Centre de Recherche en Économie et Statistique
CREST

Série des Documents de Travail

n° 2011-04

Allocating Systematic and Unsystematic Risks in a Regulatory Perspective

C. GOURIÉROUX[1]
A. MONFORT[2]

[1] CREST and University of Toronto, Canada.
[2] CREST, Banque de France and Maastricht University.

Allocating Systematic and Unsystematic Risks in a Regulatory Perspective

C., GOURIEROUX [(1)] and A., MONFORT[(2)]

(First version, September 2010, revised January, 2011)
All comments are welcome

The first author gratefully acknowledges financial support of the NSERC Canada and the chair AXA/Risk Foundation : "Large Risks in Insurance". We thank M., Summer, D., Tasche for valuable feedback. The views expressed in this paper are those of the authors and do not necessarily reflect those of the Banque de France.

[1]CREST, and University of Toronto.
[2]CREST, Banque de France, and University of Maastricht.

Allocating Systematic and Unsystematic Risks in a Regulatory Perspective

Abstract

This paper discusses the allocation of reserves among financial entities from a regulatory point of view. We introduce axioms of decentralization, additivity, compatibility with risk ordering, which should be satisfied by the allocations and we characterize the set of allocations compatible with these axioms. Then, we explain how to disentangle systematic and unsystematic risk components in these allocations. Finally, we discuss the usual relationship between basic reserve and reglementary required capital, and propose alternative solutions to the question of procyclical required capital.

Keywords : Risk Measure, Allocation, Regulation, Systematic Risk, Procyclical Effect.

1 Introduction

The definition of the required capital in Basel regulation is often presented as an important reason in the development of the recent financial crisis. The following arguments are in particular invoked :

i) The regulatory reserves were not sufficiently large to cover the (extreme) risks.

ii) They did not account for comovement of financial institutions assets and liabilities, that is, for the systematic risk factors.

iii) This regulation has a procyclical effect, instead of the expected counter-cyclical effect.

These possible drawbacks of the previous regulation explain the recent changes in both regulation and academic research. Examples are the introduction of an additional regulator focusing on systemic risk, the various stress-testing performed in US as well as in European countries, or the new measures of systemic risk introduced in the academic literature [see e.g. Adrian, Brunnermeier (2009), Acharya et alii (2010), Brownless, Engle (2010), Tarashev et alli (2010), Billio, Mamo, Pelizzon (2010)].

The aim of our paper is to discuss these questions in a detailed and critical way. Let us first recall that Basel regulation defines the required capital in two steps. In a first step each financial entity has to compute a measure of its own risk.[3] The standard risk measure used by these entities is the Value-at-Risk (VaR), which gives the maximum loss within a $\alpha\%$ confidence interval. The level α is fixed by the regulator and the risk of an individual entity is considered in isolation. Then, in a second step, the required capital is fixed from the observed individual history of VaR. A typical formula for required capital at day t is for instance :

$$RC_t = \max(VaR_t, k\frac{1}{60}\sum_{h=0}^{59} VaR_{t-k}), \tag{1.1}$$

where VaR_t denotes the VaR at horizon 1-year and the trigger parameter k depends on the technical level of the entity and is generally larger than 3. This nonlinear link function between the risk measure and the required

[3]and also such measures for each business line separately.

capital induces two regimes : in a standard risk environment for the bank, formula (1.1) reduces to : $RC_t = k\frac{1}{60}\sum_{h=0}^{59} VaR_{t-h}$. The smoothing of risk measures over 60 opening days, i.e. 3 months, is introduced to avoid an erratic evolution of the reserves. Coefficient k introduces an additional insurance against risk. When the entity becomes suddenly very risky, that is, when the current risk measure VaR_t is larger than $\frac{k}{60}\sum_{k=0}^{59} VaR_{t-h}$, the required capital becomes equal to VaR_t.

The existence of these two steps shows that any discussion has to distinguish the possible defaults of the suggested VaR risk measure and those of the link function (1.1).

We first discuss in Section 2 the subadditivity property of the standard risk measures considered in the literature, that are the Value-at-Risk, and the Distortion Risk Measures (DRM) including the Expected Shortfall (ES). These measures are used to measure the global risk, but have to be disaggregated between the entities and also to be split into systematic and unsystematic components . We introduce in Section 3 the decentralization, additivity and risk ordering axioms, which are relevant for this decomposition and characterize the allocations satisfying the three axioms. Then, we discuss the properties of the different risk contributions proposed in the literature from the regulatory point of view[4]. Section 4 considers the special case of Euler allocations and discusses their sensitivities. This marginal interpretation is used in Section 5 to derive a disaggregation formula not only in terms of entities, but also in terms of systematic and unsystematic risks, both in linear and nonlinear models. Section 6 explores the link between the required capital and the objective measures of systematic and unsystematic risks. In particular, we explain why the Through The Cycle (TTC) smoothing treatment of these components have to be performed separately to avoid the spurious procyclical effect of the standard regulation and we propose alternative solutions to avoid this procyclicity of the required capital. Section 7 concludes. Technical proofs are gathered in appendices.

[4]The research of an appropriate allocation of capital for a purpose internal to a bank , for instance to maximize shareholder value, achieve capital efficiency, or measure concentration risk in a portfolio [see e.g. Patrick et alii (1999), Dhaene, Goovaerts, Kaas (2003), Sherris (2007), Tasche (2008)] is clearly out of the scope of the present paper.

2 Subadditivity of a risk measure

The notion of subadditivity is often used in this paper. So, it is worthwile to discuss its relevance.

There exists a rather large literature interested in the suitable properties of a risk measure interpretable as a level of reserve. The pioneering papers in this field are Artzner et alii (1997), (1999), who introduce the concept of coherent risk measure. [5] Let us denote by $i, i = 1, \ldots, n$, the entity of interest (i.e. a bank in Basel regulation, an insurance company in Solvency 2), by X_i the (random) future Loss and Profit (L&P) of this entity, and by $R(X_i)$ the level of reserve corresponding to the $L\&P$ [6] [7]. As usual the individual $L\&P$ is written in dollar and not in percentage value, that is in return (see Appendix 4 for the equivalent expressions when the analysis is performed in returns). This is especially important for derivative portfolios. Indeed, derivatives such as swaps often start with a low, or even zero value [Fung, Vasicek (1997)]. Among the axioms introduced for a coherent risk measure is the subadditivity property :

$$R(\sum_{i=1}^{n} X_i) \leq \sum_{i=1}^{n} R(X_i). \tag{2.1}$$

This axiom is usually justified in the literature by the potential gain due to diversification [8]. However, it is also interesting to discuss subadditivity condition (2.1) in a regulatory perspective. Under condition (2.1), if a regulator requires the level of reserve $R(X_i)$ for entity i, the sum of these reserves is sufficient to cover the global risk, for any possible comovement between the individual $L\&P$. Typically, Basel regulation suggested the use of the

[5] See also the discussion in Goovaerts et alii (1984).

[6] For expository purpose, we omit the time index, when it is not informative. When t denotes the period, the future $L\&P$ X_{it} is known at the end of period t and random at the beginning of this period. The risk measure $R_{t-1}(X_t)$ has to be known at the beginning of this period and is in general function of the available information.

[7] The standard notation $R(X)$ may be misleading, since the level of reserve depends on the distribution of X, not on the variable itself.

[8] The idea that diversification is profitable and the associated subadditivity condition of the global risk measure are questionable. Diversification can be profitable for standard independent risks X_i, "since due to the Law of Large Numbers, the risk (per invested money unit) diminishes " [Dhaene, Goovaerts, Kaas (2003)]. However, undiversified portfolios can be preferred when the Law of Large Numbers does not apply, for instance in presence of common risk factors, or for extreme risks [see e.g. Gourieroux, Monfort (2004)].

VaR, or of the expected shortfall at a given level α for each entity. This is a stand-alone contribution, where the portfolio of each entity is considered in isolation. If these risk measures are subadditive [9], the two first arguments invoked in the introduction for the amplifying role of the regulation in the recent financial crisis are misleading.

Let us illustrate these questions in the Gaussian case :

$$(X_1, \ldots, X_n) \sim N(m, \Sigma). \tag{2.2}$$

The α-quantile (i.e. the α-VaR) of the individual $L\&P$ is :

$$q_\alpha(X_i) = m_i + q_\alpha \sigma_{ii}^{1/2}, \tag{2.3}$$

where q_α denotes the α-quantile of the standard normal distribution. Since X_i is a loss and profit, α is large (for instance $\alpha = 0.95$ or 0.99), and q_α is positive. The α-quantile of the aggregated $L\&P, X = \sum_{i=1}^{n} X_i$, is :

$$q_\alpha(X) = \sum_{i=1}^{n} m_i + q_\alpha(\sum_{i=1}^{n} \sigma_{ii} + \sum_{i \neq j} \sigma_{ij})^{1/2}. \tag{2.4}$$

By noting that the standard deviation is a quadratic norm, $\| \,.\, \|_2$ say, we see immediately that :

$$\begin{aligned} q_\alpha(X) &= \sum_{i=1}^{n} m_i + q_\alpha \| \sum_{i=1}^{n} X_i \|_2 \\ &\leq \sum_{i=1}^{n} m_i + q_\alpha \sum_{i=1}^{n} \| X_i \|_2 = \sum_{i=1}^{n} q_\alpha(X_i). \end{aligned} \tag{2.5}$$

Thus, the subadditivity property is satisfied by the VaR in the multivariate Gaussian framework.

To summarize, the previous regulatory policy is a prudential approach, which provides enough reserve to cover the total risk (at least in this Gaussian framework and if all banks' risk models work accurately). How to understand

[9] It is known that the expected shortfall is subadditive. The VaR is subadditive, if some additional assumptions are introduced on the risk distribution, for instance in a Gaussian framework with or without stochastic volatility.

the misleading assertion that the levels of reserve computed in isolation underestimate the risk since they do not account for the possible comovement between individual risks? This is likely a consequence of the different additivity properties of the variance and standard error, respectively. More precisely, if the risks are positively correlated $\sigma_{ij} \geq 0, \forall i \neq j$, we have :

$$V(\sum_{i=1}^{n} X_i) \geq \sum_{i=1}^{n} V(X_i),$$

whereas the VaR involves the standard deviation and inequality in the reverse direction :

$$[V(\sum_{i=1}^{n} X_i)]^{1/2} \leq \sum_{i=1}^{n}[V(X_i)]^{1/2}.$$

Thus, the variance, which cannot be considered as a level of reserve since its unit is \$ squared and not \$, is superadditive, whereas the standard error is subadditive.

The inequalities above become equalities in two extreme cases of dependence. We have $V(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} V(X_i)$, when the variables are independent, whereas $[V(\sum_{i=1}^{n} X_i)]^{1/2} = \sum_{i=1}^{n}[V(X_i)]^{1/2}$, when the variables are linearly dependent.

3 Disaggregation of risk measures

The aim of the recent literature on risk measures and systematic risk is twofold. First, the contributions of each entity to the total reserve needed to hedge the global risk have to be defined. Second, this individual contribution has to highlight the reserve needed to cover the exposure to systematic risk and unsystematic (or idiosyncratic) risk. More precisely, let us consider a global risk measured by the global $L\&P$: X, which is the sum of the $L\&P's$ of the individual entities $X = \sum_{i=1}^{n} X_i$. A global reserve function $R(X)$, depending on the probability distribution of X, has been defined. It has to be assigned to the different entities :

$$R(X) = \sum_{i=1}^{n} R(X, X_i), \tag{3.1}$$

say, where $R(X, X_i)$ denotes the reserve of entity i.

