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Asset Pricing with Second-Order Esscher Transforms

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Abstract

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The purpose of the paper is to introduce, in the class of discrete time no-arbitrage asset pricing models, a wider bridge between the historical and the risk-neutral state vector dynamics and to preserve, at the same time, its tractability and flexibility. This goal is achieved by introducing the notion of Exponential-Quadratic stochastic discount factor (SDF) or, equivalently, the notion of Second-Order Esscher Transform. Then, focusing on security market models, this approach is developed in three important multivariate stochastic frameworks: the conditionally Gaussian framework, the conditionally Mixed-Normal and the conditionally Gaussian Switching Regimes framework.

In the conditionally multivariate Gaussian case, our approach determines a risk-neutral mean as a function of (the short rate and of) the risk-neutral variance-covariance matrix which is different from the historical one. The conditionally mixed-normal Gaussian case provides a first generalization of the Gaussian setting, in which the risk-neutral variance-covariance matrices and mixing weights of all components (in the finite mixture) can be different from the historical ones. The Gaussian switching regime case introduces further flexibility given the serial dependence of regimes and the introduction of the regime indicator function in the exponential-quadratic SDF. We also develop switching regime models which include (in the factor's conditional mean and conditional variance) additive impacts of the present and past regimes and we stress their interpretation in terms of general "discrete-time jump-diffusion" models in which the risk included in the first and second moment of jumps is priced.

Even if we focus on security market models, we do not make any particular assumption about the state vector and therefore this approach could be used not only in option pricing models, but also for instance in interest rate and credit risk models.

Keywords: Second-Order Esscher Transform, Exponential-Quadratic Stochastic Discount Factor, No-Arbitrage Asset Pricing Models, Security Market Economies.

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1 Introduction

Discrete time asset pricing models are now widespread in the economic and financial literature and they are successfully used in many research fields, like bond and option pricing, longevity risk, liquidity and credit risk modelling, as well as exchange rate and macro-finance modelling. This large class of models contains two important families following two different (in general) asset pricing modelling principles : the first one is built on the notion of stochastic discount factor (SDF), while the second one is based on the concept of (local) risk-neutral valuation relationship (RNVR or LRNVR).

The first set of models invokes the absence of arbitrage opportunity in order to typically introduce an exponential-affine (in the factor) SDF which provides a bridge between the historical world and the risk-neutral one [see Gourieroux and Monfort (2007)]. Since the three mathematical objects specifying the models, namely the historical and the risk-neutral (R.N.) dynamics of the state vector and the one-period SDF, are linked together, three modelling strategies naturally appear (the so-called Direct Modelling, Risk-Neutral Constrained Direct Modelling and Back Modelling strategies). In each of them two objects are specified (and, possibly, the short rate if it is not assumed to be exogenous or a known function of the state vector) and the third one is obtained as a byproduct. This general discrete time no-arbitrage asset pricing setting, formalized by Bertholon, Monfort and Pegoraro (2008) [BMP (2008), hereafter], has shown its large flexibility in various contexts [see Monfort and Pegoraro (2007b) for an application to yield curve modelling, Gourieroux, Monfort and Polimenis (2006) for an application to credit risk analysis, Gourieroux and Monfort (2008) for longevity risk, and Gourieroux, Monfort and Sufana (2010) for exchange rate risk].

In the second set of no-arbitrage models the vector of state variables is made only of asset returns and a RNVR or LRNVR is introduced imposing that: *i*) the historical and risk-neutral dynamics belong to the same parametric families; *ii*) the R.N. expectation of the (arithmetic) returns of the basic assets are equal to the risk-less (arithmetic) returns; *iii*) the historical and risk-neutral variance-covariance matrix of the state-vector, conditional to the past, are the same functions of the past. Then, this RNVR or LRNVR are usually justified by a combination of assumptions on agents preferences and on probability distributions [see Rubinstein (1976), Brennan (1979), Duan (1995), Camara (1999, 2003)].

The assumptions made in both approaches obviously reduce the set of possible admissible pairs of historical and risk-neutral dynamics. For instance, in the first approach, even if the assumption of an exponential-affine SDF is well justified in the literature, in particular in consumption-based asset pricing models, in terms of minimal entropy martingale measure, in terms of discretization of continuous time security market models and for tractability of the pricing formula[3], it is not imposed by the absence of arbitrage opportunity principle which only requires the positivity of the pricing kernel and some internal consistency conditions. Among the consequences of this assumption let us mention the fact that, in conditionally Gaussian models, the historical and risk-neutral conditional variance-covariance matrices of the state vector are the same function of the past, like in the LRNVR approach.

[3]See, among others, Bansal and Yaron (2004), Bertholon, Monfort and Pegoraro (2006), Campbell and Cochrane (1999), Darolles, Gourieroux and Jasiak (2006), Föllmer and Schied (2004), Garcia, Meddahi and Tedongap (2006), Gourieroux, Jasiak and Sufana (2004), Gourieroux, Monfort and Polimenis (2003, 2006), Monfort and Pegoraro (2007b), Pegoraro (2006), Polimenis (2001), Stutzer (1995).

In this paper we adopt the first kind of approach and we introduce a wider bridge between the historical and the risk-neutral probability. More precisely, we first recall that the assumption of an exponential-affine SDF can also be viewed as the assumption that the R.N. dynamics is a conditional Esscher transform of the historical dynamics and vice versa [see Gerber and Shiu (1994), Buhlmann, Delbaen, Embrechts and Shiryaev (1996, 1998), Siu, Tong and Yang (2004), Christoffersen, Jacobs and Ornthanalai (2008)]. Then, we introduce the notion of Exponential-Quadratic SDF or, equivalently, the notion of Second-Order Esscher Transform. Then, focusing on security market models, this approach is developed in three important multivariate stochastic frameworks: the conditionally Gaussian framework, the conditionally mixed-normal and the conditionally Gaussian Switching Regimes framework.

In the conditionally multivariate Gaussian case, our approach determines a risk-neutral mean as a function of (the short rate and of) the risk-neutral variance-covariance matrix which is, at the same time, different from the historical one because of the second-order stochastic risk-sensitivity vector appearing in the SDF. In this way, we extend to a general multivariate asset pricing (SDF-based) framework the results of Christoffersen, Elkhami, Feunou and Jacobs (2009) proposed in a scalar setting and based on the particular LRNVR principle. In order to provide a more precise interpretation of the first-order and second-order stochastic risk-sensitivity vectors specifying the exponential-quadratic SDF, we calculate (in this Gaussian setting) the one-period risk premium and we compare it to the first-order risk premium generated by the exponential-affine SDF. We also calculate the Second-Order Black and Scholes pricing formula for European Call options and we find that it is a generalization of the classical Black and Scholes one in which the historical conditional variance is now replaced by the risk-neutral conditional one, function of the (constant) second-order risk-sensitivity parameter. The above mentioned results clearly generalize the widely known continuous time (Girsanov-based) and discrete time no-arbitrage asset pricing concepts established since the papers of Black and Scholes (1973), Merton (1973, 1976) and Vasicek (1977).

The conditionally mixed-normal Gaussian case provides a first generalization of the Gaussian setting, in which the risk-neutral variance-covariance matrices and mixing weights of all components (in the finite mixture) can be different from the historical ones. The Gaussian switching regime case introduces, first, further flexibility in the historical dynamics of the factor, given the serial dependence of the regimes. Second, the introduction of the regime indicator function in the exponential-quadratic SDF leads to an explicit pricing of regime-shift risk. Moreover, this modelling allows, for instance, to use in a pricing context various kinds of switching GARCH models which have been successfully used in the historical world [see Hamilton and Susmel (1994), Gray (1996), Klaassen (2002), Hass, Mittnick and Paolella (2004)].

We also develop switching regime models which include (in the factor's conditional mean and conditional variance) additive impacts of the present and past regimes and we stress their interpretation in terms of general "discrete-time jump-diffusion" models. More precisely, we specify a regime-switching security market model with serially dependent (contemporaneous and lagged) jumps able to replicate clusters with time-varying persistence. In addition, the introduction of the quadratic term in the SDF gives the possibility to price the risk provided by the first and second moment of jumps with Gaussian stochastic amplitude.

It is worth noting that, even if the paper focus on security market models, we do not make any particular assumption about the state vector and therefore this SDF-based approach (contrary to the RNVR and LRNVR ones) could be used not only in option pricing models, but also for instance in interest rate and credit risk models.

The paper is organized as follows. In Section 2 we define the Second Order Esscher Transform of a probability density function and we show, thanks to some example, how it generalizes the family of probability distributions generated by the classical (First-Order) Esscher Transform. Section 3 presents the Exponential-Quadratic Stochastic Discount Factor modelling principle in a multivariate setting, and shows how the associated change of probability measure is given by a conditional Second-Order Esscher Transform. Section 4 and 5 deal with, respectively, multivariate conditionally Gaussian and Mixed-Normal economies both specified following the Direct and Back Modelling strategy. In Section 6 we focus on General Switching Regime economies. First, in Section 6.1 we determine the conditional Second Order Esscher Transform of a general conditionally Gaussian switching regime process. Second, in Section 6.2, we apply this result to security market models, following the Direct and Back Modelling strategies defined in BMP (2008). Finally, in Section 6.3 we focus on the Additive Regime Switching Economy and we show that a particular Additive Regime Switching model can be re-parametrized as an observationally equivalent generalization of the continuous-time jump-diffusion model. Section 7 concludes and Appendices gather the proofs.

2 Esscher Transforms

Let us consider a probability $\mathbb{P}$ defined on $\mathbb{R}^n$, and f its probability density function (p.d.f.) with respect to some measure ν. For sake of completeness we briefly recall the definition of the Esscher Transform (called here First-Order Esscher Transform) and we give some examples [see Gerber and Shiu (1994)].

2.1 First-Order Esscher Transform

Definition 1 [First-Order Esscher Transform] : The First-Order Esscher Transform of $\mathbb{P}$ associated with θ_1, denoted by $F_{(\theta_1)}(\mathbb{P})$, is given by the family of probability distributions defined by the p.d.f.:

$$g(y;\theta_1) = \frac{f(y)\exp(\theta_1' y)}{\int_{\mathbb{R}^n} f(y)\exp(\theta_1' y) d\nu(y)} \tag{1}$$

or, denoting $\varphi(\theta_1) = \int_{\mathbb{R}^n} f(y)\exp(\theta_1' y) d\nu(y)$ the Laplace transform of $\mathbb{P}$:

$$g(y;\theta_1) = \frac{f(y)\exp(\theta_1' y)}{\varphi(\theta_1)} \tag{2}$$

with $\theta_1 \in \Theta_1$, Θ_1 denoting the definition set of the Laplace transform. Let us consider some examples of First-Order Esscher Transform.

i) Discrete distributions

Let us assume that ν is a counting measure on a (possibly infinite) discrete space $D \subset \mathbb{R}^n$ defined by the point masses $\{p_d, d \in D\}$. The Esscher Transform is the family of probability distributions on D with probability masses:

$$\frac{p_d \exp(\theta_1' d)}{\sum_{d \in D} p_d \exp(\theta_1' d)}\,, \quad d \in D\,, \tag{3}$$

whenever $\sum_{d \in D} p_d \exp(\theta_1' d) < \infty$.

ii) Univariate Gaussian distribution

Here ν is the Lebesgue measure on $\mathbb{R}$, and we consider the p.d.f. and the Laplace transform of a Gaussian random variable $N(\mu, \sigma^2)$:

$$\begin{aligned} f(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(y-\mu)^2\right] , \\ \varphi(\theta_1) &= \exp(\theta_1\mu + \theta_1^2\sigma^2/2) \, , \;\; \theta_1 \in \mathbb{R} \, . \end{aligned} \tag{4}$$

The associated Esscher Transform is :

$$g(y;\theta_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}\left(y - (\mu + \theta_1\sigma^2)\right)^2\right] , \tag{5}$$

that is the p.d.f. of the family of univariate Gaussian random variables $N(\mu + \theta_1\sigma^2, \sigma^2)$ with different means but the same variance as the one defined in (4).

iii) Multivariate Gaussian distribution

Here ν is the Lebesgue measure on $\mathbb{R}^n$, and we assume the following p.d.f. and associated Laplace transform:

$$\begin{aligned} f(y) &= \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}} \exp\left[-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right] , \\ \varphi(\theta_1) &= \exp(\theta_1'\mu + \theta_1'\Sigma\theta_1/2) \, , \;\; \theta_1 \in \mathbb{R}^n \, . \end{aligned} \tag{6}$$

In that case, the Esscher Transform is :

$$g(y;\theta_1) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}} \exp\left[-\frac{1}{2}(y - (\mu + \Sigma\theta_1))'\Sigma^{-1}(y - (\mu + \Sigma\theta_1))\right] , \tag{7}$$

that is the p.d.f. of the family of n-dimensional Gaussian random variable $N(\mu + \Sigma\theta_1, \Sigma)$ having different means but the same variance-covariance matrix as the starting Gaussian random variable associated to (6).

iv) Finite Mixture of Multivariate Gaussian distributions

Let us consider again, as in the previous example *iii*), that ν is the Lebesgue measure on $\mathbb{R}^n$, and let us consider the following p.d.f.:

$$\begin{aligned} & f(y) = \textstyle\sum_{j=1}^J \lambda_j \, n(y; \mu_j, \Sigma_j) \, , \\ & \text{with } \; n(y;\mu_j,\Sigma_j) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma_j}} \exp\left[-\frac{1}{2}(y-\mu_j)'\Sigma_j^{-1}(y-\mu_j)\right] , \\ & 0 \le \lambda_j \le 1, \; \textstyle\sum_{j=1}^J \lambda_j = 1 \, , \end{aligned} \tag{8}$$

and the associated Laplace transform :

$$\varphi(\theta_1) \quad = \quad \sum_{j=1}^{J} \lambda_j \exp(\theta_1' \mu_j + \theta_1' \Sigma_j \theta_1/2)\,, \quad \theta_1 \in \mathbb{R}^n\,. \tag{9}$$

In that case, the Esscher Transform is given by :

$$\begin{aligned} & g(y;\theta_1) = \sum_{j=1}^{J} \lambda_j^* \, n(y;\mu_j + \Sigma_j\theta_1, \Sigma_j)\,, \\ & \text{with } \lambda_j^* = \frac{\lambda_j \exp(\theta_1'\mu_j + \theta_1'\Sigma_j\theta_1/2)}{\sum_{j=1}^{J} \lambda_j \exp(\theta_1'\mu_j + \theta_1'\Sigma_j\theta_1/2)}\,, \\ & 0 \leq \lambda_j^* \leq 1,\ \sum_{j=1}^{J} \lambda_j^* = 1\,, \end{aligned} \tag{10}$$

that is the p.d.f. of the family of n-dimensional Finite Mixtures of J Gaussian random variables $N(\mu_j + \Sigma_j\theta_1, \Sigma_j)$, $j \in \{1, \ldots, J\}$, in which, as in the previous example, each component has (for any given θ_1) a different mean but the same variance-covariance matrix as the Gaussian components characterizing the mixture in (8) and, moreover, the weights λ_j^* are different from the initial ones λ_j.

