Centre de Recherche en Économie et Statistique
CREST

Série des Documents de Travail

n° 2011-12

Joint Econometric Modeling of Spot Electricity Prices, Forwards and Options

A. MONFORT[1]
O. FÉRON[2]

[1] Centre de Recherche en Economie et Statistiques (CREST) and University of Maastricht. 15 boulevard Gabriel Péri, 92245 Malakoff, France. monfort@ensae.fr - Tel. 33 (0)1 41 17 77 28.

[2] EDF Research and Development and FiME, Laboratoire de Finance des Marchés de l'énergie, 1 avenue du Général de Gaulle, 92141 Clamart Cedex, France. olivier-2.feron@edf.fr Tel. 33 (0)1 47 46 65 36 38.

JOINT ECONOMETRIC MODELING OF SPOT ELECTRICITY PRICES, FORWARDS AND OPTIONS

Alain MONFORT*, Olivier FÉRON**

Abstract

We propose a joint modeling of spot electricity prices , forwards prices and other derivative prices, using recent developments in discrete time asset pricing methods based on the notions of stochastic discount factor and of Compound Autoregressive (or affine) stochastic processes.We show that this approach provides quasi explicit formulae for forward and option prices , while allowing for a large flexilbility in the modeling of dynamics ,spikes and seasonality , both in the historical and the risk neutral worlds. We also propose a variety of inference techniques involving inversion methods, the Kalman filter and the Kitagawa-Hamilton filter. Finally , examples using simulations of spot and forward prices illustrate the large potentialities of our modeling.

Keywords: Electricity derivative pricing , spikes, Car processes, stochastic discount factor, Kitagawa-Hamilton filter.

1. INTRODUCTION

There exists a growing literature dealing with statistical models of the time series of spot electricity prices. Two important features of these models are seasonal patterns and spikes, namely upwards jumps shortly followed by a downward move. Some of these papers start from continuous time specification [see e;g. Deng (2000), Weron, Bierbrauer and Truck (2004), Geman and Roncoroni (2006)], others directly use a discrete time approach. In the latter set of studies the spikes are often captured by switching regime

*Centre de Recherche en Economie et Statistiques (CREST) and University of Maastricht, 15 boulevard G. Peri , 92240 Malakoff , France, monfort@ensae.fr, tel. 33 (0)1 41 17 77 28

**EDF Research and Development and FiME, Laboratoire de Finances des Marchés de l'énergie, 1 Avenue du Général de Gaulle, F-92141 Clamart Cedex, France, olivier-2.feron@edf.fr, tel. 33 (0)1 47 46 65 36 38

models driven by a hidden Markov chain similar to the ones proposed by Hamilton (1989). In these approaches the Markov chain may be homogenous [see Huisman and Mahieu (2003), De Jong (2005)] or time-varying [see Mount, Ning and Cai (2006)]. A survey paper is proposed by Bunn and Karakatsahi (2003). Another strand of literature studies the forward prices. In this field a seminal paper is the one by Lucia and Schwartz (2002), who use a basic mean reverting continuous time model with a deterministic component and a simple way to transpose this model in the risk neutral world. This paper has been followed by several papers using more advanced continuous time stochastic processes [see e.g. Henth, Kallsen and Meyer -Brandis (2005), Benth and Koekebakker (2005)].

However these two strands of literature are disconnected and the aim of this paper is to propose a global approach of the dynamics of spot electricity prices, forward prices and options which is able to model at the same time the dynamics of the spot prices and the pricing of derivatives.

This approach rests on four pillars. The first pillar is a probabilistic breakthrough: the affine or Compound Autoregressive (Car) processes [see Darolles, Jasiak and Gourieroux (2008)]. The second pillar is a mathematical tool: the Laplace transform, which is particularly well adapted to the Car class. The third pillar is the recent development of the econometrics of asset pricing in discrete time based on the notion of stochastic discount factor [see Gourieroux and Monfort (2006), and Bertholon, Pegoraro and Monfort (2006)]. The fourth pillar is the statistical approach of nonlinear state space models based on the Kitagawa-Hamilton's algorithm. In particular we will use several important results. First the regime switching models can be incorporated in the Car class. Second, the family of exponential affine SDF provides a tractable and flexible bridge between the historical world and the risk neutral world, and allows to reach a Car risk neutral dynamics even if the historical dynamics is not Car. Third, in a Car risk neutral framework, there exist explicit or quasi explicit formulas for the multihorizon Laplace transforms and the truncated multihorizon Laplace transforms, which allow for tractable pricing of forwards, futures and options.

The paper is organized as follows. In section 2 we describe the information in the economy and its historical dynamics. Section 3 introduces the notion of stochastic discount factor, its specification, and the implied risk neutral (RN) dynamics. In section 4 we present the general notions of Laplace transforms, of Car dynamics and we apply these notions to our framework with a special attention to the bridge between the historical and the RN dynamics and to the historical and RN seasonal patterns. Section 5 deals with pricing of forwards and options (on spot or forwards) which, in our approach, is quasi explicit and we stress the importance of the choice of internal consistency (IC) conditions and, in particular, that, given the non storability of electricity , IC conditions on the spot price must not be imposed. Inference methods are proposed in section 6, when estimated forward curves are available for all maturities and dates and we distinguish the case where there are quantitative latent variables, in addition to the qualitative latent variables capturing spikes, and the case where there are no quantitative latent variables; in the former case a simple two step procedure is proposed and in the latter two versions of the inversion method are considered. Section 7 studies the inference problem when estimated forward curves are not available but only prices of various forward contracts of different maturities and different delivery periods, and, in this context a sequential approach based on the Kitagawa-Hamilton filter and the extended Kalman filter is proposed. Section 8 proposes various extensions to the case of multiple lags, to the use of spot price returns (instead of prices) and to the introduction of non additive impact of the exogenous variables. Finally in section 9, an illustration on simulated spot and forward electricity prices assesses the effectiveness of affine models. Several appendices gather the proofs.

2. HISTORICAL DYNAMICS

2.1. Information

We consider a discrete time economy in which the new information of the agents at date $t, (t = 1, 2, \dots, T)$, is partitioned into three sets of variables.

The first set is the set of endogenous variables. It will contain [1] $\log S_t, S_t$ being the spot electricity price at t. Another endogenous variable will be a qualitative variable

[1] We could also choose to include the return $\log \dfrac{S_t}{S_{t-1}}$ see section 8.2.

z_t, valued in $(e_1, \ldots, e_J)$ the columns of the identity matrix of size J; this variable will, in particular, drive stochastic drifts, stochastic volatilities and spikes (see section 8.4 for an extended use of z_t). Finally the set of endogenous variables will also contain a m-vector Y_t of variables which interact with $(\log S_t, z_t)$; technically this means that Y_t will cause $(\log S_t, z_t)$ and will be caused by $(\log S_t, z_t)$, see appendix 1. At this stage we need not make any assumption about the observability of these variables by the econometrician, however we typically will assume that $\log S_t$ is observable, whereas z_t is not (see section 8.4 for an extension) and Y_t will contain observable (or constructed) variables, like forward prices at given residual maturities, or latent variables, the role of which being to make more flexible the dynamics of the variables of interest. The second set of variables, denoted by w_t^e, is the set of exogenous variables, that is to say a set of variables which may cause $(\log S_t, z_t', Y_t')'$ but are not caused by $(\log S_t, z_t, Y_t')'$. In this set we could find variables like temperature or demand. The third set of variables, denoted by w_t^i, is the set of independent variables, in the sense that the process w_t^i is independent of the process $(\log S_t, z_t', Y_t', w_t^{e'})'$. Such variables will include the short interest rate or, more generally, interest rate curves.

The whole information of the agents at t, will be denoted by $(\underline{\log S_t}, \underline{Y_t}, \underline{z_t}, \underline{w_t^e}, \underline{w_t^i})$, where we use, for instance, the notation $\underline{z_t} = (z_1', \ldots, z_t')'$.

2.2. Specification of the historical dynamics

We decompose $\log S_t$ into a function of time and of the exogenous vector w_t^e, denoted by ν_t^s, and the difference $s_t = \log S_t - \nu_t^s$. The first component ν_t^s captures the seasonal effects and, possibly, the effect of exogenous variables like temperature. Similarly we write $y_t = Y_t - \nu_t^y$, where ν_t^y is a vector function of w_t^e, and we specify the joint dynamics of $w_t = (z_t', s_t, y_t')'$ as a switching regime VAR(1). More precisely we assume that :

$$\left\{ \begin{array}{rcl} s_{t+1} & = & \mu' z_{t+1} + \varphi_1 (s_t - \mu' z_t) + \varphi_2' y_t + \varepsilon_{t+1}^s \\ \\ y_{t+1} & = & \psi_0 + \psi_1 (s_t - \mu' z_t) + \psi_2' y_t + \varepsilon_{t+1}^y \end{array} \right. \tag{1}$$

$$\text{where} \begin{pmatrix} \varepsilon_{t+1}^s \\ \varepsilon_{t+1}^y \end{pmatrix} = \Sigma^{1/2}(z_{t+1}, z_t)\varepsilon_{t+1} \tag{2}$$

ε_{t+1} being a standard Gaussian white noise process of size $(m+1)$, $\Sigma^{1/2}(z_{t+1}, z_t)$ a symmetric positive definite matrix function of z_{t+1}, z_t and where the conditional distribution of z_{t+1} given $\underline{w^e_{t+1}}, \underline{z_t}, \underline{s_t}, \underline{y_t}$, depends on z_t (and not on $z_{t-1}, z_{t-2}, \ldots$) and, possibly, of s_t and w^e_{t+1}. We also introduce the notation :

$$\pi_{ijt} = P(z_{t+1} = e_j / z_t = e_i, s_t, w^e_{t+1}) \tag{3}$$

Note that if $z_{t+1} = e_j$, $\mu' z_{t+1}$ is equal to μ_j, the jth component of μ. The dynamics of the exogenous and independent processes w^e_t and w^i_t are not specified, we just denote by $f^e(w^e_{t+1}/\underline{w^e_t})$ the conditional probability density function (p.d.f.) of w^e_{t+1} given $\underline{w^e_t}$ and $f^i(w^i_{t+1}/\underline{w^i_t})$ the conditional p.d.f. of w^i_{t+1} given $\underline{w^i_t}$. Let us now discuss more precisely the dynamics defined by equations (1) (2) (3).

First the joint conditional p.d.f. of $w^e_{t+1}, z_{t+1}, s_{t+1}, y_{t+1}$ given $\underline{w^e_t}, \underline{z_t}, \underline{s_t}, \underline{y_t}$ is factorized as (taking $z_{t+1} = e_j$ and $z_t = e_i$) :

$$f^e(w^e_{t+1}/\underline{w^e_t})\pi_{ijt} n[s_{t+1}, y_{t+1}; m_{ijt}, \Sigma(e_j, e_i)] \tag{4}$$

where $n[s_{t+1}, y_{t+1}; m_{ijt}, \Sigma(e_j, e_i)]$ is the p.d.f of the normal multivariate distribution with variance-covariance matrix $\Sigma(e_j, e_i)$ and mean :

$$m_{ijt} = \begin{pmatrix} \mu_j + \varphi_1(s_t - \mu_i) + \varphi_2' y_t \\ \\ \psi_0 + \psi_1(s_t - \mu_i) + \psi_2' y_t \end{pmatrix}$$

Note that since $s_{t+1} = \log S_{t+1} - \nu^s_{t+1}, y_{t+1} = Y_{t+1} - \nu^y_{t+1}$ where ν^s_{t+1} and ν^y_{t+1} are functions of w^e_{t+1}, the exogenous variables w^e_{t+1} and w^e_t appear in the last term of (4), when s_{t+1} and y_{t+1} are replaced by their expressions above.

