1 is absorbing, and we can denote:

$$P(d_{i,t+1} = 0|d_{i,t} = 0, \Omega_t^*) = \exp(-\lambda_{t+1}),$$

where $\lambda_{t+1} = \lambda(F_{t+1})$ does not depend on i.

Definition 1: The point process d_i has an intensity μ_t with respect to the filtration (Ω_t^*) if and only if [see e.g. Duffie, Garleanu (2001)]:

i) μ_{t+1} is Ω_t^* measurable,

ii) $P[\tau_i > t + h|\tau_i > t, \Omega_t^*] = E[\Pi_{k=1}^h \exp(-\mu_{t+k})|\Omega_t^*], \forall t, h$, where $\tau_i = \inf\{t : d_{i,t} = 1\}$ is the lifetime of entity i.

Proposition 1: Under Assumption A_0 and under the historical probability each point process d_i admits a default intensity λ_t conditional on factor process (F_t) with respect to the filtration (Ω_t^*).

Proof: In our framework, we have:

$$
\begin{aligned}
& P[\tau_i > t+h | \tau_i > t, \Omega_t^*] \\
= \ & E\{P[\tau_i > t+h | \tau_i > t, \Omega_t^*, \underline{F_{t+h}}] | \tau_i > t, \Omega_t^*\} \\
= \ & E\{\Pi_{k=1}^h P[\tau_i > t+k | \tau_i > t+k-1, \Omega_t^*, \underline{F_{t+h}}] | \tau_i > t, \Omega_t^*\} \\
= \ & E\{\Pi_{k=1}^h P[\tau_i > t+k | \tau_i > t+k-1, \Omega_t^*, \underline{F_{t+k}}] | \tau_i > t, \Omega_t^*\} \\
& \text{(by using the argument in Appendix 1)} \\
= \ & E\{\Pi_{k=1}^h P[d_{i,t+k} = 0 | d_{i,t+k-1} = 0, \Omega_t^*, \underline{F_{t+k}}] | \tau_i > t, \Omega_t^*\} \\
= \ & E[\Pi_{k=1}^h \exp(-\lambda_{t+k}) | d_{i,t} = 0, \Omega_t^*].
\end{aligned}
$$

Thus the point process d_i admits the intensity λ_t.$\square$

If λ_{t+1} is small we have approximately: $P(d_{i,t+1} = 1 | d_{i,t} = 0, \Omega_t^*) \simeq \lambda_{t+1}$. This condition ($\lambda_{t+1}$ is small) is usually satisfied if the time unit is small, that is when the discrete time approach tends to a continuous time approach. The previous approximation provides the usual interpretation of λ_{t+1} as the (continuous time) default intensity.

Let us now consider the dynamics of the individual point processes under the risk-neutral distribution. For this purpose, let us assume that the s.d.f. $m_{t,t+1}$ is of the form[5]:

$$m_{t,t+1} = \exp(\delta_0 + \delta_F' F_{t+1} + \delta_S n_{t+1}). \tag{2.11}$$

Proposition 2: Under the risk-neutral probability,

i) the point processes d_i, $i = 1, \ldots, I$, are still independent conditional on process (F_t);

ii) State 1 is still absorbing;

[5] The results that follow (Propositions 2 to 5) remain valid when δ_0 is replaced by $-r(F_t) - \Psi(\delta_F, \delta_S)$, where r is a function of F_t, that defines the riskfree short-term rate, and Ψ denotes the conditional log-Laplace transform of (F_t, n_t).

iii) We have: $Q[d_{i,t+1} = 0 | d_{i,t} = 0, \Omega_t^*] \equiv \exp(-\lambda_{t+1}^Q)$, where:

$$\lambda_{t+1}^Q = \lambda_{t+1} + \log\{\exp(-\lambda_{t+1}) + [1 - \exp(-\lambda_{t+1})]\exp(\delta_S)\}.$$

In particular, if $\delta_S = 0$, $\lambda_{t+1}^Q = \lambda_{t+1}$.

Proof: See Appendix 2.

Since δ_S is expected to be nonnegative we have $\lambda_{t+1}^Q \geq \lambda_{t+1}$ $\forall t$. Moreover, $\lambda_{t+1}^Q = \lambda_t$ $\forall t$ if and only if $\delta_S = 0$ [see also Monfort, Renne (2013)]. If λ_{t+1} is small, it is easily checked that $\lambda_{t+1}^Q \simeq \lambda_{t+1}\exp(\delta_S)$.

Proposition 3: The risk-neutral p.d.f. of F_{t+1} given Ω_t is proportional to:

$$f_t^P(F_{t+1}|\Omega_t)\exp(\delta_F' F_{t+1})[\sum_{d_{i,t+1}=0}^{1} f^P(d_{i,t+1}|F_{t+1}, d_{i,t} = 0, n_t)\exp(\delta_S d_{i,t+1})]^{(I-N_t)},$$

where $f_t^P(F_{t+1}|\Omega_t)$ denotes the historical conditional p.d.f. of F_{t+1} given Ω_t.

Proof: see Appendix 2.

Under the risk-neutral probability the distribution of F_{t+1} given Ω_t depends not only on F_t through $f_t^P(F_{t+1}|\Omega_t)$, but also on cumulated default count N_t. Therefore the sequence of counts (n_t) will generally Granger cause the factor in the risk-neutral world. However, when $\delta_S = 0$, the sum appearing in the previous formula is equal to 1 and the dependency on N_t disappears. This discussion is summarized below.

Corollary 1: Under assumption A_0 and the exponential affine specification (2.11) of the s.d.f., the default count process (n_t) does not Q-cause the factor process (F_t) if and only if $\delta_S = 0$, that is, if the default-event surprise is not priced.

Proposition 4: If $\delta_S \neq 0$, a default intensity does not exist in the risk-neutral world.

Proof: Putting $h = 1$ in Definition 1, we see that, if an intensity process exists under Q, it is necessarily equal to process λ_t^Q. Let us now consider the quantity $Q[\tau_i > t+h|\tau_i > t, \Omega_t^*]$ and show that it cannot be equal to $E^Q[\Pi_{k=1}^h \exp(-\lambda_{t+k}^Q)|\Omega_t^*]$. Indeed, since process (F_t) is no longer exogenous, we cannot replace $Q[\tau_i > t+k|\tau_i > t+k-1, \Omega_t^*, \underline{F_{t+h}}]$ by $Q[\tau_i > t+k|\tau_i > t+k-1, \Omega_t^*, \underline{F_{t+k}}]$ in the analogue of the proof of Proposition 1.□

The previous proposition has important consequences in terms of pricing of defaultable bonds. The price at date t of a defaultable bond with zero recovery rate and time-to-maturity h is:

$$B(t,h) = E_t^Q[\exp(-r_t - \ldots - r_{t+h-1})\mathbb{I}_{(d_{i,t+h}=0)}],$$

where r_t is the riskfree rate between t and $t+1$, equal to $-\log E[m_{t,t+1}|\Omega_t]$. It is easily seen, using assumption A_0 and conditioning first on Ω_t^*, that r_t is function of F_t only.

Proposition 5: If $\delta_S = 0$, we have:

$$B(t,h) = E_t^Q[\exp(-r_t - \ldots - r_{t+h-1} - \lambda_{t+1}^Q - \ldots - \lambda_{t+h}^Q)],$$

with $\lambda_t^Q = \lambda_t$.

If $\delta_S \neq 0$, the previous formula is no longer valid. It could be replaced by the following formula:

$$B(t,h) = E_t^Q[\exp(-r_t - \ldots - r_{t+h-1} - \tilde{\lambda}_{t+1,t+h}^Q - \ldots - \tilde{\lambda}_{t+h,t+h}^Q)],$$

where $\tilde{\lambda}_{t+1,t+h}^Q$ is defined by: $Q(d_{t+1} = 0|d_t = 0, \underline{F_{t+h}}) = \exp(-\tilde{\lambda}_{t+1,t+h}^Q)$. $\tilde{\lambda}_{t+1,t+h}^Q$ is doubly indexed and function of $\underline{F_{t+h}}$, and thus it is not an intensity process. It can be seen as a "forward" intensity.