2.2 Second-Order Esscher Transform

The purpose of this section is to introduce a new family of probability distributions, associated with the p.d.f. f, having the (First-Order) Esscher Transform as a subset. This new family, that we call Second-Order Esscher Transforms and which is built upon the concept of Second-Order Laplace Transform, gives the possibility, for instance, to modify not only the mean but also the variance-covariance matrix of a multivariate Gaussian distribution or the mean and the variance-covariance matrix of the components of a mixture of multivariate Gaussian distributions (see examples below). Many other examples, including switching regimes models, will be also considered.

Definition 2 [Second-Order Laplace Transform] : The Second-Order Laplace Transform of the p.d.f. $f(y)$ is :

$$\varphi_S(\theta_1, \theta_2) = \int_{\mathbb{R}^n} f(y) \exp(\theta_1' y + y'\theta_2 y) d\nu(y) \tag{11}$$

with $\theta_1 \in \mathbb{R}^n$, $\theta_2 \in \mathcal{S}_n(\mathbb{R})$ an $(n \times n)$ real symmetric matrix[4] and $\theta = (\theta_1, \theta_2) \in \Theta$, Θ being the definition set $\{(\theta_1, \theta_2) \in \mathbb{R}^n \times \mathcal{S}_n(\mathbb{R}) : \int_{\mathbb{R}^n} f(y) \exp(\theta_1' y + y'\theta_2 y) d\nu(y) < \infty\}$.

Definition 3 [Second-Order Esscher Transform] : The Second-Order Esscher Transform of $\mathbb{P}$ associated with (θ_1, θ_2), denoted by $S_{(\theta_1,\theta_2)}(\mathbb{P})$, is given by the family of probability distributions defined by the p.d.f.:

$$g(y;\theta_1,\theta_2) = \frac{f(y)\exp(\theta_1' y + y'\theta_2 y)}{\varphi_S(\theta_1,\theta_2)}\,. \tag{12}$$

Let us now present examples of Second-Order Esscher Transforms [the proofs of examples from *vi)* to *viii)* are given in Appendix 1].

v) Discrete distributions (example i) continued)

[4]Observe that the assumption $\theta_2 \in \mathcal{S}_n(\mathbb{R})$ is not a restriction since any square matrix A (say) is the sum of a symmetric matrix $(A + A')/2$ and of an antisymmetric matrix $(A - A')/2$, and since a quadratic form associated to an antisymmetric matrix is equal to zero.

Let us first consider the case of the discrete distributions introduced in example i). The associated Second-Order Esscher transform is the family of probability distributions on D with probability masses:

$$\frac{p_d \exp(\theta_1' d + d'\theta_2 d)}{\sum_{d \in D} p_d \exp(\theta_1' d + d'\theta_2 d)}, \quad d \in D, \tag{13}$$

assuming $\sum_{d \in D} p_d \exp(\theta_1' d + d'\theta_2 d) < \infty$.

vi) Univariate Gaussian distribution (example ii) continued)

The Second-Order Esscher transform of the p.d.f. of a univariate ($n = 1$) Gaussian random variable $N(\mu, \sigma^2)$ is given by:

$$g(y; \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{1-2\theta_2\sigma^2}}} \exp\left[-\left(\frac{1-2\theta_2\sigma^2}{2\sigma^2}\right)\left(y - \frac{(\mu+\theta_1\sigma^2)}{1-2\theta_2\sigma^2}\right)^2 \right], \tag{14}$$

which is, under the condition $\theta_2 < \frac{1}{2\sigma^2}$, the p.d.f. of the family of the Gaussian random variables $N\left(\frac{(\mu+\theta_1\sigma^2)}{1-2\theta_2\sigma^2}, \frac{\sigma^2}{1-2\theta_2\sigma^2}\right)$. Compared with $N(\mu, \sigma^2)$, this family has, in general, not only different means (driven by the two parameters (θ_1, θ_2)) but also different variances (driven by θ_2). Observe that any Gaussian distribution can be reached when $\theta = (\theta_1, \theta_2)$ varies in $\Theta = \mathbb{R} \times \left]-\infty, \frac{1}{2\sigma^2}\right[$.

vii) Multivariate Gaussian distribution (example iii) continued)

The Second-Order Esscher transform of the p.d.f. of a n-dimensional Gaussian random variable $N(\mu, \Sigma)$ is [see Appendix 1]:

$$\begin{aligned} g(y; \theta_1, \theta_2) = & \frac{1}{(2\pi)^{n/2}\sqrt{\det\left[(\Sigma^{-1} - 2\theta_2)^{-1}\right]}} \times \\ & \exp\left[-\frac{1}{2}(y - (I - 2\Sigma\theta_2)^{-1}(\mu + \Sigma\theta_1))'(\Sigma^{-1} - 2\theta_2)(y - (I - 2\Sigma\theta_2)^{-1}(\mu + \Sigma\theta_1)) \right], \end{aligned} \tag{15}$$

that is the p.d.f. of the family of the n-dimensional Gaussian random variable $N((I - 2\Sigma\theta_2)^{-1}(\mu + \Sigma\theta_1), (\Sigma^{-1} - 2\theta_2)^{-1})$ if $(\Sigma^{-1} - 2\theta_2)$ is assumed to be a symmetric positive definite matrix, that is $(\Sigma^{-1} - 2\theta_2) \in \mathcal{S}_n^+(\mathbb{R})$ or, equivalently, if the eigenvalues of $\theta_2\Sigma$ are smaller than $\frac{1}{2}$ that is, if $\theta_2 = \Sigma^{-1/2}ADA'\Sigma^{-1/2}$, where D is a diagonal matrix with diagonal terms smaller than $\frac{1}{2}$ and A is an orthogonal matrix. Like in the previous example, for any given (θ_1, θ_2), the Gaussian random variable generated by (12) has a different mean as well as a different variance-covariance matrix compared to (6) and any n-dimensional Gaussian distribution can be reached. When we assume $\theta_2 = 0$, the conditional mean $(I - 2\Sigma\theta_2)^{-1}(\mu + \Sigma\theta_1)$ and variance-covariance matrix $(\Sigma^{-1} - 2\theta_2)^{-1}$ degenerate to those of example *iii*).

viii) Finite Mixture of Multivariate Gaussian distributions (example iv) continued)

Given a finite mixture of n-dimensional Gaussian random variables [see example *iv*)] with p.d.f. $f(y) = \sum_{j=1}^{J} \lambda_j \, n(y; \mu_j, \Sigma_j)$, the associated family of probability density functions generated by the

Second-Order Esscher Transform is:

$$
\begin{aligned}
& g(y;\theta_1,\theta_2) = \textstyle\sum_{j=1}^{J} \lambda_j^* \, n(y; (I-2\Sigma_j\theta_2)^{-1}(\mu_j+\Sigma_j\theta_1), (\Sigma_j^{-1}-2\theta_2)^{-1})\,, \\
& \text{with } \lambda_j^* = \frac{\lambda_j \varphi_{S,j}(\theta_1,\theta_2)}{\sum_{j=1}^{J} \lambda_j \varphi_{S,j}(\theta_1,\theta_2)}\,, \\
& \varphi_{S,j}(\theta_1,\theta_2) = \textstyle\int_{\mathbb{R}^n} \exp(\theta_1 y + y'\theta_2 y) n(y;\mu_j,\Sigma_j)dy \\
& \qquad = \exp\left[-\frac{1}{2}\log\det(I-2\Sigma_j\theta_2) - \frac{1}{2}\mu_j'\Sigma_j^{-1}\mu_j + \frac{1}{2}(\Sigma_j^{-1}\mu_j+\theta_1)'(\Sigma_j^{-1}-2\theta_2)^{-1}(\Sigma_j^{-1}\mu_j+\theta_1)\right]\,, \\
& \text{and } 0 \leq \lambda_j^* \leq 1,\ \textstyle\sum_{j=1}^{J} \lambda_j^* = 1\,.
\end{aligned}
\tag{16}
$$

This is the family of p.d.f. of a n-dimensional Finite Mixture of J Gaussian random variables $N((I-2\Sigma_j\theta_2)^{-1}(\mu_j+\Sigma_j\theta_1), (\Sigma_j^{-1}-2\theta_2)^{-1})$, $j \in \{1,\ldots,J\}$, having a mean and a variance-covariance matrix different from the corresponding components in (8), as well as different mixing weights.

3 The Exponential-Quadratic Stochastic Discount Factor Modelling Principle

3.1 Information and Historical Distribution

In what follows, we consider an economy between dates 0 and T. The new information in the economy at date t is denoted by w_t, while $\underline{w}_t = (w_t, w_{t-1}, ..., w_0)$ is the entire information between 0 and t. The random variable w_t is called a factor or a state vector and its dimension is n.

The historical dynamics of w_t is defined by the conditional distribution of w_{t+1} given $\underline{w}_t$, denoted by $\mathbb{P}_{t+1}$ (say) and characterized either by the p.d.f. $f_t(w_{t+1}|\underline{w}_t)$ or the Laplace transform $\varphi_t(u|\underline{w}_t)$, or the Log-Laplace transform $\psi_t(u|\underline{w}_t) = \log[\varphi_t(u|\underline{w}_t)]$.

3.2 The Exponential-Affine Stochastic Discount Factor

Assuming existence, linearity and continuity of the pricing function, and under the absence of arbitrage opportunity principle, Hansen and Richard (1987) and Bertholon, Monfort and Pegoraro (2008) show the existence of a positive Stochastic Discount Factor (SDF) $M_{t,t+1}(\underline{w}_{t+1})$, for each $t \in \{0,\ldots,T-1\}$, such that the price at date t of the payoff $g(\underline{w}_s)$ delivered at $s > t$ is given by $p_t\left[g(\underline{w}_s)\right] = E_t\left[M_{t,t+1}...M_{s-1,s}\,g(\underline{w}_s)\right]$ [see also Cochrane (2005)].

The asset pricing literature has in general derived or specified $M_{t,t+1}(\underline{w}_{t+1})$ as an exponential-affine function of w_{t+1}. Indeed, this form naturally stands out in equilibrium models like CCAPM [see e.g. Cochrane (2005)], consumption-based asset pricing models with habit formation or with Epstein-Zin preferences [see, among others, Bansal and Yaron (2004), Campbell and Cochrane (1999), Garcia, Meddahi and Tedongap (2006), Garcia, Renault and Semenov (2006)]. Moreover, in general continuous-time security market models the discretized version of the SDF is exponential-affine [see Gourieroux and Monfort (2007)]. Finally, the exponential-affine specification is particularly well adapted to the Laplace Transform which is a central tool in discrete-time asset pricing theory [see e.g. Bertholon, Monfort and Pegoraro (2006), Darolles, Gourieroux and Jasiak (2006), Gourieroux, Jasiak and Sufana (2009), Gourieroux, Monfort and Polimenis (2003, 2006), Monfort and Pegoraro (2007a, 2007b), Pegoraro (2006), Polimenis (2001)].

More precisely, under the no-arbitrage restriction on the (predetermined) risk-free short rate r_{t+1} for the period $(t, t+1)$, one assumes :

$$M_{t,t+1} = \exp\left[-r_{t+1}(\underline{w}_t) + \alpha'_{1,t}(\underline{w}_t)w_{t+1} - \psi_t(\alpha_{1,t}|\underline{w}_t)\right] . \tag{17}$$

where $\alpha_{1,t}(\underline{w}_t)$ is the n-dimensional "factor loading" or "risk-sensitivity" vector, also called the "market price" of factor risk.

If the SDF has the exponential-affine form (17), it is well known that the Risk-Neutral (R.N.) conditional distribution of w_{t+1}, given $\underline{w}_t$ and denoted by $\mathbb{Q}_{t+1}$, has an exponential-affine (in w_{t+1}) p.d.f. with respect to $\mathbb{P}_{t+1}$ given by:

$$\begin{aligned} d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) = \frac{M_{t,t+1}(\underline{w}_{t+1})}{E_t\left[M_{t,t+1}(\underline{w}_{t+1})\right]} &= \frac{\exp\left(\alpha'_{1,t}w_{t+1}\right)}{\varphi_t(\alpha_{1,t})} , \\ &= \exp\left[\alpha'_{1,t}w_{t+1} - \psi_t(\alpha_{1,t})\right] . \end{aligned} \tag{18}$$

The R.N. conditional p.d.f. of w_{t+1} given $\underline{w}_t$ (with respect to the same measure as the corresponding conditional historical probability) is $f_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) = f_t(w_{t+1}|\underline{w}_t)d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t)$ and the R.N. conditional Log-Laplace transform is $\psi_t^{\mathbb{Q}}(u_1) = \psi_t(u_1 + \alpha_{1,t}) - \psi_t(\alpha_{1,t})$ $(u_1 \in \mathbb{R}^n)$.

