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Robust Portfolio Allocation with Systematic Risk Contribution Restrictions*

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Abstract

The standard mean-variance approach can imply extreme weights in some assets in the optimal allocation and a lack of stability of this allocation over time. To improve the robustness of the portfolio allocation, but also to better control for the portfolio turnover and the sensitivity of the portfolio to systematic risk, it is proposed in this paper to introduce additional constraints on both the total systematic risk contribution of the portfolio and its turnover. Our paper extends the existing literature on risk parity in three directions: i) we consider other risk criteria than the variance, such as the Value-at-Risk (VaR), or the Expected Shortfall; ii) we manage separately the systematic and idiosyncratic components of the portfolio risk; iii) we introduce a set of portfolio management approaches which control for the degree of market neutrality of the portfolio, for the strength of the constraint on systematic risk contribution and for the turnover.

Keywords: Asset Allocation, Portfolio Turnover, Risk Diversification, Minimum Variance Portfolio, Risk Parity Portfolio, Systematic Risk, Euler Allocation, Hedge Fund.

JEL classification: G12, C23.

1 Introduction

The gap between theory and practice is well illustrated by the example of portfolio management since Markowitz (1952) introduced the mean-variance framework. The resolution of the allocation problem by a simple quadratic optimization is the main advantage of the mean-variance approach. However, in practice this approach is implemented by replacing the theoretical mean and variance by their (unrestricted) historical counterparts, and the associated estimated mean-variance portfolios have several drawbacks: they are very sensitive to errors in the estimates of the mean and variance inputs [see e.g. Chopra (1993), Chopra and Ziemba (1993)], the resolution of a large-scale quadratic optimization problem is not straightforward [see e.g. Konno and Hiroaki (1991)], and dominant factor in the covariance matrix results in extreme weights in optimal portfolios [see e.g. Green and Hollifield (1992)]. Finally the portfolio allocations are very erratic over time, which implies significant transaction costs or liquidity risks. These drawbacks are even more pronounced when the portfolio is based on a large number of assets.

These difficulties are due mainly to the sensitivity of the mean-variance efficient portfolio allocation to the smallest eigenvalues of the variance matrix and to the poor accuracy of the inverse variance matrix with the standard estimation methods. The literature has proposed different ways to get more robust portfolio allocations, as the potential cost of a loss of efficiency. First, some robust estimation methods have been introduced, following results known in statistics[1]. Typical of such approaches are the shrinkages of the estimated variance matrix, which admit Bayesian interpretation [Garlappi, Uppal, Wang (2007), Goldfarb, Iyengar (2003), Ledoit, Wolf (2004)], the l_1- or l_2- penalizations introduced in the empirical optimization problem [see e.g. Broadie et al. (2008), DeMiguel et al. (2009)a, Fan et al. (2012)a], or the refresh time subsample approach with far more percentage of data used for any given pair of assets than for all the assets of the portfolio [Barndorff-Nielsen et al. (2008)].

Robustness can also be achieved by introducing restrictions in the empirical optimization problem even if these restrictions are not required by Financial Theory. These constraints have often

[1] It is well-known that the standard OLS estimator in a regression model $y = Xb + u$ is not robust. The expression of the OLS estimator: $\hat{b} = (X'X)^{-1}X'y$ includes the inversion of the design matrix $X'X$, and this inversion is not accurate when the explanatory variables are quasi-colinear. This lack of robustness is solved, either by considering Bayesian estimators, or by introducing l_2- penalizations, or by constraining the parameters.

simple interpretations. They can be shortselling restrictions [Frost, Savarino (1988), Chopra (1993) Jagannathan, Ma (2003)], gross exposure constraints [Fan et al. (2012)a], at the limit ”fully diversified” portfolios in terms of either budget allocations [see Elton, Gruber (1977), Duchin, Levy (2009), DeMiguel et al. (2009)b, Kritzman et al. (2010), Beleznay et al. (2012)], or contributions to total risk [see e.g. Martellini (2008), Choueifaty and Coignard (2008), Maillard et al. (2010)]. Bruder et al. (2011).

The idea of imposing additional diversification constraints is now commonly used in the asset management industry, and more enhanced strategies are grouped under the risk parity denomination. Risk parity is a general term for all investment techniques that attempt to take equal risk in the different underlyings of a portfolio. However, risk parity implementations differ considerably: investment universes, risk definitions, risk forecasting methods and risk exposures calculation can be different from one implementation to another one. Thus, risk parity is more a conceptual approach rather than a specific system, and it is in general difficult to compare the different approaches.

Many questions are raised by risk parity approaches. First, the total risk of a given portfolio is uniquely measured by its volatility, and contributions to total risk by the contribution of each underlying asset to this volatility. However in a risk parity allocation, it is more natural to define the total risk as the potential loss at the portfolio level and the contribution to total risk as the amount of initial wealth measured in risk unit on each portfolio underlying. These amounts are called risk budget in the literature [see e.g. Chow, Kritzmann (2001), Lee, Lam (2001)]. By defining risk budget through volatility contributions, Gaussian returns are implicitely assumed [see e.g. Inker (2010)]. Once the potential loss of capital for each portfolio underlying has been estimated, the portfolio can be determined. Second, the optimality of the standard risk parity portfolio, which imposes equal risk budgets on the underlying, can be discussed. This risk parity approach does not ensure that the total risk of the portfolio is optimized. Third the definition of the investment universe has a significant impact on the risk parity portfolio. In particular the risk parity portfolio allocation changes when we duplicate one asset. Thus it does not satisfy the duplication invariance property in the terminology of Choueifaty et al. (2011). Finally, risk parity approaches decrease portfolio concentration by construction in increasing the small cap weights. Then they create liquidity issues, since we have to dynamically rebalance an equity portfolio with a bigger liquidity exposure on small caps.

We develop in this paper a new implementation of the risk parity principle that circumvents the usual limitations of the current implemented ones. Our contribution to the literature is threefold. First, we use a more appropriate risk measure than the variance to account for extreme risks and give a reserve interpretation of the risk contributions in a general non Gaussian framework. Second, we introduce the risk contribution restriction on the total contribution of the portfolio to systematic risk and do not impose equal contributions to the systematic and unsystematic components of the portfolio risk. Third, we discuss the interest of such a restriction in terms of portfolio turnover and transaction costs.

The paper is organized as follows. In Section 2, we focus on the difference between the standard optimal portfolios and the associated risk parity portfolios. Section 3 considers asset returns with systematic and idiosyncratic components. Then we construct and compare different risk parity portfolios, when the parity is written on both types of components. Section 4 derives and compares optimal portfolios for different risk measures, especially the volatility, the Value-at-Risk and the Expected Shortfall. Section 5 presents empirical applications on portfolio of futures on commodities and Section 6 concludes. Some extensions and proofs are given in the appendices.

2 Portfolio Allocation with Risk Contribution Restrictions

We review in this section basic results on portfolio and risk allocations to highlight the difference between the standard optimal portfolios and the portfolios with risk contribution restrictions. We denote by $y_1, ..., y_n$ the returns of n risky assets, Y the corresponding vector of returns, μ the vector of expected returns, Ω the associated volatility matrix, and w the portfolio allocation, satisfying the standardized budget constraint $w'e = 1$, with e being a n-dimensional vector of 1. We denote by $R(w)$ the scalar risk measure associated with allocation w. The risk measure depends on allocation w through the distribution of the portfolio return $w'Y$.

2.1 Minimum Risk Portfolios

Let us focus first on the risk minimization problem. We obtain the minimum risk allocation by solving the program:

$$w^* = \underset{w'e=1}{\operatorname{argmin}}\, R(w).$$

The optimization problem above is written under the standardized budget constraint $w'e = 1$. This possibility to standardize the budget constraint exists if the risk measure is homogeneous of degree 1, that is, if:

$$R(cw) = cR(w), \quad \text{say},$$

for any positive scalar c. Indeed, the solution of an optimization problem such as:

$$w^*(c) = \min_w R(w), \;\; s.t. \;\; w'e = 1/c,$$

is equal to $w^*(c) = cw^*$. Thus, the solution with another budget restriction is deduced from the solution of the standardized optimization problem by an appropriate scaling. This solution is such that : $\frac{\partial R}{\partial w_i}(w^*) = \lambda(w^*), \forall i$, where $\lambda(w^*)$ is the Lagrange multiplier associated with the budget restriction.

2.2 Portfolios with Risk Contribution Restrictions

The recent literature on risk measures focuses on the risk contribution of each asset to the total portfolio risk. In this respect the risk contributions differ from the weights in portfolio allocations, since they also account for the effect of each individual asset on the total risk. Let us consider a global portfolio risk measured by $R(w)$. This total risk can be assigned to the different assets as:

$$R(w) = \sum_{i=1}^{n} R_i(w), \tag{2.1}$$

where $R_i(w)$ denotes the risk contribution of asset i to the risk of the whole portfolio. If the risk measure is homogenous of degree 1, we get the Euler formula:

$$R(w) = \sum_{i=1}^{n} w_i \frac{\partial R(w)}{\partial w_i}.$$

The Euler formula has an interpretation in terms of marginal contribution to global risk w.r.t. a change of scale in the portfolio allocation.[2]

This explains why it is often proposed in the literature to choose:

$$R_i(w) = w_i \frac{\partial R(w)}{\partial w_i}, \tag{2.2}$$

called the Euler allocation [Litterman (1996), p.28, Garman (1997), footnote 2, Qian (2006)]. The difference between the portfolio allocation and the risk contribution is captured by the marginal risk $\partial R(w)/\partial w_i$ [see (2.2)].

The Euler decomposition can be used to construct portfolios with constraints on the risk contributions. For instance, Equally Weighted Risk Contribution portfolios have been considered in the literature [see e.g. Scherer (2007), Maillard et al. (2010), Asness et al. (2012)], and are gaining in popularity among practitioners [Asness (2010), Sullivan (2010), Dori et al. (2011)].

This practice can be generalized by imposing the risk contributions to be proportional to some benchmarks π_i, $i = 1, ..., n$, which are not necessarily equal:

$$\frac{\partial R(w)}{\partial w} = \lambda(w)\, diag(\pi_i)\ \mathrm{vec}(1/w), \tag{2.3}$$

where $diag(\pi_i)$ is the diagonal matrix with π_i, $i = 1, \ldots, n$ as diagonal elements. That is we consider Risk Parity portfolios after an appropriate adjustment for the notion of parity.

2.3 Risk Contribution Restrictions and Portfolio Turnover

The introduction of restrictions (2.3) can be justified by the effect of trading costs. Let us assume that the investor's portfolio allocation at the beginning of the period is: $w_0 = (w_{0,1}, ..., w_{0,n})'$, and that the investor updates his portfolio to get the new allocation $w = (w_1, ..., w_n)'$. He will account for the risk $R(w)$ of the new allocation and for the trading costs when reallocating the portfolio from w_0 to w. Under no short sale constraints: $w_{0,i} \geq 0, w_i \geq 0, \forall i$, the trading cost (turnover) may be measured by:

$$T(w, w_0) = c \sum_{i=1}^{n} w_{0,i} \ln \left(\frac{w_{0,i}}{w_i} \right). \tag{2.4}$$

[2] The Euler formula is obtained by differentiating the homogeneity condition $R(cw) = cR(w)$, with respect to c and setting $c = 1$.

Indeed, when the allocation adjustments are small, we have:

$$T(w, w_0) \approx -c\left[\sum_{i=1}^{n} w_{0,i}\frac{w_i - w_{0,i}}{w_{0,i}} - \frac{1}{2}\sum_{i=1}^{n} w_{0,i}\frac{(w_i - w_{0,i})^2}{w_{0,i}^2}\right] \approx \frac{c}{2}\sum_{i=1}^{n}\frac{(w_i - w_{0,i})^2}{w_{0,i}},$$

since the two portfolios satisfy the budget constraint: $e'w = e'w_0 = 1$.

This approximation has a direct interpretation in terms of transaction costs, in which the cost for trading asset i is proportional to $1/w_{0,i}$. This assumption on trading costs can find a justification if the initial allocation corresponds to a market portfolio. Assets with the highest market weights w_{oi} are also the most liquid ones, and their trading is associated with low transaction cost. At the opposite, assets with the lowest market weights are less liquid and then trading these assets is expensive in terms of transaction costs. This cost for trading asset i is proportional to $(w_i - w_{0,i})^2$. Thus the implied market impact function for trading asset i is strictly convex.

The investor has to balance risk reduction and trading cost in his portfolio management. Thus, he can minimize a combination of both criteria, and choose:

$$w = \underset{w}{\operatorname{argmin}}\, R(w) + \lambda c \sum_{i=1}^{n} w_{0,i} \ln\left(\frac{w_{0,i}}{w_i}\right), \tag{2.5}$$

where $\lambda > 0$ is a smoothing parameter introduced to control the portfolio turnover. With $\lambda = 0$, the investment objective focuses on risk control. For high λ, the control is on the portfolio turnover, and the investment objective is to enhance the initial portfolio allocation in terms of risk control, but with a limited turnover.

The associated first-order condition is:

$$\frac{\partial R(w)}{\partial w_i} - \lambda\, c\, \frac{w_{0,i}}{w_i} = 0 \Leftrightarrow w_i \frac{\partial R(w)}{\partial w_i} = \lambda\, c\, w_{0,i}. \tag{2.6}$$

The risk contributions are proportional to the initial portfolio allocations: $\pi_i \propto w_{0,i}$. In particular, the benchmark levels of risk contributions π_i, $i = 1, ..., n$ depend on the current investor's portfolio. This approach is clearly suitable to advise investors that do not want to enhance their risk management without generating a high portfolio turnover.