Proof: We have:

$$\begin{aligned}
B(t,h) &= E^Q[\exp(-r_t - \ldots - r_{t+h-1})\mathbb{I}_{(d_{i,t+h}=0)}|\Omega_t, d_{i,t}=0] \\
&= E^Q[E^Q[\exp(-r_t - \ldots - r_{t+h-1})\mathbb{I}_{(d_{i,t+h}=0)}|\Omega_t, \underline{F_{t+h}}, d_{i,t}=0]|\Omega_t, d_{i,t}=0] \\
&= E^Q[\exp(-r_t - \ldots - r_{t+h-1})E^Q[\mathbb{I}_{(d_{i,t+h}=0)}|\Omega_t, \underline{F_{t+h}}, d_{i,t}=0]|\Omega_t, d_{i,t}=0] \\
&\quad \text{(using assumption } A_0 \text{ and the fact that } r_t \text{ is function of } F_t) \\
&= E^Q[\exp(-r_t - \ldots - r_{t+h-1}) \\
\times &\quad E^Q\{\Pi_{k=1}^h Q[\tau_i > t+k|\tau_i > t+k-1, \underline{F_{t+h}}]|\tau_i > t, \Omega_t\}]|\Omega_t, d_{i,t}=0].
\end{aligned}$$

If $\delta_S = 0$, the factor process (F_t) remains exogenous in the risk-neutral world and $Q[\tau_i > t+k|\tau_i > t+k-1, \underline{F_{t+h}}]$ can be replaced by $Q[\tau_i > t+k|\tau_i > t+k-1, \underline{F_{t+k}}]$, which is also equal to $\exp(-\lambda^Q_{t+k})$ and to $\exp(-\lambda_{t+k})$ (Proposition 2 iii)). If $\delta_S \neq 0$, the expression $Q[\tau_i > t+k|\tau_i > t+k-1, \underline{F_{t+h}}]$ is equal to $\exp(-\tilde{\lambda}^Q_{t+k,t+h})$ and the result follows.$\square$

Proposition 4 shows that an intensity can exist in the historical world without existing in the risk-neutral world (compare with Artzner, Delbaen (1955) Appendix 1, p 2759). Besides, Proposition 5 shows that by assuming the existence of an intensity in the risk-neutral world we implicitly do not price the surprise events. Nevertheless this assumption is usually done in the literature [see e.g. Lando (1998), Duffee (1999), Duffie (2005), Duffie, Singleton (1999), Jarrow, Yu (2001), Jarrow, Lando, Yu (2005)].

2.5 Relaxing the noncausality assumption

Let us now discuss how the results above are modified if we relax the noncausality assumption A_0 ii) under the historical distribution, that is, if we consider the new set of assumptions:

$$A_0^* = A_0 i) + A_0^* ii),$$

where:

Assumption A_0^* **ii)**: The conditional historical distribution of F_{t+1} given Ω_t is equal to the distribution of F_{t+1} given F_t, n_t. Thus this conditional distribution can also depend on n_t.

When the noncausality condition is not satisfied, the point processes d_i, $i = 1, \ldots, I$, no longer admit intensities under the historical probability (compare with Proposition 1). The decomposition (2.10) of the derivative price has to be modified. Indeed when the noncausality of process (d_i), that is Assumption A_0 ii), is no longer satisfied, we do not have equality (2.8) and $\Pi(\tilde{g}, h)$ becomes different from $\tilde{\Pi}(\tilde{g}, h)$. In this case we have the following decomposition of $\Pi(g, h)$:

$$\Pi(g, h) = \tilde{\Pi}(g, h) + [\Pi(\tilde{g}, h) - \tilde{\Pi}(\tilde{g}, h)] + \Pi(g - \tilde{g}, h). \tag{2.12}$$

The additional term between brackets is an adjustment term for causality of the count process and we have the decomposition:

Price=Standard Price + Causality Adjustment + Surprise Adjustment.

Finally let us discuss the expressions of the riskfree rate for the an s.d.f. proportional to $\exp(\delta_F' F_{t+1} + \delta_S n_{t+1})$. Under Assumption A_0, the riskfree rate at term h is a function of F_t and depends on the risk premium δ_S on default count. We have:

$$r_f(t, t+h) \equiv r(h, F_t, \delta_S).$$

Under Assumption A_0^*, the riskfree rate also depends on n_t. We have:

$$r_f(t, t+h) \equiv r(h, F_t, n_t, \delta_S).$$

We note two different effects of pricing the surprise. First the risk premium δ_S is always a component of the riskfree rate. Second, if the factor process is not exogenous, we observe "jumps" in the riskfree rate when default occurs in the sense that the formula valid in absence of default $r(h, F_t, 0, \delta_S)$ is replaced by $r(h, F_t, n_t, S_S)$; moreover the magnitude of these jumps depends on the number of defaults.

3 Homogenous Pool

3.1 The dynamic Poisson model

Let us illustrate the discussion above by considering a Poisson regression model for the default counts with the exogenous factors as explanatory variables [see Cameron, Trivedi (1989) for Poisson regression models]. Moreover let us consider factors which follow a compound autoregressive (CaR) dynamic [Darolles, Gourieroux, Jasiak (2006)].

Assumption A.1: i) The conditional distribution of n_{t+1} given $\underline{F_{t+1}}, \underline{n_t}$ is Poisson $\mathcal{P}(\beta' F_{t+1} + \gamma)$.

ii) The conditional Laplace transform of F_{t+1} given $\underline{F_t}$ is exponential affine in F_t:

$$E_t[\exp(v' F_{t+1})] = E[\exp(v' F_{t+1}) | \underline{F_t}, \underline{n_t}] \equiv \exp[A(1, v)' F_t + B(1, v)], \text{ for any } v \in \mathcal{V},$$

where $\mathcal{V}$ is the set of arguments v for which the Laplace transform exists and functions $A(1, .)$, $B(1, .)$ characterize the dynamics of factor F.[6]

The conditional Poisson model defined in Assumption A.1 i) is simply the aggregate of a microscopic model in which the individual defaults are independent conditional on the factor path. The conditional individual default probabilities are the same for all alive entities and are equal to $\beta' F_{t+1} + \gamma$ divided by the number of alive entities $I - N_t$. This is the doubly stochastic model or model with stochastic intensity [see Cox (1955)] written under its macroscopic version. In this respect the model extends the model

[6] The factors are often assumed nonnegative as well as the components of β and parameter γ to ensure the positivity of the default intensity. In this case $\mathcal{V} \supset (-\infty, 0)^L$, where $L = \; dim \; F_t$.

considered in Collin-Dufresne, Goldstein, Helwege (2010), in which the common frailty $F_t \equiv S$ is assumed time independent. As seen below the introduction of a dynamic frailty is needed to get an appropriate dynamic treatment of the information available to investors. Indeed, even if the dynamic frailty is observed up to time t by the investor, the investor will not know perfectly its future values; this creates a dependence between the future individual defaults, jump in the default intensities when a default occurs, and this dependence changes with the prediction horizon. This specification is very flexible to manage the term structure of default dependence.[7]

For a CaR process, the Laplace transform of the cumulated process is also exponential affine at any prediction horizon h, and we can write:

$$E_t[\exp(v' \sum_{k=1}^{h} F_{t+k})] = \exp[A(h,v)' F_t + B(h,v)],$$

where functions $A(h,v), B(h,v)$ are defined recursively (see Appendix 3).

Proposition 6: Under Assumption A.1 the joint process (F_t, n_t) is jointly compound autoregressive and, for any horizon h, we can write:

$$E_t[\exp(u'_F \sum_{k=1}^{h} F_{t+k} + u_S \sum_{k=1}^{h} n_{t+k})] = \exp[a'_F(h, u_F, u_S)' F_t + b(h, u_F, u_S)],$$

where u_F, u_S are the arguments of the Laplace transform and functions a_F and b are given by:

$$\begin{aligned} a_F(h, u_F, u_S) &= A[h, u_F + \beta(\exp u_S - 1)] \\ b(h, u_F, u_S) &= B[h, u_F + \beta(\exp u_S - 1)] + h\gamma(\exp u_S - 1). \end{aligned}$$

[7] To keep a nondenegerate default dependence, Collin-Dufresne, Goldstein, Helwege (2008) assumed that the common static frailty S is not observed by the investor. However, when time goes on, the investor updates in a Bayesian way his knowledge about S, which becomes known after a sufficiently long time. In our framework the investor cannot get the asymptotic knowledge of the future frailty values, since the frailty receives independent shocks at any future date.

Proof: We get:

$$
\begin{aligned}
& E_t[\exp(u_F' \sum_{k=1}^{h} F_{t+k} + u_S \sum_{k=1}^{h} n_{t+k})] \\
= \quad & E_t\{\exp(u_F' \sum_{k=1}^{h} F_{t+k}) E_t[\exp(u_S \sum_{k=1}^{h} n_{t+k} | \underline{F_{t+h}}]\} \text{ (by iterated expectation)} \\
= \quad & E_t\{\exp(u_F' \sum_{k=1}^{h} F_{t+k}) \Pi_{k=1}^{h} E[\exp(u_S n_{t+k}] | F_{t+k})\} \text{ (by Assumption A.1)} \\
= \quad & E_t\{\exp(u_F' \sum_{k=1}^{h} F_{t+k}) \Pi_{k=1}^{h} \exp[(\beta' F_{t+k} + \gamma)(\exp u_S - 1)]\} \text{ (by using the} \\
& \text{expression of the Laplace transform of a Poisson variable)} \\
= \quad & E_t\{\exp([u_F + \beta(\exp u_S - 1)]' \sum_{k=1}^{h} F_{t+k})\} \exp|h\gamma(\exp u_S - 1)] \\
= \quad & \exp\{A[h, u_F + \beta(\exp u_S - 1)]' F_t + B[h, u_F + \beta(\exp u_s - 1)] + h\gamma(\exp u_S - 1)\}.
\end{aligned}
$$

This is an exponential affine function of F_t. This proves that the process (F_t, n_t) is jointly affine and the expressions of a_F and b follow.□