Moreover, we would like to decompose the individual contribution into :

$$R(X, X_i) = R_s(X, X_i) + R_u(X, X_i) + R_{s,u}(X, X_i), \tag{3.2}$$

where R_s (resp. R_u) denotes the reserve for marginal systematic (resp. unsystematic) risk and $R_{s,u}$ the reserve for the cross effects. We focus in this section and the following one on decomposition (3.1) and defer the main discussion on systematic risk to Sections 5 and 6.

3.1 A set of axioms

From an axiomatic point of view, it is important to distinguish the reserve $R(X)$ for global risk, where the risk measure R is usually either a VaR, or a coherent measure satisfying a subadditivity condition, and the contributions to the total reserve $R(X, .)$. These contributions are contingent to the total risk level and should satisfy at least a decentralization axiom, an additivity axiom and a risk ordering axiom.

i) Decentralization axiom

Decentralization axiom : The individual contribution $R(X, X_i)$ of entity i depends on the joint distribution of (X, X_i), but is independent of the decomposition of $X - X_i$ into $\Sigma_{j \neq i} X_j$.

This axiom has been first introduced in Kalkbrener (2005). In a regulatory perspective, it has the advantage of allowing for a computation of $R(X, X_i)$ by entity i, while preserving a minimal confidentiality on the individual portfolios of the other entities.

More precisely, let us consider entities invested in stocks. The individual $L\&P's$ are : $X_i = Y'\gamma_i$, where Y denotes the vector of share values of the stocks, and γ_i is the portfolio composition for entity i. The reserves are usually evaluated for a crystallized portfolio, that is, with the composition γ_i^- existing at the beginning of period t (end of period $t-1$). With this practice, the regulator has to provide to entity i the type of measure $R(X, X_i)$

to consider, the past data on Y, and the sum $\sum_{j=1}^{n} \gamma_j^-$ corresponding to the global crystallized portfolio, without providing the individual information on competitors'portfolios $\gamma_j^-, \forall j \neq i$.

ii) Additivity axiom

The additivity axiom has been first introduced in Garman (1997) [see also Kalkbrener (2005), where it is called linear aggregation axiom].

Additivity axiom :

$R(X) = \sum_{i=1}^{n} R(X, X_i)$, for any decomposition of X into $X = \sum_{i=1}^{n} X_i$.

Intuitively, the total reserve should not depend on the number of entities holding the risk and of their respective sizes, whenever the sum of these $L\&P's$ stays the same. The additivity axiom has several consequences.

i) For $n = 1$, we get : $R(X) = R(X, X)$.
ii) For $n = 2, 3$, we have :
$R(X) = R(X, X_1 + X_2) + R(X, X - X_1 - X_2)$

$= R(X, X_1) + R(X, X_2) + R(X, X - X_1 - X_2)$,
which implies :

$$R(X, X_1 + X_2) = R(X, X_1) + R(X, X_2), \forall X_1, X_2. \tag{3.3}$$

This is the additivity property of the function $X_i \to R(X, X_i)$, for any given X. By imposing the additivity axiom, any merging, or demerging of entities without effect on global risk provides no spurious advantage in terms of reserve.

The discussion in Section 2 has shown that an α-quantile $q_\alpha(.)$, or more generally a coherent risk measure $R(.)$, does not satisfy the additivity condition; in general we have :

$$R(X) \neq \sum_{i=1}^{n} R(X_i).$$

Thus, such an individual measure of risk cannot be used as the contribution to global reserve, that is, we cannot choose : $R(X, X_i) = R(X_i)$. The main reason is that the standard measures of individual risk such as the VaR and the expected shortfall are not contingent to the level of global risk. By choosing an oversized level of reserve, the regulator will penalize without any economic reason the entities i such that :

$$R(X_i)/\sum_{i=1}^{n} R(X_i) \geq R(X, X_i)/R(X).$$

According to Tasche (2008) [Remark 17.2], the choice of stand-alone risk measure values as risk contributions would punish more those banks which improve the diversification of the global regulatory portfolio.

iii) Risk ordering axiom

The allocations have also to be compatible with an appropriate notion of stochastic dominance. Intuitively, the allocations have to take into account not only the individual risk of entity i, but also its hedging potential with respect to the set of other entities. Thus, we have to introduce a directional notion of stochastic dominance valid for an individual risk X_1, say, and a given global $L\&P : X$. For this purpose, let us consider the virtual decomposition of the global portfolio into X_1 and $\tilde{X}_2 = X - X_1$ obtained by aggregating the $L\&P$ of the other entities.

Definition 3.1 : Let us consider the $L\&P$: X, X_1, X_1^*. We say that X_1^* stochastically dominates X_1 at order 2 with respect to X, if and only if :

$$E[U(X_1^*, X - X_1^*)|X] \geq E[U(X_1, X - X_1)|X],$$

for any concave function U, increasing with respect to both arguments.

This is a stochastic dominance at order 2 applied to the virtual portfolio $X_1, \tilde{X}_2$, whose allocations are constrained to sum up to a given X. The directional stochastic partial ordering is denoted by $\succeq_X$.

The directional stochastic dominance can be characterized in simpler ways.

Proposition 3.2 : We have the following equivalences :
i) $X_1^* \succeq_X X_1$;
ii) $E[U(X_1^*)|X] \geq E[U(X_1)|X]$, for any concave function U;

iii) There exists a variable Z such that :

$$X_1 = X_1^* + Z, \text{ with } E(Z|X, X_1^*) = 0.$$

Proof : See Appendix 1.

Proposition 3.2 above shows that the directional stochastic dominance is equivalent to the standard stochastic dominance at order 2 applied to the conditional distribution of X_1 given X [see Rothschild, Stiglitz (1970)].

The next axiom concerns the compatibility of the contribution with the directional stochastic dominance.

Risk ordering axiom

We have $R(X, X_1^*) \leq R(X, X_1)$ for any pair of entity risks such that $X_1^* \succeq_X X_1$.

iv) Restrictions implied by the set of axioms

The decentralization and additivity axioms imply rather strong restrictions on the contributions as shown by the next Propositions.

Proposition 3.3 : Under the decentralization and additivity axioms, we have :

$$R(X, X_1 + Z) = R(X, X_1),$$

for any variable Z independent of (X, X_1) with a symmetric distribution.

Proof : We have the equalities :

$$\begin{aligned} R(X) &= R(X, X_1) + R(X, X_1) + R(X, X - 2X_1) \\ &= R(X, X_1 + Z) + R(X, X_1 - Z) + R(X, X - 2X_1). \end{aligned}$$

Since the joint distributions of the pairs $(X, X_1 + Z)$ and $(X, X_1 - Z)$ are the same under the assumptions of Proposition 3.3, we deduce :

$$R(X, X_1) = R(X, X_1 + Z).$$

QED

The result in Proposition 3.3 clearly shows the difference between a marginal measure of risk and an allocation. By passing from X_1 to $X_1 + Z$, we increase marginally the risk for the second-order stochastic dominance. However, due to the additivity and decentralization axioms, this increase has not been taken into account in the contribution. This is due to the compensation between X_1 and the $L\&P$ of the other entities.

Since some axioms imply strong restrictions on the contributions, we have to check if there exist contributions satisfying jointly the three axioms.

Proposition 3.4 : The contributions : $R(X, X_i) = E[X_i|X = R(X)]$ satisfy the three axioms.

Proof : i) We have :

$$\begin{aligned} \sum_{i=1}^{n} R(X, X_i) &= \sum_{i=1}^{n} E[X_i|X = R(X)] \\ &= E[X|X = R(X)] = R(X), \end{aligned}$$

which proves the additivity.

ii) Moreover if $X_1^* \succeq_X X_1$, we have :

$$\begin{aligned} & E[X_1|X = R(X)] \\ = \; & E[X_1^* + Z|X = R(X)], \text{ with } E(Z|X_1^*, X) = 0 \\ = \; & E\{E[X_1^* + Z|X_1^*, X]|X = R(X)\} \\ = \; & E[X_1^*|X = R(X)], \text{ by Proposition 3.2 iii).} \end{aligned}$$

Thus, $R(X, X_1^*) = R(X, X_1)$.

We deduce the compatibility with the risk preordering, at least in a wide sense.

QED

The decomposition above is valid for any choice of function $R(X)$ and in particular does not assume a priori the subadditivity of this global risk function (see the discussion in Section 6.2).

We deduce the following Corollary :

Corollary 3.5 : The contributions :

$$R_{\mu_P}(X, X_i) = \int E[X_i|X = x]\mu_P(dx),$$

where μ_P is a measure (not necessarily a probability measure), which can depend on the distribution P of X and is such that
$\int x\mu_P(dx) = R(X)$, satisfy the three axioms.

Proof : We have essentially to check the additivity condition. We get :

$$\sum_{i=1}^{n} \int E[X_i|X = x]\mu_P(dx)$$

$$= \int x\mu_P(dx) = R(X).$$

QED

Note that the contributions in Proposition 3.4 are obtained when μ_P is a point mass at $R(X)$. Corollary 3.5 also implies :

Corollary 3.6 : The contributions satisfying the three axioms are not unique.

When the distribution of X is continuous, with a strictly increasing cdf, the contributions in Corollary 3.5 can be written in terms of quantiles.

Corollary 3.7 : The contributions :

$$R(X, X_i) = \int E[X_i|X = q_\alpha(X)]\nu_P(d\alpha),$$

where ν_P is a measure, which can depend on the distribution P of X and is such that :

$\int q_\alpha(X)\nu_P(d\alpha) = R(X)$, satisfy the three axioms.

Proof : The result is obtained by applying the change of variable $x = q_\alpha(X)$.

QED

The equality $R(X) = \int q_\alpha(X)\nu_P(d\alpha)$ implies that $R(X)$ is a weighted quantile. ν_P looks like a distortion measure, except that it can depend on the distribution of X, since the allocation is contingent to X. It will be called the **allocation distortion measure** (ADM) in the rest of the paper.

More precisely, let us consider a global risk measure equal to a standard distortion risk measure (DRM) that is, a weighted combination of VaR [see Wang (2000), Acerbi (2002)] :

$$R(X) = DRM_H(X) = \int q_{\alpha^*}(X)H(d\alpha^*),$$

where H denotes a distortion (or spectral) probability measure [10] on $(0, 1)$. This measure H is fixed independently of the distribution P of global risk. This DRM is subadditive if and only if function H is convex [Wang, Young (1998)]. Corollary 3.7 is saying that we do not have necessarily to use the fixed distortion measure H defining the DRM as the allocation distortion risk measure. Indeed, a given level of global reserve, 2 billions \$, say, can be seen as the value of a VaR as well as the value of an ES, and more generally as the value of an infinite number of alternative DRM, whenever $R(X) = \int q_\alpha(X)\nu_P(dx)$. The measures to define the global reserve and to allocate it can be different.

It is important to stress that, under rather weak conditions, all the contributions are necessarily of the form given in Corollary 3.5.

Proposition 3.8 : Let us assume that the $L\&P$: X, X_i belong to a space $L^2(Y)$, where Y is a given set of random variables. If the contribution $X_i \to$

[10] The unit mass property of the distortion measure is a consequence of the certainty axiom, saying that $R(c) = c$ for a constant risk $X = c$. When the allocation distortion measure ν_P depends on risk distribution P, the certainty axiom implies that ν_P is unit mass for every P equal to a point mass at c. ν_P can have a non unit mass, otherwise.

$R(X, X_i)$ is continuous with respect to X_i for the L^2-norm, then, under Axioms 1 and 2, we have $R(X, X_i) = E[a_P(X)X_i]$, where $a_P(X)$ is a square integrable function on X, which can depend on the distribution of X and is such that $E[a_P(X)X] = R(X)$.

Proof : In the Hilbert space $L^2(Y)$, the continuous linear form are necessarily of the type :

$$R(X, X_i) = E[a(Y)X_i],$$

where $a(Y) \in L^2(Y)$ [see e.g. Rudin (1966), Chapter 4]. The special expression of function $a(Y)$ is a direct consequence of the decentralization and additivity axioms.