This solution is especially appealing in a multi-period framework. Indeed, in a myopic dynamic portfolio management, the sequence of optimization problems is:

$$w_t^* = \underset{w_t}{\operatorname{argmin}}\, R_t(w_t) + \lambda\, c_t \sum_{j=1}^{n} w_{t-1,j}^* \ln\left(\frac{w_{t-1,j}^*}{w_{t,j}}\right), \tag{2.7}$$

where the conditional risk measure $R_t(w_t)$ and the trading cost c_t depend on time. Then the risk contribution restrictions are proportional to $w^*_{t-1,i}$ and path dependent. In this dynamic framework, the λ parameter controls the speed of convergence of the current portfolio towards the time dependent minimum risk portfolio. In a stable risk environment, that is, if R_t and c_t do not depend on time, the optimal dynamic reallocation approaches the minimum risk portfolio in several steps instead of doing the reallocation at a single date. This point is especially appealing for big institutional investors that want to reallocate huge portfolios without destabilizing the markets. [3] This multi-period optimal reallocation approach is also appealing when managing portfolios of illiquid assets.

3 Portfolio Allocation with Systematic Risk Contribution Restrictions

In this section, we consider portfolio allocations constructed to monitor the systematic and idiosyncratic components of the portfolio return. This is done by imposing the risk contribution restrictions on these two components of the total risk. We consider factor models to discuss the effects of the systematic and idiosyncratic components of the risk.

3.1 Systematic and Idiosyncratic Risks

Let us assume that the asset returns follow the one-factor model:

$$y_i = \beta_i\, f + \sigma_i\, u_i, \qquad i = 1, ..., n, \tag{3.1}$$

where f is the common (or systematic) factor, β_i is the factor loading of asset i w.r.t. factor f, and u_i is the idiosyncratic (or specific) component, independent of the factor. We assume that the idiosyncratic terms are mutually independent[4], with unconditional zero mean and unit variance.

[3] This is an important criterion for food commodity markets, when the commodity is also traded for consumption by households, for instance.

[4] Any residual dependence might be captured by introducing additional common factors. This would lead to a multifactor model. We consider the one-factor model for expository purpose.

We get the following decomposition of the return covariance matrix:

$$\Omega = \beta\beta'\sigma_f^2 + \Sigma, \tag{3.2}$$

where $\Sigma = Vu = diag(\sigma^2)$, σ_f^2 is the variance of the common factor and β the vector of factor loadings. The portfolio return can be decomposed accordingly into a systematic and an idiosyncratic component as:

$$w'Y = \left(\sum_{i=1}^{n} w_i\beta_i\right) f + \sum_{i=1}^{n} w_i\sigma_i u_i. \tag{3.3}$$

The effects of the systematic and idiosyncratic components can be analyzed for both risk contributions and portfolio allocations.

3.2 Systematic and Idiosyncratic Risk Contributions

The decomposition principle (3.1) can be applied to disentangle the systematic and idiosyncratic components of the risk as follows:

$$R_i(w) = R_{is}(w) + R_{iu}(w), i = 1, \ldots, n,$$

where $R_{is}(w)$ [resp. $R_{iu}(w)$] denotes the systematic (resp. idiosyncratic) risk contribution of asset i to the total systematic (resp. idiosyncratic) component of the risk. The risk decompositions above can be aggregated to get a decomposition of the total risk as:

$$R(w) = R_s(w) + R_u(w),$$

with $R_s(w) = \sum_{i=1}^n R_{is}(w)$ and $R_u(w) = \sum_{i=1}^n R_{iu}(w)$. These decompositions are summarized in Table 1. This table shows how to pass from the assets $i = 1, ..., n$ tradable on the market, to the virtual assets f and $(u_1, ..., u_n) = u$, which are not directly tradable, that is, how to transform the decomposition of the total risk with respect to basic assets $i = 1, ..., n$ to a decomposition with respect to virtual assets. This is done by constructing an appropriate two entries table, and summing up per column instead of summing up per row [see Gourieroux, Monfort (2012)].

How to derive this thinner risk decomposition in practice, while keeping an interpretation in terms of Euler decomposition? Let us consider the virtual portfolio with allocation $w_{i,s}$ in the systematic component and $w_{i,u}$ in the idiosyncratic one. Thus the associated portfolio return becomes:

$$\left(\sum_{i=1}^{n} w_{i,s}\beta_i\right) f + \sum_{i=1}^{n} w_{i,u}\sigma_i u_i.$$

Assets	Systematic factor	Idiosyncratic error terms	Total
1			$R_1(w)$
$\vdots$			$\vdots$
i	$R_{is}(w)$	$R_{iu}(w)$	$R_i(w)$
$\vdots$			$\vdots$
n			$R_n(w)$
Total	$R_s(w)$	$R_u(w)$	$R(w)$

Table 1: Decomposition of the Global Risk Measure

This portfolio invests $w_{i,s}$ in $\beta_i f$, and $w_{i,u}$ in $\sigma_i u_i, i = 1, \dots, n$. If we denote $\tilde{w}$ the components $w_{i,s}, w_{i,u}$, the risk measure of this virtual portfolio can be written as: $\tilde{R}(w_s, w_u)$, where $\tilde{R}(w, w) = R(w)$. The extended risk measure $\tilde{R}$ is also homogenous of degree 1. Thus we can apply the Euler formula to $\tilde{R}$ and get:

$$\tilde{R}(w_s, w_u) = \sum_{i=1}^{n} w_{i,s} \frac{\partial \tilde{R}}{\partial w_{i,s}}(w_s, w_u) + \sum_{i=1}^{n} w_{i,u} \frac{\partial \tilde{R}}{\partial w_{i,u}}(w_s, w_u).$$

Then, for $w_s = w_u = w$, we deduce the thinner decomposition:

$$\tilde{R}(w) = \sum_{i=1}^{n} w_i \frac{\partial \tilde{R}}{\partial w_{i,s}}(w, w) + \sum_{i=1}^{n} w_i \frac{\partial \tilde{R}}{\partial w_{i,u}}(w, w),$$

and can define: $R_{is}(w) = w_i \frac{\partial \tilde{R}}{\partial w_{i,s}}(w, w)$, $R_{iu}(w) = w_i \frac{\partial \tilde{R}}{\partial w_{i,u}}(w, w)$. Finally, the risk measure $R(w)$ is also function of parameters β_i, σ_i, $i = 1, ..., n$, involved in the factor model and we get:

$$w_i \frac{\partial \tilde{R}}{\partial w_{i,s}}(w, w) = \frac{\partial R}{\partial \beta_i}(w), \ w_i \frac{\partial \tilde{R}}{\partial w_{i,u}}(w, w) = \frac{\partial R}{\partial \sigma_i}(w), \tag{3.4}$$

in which the dependence of R with respect to β_i, σ_i is not explicitely written for expository purpose. We get a decomposition, which highlights the effects on the total portfolio risk of shocks on either the factor, or the idiosyncratic term.

3.3 Portfolios with Systematic Risk Contribution Restrictions

In the standard portfolios with risk contribution restrictions, the constraints are written on the basic assets. The approach can be extended by considering risk contributions written on the systematic and unsystematic components of the portfolio. Let us consider the following optimization problem:

$$w(\delta, \pi) = \underset{w'e=1}{\operatorname{argmin}}\, R^2(w) + \delta\left[(1-\pi)R_s(w) - \pi R_u(w)\right]^2, \tag{3.5}$$

where $\delta \in (0, \infty)$ is a smoothing parameter. In the limiting case $\delta = \infty$, we get the optimization with a strict constraint on the contribution to systematic risk[5] : $R_s(w) = \pi R(w)$. When $\delta = 0$, we get the minimum risk portfolio.

As in Section 2.3, we can justify the introduction of this risk contribution restriction by the effect of trading costs, both on individual assets and on the factors, when derivative instruments allows investors to directly trade on the virtual factor asset. This is the case for equity investing, where the factor is usually the market portfolio.

4 Illustrations with Different Risk Measures

This section provides the closed form expressions of the minimum risk portfolios and the risk contributions for three risk measures, that are the volatility, the Value-at-Risk, and the Distorsion Risk Measures, including the Expected Shortfall.

4.1 The Volatility Risk Measure

When the risk is measured by the volatility, we get: $R(w) = (w' \,\Omega\, w)^{1/2}$.

4.1.1 Minimun variance portfolio

Let us assume that the set of assets does not include the riskfree asset, or equivalently that the volatility matrix Ω is invertible. For the volatility risk measure, we get the minimum-variance portfolio [see Markowitz (1952)], whose optimal allocation has the closed form expression:

$$w* = \frac{\Omega^{-1} e}{e' \,\Omega^{-1}\, e}.$$

[5]If the level π belongs to the domain of admissible values of $R_s(w)/R(w)$, when w varie. Otherwise, we get the portfolio allocation with a systematic risk budget the closest to π.

4.1.2 Risk contributions

We have:

$$\frac{\partial R(w)}{\partial w} = \frac{\Omega\, w}{(w'\,\Omega\, w)^{1/2}},$$

and the risk contributions are:

$$R_i(w) = \frac{w_i}{(w'\Omega w)^{1/2}} \sum_{j=1}^{n} \Omega_{i,j}\, w_j = \frac{Cov(w_i y_i, w'Y)}{V(w'Y)},$$

where $\Omega_{i,j}$ is the generic element of Ω, and $Cov(.)$ and $V(.)$ denote respectively the covariance and the variance. Thus the contribution $R_i(w)$ is the beta coefficient of the part of the portfolio invested in asset i with respect to the total portfolio.

4.1.3 Systematic and idiosyncratic risk contributions

In a single factor model, we have: $R(w) = [w'(\beta\beta'\sigma_f^2 + \Sigma)w]^{1/2}$ and the Euler risk contributions can be written as:

$$R_i(w) = \frac{w_i}{(w'\,\Omega\, w)^{1/2}} \left[\beta_i\, w'\,\beta\, \sigma_f^2 + w_i\, \sigma_i^2\right] = R_{is}(w) + R_{iu}(w),$$

where $R_{is}(w)$ is the systematic risk contribution i and $R_{iu}(w)$ is the idiosyncratic risk contribution of asset i:

$$R_{is}(w) = w_i\, \beta_i \frac{w'\beta\, \sigma_f^2}{(w'\Omega w)^{1/2}}, \qquad R_{iu}(w) = w_i \frac{w_i\, \sigma_i^2}{(w'\Omega w)^{1/2}}.$$

The expression of component $R_{is}(w)$ shows the quantity $w_i\beta_i$ invested in the systematic factor f, and the risk contribution $\frac{w'\beta\, \sigma_f^2}{(w'\Omega w)^{1/2}}$ of a unit invested in f. By adding the decompositions per asset, we get the decomposition of the total portfolio risk as:

$$R(w) = R_s(w) + R_u(w), \;\text{ with }\; R_s(w) = (w'\beta)^2 \frac{\sigma_f^2}{(w'\Omega w)^{1/2}} \;\text{ and }\; R_u(w) = \frac{w'\Sigma w}{(w'\Omega w)^{1/2}},$$

that is the standard variance decomposition equation.

As an illustration we provide in Appendix A this decomposition and its interpretation for the minimum-variance portfolio.

4.2 The α-VaR Risk Measure

The introduction of the VaR corresponds to the safety first criterion initially introduced by Roy (1952). The α-VaR risk measure is defined by:

$$R(w) = -q_\alpha(w'Y),$$

where q_α is the α-quantile of the distribution of the portfolio return. More precisely, the α-VaR is defined by the condition: $P\left[w'Y < q_\alpha(w'Y)\right] = \alpha$.

4.2.1 Minimum α-VaR portfolio

Let us first consider the Gaussian case before discussing the general framework.

- Let us assume that the set of basic assets does not include the riskfree asset and consider the allocation minimizing the α-VaR in a Gaussian framework. When the vector of returns is Gaussian with mean μ and variance-covariance Ω, the optimal allocation minimizes:

  $$-q_\alpha(w'Y) = -w'\mu - q_\alpha w'\Omega w,$$

  where q_α denotes the α-quantile[6] of the standard Gaussian distribution under the budget restriction $w'e = 1$. The minimum α-VaR portfolio allocation is then given by:

  $$w^* = \frac{\Omega^{-1}e}{e'\,\Omega^{-1}\,e} + \frac{1}{2\,q_\alpha}\Omega^{-1}\left[\mu - \frac{e'\,\Omega^{-1}\,\mu}{e'\,\Omega^{-1}\,e}e\right].$$

  This formula highlights the key role of the minimum variance portfolio as the benchmark portfolio for a very risk averse investor (when $\alpha \to 0$ and $q_\alpha \to \infty$), but also the importance of the excess expected returns.