Second, if we use the notation $s^*_t = s_t - \mu' z_t$ we see that $s_t = \mu' z_t + s^*_t$, where the dynamics of the pair (s^*_t, y_t) is given by :

$$\begin{cases} s^*_{t+1} & = & \varphi_1 s^*_1 + \varphi'_2 y_t + \varepsilon^s_{t+1} \\ \\ y_{t+1} & = & \psi_0 + \psi_1 s^*_t + \psi'_2 y_t + \varepsilon^y_{t+1} \end{cases} \tag{5}$$

In particular, z_t does not cause (s^*_t, y_t). Therefore if $\mu' z_t$ takes a large value, because z_t is in a"spike state" at date t, obviously $s_t = \mu' z_t + s^*_t$ also takes a large value but, if the probability to stay in a spike state is small, z_{t+1} is likely to be in a "non spike state" and therefore the value of $s_{t+1} = \mu' z_{t+1} + s^*_{t+1}$ is likely to decrease immediately. This would has not been the case if the first equation of (1) would have been $s_{t+1} = \mu' z_{t+1} + \varphi_1 s_t + \varphi_2 y_t + \varepsilon^s_{t+1}$ because, in this case, the large value of s_t would have heavily impacted s_{t+1} excepted if φ_1 is small, i.e. if the mean reversion is very large. In other words, in our specification there is no need to introduce an additional state in order to impose a fast return to a "normal" situation after a spike, like in Huisman and Mahieu (2003), for instance. Moreover, in our specification, it is possible to have successive upward jumps and this possibility is amplified if the probability to stay in a "spike state" i is an increasing function of s_t, for instance of the form $\dfrac{1}{1+\exp(a_i + b_i s_t)}$ where b_i is negative.

Third, since the conditional variance-covariance matrix of (s_{t+1}, y_{t+1}), given $(\underline{w^e_{t+1}}, \underline{z_{t+1}}, \underline{s_t}, \underline{y_t})$, namely $\Sigma(z_{t+1}, z_t)$, depends on z_{t+1}, z_t our specification is also able to capture stochastic volatility features.

Finally, in equation (1) we have introduced only one lag, mainly for sake of notational simplicity, but an extension to multiple lags is straightforward (see section 8.1).

3. RISK NEUTRAL DYNAMICS

3.1. Stochastic discount factor

It is known [see Bertholon, Monfort, Pegoraro (2008)] that, under standard assumptions including absence of arbitrage opportunity, the price at t of a payoff $g(\underline{w_T}, \underline{w^e_T})$ (also denoted by g_T) at $T > t$ is given by :

$$\tilde{p}_t = E_t\left(\tilde{M}_{t,t+1} \dots \tilde{M}_{T-1,T}\, g_T\right) \tag{6}$$

where the $\tilde{M}_{\tau,\tau+1}$ are positive random variables functions of the information $I_{\tau+1}$ at $\tau+1$, and E_t is the historical conditional expectation given the information I_t at time t, here $I_t = (\underline{w_t}, \underline{w_t^e}, \underline{w_t^i})$. In particular, taking $T = t+1$, and $g_T = 1$, we get :

$$E_t \tilde{M}_{t,t+1} = \exp(-r_{t+1}) \tag{7}$$

where r_{t+1} is the (geometric) short interest rate between t and $t+1$ (known at t).

3.2. Risk neutral conditional densities

The risk neutral (RN) dynamics of (w_t, w_t^e, w_t^i) is defined by the RN conditional densities :

$$f^Q(w_t, w_t^e, w_t^i / \underline{w_{t-1}}, \underline{w_{t-1}^e}, \underline{w_{t-1}^i}) =$$

$$f(w_t, w_t^e, w_t^i / \underline{w_{t-1}}, \underline{w_{t-1}^e}, \underline{w_{t-1}^i}) \tilde{M}_{t-1,t} \exp(r_t)$$

and formula (6) can be written equivalently :

$$\tilde{p}_t = E_t^Q \left[\exp(-r_{t+1} - \ldots - r_T) g_T\right] \tag{8}$$

3.3. Our specifications

If we do not want to specify the dynamics of the exogenous variables w_t^e, we can work conditionally to a future scenario of these variables (see section 8.4 for another approach).

It is shown in appendix 1 that given the exogeneity of w_t^e, the price at t of a payoff $g(\underline{w_T}, \underline{w_T^e})$ conditional to a future scenario for the exogenous variables can be written :

$$p_t = E_t \left[M_{t,t+1} \ldots M_{T-1,T} \, g(\underline{w_T}, \underline{w_T^e})\right] \tag{9}$$

where $M_{t,t+1}$ is a stochastic discount factor, function of $(\underline{w_{t+1}}, \underline{w_{t+1}^e})$ and satisfying :

$$M_{t,t+1} = \exp(-r_{t+1}) M_{t,t+1}^*(\underline{w_{t+1}}, \underline{w_{t+1}^e})$$

with :

$$E_t\left(M^*_{t,t+1}/\underline{w_t}, \underline{w^e_{t+1}}\right) = 1 \tag{10}$$

In (9) and (10) the values of the exogenous variables are considered as non random and E_t is the conditional expectation operator given $\underline{w}_t$.

Given the independence of r_t, equation (9) can also be written :

$$p_t = B(t, T-t)E_t\left[M^*_{t,t+1}\ldots M^*_{T-1,T}\, g_T(\underline{w_T}, \underline{w^e_T})\right]$$

where $B(t, T-t)$ is the price at t of a zero-coupon bond of residual maturity $T-t$, or equivalently :

$$p_t = B(t, T-t)E^Q_t[g_T(\underline{w_T}, \underline{w^e_T})] \tag{11}$$

where the Q dynamics is defined by the conditional p.d.f.

$$\begin{aligned} f^Q(w_t/\underline{w_{t-1}}, \underline{w^e_t}) &= f(w_t/\underline{w_{t-1}}, \underline{w^e_t})M^*_{t-1,t} \\ &= f(w_t/\underline{w_{t-1}}, \underline{w^e_t})M_{t-1,t}\exp(r_t) \end{aligned}$$

Here we choose a stochastic discount factor of the following type :

$$M_{t,t+1} = \exp(-r_{t+1} + \Gamma'_{t+1}\varepsilon_{t+1} - \frac{1}{2}\Gamma'_{t+1}\Gamma_{t+1} + \delta'_{t+1}z_{t+1}) \tag{12}$$

or, equivalently,

$$M^*_{t,t+1} = \exp(\Gamma'_{t+1}\varepsilon_{t+1} - \frac{1}{2}\Gamma'_{t+1}\Gamma_{t+1} + \delta'_{t+1}z_{t+1}) \tag{13}$$

where using the notation $x_t = (s_t, y'_t)'$ the "prices of risk", Γ_{t+1} and δ_{t+1} are of the form :

$$\begin{aligned} \Gamma_{t+1} &= \Gamma(z_{t+1}, z_t, x_t, w^e_{t+1}) \\ \delta_{t+1} &= \delta(z_t, s_t, w^e_{t+1}) \end{aligned}$$

and where we impose the identification constraints :

$$\sum_{j=1}^{J}\pi_{ij,t}\exp[\delta_j(e_i, s_t, w^e_{t+1})] = 1 \tag{14}$$

with $\pi_{ijt} = P(z_{t+1} = e_j / z_t = e_i, s_t, w^e_{t+1})$.

Given that ε_{t+1} is a standard gaussian white noise, and using (14), we see that $M^*_{t,t+1}$ satisfies condition (10).

Specification (12) (or (13)) shows that we are pricing the (standardized) innovations ε_{t+1} of x_{t+1} through Γ_{t+1} and the regimes z_{t+1} through δ_{t+1}; moreover, since Γ_{t+1} depends on z_{t+1}, the pricing of ε_{t+1} may depend on the regimes. In the next sections we will show how to specify Γ_{t+1} and δ_{t+1} in order to get tractable derivative pricing.

4. LAPLACE TRANSFORMS AND CAR DYNAMICS

4.1. Definition of a Car process

Definition 1 : A process w_t is Compound Autoregressive of order 1 [Car (1)], or affine, if the conditional Laplace transform of w_{t+1} given $\underline{w_t}, \varphi_t(u) = E_t \exp(u' w_{t+1})$, where u is a vector with real components, has the form :

$$\varphi_t(u) = \exp[a'(u) w_t + b(u)] \tag{15}$$

In other words, the log-Laplace transform $\psi_t(u) = a'(u) w_t + b(u)$ is affine in w_t.

This kind of process has many interesting properties [see Darolles, Jasiak, Gourieroux (2006)]. A property which is particularly important is the following :

Proposition 1 : If w_t is Car(1), the multihorizon conditional Laplace transforms, for a given $\alpha = (\alpha_1', \ldots, \alpha_H')'$ where the $\alpha_i's$ are vectors with real or complex components,

$$L_{t,h}(\alpha) = E_t \exp[\alpha'_{H-h+1} w_{t+1} + \ldots + \alpha'_H w_{t+h}]$$

$t = 1, \ldots, T, h = 1, \ldots, H$, are exponential affine functions of w_t :

$$L_{t,h}(\alpha) = \exp(c_h' w_t + d_h) \tag{16}$$

where the sequences c_h and d_h are defined by : $c_0 = 0, d_0 = 0$, and, for $h = 1, \ldots, H$:

$$\left\{ \begin{array}{rcl} c_h & = & a(\alpha_{H-h+1} + c_{h-1}) \\ \\ d_h & = & b(\alpha_{H-h+1} + c_{h-1}) + d_{h-1} \end{array} \right. \tag{17}$$

a and b being the functions defined in (15).

Proof : see appendix 2

The previous proposition allows a straighforward computation of many multihorizon Laplace transforms, in particular $E_t \exp(\alpha_0' w_{t+h}), h = 1, \ldots, H$ (take $\alpha_1 = \alpha_2 = \ldots = \alpha_{H-1} = 0, \alpha_H = \alpha_0$) and $E_t \exp(\alpha_0' w_{t+s} + \ldots + \alpha_0' w_{t+h}), h = 1, \ldots, H$ (take $\alpha_i = \alpha_0, \forall i$).

Another crucial property [see Duffie, Pan and Singleton (2000)], is the one allowing the computation of a truncated Laplace Transform. Let us introduce the notation $\tilde{w}_{t+1,h} = (w_{t+1}', \ldots, w_{t+h}')'$ and let us consider the truncated conditional real Laplace transform :

$$\tilde{\varphi}_t(u, v, \gamma) = E_t \exp(u' \tilde{w}_{t+1,h}) \mathbf{1}_{(v' \tilde{w}_{t+1,h} < \gamma)}$$

where u, v are vectors with real components. We have the following property :

Proposition 2 : Considering the untruncated complex conditional Laplace transform $\varphi_t(\lambda) = E_t \exp(\lambda' \tilde{w}_{t+1,h})$, where $\lambda = u + iv$ is a vector with complex components, we have

$$\tilde{\varphi}_t(u, v, \gamma) = \frac{\varphi_t(u)}{2} - \frac{1}{\pi} \int_0^\infty \frac{Im[\varphi_t(u+; ivx) \exp(-i\gamma x)]}{x} dx \tag{18}$$

where Im means imaginary part.

This property is important for the computation of options because truncated real Laplace transforms naturally occur in these computations, and, since the computation of the untruncated complex Laplace is easy (see property 1) in the Car(1) case, property 2 says that the additional computational burden is just a univariate integral. Note that for the computation of option prices what matters is the risk neutral dynamics, therefore the computation of such prices will be easy if the RN dynamics is Car(1), but not necessarily the historical dynamics.

Also note that the definition and the properties of a Car(1) process are easily generalized to Car(p) (see section 8)

4.2. Historical conditional Laplace transform of w_{t+1}

Using the notation $x_{t+1} = (s_{t+1}, y'_{t+1})'$, system (1) can be written

$$x_{t+1} = \mu(z_{t+1}, z_t) + \Phi x_t + \Sigma^{1/2}(z_{t+1}, z_t)\varepsilon_{t+1} \tag{19}$$

with $\mu(z_{t+1}, z_t) = \begin{pmatrix} \mu' z_{t+1} - \varphi_1 \mu' z_t \\ \\ \psi_0 - \psi_1 \mu' z_t \end{pmatrix}$

$$\Phi = \begin{pmatrix} \varphi_1 & \varphi_2' \\ \\ \psi_1 & \psi_2' \end{pmatrix}$$

Proposition 3 : The historical conditional Laplace transform of $w_{t+1} = (z'_{t+1}, x'_{t+1})'$ is :

$$\begin{aligned} \varphi_t(u) &= E_t \exp(u' z_{t+1} + v' x_{t+1}) \\ &= \exp[(A_{1t}, \ldots, A_{Jt}) z_t + v' \Phi x_t] \\ \text{where} \qquad A_{it} &= \log \left\{ \sum_{j=1}^{J} \pi_{ijt} \exp[u' e_j + v' \mu(e_j, e_i) + \frac{1}{2} v' \Sigma(e_j, e_i) v] \right\} \end{aligned}$$

Proof : see appendix 3.

It is important to note that $\varphi_t(u, v)$ is not in general exponential affine in w_t and, therefore, w_t is not Car(1), since A_{it} depends on w_t through π_{ijt}. However w_t is Car(1), if π_{ijt} does not depend on s_t, i.e. if z_t is an homogenous Markov chain.