Let us now compare the different pricing formulas, when the s.d.f. is exponential affine in both the factor and the default count:

$$
m_{t,t+1} = \exp(\delta_0 + \delta_F' F_{t+1} + \delta_S n_{t+1}). \tag{3.1}
$$

The price of the payoff $\exp(uN_{t_0+h}) = \exp(u \sum_{k=1}^{h} n_{t_0+k}) \equiv N(u)_{t_0+h}$ (say) is given by:

$$
\begin{aligned}
\Pi(N(u), h) \quad = \quad & E_{t_0}[\Pi_{k=1}^{h} m_{t_0+k-1,t_0+k} \exp(u \sum_{k=1}^{h} n_{t_0+k})] \\
= \quad & E_{t_0}\{\exp[h\delta_0 + \delta_F' \sum_{k=1}^{h} F_{t_0+k} + (\delta_S + u) \sum_{k=1}^{h} n_{t_0+k}]\} \\
= \quad & \exp\{A[h, \delta_F + \beta(\exp(\delta_S + u) - 1]' F_{t_0} + B[h, \delta_F + \beta(\exp(\delta_S + u) - 1)] \\
+ \quad & h[\delta_0 + \gamma[\exp(\delta_S + u) - 1]]\}.
\end{aligned} \tag{3.2}
$$

When the payoff is replaced by its expectation given $\underline{F_{t_0+h}}$, we get:

$$\begin{aligned}\tilde{N}(u)_{t_0+k} \equiv E[\exp(uN_{t_0+k})|\underline{F_{t_0+k}}] &= \Pi_{k=1}^{h} E[\exp(un_{t_0+k})|F_{t_0+k}] \\ &= \exp[\beta' \sum_{k=1}^{h} F_{t_0+k}(\exp u - 1) + h\gamma(\exp u - 1)].\end{aligned}$$

The price of this expected payoff is:

$$\begin{aligned}\Pi(\tilde{N}(u), h) &= E_{t_0}\{\Pi_{k=1}^{h} m_{t_0+k-1,t_0+k} E_{t_0}[\exp(uN_{t_0+h})|\underline{F_{t_0+k}}]\} \\ &= E_{t_0}\{\exp[h\delta_0 + \delta_F' \sum_{k=1}^{h} F_{t_0+k} + \delta_S \sum_{k=1}^{h} n_{t_0+k} \\ &+ \beta' \sum_{k=1}^{h} F_{t_0+k}(\exp u - 1) + h\gamma(\exp u - 1)]\} \\ &= \exp\{A[h, \delta_F + \beta(\exp u - 1) + \beta(\exp \delta_S - 1)]' F_{t_0} \\ &+ B[h, \delta_F + \beta(\exp u - 1) + \beta(\exp \delta_S - 1)] \\ &+ h[\delta_0 + \gamma(\exp u - 1) + \gamma(\exp \delta_S - 1)]\}. \qquad (3.3)\end{aligned}$$

Let us finally consider how the pricing formulas are modified when the s.d.f. depends on the factor only such that:

$$\begin{aligned}\tilde{m}_{t,t+1} &= E[\exp(\delta_0 + \delta_F' F_{t+1} + \delta_S n_{t+1})|F_{t+1}] \\ &= \exp\{\delta_0 + \gamma(\exp \delta_S - 1) + [\delta_F + \beta(\exp \delta_S - 1)]' F_{t+1}\}. \qquad (3.4)\end{aligned}$$

We easily derive the price of $N(u)_{t_0+h} = \exp(uN_{t_0+h})$ and of its expectation given $\underline{F_{t_0+h}}$ based on this modified s.d.f. We get:

$$\tilde{\Pi}(N(u), h) = \tilde{\Pi}(\tilde{N}(u), h) = \Pi(\tilde{N}(u), h). \qquad (3.5)$$

We deduce the following proposition:

Proposition 7: Under Assumption $A.1$, the term structures of the prices given in (3.2), (3.3) and (3.5) are exponential affine in F_t. The factor sensitivities are all based on the $A(h,.)$ function and derived by changing the argument u, and the risk sensitivity coefficients associated with the factor and default count, that are δ_F and δ_S, respectively, according to the derivative to be priced.

In particular for $u = 0$, we get the term structure of the riskfree zero-coupon prices:

$$\begin{aligned} B_f(t_0,h) &= \Pi(1,h) = \tilde{\Pi}(1,h) \\ &= \exp\{A[h, \delta_F + \beta(\exp\delta_S - 1)]'F_{t_0} + B[h, \delta_F + \beta(\exp\delta_S - 1)] \\ &+ h[\delta_0 + \gamma(\exp\delta_S - 1)]\}. \end{aligned}$$

3.2 Pricing individual and joint defaults

We have derived above the closed-form expression of the price of an exponential transformation of the default count: $\Pi(N(u),h)$, with $N(u) = \exp(uN)$. It is known that the price of any derivative written on N_{t_0+h} can be deduced from the prices of the derivatives with exponential payoff [see Duffie, Pan, Singleton (2000)]. Let us now explain how the pricing formula (3.2) can be used to deduce a closed-form expression for the price of the joint default of K individual contracts, that is $\Pi(d_1 \ldots d_K, h)$. We have the following result proved in Appendix 4:

Lemma 1: If $N = d_1 + \ldots + d_I$ and the indicator variables $d_i, i = 1, \ldots, I$, are exchangeable,

$$E(d_1 \ldots d_K) = \frac{E[N(N-1)\ldots(N-K+1)]}{I(I-1)\ldots(I-K+1)}, \text{ for } K \leq I.$$

This lemma can be applied to the forward-neutral probability to get the similar relationship written in term of prices.

Corollary 2: $\Pi(d_1 \ldots d_K, h) = \dfrac{\Pi[N(N-1)\ldots(N-K+1), h]}{I(I-1)\ldots(I-K+1)}$.

Moreover the price of $g(N) = N(N-1)\dots(N-K+1)$ can be deduced from the prices of exponential transforms of N.

Corollary 3: $\Pi(d_1 \dots d_K, h) = \dfrac{1}{I(I-1)\dots(I-K+1)} \left(\dfrac{d^K}{dv^K} \Pi[\exp(N \log(v)], h] \right)_{v=1}$.

The standard approaches for credit derivative pricing assume the existence of intensities under the risk-neutral probability to derive closed-form expressions of the derivative prices. This approach cannot be applied when the factor process is not exogenous under Q. Corollary 3 explains how to deal with this difficulty when the pool is homogenous with a sufficiently large size. We first derive the price of exponential functions of default counts, which admit closed-form expressions [see pricing formula (3.2)]. Then the prices of individual and joint defaults are deduced by an appropriate differentiation.

4 Heterogenous pool

The approach of Section 3 can be extended to a heterogenous pool composed of J homogenous segments with different risk characteristics. This extension allows to disentangle the effect on price of the common factor and the effect of contagion. It is also appropriate for the analysis of the default correlations within and between segments, both under the historical and risk-neutral probabilities. We will also relax the noncausality assumption of the count process for factor process under the historical distribution.

4.1 The model

Let us consider a pool which is segmented into J segments of initial size $I_j, j = 1, \dots, J$. We denote by $d_{i,j,t}$ the default indicator at date t of the individual i belonging to segment j, $i = 1, \dots, I_j$, $j = 1, \dots, J$, by $n_{j,t}, j = 1, \dots, J$, the default counts by segment and

$n_t = (n_{1,t}, \ldots, n_{J,t})'$. We assume that a given individual belongs to the same segment at all dates. For instance for corporations the segment can be defined by the industrial sector, by the size, by the domicile country, but cannot be the rating, which is varying in time. In other words, there is no transition between segments over time.

The extension of the model introduced in Section 3.1 is given below.

Assumption A.2: Model for heterogenous pool.

i) Conditional on $\Omega_t^* = (F_{t+1}, \Omega_t)$ the counts $n_{j,t+1}, j = 1, \ldots, J$ are independent with Poisson distributions: $n_{j,t+1} \sim \mathcal{P}(\beta_j' F_{t+1} + c_j' n_t + \gamma_j), j = 1, \ldots, J$.

ii) The conditional Laplace transform of F_{t+1} given Ω_t is exponential affine in F_t, n_t:

$$E_t[\exp(v' F_{t+1}) | \underline{F_t}, \underline{n_t}] = \exp[A_F(1, v)' F_t + A_S(1, v)' n_t + B(1, v)], \text{ for any } v \in \mathcal{V}.$$

Thus the conditional distribution of future default counts depends on both a dynamic frailty component and lagged default counts. This approach extends the specifications with dynamic frailty only, introduced to reproduce the observed default clustering and default dependence [see e.g. Gourieroux, Monfort and Polimenis (2006), Duffie et al. (2009)], as well as the specifications with contagion only. For instance, the introduction of the lagged default counts in the conditional distribution of $n_{j,t+1}$ given in i), is in line with Lang, Stulz (1992), Jarrow, Yu (2001), Billio et al. (2012), or with the Hawkes' (1971) specification of the mutually exciting point processes in a continuous-time framework [see e.g. Lando, Nielsen (1998), Errais, Giesecke, Goldberg (2010) for applications to credit risk]. Note that the total number of defaults N_t might also be introduced as a component of F_{t+1} (see the applications in Section 5).