- In the general framework, the returns are not necessarily Gaussian and the minimum α-VaR portfolio is the solution of the system of equations:

  $$\frac{\partial q_\alpha(w'Y)}{\partial w_i} = \lambda(w), i = 1, ..., n, \tag{4.1}$$

[6]Since α is small, q_α is negative. Thus, the α-VaR is an increasing function of the variance of the portfolio return and a decreasing function of its expected return.

where the Lagrange multiplier $\lambda(w)$ is fixed by the budget restriction $w'e = 1$. The derivative of the α-VaR is equal to [Gourieroux, Laurent, Scaillet (2000), Hallerbach (2003)]:

$$\frac{\partial q_\alpha}{\partial w_i}(w'Y) = E\left[y_i | w'Y = q_\alpha(w'Y)\right], i = 1, ..., n. \tag{4.2}$$

This derivative has no closed form expression in general and the minimum α-VaR allocation has to be computed numerically.

4.2.2 Risk contributions

When the risk is measured by the α-VaR, we get the following decomposition formula of the global conditional quantile [see Gourieroux, Monfort (2012)]:

$$q_\alpha(w'Y) = w' \frac{\partial q_\alpha(w'Y)}{\partial w} = w'E\left[Y | w'Y = q_\alpha(w'Y)\right], \tag{4.3}$$

and

$$R_i(w) = E\left[w_i y_i | w'Y = q_\alpha(w'Y)\right]. \tag{4.4}$$

It measures the part of the expected loss due to asset i when the total portfolio is in distress.

4.2.3 Systematic and idiosyncratic risk components

In the α-VaR case, the marginal effect of a change of weight of asset i can be decomposed by Equation (3.4) as :

$$w_i \frac{\partial q_\alpha(w'Y)}{\partial w_i} = \beta_i \frac{\partial q_\alpha(w'Y)}{\partial \beta_i} + \sigma_i \frac{\partial q_\alpha(w'Y)}{\partial \sigma_i}. \tag{4.5}$$

The Euler components associated with systematic and idiosyncratic risks can be explicited as follows :

$$\frac{\partial q_\alpha(w'Y)}{\partial \beta} = E\left[f | w'Y = q_\alpha(w'Y)\right], \qquad \frac{\partial q_\alpha(w'Y)}{\partial \sigma_i} = E\left[u_i | w'Y = q_\alpha(w'Y)\right].$$

In the linear factor model, the general decomposition (3.4) becomes:

$$R_{is}(w) = \beta_i E\left[f | w'Y = q_\alpha(w'Y)\right], \qquad R_{iu}(w) = \sigma_i E\left[u_i | w'Y = q_\alpha(w'Y)\right], \tag{4.6}$$

and the decomposition of the total portfolio risk is:

$$R(w) = R_s(w) + R_u(w),$$

with

$$R_s(w) = \beta' \, E\left[f|w'Y = q_\alpha(w'Y)\right], \qquad R_u(w) = \sum_{i=1}^{n} \sigma_i E\left[u_i|w'Y = q_\alpha(w'Y)\right]. \tag{4.7}$$

4.3 Distorsion Risk Measures

A Distortion Risk Measure is a weighted function of the VaRs associated with the different risk levels. It can be written as:

$$R(w) = \int VaR_\alpha(w) dH(\alpha) = -\int q_\alpha(w) dH(\alpha),$$

where H is a given distortion measure on $(0,1)$, that is, an increasing concave function. The Expected Shortfall is obtained when H is the cumulative distribution function of the uniform distribution on the interval $[0,\alpha]$ [see e.g. Wang (2000), Acerbi (2002), Acerbi, Tasche (2002)].

4.3.1 Minimum DRM portfolio

The optimal allocations have no closed form expression and have to be derived numerically. The minimum DRM portfolios solve the first-order condition [see Gourieroux *et al.* (2000)]:

$$(1 - q_\alpha(w'\,Y))\, E\left[Y|w'\,Y = q_\alpha(w'\,Y)\right] = 0.$$

4.3.2 Risk contributions

Let us for instance consider the Expected Shortfall ES_α. By definition we have:

$$ES_\alpha(w'Y) = w' E\left[Y|w'Y > q_\alpha(w'Y)\right], \tag{4.8}$$

with risk contribution [Tasche (2000)]: $R_i(w) = E\left[w_i y_i|w'Y > q_\alpha(w'Y)\right]$. Thus, the risk decompositions for VaR and ES differ by their conditioning set. These conditioning sets correspond to different definitions of portfolio distress.

5 Application

We apply in this section the portfolio managements of Section 3 above to futures on commodities.

5.1 The investment universe

We consider futures contracts on physical commodities. These assets are split into five sectors as described in Table 2.

Table 2: The Commodities

Energy	**Grains & Seeds**	**Softs**	**Live Stock**	**Metals**
brent crudeoil*	corn*	cocoa	lean hogs*	copper*
heating oil*	rice	coffee*	live cattle*	gold*
light crudeoil*	soybean oil*	cotton*		palladium
natural gas*	soybeans*	orange juice		platinum
	wheat*	sugar*		silver*

The prices are daily closing prices from 14 May, 1990 up to 24 September, 2012, and are all denominated in US$, even for metals traded in London. The physical commodity prices include the storage and transportation costs. The returns are adjusted by rolling the futures positions in order to avoid the delivery process and to get a stable time-to-maturity over time.

We provide in Table 3 summary statistics of the historical distribution of returns, that are the historical mean, volatility, skewness and kurtosis, with the historical Value-at-Risk for risk levels 1%, 5%, 95%, 99%. The VaR at levels 95% and 99% are relevant in case of short sale of the commodity. We also display the historical betas of each asset return with respect to the Dow Jones-UBS (DJUBS) commodity index.

[Insert Table 3 : Summary Statistics of Asset Returns].

We observe rather symmetric distributions, except for commodity ”brent crudeoil”, which is left skewed, and for ”cotton”, which is right skewed [7]. All distributions feature tails fatter than Gaussian tail with kurtosis up to 30-40 for ”brent crudeoil” and ”cotton”.

The betas are all nonnegative and some returns are very sensitive to changes in the market index such as ”brent crudeoil” and ”soybeans”. These large values do not reflect the composition of the DJUBS index only. Indeed this index includes currently 20 physical commodities for 7 sectors. Thus, several commodities in Table 2 are not included in the index. The commodities included in the index are marked with a ”*” in Table 2. Moreover, if the weights of included assets are fixed according to their global economic significance and market liquidity, they are capped. No commodity can compose more than 15% of the index and no sector more than 33%. For instance cocoa, coffee and cotton have similar weights in the index, but cotton has a much higher beta than the two other commodities.

This analysis can be completed by considering the historical bivariate distributions for any pair of assets. We provide in Figure 1 the historical correlations between asset returns and in Figure 2 the scatterplots of the bivariate distributions for the Grains & Seeds sector, the one-dimensional distribution being displayed on the diagonals.

[Insert Figure 1 : Historical Correlation Matrix]

[Insert Figure 2 : Scatterplots of One and Bi-dimensional Distributions for Sector Grains & Seeds]

Figure 1 and Figure 2 provide similar information on the pairwise links between asset returns. We get high correlations especially between ”oil” commodities and within the ”Live Stock” sector. Let us now focus on the sector ” Grains & Seeds” and on Figure 2, where the scatterplots are easy to interpret. We observe small positive dependence between returns for a significant number of pairs, but also strong positive links for pairs of substituable commodities such as ”soybeans” and ”soybean oil”, or ”corn” and ”wheat”. We even observe multiregimes of dependence for ”soybean”

[7] Even if we do not focus on portfolio performances in this paper, note that positive historical skewness of individual asset returns might explain some good performance properties of the equally weighted portfolio [Beleznay et al. (2012)]. The small observed skewness show that this argument will not apply to commodities.

and ”wheat”, where the scatterplot shows two regression lines.

For expository purpose, it is not possible to plot all the return dynamics and we focus in Figure 3 on the sector ”Grains & Seeds”. [8]

[Insert Figure 3 : Returns for the Sector ”Grains & Seeds”]

The evolutions can be very different in such a sector, which is clearly not homogenous. Even if we observe common volatility clustering, there is a switching trend in both mean and volatility for commodity ”soybeans” and partly for the commodity ”soybean oil” positively correlated with it. This is this change of regime in 2004, which explains the double regime dependence mentioned earlier.

5.2 Benchmark portfolio allocations

Let us now consider four portfolio allocations for the sector ”Grains & Seeds”: an equally-weighted portfolio, a minimum-variance portfolio, and two risk parity portfolios using either the volatility, or the VaR at 5%, respectively. The three first portfolios are frequently considered in the H.F. literature [see e.g. DeMiguel et al. (2009)b], and can be used as benchmarks for comparison. The fourth portfolio allocation focuses on extreme risks. The VaR and VaR contributions are estimated by kernel methods[9], the means and variances by their historical counterparts based on the 252 previous observations. For each portfolio, we provide the evolutions of the portfolio weights, of the contributions to volatility and to VaR, respectively, computed under shortselling restrictions.

[Insert Figure 4 : Evolution of Portfolio Weights]

[Insert Figure 5 : Evolution of Volatility Contributions]

[8] The analysis for the other commodity sectors are available from the authors upon request

[9] The standard Nadaraya-Watson estimator has to be adjusted to ensure that the estimated VaR and VaR contributions are compatible, that is satisfy exactly the Euler restriction.

[Insert Figure 6 : Evolution of VaR Contributions]

The main expected effect is to diminish the weights of highly risky assets for all strategies controlling the risk (see Figure 4). At the extreme, the commodity "soybeans" is not introduced in the min-variance allocation, whereas it appears underweighted for strategies based on risk contributions, which are using the "substituability" with the less risky "soybean oil". We also observe the instability over time of the weights for the min-variance portfolio, largely mentioned in the literature. On the contrary the two risk parity portfolios exhibit stable weights with a lower turnover. However the final allocation depends on the risk measure selected to write the risk contribution restrictions.

The risk parity portfolios have rather stable risk contributions for both risk measures [see Figures 5 and 6], especially when we compare their contributions to the VaR with the contribution of the equally weighted and min-variance portfolios. Whereas the min-variance portfolio shows very erratic contributions to total risk, we observe a highly risky trend in the evolution of the risk contribution for the portfolio with naive $1/n$ diversification.

Finally the Variance and VaR contributions of the two last portfolios are almost the same, even if the portfolio VaR and portfolio volatility differ significantly.

5.3 Dynamic portfolio management with control on the turnover

Let us now consider a minimum risk portfolio with a control on the portfolio turnover. The selected measure is the VaR at 5%, and the allocation at date t is the solution of the optimization problem of Section 2.3 :

$$w_t^* = \arg\min_{w_t} VaR_t(w_t) + \lambda \sum_{j=1}^{n} w_{t-1,j}^* ln\left(\frac{w_{t-1,j}^*}{w_{t,j}}\right),$$

with a constant trading cost standardized at $c = 1$. For $\lambda = 0$, we get the minimum VaR portfolio. For positive λ, we reduce the turnover and obtain dynamic portfolio managements corresponding to different aversions to turnover.

[Insert Figure 7: Portfolio Weights, VaR Contributions, Turnover, Contribution to Systematic Risk for $\lambda = 0.01$]

[Insert Figure 8: Portfolio Weights, VaR Contributions, Turnover, Contribution to Systematic Risk for $\lambda = 1$]

In Figures 7 and 8, we have reported the evolutions of the main characteristic of the optimal portfolios : the allocation, the VaR contributions, the turnover (plotted in a logarithmic scale) and the contribution to systematic risk. The figures are given for two values $\lambda = 0.01$ and $\lambda = 1$ of the aversion on trading costs. For $\lambda = 0.01$, the optimal portfolio is closed to the min-VaR portfolio. The initial portfolio is an equally weighted portfolio, as clearly seen by considering the portfolio weights of Figure 8 at the beginning of the period. We observe several effects when increasing parameter λ: i) the weights become much smoother than the VaR contributions, ii) the asset to be withdrawn from the portfolio, such as "soybean", are sold more progressively, iii) the turnover, that are the trading costs, are much smaller, smoother, with wider clusters in the last period.

Finally, even if the basic asset allocations are smoothed when increasing λ, we still observe large movements in the contribution to systematic risk. We will now try to also control for this feature.

5.4 Portfolio management with systematic risk contribution restrictions and turnover

Let us now consider the constrained optimization problem introduced in Section 3.3 with a control for turnover :

$$\begin{aligned} &\min_{w_t} VaR_t(w_t) + \delta[(1-\pi)VaR_{s,t}(w_t) - \pi VaR_{u,t}(w_t)]^2 + \lambda \sum_{j=1}^{n} w^*_{t-1,j} ln\left(\frac{w^*_{t-1,j}}{w_{t,j}}\right), \\ &\text{s.t. } w_t' e = 1, w_{it} \geq 0, i = 1, \ldots, n, \end{aligned} \tag{5.1}$$

which corresponds to a mix between the minimization of the total VaR, the constraint on the risk contribution for systematic risk and the turnover. The risk measure is the VaR at 5%, and the systematic component is driven by a single factor chosen equal to the DJUBS index return.

The optimal allocation depends on control parameters δ, π and λ :

- δ is a smoothing parameter : we get the min-VaR portfolio when $\delta = 0$, and the min-VaR portfolio with strict restriction on the systematic risk contribution when $\delta \to \infty$,

- The benchmark systematic risk contribution π takes values in (0,1). When the factor is a market index, π measures the degree of market neutrality of the portfolio, for extreme risks. When $\pi = 0$, we are looking for a portfolio with no market influence on extreme risks,

- When used, the control for turnover takes two different values : $\lambda = 0.01$ and $\lambda = 1$.