4.3. Risk Neutral conditional Laplace transform of w_{t+1}.

The Risk Neutral conditional Laplace transform of w_{t+1} is given by :

$$\begin{aligned} \varphi_t^Q(u, v) &= E_t^Q \exp(u' z_{t+1} + v' x_{t+1}) \\ &= E_t M^*_{t,t+1} \exp(u' z_{t+1} + v' x_{t+1}) \end{aligned}$$

where $M^*_{t,t+1}$ is given by (13) and (14).

Proposition 4 : The Risk Neutral conditional Laplace transform of w_{t+1} is given by :

$$\varphi_t^Q(u, v) = \exp[(\bar{A}_{1t}, \ldots, \bar{A}_{Jt}) z_t + v' \Phi x_t]$$

$$\begin{aligned} \text{with} \quad \bar{A}_{it} &= \log \left\{ \sum_{j=1}^{J} \pi_{ijt} \exp[v' \Sigma^{1/2}(e_j; e_i) \Gamma(e_j, e_i, x_t, w^e_{t+1}) \right. \\ &\left. + \frac{1}{2} v' \Sigma(e_j, e_i) v + \delta_j(e_i, s_t, w^e_{t+1}) + u' e_j + v' \mu(e_j, e_i) \right\} \end{aligned}$$

Proof : See appendix 4.

So, in general, the RN dynamics of w_{t+1} is also non Car(1).

4.4. RN seasonality and RN Car dynamics

We would like to allow for a RN seasonality, and possibly, an influence of exogenous variables which are different from those appearing in the historical world. So we introduce the notations :

$$\begin{aligned}
\tilde{s}_t &= \log S_t - \tilde{\nu}_t^s \\
\tilde{y}_t &= Y_t - \tilde{\nu}_t^y \\
\text{and} \quad \tilde{x}_t &= \begin{pmatrix} \tilde{s}_t \\ \tilde{y}_t \end{pmatrix} = x_t + \nu_t - \tilde{\nu}_t, \text{ with } \tilde{\nu}_t = \begin{pmatrix} \tilde{\nu}_t^s \\ \tilde{\nu}_t^y \end{pmatrix}, \nu_t = \begin{pmatrix} \nu_t^s \\ \nu_t^y \end{pmatrix}
\end{aligned}$$

Moreover, we would like the RN dynamics of $\tilde{w}_t = (z_t', \tilde{x}_t')'$ to be Car(1) in order to have tractable prices for various derivatives. More precisely we would like to have the following RN dynamics :

$$\tilde{x}_{t+1} = \mu^*(z_{t+1}, z_t) + \Phi^* \tilde{x}_t + \Sigma^{1/2}(z_{t+1}, z_t)\varepsilon_{t+1}^* \tag{20}$$

$$\begin{aligned}
\text{with } \mu^*(z_{t+1}, z_t) &= \begin{pmatrix} \mu^{*'} z_{t+1} - \varphi_1^* \mu^{*'} z_t \\ \psi_0^* - \psi_1^* \mu^{*'} z_t \end{pmatrix} \\
\Phi^* &= \begin{pmatrix} \varphi_1^* \ \varphi_2^{*'} \\ \\ \psi_1^* \ \psi_2^{*'} \end{pmatrix}
\end{aligned}$$

where ε_t^* is a standard Gaussian white noise under Q, and z_t is an homogenous Markov chain under Q, with transition probabilities π_{ij}^*.
Equivalently, we would like to have the system :

$$\begin{cases} \tilde{s}_{t+1} &= \mu^{*'} z_{t+1} + \varphi_1^*(\tilde{s}_t - \mu^{*'} z_t) + \varphi_2^{*'} \tilde{y}_t + \varepsilon_{t+1}^{s*} \\ \\ \tilde{y}_{t+1} &= \psi_0^* + \psi_1^*(\tilde{s}_t - \mu^{*'} z_t) + \psi_2^{*'} y_t + \varepsilon_{t+1}^{y*} \end{cases}$$

$$\text{with } \begin{pmatrix} \varepsilon_{t+1}^{s*} \\ \varepsilon_{t+1}^{y*} \end{pmatrix} = \Sigma^{1/2}(z_{t+1}, z_t)\varepsilon_{t+1}^* \tag{21}$$

where ε_t^* is a standard Gaussian white noise under Q, and z_t is an homogenous Markov chain under Q, with transition probabilities π_{ij}^*.

It turns out that from any historical dynamics defined by (1) and π_{ijt}, we can reach any RN dynamics defined by (20) and π_{ij}^*, with a s.d.f. of the form (12), provided that Γ_{t+1} and δ_{t+1} are well chosen.

Proposition 5 : If the historical dynamics is defined by system (1) and π_{ijt}, we obtain the RN dynamics defined by (20) and π_{ij}^* if we choose a s.d.f. of the form (12) with :

$$\begin{aligned} \delta_j(e_i, s_t, w_{t+1}^e) &= \log \frac{\pi_{ij}^*}{\pi_{ijt}} \\ \\ \Gamma(e_j, e_i, x_t, w_{t+1}^e) &= \Sigma^{-1/2}(e_j, e_i)[(\Phi^* - \Phi)x_t \\ \\ &+ \Phi^*(\nu_t - \tilde{\nu}_t) + \mu^*(e_j, e_i) - \mu(e_j, e_i)] \end{aligned}$$

Proof : see appendix 5.

It is worth noting that the $\delta_j' s$ of the previous proposition automatically satisfy identification constraints (14).

In this context, we can easily compute the RN conditional Laplace transform of $\tilde{w}_{t+1} = (z_{t+1}', \tilde{x}_{t+1})'$.

Proposition 6 : Under the condition of proposition 5, the RN conditional Laplace transform of $\tilde{w}_{t+1} = (z'_{t+1}, \tilde{x}'_{t+1})'$ is

$$\begin{aligned}
\tilde{\varphi}^Q(u,v) &= E_t^Q \exp(u' z_{t+1} + v' \tilde{x}_{t+1}) \\
&= \exp[(A_1^* \dots A_J^*) z_t + v' \Phi^* \tilde{x}_t] \\
\text{with } : A_i^* &= \log \left\{ \sum_{j=1}^{J} \pi_{ij}^* \exp[u' e_j + v' \mu^*(e_j, e_i) + \frac{1}{2} v' \Sigma(e_j, e_i) v \right\}
\end{aligned} \tag{22}$$

So this conditional Laplace transform is of the form $\exp[a'(u,v)\tilde{w}_t + b(u,v)]$ given in definition 1, with

$$a'(u,v) = (A_1^* \dots, A_J^*, v'\Phi^*) \text{ and } b(u,v) = 0.$$

5. PRICING

5.1. Forward prices

The forward price at t and of residual maturity h, denoted by $F(t,h)$, is such that the payoff $S_{t+h} - F(t,h)$ delivered at $t+h$ has a price equal to zero at t, and therefore according to formula (11) :

$$0 = B(t,h) E_t^Q \left[S_{t+h} - F(t,h) \right]$$

and,

$$F(t,h) = E_t^Q \left[S_{t+h} \right] \tag{23}$$

This formula is based on formula (11) which in turn uses the independence between the short rate r_t and the other variables of the system (under the historical and the RN dynamics); therefore it is natural to obtain the identity between $F(t,h)$ and the general formula of a future price, which is a well-known result in this context.

Using the formula :

$$\begin{aligned} S_{t+h} &= \exp[\log(S_{t+h})] \\ &= \exp(\tilde{\nu}^s_{t+h} + \tilde{s}_{t+h}) \end{aligned}$$

we get :

$$F(t,h) = \exp(\tilde{\nu}^s_{t+h}) E_t^Q \left[\exp(\tilde{s}_{t+h})\right] \tag{24}$$

Moreover $\tilde{s}_t$ is the $J+1$th component of $\tilde{w}_t$ which is Car(1) under Q, so we have to compute $E_t^Q \exp(e'_{J+1}\tilde{w}_{t+h})$ where e_{J+1} is the vector selecting the $J+1$th component. Using Proposition 1 and formula (24) we get the following result.

Proposition 7 : The forward price at time and of residual maturity $h, h = 1, \ldots, H$, is

$$F(t,h) = \exp(\tilde{\nu}^s_{t+h}) + c'_h \tilde{w}_t)$$

where c_h is obtained recursively from

$$\begin{aligned} c_1 &= a(e_{J+1}) \\ c_h &= a(c_{h-1}),\ h = 2, \ldots, H. \end{aligned}$$

where a is function defined in proposition 6, or equivalently,

$$\log F(t,h) = \tilde{\nu}^s_{t+h} + c^s_h \tilde{s}_t + c^{y\prime}_h \tilde{y}_t + c^{z\prime}_h z_t$$

where c'_h has been partitioned into $c'_h = (c^{z\prime}_h, c^s_h, c^{y\prime}_h)$

Proof : It is a direct consequence of proposition 6 and proposition 1 with $\alpha_H = e_{J+1}, \alpha_h = 0, h = 1, \ldots, H-1$.

Therefore $\log F(t,h)$ is the sum of four components. The first component $\tilde{\nu}^s_{t+h}$ contains the risk neutral deterministic part of $\log S_{t+h}$ in particular the risk neutral seasonal component of $\log S_{t+h}$. The second component $c^s_h \tilde{s}_t$ measures the impact of $\log S_t$ purged from its RN exogenous part. The third part is similar but for the factor Y_t. The last part $c^{z\prime}_h z_t$ is a constant depending on the regime. If we now introduce the historical decomposition of $\log S_t$ and Y_t namely $\log S_t = \nu^s_t + s_t, Y_t = \nu^y_t + y_t$, the decomposition of $\log F(t,h)$ become the following :

Corollary 1

$$\begin{aligned}\log F(t,h) &= \tilde{\nu}^s_{t+h} + c^s_h(\nu^s_t - \tilde{\nu}^s_t) + c^{y\prime}_h(\nu^y_t - \tilde{\nu}^y_t) \\ &+ c^s_h s_t + c^{y\prime}_h y_t + c^{z\prime}_h z_t\end{aligned}$$

Proof : It is a direct consequence of the identities $\tilde{s}_t = \nu^s_t - \tilde{\nu}^s_t + s_t, \tilde{y}_t = \nu^y_t - \tilde{\nu}^y_t + y_t$.

Therefore the historical exogenous part of $\log F(t,h)$, is :

$$m(t,h) = \tilde{\nu}^s_{t+h} + c^s_h(\nu^s_t - \tilde{\nu}^s_t) + c^{y\prime}_h(\nu^y_t - \tilde{\nu}^y_t) \tag{25}$$

In particular, if the various exogenous parts only capture seasonality, the historical seasonally adjusted values of $F(t,h)$ are :

$$\begin{aligned}f(t,h) &= \exp[\log F(t,h) - m(t,h)] \\ &= \exp(c^s_h s_t + c^{y\prime}_h y_t + c^{z\prime}_h z_t)\end{aligned} \tag{26}$$

If we assume that the seasonal patterns are identical in the historical and the RN world [see Lucia and Scharwtz (2000)] i.e. $\tilde{\nu}^s_t = \nu^s_t, \tilde{\nu}^y_t = \nu^y_t$, we get $m(t,h) = \nu_{1,t+h}$, that is the seasonal part of $\log F(t,h)$ is the same as the one of $\log S_{t+h}$, and, therefore this seasonal part only depends on the delivery date $t+h$ and not on the present date t.