Conversely, the p.d.f. of the conditional historical distribution with respect to the R.N. one is given by :

$$\begin{aligned} d_t^{\mathbb{P}}(w_{t+1}|\underline{w}_t) = \frac{1}{d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t)} &= \exp\left[-\alpha'_{1,t}w_{t+1} + \psi_t(\alpha_{1,t})\right] , \\ &= \exp\left[-\alpha'_{1,t}w_{t+1} - \psi_t^{\mathbb{Q}}(-\alpha_{1,t})\right] , \end{aligned} \tag{19}$$

since $\psi_t^{\mathbb{Q}}(-\alpha_{1,t}) = -\psi_t(\alpha_{1,t})$. From Definition 1, relations (18) and (19) we have the following:

Proposition 1 : If we consider the exponential-affine stochastic discount factor $M_{t,t+1}$, the risk-neutral conditional distribution $\mathbb{Q}_{t+1}$ of w_{t+1}, conditionally to $\underline{w}_t$, is the conditional First-Order Esscher Transform of $\mathbb{P}_{t+1}$ associated with $\alpha_{1,t}$, that is $\mathbb{Q}_{t+1} = F_{(\alpha_{1,t})}(\mathbb{P}_{t+1})$. Conversely, the historical conditional distribution $\mathbb{P}_{t+1}$ is the conditional First-Order Esscher Transform of $\mathbb{Q}_{t+1}$ associated with $-\alpha_{1,t}$, that is $\mathbb{P}_{t+1} = F_{(-\alpha_{1,t})}(\mathbb{Q}_{t+1})$.

3.3 The Exponential-Quadratic Stochastic Discount Factor

The purpose of this section is to generalize the classical exponential-affine SDF change of probability (i.e., the conditional First-Order Esscher Transform) presented in the previous section by means of the conditional Second-Order Esscher Transform that is, by introducing the following exponential-quadratic SDF:

$$M_{t,t+1}^{(S)} = \exp\left[-r_{t+1}(\underline{w}_t) + \alpha'_{1,t}(\underline{w}_t)w_{t+1} + w'_{t+1}\alpha_{2,t}(\underline{w}_t)w_{t+1} - \psi_{S,t}(\alpha_{1,t}, \alpha_{2,t}|\underline{w}_t)\right] , \tag{20}$$

with $\psi_{S,t}(\alpha_{1,t}, \alpha_{2,t}|\underline{w}_t) = \log \varphi_{S,t}(\alpha_{1,t}, \alpha_{2,t}|\underline{w}_t)$, $\varphi_{S,t}(\alpha_{1,t}, \alpha_{2,t}|\underline{w}_t) = E_t[\exp(\alpha'_{1,t}w_{t+1} + w'_{t+1}\alpha_{2,t}w_{t+1})]$ the conditional second-order Log-Laplace transform and where $\alpha_{2,t}$ is a (time-varying) $(n \times n)$ symmetric matrix $(\alpha_{2,t} \in \mathcal{S}_n(\mathbb{R}))$. The functions $\alpha_{1,t}$ and $\alpha_{2,t}$ are called risk-sensitivity coefficients.

In that case, the Risk-Neutral (R.N.) conditional distribution $\mathbb{Q}_{t+1}$ of w_{t+1} given $\underline{w_t}$, has an exponential-quadratic (in w_{t+1}) p.d.f. with respect to $\mathbb{P}_{t+1}$ given by:

$$d_t^{\mathbb{Q},S}(w_{t+1}|\underline{w}_t) = \frac{M^{(S)}_{t,t+1}(\underline{w}_{t+1})}{E_t\left[M^{(S)}_{t,t+1}(\underline{w}_{t+1})\right]} = \exp\left[\alpha'_{1,t}w_{t+1} + w'_{t+1}\alpha_{2,t}w_{t+1} - \psi_{S,t}(\alpha_{1,t},\alpha_{2,t})\right], \quad (21)$$

and, therefore, the R.N. conditional p.d.f. (with respect to the same measure as the corresponding conditional historical probability) is $f_t^{\mathbb{Q},S}(w_{t+1}|\underline{w}_t) = f_t(w_{t+1}|\underline{w}_t)d_t^{\mathbb{Q},S}(w_{t+1}|\underline{w}_t)$ and the R.N. conditional second-order Log-Laplace transform is:

$$\psi^{\mathbb{Q}}_{S,t}(u_1,u_2) = \psi_{S,t}(u_1+\alpha_{1,t}, u_2+\alpha_{2,t}) - \psi_{S,t}(\alpha_{1,t},\alpha_{2,t})\,, \ \ u_1 \in \mathbb{R}^n\,, \ \ u_2 \in \mathcal{S}_n(\mathbb{R})\,. \quad (22)$$

Conversely, the p.d.f. of the conditional historical distribution with respect to the R.N. one is given by :

$$\begin{aligned} d_t^{\mathbb{P},S}(w_{t+1}|\underline{w}_t) = \frac{1}{d_t^{\mathbb{Q},S}(w_{t+1}|\underline{w}_t)} &= \exp\left[-\alpha'_{1,t}w_{t+1} - w'_{t+1}\alpha_{2,t}w_{t+1} + \psi_{S,t}(\alpha_{1,t},\alpha_{2,t})\right] \\ &= \exp\left[-\alpha'_{1,t}w_{t+1} - w'_{t+1}\alpha_{2,t}w_{t+1} - \psi^{\mathbb{Q}}_{S,t}(-\alpha_{1,t},-\alpha_{2,t})\right], \end{aligned} \quad (23)$$

since $\psi^{\mathbb{Q}}_{S,t}(-\alpha_{1,t},-\alpha_{2,t}) = -\psi_{S,t}(\alpha_{1,t},\alpha_{2,t})$. We get the following :

Proposition 2 : If we consider the exponential-quadratic stochastic discount factor $M^{(S)}_{t,t+1}$, the risk-neutral conditional distribution $\mathbb{Q}_{t+1}$ of w_{t+1}, conditionally to $\underline{w_t}$, is the conditional Second-Order Esscher Transform of $\mathbb{P}_{t+1}$ associated with $(\alpha_{1,t},\alpha_{2,t})$, that is $\mathbb{Q}_{t+1} = S_{(\alpha_{1,t},\alpha_{2,t})}(\mathbb{P}_{t+1})$. Conversely, the historical conditional distribution $\mathbb{P}_{t+1}$ is the conditional Second-Order Esscher Transform of $\mathbb{Q}_{t+1}$ associated with $(-\alpha_{1,t},-\alpha_{2,t})$, that is $\mathbb{P}_{t+1} = S_{(-\alpha_{1,t},-\alpha_{2,t})}(\mathbb{Q}_{t+1})$.

3.4 Internal Consistency Conditions

The no-arbitrage discrete-time asset pricing setting based on an exponential-affine SDF $M_{t,t+1}$, conveniently provides explicit conditions, through the historical and R.N. Log-Laplace transforms ψ_t and $\psi_t^{\mathbb{Q}}$, to guarantee the internal consistency of the model [see BMP (2008) for details]. These Internal Consistency Conditions (ICC) are easily extended to the case of an exponential-quadratic SDF $M^{(S)}_{t,t+1}(\underline{w}_{t+1})$. Let us consider, for instance, the situation in which the factor w_{t+1} contains (at least) a geometric stock return and in which the short rate r_{t+1} is exogenous. If $w_{j,t+1} = e'_j w_{t+1}$ is a scalar geometric return (e_j being the j^{th} column of the identity matrix $I_{n\times n}$) we must have:

$$\begin{aligned} &\exp(-r_{t+1})E_t^{\mathbb{Q}}[\exp(w_{j,t+1})] = 1 \\ \iff\ & r_{t+1} = \psi^{\mathbb{Q}}_{S,t}(e_j,0) \\ \iff\ & r_{t+1} = \psi_{S,t}(\alpha_{1,t}+e_j,\alpha_{2,t}) - \psi_{S,t}(\alpha_{1,t},\alpha_{2,t})\,. \end{aligned} \quad (24)$$

4 Conditionally Gaussian Economies

4.1 Direct Modelling

Let us assume that the factor w_t is a n-dimensional vector of geometric stock returns of risky assets, that is $w_{i,t+1} = \log(S_{i,t+1}/S_{i,t})$ for each $i \in \{1,\dots,n\}$, where $S_{i,t}$ is the price at t of asset i. If we follow the Direct Modelling strategy formalized by Bertholon, Monfort and Pegoraro (2008), we first have to specify the historical dynamics $(\mathbb{P}_{t+1})$ of w_{t+1}. Assuming conditional normality, that is:

$$w_{t+1}|\,\underline{w}_t \overset{\mathbb{P}}{\sim} N\left(\mu_t, \Sigma_t\right)\,, \tag{25}$$

we have to choose μ_t and Σ_t (including, for instance, VAR and VARMA models with GARCH-type noise). Second, we have to specify $\alpha_{1,t}$ and $\alpha_{2,t}$ and to impose the ICC (24) :

$$r_{t+1} = \psi_{S,t}(e_i + \alpha_{1,t}, \alpha_{2,t}) - \psi_{S,t}(\alpha_{1,t}, \alpha_{2,t})\,, \tag{26}$$

where

$$\psi_{S,t}(u_1, u_2) = -\tfrac{1}{2}\log\det\left(I - 2\Sigma_t u_2\right) - \tfrac{1}{2}\mu_t'\Sigma_t^{-1}\mu_t + \tfrac{1}{2}(\Sigma_t^{-1}\mu_t + u_1)'(\Sigma_t^{-1} - 2u_2)^{-1}(\Sigma_t^{-1}\mu_t + u_1) \tag{27}$$

which implies :

$$\begin{aligned} r_{t+1} &= \tfrac{1}{2}(\Sigma_t^{-1}\mu_t + e_i + \alpha_{1,t})'(\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}(\Sigma_t^{-1}\mu_t + e_i + \alpha_{1,t}) \\ &\qquad -\tfrac{1}{2}(\Sigma_t^{-1}\mu_t + \alpha_{1,t})'(\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}(\Sigma_t^{-1}\mu_t + \alpha_{1,t}) \\ &= \tfrac{1}{2}e_i'(\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}e_i + e_i'(I - 2\Sigma_t\alpha_{2,t})^{-1}(\mu_t + \Sigma_t\alpha_{1,t}) \quad \forall\, i \in \{1, \ldots, n\}\,, \end{aligned} \tag{28}$$

that is :

$$\tfrac{1}{2}\,vdiag\left[(\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\right] + (I - 2\Sigma_t\alpha_{2,t})^{-1}(\mu_t + \Sigma_t\alpha_{1,t}) = r_{t+1}e\,, \tag{29}$$

where e denotes the n-dimensional unitary vector. The specification of the historical dynamics (25) and of the exponential-quadratic SDF (20) implies the following R.N. dynamics $(\mathbb{Q}_{t+1})$:

$$w_{t+1}|\,\underline{w}_t \overset{\mathbb{Q}}{\sim} N\left[(I - 2\Sigma_t\alpha_{2,t})^{-1}(\mu_t + \Sigma_t\alpha_{1,t}), (\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\right]\,, \tag{30}$$

that is, $\mathbb{Q}_{t+1} = S_{(\alpha_{1,t},\alpha_{2,t})}(\mathbb{P}_{t+1})$. If we impose to (30) the ICC (29), we find that the R.N. dynamics compatible with no-arbitrage restrictions is:

$$N\left[r_{t+1}e - \tfrac{1}{2}\,vdiag\left((\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\right), (\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\right]\,. \tag{31}$$

It is important to stress that this new exponential-quadratic SDF change of probability measure induces (with respect to the exponential-affine one) a different R.N. conditional mean $\mu_t^{\mathbb{Q}} = r_{t+1}e - \frac{1}{2}\,vdiag\left((\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\right)$ (observe that μ_t disappears like in the classical exponential-affine setting) and a different R.N. conditional variance-covariance matrix $\Sigma_t^{\mathbb{Q}} = (\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}$ because of the second-order risk-sensitivity function $\alpha_{2,t}$. We also find that the risk-sensitivity vectors $\alpha_{1,t}$ and $\alpha_{2,t}$, characterizing the SDF, are given by :

$$\begin{aligned} \alpha_{2,t} &= \frac{\Sigma_t^{-1} - (\Sigma_t^{\mathbb{Q}})^{-1}}{2}\,, \\ \text{and}\ \ \alpha_{1,t} &= (\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} - \Sigma_t^{-1}\mu_t\,. \end{aligned} \tag{32}$$

So α_{2t} is a measure of the variance-covariance rise when moving from the historical to the risk-neutral world, while α_{1t} is a measure of the increase of the weighted expected mean. It is important to highlight that relation (29) makes $\alpha_{1,t}$ a function of $\alpha_{2,t}$ and the latter can be any function of the date t information such that $\Sigma_t^{\mathbb{Q}} \in \mathcal{S}_n^+(\mathbb{R})$.

4.2 Risk Premium and Second-Order Black and Scholes Pricing Formula

In order to provide a more precise interpretation of the risk-sensitivity functions $\alpha_{1,t}$ and $\alpha_{2,t}$, let us first consider the scalar case ($n = 1$), studied by Christoffersen, Elkhami, Feunou and Jacobs (2009) in the RNVR setting. Under the risk-neutral probability we have:

$$
\begin{aligned}
& w_{t+1} | \underline{w}_t \overset{\mathbb{Q}}{\sim} N\left[\mu_t^{\mathbb{Q}}(\alpha_{2,t}), (\sigma_t^{\mathbb{Q}})^2(\alpha_{2,t})\right] , \\
& \text{where } \ \mu_t^{\mathbb{Q}}(\alpha_{2,t}) = r_{t+1} - \tfrac{1}{2}\,(\sigma_t^{\mathbb{Q}})^2(\alpha_{2,t}), \quad (\sigma_t^{\mathbb{Q}})^2(\alpha_{2,t}) = \frac{\sigma_t^2}{(1-2\sigma_t^2\alpha_{2,t})} ,
\end{aligned}
\tag{33}
$$

and thus, if we define the risk premium between t and $t+1$ in the following way:

$$
\lambda_{t,t+1} = \log E_t[\exp(y_{t,t+1})] - r_{t+1} ,
$$

then, from (29), we can write:

$$
\begin{aligned}
\lambda_{t,t+1} &= \mu_t + \tfrac{1}{2}\sigma_t^2 - r_{t+1} \\
&= \left[\mu_t - \mu_t^{\mathbb{Q}}(\alpha_{2,t})\right] + \tfrac{1}{2}\left[\sigma_t^2 - (\sigma_t^{\mathbb{Q}}(\alpha_{2,t}))^2\right] \\
&= \lambda_{t,t+1}^F + \left[\mu_t^{\mathbb{Q}}(0) - \mu_t^{\mathbb{Q}}(\alpha_{2,t})\right] + \tfrac{1}{2}\left[\sigma_t^2 - (\sigma_t^{\mathbb{Q}}(\alpha_{2,t}))^2\right] ,
\end{aligned}
\tag{34}
$$

where $\lambda_{t,t+1}^F := \mu_t - \mu_t^{\mathbb{Q}}(0) = -\,\alpha_{1,t}\,\sigma_t^2$ denotes the (first-order) risk premium associated to an exponential-affine SDF ($\alpha_{2,t} = 0$). Relation (34) shows the role played by $\alpha_{2,t}$, that is, the consequences on the asset risk premium played by the introduction of a quadratic term in the SDF:

i) if we assume $\alpha_{2,t} = 0$ (an exponential-affine SDF) we find $\lambda_{t,t+1} = \lambda_{t,t+1}^F$, that is, the risk premium is (classically) determined comparing only historical and risk-neutral factor conditional means and $-\alpha_{1,t}$ can be interpreted as a first moment-based risk premium per unit of conditional variance;

ii) if $\alpha_{2,t} \neq 0$, relation (34) tell us that the size of $\lambda_{t,t+1}$ differs from $\lambda_{t,t+1}^F$ because of $\sigma_t^2 \neq (\sigma_t^{\mathbb{Q}}(\alpha_{2,t}))^2$ and $\mu_t^{\mathbb{Q}}(\alpha_{2,t}) \neq \mu_t^{\mathbb{Q}}(0)$. This means that $\alpha_{2,t}$ introduces in the risk premium not only a second moment-based source of risk information but it also modifies, at the same time, the role played by the first moment-based source of risk.