5.4.1 Influence of both the parameters π and δ with $\lambda = 0.01$

[Insert Figure 9 : Marginal Allocations for Rice and Wheat]

In Figure 9, the allocations for commodities ”rice” and ”wheat” are provided as function of δ and π. They are computed for the last available date : September, 24, 2012, and are based on the 252 days preceding the computation date. As expected the allocations of commodities with a large (resp. small) beta diminish (resp. increase), when we get closer to market neutrality. The surfaces feature convexity property, which means that we have no fund Separation Theorem.
Figure 10 provides the evolution according to π and δ of the portfolio systematic risk contribution. As expected, as π increases, and whatever is the value for δ, the systematic risk contribution in the portfolio increases as well.

Figure 11 complements the previous figure and illustrates how the individual systematic risk contributions account for the portfolio systematic risk for some fixed values of π and varying values of δ. When $\delta = 0$, we get a small contribution to systematic risk as expected in the min-VaR portfolio. For large δ, the total contribution to systematic risk is close to the benchmark systematic contribution π. When $\pi = 1$, the individual systematic risk contributions equalize themselves as δ increases and sum-up to the total systematic risk of the portfolio, i.e. 51.8% (see Table 4). The three last columns of Table 4 provide the two entries table of contributions to total risk, that is the estimated counterpart of Table 1, derived for the min-VaR portfolio.

Globally, we observe that the introduction of the (smoothed) constraint on the systematic risk contribution has implied rather similar individual systemic risk contributions, despite very different betas. In fact the introduction of this restrictions on the factor component is stabilizing the allocations and risk contributions [see McKinlay, Pastor (2000) for a similar effect when a factor structure is partially taken into account].

[Insert Figure 10 : Contribution of the Systematic Risk to Total Risk]

[Insert Figure 11 : Relative Contributions to Systematic Risk]

5.4.2 Dynamic evolution of the weights and the systematic risk contributions for fixed values of π, δ and λ

We provide in Figures 12-15 the dynamic evolution of the weights and of the systematic risk contributions to total risk for the portfolios obtained with equation (5.1) for different values of the couple (π,δ) and in the two cases $\lambda = 0.01$ and $\lambda = 1$.

[Insert Figure 12 : Portfolio Allocations for $\lambda = 0.01$]

[Insert Figure 13 : Relative Contributions to Systematic Risk for $\lambda = 0.01$]

[Insert Figure 14 : Portfolio Allocations for $\lambda = 1$]

[Insert Figure 15 : Relative Contributions to Systematic Risk for $\lambda = 1$]

For the small value of $\lambda = 0.01$, it seems quite possible to control the market neutrality of the final portfolio by changing the values of π and δ. The higher are π and δ values, the higher are the allocations of the assets with a higher β (this is the case for Rice and Wheat on Figure 12). On the opposite, when the manager focuses on reducing the turnover of its portfolio (e.g. when $\lambda = 1$), then it seems very difficult to balance this condition whatever are the values of π and δ.

Note also that a portfolio management which controls for extreme risk does not necessarily imply a "diversification" in terms of portfolio allocation. It may be less risky to allocate the budget in a small number of assets. This phenomena is clearly seen on the top right panels of Figure 12 at the end of the period.

When we look at Figure 13, we note that the solicited level of systematic risk contribution is reached rather slowly, with fluctuation at the end of the period. This solicited level cannot be reached when the control parameter λ is equal to 1 as it is shown on Figure 15. Indeed, whatever the π is, we get the same dynamic pattern of the systematic risk contributions. For such a level ($\lambda = 1$), the tradeoff between the control on the systematic risk contributions and the turnover is clearly in favor of this latter and we get then stuck with the initial portfolio, which is, in the case of Figure 14, set as being the equally-weighted portfolio. This example shows that the calibration of the different control parameters is essential and yields to quite different portfolio profiles.

6 Concluding Remarks

We have introduced in this paper a unified optimization framework for asset allocation, which provides a mix between risk minimization, weakened risk contribution restrictions and turnover. These allocation techniques include the most well-known allocation procedures, such as the mean-variance and the minimum-variance allocation as well as the equally weighted and risk parity portfolios.

There exist at least four reasons for considering such a mix focusing on the systematic component of the risk :

- the first one is to account for transaction costs, when looking for the portfolio adjustment. In this respect the introduction of constraints on the risk contributions can have such an interpretation,
- the second one is to account for the regulation for financial stability, that is, for the introduction of constraints on the budgets allocated to the different types of assets, according to their individual risk, but also to the capital required for systematic risk, which is based on the risk contribution. This justifies a restriction written on the systematic component of the portfolio,

- the third one is the possibility to manage the degree of market neutrality of the portfolio,
- finally, the standard mean-variance approach applied to a large number of assets is very sensitive to small changes in the inputs, especially to the estimate of the volatility-covolatility matrix of asset returns. The introduction of budget and/or risk contributions on either asset classes, or types of risks (systematic vs unsystematic) will robustify such an approach as well as the accounting for turnover.

However, if such a mix is needed, there is no general method to select an optimal mix, which might depend on the preference of the investor, but also on the liquidity features and on the potential regulation. In this framework, the best approach consists in considering different mix, to apply them empirically for portfolio allocation and compare the properties of the associated portfolios in terms of stability over time of budget allocations, risk contributions and performances.

Our approach is easily extended to other type and number of factors. At the limit, these factors on which the risk budgeting constraints are written might be at the disposal of the portfolio manager and be selected to create oriented portfolio managements [see e.g. Meucci (2007)].

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A Minimum variance portfolio and systemic risk contribution

Let us decompose the minimum-variance allocation to disentangle the effects of systematic and unsystematic components. This portfolio is given by:

$$w^* = \frac{\Omega^{-1}e}{e'\Omega^{-1}e} = \frac{(\beta\beta'\sigma_f^2 + \Sigma)^{-1}e}{e'(\beta\beta'\sigma_f^2 + \Sigma)^{-1}e}.$$

The inverse $(\beta\beta'\sigma_f^2 + \Sigma)^{-1}$ admits the explicit expression:

$$(\beta\beta'\sigma_f^2 + \Sigma)^{-1} = \Sigma^{-1} - \frac{\sigma_f^2\,\Sigma^{-1}\beta\beta'\Sigma^{-1}}{1 + \sigma_f^2\,\beta'\Sigma^{-1}\beta}.$$

We deduce that:

$$w^* = \frac{\Sigma^{-1}e + \sigma_f^2[\beta'\Sigma^{-1}\beta\Sigma^{-1}e - \beta'\Sigma^{-1}e\Sigma^{-1}\beta]}{e'\Sigma^{-1}e + \sigma_f^2[\beta'\Sigma^{-1}\beta e'\Sigma^{-1}e - (\beta'\Sigma^{-1}e)^2]}. \tag{A.1}$$

Thus, w^* is a weighted average of the optimal allocation in the idiosyncratic virtual assets, i.e. $w_k^* = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e}$, and of the optimal allocation in the systematic virtual asset, i.e.

$$w_s^* = \frac{\beta'\Sigma^{-1}\,\beta\,\Sigma^{-1} - \beta'\,\Sigma^{-1}\,e\,\Sigma^{-1}\,\beta}{\beta'\,\Sigma^{-1}\,\beta\,e'\,\Sigma^{-1} - (\beta'\,\Sigma^{-1}\,e)^2}.$$

Symmetrically, we get a decomposition of the total risk of the minimum-variance portfolio into its systematic and unsystematic risk contributions. We get:

$$R_s(w^*) = \frac{(e'\Omega^{-1}\beta)^2\sigma_f^2}{(e'\Omega^{-1}e)^{3/2}}, \qquad R_u(w^*) = \frac{e'\Omega^{-1}\Sigma\Omega^{-1}e}{(e'\Omega^{-1}e)^{3/2}}. \tag{A.2}$$

Table 3: Summary Statistics of Asset Returns

NAME	Beta	Ann.Ret	Ann.Vol	Min.Ret	Max.Ret	Skew	Kurt	MaxDD	VaR1%	VaR5%	VaR95%	VaR99%
brentcrudeoil	126.90	10.50	26.17	-30.46	12.22	-1.20	29.66	-73.92	-4.43	-2.43	2.50	4.19
heatingoil	96.46	5.67	19.79	-17.88	7.67	-0.51	14.58	-64.51	-3.35	-1.88	1.97	3.40
lightcrudeoil	93.24	4.67	18.86	-18.42	8.65	-0.78	18.81	-64.82	-3.41	-1.77	1.78	3.33
naturalgas	55.49	-7.47	19.07	-7.49	12.89	0.53	11.92	-92.03	-3.52	-2.01	1.77	3.91
corn	57.62	1.11	18.56	-11.44	12.55	0.31	15.15	-68.32	-3.65	-1.66	1.75	3.78
rice	20.57	-1.62	12.81	-4.69	3.58	-0.04	6.65	-54.00	-2.46	-1.25	1.31	2.52
soybeanoil	53.87	1.86	15.62	-5.45	6.06	0.13	7.14	-54.04	-2.73	-1.45	1.61	2.79
soybeans	103.65	12.92	34.80	-14.98	14.34	0.04	6.70	-73.75	-6.02	-3.35	3.42	6.15
wheat	43.69	-0.84	15.66	-7.95	7.34	0.17	12.59	-64.82	-3.12	-1.38	1.49	2.98
leanhogs	14.98	-2.17	12.48	-5.14	4.98	-0.05	5.95	-59.34	-2.20	-1.31	1.25	2.11
livecattle	11.97	0.67	8.17	-5.67	2.57	-0.36	7.96	-30.31	-1.36	-0.83	0.85	1.31
copper	90.92	10.30	27.48	-12.58	11.54	-0.04	6.26	-64.31	-4.81	-2.62	2.77	4.65
gold	38.81	5.30	13.57	-6.06	8.53	0.03	12.23	-40.03	-2.68	-1.29	1.31	2.24
palladium	56.89	10.26	26.68	-10.69	13.32	0.00	8.95	-83.94	-5.03	-2.54	2.66	4.53
platinum	50.18	7.62	19.85	-8.47	9.39	-0.28	7.92	-63.67	-3.55	-1.94	1.87	3.22
silver	77.05	9.67	24.67	-16.50	12.51	-0.68	13.34	-56.34	-4.76	-2.30	2.31	4.24
cocoa	33.07	0.42	21.05	-8.09	7.97	-0.03	7.39	-60.72	-3.89	-2.05	2.09	3.72
coffee	39.63	1.41	21.71	-9.68	17.72	0.68	16.13	-65.29	-3.75	-2.03	2.14	3.82
cotton	85.34	5.20	37.37	-21.74	34.66	1.18	38.31	-95.66	-6.81	-2.75	2.94	6.95
orange_juice	23.69	1.42	29.67	-16.58	20.04	0.63	15.51	-82.78	-5.39	-2.74	2.63	5.60
sugar	67.78	14.42	37.95	-11.91	19.61	0.27	7.97	-65.71	-7.00	-3.60	3.93	6.66
DJUBS Index	100.00	0.01	15.62	-8.76	5.81	-0.25	7.25	-57.13	-2.74	-1.53	1.55	2.61