QED

The condition $X_i \in L^2(Y)$ means that the portfolios of interest are written on some basic assets Y, possibly including derivatives with nonlinear (square integrable) payoffs.

The allocations in Proposition 3.8 are the same as the allocations in Corollary 3.5. Indeed, we have :

$$\begin{aligned} E[a_P(X)X_i] &= E[a_P(X)E(X_i|X)] \\ &= \int E[X_i|X = x]a_P(x)P(dx), \end{aligned}$$

where $P(dx)$ is the historical distribution of X. This expression is of the form given in Corollary 3.5.

Since a_P is a measure density, $R(X, X_i)$ is simply a weighted risk allocation in the terminology of Furman, Zitikis (2008). When a_P is positive, the allocation can be interpreted as the value of X_i obtained by applying the pricing operator a_P function of the distribution of X. This corresponds to the premium calculation principle introduced in Gerber (1979).

Other axioms have been considered in the literature. For instance Kalkbrener (2005) proved the uniqueness of the allocation under an additional continuity assumption and the fact [11] that :

Diversification axiom :

$$R(X, X_i) \leq R(X, X) = R(X), \forall X_i.$$

Moreover, under these additional conditions, the global reserve $R(X)$ is necessarily a subadditive function.

We do not introduce this diversification axiom. Indeed, since :

$$R(X) = R(X, X_1) + R(X, X_2), \text{ when } X_1 + X_2 = X,$$

this axiom implies the nonnegativity of the reserve contribution. It is important to leave open the possibility of a negative contribution [12], when a given entity is more prudential than deemed necessary. Typically, if the entity portfolio includes mainly riskfree asset, the contribution will be negative, which means the authorization for an increased leverage without requiring positive reserve in liquid riskfree asset.

Tasche (2008) proved the uniqueness of the allocation under an additional condition of compatibility between the global and individual Return On Risk Adjusted Capital (RORAC). Such a performance criterion may be appropriate for a capital allocation to business units, but is difficult to interpret from a regulatory point of view [13].

In fact, as for the notion of coherent risk measure, the axiomatic has to restrict the set of possibilities to easily interpretable allocations, not necessarily to provide a unique solution.

3.2 Related literature

There exist two streams of literature for defining risk measures contingent to the level of global risk. Some authors focus on the direct introduction of the contingent reserve levels $R(X, X_i)$. The CoVaR [Adrian, Brunnermeier

---

[11] A similar diversification condition is introduced in Hesselager, Andersson (2002), or Furman, Zitikis (2008).

[12] see e.g. Uryasev, Theiler, Serrano (2010) for a practical example of negative allocation.

[13] The definition of the RORAC in Tasche (2008) as $RORAC(X) = E(X)/R(X)$ seems rather restrictive. The recent literature on portfolio management has typically preferred the ratio $q_{1/2}(X)/q_\alpha(X)$, that is, a median instead of a mean in the numerator when $R(X)$ is a VaR.

(2009)], or the computation of the reserve based on the Shapley value [Denault (2001), Koyluoglu, Stocker (2002), Taraschev et alii (2009)] belong to to this literature. Other authors directly introduce a decomposition formula (3.1) of the total reserve, and then try to interpret ex-post the elements $R(X, X_i)$ in this decomposition [see e.g. Tasche (2000), (2001), Acharya et alii (2010)]. Let us briefly describe these approaches.

i) The CoVaR and its limitations

Adrian and Brunnermeier (2009) propose to analyze the risk of the entities (resp. the system), when the system (resp. entity) is in distress.

More precisely, let us denote by $q_\alpha(X)$ the α-quantile corresponding to the system[14]. The CoVaR for entity i and confidence level α when the system is in distress is defined by :

$$P[X_i < CoVaR_{i|s,\alpha}(X) | X = q_\alpha(X)] = \alpha. \tag{3.4}$$

Similarly, let us denote by $q_\alpha(X_i)$ the α-quantile corresponding to entity i, the CoVaR for the system and confidence level α, when i is in distress is defined by :

$$P[X < CoVaR_{s|i,\alpha}(X) | X_i = q_\alpha(X_i)] = \alpha. \tag{3.5}$$

The two CoVaR above are generally different, leading to alternative definitions of the contribution of entity i to systematic risk. For instance, the authors define the systematic component of the reserve of entity i as :

$$R_s(X, X_i) = CoVaR_{s|i,\alpha}(X) - q_\alpha(X), \tag{3.6}$$

and, implicitly, add this quantity, called Δ CoVaR, to the usual individual VaR considered as the second (unsystematic) component :

$$R_u(X, X_i) = q_\alpha(X_i). \tag{3.7}$$

Thus the underlying total reserve for entity i becomes :

$$R(X, X_i) = CoVaR_{s|i,\alpha}(X) - q_\alpha(X) + q_\alpha(X_i).$$

Alternatively, it would be possible to choose :

[14] A similar approach can be based on another risk measure such as the Expected Shortfall [see e.g. Kim (2010)].

$$R_s^*(X, X_i) = CoVaR_{i|s,\alpha}(X) - q_\alpha(X_i), \tag{3.8}$$

with $R_u^*(X, X_i) = q_\alpha(X_i)$. In this case the total reserve for entity i is :

$$R^*(X, X_i) = CoVaR_{i|s,\alpha}(X).$$

Choosing the causality direction, that is, choosing between definition (3.5)-(3.6) and definition (3.7)-(3.6), is a first difficulty. Moreover, the CoVaR approach features other drawbacks. For instance :

(*) The CoVaR cannot be computed for identical entities (clones in Adrian, Brunnermeier terminology). Indeed, let us assume $X_1 = X_2$, for instance, and $X = 2X_1$. Then, the conditional distribution of X given X_1 (resp. X_1 given X) is discrete, which does not allow for the uniqueness of the α-quantile, except for limiting confidence levels $\alpha = 0$, or 1.

(**) The implicit choice $R_u(X, X_i) = q_\alpha(X_i)$ is not appropriate. Indeed, we have seen in Section 2, that, in general, $\sum_{i=1}^{n} q_\alpha(X_i) > q_\alpha(X)$; this means that the individual $\alpha-$quantiles are more than sufficient to cover the total risk and in particular its systematic component. Thus, $q_\alpha(X_i)$ already includes some reserve for systematic risk.

(***) In the approach (3.8), we get :

$$R^*(X, X_i) = CoVaR_{i|s,\alpha}(X),$$

The CoVaR does not satisfy the additivity axiom since the sum of CoVaR can be different for different allocations $X_i, i = 1, \ldots, n$ of a same portfolio $X = \sum_{i=1}^{n} X_i$.

ii) The Shapley value

A Shapley value [Shapley (1953)] is a fair allocation of gains obtained by cooperation among several actors. Let us assume that all actors $i = 1, \ldots, n$

accept to cooperate and introduce a superadditive value function $v(S)$, which measures the gain of this cooperation for a coalition $S \subset \{1, \ldots, n\}$. The superadditivity condition :

$$v(S \cup T) \geq v(S) + v(T),$$

expresses the fact that cooperation can only be profitable.

The Shapley value is one way to distribute the total gains of the players, if they all collaborate, by demanding for each actor i a contribution $v(S \cup \{i\}) - v(S)$ as a fair compensation to join coalition S. The Shapley value is defined as a mean of these compensations over all possible coalitions :

$$V_i = \sum_{S \subset \{1,\ldots,n\} \setminus \{i\}} \left\{ \frac{|S|!(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)] \right\}, \qquad (3.9)$$

where $|S|$ denotes the number of actors in coalition S. Denault (2001), Koyluoglu, Stocker (2002), Tarashev et alii (2009) (with the Varying Tail Events procedure) propose the Shapley value as a fair allocation of the reserve with $v(S) = -R(\Sigma_{i \in S} X_i)$, and R a risk measure such as a VaR, or an expected shortfall.

In a regulatory perspective, the drawback of this approach is threefold[15] :

(*) It considers as given the oversized VaR measure, for instance, proposed by the regulator, without trying to correct this overevaluation of risks.

(**) It assumes a total cooperation of the entities, which are in practice competitors.

(***) The Shapley allocation does not satisfy the decentralization, i.e. confidentiality, axiom (which intuitively is not compatible with a total cooperation).

In fact, as mentioned in Denault (2001), the Shapley principle is appropriate for reallocating in a fair manner the oversizing of reserves in the current regulation, but not in suppressing in a fair manner this oversizing.

[15] However, this approach can be appropriate for the internal allocation of the economic capital of a bank between its business lines, or departments.

In a regulatory perspective, such a Shapley reallocation could only be done by the regulator itself (due to confidentiality restriction). This would lead to a highly centralized computation of VaRs by the regulator itself, but is clearly not implementable in practice. First, the regulator does not possess the technical departments to make such computations for all entities. Second, such a centralized approach contradicts the spirit of the second Pillar of Basel regulation, where the entities have to learn how to manage and control their internal risks by themselves.

iii) The disaggregation approaches

In Tasche (2000), (2001) the global risk measure is for instance an α-quantile. Then it is noted that:

$$\begin{aligned} R(X) = q_\alpha(X) &= E[X|X = q_\alpha(X)] \\ &= \sum_{i=1}^{n} E[X_i|X = q_\alpha(X)]. \end{aligned} \tag{3.10}$$

The contribution of entity i to global reserve level is defined as :

$$R(X, X_i) = E[X_i|X = q_\alpha(X)].$$

This is exactly the allocation principle in Proposition 3.4 (see Appendix 4 i) for the equivalent expression written in returns).

Similarly, the Expected Shortfall[16] (ES) as a risk measure for the global risk can be decomposed as [Tasche (2000)] :

$$\begin{aligned} R(X) = ES_\alpha(X) &= E[X|X > q_\alpha(X)] \\ &= \sum_{i=1}^{n} E[X_i|X > q_\alpha(X)]. \end{aligned} \tag{3.11}$$

Then, the contribution of institution i to global reserve level is defined as :

[16] Also called Conditional VaR (CVaR) in a part of the literature [see e.g. Rockafellar, Uryasev (2002), Acerbi, Tasche (2002)], or TailVaR, or Conditional Tail Expectation (CTE) [Kim (2007)].

$$R(X, X_i) = E[X_i | X > q_\alpha(X)].$$

This allocation is called Tail Conditional Expectation (TCE) allocation. TCE allocations have been computed for several multivariate distribution families, as the Gaussian family [Panjer (2002)], the elliptical distributions [Landsman, Valdez (2003)], the multivariate skew-elliptical [Cai, Tan (2005)], or multivariate Pareto [Chiragiev, Landsman (2007)].

Decompositions (3.10) and (3.11) above differ essentially by the conditioning set. They stress that an additive decomposition involves both conditioning with respect to system distress [as in definition (3.4) of the CoVaR] and conditional expectations (instead of conditional quantiles as in the CoVaR approach) to ensure the additivity property.

Note also that decomposition formulas (3.10) and (3.11) are valid if the entities use the same information set at the beginning of the period.

However, there exists a multiplicity of decompositions of the global risk, for example of the expected shortfall (see Corollary 3.5). For instance, by applying the allocation principle in Proposition 3.4, we derive an alternative set of allocations of the expected shortfall :

$$\tilde{R}(X, X_i) = E[X_i | X = ES_\alpha(X)]. \tag{3.12}$$

If the distribution of X is continuous with a strictly positive density, $q_\alpha(X)$ and $ES_\alpha(X)$ are both continuous strictly increasing function of α. Thus, TCE allocation $E[X_i | X = ES_\alpha(X)]$ in (3.13) is equal to a VaR allocation (3.10) : $E[X_i | X = q_{\alpha^*}(X)]$ associated with another confidence level α^*.

Both sets of allocations (3.11) and (3.12) of the expected shortfall satisfy the three axioms of Section 3.1.