This result is easily generalized to the multivariate case and we get:

$$
\begin{aligned}
\lambda_{t,t+1} &= \mu_t + \tfrac{1}{2}\, vdiag\, \Sigma_t - r_{t+1}\, e \\
&= \mu_t - \mu_t^{\mathbb{Q}}(\alpha_{2,t}) + \tfrac{1}{2}\, vdiag\, (\Sigma_t - \Sigma_t^{\mathbb{Q}}(\alpha_{2,t})) \\
&= \lambda_{t,t+1}^F + \left[\mu_t^{\mathbb{Q}}(0) - \mu_t^{\mathbb{Q}}(\alpha_{2,t})\right] + \tfrac{1}{2}\, vdiag\, \left[\Sigma_t - (\Sigma_t^{\mathbb{Q}}(\alpha_{2,t}))\right] ,
\end{aligned}
\tag{35}
$$

with $\lambda_{t,t+1}^F := (\mu_t - \mu_t^{\mathbb{Q}}(0)) = -\,\Sigma_t\,\alpha_{1,t}$ denoting now the n-dimensional (first-order) risk premium we have when $\alpha_{2,t} = 0$.

It is also relevant to observe from relation (33) that, when considering the particular static case ($r_{t+1} = r, \sigma_t = \sigma, \alpha_{2,t} = \alpha_2$), we immediately find a (discrete-time) generalization of the Black and Scholes (1973) setting and an associated European Call option pricing formula $C_{BS}(t, h; K, S_t, r, \sigma^2)$ (say), where K is the strike price and h denotes the residual maturity. Indeed, the Gaussian stock return risk-neutral dynamics, namely $IIN\left[r - (\sigma^{\mathbb{Q}})^2(\alpha_2)/2, (\sigma^{\mathbb{Q}})^2(\alpha_2)\right]$, immediately delivers the following explicit Second-Order Black and Scholes pricing formula (for European Call options):

$$C_{BS}^{(S)}(t, h; K, S_t, r, \sigma^2, \alpha_2) = C_{BS}(t, h; K, S_t, r, (\sigma^{\mathbb{Q}})^2(\alpha_2)) \,, \tag{36}$$

in which α_2 is an additional degree of freedom with respect to the classical Black and Scholes one ($\alpha_2 = 0$ implies $C_{BS}^{(S)}(t, h; K, S_t, r, \sigma^2, 0) = C_{BS}(t, h; K, S_t, r, \sigma^2)$). Moreover, this source of flexibility can be further exploited by specifying $\alpha_{2,t}$ as a deterministic function of time, still leading to an explicit pricing formula.

It is also clear that we can easily propose, in a dynamic setting, richer Call option pricing formulas once we assume σ_t^2 and $\alpha_{2,t}$ functions of the date t information. In that case, the pricing formula has no longer a closed form but it can be easily determined by simulation for any residual maturity h.

4.3 Back Modelling

Let us maintain the conditionally Gaussian setting of the previous section, but let us now adopt the Back Modelling strategy of Bertholon, Monfort and Pegoraro (2008). More precisely, we assume that the R.N. dynamics ($\mathbb{Q}_{t+1}$) of w_{t+1} is given by:

$$w_{t+1} | \, \underline{w_t} \overset{\mathbb{Q}}{\sim} N\left(\mu_t^{\mathbb{Q}}, \Sigma_t^{\mathbb{Q}}\right) \,, \tag{37}$$

with the associated conditional second-order Log-Laplace transform

$$\begin{aligned} \psi_{S,t}^{\mathbb{Q}}(u_1, u_2) &= -\tfrac{1}{2}\log\det\left(I - 2\Sigma_t^{\mathbb{Q}} u_2\right) - \tfrac{1}{2}\mu_t^{\mathbb{Q}\prime}(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} \\ &\quad + \tfrac{1}{2}[(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} + u_1]'[(\Sigma_t^{\mathbb{Q}})^{-1} - 2u_2]^{-1}[(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} + u_1] \,, \end{aligned} \tag{38}$$

and we impose the ICC $\psi_{S,t}^{\mathbb{Q}}(e_i, 0) = r_{t+1}$ for all $i \in \{1, \ldots, n\}$, that is:

$$\begin{aligned} r_{t+1} &= -\tfrac{1}{2}\mu_t^{\mathbb{Q}\prime}(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} + \tfrac{1}{2}[(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} + e_i]'\Sigma_t^{\mathbb{Q}}[(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} + e_i] \\ &= \tfrac{1}{2}e_i'\Sigma_t^{\mathbb{Q}}e_i + e_i'\mu_t^{\mathbb{Q}} \quad \forall\, i \in \{1, \ldots, n\} \,. \end{aligned} \tag{39}$$

From (39) we have $\mu_t^{\mathbb{Q}} = r_{t+1}e - \frac{1}{2}vdiag\Sigma_t^{\mathbb{Q}}$ and, therefore, we find the no-arbitrage risk-neutral dynamics:

$$N\left[r_{t+1}e - \tfrac{1}{2}\, vdiag\, \Sigma_t^{\mathbb{Q}}, \Sigma_t^{\mathbb{Q}}\right] \,. \tag{40}$$

The associated historical dynamics $\mathbb{P}_{t+1}$ is given, for *any* $(\alpha_{1,t}, \alpha_{2,t})$, by $\mathbb{P}_{t+1} = S_{(-\alpha_{1,t}, -\alpha_{2,t})}(\mathbb{Q}_{t+1})$ and we have:

$$w_{t+1} | \, \underline{w_t} \overset{\mathbb{P}}{\sim} N\left[(I + 2\Sigma_t^{\mathbb{Q}}\alpha_{2,t})^{-1}(r_{t+1}e - \tfrac{1}{2}\, vdiag\, \Sigma_t^{\mathbb{Q}} - \Sigma_t^{\mathbb{Q}}\alpha_{1,t}), ((\Sigma_t^{\mathbb{Q}})^{-1} + 2\alpha_{2,t})^{-1}\right] \,. \tag{41}$$

So, for any given R.N. dynamics, the historical dynamics is also conditionally Gaussian and any conditional mean and any conditional variance-covariance matrix can be reached[5].

[5]It is important to highlight that our class of Gaussian security market models can be easily extended to a general

5 Conditionally Mixed-Normal Economies

The purpose of this section is to extend the results of the previous section, based on a Gaussian distributed n-dimensional factor, to the case of a finite mixture of conditionally multivariate Gaussian processes. We first follow the Direct Modelling strategy and then the Back Modelling one.

5.1 Direct Modelling

Let us assume that the historical p.d.f. of w_{t+1}, conditionally to $\underline{w_t}$, is:

$$f_t(w_{t+1}|\underline{w_t}) = \sum_{j=1}^{J} \lambda_{j,t}\, n(w_{t+1}|\underline{w_t};\mu_{j,t},\Sigma_{j,t}) \,,$$

$$\text{with } n(w_{t+1}|\underline{w_t};\mu_{j,t},\Sigma_{j,t}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma_{j,t}}} \exp\left[-\frac{1}{2}(w_{t+1}-\mu_{j,t})'\Sigma_{j,t}^{-1}(w_{t+1}-\mu_{j,t})\right] \,, \tag{42}$$

where $\lambda_{j,t}$ is a function of $\underline{w_t}$ satisfying:

$$0 \le \lambda_{j,t} \le 1,\ \sum_{j=1}^{J} \lambda_{j,t} = 1 \,. \tag{43}$$

This family of historical dynamics contains all univariate AR, ARMA and GARCH models with mixed-normal distributed innovations able, therefore, to completely span the skewness-kurtosis domain of maximal size [see Bertholon, Monfort and Pegoraro (2006) for a formal proof] as well as their multivariate analogues. The conditional second-order historical Laplace transform is given by:

$$\begin{aligned}
\varphi_{S,t}(u_1,u_2) &= \sum_{j=1}^{J} \lambda_{j,t} \int \exp(u_1' w_{t+1} + w_{t+1}' u_2 w_{t+1}) n(w_{t+1}|\underline{w_t};\mu_{j,t},\Sigma_{j,t}) dw_{t+1} \\
&= \sum_{j=1}^{J} \lambda_{j,t}\varphi_{j,S,t}(u_1,u_2) \\
\text{with } \varphi_{j,S,t}(u_1,u_2) &= \exp\Big[-\tfrac{1}{2}\log\det(I-2\Sigma_{j,t}u_2) - \tfrac{1}{2}\mu_{j,t}'\Sigma_{j,t}^{-1}\mu_{j,t} \\
&\qquad + \frac{1}{2}(\Sigma_{j,t}^{-1}\mu_{j,t}+u_1)'(\Sigma_{j,t}^{-1}-2u_2)^{-1}(\Sigma_{j,t}^{-1}\mu_{j,t}+u_1)\Big]
\end{aligned} \tag{44}$$

and given the exponential-quadratic (in the factor w_{t+1}) stochastic discount factor:

$$M_{t,t+1}^{(S)} = \exp\left[-r_{t+1} + \alpha_{1,t}' w_{t+1} + w_{t+1}'\alpha_{2,t} w_{t+1} - \psi_{S,t}(\alpha_{1,t},\alpha_{2,t})\right] \,,$$

the ICC associated with the i^{th} geometric return $w_{i,t+1}$ is given by:

$$\varphi_{S,t}(\alpha_{1,t}+e_i,\alpha_{2,t}) = \exp(r_{t+1})\varphi_{S,t}(\alpha_{1,t},\alpha_{2,t}) \,,\ \ i \in \{1,\ldots,n\} \,. \tag{45}$$

class of Gaussian asset pricing models, in which the risk driven by any (observable and/or latent) Gaussian-distributed factor w_{t+1} is priced by means of an exponential-quadratic SDF. In the case of interest rate models, for instance, it could be relevant to study the relationships with the class of affine-quadratic yield curve models proposed by Chen, Filipovic and Poor (2004), Cheng and Scaillet (2007), Gourieroux, Jasiak and Sufana (2009), Gourieroux and Sufana (2003) and Leippold and Wu (2002, 2003). This is the object of ongoing and future research works.

Thus, the risk-neutral dynamics $\mathbb{Q}_{t+1}$ of w_{t+1}, generated by the conditional Second-Order Esscher transform of $\mathbb{P}_{t+1}$ associated with $(\alpha_{1,t}, \alpha_{2,t})$ $[\mathbb{Q}_{t+1} = S_{(\alpha_{1,t},\alpha_{2,t})}(\mathbb{P}_{t+1})]$, is characterized by the following p.d.f.:

$$
\begin{aligned}
&f_t^{\mathbb{Q}}(w_{t+1}|\underline{w_t}) = \textstyle\sum_{j=1}^{J} \lambda_{j,t}^*(\alpha_{1,t}, \alpha_{2,t})\, n \left[w_{t+1}|\underline{w_t}\,; (\Sigma_{j,t}^{-1} - 2\alpha_{2,t})^{-1}(\Sigma_{j,t}^{-1}\mu_{j,t} + \alpha_{1,t}), (\Sigma_{j,t}^{-1} - 2\alpha_{2,t})^{-1}\right] \,, \\
&\text{with } \lambda_{j,t}^*(\alpha_{1,t}, \alpha_{2,t}) = \frac{\lambda_{j,t}\varphi_{j,S,t}(\alpha_{1,t}, \alpha_{2,t})}{\sum_{j=1}^{J} \lambda_{j,t}\varphi_{j,S,t}(\alpha_{1,t}, \alpha_{2,t})} \,, \\
&0 \leq \lambda_{j,t}^*(\alpha_{1,t}, \alpha_{2,t}) \leq 1,\ \textstyle\sum_{j=1}^{J} \lambda_{j,t}^*(\alpha_{1,t}, \alpha_{2,t}) = 1 \,.
\end{aligned}
\tag{46}
$$

and where $(\alpha_{1,t}, \alpha_{2,t})$ satisfy (45). We observe that, as indicated in example *viii*), the exponential-quadratic change of probability measure, applied to (42), modifies not only each conditional mean but also each conditional variance-covariance matrix of the Gaussian components. Moreover, the risk-neutral mixing weights $\lambda_{j,t}^*(\alpha_{1,t}, \alpha_{2,t})$ are different from the historical ones $(\lambda_{j,t})$.