We deduce the following Proposition, which extends Proposition 6.

Proposition 8: Under Assumption A.2 the joint process (F_t, n_t) is jointly compound autoregressive. For any horizon h, we can write:

$$
\begin{aligned}
& E_t[\exp(u_F' \sum_{k=1}^{h} F_{t+k} + u_S' \sum_{k=1}^{h} n_{t+k})] \\
= \quad & \exp[a_F'(h, u_F, u_S) F_t + a_S'(h, u_F, u_S) n_t + b(h, u_F, u_S)],
\end{aligned}
$$

where

$$
\begin{aligned}
a_F(1, u_F, u_S) &= A_F[1, u_F + \sum_{j=1}^{J} \beta_j (\exp u_{jS} - 1)], \\
a_s(1, u_F, u_S) &= A_s[1, u_F + \sum_{j=1}^{J} \beta_j (\exp u_{jS} - 1)] + \sum_{j=1}^{J} c_j (\exp u_{jS} - 1), \\
b(1, u_F, u_S) &= B[1, u_F + \sum_{j=1}^{J} \beta_j (\exp u_{jS} - 1)] + \sum_{j=1}^{T} \gamma_j (\exp u_{jS}' - 1),
\end{aligned}
$$

and similar functions for other horizons h are deduced by recursion [see Appendix 3].

Proof: Indeed we have:

$$
\begin{aligned}
& E_t[\exp(u_F' F_{t+1} + u_S' n_{t+1})] \\
= \quad & E_t[\exp(u_F' F_{t+1}) \exp(\sum_{j=1}^{J} u_{jS} n_{j,t+1})] \\
= \quad & E_t\{\exp[u_F' F_{t+1} + \sum_{j=1}^{J} [\beta_j' F_{t+1} + c_j' n_t + \gamma_j][\exp(u_{jS}) - 1]\} \\
= \quad & E_t \exp\{[u_F + \sum_{j=1}^{J} \beta_j (\exp u_{jS} - 1)]' F_{t+1}\} \exp[\sum_{j=1}^{J} c_j' n_t (\exp u_{jS} - 1)] \\
& \exp(\sum_{j=1}^{J} \gamma_j [\exp(u_{jS}) - 1]) \\
= \quad & \exp\{A_F'[1, u_F + \sum_{j=1}^{J} \beta_j (\exp u_{jS} - 1)] F_t + A_s'[1, u_F + \sum_{j=1}^{J} \beta_j (\exp u_{jS} - 1)] n_t \\
+ \quad & \sum_{j=1}^{J} c_j' (\exp u_{jS} - 1) n_t + B[1, u_t + \sum_{j=1}^{J} \beta_j \exp(u_{jS} - 1)] \\
+ \quad & \sum_{j=1}^{J} \gamma_j [\exp(u_{jS}) - 1]\}.
\end{aligned}
$$

The result follows by identification.□

The dynamic model described in Assumption A.2 is easily interpretable, when the default counts do not cause the factor process, that is, when $A_S(1,v) = 0, \forall v$. Factor (F_t) representes the exogenous shocks with joint effect on the probabilities of default, whereas the matrix C with rows $c_j', j = 1,\ldots,J$ characterizes the contagion phenomena. This matrix gives the segments connected by possible contagion effects, but also the direction and magnitude of the contagion [for such an interpretation, see e.g. Billio, Getmansky, Lo, Pellizon (2012) for a model with contagion only, Darolles, Gagliardini, Gourieroux (2012) with a model including both dynamic frailty and contagion].

4.2 Pricing formulas

Let us now extend Proposition 7. We denote $N_{t+h} = \sum_{k=1}^{h} n_{t+k}$ and consider the s.d.f. function of the surprises of default events in each segment:

$$m_{t,t+1} = \exp[\delta_0 + \delta_F' F_{t+1} + \delta_S' n_{t+1}].$$

Proposition 9: Under Assumption A.2, the price at date t_0 of the exponential payoff $N(u)_{t_0+h} = \exp(u' N_{t_{0+h}})$ is given by:

$$\begin{aligned} \Pi(N(u),h) &= \exp\{a_F(h,\delta_F,\delta_S+u)'F_{t_0} + a_s(h,\delta_F,\delta_S+u)'n_{t_0} \\ &+ b(h,\delta_F,\delta_S+u) + h\delta_0]. \end{aligned} \tag{4.1}$$

Proof: We have:

$$\begin{aligned} \Pi(N(u),h) &= E_{t_0}[\Pi_{k=1}^h m_{t_0+k-1,t_0+k} \exp(u' N_{t_0+h})] \\ &= E_{t_0}\{\exp[h\delta_0 + \delta_F' \sum_{k=1}^{h} F_{t_0+k} + (\delta_S+u)' \sum_{k=1}^{h} n_{t_0+k}]\} \\ &= \exp\{a_F(h,\delta_F,\delta_s+u)'F_{t_0} + a_S(h,\delta_F,\delta_S+u)'n_{t_0} \\ &+ b(h,\delta_F,\delta_S+u) + h\delta_0\}.\square \end{aligned}$$

Thus the derivative prices, including the riskfree zero-coupon bonds, depend on the surprise on credit events in different ways: i) by means of the risk premium components of vector δ_S, and ii) by the current default counts n_{j,t_0}, $j = 1, \ldots, J$, in the different segments. These effects can be more or less important according to the form of functions a_F, a_S and b, that is, according to the sensitivity parameters β_j and the contagion parameters c_j.

5 An illustration

In this section, we illustrate the interest of the models introduced in the previous sections. We first consider a homogenous pool and show that the mispricing due to the omission of the default-event surprise can be significant. Then we analyze the propagation of the effect of default events in a model with different segments.

5.1 Homogenous model

Let us first consider the homogenous pool of Section 3. The factor F_t is given by $[F_{1,t}, F_{1,t-1}, F_{2,t}]$, where the processes $(F_{1,t})$ and $(F_{2,t})$ are independent auto-regressive gamma (ARG) processes with parameters ρ_i, ν_i and μ_i, $i \in \{1, 2\}$ (see Appendix 3 for the definition of an ARG process). Adding a lagged value of $F_{1,t}$ in the factor F_t allows for more flexible specifications of the s.d.f. and hence of the term structure of yields. In particular, this feature is important to generate realistic fluctuations of term structures of riskfree rates and associated term premia. We set $\beta = [0, 0, 1]'$ and $\gamma = 0$, implying that $F_{2,t}$ is the expectation of default count n_t conditional on F_t, since $n_t \sim \mathcal{P}(F_{2,t})$.

Denoting by $\delta_{F,1}$, $\delta_{F,-1}$ and $\delta_{F,2}$ the three entries of δ_F, the riskfree short-term rate between dates t and $t+1$ is affine:[8]

$$r_t = \kappa_0 + \kappa_F' F_t$$

[8]This is easily deduced from the facts that (a) $r_t = -\log E(m_{t,t+1})$, (b) $m_{t,t+1}$ is exponential affine in (F_{t+1}, n_{t+1}) and (c) the latter vector is Car(1).

with sensitivity coefficients given by:

$$\begin{cases} \kappa_0 &= -\delta_0 + \mu_1 \log(1 - \nu_1 \delta_{F,1}) + \mu_2 \log(1 - \nu_2 \left[\delta_{F,2} + \beta(\exp \delta_S - 1)\right]) \\ \kappa_F &= -\left[\begin{array}{ccc} \frac{\rho_1 \delta_{F,1}}{1-\nu_1\delta_{F,1}} + \delta_{F,-1}, & 0, & \frac{\rho_2\left[\delta_{F,2}+\beta(\exp\delta_S - 1)\right]}{1-\nu_2\left[\delta_{F,2}+\beta(\exp\delta_S-1)\right]} \end{array} \right]'. \end{cases}$$

The other parameter values are: $\delta_F = [0.5, -0.43,\ -0.1]'$, $\delta_S = 0.1003$, $\delta_0 = -0.27$, $\rho_1 = 0.8$, $\nu_2 = 0.1$, $\mu_1 = 5$, $\rho_2 = 0.9$, $\nu_2 = 0.3$, $\mu_2 = 0.4$ and the pool includes 200 entities. The parameters are such that, conditionally on F_t, the s.d.f. $m_{t,t+1}$ is positively related to the surprise on defaults, simply given by $n_t - F_{2,t}$. Moreover, the short-term riskfree rate is:

$$r_t = 0.01 + \frac{1}{100}.F_{1,t} - 0.005.F_{2,t}. \tag{5.1}$$

Since the support of $F_{1,t}$ and $F_{2,t}$ is $[0, +\infty[$ and these two variables are independent, the short-term rate is not necessarily positive. However, negative short-term rates remain rare events for the selected parameter values.[9]

Figure 1 displays the results of a 200-period simulation of the model. The first panel shows that the riskfree short-term rate r_t is mainly driven by $F_{1,t}$, except during episodes of high default risk, i.e. when $F_{2,t}$ is large ($F_{2,t}$ is plotted in the second panel). The third panel shows the associated time series of long-term riskfree and individual defaultable-bond yields, with the same time-to-maturity h. The long-term riskfree rates are negatively affected when default risk, proxied by $F_{2,t}$, rises. This result comes from the facts that r_t is negatively related to $F_{2,t}$ (see equation 5.1) and that $F_{2,t}$ is persistent under the risk-neutral measure. On the contrary, defaultable-bond yields are positively related to $F_{2,t}$, the negative effect of $F_{2,t}$ on the riskfree long-term yield being more than compensated by the positive effect of $F_{2,t}$ on the credit spread (that is the spread

[9]Nevertheless, negative short-term rates have been observed in the recent post-crisis period, notably on sovereign T-bills.

between the defaultable and the riskfree bond).