5.2. Internal Consistency (IC) conditions

We have seen that the price at t of a payoff g_T delivered at $T > t$ is given, in our framework, by (see equation (11)) :

$$p_t = B(t, T-t) E^Q_t [g_T(w_T, \underline{w^e_T})]$$

The theoretical price p_t given by the model is therefore a function of $(\underline{w_t}, \underline{w^e_T})$ and of the parameters θ of the model. In some cases this function $p_t(\underline{w_t}, \underline{w^e_T}, \theta)$ is also completely known by the econometrician and given by $p^0_t(\underline{w_t}, \underline{w^e_T})$, say. In this case the model must satisfy the IC conditions implied by :

$$p_t(\underline{w_t}, \underline{w_T^e}, \theta) = p_t^0(\underline{w_t}, \underline{w_T^e}) \; \forall \underline{w_t}, \underline{w_T^e}, \theta$$

For instance, if some components of Y_t are equal to $\log F(t, h_i)$, for some h_i, we have for each h_i, the identity :

$$0 = E_t^Q (S_{t+h_i} - \exp(Y_{it}))$$

where Y_{it} is the component of Y_t equal to $\log F(t, h_i)$, which implies, using corollary 1 and formula (23) :

$$\begin{aligned} Y_{it} &= m(t, h_i) + c_{h_i}^s s_t + c_{h_i}^{y\,\prime} y_t + c_{h_i}^{z\,\prime} z_t \\ &= \nu_t^{y_i} + y_{it} \end{aligned}$$

which implies :

$$\begin{aligned} c_{h_i}^z &= 0 \\ c_{h_i}^s &= 0 \\ c_{h_i}^y &= e_i \text{ (the vector selecting } y_{it} \text{ in } y_t) \\ m(t, h_i) &= \nu_t^{y_i} \end{aligned}$$

Since $m(t, h_i) = \tilde{\nu}_{t+h_i}^s + \nu_t^{y_i} - \tilde{\nu}_t^{y_i}$, the last condition is $\tilde{\nu}_{t+h_i}^s = \tilde{\nu}_t^{y_i}$

If we are working directly with seasonally adjusted variables $y_{it} = \log f(t, h_i)$, we may assume the equality :

$$y_{it} = c_{h_i}^s s_t + c_{h_i}^{y\,\prime} y_t + c_{h_i}^{z\,\prime} z_t$$

and therefore only impose the IC conditions $c_{h_i}^s = 0, c_{h_i}^y = e_i, c_{h_i}^z = 0$.

Note that if electricity was a tradable and storable asset, we should impose the constraint :

$$S_t = \exp(-r_{t+1}) \, E_t^Q S_{t+1}$$

which would imply :

$$S_t = B(t, T-t) E_t^Q S_T$$

and $F(t, T-t) = \dfrac{S_t}{B(t, T-t)}$

However, since electricity is not storable we do not impose this condition.

5.3. Pricing options on spot or forward prices

The price at t of an option written on the spot price, with residual maturity h and stike K, is :

$$\begin{aligned}
C_S(t, K, h) &= B(t, t+h) E_t^Q (S_{t+h} - K)^+ \\
&= B(t, t+h) E_t^Q [\exp(\tilde{\nu}^s_{t+h} + \tilde{s}_{t+h}) - K]^+ \\
&= B(t, t+h) \exp(\tilde{\nu}^s_{t+h}) E_t^Q [\exp(\tilde{s}_{t+h}) - K \exp(-\tilde{\nu}^s_{t+h})]^+ \\
&= B(t, t+h) \exp(\tilde{\nu}^s_{t+h}) \{ E_t^Q [\exp(\tilde{s}_{t+h}) \mathbb{1}_{(\tilde{s}_{t+h} > \log K - \tilde{\nu}^s_{t+h})} \} \\
&- K \exp(-\tilde{\nu}^s_{t+h}) Q(\tilde{s}_{t+h} > \log K - \tilde{\nu}^s_{t+h}) \}
\end{aligned}$$

So we have the following result :

Proposition 8 : The price of an option on S_{t+h} is

$$C_S(t, K, h) = B(t, t+h)[\exp(\tilde{\nu}^s_{t+h}) \tilde{\varphi}_t^Q(1, -1, \gamma(t, h, K)) - K \tilde{\varphi}_t^Q(0, -1, \gamma(t, h, K))$$

where :

$$\begin{aligned}
\tilde{\varphi}_t^Q(u, v, \gamma) &= E_t^Q [\exp(u \tilde{s}_{t+h}) \mathbb{1}_{(v \tilde{s}_{t+h} < \gamma)}] \\
\text{and} \quad \gamma(t, h, K) &= -\log K + \tilde{\nu}^s_{t+h}
\end{aligned}$$

The truncated conditional Laplace transform $\tilde{\varphi}_t^Q(u, v, \gamma)$ can be computed from the conditional complex Laplace transform $\varphi_t(\lambda) = E_t^Q \exp(\lambda \tilde{s}_{t+h})$ using proposition 2, and $\varphi_t(\lambda)$ can be computed using proposition 6.

Similarly the price at t of an option written on the forward price
$F(t+h, k)$ and strike K is :

$$C_F(t, k, h, K) = B(t, t+h) E_t^Q [F(t+h, k) - K]^+$$

and using proposition 7 :

$$\begin{aligned}
C_F(t,k,h,K) &= B(t,t+h)E_t^Q[\exp(\tilde{\nu}^s_{t+h+k}+c_k'\tilde{w}_{t+h})-K]^+ \\
&= B(t,t+h)\exp(\tilde{\nu}^s_{t+h+k})E_t^Q[\exp(c_k'\tilde{w}_{t+h})-K\exp(-\tilde{\nu}^s_{t+h+k})]^+ \\
&= B(t,t+h)\exp(\tilde{\nu}^s_{t+h+k})\{E_t^Q[\exp(c_k'\tilde{w}_{t+h})\mathbb{1}_{(c_k'\tilde{w}_{t+h}>\log K-\tilde{\nu}^s_{t+h+k})}]\} \\
&- K\exp(-\tilde{\nu}^s_{t+h+k})\mathbb{1}_{(c_k'\tilde{w}_{t+h}>\log K-\tilde{\nu}^s_{t+h+k})}
\end{aligned}$$

Therefore we have a similar result :

Proposition 9 : The price of an option written on $F(t+h,k)$ is :

$$\begin{aligned}
C_F(t,k,h,K) = B(t,t+h) \quad & [\exp(\tilde{\nu}^s_{t+h+k})\tilde{\varphi}_t^Q(c_k,-c_k,\gamma(t,h,k,K))... \\
& ... - K\tilde{\varphi}_t^Q(0,-c_k,\gamma(t,h,k,K)]
\end{aligned}$$

where :

$$\begin{aligned}
\tilde{\varphi}_t^Q(u,v,\gamma) &= E_t^Q[\exp(u'\tilde{w}_{t+h})\mathbb{1}_{(v'\tilde{w}_{t+h}<\gamma)}] \\
\gamma(t,h,k,K) &= -\log K+\tilde{\nu}^s_{t+h+k}
\end{aligned}$$

Again $C_F(t,k,h,K)$ can be computed using propositions 2 and 6.

6. INFERENCE BASED ON ESTIMATED FORWARD CURVES

In this section we assume that at date $t=1,\dots,T$ seasonally adjusted forward prices $f(t,h), h\in\mathcal{H}$ have been estimated. In this context we consider first a model without quantitative latent variable and then a model with quantitative latent variables.

6.1. A model without latent quantitative variables

We consider the historical dynamics given by equations (27), (28),(29) :

$$\left\{\begin{aligned}
s_{t+1} &= \mu' z_{t+1}+\varphi_1(s_t-\mu' z_t)+\varphi_2' y_t+\varepsilon^s_{t+1} \\
y_{t+1} &= \psi_0+\psi_1(s_t-\mu' z_t)+\psi_2' y_t+\varepsilon^y_{t+1}
\end{aligned}\right. \tag{27}$$

$$\text{where } : \begin{pmatrix}\varepsilon^s_{t+1}\\ \varepsilon^y_{t+1}\end{pmatrix} = \Sigma^{1/2}(z_{t+1},z_t)\varepsilon_{t+1}, \varepsilon_t\sim IIN(0,I) \tag{28}$$

and z_t is valued in $\{e_1, \ldots, e_J\}$ with

$$\pi_{ijt} = P(z_{t+1} = e_j / z_t = e_i, s_t, w_{t+1}^e) \tag{29}$$

We assume that y_t is a vector of m estimated seasonally adjusted log-forward prices $\log f(t,h), h \in \mathcal{H}_0$.

We denote by θ the vector of parameters appearing in (27), (28), (29). This vector θ can be estimated by the Maximum Likelihood method using the Kitagawa-Hamilton algorithm (see Hamilton (1989)), since the only latent variable is z_t. If we assume, moreover, that $\Sigma^{1/2}(z_{t+1}, z_t)$ is block-diagonal of the form $\begin{bmatrix} \sigma(z_{t+1}, z_t) & 0 \\ 0 & \Sigma_y^{1/2} \end{bmatrix}$, we can first estimate the parameters appearing in the first equation of (27), in $\sigma(z_{t+1}, z_t)$ and the $\pi'_{ijt}s$, by the ML method only based on the first equation of (27). Then we can estimate, in a second step, the parameters in the second set of equations of (27) and Σ_y, by Ordinary Least Squares (OLS) once z_t has been replaced by its smoothed value, based on the Kim (1994) smoothing algorithm, and on the estimations of the first step.

Note that if y_t does note cause (s_t, z_t), that is if $\varphi_2 = 0$, the first step estimation only necessitates the estimation of the joint dynamics of (s_t, z_t), only based on the observations of s_t.

Let us now consider the estimation of the parameters, denoted by θ^*, appearing in the RN dynamics characterized by system (21) and the transition probabilities π^*_{ij}. This vector θ^* will be estimated from the observations of the log-forward prices not used in the estimation of θ, $\log f(t,h), h \in \mathcal{H} - \mathcal{H}_0$, and from the theoretical values of these log-forward prices given by (26) :

$$\log f(t,h) = c_h^s s_t + c_h^{y\prime} y_t + c_h^{z\prime} z_t$$

More precisely, taking θ at its estimated value and the $z'_t s$ at their smoothed values $\hat{z}_t$ we minimize with respect to θ^* :

$$\sum_{t=1}^{T} \sum_{h \in \mathcal{H} - \mathcal{H}_0} [\log f(t,h) - c_h^s s_t - c_h^{y\prime} y_t - c_h^{z\prime} \hat{z}_t]^2 \tag{30}$$

under the IC constraints :

$$c_h^z = 0, c_h^s = 0, c_h^y = e_h \text{ for } h \in \mathcal{H}_0$$

and e_h being the vector selecting the component of y_t equal to $\log f(t,h)$.

For instance if $m = 2$, that is if y_t contains two seasonally adjusted log-forward prices, and if $J = 2$, the number of components of

$$\theta^* = [\mu^{*'}, \varphi_1^*, \varphi_2^{*'}, \psi_0^{*'}, \psi_1^{*'}, (vec\psi_2^*)', \pi_{ij}^*]$$

is 15 (taking into account the constraints on the π_{ij}^*) and the number of IC constraints is 10, so there are 5 degrees of freedom in the minimization.

Not that, in principle, it would be also possible to use raw data $\log S_t$ and $\log F(t,h)$ instead of seasonally adjusted data. In the first step we could estimate at the same time θ and the parameters appearing in ν_t^s and ν_t^y by replacing in system (27) s_t by $\log S_t - \nu_t^s$ and y_t by $Y_t - \nu_t^y, Y_t$ being the vector of components $\log F(t,h), h \in \mathcal{H}_0$.

Then it would be possible to estimate θ^* and the parameters appearing in $\tilde{\nu}_t^s$ and $\tilde{\nu}_t^y$ by replacing the objective function of the minimization (30)by :

$$\sum_{t=1}^{T} \sum_{h \in \mathcal{H} - \mathcal{H}_0} \left[\log F(t,h) - \tilde{\nu}_{t+h}^s - c_h^s(\nu_t^s - \tilde{\nu}_t^s) - c_h^{y\prime}(\nu_t^y - \tilde{\nu}_t^y) - c_h^s s_t - c_h^{y\prime} y_t - c_h^{z\prime} z_t \right]^2 \tag{31}$$

in which θ, ν_t^s, ν_t^y, z_t are replaced by their estimations, and where we take into account the same IC as before and the additional constraints $\tilde{\nu}_{t+h_i}^s = \tilde{\nu}_t^{y_i}$, where $h_i \in \mathcal{H}_0$ and $\tilde{\nu}_t^{y_i}$ is the corresponding component of $\tilde{\nu}_t^y$.

For instance if the deterministic components $\tilde{\nu}_t^s, \tilde{\nu}_t^{y_i}$ are constants depending on the month corresponding to date t, this means that the RN deterministic component of $F(t, h_i)$, must be identical to that of $\log S_{t+h_i}$.

6.2. A model with latent quantitative variables : the inversion method

Let us assume that y_t is partitioned into (y_{1t}', y_{2t}'), where y_{1t} is observed whereas y_{2t} is not, and let us denote by p_1 and p_2 the sizes of y_{1t} and y_{2t}, with $p_1 + p_2 = p$.