5.2 Back Modelling

Let us follow now the Back Modelling strategy and let us assume, first, a risk-neutral dynamics $\mathbb{Q}_{t+1}$ described by the following p.d.f.:

$$
\begin{aligned}
&f_t^{\mathbb{Q}}(w_{t+1}|\underline{w_t}) = \textstyle\sum_{j=1}^{J} \lambda_{j,t}^{\mathbb{Q}}\, n \left(w_{t+1}|\underline{w_t}\,; \mu_{jt}^{\mathbb{Q}}, \Sigma_{j,t}^{\mathbb{Q}}\right) \,, \\
&\text{with } 0 \leq \lambda_{j,t}^{\mathbb{Q}} \leq 1,\ \textstyle\sum_{j=1}^{J} \lambda_{j,t}^{\mathbb{Q}} = 1 \,.
\end{aligned}
\tag{47}
$$

The conditional second-order risk-neutral Laplace transform is given by:

$$
\begin{aligned}
\varphi_{S,t}^{\mathbb{Q}}(u_1, u_2) &= \sum_{j=1}^{J} \lambda_{j,t}^{\mathbb{Q}}\varphi_{j,S,t}^{\mathbb{Q}}(u_1, u_2) \\
&= \sum_{j=1}^{J} \lambda_{j,t}^{\mathbb{Q}} \exp\left[-\frac{1}{2}\log\det(I - 2\Sigma_{j,t}^{\mathbb{Q}}u_2) - \frac{1}{2}\mu_{j,t}^{\mathbb{Q}}{}'(\Sigma_{j,t}^{\mathbb{Q}})^{-1}\mu_{j,t}^{\mathbb{Q}}\right. \\
&\qquad \left. + \frac{1}{2}((\Sigma_{j,t}^{\mathbb{Q}})^{-1}\mu_{j,t}^{\mathbb{Q}} + u_1)'((\Sigma_{j,t}^{\mathbb{Q}})^{-1} - 2u_2)^{-1}((\Sigma_{j,t}^{\mathbb{Q}})^{-1}\mu_{j,t}^{\mathbb{Q}} + u_1)\right]
\end{aligned}
\tag{48}
$$

and the internal consistency conditions are:

$$
\begin{aligned}
\exp(r_{t+1}) &= \varphi_{S,t}^{\mathbb{Q}}(e_i, 0) \\
&= \sum_{j=1}^{J} \lambda_{j,t}^{\mathbb{Q}} \exp\left[-\frac{1}{2}\mu_{j,t}^{\mathbb{Q}}{}'(\Sigma_{j,t}^{\mathbb{Q}})^{-1}\mu_{j,t}^{\mathbb{Q}} + \frac{1}{2}((\Sigma_{j,t}^{\mathbb{Q}})^{-1}\mu_{j,t}^{\mathbb{Q}} + e_i)'\Sigma_{j,t}^{\mathbb{Q}}((\Sigma_{j,t}^{\mathbb{Q}})^{-1}\mu_{j,t}^{\mathbb{Q}} + e_i)\right] \\
&= \sum_{j=1}^{J} \lambda_{j,t}^{\mathbb{Q}} \exp\left[\frac{1}{2}e_i'\Sigma_{j,t}^{\mathbb{Q}}e_i + e_i'\mu_{j,t}^{\mathbb{Q}}\right] \,,\ \ i \in \{1, \ldots, n\} \,.
\end{aligned}
\tag{49}
$$

In the univariate case, restriction (49) reduces to:

$$1 \quad = \quad \sum_{j=1}^{J} \lambda_{j,t}^{\mathbb{Q}} \exp\left(\frac{1}{2}(\sigma_{j,t}^{\mathbb{Q}})^2 + \mu_{j,t}^{\mathbb{Q}} - r_{t+1}\right) \,, \quad i \in \{1, \dots, n\} \,.$$

The historical factor dynamics $\mathbb{P}_{t+1}$, given by $\mathbb{P}_{t+1} = S_{(-\alpha_{1,t},-\alpha_{2,t})}(\mathbb{Q}_{t+1})$, is again for any $(\alpha_{1,t}, \alpha_{2,t})$ a finite mixture of conditionally multivariate Gaussian processes with p.d.f. given by:

$$\begin{aligned} & f_t(w_{t+1}|\underline{w_t}) = \textstyle\sum_{j=1}^{J} \lambda_{j,t}(-\alpha_{1,t}, -\alpha_{2,t}) \times \\ & \qquad n\left[w_{t+1}|\underline{w_t}\,; (I + 2\,\Sigma_{j,t}^{\mathbb{Q}}\,\alpha_{2,t})^{-1}(\mu_{j,t}^{\mathbb{Q}} - \Sigma_{j,t}^{\mathbb{Q}}\alpha_{1,t}), ((\Sigma_{j,t}^{\mathbb{Q}})^{-1} + 2\alpha_{2,t})^{-1}\right] \,, \\ & \text{with } \; \lambda_{j,t}(-\alpha_{1,t}, -\alpha_{2,t}) = \frac{\lambda_{j,t}^{\mathbb{Q}}\varphi_{j,S,t}^{\mathbb{Q}}(-\alpha_{1,t}, -\alpha_{2,t})}{\sum_{j=1}^{J}\lambda_{j,t}^{\mathbb{Q}}\varphi_{j,S,t}^{\mathbb{Q}}(-\alpha_{1,t}, -\alpha_{2,t})} \,, \\ & 0 \le \lambda_{j,t}(-\alpha_{1,t}, -\alpha_{2,t}) \le 1, \; \textstyle\sum_{j=1}^{J}\lambda_{j,t}(-\alpha_{1,t}, -\alpha_{2,t}) = 1 \,, \end{aligned} \tag{50}$$

and again each Gaussian p.d.f. has a conditional mean and conditional variance-covariance matrix which are different from the R.N. ones.

6 Conditionally Gaussian Switching Regime Economies

6.1 Conditional Second-Order Esscher Transform of a General Conditionally Gaussian Switching Regime Process

Let us consider the $(J+1)$-dimensional factor $w_{t+1} = (y_{t+1}, z'_{t+1})'$, where y_{t+1} is a scalar geometric return between t and $t+1$ and z_{t+1} is a J-state variable valued in $\mathcal{E} = \{e_1, \dots, e_J\}$, where e_j is the j^{th} column of a $(J \times J)$ identity matrix (the generalization to a vector of returns is straightforward). We assume that the historical dynamics of w_{t+1} is described by the following general regime-switching model:

$$\begin{aligned} & y_{t+1} = \mu_t(\underline{y_t}, z_t, z_{t+1}) + \sigma_t(\underline{y_t}, z_t, z_{t+1})\varepsilon_{t+1} \\ & \varepsilon_{t+1}|z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{P}}{\sim} N(0,1) \\ & \mathbb{P}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi_{i,j}(\underline{y_t}) = \pi_{i,j,t} \quad \forall (e_i, e_j) \in \mathcal{E} \times \mathcal{E} \,. \end{aligned} \tag{51}$$

This family contains, for instance, the regime-switching ARCH and GARCH specifications proposed, respectively, by Hamilton and Susmel (1994), Gray (1996), Klaassen (2002) and Hass, Mittnik and Paolella (2004). The historical distribution $\mathbb{P}_{i,t+1}$ (say) of $(y_{t+1}, z'_{t+1})'$, conditionally to $\underline{y_t}$ and $z_t = e_i$, has a p.d.f. given by:

$$f_t(y_{t+1}, e_j|\underline{y_t}, z_t = e_i) \quad = \quad n\left[y_{t+1}; \mu_t(\underline{y_t}, e_i, e_j), \sigma_t^2(\underline{y_t}, e_i, e_j)\right] \pi_{i,j,t} \,. \tag{52}$$

Now, let us determine the second-order Laplace transform of $w_{t+1} = (y_{t+1}, z'_{t+1})'$, conditionally to $(\underline{y_t}, \underline{z_t})$. By definition, we have:

$$\varphi_{S,t}(\widetilde{u}_1, \widetilde{u}_2) \quad = \quad E\left[\exp\left(\widetilde{u}_1' w_{t+1} + w_{t+1}'\widetilde{u}_2 w_{t+1}\right) \,|\, \underline{w_t}\right] \,, \tag{53}$$

where $\widetilde{u}_1 \in \mathbb{R}^{J+1}$ and $\widetilde{u}_2$ is a $(J+1)\times(J+1)$ symmetric matrix. However, given the very specific range of z_{t+1}, i.e. $\mathcal{E} = \{e_1, \ldots, e_J\}$, the parametrization $\widetilde{u}_1$ and $\widetilde{u}_2$ is redundant. First, for any $(J \times J)$ matrix a quadratic term of the type $z'_{t+1} A z_{t+1}$ is linear in z_{t+1} (and equal to $vdiag(A)' z_{t+1}$) and, therefore, we can ignore the quadratic term in z_{t+1} included in $w'_{t+1}\widetilde{u}_2 w_{t+1}$. Second, any linear term of the form $a y_{t+1}$ can be incorporated into a cross-product term of the form $b' z_{t+1} y_{t+1}$ and, therefore, we can ignore the linear term in y_{t+1} included in $\widetilde{u}'_1 w_{t+1}$. Finally, the second-order Laplace transform of $w_{t+1} = (y_{t+1}, z'_{t+1})'$ is:

$$\varphi_{S,t}(u_1, u_2) \quad = \quad E\left[\exp\left(u'_1 z_{t+1} + u'_{2,1} z_{t+1} y_{t+1} + u_{2,2} y^2_{t+1}\right) \mid \underline{y_t}, \underline{z_t}\right] , \tag{54}$$

where $u_2 = (u'_{2,1}, u'_{2,2})'$. Using the notation $\mu_{i,j,t} = \mu_t(\underline{y_t}, e_i, e_j)$ and $\sigma_{i,j,t} = \sigma_t(\underline{y_t}, e_i, e_j)$, we obtain from (54):

$$\varphi_{S,t}\left(u_1, u_2\right) = \widetilde{\varphi}_{S,t}\left(u_1, u_2\right)' z_t ,$$

$$\text{with} \quad \widetilde{\varphi}_{S,t}\left(u_1, u_2\right) = \left[\widetilde{\varphi}_{S,t,1}\left(u_1, u_2\right), \ldots, \widetilde{\varphi}_{S,t,J}\left(u_1, u_2\right)\right]' ,$$

$$\begin{aligned} \text{and} \quad \widetilde{\varphi}_{S,t,i}\left(u_1, u_2\right) &= E\left[\exp\left(u'_1 z_{t+1} + u'_{2,1} z_{t+1} y_{t+1} + u_{2,2} y^2_{t+1}\right) \mid z_t = e_i, \underline{z_{t-1}}, \underline{y_t}\right] \\ &= \sum_{j=1}^{J} \pi_{i,j}(\underline{y_t}) \exp(u'_1 e_j) \widetilde{\varphi}_{S,t,i,j}\left(u'_{2,1} e_j, u_{2,2}\right) , \end{aligned} \tag{55}$$

and where :

$$\begin{aligned} \widetilde{\varphi}_{S,t,i,j}\left(u'_{2,1} e_j, u_{2,2}\right) &= E\left[\exp\left(u_{2,2} y^2_{t+1} + u'_{2,1} e_j y_{t+1}\right) \mid z_{t+1} = e_j, z_t = e_i, \underline{z_{t-1}}, \underline{y_t}\right] \\ &= \int_{\mathbb{R}} n(y_{t+1}; \mu_{i,j,t}, \sigma^2_{i,j,t}) \exp\left[(u'_{2,1} e_j) y_{t+1} + u_{2,2} y^2_{t+1}\right] dy_{t+1} . \end{aligned} \tag{56}$$

The p.d.f. of the conditional Second-Order Esscher transform $S_{\theta_1,\theta_2}(\mathbb{P}_{i,t+1})$ is obtained, first, by multiplying the p.d.f. (52) by $\exp(\theta'_1 z_{t+1} + \theta'_{2,1} z_{t+1} y_{t+1} + \theta_{2,2} y^2_{t+1})$ and then, this product is normalized by $\varphi_{S,t}(\theta_1, \theta_2)$. So, we obtain the following result:

Proposition 3 : The p.d.f. of the family of probability distributions $\mathbb{P}^*_{i,t+1}$ (say) generated by the conditional Second-Order Esscher transform $S_{(\theta_1,\theta_2)}(\mathbb{P}_{i,t+1})$ applied to the p.d.f. (52) is given by:

$$\begin{aligned} & g_t(y_{t+1}, e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) \\ = \quad & \frac{\pi_{i,j,t} \exp\left(\theta'_1 e_j + \theta'_{2,1} e_j y_{t+1} + \theta_{2,2} y^2_{t+1}\right) n(y_{t+1}; \mu_{i,j,t}, \sigma^2_{i,j,t})}{\varphi_{S,t}(\theta_1, \theta_2)} \\ = \quad & \pi^*_{i,j,t}\, n\left(y_{t+1}; \frac{\mu_{i,j,t} + \sigma^2_{i,j,t} \theta'_{2,1} e_j}{1 - 2\sigma^2_{i,j,t}\theta_{2,2}}, \frac{\sigma^2_{i,j,t}}{1 - 2\sigma^2_{i,j,t}\theta_{2,2}}\right) , \end{aligned} \tag{57}$$

where

$$\pi^*_{i,j,t} = \pi^*_{i,j}(\underline{y_t}) = \frac{\pi_{i,j,t}\exp(\theta_1' e_j)\widetilde{\varphi}_{S,t,i,j}(\theta_{2,1}' e_j, \theta_{2,2})}{\sum_{j=1}^{J} \pi_{i,j,t}\exp(\theta_1' e_j)\widetilde{\varphi}_{S,t,i,j}(\theta_{2,1}' e_j, \theta_{2,2})},$$

$$\text{with} \quad \widetilde{\varphi}_{S,t,i,j}(u_1, u_2) = \exp\left[-\frac{1}{2}\log\left(1 - 2\sigma^2_{i,j,t}u_1\right) - \frac{1}{2}\frac{\mu^2_{i,j,t}}{\sigma^2_{i,j,t}} + \frac{1}{2}\frac{(\mu_{i,j,t} + \sigma^2_{i,j,t}u_1)^2}{(\sigma^2_{i,j,t} - 2\sigma^4_{i,j,t}u_2)}\right].$$

[Proof : see Appendix 2].