[Insert Figure 1: Evolutions of Factors and Rates]

Figure 2 shows the term structure of riskfree rates at date 100, and the term structure of riskfree rates that would prevail if agents were risk-neutral (line with diamonds), i.e. under the expectation hypothesis.[10] The difference between the two curves provides the term premia, that are the premia demanded by risk-averse investors to bear the interest-rate risk. These term premia exist even if the payoff of the bond is 1 and therefore not subject to default. Indeed, the riskfree bonds are credit-riskfree, but their returns still depend on the fluctuations of the future riskfree short term rate, and are therefore risky in that sense.

[Insert Figure 2: Term-Structure of Riskfree Rates]

Figure 3 displays the term structure of default probabilities of one given entity at date $t = 100$ (line with diamonds). For instance, based on the information available at that date, the probability that entity i defaults before 10 periods is about 5%. CDS prices associated with this default event are plotted on the same chart (solid black line). In order to make these prices comparable with the probabilities of default (the diamonds), the reported CDS prices are forward prices, obtained by dividing $\Pi(d_i, h)$ by $\Pi(1, h)$. This standardization avoids the discounting effects that are implicitly present in the CDS pricing formula given in Corollary 3 where $\Pi(d_i, h)$ is paid upfront at date t, for the payoff $d_{i,t+h}$ settled at date $t + h$. Even with that adjustment, the reported forward CDS prices do not correspond to the risk-neutral probabilities of default. As an illustration, the chart shows that the value of the 20-period forward CDS price is 14%: this means that, at date $t = 100$, the price of $0.14 - d_{i,t+20}$ is zero.

The plot suggests that the implied probabilities of default derived from CDS prices are higher than the physical ones. Further, we also display the (forward) CDS prices

[10] Under the expectation hypothesis, the riskfree rates are given by $-1/h \times \log E_t^{\mathbb{P}}[\exp(-r_t - \ldots - r_{t+h-1})]$.

that one would obtain by using the projected s.d.f. $\tilde{m}_{t,t+1}$ instead of the true one $m_{t,t+1}$ (dashed line). The spreads between "true" CDS prices and the latter define the credit-event risk premia; these premia correspond to the additional remuneration required by investors to hold assets whose payoffs are negatively correlated to surprisingly high default rates. In our example, about half of the total credit-risk premia (the spread between the risk-neutral and physical probabilities of default) are accounted for by these credit-event risk premia and this proportion weakly depends on the time-to-maturity.

[Insert Figure 3: CDS prices and Probabilities of Default]

5.2 Contagion

This subsection provides an illustration of the model for heterogenous pool introduced in Section 4. We consider a simple setting, but the approach remains tractable when dealing with larger systems and more complicated exposure setups.

Six homogenous segments are involved, each of them being constituted of 100 entities. The factor F_t is equal to $[F_{B,t}, F'_{N,t}]'$; $(F_{B,t})$ is a sequence of i.i.d. Bernoulli variables with parameter $\nu = 0.05$. The process $(F_{N,t})$, of dimension 6, keeps memory of past default counts in the different segments. Specifically, we have:

$$F_{N,j,t} = \rho F_{N,j,t-1} + n_{j,t-1}, \quad j = 1, \dots, 6,$$

where the smoothing parameter ρ is chosen independent of the segment. If ρ is equal to one and $F_{N,t_0} = 0$, then $F_{N,t}$ gives the cumulated number of defaults between t_0 and $t-1$ in the different segments. When $0 < \rho < 1$, $F_{N,t}$ keeps track of the number of past defaults, but underweights the oldest ones. We use $\rho = 0.8$ in the numerical example presented below.

Conditional on $\Omega_t^* = (F_{t+1}, \Omega_t)$, the counts $n_{j,t+1}$, $j = 1, \dots, 6$, follow independent

Poisson distributions:

$$\begin{cases} n_{1,t+1} & \sim \mathcal{P}(0.4 \times F_{N,6,t} + F_{B,t}), \\ n_{j,t+1} & \sim \mathcal{P}(0.4 \times F_{N,j-1,t}), \quad \text{if } j > 1. \end{cases} \tag{5.2}$$

This structure defines a circular network of segments where the probability of experiencing defaults in segment j depends on the number of recent defaults in segment $j-1$ (or in segment 6 for $j = 1$).

[Insert Figure 4: Evolutions of Factors and Default Counts]

Figure 4 displays simulated trajectories of the processes (F_t) and (n_t). We initialize the simulation with $F_{N,1} = 0$. At date 5, we get the high value of factor F_B ($F_{B,5} = 1$) that generates two defaults in segment 1. This implies an increase in factor $F_{N,1,t}$, which induces one default in segment 2 at date 6, and so on. Even in the absence of new shock on $F_{B,t}$, defaults occur again in segment 1 at date 17 because of propagation across segments till segment 6 (recall that segment 1 is exposed to segment 6, see equation 5.2). After the 30^{th} period, default intensities fade and the $F_{N,i,t}$'s are all back to small values. In the absence of a new shock on F_B, there is no additional default. A new default wave is triggered after the 40^{th} period, due to a shock on F_B that translates into three defaults in segment 1 and so on.

[Insert Figure 5: CDS prices and Probabilities of Default]

Figure 5 illustrates the implications of the model with contagion in terms of forecasting and pricing. We focus on two dates ($t = 1$ and $t = 45$) and two segments (1 and 4). For each segment and date, two charts are provided:

- The upper chart presents the same kinds of curves as the ones plotted in Figure 3: the black solid line indicates the probabilities that entity i defaults between t and $t+h$ (where t is the current date, i.e. either 1 or 45); the grey solid line is the forward CDS price as defined above and the dotted grey line is the forward CDS price computed with the s.d.f. $\tilde{m}_{t,t+1}$ that ignores the pricing of default events.

- The lower chart presents the first differences of the previous curves (with respect to horizon h). Therefore, this chart focuses on the event of a default of entity i at specific future dates $t+h$, for h between 1 and 15: the black solid line indicates the probabilities of default of entity i at date $t+h$ and the grey solid line reflects the cost of insuring against a default of entity i exactly at date $t+h$.

The prices are obtained with a s.d.f. $m_{t,t+1}$ where only two parameters among δ_0, δ_F and δ_S are non zero (see equation 3.1): the entry of δ_S associated with the first segment is set to 0.1 and the one of δ_F associated to $F_{B,t}$ is set to -0.1. As in the previous example, the spread between the grey solid line and the dotted solid line is accounted for by credit-event risk premia.

At date 1, the default probabilities for segment-1 entities in any of the next seven periods is equal 0.05% (see Panel A in Figure 3). To understand that, recall that the number of defaults in segment 1, conditional on F_t, follows a Poisson distribution $\mathcal{P}(0.4 \times F_{N,6,t} + F_{B,t})$. Therefore, we can have a default in segment 1 at date t only if either $F_{B,t} > 0$, or $F_{N,6,t} > 0$. Further, since there cannot be any default in segment j ($j > 1$) without previous defaults in segment $j-1$, $F_{N,6,t}$ necessarily remains at zero for at least 6 periods when $F_{N,t} = 0$. In the latter case, the default probability for any entity in segment 1 at dates $t+h$ for $h < 7$ is constant and equal to 0.05%.[11] Beyond that horizon, the probability of default increases because of possible contagion along the lines described above.

The other plots in Figure 5 show that various profiles of expected probabilities of default can be obtained in that framework. Let us look at Panel D, that corresponds to the probabilities of default of segment-4 entities in future dates $t+h$, as expected from date $t = 45$. This chart suggests that the probabilities of default are decreasing

[11] 0.05% is then the expected value of $n_{t+h}/100$, which is equal to the expected value of $F_{B,t+h}/100$.

in the next 6 periods, but increase beyond that horizon. This stems from the fact that the expectation of $n_{4,t}$ conditional on future F_t is equal to $F_{3,t}$ and that, based on the information available at date $t = 45$, $F_{3,t+h}$ is expected to decrease in the next six periods. However, one default occurs in segment 4 at date 45 and this default could propagate across the different segments and generates a new wave of defaults that would take 6 periods before affecting segment 4 again. This contributes to the increase (with respect to h) in the expected probabilities of default in segment 4 beyond $t + 6$.

The propagation schemes are summarized in the left-hand-side plots of Figure 5. They provide the direction of propagation and indicate the number of defaults, when defaults occur.

This illustration shows how the model for heterogenous pool with both dynamic frailty and contagion is able to reproduce stylized facts highlighted in the literature such as the increase in CDS spreads and the increase in default correlation responding to a borrower bankruptcy [see e.g. Jorion, Zhang (2009)]. This model is even more flexible since it is able to analyze these responses in a dynamic way.