3.3 Gaussian L&P

If the $LSP's$ are jointly Gaussian, we have the following Property :

Proposition 3.9 : In the Gaussian framework, all allocations of Corollary 3.5 coincide.

Proof : In the Gaussian case, we have :

$$E(X_i|X=x) = E(X_i) + \frac{Cov(X_i,X)}{V(X)}[x - E(X)].$$

We deduce immediately the expressions of the other allocations in Corollary 3.5. We have :

$$\begin{aligned} R_\mu(X,X_i) &= E(X_i) + \frac{Cov(X_i,X)}{V(X)}[\int x\mu_P(dx) - E(X)] \\ &= EX_i + \frac{Cov(X_i,X)}{V(X)}[R(X) - E(X)], \end{aligned}$$

and we obtain the result.

QED

Loosely speaking, we get the uniqueness of all allocations satisfying the three axioms under the additional Gaussian assumption.[17]

Expression $R_\mu(X,X_i)$ shows the hedging interpretation of the allocation. It is the sum of the expected individual $L\&P$ and an hedging premium, equal to a sensitivity coefficient [i.e. the standard beta $Cov(X_i,X)/V(X)$] multiplied by the excess global risk $R(X) - EX$ (often called global Economic Capital).

In the particular case of the VaR : $R(X) = q_\alpha(X)$, we get :

$$\begin{aligned} q_\alpha(X) &= E(X) + q_\alpha[V(X)]^{1/2}, \\ \text{and} \quad R(X,X_i) &= EX_i + q_\alpha \mathrm{cov}(X_i,X)/V(X), \end{aligned}$$

where q_α is the α-quantile of the standard normal. This risk allocation is called Component VaR by Garman (1996).

4 Sensitivity analysis

In this section, we consider the effects on the global reserve and contributions of some changes in individual L&P. Let us assume given benchmark $L\&P's$:

[17] More generally, whenever $E(X_i|X=x)$ is linear affine in x.

$X_1, \ldots, X_n, X = \sum_{i=1}^{n} X_i$ and modify the respective sizes of the entities portfolios to $\lambda_1 X_1, \ldots, \lambda_n X_n$, with $\lambda_i \geq 0, \forall i$. The total $L\&P$ becomes $\sum_{i=1}^{n} \lambda_i X_i$ and the associated global reserve is a function of $\wedge = (\lambda_1, \ldots, \lambda_n)'$ and the joint distribution of $X_1, \ldots, X_n$. We focus on the dependence in $\wedge$ and denote :

$$R^*(\wedge) = R(\sum_{i=1}^{n} \lambda_i X_i). \tag{4.1}$$

In particular $q_\alpha^*(\wedge), ES_\alpha^*(\wedge), DRM^*(\wedge)$ are the VaR, ES and DRM evaluated at this modified portfolio.

When $\wedge = e = (1, \ldots, 1)'$, the total reserve is $R^*(e) = R(X)$.

4.1 Euler allocation

As noted in Litterman (1996), p28, and Garman (1997), footnote 2, if the risk measures are homogenous of degree 1, that is, satisfy the condition :

$$R(\lambda X) = \lambda R(X), \forall \lambda > 0, \tag{4.2}$$

or, equivalently, $R^*(\lambda e) = \lambda R^*(e), \forall \lambda > 0, \forall \wedge$, we get the Euler condition :

$$R^*(e) = \sum_{i=1}^{n} \frac{\partial R^*(e)}{\partial \lambda_i}, \tag{4.3}$$

(obtained by differentiating both sides of equation (4.2) with respect to λ). This provides a decomposition of a global risk measure as the sum of its sensitivities corresponding to shocks performed separately on each entity. This justifies the terminology Euler allocation [18] used in McNeil et alii (2005), Section 6.3, and Tasche (2007).

Let us consider this decomposition for a global risk defined by a distortion risk measure.

[18] The Euler allocation has an interpretation in terms of game theory and correspond to an Aumann-Shapley value [see Denault (2001)].

$$DRM_H(X) = \int q_{\alpha^*}(X) H(d\alpha^*), \tag{4.4}$$

where H denotes a distortion (probability) measure on (0,1).

The VaR and more generally the DRM are homogenous functions of degree 1. Thus, the total reserve can be decomposed into :

$$DRM_H(X) = \sum_{i=1}^{n} DM_{H,i}, \tag{4.5}$$

where the marginal expected distortion risk measures are given by :

$$DRM_{H,i} = \int q_{\alpha^*,i} H(d\alpha^*), \text{with } \; q_{\alpha,i} = \frac{\partial q^*_\alpha(e)}{\partial \lambda_i}. \tag{4.6}$$

The following result has been derived in Gourieroux, Laurent, Scaillet (2000) [see also the beginning of Appendix 2, formula a.3].

Proposition 4.1 :

$$q_{\alpha,i} = \frac{\partial q^*_\alpha(e)}{\partial \lambda_i} = E[X_i | X = q_\alpha(X)].$$

From Proposition 4.1, we deduce that the allocation considered in (3.10) is a Euler allocation. The Euler allocation of a distortion risk measure is given by :

$$DRM_{H,i} = \int E[X_i | X = q_{\alpha *}(X)] H(d\alpha^*). \tag{4.7}$$

This Euler allocation applied to DRM satisfies the three axioms of Section 3.1. However, it is rather restrictive, since it **corresponds to the choice of an allocation distortion measure equal to the distortion measure itself.**

The Euler decomposition formula (4.7) is in particular valid for the expected shortfall[19], for which the distortion measure is the uniform distribution on the interval $(\alpha, 1)$ [Wang (2000), Acerbi, Tasche (2002)] :

[19] The expected shortfall is the smallest coherent risk measure dominating the VaR [Delbaen (1998), Theorem 6.10].

$$ES_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 q_{\alpha*}(X)d\alpha^*, \tag{4.8}$$

It can be checked that the marginal expected shortfall is equal to :

$$\begin{aligned} MES_i &= \frac{1}{1-\alpha}\int_\alpha^1 E[X_i|X = q_{\alpha*}(X)]d\alpha^* \\ &= E[X_i|X > q_\alpha(X)]. \end{aligned} \tag{4.9}$$

This simplified expression of the marginal expected shortfall has been first derived by Tasche (2000) [see also Scaillet (2004), Fermanian, Scaillet (2005) and Appendix 2]. Thus, TCE allocation (3.12) [and not decomposition (3.13)] corresponds to the relative Euler allocation of the expected shortfall.

The Euler allocations of global risk depend on the selected distortion measure, for instance of the choice between the VaR and the expected shortfall [see e.g. Kurth, Tasche (2002), for a theoretical comparison based on the CreditRisk$^+$ model]. However, there exist important cases in which the relative Euler allocations do not depend on the selected DRM as shown in the example below.

Example : A stochastic volatility model.

Let us assume : $X_i = \sigma(f)u_i, i = 1, \ldots, n$, where $u_1, \ldots, u_n$ are independent Gaussian variables, $u_i \sim N(0, \sigma_i^2)$ and, f is a factor independent of these error terms. We have :

$$\begin{aligned} \sum_{i=1}^n \lambda_i X_i &= \sigma(f)\sum_{i=1}^n \lambda_i u_i \\ &= \sigma(f)(\sum_{i=1}^n \lambda_i^2\sigma_i^2)^{1/2}v, \end{aligned}$$

where v is a standard Gaussian variable. Since a quantile function is homogenous of degree 1, we have :

$$q_\alpha^*(\wedge) = (\sum_{i=1}^{n} \lambda_i^2 \sigma_i^2)^{1/2} q_\alpha[\sigma(f)v],$$

$$\text{and} \quad \frac{\partial q_\alpha^*(e)}{\partial \lambda_i} = \frac{\sigma_i^2}{\sum_{j=1}^{n} \sigma_j^2} q_\alpha^*(e) = E[X_i | X = q_\alpha(X)].$$

Thus, the relative Euler allocations are $\sigma_i^2 / \sum_{j=1}^{n} \sigma_j^2$, and are independent of α. The relative allocations are also the same for any DRM.

Note however that the homogeneity assumption of the global risk measure is questionable, especially if the regulation is used for economic policy. As an illustration, let us assume that the portfolios of interest include the different types of credits. From a macroeconomic point of view, there exists an optimal level for the global amount of credit to be distributed in the economy. The global risk measure has to be chosen as an incentive to reach this optimal level. The cost of the reserve has to be small, if the current amount of credit is below this optimal level, large, otherwise. Mathematically, we expect function $\lambda \to R(\lambda X)/\lambda$ to be increasing in λ, not constant. For instance $R_c(X) = E(X) + E[(X-c)^+]$, where c is the "optimal level" would satisfy this condition.[20] [21]

4.2 Euler allocation sensitivities

The Euler allocation of the global reserve involves the first-order expansion of the risk measure with respect to a uniform change of portfolio sizes

[20]The homogeneity assumption is sometimes justified by an invariance of the risk measure with respect to a change of money unit. We note that function $R_c(X)$ satisfies this latter condition which involves the changes $X \to \lambda X$ and $c \to \lambda c$, since the optimal level is also written in money unit, but is not homogenous of degree 1 in X for fixed c.

[21]For a similar reason, we have not introduced the translation axiom $R(X+c) = R(X)+c$, for any constant c. Indeed, the economic policy has also to manage the distribution of credit in the economy between the very risky projects (start up) and less risky ones. Therefore, it can be natural to penalize some non risky distribution of credits. In such a case, we expect $R(c) > c$, for a large positive deterministic $L\&P$.

$(X_1,\ldots,X_n) \to (\lambda X_1,\ldots,\lambda X_n)$. Let us now consider the second-order expansion for a non uniform change of portfolio allocation :

$$(X_1,\ldots,X_n) \to (\lambda_1 X_1,\ldots,\lambda_n X_n),$$

where $\wedge = (\lambda_1,\ldots,\lambda_n)'$ is close to $e = (1,\ldots,1)'$.

The expansion of a distortion risk measure is :

$$\begin{aligned} DRM_H^*(\wedge) &\simeq DRM_H^*(e) + \frac{\partial DRM_H^*(e)}{\partial \wedge'}(\wedge - e) + \frac{1}{2}(\wedge - e)'\frac{\partial^2 DRM_H^*(e)}{\partial \wedge \partial \wedge'}(\wedge - e) \quad (4.10) \\ &= \frac{\partial DRM_H^*(e)}{\partial \wedge'} \wedge + \frac{1}{2}(\wedge - e)'\frac{\partial^2 DRM_H^*(e)}{\partial \wedge \partial \wedge'}(\wedge - e). \quad (4.11) \end{aligned}$$

Since the first-order partial derivative $\frac{\partial DRM_H^*(\wedge)}{\partial \lambda_i}$ is equal to the Euler allocation $DRM_{H,i}(\wedge)$ of the global risk measure to entity i, when the individual $L\&P's$ are $\lambda_i X_i, i = 1,\ldots,n$, we deduce that :

$$\frac{\partial^2 DRM_H^*(e)}{\partial \lambda_i \partial \lambda_j} = \left(\frac{\partial}{\partial \lambda_i}\left[\frac{\partial}{\partial \lambda_j} DRM_H^*(\wedge)\right]\right)_{\wedge = e} = \left[\frac{\partial}{\partial \lambda_i} DRM_{H,j}^*(\wedge)\right]_{\wedge = e}, \quad (4.12)$$

and we have the following result :

Proposition 4.2 : The marginal change in the Euler allocation $DRM_{H,j}$ associated with a change of size of entity i is such that :

$$DRM_{H,j}^{(i)} \equiv \left[\frac{\partial DRM_{H,j}^*(\wedge)}{\partial \lambda_i}\right]_{\wedge = e} = \frac{\partial^2 DRM_H^*(e)}{\partial \lambda_i \partial \lambda_j}, \forall i,j.$$

In particular,

i) It is equal to the second-order cross-derivative of the global risk measure with respect to changes in portfolio values of entities i and j.

i) The marginal reserve allocations satisfy the symmetry property :

$$DRM_{H,j}^{(i)} = DRM_{H,i}^{(j)}, \forall i,j.$$

iii) They are such that :

$$\sum_{i=1}^{n}\sum_{j=1}^{n} DRM_{H,j}^{(i)} = 0.$$

Proof : Parts i) and ii) are direct consequences of equality (4.11) and of the symmetry of an Hessian matrix. Property iii) is a consequence of the homogeneity property. Indeed, we know that :

$$DRM_H^*(\lambda e) = \lambda DRM_H^*(e), \forall \lambda,$$

and, by applying the second-order expansion with $\wedge = \lambda e$, we deduce that :

$$e' \frac{\partial^2 DRM_H^*(e)}{\partial \wedge \partial \wedge'} e = 0,$$

that is, $\sum_{i=1}^{n}\sum_{j=1}^{n} DRM_{H,j}^{(i)} = 0.$

QED

Moreover, if the distortion risk measure DRM_H is subadditive, the Hessian is positive semi-definite,[22] and, by Cauchy-Schwarz inequality, we have the equivalences :

$$e' \frac{\partial^2 DRM_H^*(e)}{\partial \wedge \partial \wedge'} e = 0 \Longleftrightarrow \frac{\partial^2 DRM_H^*(e)}{\partial \wedge \partial \wedge'} e = 0 \Leftrightarrow e' \frac{\partial^2 DRM_H^*(e)}{\partial \wedge \partial \wedge'} = 0.$$

We deduce the following restrictions on the allocation sensitivities.