From Proposition 3 we see that the joint $\mathbb{P}^*$-distribution of $(y_{t+1}, z_{t+1}')'$, conditionally to $\underline{y_t}, \underline{z_t}$, is:

$$y_{t+1} = \mu^*_t(\underline{y_t}, z_t, z_{t+1}) + \sigma^*_t(\underline{y_t}, z_t, z_{t+1})\xi_{t+1}, \tag{58}$$

where $\mu^*_t(\underline{y_t}, e_i, e_j)$ and $\sigma^*_t(\underline{y_t}, e_i, e_j)$ are respectively given by:

$$\mu^*_{i,j,t} = \frac{\mu_{i,j,t} + \sigma^2_{i,j,t}\theta_{2,1}' e_j}{1 - 2\sigma^2_{i,j,t}\theta_{2,2}} \quad \text{and} \quad \sigma^*_{i,j,t} = \left(\frac{\sigma^2_{i,j,t}}{1 - 2\sigma^2_{i,j,t}\theta_{2,2}}\right)^{1/2}, \tag{59}$$

and where:

$$\begin{aligned} &\mathbb{P}^*(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi^*_{i,j}(\underline{y_t}) = \pi^*_{i,j,t}, \quad \forall (e_i, e_j) \in \mathcal{E} \times \mathcal{E} \\ &\xi_{t+1} | z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{P}^*}{\sim} N(0,1). \end{aligned} \tag{60}$$

6.2 Asset Pricing Modelling Strategies

6.2.1 Direct Modelling

The purpose of this section and the following one is to deal with the specification of a security market model when the investor's information at each date t is the $(J+1)$-dimensional factor $w_t = (y_t, z_t')'$ introduced in the previous section, and when the SDF $M^{(S)}_{t,t+1}$ is an exponential-quadratic function of the stock return y_{t+1} and of the Markov chain z_{t+1}, that is:

$$M^{(S)}_{t,t+1} \;=\; \exp\left[-r_{t+1} + \alpha_{1,t}' z_{t+1} + \alpha_{2,1,t}' z_{t+1} y_{t+1} + \alpha_{2,2,t} y^2_{t+1} - \widetilde{\psi}_{S,t}(\alpha_{1,t}, \alpha_{2,t})' z_t\right], \tag{61}$$

where $\widetilde{\psi}_{S,t}(\alpha_{1,t}, \alpha_{2,t}) = \log \widetilde{\varphi}_{S,t}(\alpha_{1,t}, \alpha_{2,t})$, and denoting $\alpha_{2,t} = (\alpha_{2,1,t}', \alpha_{2,2,t})'$. This means that a conditional Second-Order Esscher transform is used to move from the historical to the risk-neutral world and vice versa.

It is important to highlight that this asset pricing setting provides two important generalizations with respect to the model presented in Section 5: *i*) the regime indicator function z_{t+1} is a Markov chain and not an i.i.d. process and *ii*) z_{t+1} is introduced in $M^{(S)}_{t,t+1}$ and thus regime-shift risk is priced (via $\alpha_{1,t}$ and $\alpha_{2,1,t}$). Note that this second generalization is also easily introduced in the Mixed-Normal economy by simply assuming a SDF as in (61) (instead of the pricing kernel (20) with $w_{t+1} = y_{t+1}$) in which z_{t+1} is such that for all $i \in \{1, \dots, J\}$, and for any given $j \in \{1, \dots, J\}$, $\pi_{ij,t} = \pi_{j,t}$. In this section we consider the case where the Direct Modelling strategy is adopted, while in the following section we will derive the model on the basis of the Back Modelling strategy.

First, the historical dynamics ($\mathbb{P}_{t+1}$) of $(y_{t+1}, z'_{t+1})'$, conditionally to $(\underline{y_t}, \underline{z_t})$, is given by (51) and, second, the SDF is assumed to be (61). The ICC requires the following constraint on model parameters and risk-sensitivity vectors:

$$\left[\widetilde{\psi}_{S,t}(\alpha_{1,t}, \alpha_{2,1,t} + e, \alpha_{2,2,t}) - \widetilde{\psi}_{S,t}(\alpha_{1,t}, \alpha_{2,1,t}, \alpha_{2,2,t})\right]' z_t = r_{t+1}\,, \quad \forall(\underline{y_t}, \underline{z_t})\,, \tag{62}$$

where e is the J-dimensional vector whose components are equal to 1. The R.N. dynamics ($\mathbb{Q}_{t+1}$) of $(y_{t+1}, z'_{t+1})'$, conditionally to $(\underline{y_t}, \underline{z_t})$, is defined by the family of probability distributions generated by the conditional Second-Order Esscher transform of $\mathbb{P}_{t+1}$ associated with $(\alpha_{1,t}, \alpha_{2,t})'$ [$\mathbb{Q}_{t+1} = S_{(\alpha_{1,t},\alpha_{2,t})}(\mathbb{P}_{t+1})$], and it is given by:

$$\begin{aligned}
& y_{t+1} = \mu_t^{\mathbb{Q}}(\underline{y_t}, z_t, z_{t+1}) + \sigma_t^{\mathbb{Q}}(\underline{y_t}, z_t, z_{t+1})\xi_{t+1}\,, \\
& \text{with } \ \mu_t^{\mathbb{Q}}(\underline{y_t}, e_i, e_j) = \frac{\mu_{i,j,t} + \sigma^2_{i,j,t}\alpha'_{2,1,t}e_j}{1 - 2\sigma^2_{i,j,t}\alpha_{2,2,t}}\,, \quad \sigma_t^{\mathbb{Q}}(\underline{y_t}, e_i, e_j) = \left(\frac{\sigma^2_{i,j,t}}{1 - 2\sigma^2_{i,j,t}\alpha_{2,2,t}}\right)^{1/2}\,, \\
& \qquad \xi_{t+1} | z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{Q}}{\sim} N(0,1)\,, \\
& \qquad \mathbb{Q}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi^{\mathbb{Q}}_{i,j}(\underline{y_t}) = \pi^{\mathbb{Q}}_{i,j,t}\,, \ \ \forall(e_i, e_j) \in \mathcal{E} \times \mathcal{E}
\end{aligned} \tag{63}$$

and where

$$\pi^{\mathbb{Q}}_{i,j,t} = \frac{\pi_{i,j,t}\exp(\alpha'_{1,t}e_j)\widetilde{\varphi}_{S,t,i,j}(\alpha'_{2,1,t}e_j, \alpha_{2,t})}{\sum_{j=1}^{J}\pi_{i,j,t}\exp(\alpha'_{1,t}e_j)\widetilde{\varphi}_{S,t,i,j}(\alpha'_{2,1,t}e_j, \alpha_{2,t})}\,. \tag{64}$$

6.2.2 Back Modelling

Following the Back Modelling strategy, we first assume that the R.N. dynamics ($\mathbb{Q}_{t+1}$) of $(y_{t+1}, z'_{t+1})'$, conditionally to $(\underline{y_t}, \underline{z_t})$, be given by:

$$\begin{aligned}
& y_{t+1} = \mu_t^{\mathbb{Q}}(\underline{y_t}, z_t, z_{t+1}) + \sigma_t^{\mathbb{Q}}(\underline{y_t}, z_t, z_{t+1})\xi_{t+1}\,, \\
& \mathbb{Q}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi^{\mathbb{Q}}_{i,j}(\underline{y_t}) = \pi^{\mathbb{Q}}_{i,j,t}\,, \ \ \forall(e_i, e_j) \in \mathcal{E} \times \mathcal{E} \\
& \xi_{t+1} | z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{Q}}{\sim} N(0,1)\,.
\end{aligned} \tag{65}$$

Second, we impose the ICC

$$\widetilde{\psi}^{\mathbb{Q}}_{S,t}(0, e, 0)' z_t = r_{t+1}\,, \tag{66}$$

where $\widetilde{\psi}^{\mathbb{Q}}_{S,t}(u_1, u_{2,1}, u_{2,2}) = \log \widetilde{\varphi}^{\mathbb{Q}}_{S,t}(u_1, u_{2,1}, u_{2,2})$. Once risk-neutral parameters are constrained in order to satisfy (66), we can apply the change of probability measure associated with the exponential-quadratic SDF (61), with the important difference (with respect to the Direct Modelling) that now the risk-sensitivity vectors $(\alpha'_{1,t}, \alpha'_{2,1,t}, \alpha_{2,2,t})'$ can be specified, without any constraint, as any non-linear function of the information.

The historical dynamics $(\mathbb{P}_{t+1})$ of $(y_{t+1}, z'_{t+1})'$, conditionally to $(\underline{y_t}, \underline{z_t})$, is the family of probability distributions generated by the conditional Second-Order Esscher transform of $\mathbb{Q}_{t+1}$ associated with $(-\alpha'_{1,t}, -\alpha'_{2,t})'$ $[\mathbb{P}_{t+1} = S_{(-\alpha_{1,t}, -\alpha_{2,t})}(\mathbb{Q}_{t+1})]$, and it is given by:

$$
\begin{aligned}
& y_{t+1} = \mu_t(\underline{y_t}, z_t, z_{t+1}) + \sigma_t(\underline{y_t}, z_t, z_{t+1})\varepsilon_{t+1} \\
& \text{with } \mu_t(\underline{y_t}, e_i, e_j) = \frac{\mu^{\mathbb{Q}}_{i,j,t} - (\sigma^{\mathbb{Q}}_{i,j,t})^2\,\alpha'_{2,1,t}e_j}{1 + 2(\sigma^{\mathbb{Q}}_{i,j,t})^2\,\alpha_{2,2,t}}\,,\quad \sigma_t(\underline{y_t}, e_i, e_j) = \left(\frac{(\sigma^{\mathbb{Q}}_{i,j,t})^2}{1 + 2(\sigma^{\mathbb{Q}}_{i,j,t})^2\,\alpha_{2,2,t}}\right)^{1/2}, \\
& \qquad \varepsilon_{t+1}|z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{P}}{\sim} N(0,1)\,, \\
& \qquad \mathbb{P}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi_{i,j}(\underline{y_t}) = \pi_{i,j,t} \quad \forall (e_i, e_j) \in \mathcal{E} \times \mathcal{E}
\end{aligned}
\tag{67}
$$

and where

$$
\pi_{i,j,t} = \frac{\pi^{\mathbb{Q}}_{i,j,t} \exp(-\alpha'_{1,t}e_j)\widetilde{\varphi}^{\mathbb{Q}}_{S,t,i,j}(-\alpha'_{2,1,t}e_j, -\alpha_{2,t})}{\sum_{j=1}^{J} \pi^{\mathbb{Q}}_{i,j,t} \exp(-\alpha'_{1,t}e_j)\widetilde{\varphi}^{\mathbb{Q}}_{S,t,i,j}(-\alpha'_{2,1,t}e_j, -\alpha_{2,t})}\,. \tag{68}
$$

6.3 The Additive Regime Switching Economy

6.3.1 The Conditionally Gaussian Additive Regime Switching Model

The general regime-switching historical dynamics of $w_{t+1} = (y_{t+1}, z'_{t+1})'$, introduced in Section 6.1, was given by:

$$
\begin{aligned}
& y_{t+1} = \mu_t(\underline{y_t}, z_t, z_{t+1}) + \sigma_t(\underline{y_t}, z_t, z_{t+1})\varepsilon_{t+1} \\
& \varepsilon_{t+1}|z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{P}}{\sim} N(0,1) \\
& \mathbb{P}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi_{i,j}(\underline{y_t}) = \pi_{i,j,t} \quad \forall (e_i, e_j) \in \mathcal{E} \times \mathcal{E}\,.
\end{aligned}
\tag{69}
$$

Now, let us assume that $\mu_t(\underline{y_t}, z_t, z_{t+1})$ and $\sigma^2_t(\underline{y_t}, z_t, z_{t+1})$ are *additive* in z_t and z_{t+1}, i.e., they are of the form:

$$
\begin{cases}
\mu_t(\underline{y_t}, z_t, z_{t+1}) = \mu_{0,t} + \mu'_{1,t}z_t + \mu'_{2,t}z_{t+1}\,, \\
\sigma^2_t(\underline{y_t}, z_t, z_{t+1}) = \sigma^2_{0,t} + {\sigma^2_{1,t}}'z_t + {\sigma^2_{2,t}}'z_{t+1}\,,
\end{cases}
\tag{70}
$$

where $\mu_{i,t}$ and $\sigma^2_{i,t}$ may be functions of $\underline{y_t}$, for all $i \in \{0,1,2\}$. With obvious notation we can also write:

$$
\begin{cases}
\mu_t(\underline{y_t}, e_i, e_j) = \mu_{0,t} + \mu_{1,t,i} + \mu_{2,t,j}\,, \\
\sigma^2_t(\underline{y_t}, e_i, e_j) = \sigma^2_{0,t} + \sigma^2_{1,t,i} + \sigma^2_{2,t,j}\,.
\end{cases}
\tag{71}
$$

Applying the general result of Section 6.1, we immediately find that the joint $\mathbb{P}^*$-distribution of $(y_{t+1}, z'_{t+1})'$, conditionally to $\underline{y_t}, \underline{z_t}$, is:

$$\begin{aligned} & y_{t+1} = \mu^*_t(\underline{y_t}, z_t, z_{t+1}) + \sigma^*_t(\underline{y_t}, z_t, z_{t+1})\xi_{t+1} , \\ & \mathbb{P}^*(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi^*_{i,j}(\underline{y_t}) = \pi^*_{i,j,t} \quad \forall (e_i, e_j) \in \mathcal{E} \times \mathcal{E} , \\ & \varepsilon_{t+1} | z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{P}^*}{\sim} N(0,1) , \end{aligned} \tag{72}$$

where

$$\pi^*_{i,j,t} = \pi^*_{i,j}(\underline{y_t}) = \frac{\pi_{i,j,t} \exp(\theta'_1 e_j) \widetilde{\varphi}^{(a)}_{S,t,i,j}(\theta'_{2,1} e_j, \theta_{2,2})}{\sum_{j=1}^{J} \pi_{i,j,t} \exp(\theta'_1 e_j) \widetilde{\varphi}^{(a)}_{S,t,i,j}(\theta'_{2,1} e_j, \theta_{2,2})} ,$$

with $\widetilde{\varphi}^{(a)}_{S,t,i,j}$ obtained from $\widetilde{\varphi}_{S,t,i,j}$ in Proposition 3 by replacing $\mu_{i,j,t}$ and $\sigma^2_{i,j,t}$ by their expressions in (71), and where $\mu^*_t(y_t, e_i, e_j)$ and $\sigma^*_t(\underline{y_t}, e_i, e_j)$ are respectively given by:

$$\begin{aligned} \mu^*_{i,j,t} &= \frac{\mu_{0,t} + \mu_{1,t,i} + \mu_{2,t,j} + (\sigma^2_{0,t} + \sigma^2_{1,t,i} + \sigma^2_{2,t,j})\theta'_{2,1} e_j}{1 - 2(\sigma^2_{0,t} + \sigma^2_{1,t,i} + \sigma^2_{2,t,j})\theta_{2,2}} , \\ \sigma^*_{i,j,t} &= \left(\frac{\sigma^2_{0,t} + \sigma^2_{1,t,i} + \sigma^2_{2,t,j}}{1 - 2(\sigma^2_{0,t} + \sigma^2_{1,t,i} + \sigma^2_{2,t,j})\theta_{2,2}} \right)^{1/2} , \end{aligned} \tag{73}$$

which are no longer additive in (z_t, z_{t+1}). Observe that, if $\theta_{2,2} = 0$, $(\sigma^*_{i,j,t})^2$ is additive while $\mu^*_{i,j,t}$ is not, except the case in which we also have $\theta_{2,1} = 0$.