6 Concluding remarks

By neglecting default-event surprises, the standard approaches for pricing credit derivatives can imply some mispricing. This mispricing is related to the causality from the default count process to the factor process. Moreover we have seen that the default intensity no longer exists under the risk-neutral probability when the surprise on default event is priced.

The analysis is easily extended to heterogenous pools of credits, where defaults can be driven by a dynamic frailty as well as by past default counts in the different segments. This model is appropriate for disentangling the effects of exogenous shocks from contagion effects. The illustration shows how shocks propagate in the system and the implications of this propagation on derivative prices.

Finally, note that credit-derivative pricing is still an important topic, even if traded volumes of these derivatives have decreased in the aftermath of the recent financial crisis.[12] First, this volume remains significant. Second, coherent pricing formulas are also useful from a regulating point of view, in particular to compute the required capital for financial institutions. Indeed, for rather illiquid assets, the usual mark-to-market (fair value) approach is progressively replaced by mark-to-model values. The model considered in this paper can serve this purpose.

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[12] According the BIS (http://www.bis.org/statistics/otcder/dt21.csv), the amounts outstanding of over-the-counter traded CDS was larger than \$45tr in mid-2008. Between mid-2009 and the end of 2012, this amount lied between \$20tr and \$25tr.

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Appendix 1
Conditional Independence

Lemma A.1: The process (n_t) admits independent components conditional on the factor process (F_t).

Proof: Let us consider the Sims' characterization of the noncausality of process (n_t). We get:

$$f(n_t|\underline{n_{t-1}}, \underline{F_T}) = f(n_t|\underline{n_{t-1}}, \underline{F_t}),\ \forall t \leq T.$$

Moreover by Assumption A_0 ii), we get:

$$f(n_t|\underline{n_{t-1}}, \underline{F_t}) = f(n_t|F_t).$$

Thus we deduce that $f(n_t|\underline{n_{t-1}}, \underline{F_T})$ does not depend on $\underline{n_{t-1}}$, which characterizes the independence of $n_1, \ldots, n_T$ given $\underline{F_T}$.$\square$

The same approach can be followed to prove the conditional independence of the individual point processes $d_i = (d_{i,t}), i = 1, \ldots, I$, conditional on the factor process.

Let us now consider the projected s.d.f. We get:

$$\begin{aligned} & E[\Pi_{k=1}^{h} m(F_{t_0+k}, n_{t_0+k})|\underline{F_{t_0+h}}] \\ = \quad & \Pi_{k=1}^{h} E[m(F_{t_0+k}, n_{t_0+k})|\underline{F_{t_0+h}}] \text{ (by conditional independence)} \\ = \quad & \Pi_{k=1}^{h} E[m(F_{t_0+k}, n_{t_0+k})|F_{t_0+k}] \text{ (by Assumption } A_0 i)). \end{aligned}$$

Appendix 2
Proofs of Propositions 2 and 3

i) Proof of Proposition 2

We have:

$$\begin{aligned} f^P(F_{t+1}, d_{t+1}|\Omega_t) &= f^P(F_{t+1}|\Omega_t) f^P(d_{t+1}|F_{t+1}, \Omega_t) \\ &= f^P(F_{t+1}|\Omega_t) \prod_{i \in PaR_t} f^P(d_{i,t+1}|F_{t+1}, d_{i,t} = 0, n_t), \end{aligned}$$

by the independence of the individual point processes given (F_t).

The risk-neutral conditional p.d.f of (F_{t+1}, d_{t+1}) given Ω_t is proportional to:

$$f^P(F_{t+1}|\Omega_t)\exp(\delta_F' F_{t+1}) \prod_{i \in PaR_t} f^P(d_{i,t+1}|F_{t+1}, d_{i,t}=0, n_t)\exp(\delta_S d_{i,t+1}).$$

Therefore, we have:

$$f^Q(d_{i,t+1}|F_{t+1}, \Omega_t, d_{i,t}=0) \propto \prod_{i \in PaR_t} f^P(d_{i,t+1}|F_{t+1}, d_{i,t}=0, n_t)\exp(\delta_S d_{i,t+1}),$$

where the proportionality coefficient depends on the conditioning variables. This implies that under the risk-neutral probability:

i) the point processes $d_i, i = 1, \ldots, I$ are independent conditional on (F_t), due to the multiplicative decomposition of the joint density;

ii) state 1 is still absorbing;

iii) the risk-neutral p.d.f. $f^Q(d_{i,t+1}|F_{t+1}, d_{i,t}=0, n_t)$ is the Esscher transform [Esscher (1932), Gerber, Shin (1994)] of the historical p.d.f. $f^P(d_{i,t+1}|F_{t+1}, d_{i,t}=0, n_t)$ associated with parameter δ_S.

In particular $f^Q(d_{i,t+1}=0|F_{t+1}, d_{i,t}=0, n_t) \equiv \exp(-\lambda^Q_{t+1})$ is proportional to $f^P(d_{i,t+1}=0|F_{t+1}, d_{i,t}=0, n_t) \equiv \exp(-\lambda_{t+1})$ and $f^Q(d_{i,t+1}=1|F_{t+1}, d_{i,t}=0, n_t)$ is proportional to $[1-\exp(-\lambda_{t+1})]\exp(\delta_S)$.

Therefore we get:

$$\exp(-\lambda^Q_{t+1}) = \frac{\exp(-\lambda_{t+1})}{\exp(-\lambda_{t+1}) + [1-\exp(-\lambda_{t+1})]\exp(\delta_S)}$$

and Proposition 2 follows.□

ii) Proof of Proposition 3

The proof of Proposition 3 is derived by noting that the risk-neutral conditional p.d.f of F_{t+1} given Ω_t is obtained by summing on d_{t+1} the joint risk-neutral conditional p.d.f. of (F_{t+1}, d_{t+1}) given Ω_t. We get:

$$f^P(F_{t+1}|\Omega_t)\exp(\delta_F' F_{t+1})[\sum_{d_{i,t+1}=0}^{1} f^P(d_{i,t+1}|F_{t+1}, d_{i,t}=0, n_t)\exp(\delta_S d_{i,t+1})]^{(I-N_t)}.\square$$

Appendix 3
Recursive Formulas for CaR Processes

We recall in this appendix the recursive formulas for computing the Laplace transform of a (multidimensional) CaR process at the different prediction horizons [see e.g. Darolles, Gourieroux, Jasiak (2006)]. We write these formulas for a process (Y_t), which will be either $Y_t = F_t$, or $Y_t = (F_t', n_t')'$ in our applications.

Proposition A.1: For a CaR process such that:

$$E_t[\exp(u'Y_{t+1})] = \exp[a(1,u)'Y_t + b(1,u)],$$

we also have:
$$E_t[\exp(u'\sum_{k=1}^{h} Y_{t+k})] = \exp[a(h,u)'Y_t + b(h,u)],$$
where the functions $a(h,u), b(h,u)$ satisfy the recursive equations:

$$\begin{aligned} a(h,u) &= a[1, u + a(h-1,u)], && \text{(a.1)} \\ b(h,u) &= b[1, u + a(h-1,u)] + b(h-1,u). && \text{(a.2)} \end{aligned}$$

Proof: The recursive formulas are easily derived by applying the iterated expectation theorem. We get:

$$\begin{aligned} & E_t[\exp(u'\sum_{k=1}^{h} Y_{t+k})] \\ = \; & E_t\{\exp(u'Y_{t+1})E_{t+1}[\exp(u'\sum_{k=1}^{h-1} Y_{t+1+k})]] \\ = \; & E_t\{\exp[u'Y_{t+1} + a(h-1,u)'Y_{t+1} + b(h-1,u)]\} \\ = \; & \exp[a[1, u + a(h-1,u)]'Y_t + b[1, u + a(h-1,u)] + b(h-1,u)]. \end{aligned}$$

The recursive formulas of the Proposition are deduced by identification.□

The recursive formulas (a.1) - (a.2) can also be used to deduce recursively the expressions of the derivatives w.r.t. argument u. We get:

$$\begin{aligned} \frac{\partial a(h,u)}{\partial u'} &= \frac{\partial a}{\partial u'}[1, u + a(h-1,u)][Id + \frac{\partial a}{\partial u'}(h-1,u)], && \text{(a.3)} \\ \frac{\partial b(h,u)}{\partial u'} &= \frac{\partial b}{\partial u'}[1, u + a(h-1,u)](Id + \frac{\partial a}{\partial u'}(h-1,u)] + \frac{\partial b}{\partial u'}(h-1,u). && \text{(a.4)} \end{aligned}$$

Example: The autoregressive gamma (ARG) process.

This process is the time-discretized Cox, Ingersoll, Ross process [Cox, Ingersoll, Ross (1985)]. The conditional Laplace transform of the ARG process is:

$$E_t[\exp(uY_{t+1})] = \exp[\frac{\rho u}{1-\nu u} Y_t - \mu \log(1-\nu u)].$$

Thus we have: $a(1,u) = \dfrac{\rho u}{1-\nu u}, b(1,u) = -\mu \log(1-\nu u)$.