Let us consider a set of seasonally adjusted log forward prices $\log f(t, h), h \in \mathcal{H}_2 \subset \mathcal{H}$ not appearing in y_{1t} and such that the cardinal of $\mathcal{H}_2$ is m_2. Denoting the vector of components $\log f(t, h), h \in \mathcal{H}_2$ by $\bar{y}_{2t}$, we get from (26), with obvious notations :

$$\bar{y}_{2t} = c_1 s_t + C_1 y_{1t} + C_2 y_{2t} + C_3 z_t \tag{32}$$

Introducing the notation $\bar{x}_t = (s_t, y'_{1t}, \bar{y}'_{2t})'$ and, as above, $x_t = (s_t, y'_{1t}, y'_{2t})'$, we get :

$$\bar{x}_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ c_1 & C_1 & C_2 \end{pmatrix} x_t + \begin{pmatrix} 0 \\ 0 \\ C_3 \end{pmatrix} z_t \tag{33}$$

or :

$$\bar{x}_t = \bar{C} x_t + \bar{C}_3 z_t \tag{34}$$

Moreover system (1) can be written, using the notations in (19) :

$$x_{t+1} = \mu(z_{t+1}, z_t) + \Phi x_t + \Sigma^{1/2}(z_{t+1}, z_t)\varepsilon_{t+1} \tag{35}$$

and, therefore, using (34) we get :

$$\bar{C}^{-1}(\bar{x}_{t+1} - \bar{C}_3 z_{t+1}) = \mu(z_{t+1}, z_t) + \Phi \bar{C}^{-1}(\bar{x}_t - \bar{C}_3 z_t) + \Sigma^{1/2}(z_{t+1}, z_t)\varepsilon_{t+1} \tag{36}$$

or :

$$\bar{x}_{t+1} = \bar{\mu}(z_{t+1}, z_t) + \bar{\Phi}\bar{x}_t + \bar{\Sigma}^{1/2}(z_{t+1}, z_t)\varepsilon_{t+1} \tag{37}$$

with :

$$\bar{\mu}(z_{t+1}, z_t) = \bar{C}\mu(z_{t+1}, z_t) + \bar{C}_3 z_{t+1} - \bar{C}\Phi\bar{C}^{-1}\bar{C}_3 z_t$$

$$\bar{\Phi} = \bar{C}\Phi\bar{C}^{-1}, \bar{\Sigma}^{1/2}(z_{t+1}, z_t) = \bar{C}\Sigma^{1/2}(z_{t+1}, z_t)$$

System (37) is similar to (35), with an important difference : $\bar{x}_t$ is fully observable.

Denoting by y_t^*, the set of $\log f(t,h)$ not appearing in y_{1t} and $\bar{y}_{2t}$, and assuming that y_t^* is observed with Gaussian errors we have :

$$y_t^* = C^* x_t + C_3^* z_t + \xi_t$$

$\xi_t \sim IIN(0, \sigma^2)$ or :

$$y_t^* = C^* \bar{C}^{-1}(\bar{x}_t - \bar{C}_3 z_t) + C_3^* z_t + \xi_t \tag{38}$$

Equations (37) and (38) provide a system in which the only latent variable is z_t and, therefore, this system can be estimated by the ML method and the Kitagawa-Hamilton algorithm, providing simultaneously estimators of the historical and RN dynamics. If y_{1t} contains log forward prices, IC conditions must be imposed like in the previous section.

6.3. A model with latent quantitative variable : a two step approach

The previous method may be time consuming since it involves nested recursive computations implied by the computation of the theoretical forward prices on the one hand, and by the Kitagawa-Hamilton algorithm on the other hand.

A simpler approach would be to start from equation (37), to estimate it with unconstrained parameters and the specification of the π_{ijt}, and to compute smoothed values of the z_t. In a second step the values of z_t would be replaced by $\hat{z}_t$ and, since the only latent variables would be the y_{2t}, the inversion technique would lead to an explicit formula for the likelihood function of (37) and (38).

7. INFERENCE BASED ON OBSERVED FORWARD PRICES

Let us assume that at each date t we observe N_t forward contracts with delivery periods $(t + H_{it}, t + H_{it} + D_i), i = 1, \ldots, N_t$. The forward price of this contract, denoted by $\tilde{F}(t, H_{it}, D_i)$ is such that the price at t of the payoffs $(S_{t+H_{it}} - \tilde{F})$ at $t + H_{it}, \ldots, (S_{t+H_{it}+D_i-1} - \tilde{F})$ at $t + H_{it} + D_i$ is equal to zero, i.e.

$$0 = \sum_{h=H_{it}}^{H_{it}+D_i-1} B(t,h) E_t^Q (S_{t+h} - \tilde{F})$$

and, therefore :

$$\tilde{F}(t, H_{it}, D_i) = \frac{\sum_{h=H_{it}}^{H_{it}+D_i-1} B(t,h)F(t,h)}{\sum_{h=H_{it}}^{H_{it}+D_i-1} B(t,h)}$$

which might be approximated by :

$$\tilde{F}(t, H_{it}, D_i) = \frac{1}{D_i} \sum_{h=H_{it}}^{H_{it}+D_i-1} F(t,h)$$

From proposition 7 and corollary 1 we know that :

$$\log F(t,h) = m(t,h) + c_h^s s_t + c_h^{y\prime} y_t + c_h^{z\prime} z_t \tag{39}$$

where $m(t,h) = \tilde{\nu}^s_{t+h} + c_h^s(\nu_t^s - \tilde{\nu}_t^s) + c_h^{y\prime}(\nu_t^y - \tilde{\nu}_t^y)$ where $\nu_t^s, \tilde{\nu}_t^s$ are the historical and RN deterministic components of $\log S_t$, and $\nu_t^y, \tilde{\nu}_t^y$ are the corresponding components of Y_t.

In this context the stochastic processes $\{F(t,h), t = 1, \dots, T; h \text{ fixed}\}$ are no longer observable and we may consider that the only observable stochastic process is S_t. The other variables Y_t are latent, we assume that $\nu_t^y = \tilde{\nu}_t^y = 0$, and therefore, we have $y_t = Y_t$ and :

$$m(t,h) = \tilde{\nu}^s_{t+h} + c_h^s(\nu_t^s - \tilde{\nu}_t^s) \tag{40}$$

Note that, since the y_t are latent, identification conditions must be imposed in the second set of equations in (1) and (21). For instance, we can impose $\psi_0 = \psi_0^* = 0$, ψ_1 and ψ_1^* triangular and the variance-covariance matrix of ε_{t+1}^y and ε_{t+1}^{y*} equal to the identity matrix.

The forward prices $\tilde{F}(t, H_{it}, D_i)$ become :

$$\tilde{F}(t, H_{it}, D_i) = \frac{1}{D_i} \sum_{h=H_{it}}^{H_{it}+D_i-1} \exp[m(t,h) + c_h^s s_t + c_h^{y\prime} y_t + c_h^{z\prime} z_t] \tag{41}$$

Note that, if we use the raw data $\log S_t$, instead of $s_t = \log S_t - \nu_t^s$ formula (41) becomes :

$$\tilde{F}(t, H_{it}, D_i) = \frac{1}{D_i} \sum_{h=H_{it}}^{H_{it}+D_i-1} \exp[\tilde{m}(t,h) + c_h^s \log S_t + c_h^{y\prime} y_t + c_h^{z\prime} z_t] \tag{42}$$

with :

$$\tilde{m}(t,h) = \tilde{\nu}_{t+h}^s - c_h^s \tilde{\nu}_t^s \tag{43}$$

In any case $\log \tilde{F}(t, H_{it}, D_i)$ is no longer linear in the random variables of interest and this new feature makes inference more complicated.

However inference is still tractable in some situations. If we assume for instance that the latent variable y_t does not appear in the first equation of system (1) (i.e. $\varphi_2 = 0$), and that $\Sigma^{1/2}(z_{t+1}, z_t)$ is block-diagonal, we can estimate the first equation of system (1) :

$$s_{t+1} = \mu' z_{t+1} + \varphi_1(s_t - \mu' z_t) + \sigma(z_{t+1}, z_t)\varepsilon_{t+1}$$

(where σ is the $(1,1)$ entry of $\Sigma^{1/2}$) using the Kitagawa-Hamilton algorithm. Then replacing the $z_t's$ by their smoothed values $\hat{z}_t$ we get the system :

$$\left\{ \begin{array}{lcl} s_t & = & \mu' \hat{z}_t + \varphi_1(s_{t-1} - \mu' \hat{z}_{t-1}) + \sigma(\hat{z}_t, \hat{z}_{t-1})\varepsilon_t^s \\ \\ y_t & = & \psi_0 + \psi_1(s_{t-1} - \mu' \hat{z}_{t-1}) + \psi_2' y_{t-1} + \Sigma_{22}^{1/2}(\hat{z}_t, \hat{z}_{t-1})\varepsilon_t^y \end{array} \right. \tag{44}$$

(where $\Sigma_{22}^{1/2}$ is the south-east block of $\Sigma^{1/2}$).

Adding Gaussian error terms ξ_t in (41) and replacing z_t by $\hat{z}_t$ we have :

$$\tilde{F}(t, H_{it}, D_i) = \frac{1}{D_i} \sum_{h=H_{it}}^{H_{it}+D_i-1} \exp[m(t,h) + c_h^s s_t + c_h^{y\prime} y_t + c_h^{z\prime} \hat{z}_t] + \xi_t \tag{45}$$

$$\xi_t \sim IIN(0, \omega^2 I), i = 1, \ldots, N_t$$

Equations (44) and (45) constitute a state space model, in which the latent variable is y_t. The transition equations are the second set of (44) and, therefore, linear, whereas the measure equations are (45) and the first equation of (44) and, therefore, nonlinear.

The parameters appearing in this nonlinear state-space system can be estimated using the extended Kalman filter.

Another possibility would be to estimate simultaneously the RN parameters and the y_t's, by minimizing with respect to these two sets of variables the sum of the squared differences between the observed values of the $\tilde{F}(t, H_{it}, D_i)$ and their theoretical formulas. In these formulas the z_t's might be replaced by smoothed values obtained from a first stage estimation of the univariate dynamics of (s_t, z_t) based, for instance, on a switching AR(p).

8. POSSIBLE EXTENSIONS

8.1. Multiple lags

In the previous section we have assumed that only one lag appears in all the equations. However it would be straightforward to introduce more lags, using, in particular, the fact that a Car(p) process can be transformed into a Car(1) by extending the size of the process. It is, however, preferable not to introduce lags in the latent variables y_{2t} in order to keep the simplicity of the inversion technique proposed in section 6.2. Since a priori insights on the dynamics of latent variables are not in general available, this constraint is not really restrictive.

8.2. Use of the spot price returns

Instead of using the variable $s_t = \log S_t$, we could use the return variables $s_t = \log \dfrac{S_t}{S_{t-1}}$. The only change would be in the expression of $F(t,h)$. Putting again $\tilde{s}_t = s_t - \tilde{\nu}_t^s$, we would have :

$$\begin{aligned} F(t,h) &= E_t^Q S_{t+h} \\ &= S_t E_t^Q \exp(s_{t+1} + \ldots + s_{t+h}) \\ &= S_t \exp(\tilde{\nu}_{t+1}^s + \ldots + \tilde{\nu}_{t+h}^s) E_t^Q \exp(\tilde{s}_{t+1} + \ldots + \tilde{s}_{t+h}) \end{aligned} \tag{46}$$

If the process $\tilde{s}_t$ is Car(1) in the RN world, the multihorizon Laplace transform appearing in (46) can be easily computed using proposition 1 using $\alpha_h = e_{J+1}, h = 1, \ldots, H$.

8.3. Non additive impact of the exogenous variables

We have assumed that, in the RN world, the exogenous variables only appear in the systematic part of $\tilde{\nu}_t^s$ and $\tilde{\nu}_t^y$ of the additive decomposition $\log S_t = \tilde{\nu}_t^s + \tilde{s}_t$ and $y_t = \tilde{\nu}_t^y + \tilde{y}_t$. In particular we have assumed that the RN transition probabilities π_{ij}^* do not depend on exogenous variables, contrary to their historical counterpart π_{ijt}. The introduction of non additive exogenous variables would lead to a non homogenous Car(1) dynamics of $\tilde{w}_t$ characterized by a conditional Laplace transform of the form

$$\exp[a'_{t+1}(u)w_t + b_{t+1}(u)]$$

where $a_{t+1}(.)$ and $b_{t+1}()$ depend on time through the exogenous variables.

It turns out that, in this context, the multihorizon conditional Laplace transforms can still be computed recursively, using a generalized algorithm which is forward in h and backward in t (see appendix 6 and Gourieroux Monfort Polimenis (2006) for details).