6.3.2 The Generalized Discrete-Time "Jump-Diffusion" Case

The purpose of this section is to show that a particular Additive Regime Switching model can be re-parametrized as a discrete-time generalization of the well known continuous-time jump-diffusion model. Let us impose to the conditional mean and variance in (70) the following specification:

$$\begin{aligned} \mu_t(\underline{y_t}, z_t, z_{t+1}) &= \mu_{0,1} + \mu_{0,2} y_t + \mu'_2 z_{t+1} , \\ \sigma^2_t(\underline{y_t}, z_t, z_{t+1}) &= \sigma^2_1 + \sigma^{2\prime}_2 z_{t+1} , \end{aligned} \tag{74}$$

and let us assume for instance, for identification reasons, that the first component of μ_2 and σ^2_2 are equal to zero (i.e., $\mu_{2,1} = \sigma^2_{2,1} = 0$). Then, model (69) can be written in the following observationally equivalent way:

$$\begin{aligned} & y_{t+1} = \mu_{0,1} + \mu_{0,2} y_t + x'_{t+1} z_{t+1} + \sigma_1 \varepsilon_{t+1} , \\ & \mathbb{P}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi_{i,j}(\underline{y_t}) = \pi_{i,j,t} \quad \forall (e_i, e_j) \in \mathcal{E} \times \mathcal{E} , \\ & \varepsilon_{t+1} | z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{P}}{\sim} N(0,1) , \end{aligned} \tag{75}$$

where:

$$
\begin{aligned}
& x_{t+1} = \mu_2 + \Sigma_2 \eta_{t+1} \,, \\
& \text{with } \; x_{t+1} = (x_{1,t+1}, \ldots, x_{J,t+1})' \,, \\
& \text{and } \; \Sigma_2 = diag(\sigma_{2,1}, \ldots, \sigma_{2,J}) \,, \; \eta_{t+1} \sim IIN(0, I_J) \,,
\end{aligned}
\tag{76}
$$

that is,

$$
x_{j,t+1} \,|\, \underline{y_{t+1}}, \underline{z_{t+1}} \overset{\mathbb{P}}{\sim} N(\mu_{2,j}, \sigma^2_{2,j}) \;\; \forall j \in \{1, \ldots, J\} \,, \text{ independently} \,. \tag{77}
$$

The process x_{t+1} introduces, within regimes, stochastic Gaussian components selected by the regime indicator function z_{t+1}. In other words, the process x_{t+1} introduces discrete-time "jumps" in the level of the stock return, and the Markov chain z_{t+1} makes the time series of that jumps serially dependent. Moreover, it is easy to verify that, under the probability measure $\mathbb{P}^*$ induced by the conditional Second-Order Esscher transform, the process x_{t+1} is characterized, conditionally to $\underline{y_{t+1}}, \underline{z_{t+1}}$, by the following distribution:

$$
\begin{aligned}
& x_{j,t+1} \,|\, \underline{y_{t+1}}, \underline{z_{t+1}} \overset{\mathbb{P}^*}{\sim} N\left(\mu^*_j, (\sigma^*_j)^2\right) \;\; \forall j \in \{1, \ldots, J\} \,, \text{ independently} \,, \\
& \text{where } \; \mu^*_j = \frac{\mu_{2,j} + \sigma^2_{2,j} \theta'_{2,1} e_j}{1 - 2\left((\sigma^2_1 + \sigma^2_{2,j}\right)\theta_{2,2}} \,, \\
& \sigma^*_j = \left(\frac{\sigma^2_{2,j}}{1 - 2\left(\sigma^2_1 + \sigma^2_{2,j}\right)\theta_{2,2}}\right)^{1/2} \,,
\end{aligned}
\tag{78}
$$

and

$$
\mathbb{P}^*(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi^*_{i,j}(\underline{y_t}) = \pi^*_{i,j,t} \quad \forall (e_i, e_j) \in \mathcal{E} \times \mathcal{E} \,,
$$

$$
\pi^*_{i,j,t} = \pi^*_{i,j}(\underline{y_t}) = \frac{\pi_{i,j,t} \exp(\theta'_1 e_j) \widetilde{\varphi}^{(a)}_{S,t,i,j}(\theta'_{2,1} e_j, \theta_{2,2})}{\sum_{j=1}^{J} \pi_{i,j,t} \exp(\theta'_1 e_j) \widetilde{\varphi}^{(a)}_{S,t,i,j}(\theta'_{2,1} e_j, \theta_{2,2})} \,.
$$

This means that, when we move from $\mathbb{P}$ to $\mathbb{P}^*$, the Gaussian-type stochastic amplitude x_{t+1} is characterized not only by a different mean but also by a different variance as proposed, in the continuous-time setting, by Broadie, Chernov and Johannes (2007) [see also, in an i.i.d. scalar setting, Backus, Chernov and Martin (2009)]. It is also important to observe that, conditionally to $(\underline{y_t}, z_t = e_i, \underline{z_{t-1}})$, the stochastic process $x'_{t+1} z_{t+1}$ follows, under $\mathbb{P}$, a mixture of J Gaussian random variables $N(\mu_{2,j}, \sigma^2_{2,j})$ each with mixing weight $\pi_{i,j,t}$, for $j \in \{1, \ldots, J\}$. Under $\mathbb{P}^*$, the J Gaussian components of the mixture are $N(\mu^*_j, (\sigma^*_j)^2)$ with mixing weights $\pi^*_{i,j,t}$, for $j \in \{1, \ldots, J\}$.

6.3.3 General Discrete-Time Serial Dependent Additive Jumps

In Section 6.3.1 we have introduced the conditionally Gaussian Additive Regime Switching model characterized by additive conditional mean and conditional variance :

$$\begin{cases} \mu_t(\underline{y_t}, z_t, z_{t+1}) = \mu_{0,t} + \mu_{1,t}' z_t + \mu_{2,t}' z_{t+1} \,, \\ \\ \sigma_t^2(\underline{y_t}, z_t, z_{t+1}) = \sigma_{0,t}^2 + \sigma_{1,t}^2{}' z_t + \sigma_{2,t}^2{}' z_{t+1} \,. \end{cases} \tag{79}$$

In Section 6.3.2 we have seen that a particular specification of (79), characterized by the following constraints:

$$\begin{cases} \mu_{0,t}(\underline{y_t}) & = & \mu_{0,1} + \mu_{0,2} y_t \,, \;\; \mu_{1,t} = 0 \,, \;\; \mu_{2,t} = \mu_2 \,, \\ \\ \sigma_{0,t}^2 & = & 0 \,, \;\; \sigma_{2,t}^2 = \sigma_2^2 \,, \;\; \sigma_{1,t,i}^2 = \sigma_1^2 \quad \forall i \in \{1, \ldots, J\} \,, \end{cases} \tag{80}$$

induces discrete-time jumps with the following important features, not generally shared by continuous-time jump-diffusion models: *i*) the time series of jumps is serially dependent since the Gaussian stochastic amplitude $x_{j,t+1}$ is selected, at each date $t+1$, by the Markov chain z_{t+1}; *ii*) the transition probability of z_{t+1} can be non-homogeneous, as assumed in (75), giving therefore the possibility to describe clusters in jumps with time-varying persistence; *iii*) the distribution of $x_{t+1}' z_{t+1}$, conditionally to $(\underline{y_t}, \underline{z_t})$, is a mixture of J Gaussian distributions while, when the typical compound Poisson structure for jump innovations is used, the number of jumps per period is a Poisson distribution, with potentially time-varying intensity, and therefore the jump component is a particular infinite mixture of conditionally Gaussian processes [see Maheu and McCurdy (2004), Christoffersen, Jacobs and Ornthanalai (2008)].

Now, if we assume the additive specification (79) without constraints (80), the associated regime switching model will be able to generate serially dependent contemporaneous and lagged discrete-time shifts, in the stock return level and variance, respectively selected by z_{t+1} and z_t. More precisely, the additional features are the following: *iv*) once the J-dimensional vectors $\mu_{2,t}$ and $\sigma_{2,t}^2$ are specified as function of the information at date t, the conditional mean and variance of the Gaussian stochastic amplitude, selected by z_{t+1}, may be functions of $\underline{y_t}$ (in particular, switching GARCH effects might be introduced); *v*) if we assume, in addition, that also the J-dimensional vectors $\mu_{1,t}(\neq 0)$ and $\sigma_{1,t}^2$ (with $\sigma_{1,t,i}^2 \neq \sigma_{1,t,j}^2 \forall i \neq j,\, i,j \in \{1, \ldots J\}$) are functions of the information at date t, then the mean and variance of y_{t+1}, conditionally to $(\underline{y_t}, \underline{z_t})$, will be affected by an additional source of time variation and of serial dependence (induced by z_t).

6.3.4 A Risk-Neutral Affine Additive Regime Switching Pricing Model with Non-Linear Market Price of Risks

The purpose of this section is to present a security market model able to propose at the same time a tractable (explicit or quasi explicit) pricing formula and non-linear risk-sensitivity coefficients and, thus, a flexible historical dynamics. We follow the Back Modelling strategy, outlined in Section 6.2.2, starting from the following Compound Autoregressive of order 1 [Car(1)] dynamics for $(y_{t+1}, z_{t+1}')'$

satisfying the internal consistency condition $E_t^{\mathbb{Q}}[\exp(y_{t+1})] = \exp(r_{t+1})$ [see Appendix 3 for a proof]:

$$
\begin{aligned}
& y_{t+1} = r_{t+1} - \left[\lambda(\mu_2^{\mathbb{Q}}, \sigma_1^{\mathbb{Q}}, \sigma_2^{\mathbb{Q}}, \pi^{\mathbb{Q}}) + \tfrac{1}{2}(\sigma_1^{\mathbb{Q}})^2\right]' z_t + \mu_2^{\mathbb{Q}\,\prime} z_{t+1} + [(\sigma_1^{\mathbb{Q}})' z_t + (\sigma_2^{\mathbb{Q}})' z_{t+1}]\xi_{t+1} \,, \\
& \xi_{t+1} | z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{Q}}{\sim} N(0,1) \,, \\
& \mathbb{Q}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{i,j}^{\mathbb{Q}} \,,
\end{aligned}
\tag{81}
$$

where $\lambda_i(\mu_2^{\mathbb{Q}}, \sigma_1^{\mathbb{Q}}, \sigma_2^{\mathbb{Q}}, \pi^{\mathbb{Q}}) = \log \sum_{j=1}^{J} \pi_{i,j}^{\mathbb{Q}} \exp\left(\mu_{2,j}^{\mathbb{Q}} + \frac{1}{2}(\sigma_{2,j}^{\mathbb{Q}})^2 + \sigma_{1,i}^{\mathbb{Q}}\sigma_{2,j}^{\mathbb{Q}}\right)$. It is well known that this risk-neutral Car(1) dynamics provides quasi explicit formulas for many derivatives prices.

We specify the following exponential-quadratic SDF:

$$
M_{t,t+1}^{(S)} = \exp\left[-r_{t+1} + \alpha_{1,t}' z_{t+1} + \alpha_{2,1,t}' z_{t+1} y_{t+1} + \alpha_{2,2,t} y_{t+1}^2 - \widetilde{\psi}_{S,t}(\alpha_{1,t}, \alpha_{2,t})' z_t\right] \,, \tag{82}
$$

and the historical dynamics ($\mathbb{P}_{t+1}$) of $(y_{t+1}, z_{t+1}')'$, conditionally to $(\underline{y_t}, \underline{z_t})$ is the family of probability distributions generated by the conditional Second-Order Esscher Transform of $\mathbb{Q}_{t+1}$ associated with the non-linear risk-sensitivity coefficients $(-\alpha_{1,t}', -\alpha_{2,t}')'$ [$\mathbb{P}_{t+1} = S_{(-\alpha_{1,t}, -\alpha_{2,t})}(\mathbb{Q}_{t+1})$]. More precisely, the historical dynamics (67) takes the following particular form:

$$
\begin{aligned}
& y_{t+1} = \mu_t(\underline{y_t}, z_t, z_{t+1}) + \sigma_t(\underline{y_t}, z_t, z_{t+1})\varepsilon_{t+1} \\
& \varepsilon_{t+1} | z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{P}}{\sim} N(0,1) \,,
\end{aligned}
\tag{83}
$$

with

$$
\begin{aligned}
& \mu_t(\underline{y_t}, e_i, e_j) = \frac{r_{t+1} - \left[\lambda_i(\mu_2^{\mathbb{Q}}, \sigma_1^{\mathbb{Q}}, \sigma_2^{\mathbb{Q}}, \pi^{\mathbb{Q}}) + \frac{1}{2}(\sigma_{1,i}^{\mathbb{Q}})^2\right] + \mu_{2,j}^{\mathbb{Q}} - [(\sigma_{1,i}^{\mathbb{Q}})^2 + (\sigma_{2,j}^{\mathbb{Q}})^2]\alpha_{2,1,t}' e_j}{1 + 2[(\sigma_{1,i}^{\mathbb{Q}})^2 + (\sigma_{2,j}^{\mathbb{Q}})^2]\alpha_{2,2,t}} \\
& \sigma_t(\underline{y_t}, e_i, e_j) = \left(\frac{(\sigma_{1,i}^{\mathbb{Q}})^2 + (\sigma_{2,j}^{\mathbb{Q}})^2}{1 + 2[(\sigma_{1,i}^{\mathbb{Q}})^2 + (\sigma_{2,j}^{\mathbb{Q}})^2]\alpha_{2,2,t}}\right)^{1/2} \,,
\end{aligned}
\tag{84}
$$