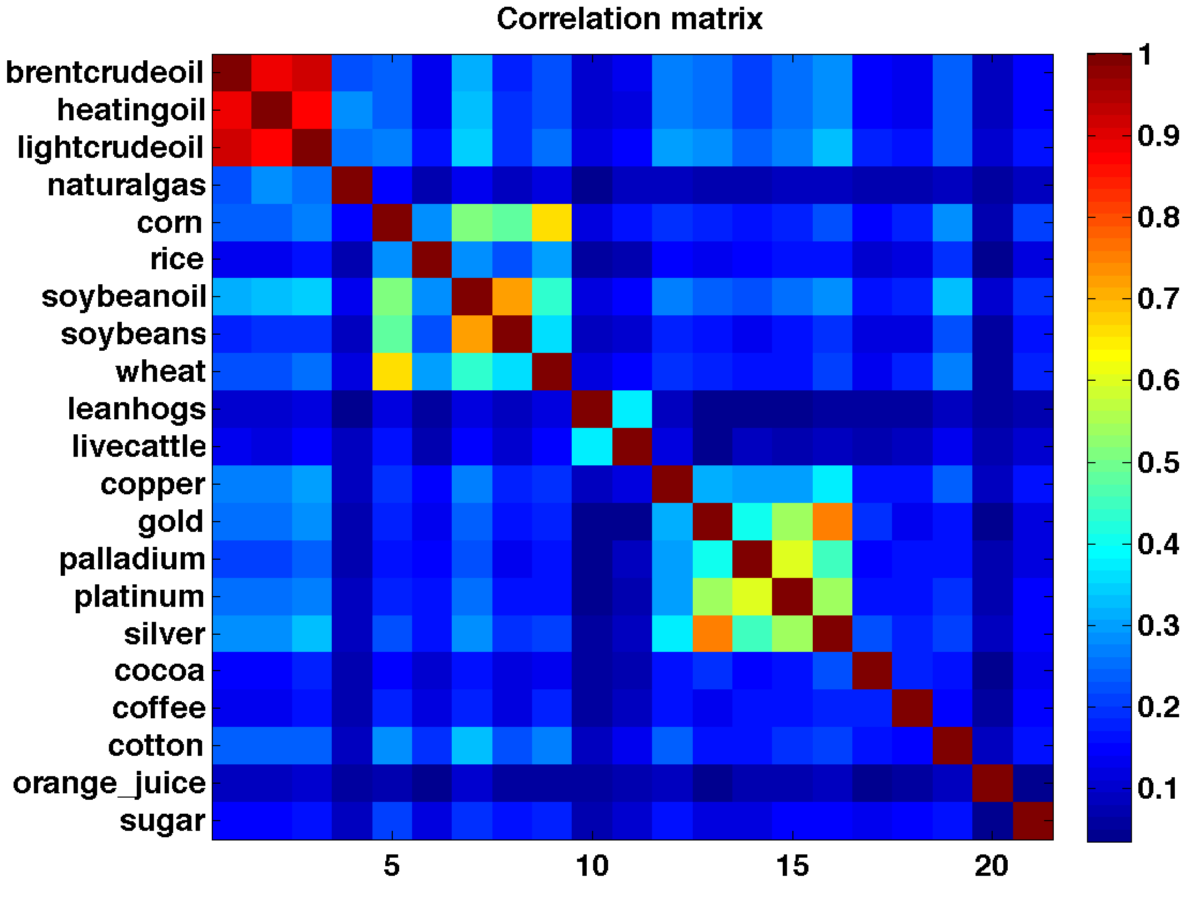


Figure 1: Historical Correlation Matrix

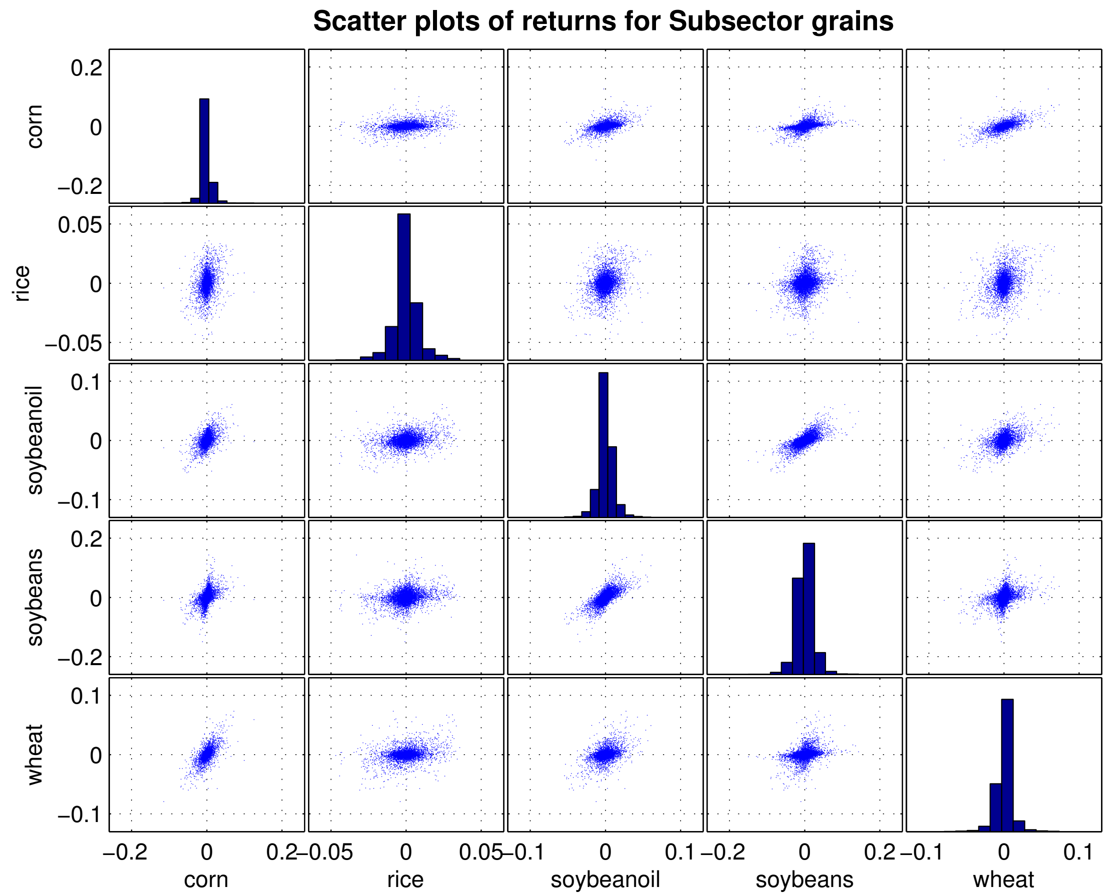


Figure 2: Scatterplots of One-and Bi-dimensional Distributions for the Sector Grains & Seeds