Appendix 4
Characterization of the joint probability of defaults

Lemma A.2: If $N = d_1 + \ldots + d_I$, where the variables $d_i, i = 1, \ldots, I$ are exchangeable, we have, for $K \leq I$:

$$P[d_1 = \ldots = d_K = 1] = E(d_1 \ldots d_K) = \frac{E[N(N-1)\ldots(N-K+1)]}{I(I-1)\ldots(I-K+1)}.$$

Proof: i) Let us first consider the case of independent defaults. Then $N \sim \mathcal{B}(I, p)$, where $p = P[d_1 = 1]$. It is easily deduced from the moment generating function of the binomial distribution [see e.g. Johnson, Kemp, Kotz (2005)] that.

$$\begin{aligned} E[N(N-1)\ldots(N-K+1)] &= I(I-1)\ldots(I-K+1)p^K \\ &= I(I-1)\ldots(I-K+1)E(d_1 \ldots d_K). \end{aligned}$$

ii) In the general framework with possible default dependence, we know by de Finetti's theorem [see e.g. Feller (1971)], that any exchangeable sequence $d_1, \ldots, d_I$ of $\{0, 1\}$ variable is such that there exists a latent variable Z, say, such that $d_1, \ldots, d_I$ are i.i.d. Bernoulli with probability $p(Z)$ conditional on Z.

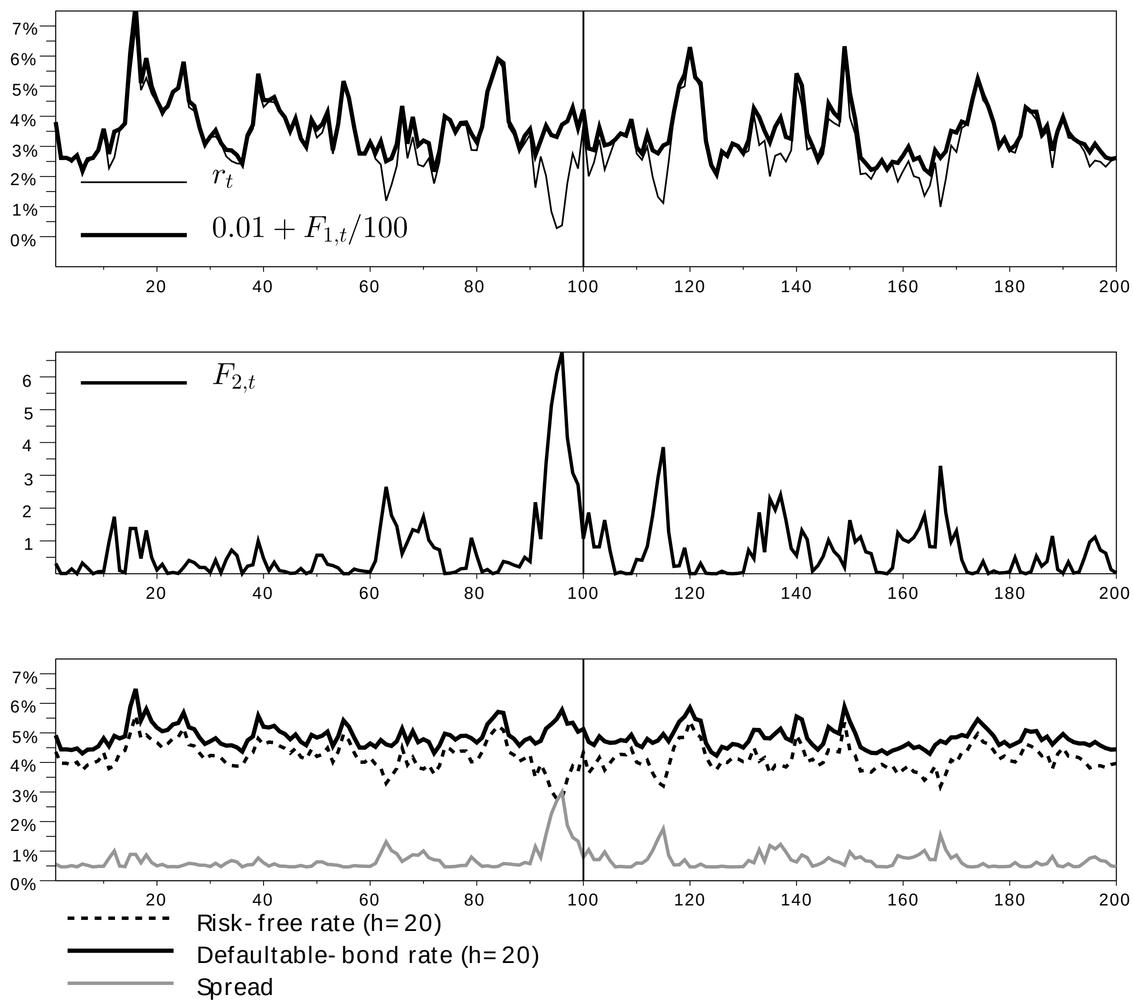
We deduce that:

$$E[N(N-1)\ldots(N-K+1)|Z] = I(I-1)\ldots(I-K+1)E[d_1 \ldots d_K|Z],$$

and by taking the expectation of both sides:

$$E[N(N-1)\ldots(N-K+1)] = I(I-1)\ldots(I-K+1)E(d_1 \ldots d_K).\square$$

Figure 1: **Homogenous Pool: Evolutions of Factors and Rates**



This figure displays simulation results for the model described in subsection 5.1. The processes $(F_{1,t})$ and $(F_{2,t})$ are independent ARG processes. The short term riskfree rate is given by $r_t = 0.01 + 0.01.F_{1,t} - 0.005.F_{2,t}$. The lowest panel displays the long-term rates (maturity of 20 periods), the riskfree rate (dashed line) and the defaultable-bond rate (solid black line). The long-term riskfree rate is negatively related to the default intensity $F_{2,t}$, whereas the defaultable-bond rate is positively related to it.

Figure 2: **Homogenous Pool: Term Structure of Riskfree Rates**

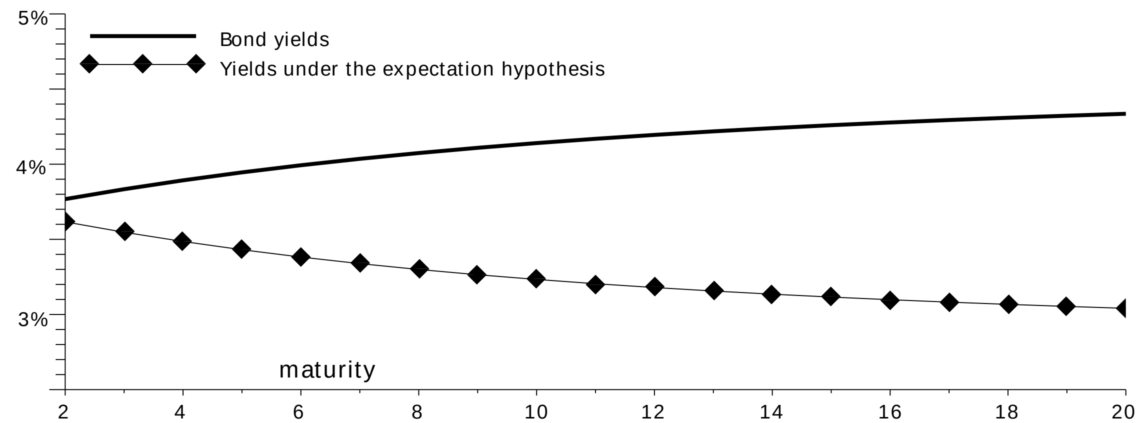


Figure 2 displays the term-structure of riskfree rates at date $t = 100$: the diamonds show the long-term interest rates under the expectation hypothesis (if the s.d.f. $m_{t,t+1}$ were equal to $\exp(-r_t)$). The difference between the two curves provides the term premia.