8.4. Dynamic specification of the exogenous variables

Up to know we have worked conditionally to future scenarios for the exogenous variables and, therefore, we dit not have to specify their dynamics. However if we do not wish to consider scenarios of exogenous variables we will have to incorporate them in the w_t vector and specify their historical as well as their risk neutral dynamics.

The simplest solution is to consider that these exogenous variables are a subvector of y_t, which is observable. Nothing is changed in the pricing and estimation results, the only particular feature is that, in system (1), the right hand side of the equation corresponding to the exogenous variables only contain past values of these variables and the error term is independent of the others and of the process z_t. In other words there is no feedback from all the other variables towards the exogenous variables.

The drawback of the previous solution is that the dynamics of the exogenous variables is assumed to be linear autoregressive and, moreover, the impact of these variables on $\log S_t$ is assumed to be linear. The latter assumption is not necessarily satisfactory if the exogenous variable is the temperature or the difference between the temperature and a "normal" level. In this case an alternative modeling, which will allow to stay in the Car domain, at least in the risk neutral world, is to assume that the exogenous variable of interested has been discretized. If we denote by z_t^e this observable discrete value process, and if we denote by $\tilde{z}_t$ the latent discrete value process aiming at capturing the spikes, the overall discrete value process introduced at the beginning of this study becomes $z_t = \tilde{z}_t \otimes z_t^e$. In other words z_t is valued in the set of vectors obtained as the Kronecker product of vectors $e_k \otimes e_l$, e_k being of size K and e_l of size $L, i.e.$ in the set of vectors e_q of size KL. If this process is an homogenous Markov chain the process w_t is Car(1) in the historical world, and also in the RN world if we use the same kind of s.d.f. as in section 3.1, and like previously this process could be Car (1) in the RN world even if it is not in the historical world. Attractive features of this modeling are that the impact of z_t^e on $\log S_t$ is nonlinear, that we could introduce different causality schemes and that the estimation procedure could be made sequentially.

Let us consider the case where $\tilde{z}_t$ has 2 states, whereas z_t^e has 3 states. In this case z_t has 6 states and an unconstrained Markov dynamics would have 30 parameters. The other extreme case would be to assume that $\tilde{z}_t$ and z_t^e are two independent Markov chains, in this case the number of parameters would be $2+6=8$ parameters and, moreover, the 6 parameters of the dynamics of z_t^e could be estimated separately since z_t^e is observable. Several intermediate cases are also of interest. Indeed we have :

$$
\begin{aligned}
& P(\tilde{z}_t = e_j, z_t^e = e_j^* / \tilde{z}_{t-1} = \tilde{e}_i, z_{t-1}^e = e_i^*) \\
= \quad & P(\tilde{z}_t = e_j / z_t^e = e_j^*, \tilde{z}_{t-1} = \tilde{e}_i, z_{t-1}^e = e_i^*) P(z_t^e = e_j^* / \tilde{z}_{t-1} = \tilde{e}_i, z_{t-1}^e = e_i^*)
\end{aligned}
$$

The first term of the RHS depends on 18 parameters and the second of 12 parameters, the total beeing 30 as mentioned above. Since z_t^e is exogenous it is natural to assume that $\tilde{z}_{t-1}$ does not appear in the second term of the RHS which, therefore, only depends on 6 parameters which can be estimated separately. If we assume moreover that z_{t-1}^e does not appear in the first term of the RHS, that is to say that z_t^e causes $\tilde{z}_t$ only instantaneously, this first term only depends on 6 parameters, which can be estimated by the Kitagawa Hamilton algorithm. Moreover the term $\mu' z_{t+1}$ in (1) could be specified additively : $\mu' z_{t+1} = \tilde{\mu} \tilde{z}_{t+1} + \mu^{*'} z_{t+1}^e$. For the model to be identified we would have to impose a constraint on $(\mu', \mu^{*'})$, for instance that one component of $\tilde{\mu}$, or μ^* is zero. Therefore in the example above the number of parameters appearing in $\mu' z_{t+1}$ would be 4.

	Historical	Risk neutral
μ_1	0	0
μ_2	0.5	0
φ_1	0.6	0.5450
φ_2	0.1	-0.4450
ψ_0	0	0
ψ_1	0.5	-0.4450
ψ_2	0.1	0.5450
Σ_1	$\begin{bmatrix} 0.05 & 0.0112 \\ 0.0112 & 0.01 \end{bmatrix}$	$\begin{bmatrix} 0.05 & 0.0112 \\ 0.0112 & 0.01 \end{bmatrix}$
σ_2	$\begin{bmatrix} 0.5 & 0.0884 \\ 0.0884 & 0.0625 \end{bmatrix}$	$\begin{bmatrix} 0.5 & 0.0884 \\ 0.0884 & 0.0625 \end{bmatrix}$
P	$\begin{bmatrix} 0.99 & 0.01 \\ 0.80 & 0.20 \end{bmatrix}$	$\begin{bmatrix} 0.99 & 0.01 \\ 0.80 & 0.20 \end{bmatrix}$

Table 1: Characteristics of historical and risk neutral dynamics

9. Illustration

Let us give some examples of simulated spot and forward electricity prices generated by affine models. The objective is not to explore the whole set of possibilities of affine models. Here, we illustrate how simple models can represent spikes in spot prices and many forms of forward curves.

9.1. Simulated spot prices

In this section we consider a regime switching VAR(1) model described by (1) with a latent factor y_t and two regimes: the first regime is persistent and represent "normal" spot prices, while the second aims at representing spikes. The values of the historical and risk neutral parameters are given in table 1. Figure 1 shows behavior variations due to the transition probability matrix and specifically to the probability of obtaining a spike. As expected the frequency of simulated spikes is directly related to the values of the transition probability.

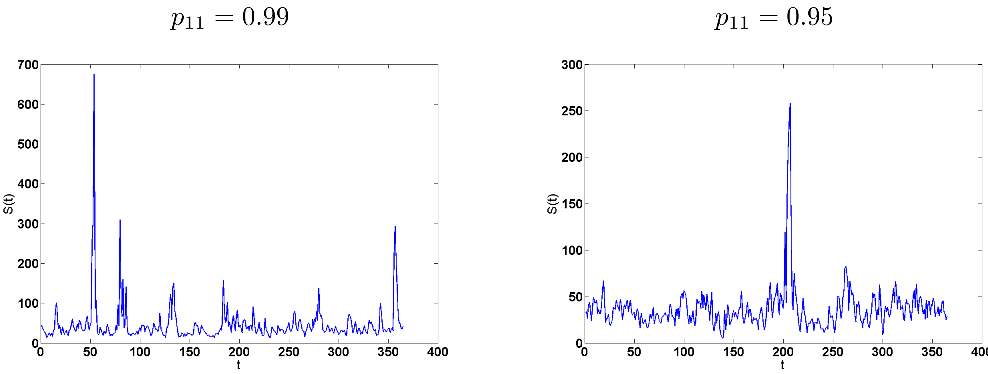

Figure 1: Simulated spot price in function of transition probabilities. The model is a VAR(2) composed of 2 regimes

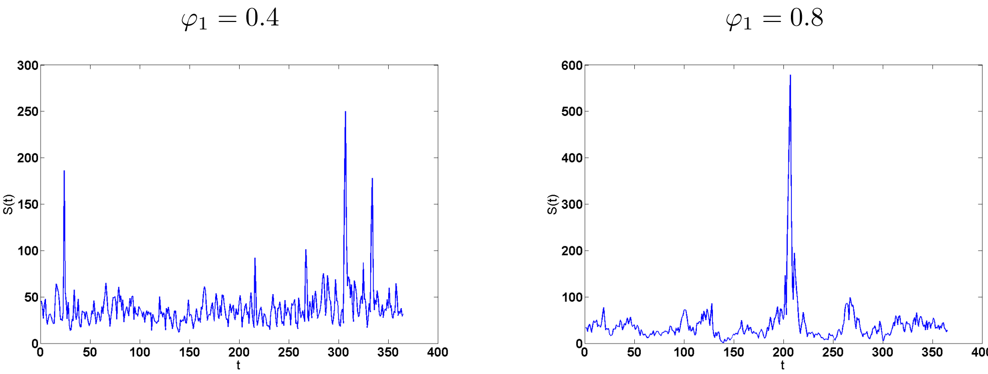

Figure 2: Simulated spot price in function of φ_1. The model is a VAR(2) composed of 2 regimes

Figure 2 shows behavior variations due to the auto-regressive coefficient φ_1. This result shows the capability of the model to differently represent the effect of a spike on subsequent dates.

9.2. Simulated Forward prices

We are now interested in simulating forward prices from affine models. We first consider $\tilde{\nu}_t^s = \tilde{\nu}_t^y = 0$. Figure 3 shows different forward curves $F(t, h)$ obtained by simulating model (1) with several values of parameters. Paramter μ_1 impacts the beginning of the curve and the long term equilibrium value. Parameter p_{11} impacts the long term behaviour of the curve. The value of observation X_t also impact the beginning of the

curve. The persistence of this impact is related to the eigen values of the matrix of polynomials in the lag operator of the risk neutral dynamics. Indeed, figure 3c shows forward curves when the highest eigen value is 0.95 whereas figure 3d shows forward curves when the higher eigen value is 0.99, where the value of X_t is more persistent.
These results illustrate the capability of affine models to represent a large set of possible foward curves.
Also note that a risk neutral deterministic part, say $\tilde{\nu}_t^s$, can be introduced to represent, for example, seasonality in the forward curve. Indeed, consider $\tilde{\nu}_t^s$ as a product of a daily and a monthly coefficients:

$$\tilde{\nu}_t = D_t M_t, \tag{47}$$

where $D_t = C_i^d$ if t corresponds to the ith day of the week, and $M_t = C_j^m$ if t belongs to the jth month of the year.
Figure 4 shows examples of simulated forward curves from affine models when all effects (parameters and seasonality) are mixed. This result shows the effectiveness of affine models in well representing forward prices.

10. CONCLUDING REMARKS

Building on recent developments of econometrics of asset pricing, we have proposed flexible and tractable modeling strategies for the joint behavior of spot electricity prices, forward prices and option prices. We have considered different types of data : seasonally adjusted versus raw spot prices, estimated forward curves versus forward contracts prices, different statistical specifications and different estimation strategies. This general framework should provide a guideline for various kinds of applications.