Corollary 4.3 : If the cumulative function H of the distortion measure is convex[23], the matrix of Euler allocation sensitivities is positive semi-definite, and

[22] Since subadditivity and homogeneity of degree 1 imply convexity.

[23] The convexity of cumulative distribution function H is equivalent to the convexity of the DRM_H with respect to risk variable X, or equivalently to its subadditivity, since a DRM is homogenous of degree 1.

$$\sum_{j=1}^{n} DRM_{H,j}^{(i)} = 0, \forall i = 1, \ldots, n,$$

$$\sum_{i=1}^{n} DRM_{H,i}^{(j)} = 0, \forall j = 1, \ldots, n,$$

The results above are well-illustrated for the expected shortfall, whose second-order derivative with respect to the changes of portfolio values is proportional to a variance-covariance matrix. More precisely, we have :

$$\frac{\partial^2 ES^*(e)}{\partial \wedge \partial \wedge'} = \frac{p[q_\alpha(x)]}{\alpha} V \left[\left(\begin{array}{c} X_1 \\ \vdots \\ X_n \end{array} \right) | X = q_\alpha(X) \right], \tag{4.13}$$

where p is the density of X [see e.g. Bertsimas, Lauprete, Samarov (2004), formula (9)].

However, the restrictions on allocations given in Corollary 4.3 valid for an Euler allocation and a subadditive DRM, are difficult to interpret from an economic point of view. They are consequences of the specific choice of the ADM. They are not valid in general for the other allocations satisfying the three axioms given in Corollary 3.5, when the allocation distortion measure differs from the distortion measure.

4.3 Gaussian L&P

Let us finally consider the effect on VaR allocations given in Section 3.3 of a change of size for entity $i : X_i \to \lambda_i X_i$, the values for the other entities being fixed, for jointly Gaussian $L\&P$.

Proposition 4.4 : We have :

$$\frac{\partial^2 q_\alpha^*(e)}{\partial \wedge \partial \wedge'} = q_\alpha \left\{ \frac{\Sigma}{(e'\Sigma e)^{1/2}} - \frac{\Sigma e e' \Sigma}{(e'\Sigma e)^{3/2}} \right\}.$$

Proof : See Appendix 3.

We check that, for any δ :

$$\delta' \frac{\partial^2 q^*_\alpha(e)}{\partial \wedge \partial \wedge'} \delta = q_\alpha \left\{ \frac{\delta' \Sigma \delta}{(e'\Sigma e)^{1/2}} - \frac{(\delta' \Sigma e)^2}{(e'\Sigma e)^{3/2}} \right\} \geq 0,$$

by Schwarz inequality, and that

$$\frac{\partial^2 q_\alpha(e)}{\partial \wedge \partial \wedge'} e = 0.$$

Indeed, in the Gaussian case, the VaR satisfies the subadditivity property (see Section 2), and therefore Corollary 4.3.

Thus, an increase of size of entity i has a negative effect on the allocation of entity j, if and only if $\sigma_{ij} \leq \frac{\sigma_{i.}\sigma_{.j}}{\sigma_{..}}$, where $\sigma_{i.} = \sum_{j=1}^{n} \sigma_{ij}, \sigma_{.j} = \sum_{i=1}^{n} \sigma_{ij}, \sigma_{..} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}$, which differs from the intuitive condition $\sigma_{ij} < 0$.

The allocations and allocation sensitivities for other distortion risk measures have similar expressions after substituting the distortion measure DRM_α of the standard normal distribution to the quantile q_α of the standard normal in all expressions above.

5 Contribution of systematic risk

The disaggregation approaches provide contributions of individual entities satisfying the axioms, but do not try in general to reallocate the global risk between systematic and unsystematic components. The aim of this section is to explain how the allocation principle can be applied to disentangle the systematic and unsystematic components of the risk. We first consider models with linear factors driving the systematic risk which are usually considered when we focus on market risk, that is the risk in the trading book. Then, the approach is extended to nonlinear factors. Nonlinear factor models are involved whenever options and/or credit risks are considered. Thus, the analysis of nonlinear factor effects has to be taken into account not only when considering the banking book, but also in a joint analysis of the trading and banking books, currently treated separately in the regulation. Indeed several factors, such as the riskfree interest rate, the business cycle, or the price of the real estate have effects on both books [see the discussion in Breuer et alii (2010)].

We first discuss the disaggregation of the reserves into systematic and unsystematic components, when

i) the individual $L\&P's$ are linearly decomposed;

ii) the global risk measure is a VaR;

ii) the allocation to entities are Euler allocations.

Then, we extend this approach to more general frameworks.

5.1 Linear factor model, VaR global risk measure and Euler allocation

Let us first consider a linear factor model. The individual $L\&P's$ can be decomposed as :

$$X_i = \sum_{k=1}^{K} \beta_{ik} f_k + \gamma_i u_i, \tag{5.1}$$

where $f_1, \ldots, f_K$ are systematic factors and u_i idiosyncratic terms with $K < n$. These factors are random at the beginning of period t, and observable at the end of this period. For instance, for fixed income derivatives, the main risk factors can be the interest rate level, slope and curvature, the spreads over T-bond rates, the exchange rates... The total P&L can be decomposed as :

$$X = \sum_{k=1}^{K} \left[(\sum_{i=1}^{n} \beta_{ik}) f_k \right] + \sum_{i=1}^{n} \gamma_i u_i. \tag{5.2}$$

The α-quantile of X is a function :

$$q_\alpha(X) = q_\alpha^* \left(\sum_{i=1}^{n} \beta_{i1}, \ldots, \sum_{i=1}^{n} \beta_{iK}, \gamma_1, \ldots, \gamma_n, \theta \right), \tag{5.3}$$

where θ denotes the parameters characterizing the joint distribution of $f_1, \ldots, f_K, u_1 \ldots, u_n$.

i) Euler allocation

The marginal effect of an homothetic change of exposure of the entities passing from X_i to $\lambda X_i = \sum_{k=1}^{n} \lambda \beta_{ik} f_k + \lambda \gamma_i u_i, i = 1, \ldots, n$ is :

$$\left[\frac{dq_\alpha(\lambda X)}{d\lambda}\right]_{\lambda=1} = \sum_{k=1}^{K}(\sum_{i=1}^{n}\beta_{ik})\frac{\partial q_\alpha(X)}{\partial \beta_k} + \sum_{i=1}^{n}\gamma_i \frac{\partial q_\alpha(X)}{\partial \gamma_i}. \tag{5.4}$$

where $\beta_k = \sum_{i=1}^{n} \beta_{ik}$.

The composite term :

$$\sum_{k=1}^{K} \beta_{ik} \frac{\partial q_\alpha(X)}{\partial \beta_k} + \gamma_i \frac{\partial q_\alpha(X)}{\partial \gamma_i} \equiv \frac{\partial q_\alpha^*(e)}{\partial \lambda_i}, \tag{5.5}$$

measures the VaR Euler contribution of entity i. The Euler allocation can be decomposed from (5.5) to highlight the effects of systematic factors and idiosyncratic term.

In some sense, decomposition formulas (5.4)-(5.5) explain how to pass from Euler allocations computed by entity, to Euler allocations computed by "virtual business lines" associated with the different risk factors, as summarized in Table 1. In other words, we propose to treat in a symmetric way the contributions to global risk of both risk factors and entities.

Table 1 : Euler decomposition of global VaR

entity risk factor	1	i	n	risk contribution of the factors
f_1 $\vdots$ f_K u_1 $\vdots$ u_n				$\sum_{i=1}^{n} \beta_{iK} \frac{\partial q_\alpha(X)}{\partial \beta_K}$ $\gamma_n \frac{\partial q_\alpha(X)}{\partial \gamma_n}$
risk contribution of the entity		$\frac{\partial q^*_\alpha(e)}{\partial \lambda_i}$		$VaR(X)$

Formula (5.5) involves the following partial derivatives of the $\alpha-$ quantile :

$$\begin{aligned} \frac{\partial q^*_\alpha(e)}{\partial \lambda_i} &= E[X_i|X = q_\alpha(X)], && (5.6) \\ \frac{\partial q_\alpha(X)}{\partial \beta_k} &= E[f_k|X = q_\alpha(X)], && (5.7) \\ \frac{\partial q_\alpha(X)}{\partial \gamma_i} &= E[u_i|X = q_\alpha(X)]. && (5.8) \end{aligned}$$

This provides another interpretation of Euler decomposition of the global risk in terms of sensitivity to risk exposure. We have :

$$\begin{aligned} q_\alpha(X) &= \sum_{i=1}^{n} E[X_i|X = q_\alpha(X)] = \sum_{i=1}^{n} \frac{\partial q^*_\alpha(e)}{\partial \lambda_i} \\ &= \sum_{i=1}^{n} \left\{ \sum_{k=1}^{K} \beta_{ik} \frac{\partial q_\alpha}{\partial \beta_k}(X) + \gamma_i \frac{\partial q_\alpha}{\partial \gamma_i}(X) \right\}. \qquad (5.9) \end{aligned}$$

We get as a by-product, the Euler components associated with systematic and unsystematic risks, respectively, as :

$$
\begin{aligned}
R_s(X, X_i) &= \sum_{k=1}^{K} \beta_{ik} \frac{\partial q_\alpha}{\partial \beta_k}(X) = \sum_{k=1}^{K} \beta_{ik} E[f_k | X = q_\alpha(X)], && (5.10)\\
R_u(X, X_i) &= \gamma_i \frac{\partial q_\alpha}{\partial \gamma_i}(X) = \gamma_i E[u_i | X = q_\alpha(X)], && (5.11)
\end{aligned}
$$

and the additivity property is also satisfied by these components :

$$
\begin{aligned}
R(X) &= R_s(X) + R_u(X), && (5.12)\\
\text{with } R_s(X) &= \sum_{i=1}^{n} R_s(X, X_i), R_u(X) = \sum_{i=1}^{n} R_u(X, X_i). && (5.13)
\end{aligned}
$$

ii) Large number of entitites

Finally, let us discuss the case of large n. If n is large, and the entities of similar sizes, we deduce from the Law of Large Numbers (LLN) that the idiosyncratic terms can be diversified, whereas the systematic factors cannot be. For expository purpose, let us consider a single factor model with $\gamma_i = 1, \forall i$. We have :

$$X = (\sum_{i=1}^{n} \beta_i) f + \sum_{i=1}^{n} u_i.$$

Let us assume that the beta coefficients are i.i.d. with a positive mean $E(\beta) > 0$, and are independent of factor f and idiosyncratic errors $u_i, i = 1, \ldots, n$. Let us also assume that these errors are independent with zero mean $E(u_i) = 0$. The Euler contribution to the VaR is equal to :

$$E[X_i | X = q_\alpha(X)] = E[X_i | X/n = q_\alpha(X/n)],$$

since the quantile function is homogenous of degree 1.[24]

By the LLN, we deduce that :

$$\lim_{n \to \infty} (X/n) = E(\beta) f.$$

[24] Note that X depends on size n, but this dependence is not indicated for expository purpose.