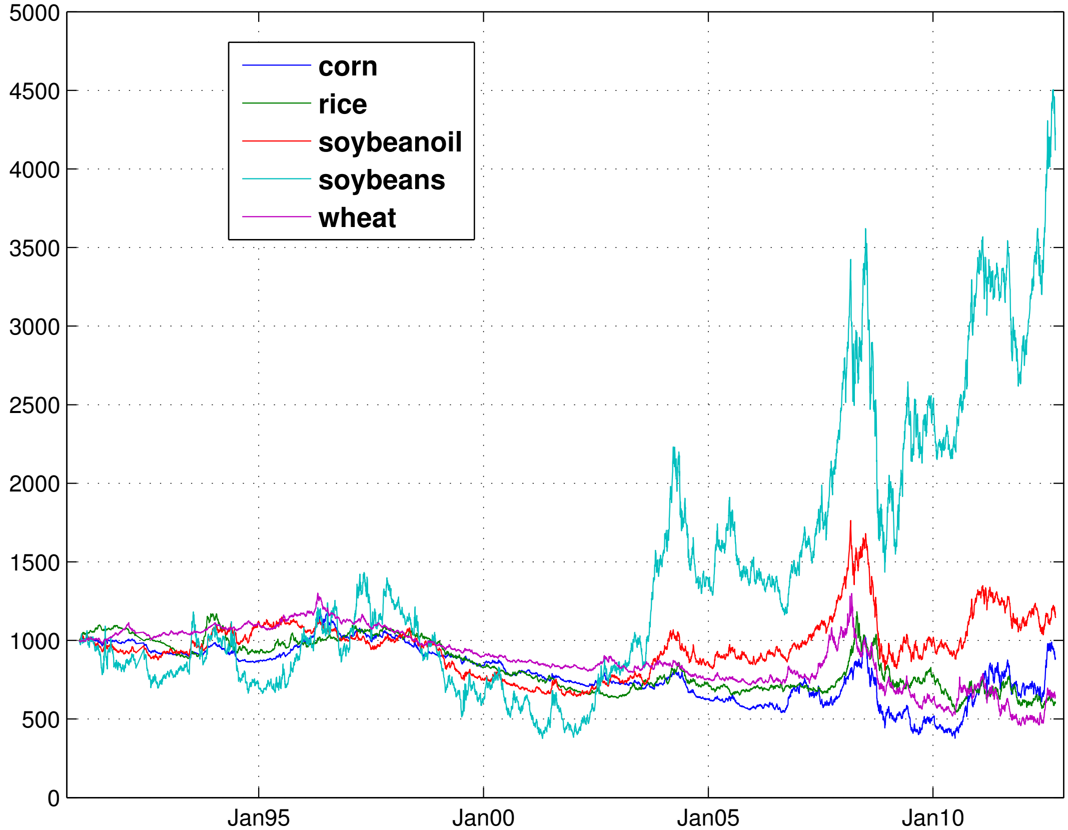


Figure 3: Returns for the Sector Grains & Seeds

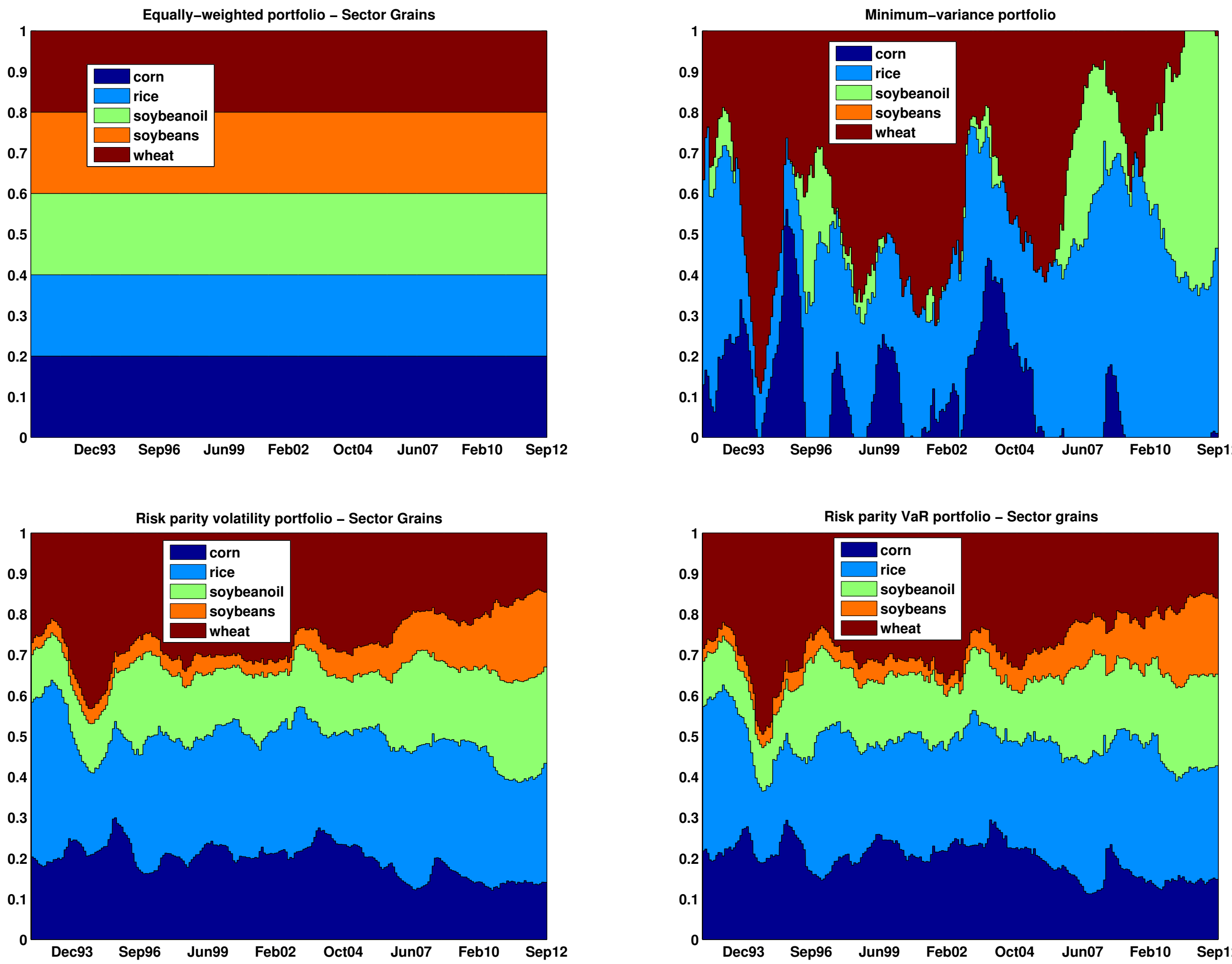


Figure 4: Evolution of Portfolio Weights for the Sector Grains & Seeds

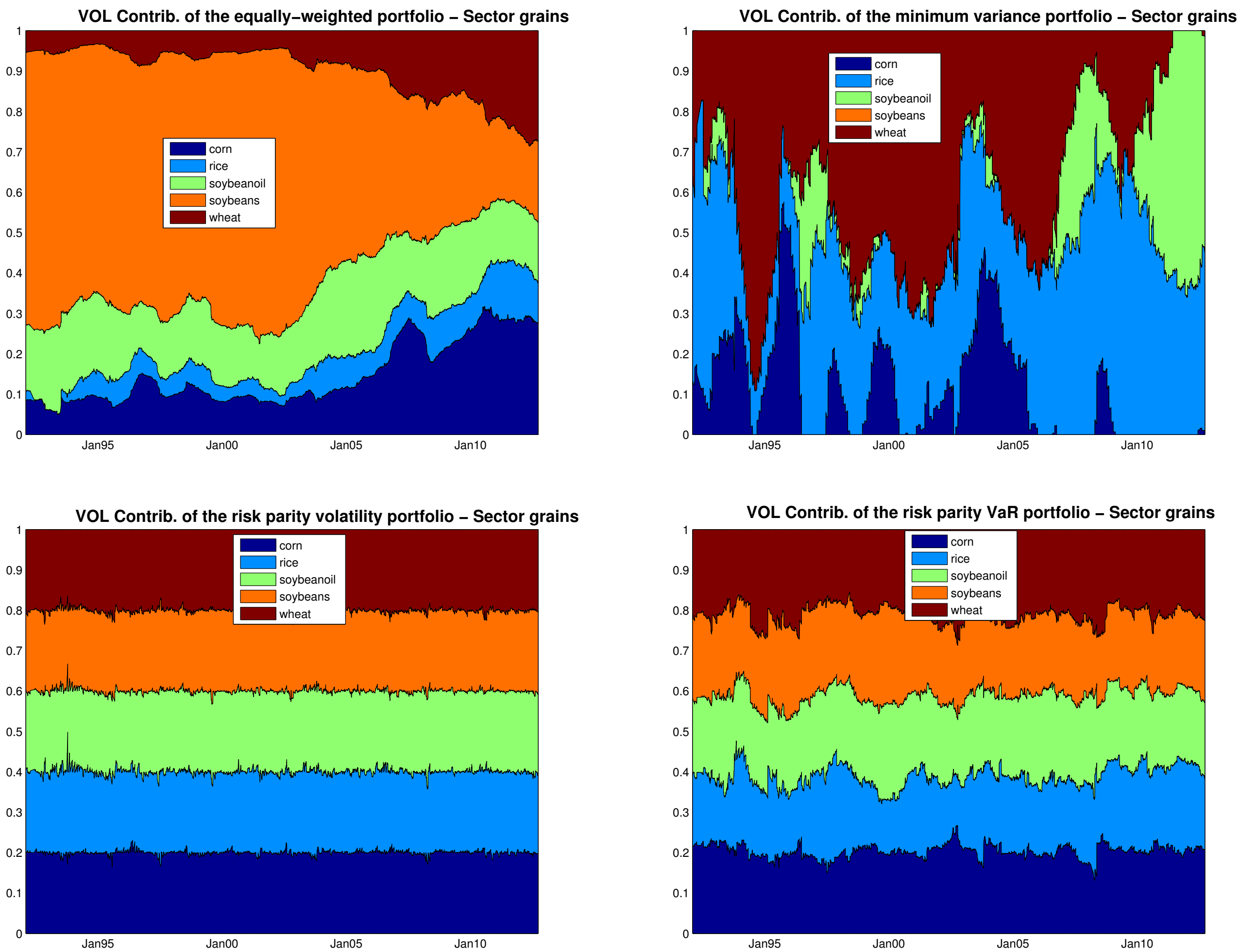


Figure 5: Evolution of Volatility Contributions in the Sector Grains & Seeds

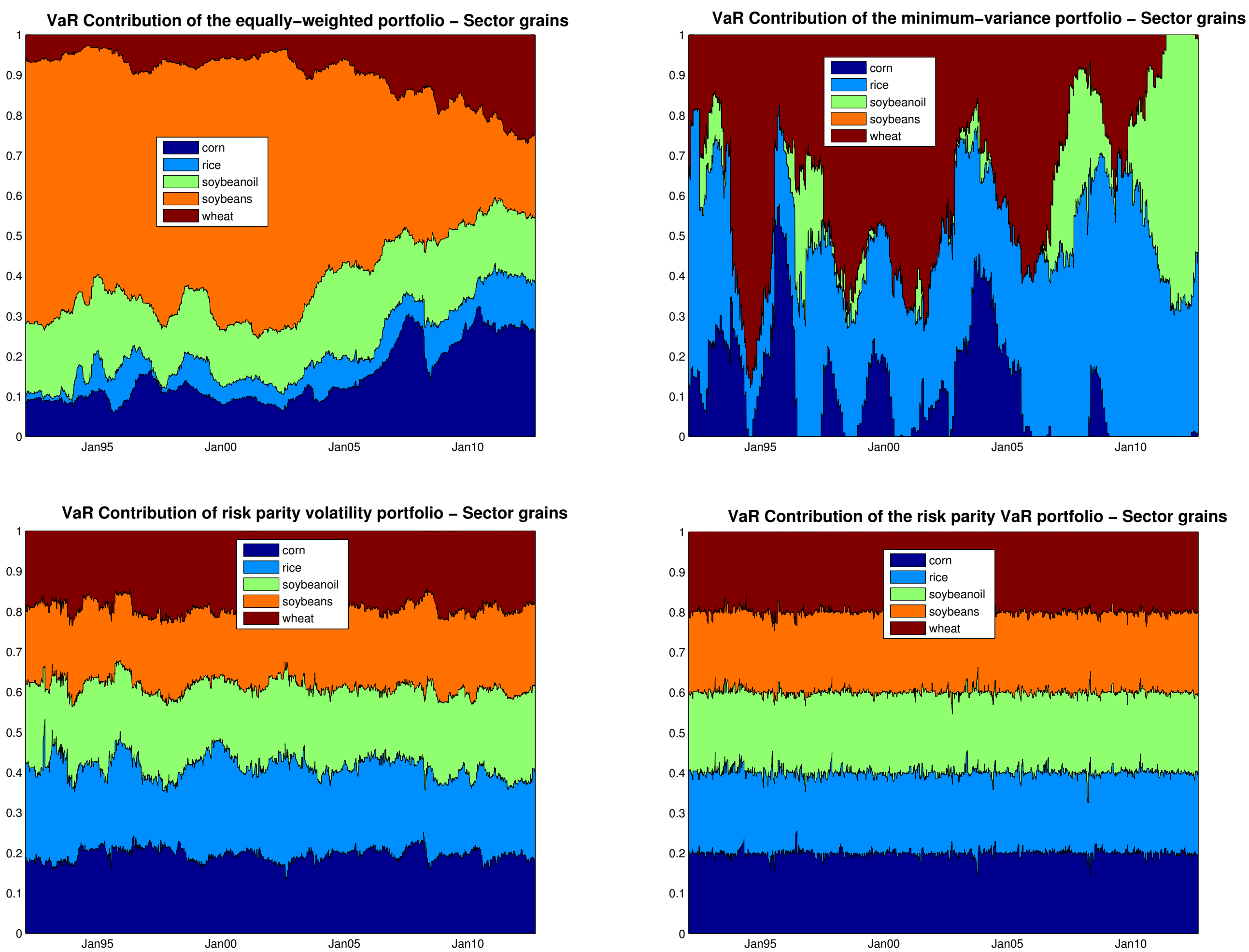


Figure 6: Evolution of VaR Contributions in the Sector Grains & Seeds

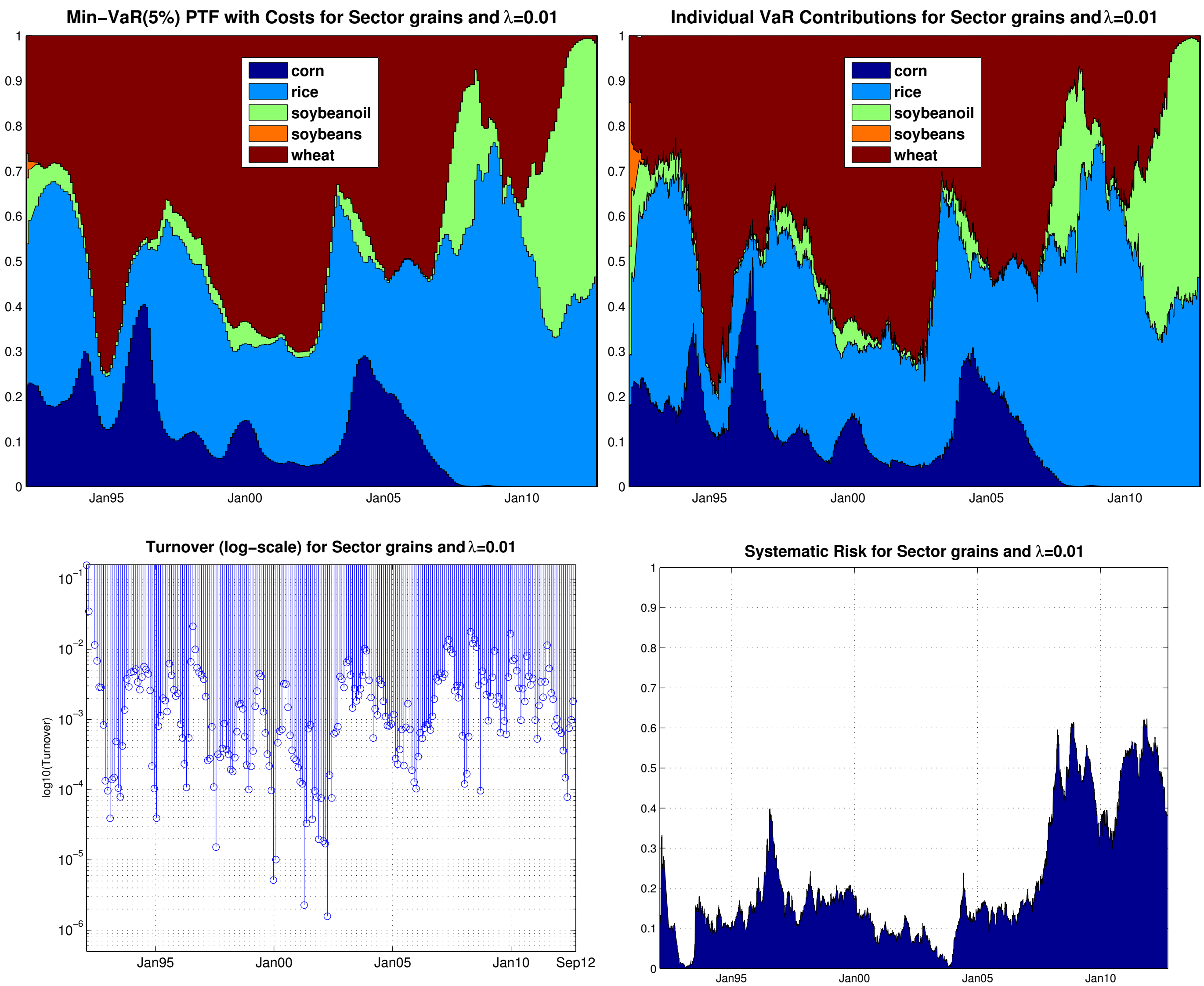


Figure 7: Weights, VaR contributions, turnover (in a logarithmic scale) and systematic risk of the 5%-VaR portfolio (see Equation (2.7)) for a unitary value of the costs ($c = 1$) and a small value of $\lambda = 0.01$ obtained for the sector Grains & Seeds

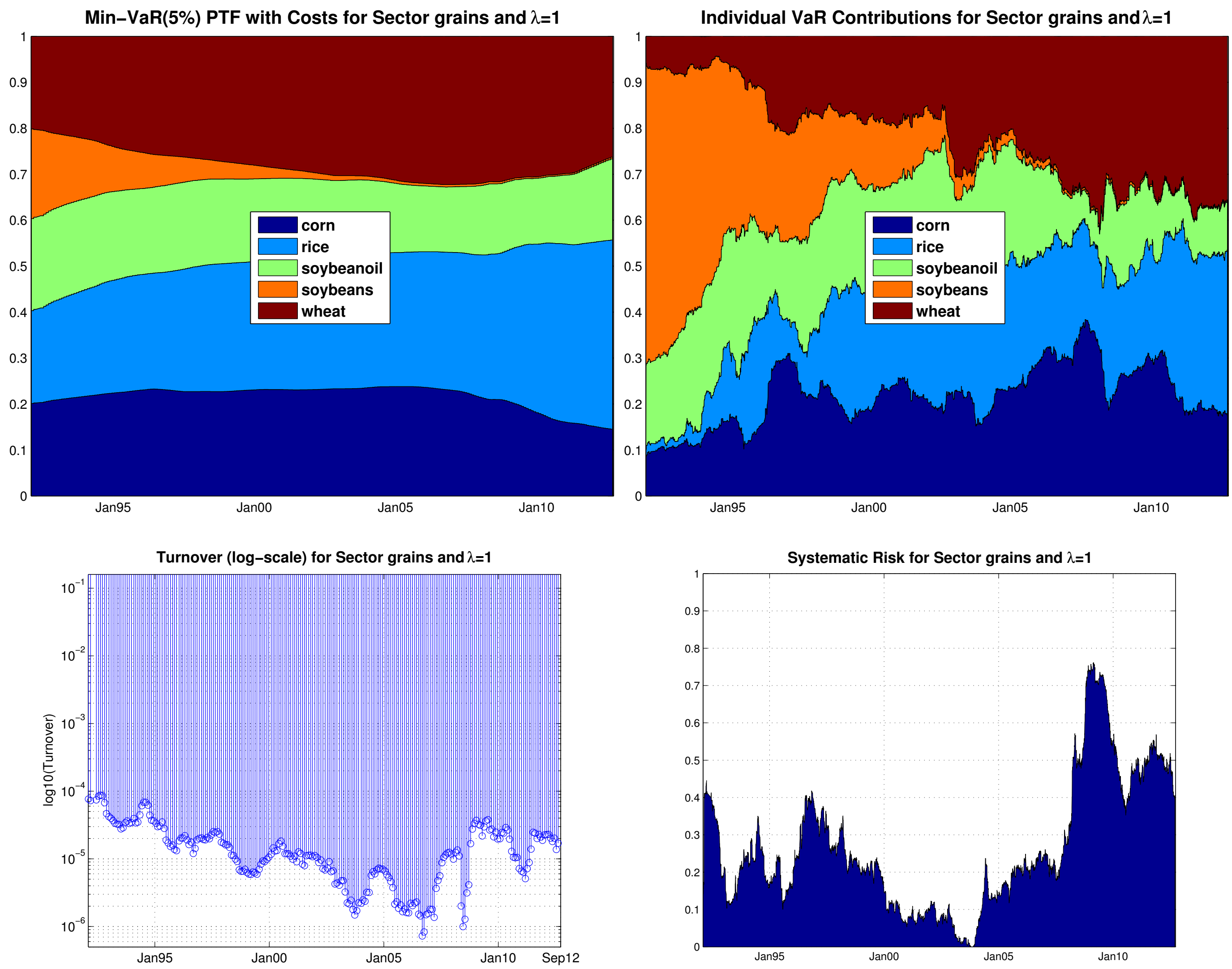
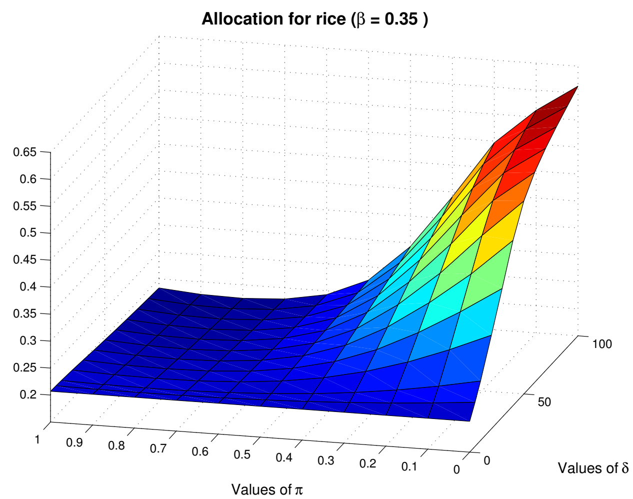