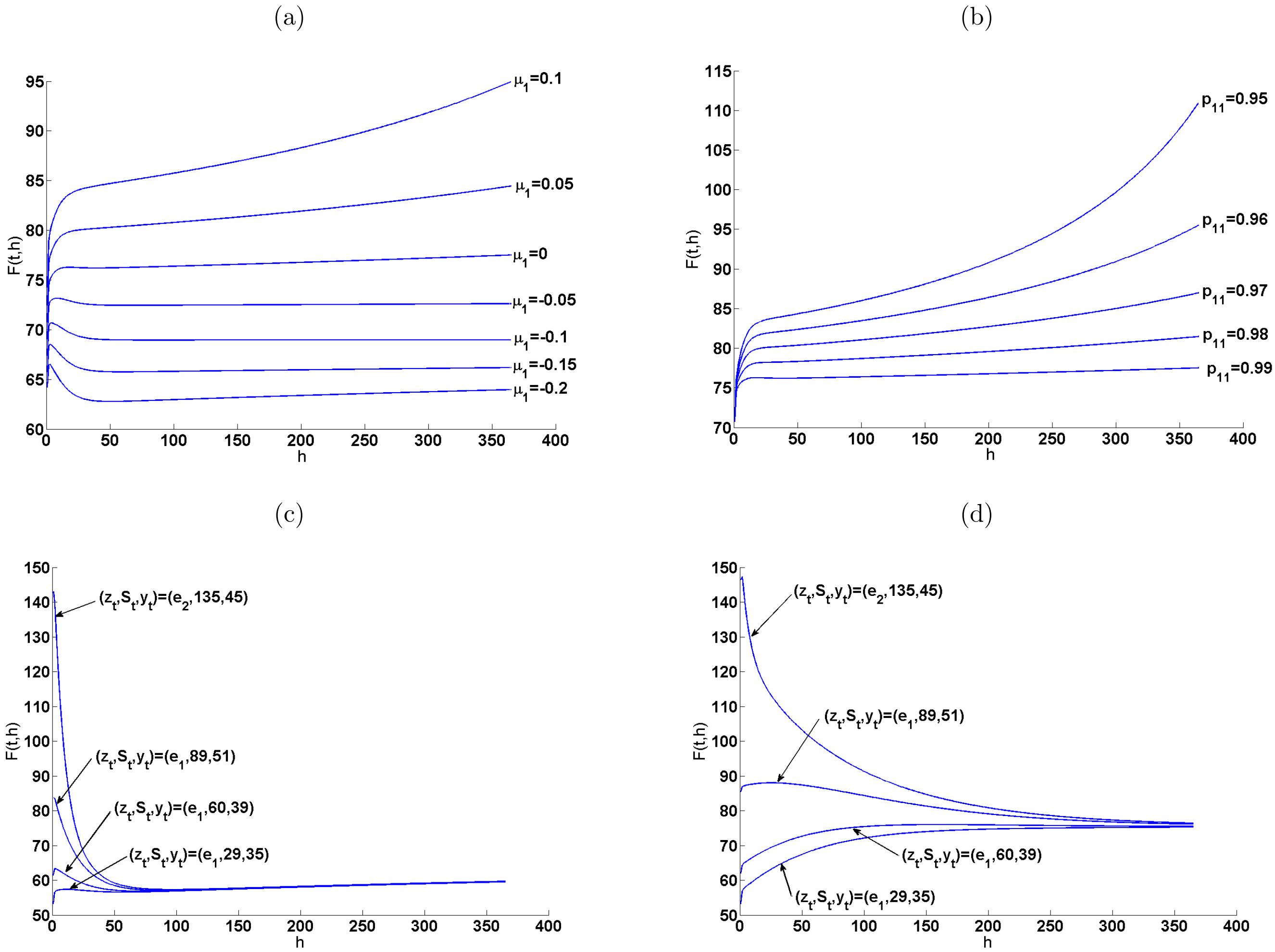

Figure 3: Simulated Forward curves in function of (a) mean μ_1, (b) transition probability p_{11}, and value of observation X_t when higher root of AR-polynomial is (c) 0.95 and (d)0.99.

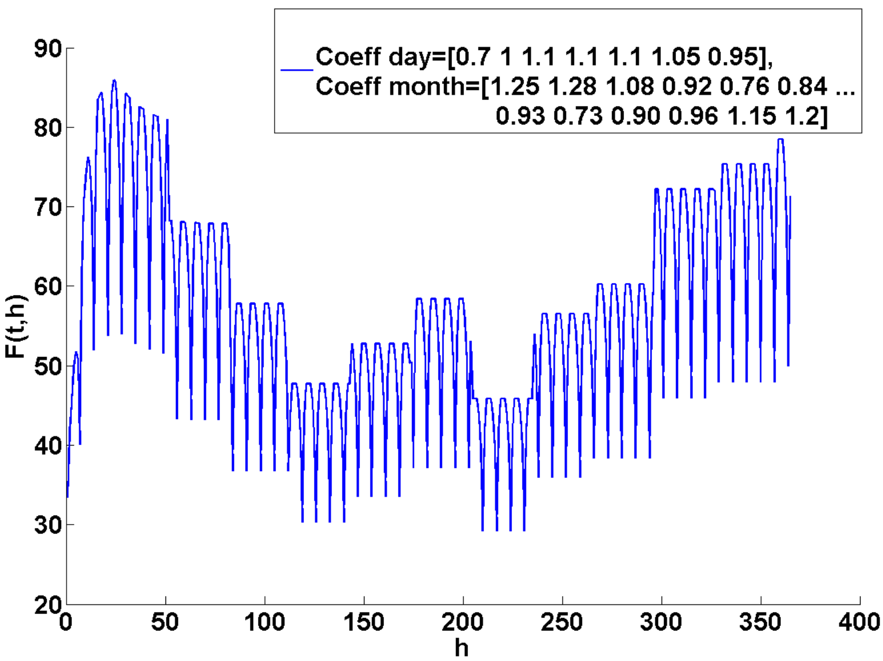

Figure 4: An example of realistic forward curve.

APPENDIX 1

NONCAUSALITY AND INDEPENDENCE IN THE HISTORICAL AND THE RISK NEUTRAL WORLDS

i) Non causality in the historical world.

Let us consider the stochastic process $(w_t', w_t^{e'})', t = 1, \dots, T$. by definition, w_t does not (Granger) cause w_t^e if :

$$f(w_t^e / \underline{w_{t-1}}, \underline{w_{t-1}^e}) = f(w_t^e / \underline{w_{t-1}^e}), \forall t$$

that is, if w_t^e and $\underline{w_{t-1}}$ are independent conditionally to $\underline{w_{t-1}^e}$.

This definition is equivalent to the Sims definition :

$$f(w_t / \underline{w_{t-1}}, \underline{w_T^e}) = f(w_t / \underline{w_{t-1}}, \underline{w_t^e}), \forall t, T$$

that is : w_t and the future values of w^e are independent conditionally to $\underline{w_{t-1}}, \underline{w_t^e}$.

Indeed the joint p.d.f. of $\underline{w_T}, \underline{w_T^e}$ can always be written :

$$\begin{aligned} f(\underline{w_T}, \underline{w_T^e}) &= \Pi_{t=1}^T f(w_t / \underline{w_{t-1}}, \underline{w_T^e}) \Pi_{t=1}^T f(w_t^e / \underline{w_{t-1}^e}) \\ &= \Pi_{t=1}^T f(w_t / \underline{w_{t-1}}, \underline{w_t^e}) \Pi_{t-1}^T f(w_t^e / \underline{w_{t-1}}, \underline{w_{t-1}^e}) \end{aligned}$$

and the result follows.

Thus, if w_t does not cause w_t^e we have

$$f(\underline{w_T}, \underline{w_T^e}) = \Pi_{t=1}^T f(w_t / \underline{w_{t-1}}, \underline{w_t^e}) \Pi_{t=1}^T f(w_t^e / \underline{w_{t-1}^e})$$

Although the notion of (strong) exogeneity is not perfectly equivalent to the notion of non-causality (see Gourieroux-Monfort 1996, chapter 1, volume 1, for the details), we will not distinguish them here.

ii) Non causality and independence in the risk neutral world

We consider the processes w_t, w_t^e and w_t^i introduced in section 2.2. Let us assume that the SDF between $t-1$ and t is factorized as follows :

$$\tilde{M}_{t-1,t} = \exp(-r_t) M^*_{t-1,t}(\underline{w_t}, \underline{w^e_t}) M^e_{t-1,t}(\underline{w^e_t}) M^i_{t-1,t}(\underline{w^i_t})$$

where

$$E(M^*_{t-1,t} / \underline{w_{t-1}}, \underline{w^e_t}) = 1$$

$$E(M^e_{t-1,t} / \underline{w^e_{t-1}}) = 1$$

$$E(M^i_{t-1,t} / \underline{w^i_{t-1}}) = 1$$

and r_t is the short rate between $t-1$ and t, (which is a function of $\underline{w^i_{t-1}}$).

Using the iterated expectation formula we get :

$$E(\tilde{M}_{t-1,t} / \underline{w_{t-1}}, \underline{w^e_{t-1}}, \underline{w^i_{t-1}}) = \exp(-r_t)$$

Moreover the joint conditional RN p.d.f of (w_t, w^e_t, w^i_t) given $(\underline{w_{t-1}}, \underline{w^e_{t-1}}, \underline{w^i_{t-1}})$ is obtained as the product of the historical one :

$$f(w_t / \underline{w_{t-1}}, \underline{w^e_t}) f^e(w^e_t / \underline{w^e_{t-1}}) f^i(w^i_t / \underline{w^i_{t-1}})$$

by $M^*_{t-1,t}, M^e_{t-1,t} M^i_{t-1,t}$, and we get :

$$f^Q(w_t / \underline{w_{t-1}}, \underline{w^e_t}) (f^e)^Q (w^e_t / \underline{w^e_{t-1}}) (f^i)^Q (w^i_t / \underline{w^i_{t-1}})$$

where :

$$\begin{aligned} f^Q &= f \times M^*_{t-1,t} \\ (f^e)^Q &= f^e \times M^e_{t-1,t} \\ (f^i)^Q &= f^i \times M^i_{t-1,t} \end{aligned}$$

So the exogeneity of w^e_t and the independence of w^i_t are preserved in the RN world.

iii Pricing

The pricing at t of a payoff $g(\underline{w_T}, \underline{w_T^e})$ delivered at $T > t$ is given by :

$$\tilde{p}_t = E_t^Q \exp(-r_{t+1} - \ldots - r_T) g(\underline{w_T}, \underline{w_T^e})$$

where E_t^Q is the RN conditional expectation given $(\underline{w_t}, \underline{w_t^e}, \underline{w_t^i})$.

Since w_t^i is Q-independent of (w_t, w_t^e) we get :

$$\tilde{p}_t = B(t, T-t) E_t^Q g(\underline{w_T}, \underline{w_T^e})$$

where $B(t, T-t)$ is the price at t of the zero-coupon bond of residual maturity $T - t$. Moreover if we fix a scenario $\underline{w_T^e}$ for the future exogenous variables, the conditional price is :

$$p_t = B(t, T-t) E^Q[g(\underline{w_T}, \underline{w_T^e}) / \underline{w_t}, \underline{w_T^e}]$$

and given the exogeneity of w^e and using Sims' version of the non-causality mentioned in (i), this conditional expectation is obtained by fixing $\underline{w_T^e}$ at scenario values and using the p.d.f. :

$$\prod_{\tau=t+1}^{T} f^Q(w_\tau / \underline{w_{\tau-1}}, \underline{w_\tau^e})$$

i.e by considering $w_{t+1}^e, \ldots, w_T^e$ as non random and fixed at their scenario values.

Also note that the conditional price can be written, using the historical dynamics and the stochastic discount factor $M_{t-1,t} = \exp(-r_t) M_{t-1,t}^*$

$$\begin{aligned} p_t &= B(t, T-t) E[M_{t,t+1}^* \ldots M_{T-1,T}^* g(\underline{w_T}, \underline{w_T^e}) / \underline{w_T}, \underline{w_T^e}] \\ &= E[M_{t,t+1} \ldots M_{T-1,T} g(\underline{w_T}, \underline{w_T^e}) / \underline{w_t}, \underline{w_T^e}, \underline{w_t^i}] \end{aligned}$$

In other words, we can work with a stochastic discount factor $M_{t-1,t}$ of the form $\exp(-r_t) M_{t-1,t}^*$, with $E(M_{t-1,t}^* / \underline{w_{t-1}}, \underline{w_t^e}) = 1$.

APPENDIX 2
PROOF OF PROPOSITION 1

$L_{t,1}(\alpha) = E_t \exp(\alpha'_H w_{t+1}) = \exp[a(\alpha_H)' w_t + b(\alpha_H)]$

and since $c_1 = a(\alpha_H), d_1 = b(\alpha_H)$, the formula is true for $h = 1, \forall t$.

Let us assume that it is true for $h-1, \forall t$, we have :

$$
\begin{aligned}
L_{t,h}(\alpha) &= E_t[\exp(\alpha'_{H-h+1} w_{t+1}) E_{t+1} \exp(\alpha'_{H-h+2} w_{t+2} + \ldots + \alpha'_H w_{t+h})] \\
&= E[\exp(\alpha'_{H-h+1} w_{t+1}) L_{t+1,h-1}(\alpha)] \\
&= E_t[\exp(\alpha'_{H-h+1} w_{t+1} + c'_{h-1} w_{t+1} + d_{h-1})] \\
&= \exp[a(\alpha_{H-h+1} + c_{h-1})' w_t + b(\alpha_{H-h+1} + c_{h-1}) + d_{h-1}]
\end{aligned}
$$

and the result follows.