Thus, the idiosyncratic part has been diversified, whereas the effect of systematic risk persists asymptotically.

When $n = \infty$, the Euler allocation of the VaR becomes :

$$\begin{aligned}
\lim_{n\to\infty} E[X_i|X = q_\alpha(X)] &= \lim_{n\to\infty} E[X_i|X/n = q_\alpha(X/n)] \\
&= E[X_i|E(\beta)f = q_\alpha[E(\beta)f]] \\
&= E[\beta_i f + u_i|E(\beta)f = q_\alpha[E(\beta)f] \\
&= E[\beta_i f|E(\beta)f = q_\alpha[E(\beta)f] \\
&= \lim_{n\to\infty} E[\beta_i f|X = q_\alpha(X)] \\
&= \lim_{n\to\infty} E[X_{s,i}|X = q_\alpha(X)].
\end{aligned}$$

In this limiting case, the Euler allocation for entity i and the Euler allocation for its systematic component coincide. Moreover, it is equivalent to condition on X, on the factor summary $E(\beta)f$, or on factor f itself.

The derivation above is important to understand the systematic contribution used in Acharya et alii (2010), Brownless, Engle (2010), which underlies the daily updated systematic risk ranking diffused by NYU Stern's Volatility Lab. [www.systemicrisisranking.stern.nyu.edu], defined by [25] $E[X_i|X = q_\alpha(X)]$. This definition is valid under the assumptions above, i.e. when the unsystematic component can be diversified and when factor f can be identified to the market portfolio.

For n large, but finite, it is easily checked by applying granularity theory [Gagliardini, Gourieroux (2010)], that the expressions :

$$E[X_{s,i}|X = q_\alpha(X)] = E[\beta_i f|X = q_\alpha(X)],$$

and $E[X_i|X = q_\alpha(X)]$ differ.

[25] Their analysis concerns stock market. X_i is the capitalization in stock i, whereas X is the market portfolio value.

5.2 General allocations in a linear factor model

The interpretation of Euler allocation discussed in Table 1 provides a principle of allocation between systematic and unsystematic risks in a general framework. The idea is to choose as new entities the (initial entity) $\times$ (type) of risk. The $L\&P$ for entity i and systematic risk (resp. unsystematic risk) is : $X_{s,i} = \sum_{k=1}^{K} \beta_{ik} f_k$(resp. $X_{u,i} = \gamma_i u_i$). The components of the total $L\&P$ are defined accordingly by : $X_s = \sum_{i=1}^{n} X_{si}$ and $X_u = \sum_{k=1}^{n} X_{u,i}$. Then, for a given ADM μ_P, the allocations are defined by :

$$
\begin{aligned}
R_{\mu,s}(X, X_i) &= R_\mu(X, X_{si}), \\
R_{\mu,u}(X, X_i) &= R_\mu(X, X_{ui}) \\
R_\mu(X, X_i) &= R_{\mu,s}(X, X_i) + R_{\mu,u}(X, X_i), \\
R_s(X) &= \sum_{i=1}^{n} R_{\mu,s}(X, X_i) = R_\mu(X, X_s), \\
R_u(X) &= \sum_{i=1}^{n} R_{\mu,u}(X, X_i) = R_\mu(X, X_u).
\end{aligned}
$$

5.3 Nonlinear factor model

The allocation of the global reserve among the entities can be done for both linear and nonlinear factor models. However, the allocation between systematic and unsystematic components is less than obvious if :

$$X_i = g_i(f, u), \tag{5.14}$$

where (multidimensional) factor f and idiosyncratic term u are independent, due to the presence of cross effects.

However, it is possible to decompose the individual $L\&P$ as :

$$\begin{aligned} X_i &= E(X_i|f) + [E(X_i|u) - E(X_i)] + [X_i - E(X_i|f) - E[X_i|u] + E(X_i)] \\ &\equiv X_{s,i} + X_{u,i} + X_{u,s,i}, \text{say}, \end{aligned} \tag{5.15}$$

where $X_{s,i}, X_{u,i}, X_{s,u,i}$ are the principle systematic and unsystematic effects, and the cross effect, respectively [26] [27]. Even if the systematic factor f and the idiosyncratic term u are independent, interaction effects will appear in the risk contributions due to the nonadditive decomposition.

Then, the allocation of entity i can be decomposed as :

$$R(X, X_i) = R_s(X, X_i) + R_u(X, X_i) + R_{s,u}(X, X_i), \tag{5.16}$$

where :

$$\begin{aligned} R_s(X, X_i) &= E[X_{s,i}|X = q_\alpha(X)] = E[E(X_i|f)|X = q_\alpha(X)], \\ R_u(X, X_i) &= E[X_{u,i}|X = q_\alpha(X)] = E[E(X_i|u)|X = q_\alpha(X)] - E(X_i), \\ R_{s,u}(X, X_i) &= R(X, X_i) - R_s(X, X_i) - R_u(X, X_i). \end{aligned}$$

This decomposition does not depend on the selected representations of the factor and the idiosyncratic term, that is, the decomposition is invariant when f or u is transformed by a one-to-one transformation.

Example : As an illustration, let us consider models with stochastic drift and volatility driven by a same factor f :

$$X_i = m_i(f) + \sigma_i(f) u_i, i = 1, \ldots, n,$$

where f is independent of $u = (u_1, \ldots, u_n)'$, and the errors are iid zero mean. We have :

[26] As in Section 5.1, we have implicitly included the expected $L\&P$ in the systematic component.

[27] This type of decomposition can also be used to distinguish the effects of dependent systematic factors [see Rosen, Saunders (2010), Section 4.5]

$$R_s(X, X_i) = E[m_i(f)|X = q_\alpha(X)],$$

$$R_u(X, X_i) = E[E\{\sigma_i(f)\}u_i|X = q_\alpha(X)],$$

$$R_{s,u}(X, X_i) = E[(\sigma_i(f) - E\{\sigma_i(f)\})u_i|X = q_\alpha(X)].$$

This example shows that in a nonlinear model, the effect of the systematic factor is captured by both R_s and $R_{s,u}$. Their relative magnitude can be highly different for the different entities. For instance in a basic stochastic volatility model, where $m_i(f) = 0$, only the cross effect matters.

6 Required Capital

The benchmark reserves introduced in the previous sections are simply objective risk measures. They vary according to the risk cycle, and are the basis for defining the required capital. The link function [see equation (1.1) for a typical example] is a crucial element of the regulatory policy. Whereas the trigger parameter is a control variate for more or less prudential policy, the smoothing can be used to decrease, or increase the effects of cycles.

The recent financial crisis revealed important drawbacks of a link function like (1.1) :
i) The trigger parameter k has been taken fixed, independently of the market environment. We would have expected a reduced trigger during a liquidity crisis.
ii) As already mentioned in the introduction, the link function implies two regimes that are a smoothed and an unsmoothed regime, the latter one appearing with a large increase of the risk of entity i. However, the consequences are not the same if this risk increase is due to an idiosyncratic shock, or to a shock on a systematic factor. In the first situation, there is an additional demand of liquid asset by entity i, which can be easily satisfied by the market. In the second situation, there is the demand for liquid asset by several entities together, which may force financial institutions to deliver at fire-sale prices, creates the deleveraging spiral, (that is, selling assets to reduce the debt (also called margin/haircut spiral in Brunnermeier, Pedersen (2009), CGFS (2010)). and accentuates the cycles and the crisis.

6.1 Change of the link function

Clearly a drawback of a link function like (1.1) is the lack of distinction between systematic and unsystematic risk, that is, of the micro and macro prudential approaches [Borio (2004)]. Instead of a formula of the type :

$$RC_{i,t} = \max[R_t(X, X_i), k_t \frac{1}{60} \sum_{h=0}^{59} R_{t-h}(X, X_i)], \tag{6.1}$$

an improved formula has to separate the two types of risks, in order to take into account the fact that the systematic components are almost comonotonic, and has to apply the two regime formula to the unsystematic component only. [28] An alternative to formula (6.1) might be :

$$\begin{aligned} RC_{i,t} &= \max[R_{u,t}(X, X_i), k_{u,t} \frac{1}{60} \sum_{h=0}^{59} R_{u,t-h}(X, X_i)] \\ &+ k_{s,t} \frac{1}{H} \sum_{h=0}^{H-1} R_{s,t-h}(X, X_i) \equiv RC_{i,t}^u + RC_{i,t}^s. \end{aligned} \tag{6.2}$$

The first component concerning the unsystematic risk is sufficient to penalize the risky investments specific to a given entity, while avoiding a too volatile evolution of the associated required capital. The second component has to be linear, since the quantile function is linear for comonotonic risks. Moreover, the smoothing window H is introduced for another purpose, not for avoiding a too volatile required capital, but for obtaining a countercyclical effect. In this respect, the smoothing window for systematic risk has too be much larger than the usual 3-month (i.e. $H = 60$), and able to cover a significant part of the cycle (one year for instance). Finally, the trigger parameter $k_{s,t}$ has to be much smaller than $k_{u,t}$, and dependent of the position within the cycle, that is smaller in the bottom of the cycle, larger in its top, to avoid the spurious creation of a liquidity gap.

The required capital $RC_{i,t}$ has to be provided to the regulators in liquid, high rated assets. Intuitively the components $RC_{i,t}^s$ and $RC_{i,t}^u$ have to be included in different accounts of the Central Bank : the unsystematic component should be in an account specific of entity i, but all contributions for

[28]To simplify the discussion, we assume a linear factor model, therefore a zero cross term : $R_{s,u} = 0$.

systematic risk might be mutualized at least at the country level. Indeed, this account will serve to insure the global system against systematic risk, which is an example of catastrophic event. If an entity is close to failure due to a systematic effect, the total reserve for systematic component can be used to avoid the failure of the entity and also some potential failures of the other ones by contagion.

Even if the application of different link functions to the systematic and unsystematic risk components seems relevant, its implementation will encounter the same difficulties than the stress testing. Indeed, the systematic factors have to be defined in a same way for all the entities by the regulators and there are many common risk factors which can be considered. Instead of looking for an exhaustive set of factors, it can be preferable to select a well-chosen limited set, focusing on the most important factors for systemic risk.

6.2 Change of global risk measure

The usual regulatory approach distinguishes the underlying basic allocation and the required capital. As noted in Section 6.1 above, the link function can be adjusted to avoid the procyclical effect of the required capital. However, this two step approach lacks coherency. For instance, the additivity property satisfied by the basic allocations is not satisfied by the required capital, since the link function is nonlinear. Is it possible to develop a one-step coherent approach ?