Figure 8: Weights, VaR contributions, turnover (in a logarithmic scale) and systematic risk of the 5%-VaR portfolio (see Equation (2.7)) for a unitary value of the costs ($c = 1$) and a quite high value of $\lambda = 1$ obtained for the sector Grains & Seeds



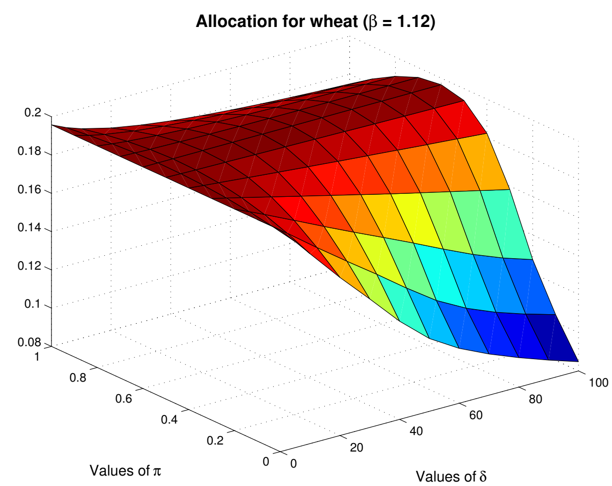


Figure 9: Marginal allocations as of 24-Sept-2012 for rice and wheat (having respectively the smallest and the largest estimated β) for various values of the parameters π and δ and with a turnover control parameter $\lambda = 0.01$. These allocations are obtained by minimizing equation (3.5) where $R(w)$ is the α-VaR risk measure with $\alpha = 5\%$

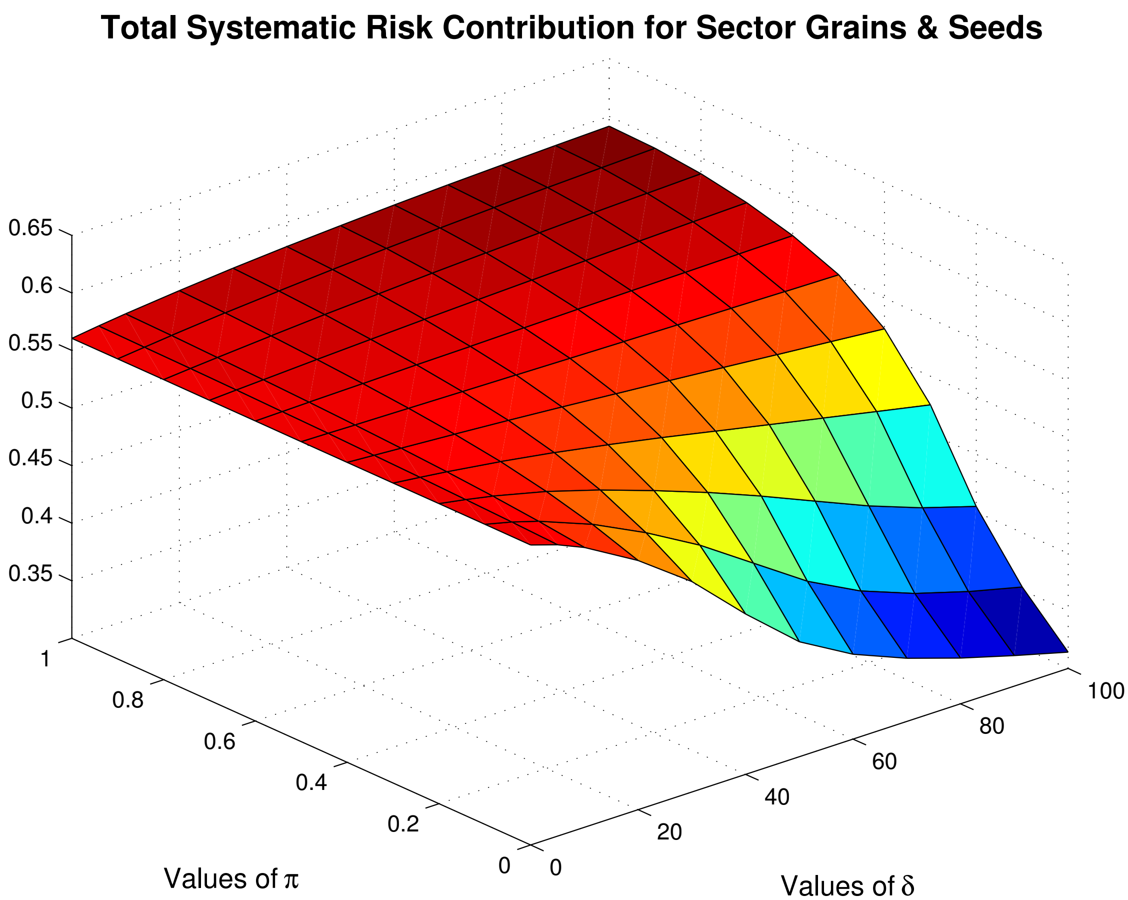


Figure 10: Contribution of the systematic risk to the total risk for the Minimum 5%-VaR portfolio with $\lambda = 0.01$ and for the Sector Grains & Seeds.

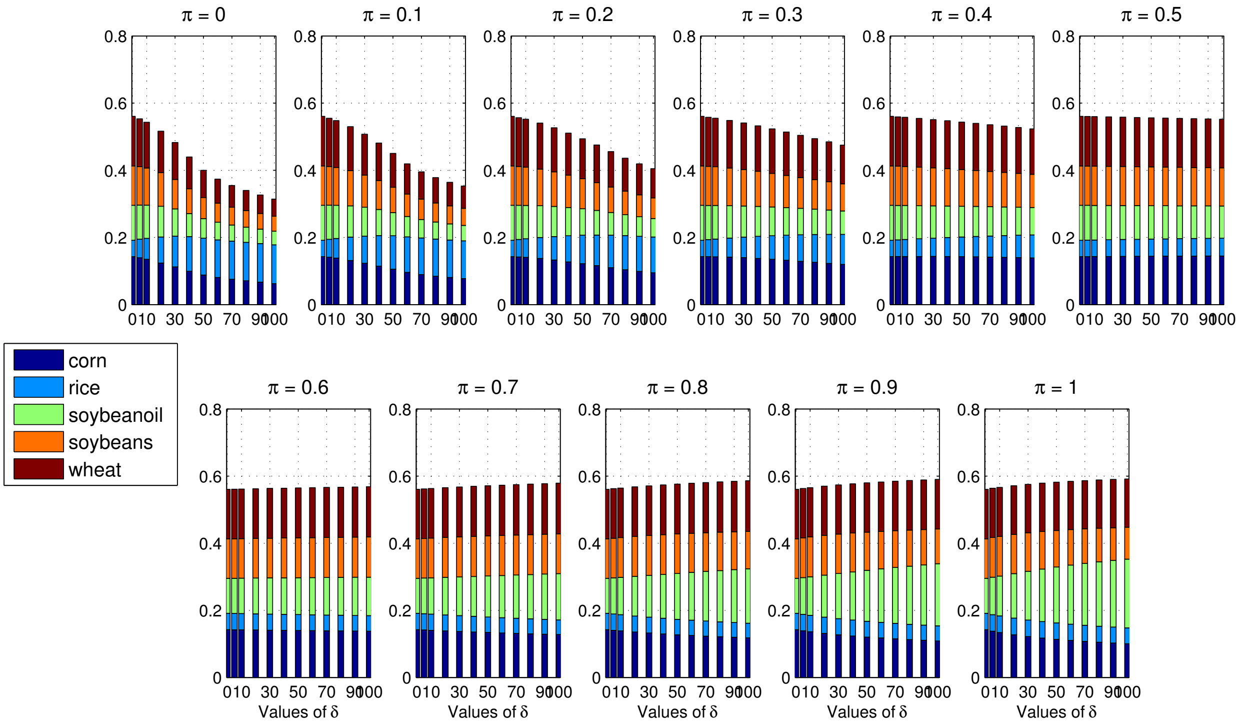


Figure 11: Relative Contributions to Systematic Risk for some fixed values of π, varying values of δ and a turnover control parameter $\lambda = 0.01$, as of 24-Sept-2012 in the Sector Grains & Seeds.

Table 4: Decomposition of Total Risk for a min-VaR (5%) portfolio with a turnover control parameter $\lambda = 0.01$ in the sector Grains & Seeds as of 24-Sept-2012.

Assets	**Beta**	**Risk parity VaR weight**	**Systematic Factor**	**Idiosyncratic Error**	**Total**
Corn	1.09	11.9%	9.3%	6.9%	16.2%
Rice	0.35	34.4%	8.7%	18.2%	26.8%
SoybeanOil	0.76	24.4%	13.2%	8.8%	22.1%
Soybeans	0.87	16.3%	10.2%	8%	18.2%
Wheat	1.12	13%	10.4%	6.3%	16.7%
Total		100%	51.8%	48.2%	100%

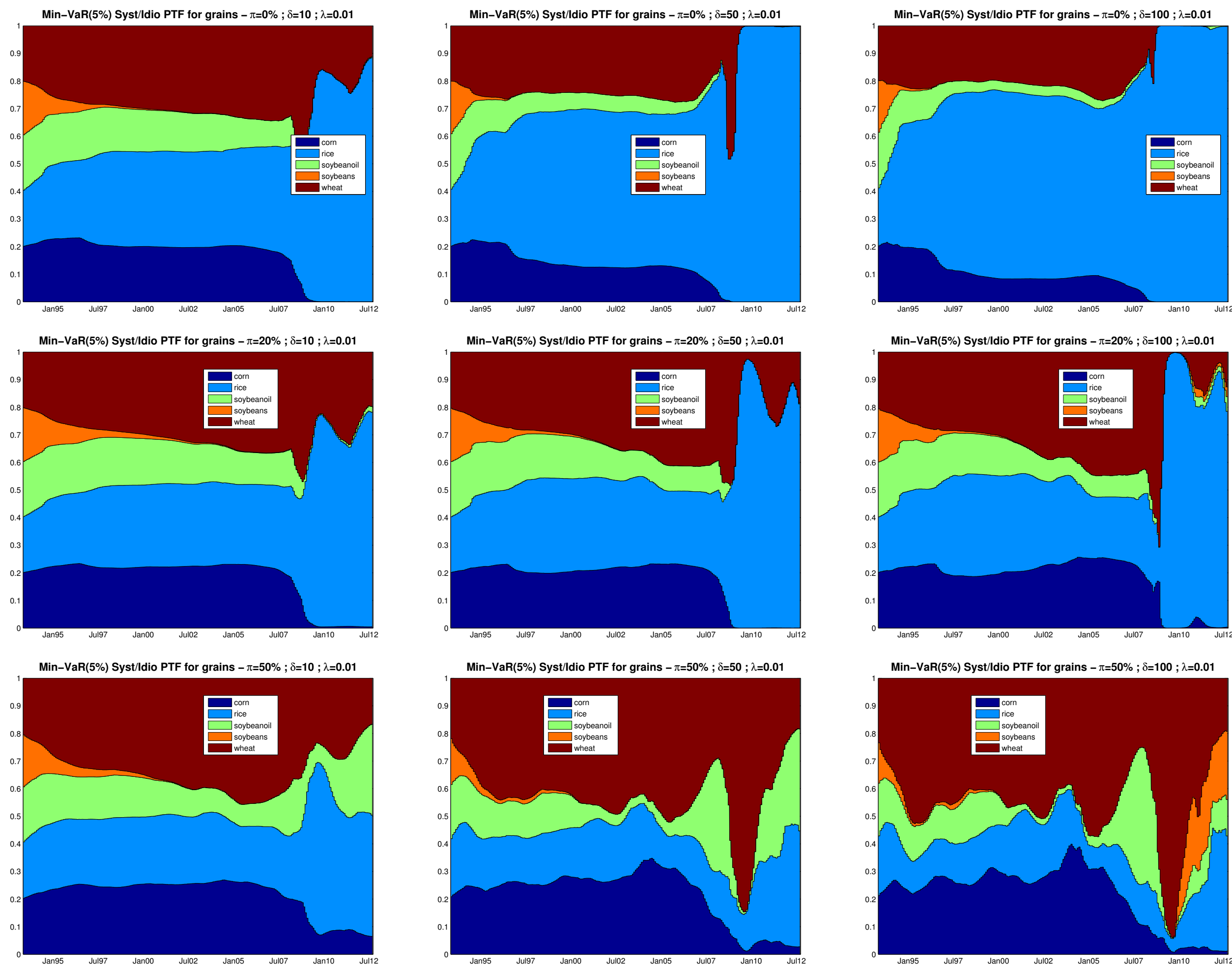


Figure 12: Allocations of the 5%-VaR portfolio in sector Grains & Seeds including the transaction costs balanced with a quite low parameter $\lambda = 0.01$ and for different couples of parameters (π, δ). Each row is for a fixed value of $\pi \in \{0; 0.2; 0.5\}$, and each column is for a fixed value of $\delta \in \{10; 50; 100\}$.