Figure 3: **Homogenous Pool: CDS Prices and Probabilities of Default**

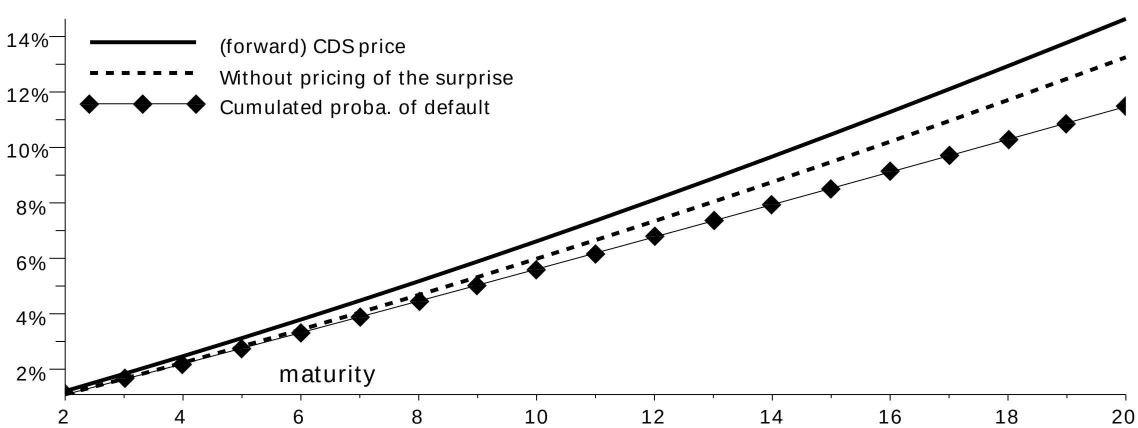
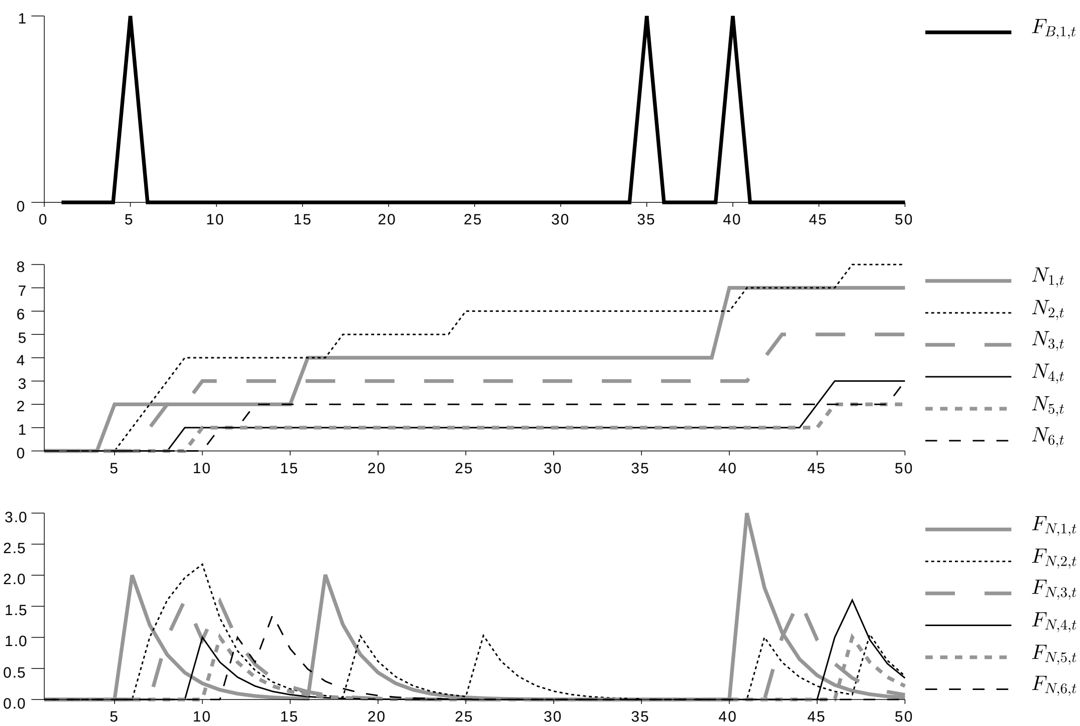


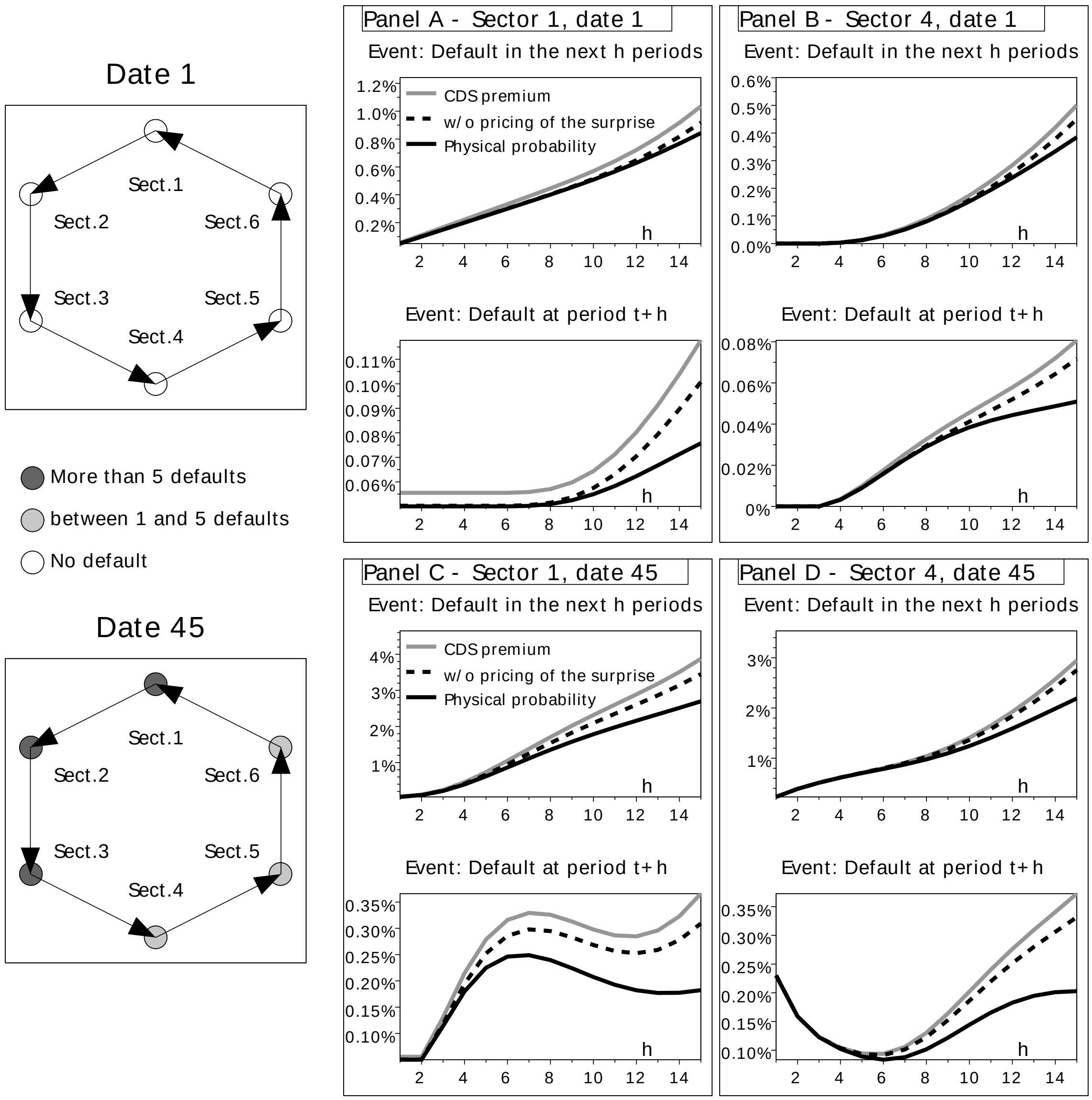
Figure 3 displays the term structure of default probabilities of one given entity at date $t = 100$ (diamonds). The solid line corresponds to forward CDS prices: for instance, the CDS price of 0.14 for $h = 20$ means that the price, at date $t = 100$, of the payoff $0.14 - d_{i,t+20}$ is zero, the payoff being settled at date $t + 20$. About half of the credit-risk premia are accounted for by credit-event premia and this proportion weakly depends on the time-to-maturity.

Figure 4: **Heterogenous Pool: Evolutions of Factors and Default Counts**



This figure displays simulated paths of (F_t) and (N_t). $F_{B,t}$ is drawn from a Bernoulli distribution with parameter $\nu = 5\%$. Conditional on $\Omega_t^* = (F_{t+1}, \Omega_t)$, the default counts $n_{i,t+1}$ follow Poisson distributions: $n_{1,t+1} \sim \mathcal{P}(0.4 \times F_{N,6,t} + F_{B,t})$ and $n_{i,t+1} \sim \mathcal{P}(0.4 \times F_{N,i-1,t})$ for $i > 1$. In addition, $F_{N,i,t} = 0.8 \times F_{N,i,t-1} + n_{i,t-1}$. A high value of factor F_B may immediately generate defaults in segment 1, and these defaults propagate to the other segments by contagion.

Figure 5: **Heterogenous Pool: CDS Prices and Probabilities of Default**



The first column of the chart describes the structure of the segment exposures and gives indications on the number of defaults per segment (white: no default, grey: between 1 and 5 defaults, black: more than 5 defaults). For each segment and each date, two charts are provided: (a) Upper chart: the black solid line indicates the probabilities that entity i defaults between t and $t+h$ (where t is either 1, or 45); the grey solid line plots the forward prices of CDS and the dotted grey line corresponds to the forward price of CDS computed with the s.d.f. $\tilde{m}_{t,t+1}$ that ignores the pricing of default events. (b) For each panel, the lower chart displays the first differences of the curves plotted on the upper chart and is related to the event of a default of entity i at specific future dates $t+h$ (for h between 1 and 15), the black solid line indicates the probabilities (under the physical measure) of default of entity i at date $t+h$ and the grey solid line reflects the cost of insuring against a default of entity i.