A possible answer is to define more precisely the global risk function $R(X)$ in a regulatory perspective. Let us consider the framework of Section 5, with underlying factors driving the systematic risk. These factors have to be known and observed ex-post by the regulators. Thus, the regulators have an augmented information set including both global $L\&P, X$, and factors F. Their risk measures should not only take into account the level of X, but also the comparison of this level with the position in the cycle, function of F. In other words, the assumption, that the global reserve $R(X)$ depends on the distribution of X only, is likely not appropriate in a regulatory perspective. This global measure should depend on the joint distribution of F, X, and a better notation would be $R(F, X)$. We have seen that the allocation problem is an hedging problem w.r.t. global $L\&P$ X. Similarly, the choice of an appropriate level of global reserve is also an hedging problem, but w.r.t. to a real economic benchmark and not the problem of controlling the stand alone risk of X. Typically, for a mortgage portfolio the optimal amount of mortgage to be distributed should depend on the real estate cycle. For instance the global reserve could be $q_{\alpha(Q)}(X)$, where Q denotes the distribution of F, or $R_c(F, X) = E(X) + E([X - c(F)]^+)$, that is, the global risk measure could change along the real estate cycle.

A similar remark can be done for the allocations. The result of Proposition 3.8 is still valid, but with an increased information set. More precisely, function $a_P(X)$ should now be replaced by a function $a_P(X, F)$, where P denotes the joint distribution of (X, F).

7 Concluding remarks

The aim of this paper was to survey and complete the current literature on capital allocation in a regulatory perspective and with special attention to systematic risk. The main message of the literature and of this paper is to avoid a crude use of a coherent risk measure such as a VaR, or an ES for computing the reserve both at the individual level and at the global level. More precisely,

i) An allocation problem is different from a risk measurement problem. The allocations are contingent to the level of global risk and have to satisfy some basic axioms.

ii) The axioms are not sufficient to define a unique allocation, and it could be important to distinguish the distortion risk measure underlying the measure of global risk from the allocation distortion risk measure explaining how to allocate the global reserve.

iii) The allocation by ADM seems applicable to any type of global hedging measure and also to disentangle marginal systematic and unsystematic risk components and cross effects.

iv) If the regulation has a purpose of economic policy, that is, if the monetary policy is not enough for the Central Bank, a coherent risk measure for global risk is likely not appropriate. The subadditivity axiom, the axiom of homogeneity... are likely not appropriate for regulatory purpose. The risk measure has to be chosen in relation with the economic environment, especially with the position in the business or real estate cycle. The regulator faces an hedging problem, not the problem of managing the stand alone global risk.

The main questions that are still to be solved are the following ones :

i) What is (are) the objective(s) of the regulator ? In particular, is he/she partly in charge of economic policy ? What is seen in our paper is that the definition of the required capital is an important instrument to control the quantity of credits and its distribution among firms, households, real estates, but also the leveraging among banks... Several Central Banks have for official objective the control of inflation by means of a prime rate. It seems important to debate of the control of the credit distribution and leveraging by means of the required capital [see e.g. Hellwig (2010) for a polemical discussion of the role of regulation].

ii) Once the objective is well-defined, how to choose the global level of reserve $R(F, X)$, which will be likely different from a global VaR, or from the sum of VaR's of the different entities ?

iii) Once the objective and global level of reserve are well-defined, how to choose the Allocation Distortion Measure, that is, the way of allocating the global reserve between the entities, between systematic, unsystematic and cross components?

To summarize, at a time where the databases, the statistical tools, the marginal and hedging risk measures, are almost in place, the different possible objectives of the regulation and their consequences on the real economy and on the required capital have now to be evaluated and compared.

R E F E R E N C E S

Acerbi, C. (2002) : "Spectral Measures of Risk : A Coherent Representation of Subjective Risk Aversion", Journal of Banking and Finance, 26, 1505-1518.

Acerbi, C., and T., Tasche (2002) : "On the Coherence of Expected Shortfall", Journal of Banking and Finance, 26, 1487-1503.

Acharya, V. Pedersen, L., Philippon, T., and M., Richardson (2010) : "Measuring Systemic Risk", Technical Report, Department of Finance, NYU.

Adrian, T., and M., Brunnermeier (2009) : "CoVaR", Federal Reserve Bank of New-York, [revised version of the Technical Report 348, in (2008)].

Artzner, P., Delbaen, F., Eber, J., and D., Heath (1997) : "Thinking Coherently", Risk, 10, 68-71.

Artzner, P., Delbaen, F., Eber, J., and D., Heath (1999) : "Coherent Measures of Risk", Mathematical Finance, 9, 203-228.

Bertsimas, D., Lauprete, G., and A., Samarov (2004) : "Shortfall as a Risk Measure : Properties, Optimization and Applications", Journal of Economic Dynamics and Control, 28, 1353-1381.

Billio, M., Mamo, K., and L., Pelizzon (2010) : "Hedge Fund Tail Risk and Marginal Risk Contribution in Fund of Hedge Funds", DP University of Venice.

Borio, C. (2004) : "Market Distress and Vanishing Liquidity : Anatomy and Policy Options", DP 158.

Breuer, T., Jandacka, M., Rheinberger, K., and M., Summer (2010) :"Does Adding up of Economic Capital for Market and Credit Risk Amount to Conservative Risk Assessment ?", Journal of Banking and Finance, 34, 703-712.

Brownless, C., and R., Engle (2010) : "Volatility, Correlation, and Tails for Systemic Risk Measurement", DP Stern Business School.

Brunnermeier, M., and L., Pedersen (2009) : "Market Liquidity and Funding Liquidity", The Review of Financial Studies, 22, 2201-2238.

Burgisser, P., Kurth, A., Wagner, A., and M., Wolf (1999) : " Integrating Correlations", Risk, 12, 57-60.

Cai, J., and S., Tan (2005) : "CTE and Capital Allocation Under the Skew Elliptical Distribution", DP.

Chiragiev, A., and Z., Landsman (2007) : "Multivariate Pareto Portfolios : TCE. Based Capital Allocation and Divided Differences", Scandinavian Actuarial Journal, 4, 261-280.

Committee on the Global Financial System (2010) : "The Role of Margin Requirements and Haircuts in Procyclicality", CGFS Paper 36, Bank of International Settlements.

Delbaen, F. (1998) : "Coherent Risk Measures on General Probability Spaces", DP ETH Zurich.

Denault, M. (2001) : "Coherent Allocation of Risk Capital", Journal of Risk, 4, 1-34.

Dhaene, J., Goovaerts, M., and R., Kaas (2003) : "Economic Capital Allocation Derived From Risk Measures", North American Actuarial Journal, 7, 1-16.

Dhaene, J., Laeven, R., Vanduffel, S., Darkiewicz, G., and M., Goovaerts (2004) : "Can a Coherent Risk Measure be Too Subadditive ?", Amsterdam-Univ-DP.

Fermanian, J.D., and O., Scaillet (2005) : "Sensitivity Analysis of VaR and Expected Shortfall for Portfolio Under Netting Agreements", Journal of Banking and Finance, 29, 927-958.

Fong, T., and O., Vasicek (1997) : "A Multidimensional Framework for Risk Analysis", Financial Analysts Journal, July/August, 51-57.

Furman, E., and R., Zitikis (2008) : "Weighted Risk Capital Allocations", Insurance : Mathematics and Economics, 43, 263-269.

Gagliardini, P., and C., Gourieroux (2010) : "Granularity Theory with Application to Finance and Insurance", mimeo, 200p.

Garman, M. (1996) : "Improving on VaR", Risk, May, 61-63.

Garman, M. (1997) : "Taking VaR to Pieces", RISK, 10, October, p70-71.

Gerber, H. (1979) : "An Introduction to Mathematical Risk Theory", Univ.of Pennsylvania, Philadelphia.

Goovaerts, M., De Vylder, F., and J., Haezendonck (1984) : "Insurance Premiums", Amsterdam, North-Holland, p111-140.

Gourieroux, C., Laurent, J.P., and O., Scaillet (2000) : "Sensitivity Analysis of Value-at-Risk", Journal of Empirical Finance, 7, 225-245.

Gourieroux, C., and A., Monfort (2004) : "Infrequent Extreme Risks", Geneva Papers on Risk and Insurance Theory, 29, 5-22.

Gray, D., and A., Jobst (2009) : " Tail Dependence Measures of Systemic Risk Using Equity Option Data Implications for Financial Stability", DP International Monetary Found, Washington DP.

Haaf, H., and D., Tasche (2002) : "Credit Portfolio Measurements", GARP Risk Review, p43-47, July/August.

Hallerbach, W. (2002) : "Decomposing Portfolio Value-at-Risk : A General Analysis", Journal of Risk, 5, 1-18.

Hellwig, M. (2010) :"Capital Regulation After the Crisis : Business as Usual N", DP, Max Planck Institute.

Hesselager, O., and U., Andersson, (2002) : "Risk Sharing and Capital Allocation", Tryg Insurance DP.

Huang, X., Oosterlee, C., and M., Mesters (2007) : "Computation of VaR and VaR Contribution in the Vasicek Portfolio Credit Loss Model : A Comparative Study", Journal of Credit Risk, 3, 75-96.

Kalkbrener, M. (2005) : "An Axiomatic Approach to Capital Allocation", Mathematical Finance, 15, 425-437.

Kim, J. (2010) : "Measuring and Managing Systemic Risk", Univ. of Waterloo DP.

Koyluoglu, H., and J., Stoker (2002) : "Honour Your Contributions", Risk, 15, 90-94.

Kurth, A., and D., Tasche (2003) : "Credit Risk Contributions to Value-at-Risk and Expected Shortfall", Risk, 16, March., 84-88.

Landsman, Z., and A., Valdez (2005) : "Tail Conditional Expectation for Exponential Dispersion Models", Astin Bulletin, 35, 189-209.

Lehar, A. (2005) : "Measuring Systemic Risk : A Risk Management Approach", Journal of Banking and Finance, 29.

Litterman, R. (1996) : "Hot Spots and Hedges", The Journal of Portfolio Management, 22, 52-75.

Martin, R., and D., Tasche (2007) : "Shortfall : A Tail of Two Parts', Risk, 20, 84-89.

Martin, R., Thompson, K., and C., Browne (2001) : "VaR : Who Contributes and How Much ?", Risk, 14, 99-102.

Matten, C. (1996) : "Managing Bank Capital", Wiley, Chichester.

McNeil, A., Frey, R., and P., Embrechts (2005) : "Quantitative Risk Management", Princeton University Press.

Panjer, H., (2002) : "Solvency and Capital Allocation", University of Waterloo, DP 01-14.

Patrick, G., Bernegger, S., and M., Ruegg (1999) : "The Use of Risk Adjusted Capital to Support Business Decision Making", Swiss Reinsurance Company DP.

Rockafellar, R., and S., Uryasev (2002) : "Conditional Value-at-Risk for General Loss Distributions", Journal of Banking and Finance, 26, 1443-1471.

Rosen, D., and D., Saunders (2010) : "Risk Factor Contributions in Portfolio Credit Risk Models", Journal of Banking and Finance, 34, 336-349.

Rothschild, M., and J., Stiglitz (1970) : "Increasing Risk I : A Definition", Journal of Economic Theory", 2, 225-243.

Rudin, W. (1966) : "Real and Complex Analysis", Mc Graw Hill.

Scaillet, O., (2004) : "Nonparametric Estimation and Sensitivity Analysis of Expected Shortfall, Mathematical Finance, 14, 155-129.

Shapley, L. (1953) : "A Value for n-person Games", in Contributions to the Theory of Games, Volume II, by H., Kuhn and A., Tucker, editors, Annals of Mathematical Studies, 28, 307-317.

Sherris, M. (2007) : "Risk Based Capital and Capital Allocation in Insurance", Australian School of Business DP.

Tarashev, N., Borio, C., and K., Tsatsaronis (2009) : "Allocating Systematic Risks to Individual Institutions : Methodology and Policy Application", DP, BIS. Quartely Review.

Tasche, D. (2000) : "Risk Contributions and Performance Measurement", DP University of Munich.