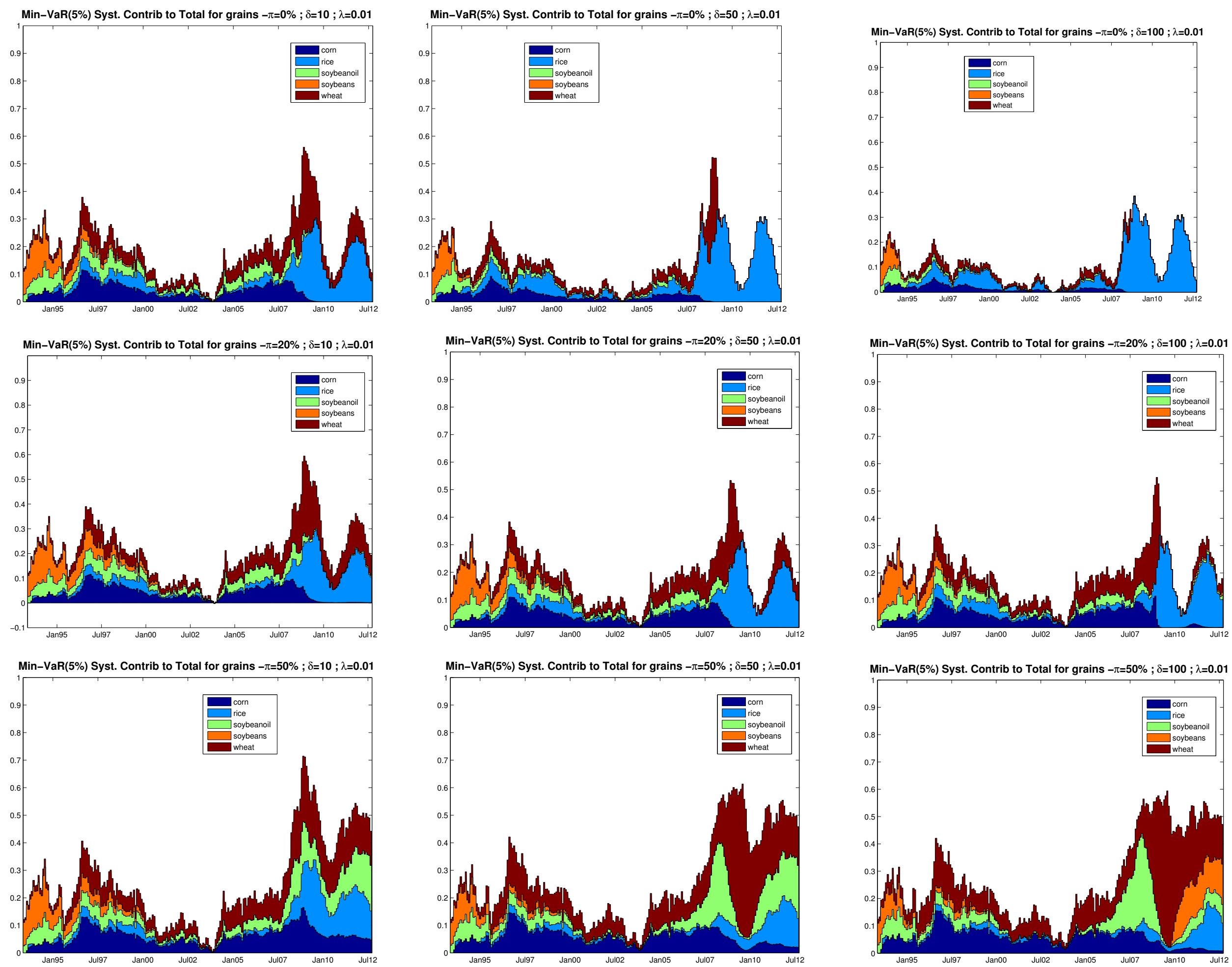


Figure 13: Systematic risk contributions of the 5%-VaR portfolio in sector Grains & Seeds including the transaction costs balanced with a quite low parameter $\lambda = 0.01$ and for different couples of parameters (π, δ). Each row is for a fixed value of $\pi \in \{0; 0.2; 0.5\}$, and each column is for a fixed value of $\delta \in \{10; 50; 100\}$.

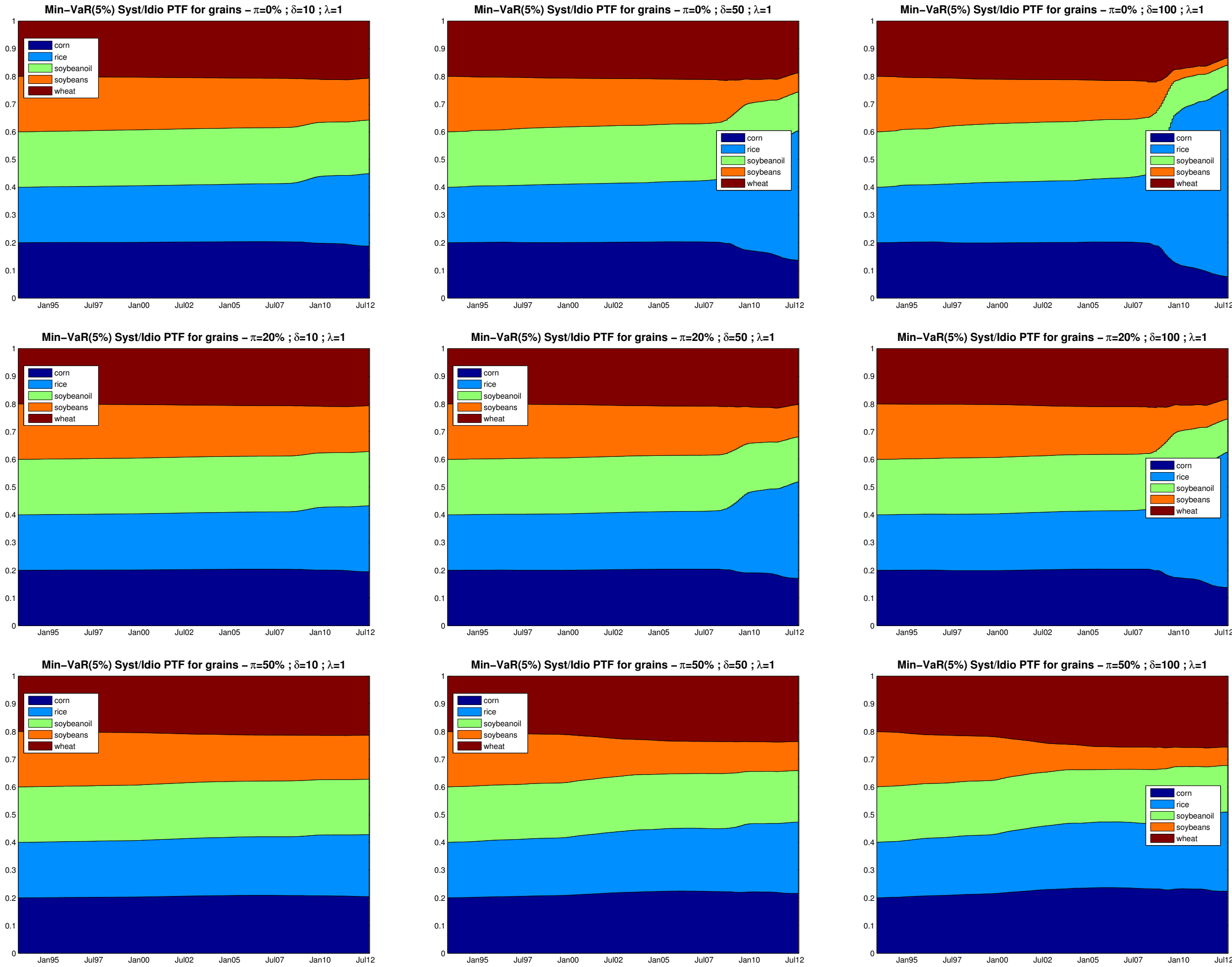


Figure 14: Allocations of the 5%-VaR portfolio in sector Grains & Seeds including the transaction costs balanced with a quite high parameter $\lambda = 1$ and for different couples of parameters (π, δ). Each row is for a fixed value of $\pi \in \{0; 0.2; 0.5\}$, and each column is for a fixed value of $\delta \in \{10; 50; 100\}$.

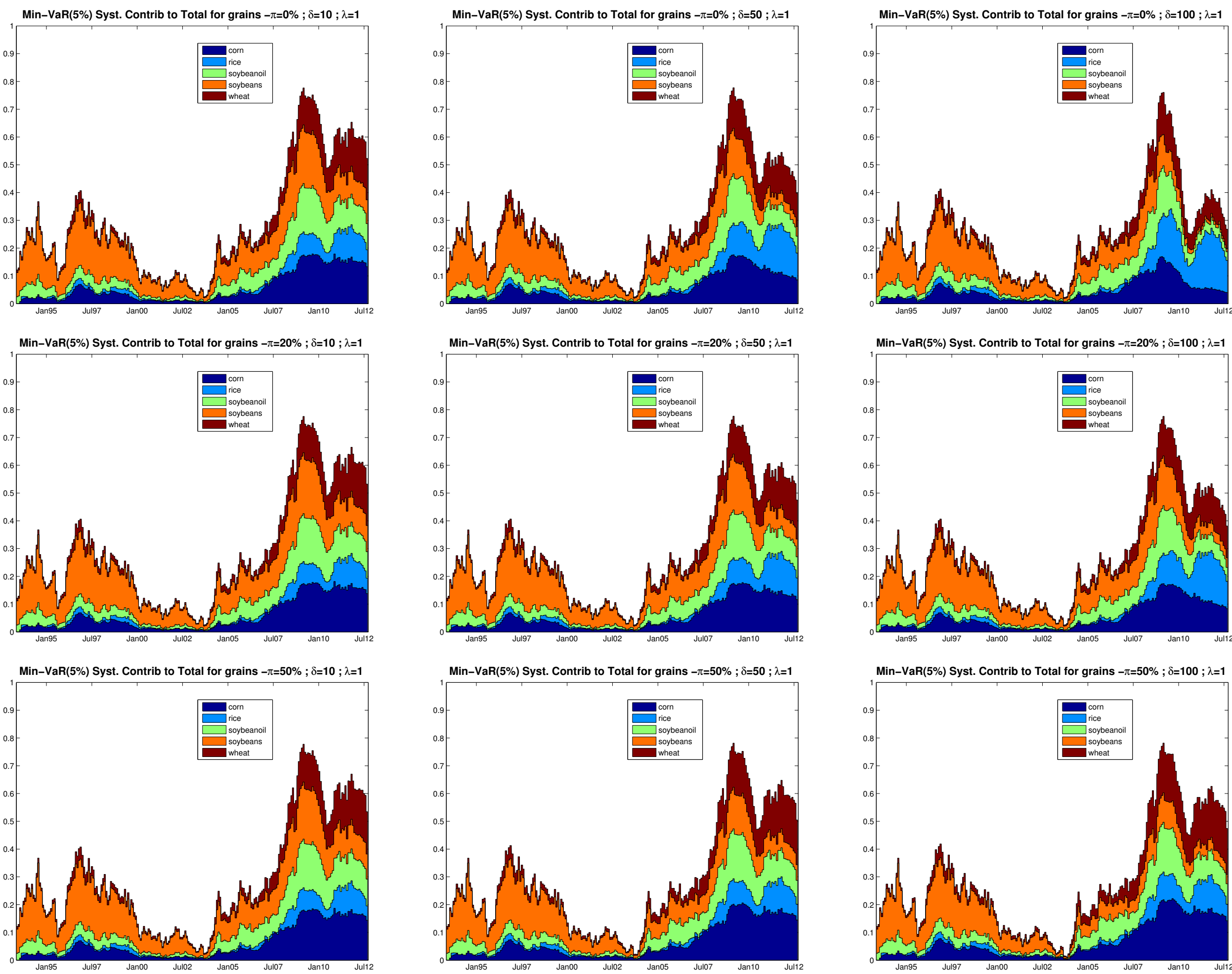


Figure 15: Systematic risk contributions of the 5%-VaR portfolio in sector Grains & Seeds including the transaction costs balanced with a quite high parameter $\lambda = 1$ and for different couples of parameters (π, δ). Each row is for a fixed value of $\pi \in \{0; 0.2; 0.5\}$, and each column is for a fixed value of $\delta \in \{10; 50; 100\}$.