APPENDIX 3
HISTORICAL CONDITIONAL LAPLACE TRANSFORM OF w_{t+1}

The historical conditional Laplace transform of $w_{t+1} = (z'_{t+1}, x'_{t+1})'$ is

$$\varphi_t(u,v) = E_t \exp(u' z_{t+1} + v' x_{t+1})$$

$$= E_t \exp[u' z_{t+1} + v' \mu(z_{t+1}, z_t) + v' \Phi x_t + v' \Sigma^{1/2}(z_{t+1}) \varepsilon_{t+1}]$$

Taking first the conditional expectation given $(\underline{w_t}, z_{t+1})$ we get :

$$
\begin{aligned}
\varphi_t(u,v) &= \exp(v' \Phi x_t) E_t \exp[u' z_{t+1} + v' \mu(z_{t+1}, z_t) + \frac{1}{2} v' \Sigma(z_{t+1}, z_t) v] \\
&= \exp[v' \Phi x_t + (A_{1t}, \ldots, A_{Jt}) z_t]
\end{aligned}
$$

where

$$A_{it} = \log \left\{ \sum_{j=1}^{J} \pi_{ijt} \exp \left[u' e_j + v' \mu(e_j, e_i) + \frac{1}{2} v' \Sigma(e_j, e_i) v \right] \right\}$$

APPENDIX 4

RISK NEUTRAL CONDITIONAL LAPLACE TRANSFORM OF w_{t+1}

The RN conditional Laplace transform of w_{t+1} is :

$$
\begin{aligned}
\varphi_t^Q(u,v) &= E_t^Q \exp(u'z_{t+1} + v'x_{t+1}) \\
&= E_t M^*_{t,t+1} \exp(u'z_{t+1} + v'x_{t+1}) \\
&= E_t \exp(\Gamma'_{t+1}\varepsilon_{t+1} - \frac{1}{2}\Gamma'_{t+1}\Gamma_{t+1} + \delta'_{t+1}z_{t+1} + u'z_{t+1} + v'x_{t+1})
\end{aligned}
$$

with $\Gamma_{t+1} = \Gamma(z_{t+1}, z_t, x_t, w^e_{t+1})$

$\delta_{t+1} = \delta(z_t, s_t, w^e_{t+1})$

and $\sum_{j=1}^{J} \pi_{ijt} \exp[\delta_j(e_i, s_t, w^e_{t+1})] = 1, \forall e_i, s_t, w^e_{t+1}$

Using the expression of x_{t+1} given by (19) we get :

$$
\begin{aligned}
\varphi_t^Q(u,v) &= E_t \exp[\Gamma'_{t+1}\varepsilon_{t+1} - \frac{1}{2}\Gamma'_{t+1}\Gamma_{t+1} + \delta'_{t+1}z_{t+1} + u'z_{t+1} + v'[\mu(z_{t+1}, z_t) + \Phi x_t \\
&\quad + \Sigma^{1/2}(z_{t+1}, z_t)\varepsilon_{t+1}] \\
&= \exp(v'\Phi x_t) \times \\
&\quad E_t \exp[-\frac{1}{2}\Gamma'_{t+1}\Gamma_{t+1} + \frac{1}{2}(\Gamma_{t+1} + \Sigma^{1/2}_{t+1}v)'(\Gamma_{t+1} + \Sigma^{1/2}_{t+1}v) + \delta'_{t+1}z_{t+1} + u'z_{t+1} + v'\mu_{t+1} \\
&= \exp(v'\Phi x_t) E_t \exp[v'\Sigma^{1/2}_{t+1}\Gamma_{t+1} + \frac{1}{2}v'\Sigma_{t+1}v + \delta'_{t+1}z_{t+1} + u'z_{t+1} + v'\mu_{t+1}] \\
&= \exp[v'\Phi x_t + (\bar{A}_{1t} \dots \bar{A}_{Jt})z_t]
\end{aligned}
$$

with :

$$\bar{A}_{it} = \log \left\{ \sum_{j=1}^{J} \pi_{ijt} \exp \left[v'\Sigma^{1/2}(e_j, e_i)\Gamma(e_j, e_i, x_t, w_{t+1}^e) \right.\right.$$

$$\left.\left. + \frac{1}{2} v'\Sigma(e_j, e_i)v + \delta_j(e_i, s_t, w_{t+1}^e) + u'e_j + v'\mu(e_j, e_i) \right] \right\}$$

APPENDIX 5
RN DYNAMICS

The RN conditional Laplace transform of $\tilde{w}_{t+1} = (\tilde{x}_{t+1}, z_{t+1})$ given by (20), and the transition matrix π_{ij}^* is :

$$E_t^Q \exp(u'z_{t+1} + v'\tilde{x}_{t+1})$$

$$= \exp(v'\Phi^*\tilde{x}_t)E_t^Q \exp[u'z_{t+1} + v'\mu^*(z_{t-1}, z_t) + v'\Sigma^{1/2}(z_{t+1}, z_t)\varepsilon_{t+1}^*]$$

$$= \exp[(A_1^* \dots A_J^*)z_t + v'\Phi^*\tilde{x}_t]$$

with $A_i^* = \log\{\sum_{j=1}^{J} \pi_{ij}^* \exp[u'e_j + v'\mu^*(e_j, e_i) + \frac{1}{2}v'\Sigma(e_j, e_i)v]\}$

Therefore the RN conditional Laplace transform of w_{t+1} implied by (20) and π_{ij}^* is, using $\tilde{x}_t = x_t + \nu_t - \tilde{\nu}_t$:

$$\exp[v'\Phi^*(\nu_t - \tilde{\nu}_t) + v'\Phi^*x_t + (A_1' \dots A_J^*)z_t]$$

Comparing with $\varphi_t^Q(u, v)$ given in proposition 4, we see that they are identical if :

i) $\pi_{ijt} \exp[\delta_j(e_i, s_t, w_{t+1}^e)] = \pi_{ij}^*$

ii) $\Sigma^{1/2}(e_j, e_i)\Gamma(e_j, e_i, x_t, w_{t+1}^e) = (\Phi^* - \Phi)x_t + \Phi^*(\nu_t - \tilde{\nu}_t) + \mu^*(e_j, e_i) - \mu(e_j, e_i)$

So, for a given historical dynamics characterized by π_{ijt}, Φ, μ and ν_t and any wished RN dynamics characterized by $\pi_{ij}^*, \Phi^*, \mu^*$ and $\tilde{\nu}_t$, we can adjust the SDF in an appropriate way by choosing the δ_j satisfying i) and Γ satisfying ii) that is :

$$\delta_j(e_i, s_t, w^e_{t+1}) = \log \frac{\pi^*_{ij}}{\pi_{ijt}}$$

and

$$\Gamma(e_j, e_i, x_t, w^e_{t+1}) =$$

$$\Sigma^{-1/2}(e_j, e_i)[(\Phi^* - \Phi)x_t + \Phi^*(\nu_t - \tilde{\nu}_t) + \mu^*(e_j, e_i) - \mu(e_j, e_i)]$$

Note that the δ_j thus defined automatically satisfy constraints (14).

APPENDIX 6

EXTENSION OF PROPOSITION 1 TO THE NON HOMEGENOUS CASE

We want to compute :

$$L_{t,h}(\alpha) = E_t[\exp(\alpha'_{H-h+1} w_{t+1} + \ldots + \alpha'_H w_{t+h})]$$

$$\forall t = 1, \ldots, t, h = 1, \ldots, H, \text{ with } \alpha = (\alpha'_1, \ldots, \alpha'_H)'$$

when

$$E_t[\exp(u' w_{t+1})] = \exp[a'_{t+1}(u) w_t + b_{t+1}(u)]$$

We want to show :

$$L_{t,h}(\alpha) = \exp(c'_{t,h} w_t + d_{t,h}), \forall t, h$$

This formula is true of $h = 1$ and any t since :

$$\begin{aligned} L_{t,1}(\alpha) &= E_t[\exp(\alpha'_H w_{t+1})] \\ &= \exp[a'_{t+1}(\alpha_H) w_t + b_{t+1}(\alpha_H)] \end{aligned}$$

and we have :$c_{t,1} = \alpha_{t+1}(\alpha_H), d_{t,1} = b_{t+1}(\alpha_H)$.

Moreover assuming that the formula is true for $h - 1$ and any t we have :

$$
\begin{aligned}
L_{t,h}(\alpha) &= E_t[\exp(\alpha'_{H-h+1}w_{t+1})E_{t+1}\exp(\alpha'_{H-h+2}w_{t+2}+\ldots+\alpha'_H w_{t+h})] \\
&= E_t[\exp(\alpha'_{H-h+1}w_{t+1})L_{t+1,h-1}(\alpha)] \\
&= E_t[\exp(\alpha'_{H-h+1}w_{t+1}+c'_{t+1,h-1}w_{t+1}+d_{t+1,h-1}] \\
&= \exp[a'_{t+1}(\alpha_{H-h+1}+c_{t+1,h-1})w_t+b_{t+1}(\alpha_{H-h+1}+c_{t+1,h-1})+d_{t+1,h}
\end{aligned}
$$

and we get

$$
\begin{aligned}
c_{t,h} &= a_{t+1}(\alpha_{H-h+1}+c_{t+1,h-1}) \\
d_{t,h} &= b_{t+1}(\alpha_{H-h+1}+c_{t+1,h-1})+d_{t+1,h-1}
\end{aligned}
$$

Note that we get the right values of $c_{t,1}$ and $d_{t,1}$ by taking $c_{t,0}=0$ and $d_{t,0}=0$ for all t.

Computation

• Starting from $c_{T+H,0}=0$ we get $c_{T,H}$ after H iterations on $c_{t,h}$

• Starting form $c_{T+H-1},0=0$ we get $c_{T,H-1}$ and $c_{T-1,H}$

etc

• Starting from $c_{t,0}=0$ we get $c_{t-1,1},c_{t-2,2},\ldots,\ c_{t-H,H},\forall t\in\{H+1,\ldots,T+1\}$

• Starting from $c_{t,0}=0$ we get $c_{t-1,1},\ldots,c_{1,t-1}\forall t\in\{2,\ldots,H\}$.

R E F E R E N C E S

Benth F.E., Kallsen J and T. Meyer-Brandis (2005) : "A non Gaussian Ornstein-Ulhenbeck Process for Electricity Spot Price Modeling and Derivative Pricing", Discussion Paper Center for Mathematics, University of Oslo n°14/2005.

Benth F.E. and S. Koekebakker (2005) : "Stochastic Modeling of Financial Electricity Contracts", Dpt of Pure Mathematics, University of Oslo, Discussion paper n°024.

Bertholon H., A. Monfort and F. Pegoraro (2008) : "Econometric Asset Pricing Modelling", Journal of Financial Econometrics, 6, 4, 407-458.

Bunn D.W. and N. Karakatsani (2003) : "Forecasting Electricity Prices", London Business School, Working Paper.

Darolles, S., C. Gourieroux, and J. Jasiak (2006) : "Structural Laplace Transform and Compound Autoregressive Models", Journal of Time Series Analysis, 24 (4), 477-503.

Deng S. (2000) "Stochastic Models of Energy Commodity Prices and Their Applications : Mean Reversion with Jumps and Spikes", Working paper, University of California Energy Institute.

De Jong C. (2005) : "The Nature of Power Spikes : a Regime Switching Approach", Working Paper, Erasmus University Rotterdam n° ERS - 2005-052.

Duffie, D., J. Pan, and K. Singleton (2000) : "Transform Analysis and Asset Pricing for Affine Jump Diffusions, Econometrica, 68, 1343-1376.

Geman H. and A. Roncoroni (2006) : "Understanding the Fine Structure of Electricity Prices", Journal of Business, 79, 3, 1225-1261.

Gourieroux, C., and A. Monfort (2007) : "Econometric Specifications of Stochastic Discount Factor Models", Journal of Econometrics, 136, 509-530.

Gourieroux, C., A. Monfort, and V. Polimenis (2006) : "Affine Model for Credit Risk Analysis", Journal of Financial Econometrics, 4(3), 494-530.

Gourieroux, C., and A. Monfort (1996) : *Statistics and Econometrics Models* (2 volumes), Cambridge University Press.

Hamilton J. (1989) : "A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle", Econometrica, 57, 2, 357-384.

Huisman R. and R. Mahieu (2003) : "Regime Jumps in Electricity Prices", Energy Economics, 25, 5, 425-434.

Huisman R. (2008) : "The Influence of Temperature on Spike Probability and Day-ahead Power Price", Energy Economics, 30, 5, 2697-2704.

Kim, C. J. (1994) : "Dynamic Linear Models with Markov Switching", Journal of Econometrics, 60, 1–22.

Lucia J. and E. Schwartz (2002) : "Electricity Prices and Power Derivatives : Evidence from the Nordic Power Exchange", Review of Derivative Research, 5, 5-50.

Mount T., Ning Y. and X. Cai (2005) : "Predicting Price Spikes in Electricity Markets Using a Regime-Switching Model with Time-Varying Parameters", Energy Economics, 28, 1, 62-80.

Monfort, A., and F. Pegoraro (2007) : "Switching VARMA Term Structure Models", Journal of Financial Econometrics 5(1), 105-153.

Weron R., Bierbrauer M. and S. Truck (2004) "Modelling Electricity Prices : Jump Diffusion and Regime Switching", Physica, 336, 39-48.