Tasche, D. (2008) : "Capital Allocation to Business Units and Sub-Portfolio : the Euler Principle", in A. Resti, ed., Pillar II in the New Basel Accord : The Challenge of Economic Capital, p423-453, Incisive Financial Publishing Lloyds TSB Bank, DP.

Uryasev, S., Theiler, U., and G., Serraino (2010) : "Risk Return Optimization with Different Risk-Aggregation Strategies", The Journal of Risk Finance, 11, 129-146.

Wang, S. (1996) : "Premium Calculation by Transforming the Layer Premium Density", ASTIN Bulletin, 26, 71-92.

Wang, S. (2000) : "A Class of Distortion Operators for Pricing Financial and Insurance Risks", Journal of Risk and Insurance, 67, 15-36.

Wang, S., and V., Young (1998) : "Ordering Risks : Expected Utility Theory Versus Yaari's Dual Theory of Risk", Insurance : Mathematics and Economics, 22, 145-161.

Yamai, Y., and T., Yoshiba (2001) : "Comparative Analysis of Expected Shortfall and VaR : Their Estimation Error, Decomposition and Optimization", IMES Discussion Paper, 2001-E-12, Bank of Japan.

Zoller, R. (1996) : "Marginal Value-at-Risk", in M. Schroder, ed., Quantitative Verfahren in Finanzmarketberreich, p 155-132, Nomos. Verl.Ges.

A P P E N D I X 1

Proof of Proposition 3.2

Equivalence between i) and ii)

First note that the function :

$$y \to U(y, x-y),$$

where U is concave is itself concave. Therefore ii) implies i).

Conversely, i) implies ii). Indeed let us denote by a a value for which the one-dimensional concave function U is maximal. If a is finite, U can be written as :

$$\begin{aligned} U(y) &= U_1(y) + U_2(y) + U(a), \\ \text{with} \quad U_1(y) &= 0, \text{ if } y \leq a, \\ &\quad U(y) - U(a), \text{otherwise}, \\ U_2(y) &= U(y) - U(a), \text{ if } y \leq a, \\ &\quad 0, \text{ otherwise}, \end{aligned}$$

U_1 (resp. U_2) is a decreasing concave (resp. increasing concave) function. The result is deduced by noting that the function :

$$(y_1, y_2) \to U_1(x - y_1) + U_2(y_2) + U(a),$$

is concave, increasing w.r.t to both y_1 and y_2.

If $a = +\infty$ [resp. $-\infty$], the result is deduced by nothing that the function : $(y_1, y_2) \to U(y_2)$ [resp. $(y_1, y_2) \to U(x - y_1)$] is concave, increasing w.r.t both y_1, y_2.

Equivalence between ii) and iii)

This equivalence is given in Rothschild, Stiglitz (1970), Theorem 2.

QED

APPENDIX 2

Explicit expression of the marginal expected shortfall

We have to prove that :

$$\begin{aligned}
\frac{\partial E[\beta X + Y | \beta X + Y > q_\alpha(\beta)]}{\partial \beta} &= E[X | \beta X + Y > q_\alpha(\beta)], & \text{(a.1)}\\
\text{where } P[\beta X + Y > q_\alpha(\beta)] &= 1 - \alpha, \forall \beta, & \text{(a.2)}
\end{aligned}$$

and $q_\alpha(\beta) = q_\alpha(\beta X + Y)$, say.

Let us assume that the joint distribution of (X, Y) is continuous with probability density function [29] $f(x, y)$. Equality (a.2) can be written as :

$$\int \left[\int_{-\beta x + q_\alpha(\beta)}^{\infty} f(x, y) dy \right] dx = 1 - \alpha, \forall \beta.$$

Thus, by differentiating with respect to β, we get :

$$\int [x - \frac{\partial q_\alpha(\beta)}{\partial \beta}] f[x, q_\alpha(\beta) - \beta x] dx = 0, \forall \beta, \qquad \text{(a.3)}$$

which implies $\dfrac{\partial q_\alpha(\beta)}{\partial \beta} = E[X | \beta X + Y = q_\alpha(\beta)]$.

The expected shortfall for $\beta X + Y$ is :

$$\begin{aligned}
& ES(\beta) \\
& = E[\beta X + Y | \beta X + Y > q_\alpha(\beta)] \\
& = \frac{1}{1 - \alpha} \int \left[\int_{-\beta x + q_\alpha(\beta)}^{\infty} (\beta x + y) f(x, y) dy \right] dx
\end{aligned}$$

Its derivative with respect to β is equal to :

[29] The general proof valid for any type of joint distribution for (X, Y) has been given in Tasche (2000), Lemma (5.6).

$$
\begin{aligned}
\frac{\partial ES(\beta)}{\partial \beta} &= \frac{1}{1-\alpha}\int \left[\int_{-\beta x + q_\alpha(\beta)}^{\infty} x f(x,y)dy\right] dx \\
&+ \frac{1}{1-\alpha}\int \left[x - \frac{\partial q_\alpha(\beta)}{\partial \beta}\right] q_\alpha(\beta) f[x, q_\alpha(\beta) - \beta x] dx \\
&= \frac{1}{1-\alpha}\int \left[\int_{-\beta x + q_\alpha(\beta)}^{\infty} x f(x,y)dy\right] dx \;\; \text{[from (a.3)]} \\
&= E[X|\beta X + Y > q_\alpha(\beta)],
\end{aligned}
$$

with is equation (a.1).

A P P E N D I X 3

Proof of Proposition 4.4

Let us consider the change $(X_1, \ldots, X_n) \to (\lambda, X_1, \ldots, \lambda X_n)$ and denote $q^*_\alpha(\wedge) = q_\alpha(\sum_{i=1}^{n} \lambda_i X_i)$. We have :

$$q^*_\alpha(\wedge) = \wedge' E(X) + q_\alpha [\wedge' V(X) \wedge]^{1/2}.$$

We deduce :

$$\begin{aligned}
\frac{\partial q^*_\alpha(\wedge)}{\partial \wedge} = E(X) \quad &+ \quad q_\alpha \frac{V(X)\wedge}{[\wedge' V(X)\wedge]^{1/2}}, \\
\frac{\partial^2 q^*_\alpha(\wedge)}{\partial \wedge \partial \wedge'} \quad &= \quad q_\alpha \left[\frac{V(X)}{[\wedge' V(X)\wedge]^{1/2}} - \frac{V(X) \wedge \wedge' V(X)}{[\wedge' V(X)\wedge]^{3/2}} \right].
\end{aligned}$$

The result in Proposition 4.4 is derived by choosing $\wedge = e$.

A P P E N D I X 4

Analysis in returns

i) Disaggregation of the VaR written in returns

The reserves are defined in subsection 3.2. iii) from the asset values, but can be alternatively written in terms of returns. For such a return analysis, the time index has to be explicitly introduced. Let us denote by $X_{i,t}, X_t$ the asset values at the end of period t. The associated returns are such that :

$$r_{i,t} = X_{i,t}/X_{i,t-1} - 1, r_t = X_t/X_{t-1} - 1.$$

The quantiles have now to take explicitly into account the information set available at the end of period $t-1$. They will be denoted by $q_{\alpha,t-1}$. The decomposition formula of the global conditional quantile becomes :

$$\begin{aligned} & E_{t-1}[X_t|X_t = q_{\alpha,t-1}(X_t)] \\ = \; & X_{t-1}\{1 + E_{t-1}(r_t|r_t = q_{\alpha,t-1}(r_t))\} \\ = \; & \sum_{i=1}^{n} X_{i,t-1}\{1 + E_{t-1}[r_{i,t}|r_t = q_{\alpha,t-1}(r_t)]\}, \end{aligned}$$

or equivalently :

$$q_{\alpha,t-1}(r_t) = E_{t-1}[r_t|r_t = q_{\alpha,t-1}(r_t)] = \sum_{i=1}^{n} w_{i,t-1}E_{t-1}[r_{i,t}|r_t = q_{\alpha,t-1}(r_t)], \tag{a.4}$$

where $w_{i,t-1} = X_{i,t-1}/X_{t-1}$ are time varying weights. Formula (a.4) provides the decomposition written in terms of returns [see e.g. Acharya et alii (2010), Brownless, Engle (2010)].

ii) Sensitivity of the return expected shortfall

It is usual to compute a VaR, or an ES, in terms of returns instead of portfolio values. We have seen in Equation (a.4) that the decomposition of the global reserve in terms of values has its analogue in terms of returns. For instance, the return expected shortfall is given by :

$$\begin{aligned} ES_{\alpha}^{r} &= E_{t-1}[r_t | r_t > q_{\alpha,t-1}(r_t)] \\ &= \sum_{i=1}^{n} w_{i,t-1} E_{t-1}[r_{i,t} | r_t > q_{\alpha,t-1}(r_t)], \end{aligned} \tag{a.5}$$

where $w_{i,t-1} = X_{i,t-1}/X_{t-1}$ are the weights in value.

Let us now consider a small change in the portfolio value of entity i, passing from $X_{i,t}$ to $\lambda X_{i,t}$, the values associated with the other entities being fixed. We have the following Lemma :

Lemma : The marginal effect of this change on the return expected shortfall is :

$$\frac{\partial ES_{\alpha}^{r,*}(e)}{\partial \lambda_i} = w_{i,t-1}\{1 + E_{t-1}[r_{i,t} | r_t > q_{\alpha,t-1}(r_t)]\} - 1.$$

Proof : The effect on the expected shortfall of a small change in the portfolio value of entity i is :

$$\begin{aligned} \frac{\partial ES_{\alpha}^{*}(e)}{\partial \lambda_i} &= E_{t-1}[X_{it} | X_t > q_{\alpha,t-1}(X_t)] \\ &= X_{i,t-1}\left[1 + E_{t-1}[r_{i,t} | r_t > q_{\alpha,t-1}(r_t)]\right]. \end{aligned}$$

The expected shortfall on return is :

$$\begin{aligned} ES_{\alpha}^{r} &= E_{t-1}[r_t | r_t > q_{\alpha,t-1}(r_t)] \\ &= \frac{1}{X_{t-1}} E_{t-1}[X_t | X_t > q_{\alpha,t-1}(X_t)] - 1 \\ &= \frac{ES_{\alpha}}{X_{t-1}} - 1. \end{aligned}$$

Thus, the derivative of the return expected shortfall is :

$$\frac{\partial ES_{\alpha}^{r,*}(e)}{\partial \lambda_i} = \frac{X_{i,t-1}}{X_{t-1}}\left\{1 + E_{t-1}[r_{i,t} | r_t > q_{\alpha,t-1}(r_t)]\right\}.$$

QED

The component $ES^r_{\alpha,i} = E_{t-1}[r_{i,t}|r_t > q_{\alpha,t-1}(r_t)]$ has a complicated interpretation in terms of sensitivity, since :

$$ES^r_{\alpha,i} = E_{t-1}[r_{i,t}|r_t > q_{\alpha,t-1}(r_t)] = \frac{1}{w_{i,t-1}} \left\{ 1 + \frac{\partial ES^{r,*}_{\alpha}(e)}{\partial \lambda_i} \right\}. \qquad \text{(a.6)}$$

The interpretation above differs from the sensitivity interpretation proposed in Acharya et alii (2010), eq. 4., where the quantity $ES^r_{\alpha,i}$ is said to be equal to $\frac{\partial}{\partial w_{i,t-1}} ES^r_{\alpha}$. The main reason for this difference is the following one : whereas changes of portfolio values are easily interpretable and allow for changes specific of the entities, this is no longer the case for changes of percentage shares $w_{i,t-1}$. Indeed, these percentage shares sum up to one. Thus a change of $w_{i,t-1}$ has to be compensated by appropriate changes on the other entities and the expression of the derivative $\frac{\partial}{\partial w_{i,t-1}}$ depends on the kind of compensation which is assumed. To summarize the formula in Acharya et alii (2010) is misleading.