where

$$
\mathbb{P}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \pi_{i,j}(\underline{y_t}) = \pi_{i,j,t} \quad \forall (e_i, e_j) \in \mathcal{E} \times \mathcal{E}
$$

$$
\pi_{i,j,t} = \frac{\pi_{i,j,t}^{\mathbb{Q}} \exp(-\alpha_{1,t}' e_j) \widetilde{\varphi}_{S,t,i,j}^{\mathbb{Q}}(-\alpha_{2,1,t}' e_j, -\alpha_{2,2,t})}{\sum_{j=1}^{J} \pi_{i,j,t}^{\mathbb{Q}} \exp(-\alpha_{1,t}' e_j) \widetilde{\varphi}_{S,t,i,j}^{\mathbb{Q}}(-\alpha_{2,1,t}' e_j, -\alpha_{2,2,t})} \,. \tag{85}
$$

and where

$$\widetilde{\varphi}^{\mathbb{Q}}_{S,t,i,j}(-\alpha'_{2,1,t}e_j, -\alpha_{2,2,t}) = \exp\left\{-\frac{1}{2}\log\left[1 + 2\left[(\sigma^{\mathbb{Q}}_{1,i})^2 + (\sigma^{\mathbb{Q}}_{2,j})^2\right](\alpha'_{2,1,t}e_j)\right]\right.$$

$$-\frac{1}{2}\frac{\left(r_{t+1} - \left[\lambda_i(\mu^{\mathbb{Q}}_2, \sigma^{\mathbb{Q}}_1, \sigma^{\mathbb{Q}}_2, \pi^{\mathbb{Q}}) + \frac{1}{2}(\sigma^{\mathbb{Q}}_{1,i})^2\right] + \mu^{\mathbb{Q}}_{2,j}\right)^2}{[(\sigma^{\mathbb{Q}}_{1,i})^2 + (\sigma^{\mathbb{Q}}_{2,j})^2]}$$

$$\left.+\frac{1}{2}\frac{\left(r_{t+1} - \left[\lambda_i(\mu^{\mathbb{Q}}_2, \sigma^{\mathbb{Q}}_1, \sigma^{\mathbb{Q}}_2, \pi^{\mathbb{Q}}) + \frac{1}{2}(\sigma^{\mathbb{Q}}_{1,i})^2\right] + \mu^{\mathbb{Q}}_{2,j} - [(\sigma^{\mathbb{Q}}_{1,i})^2 + (\sigma^{\mathbb{Q}}_{2,j})^2]\alpha'_{2,1,t}e_j\right)^2}{[(\sigma^{\mathbb{Q}}_{1,i})^2 + (\sigma^{\mathbb{Q}}_{2,j})^2] + 2[(\sigma^{\mathbb{Q}}_{1,i})^2 + (\sigma^{\mathbb{Q}}_{2,j})^2]^2\alpha_{2,2,t}}\right\}.$$

Thus, it is clear from relations (84) and (85) that, since the sensitivity factors $\alpha_{1,t}$, $\alpha_{2,1,t}$ and $\alpha_{2,2,t}$ can be specified as any functions of the information at time t, we obtain a very large set of historical dynamics.

7 Conclusions and Further Developments

In this paper we have proposed, working with discrete time no-arbitrage asset pricing models, to widen the bridge between the historical and the risk-neutral factor distribution, while keeping, respectively, flexible and tractable the modelling of both dynamics. The key tools behind this more general change of probability measure are the Second-Order Esscher Transform and the Second-Order Laplace Transform. The associated change of probability measure is thus generated by an Exponential-Quadratic Stochastic Discount Factor, specified by first-order and second-order stochastic risk-sensitivity vectors.

We have shown the large flexibility of this new approach in the case of conditionally Gaussian dynamics, conditionally Mixed-Normal dynamics and conditionally Gaussian Switching Regime dynamics. These classes provide a large variety of security market models. In particular, Gaussian switching regime models show several degrees of flexibility both under the historical and risk-neutral probability, given the serial dependence of regimes and the introduction of the regime indicator function in the linear and quadratic term of the SDF.

Our approach can be coupled with a Back Modelling strategy assuming a Car risk-neutral factor dynamics and then obtaining an historical dynamics by means of a Second-Order Esscher Transform with risk-sensitivity coefficients specified as any functions of the state vector. In this case we have at the same time explicit or quasi explicit pricing formulas for several derivative assets and a very large set of possible historical dynamics.

Although we have illustrated our approach using security market models, our results are much more general that the RNVR or LRNVR ones, since they could be applied in many other asset pricing contexts like yield curve and credit risk models, longevity risk and exchange rate models. We leave these developments to future research.

Appendix 1
Computation of Second-Order Esscher Transforms

Computation of the Second-Order Esscher Transform of a Gaussian distribution

The proofs of the examples presented in Section 2.2 are based on the following result. If we consider the p.d.f. of a n-dimensional Gaussian random variable $N(\mu, \Sigma)$:

$$
\begin{aligned}
f(y) &= \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}} \exp\left[-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right] , \\
&\propto \exp\left[-\tfrac{1}{2}y'\Sigma^{-1}y + \mu'\Sigma^{-1}y\right]
\end{aligned}
\tag{A.1}
$$

then, from Definition 3 we have :

$$
\begin{aligned}
g(y;\theta_1,\theta_2) &\propto \exp\left[-\tfrac{1}{2}y'\Sigma^{-1}y + \mu'\Sigma^{-1}y + \theta_1 y + y'\theta_2 y\right] , \\
&\propto \exp\left[-\tfrac{1}{2}y'\left(\Sigma^{-1}-2\theta_2\right)y + \left(\Sigma^{-1}\mu+\theta_1\right)' y\right] , \\
&\propto \exp\left[-\tfrac{1}{2}y'\left(\Sigma^{-1}-2\theta_2\right)y + \left(\Sigma^{-1}\mu+\theta_1\right)'\left(\Sigma^{-1}-2\theta_2\right)^{-1}\left(\Sigma^{-1}-2\theta_2\right)y\right] ,
\end{aligned}
\tag{A.2}
$$

and, therefore, $g(y;\theta_1,\theta_2)$ is the p.d.f. of the n-dimensional Gaussian random variable

$$
N\left[\left(\Sigma^{-1}-2\theta_2\right)^{-1}\left(\Sigma^{-1}\mu+\theta_1\right), \left(\Sigma^{-1}-2\theta_2\right)^{-1}\right]
\tag{A.3}
$$

proving relation (15) of example vii), and relation (14) of example vi) when $n = 1$.

Computation of the Second-Order Laplace Transform of a Gaussian distribution

From relations (12) and (15) we see that the Second-Order Laplace Transform of the Gaussian random vector $y \sim N(\mu, \Sigma)$ is given by:

$$
\begin{aligned}
\varphi_S(\theta_1,\theta_2) &= \int_{\mathbb{R}^n} f(y)\exp(\theta_1' y + y'\theta_2 y)dy \\
&= \frac{f(y)\exp(\theta_1' y + y'\theta_2 y)}{g(y;\theta_1,\theta_2)} \\
&= \det\left(I-2\Sigma\theta_2\right)^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}\mu'\Sigma^{-1}\mu + \tfrac{1}{2}(\Sigma^{-1}\mu+\theta_1)'(\Sigma^{-1}-2\theta_2)^{-1}(\Sigma^{-1}\mu+\theta_1)\right] .
\end{aligned}
\tag{A.4}
$$

If we consider the case of a scalar ($n = 1$) Gaussian random variable $N(\mu, \sigma^2)$, the Second-Order Gaussian Laplace Transform (A.4) takes the following particular form:

$$
\begin{aligned}
\varphi_S(\theta_1,\theta_2) &= \int_{\mathbb{R}} f(y)\exp(\theta_1 y + \theta_2 y^2)dy \\
&= (1-2\sigma^2\theta_2)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\frac{\mu^2}{\sigma^2} + \frac{1}{2}\left(\frac{\sigma^2}{1-2\sigma^2\theta_2}\right)\left(\frac{\mu}{\sigma^2}+\theta_1\right)^2\right] .
\end{aligned}
\tag{A.5}
$$

Computation of the Second-Order Esscher Transform of a mixture of Gaussian distributions

Denoting by $n(y;\mu_j,\Sigma_j)$ the p.d.f. of the Gaussian random vector $y \sim N(\mu_j,\Sigma_j)$, we want to find the Second-Order Esscher Transform of the density:

$$\sum_{j=1}^{J} \lambda_j \, n(y;\mu_j,\Sigma_j)\,, \tag{A.6}$$

which is given, following Definition 3, by the family of probability distributions with p.d.f.:

$$g(y;\theta_1,\theta_2) = \frac{\displaystyle\sum_{j=1}^{J} \lambda_j \, \exp(\theta_1' y + y'\,\theta_2\,y)\, n(y;\mu_j,\Sigma_j)}{\displaystyle\sum_{j=1}^{J} \lambda_j \, \varphi_{S,j}(\theta_1,\theta_2)}\,, \tag{A.7}$$

where $\varphi_{S,j}(\theta_1,\theta_2)$ is the Second-Order Laplace Transform of $y \sim N(\mu_j,\Sigma_j)$ given by (A.4) with $\mu = \mu_j$ and $\Sigma = \Sigma_j$. From the results proved above we obtain:

$$\begin{aligned} g(y;\theta_1,\theta_2) &= \sum_{j=1}^{J} \lambda_j^* \, n\left[y; \left(\Sigma_j^{-1} - 2\theta_2\right)^{-1} \left(\Sigma_j^{-1}\mu_j + \theta_1\right), \left(\Sigma_j^{-1} - 2\theta_2\right)^{-1}\right], \\ \text{with } \lambda_j^* &= \frac{\lambda_j \, \varphi_{S,j}(\theta_1,\theta_2)}{\displaystyle\sum_{j=1}^{J} \lambda_j \, \varphi_{S,j}(\theta_1,\theta_2)} \end{aligned} \tag{A.8}$$

proving relation (16) of example *viii*).

Appendix 2
Proof of Proposition 3

The purpose of this appendix is to derive the p.d.f. of the family of probability distributions generated by the conditional Second-Order Esscher transform $S_{\theta_1,\theta_2}(\mathbb{P}_{i,t+1})$ applied to the p.d.f. (52). Following Definition 3, we have:

$$
\begin{aligned}
& g_t(y_{t+1}, e_j | \underline{y_t}, z_t = e_i) \\
= \; & \frac{\pi_{i,j,t} \exp\left(\theta_1' e_j + \theta_{2,1}' e_j y_{t+1} + \theta_{2,2} y_{t+1}^2\right) \; n(y_{t+1}; \mu_{i,j,t}, \sigma_{i,j,t}^2)}{\sum_{j=1}^{J} \int_{\mathbb{R}} \pi_{i,j,t} \exp\left(\theta_1' e_j + \theta_{2,1}' e_j y_{t+1} + \theta_{2,2} y_{t+1}^2\right) \; n(y_{t+1}; \mu_{i,j,t}, \sigma_{i,j,t}^2) \; dy_{t+1}} \\
= \; & \frac{\pi_{i,j,t} \exp(\theta_1' e_j) \, n(y_{t+1}; \mu_{i,j,t}, \sigma_{i,j,t}^2) \exp\left(\theta_{2,1}' e_j y_{t+1} + \theta_{2,2} y_{t+1}^2\right)}{\sum_{j=1}^{J} \pi_{i,j,t} \exp(\theta_1' e_j) \left[\int_{\mathbb{R}} n(y_{t+1}; \mu_{i,j,t}, \sigma_{i,j,t}^2) \exp\left(\theta_{2,1}' e_j y_{t+1} + \theta_{2,2} y_{t+1}^2\right) \; dy_{t+1}\right]} .
\end{aligned}
\tag{A.9}
$$

Now, given the result presented in example *vi*) and in Appendix 1, the Second-Order Laplace transform (56) is given by:

$$
\widetilde{\varphi}_{S,t,i,j}\left(\theta_{2,1}' e_j, \theta_{2,2}\right) = \exp\left[-\frac{1}{2}\log\left(1 - 2\sigma_{i,j,t}^2 \theta_{2,2}\right) - \frac{1}{2}\frac{\mu_{i,j,t}^2}{\sigma_{i,j,t}^2} + \frac{1}{2}\frac{(\mu_{i,j,t} + \sigma_{i,j,t}^2 \theta_{2,1}' e_j)^2}{(\sigma_{i,j,t}^2 - 2\sigma_{i,j,t}^4 \theta_{2,2})}\right] , \tag{A.10}
$$

and, therefore, relation (A.9) can be written as follows:

$$
\begin{aligned}
& g_t(y_{t+1}, e_j | \underline{y_t}, z_t = e_i) \\
= \; & \frac{\pi_{i,j,t} \exp\left(\theta_1' e_j\right) \widetilde{\varphi}_{S,t,i,j}\left(\theta_{2,1}' e_j, \theta_{2,2}\right) \; n\left(y_{t+1}; \frac{\mu_{i,j,t} + \sigma_{i,j,t}^2 \theta_{2,1}' e_j}{1 - 2\sigma_{i,j,t}^2 \theta_{2,2}}, \frac{\sigma_{i,j,t}^2}{1 - 2\sigma_{i,j,t}^2 \theta_{2,2}}\right)}{\sum_{j=1}^{J} \pi_{i,j,t} \exp\left(\theta_1' e_j\right) \widetilde{\varphi}_{S,t,i,j}\left(\theta_{2,1}' e_j, \theta_{2,2}\right)} \\
= \; & \pi_{i,j,t}^* \, n\left(y_{t+1}; \frac{\mu_{i,j,t} + \sigma_{i,j,t}^2 \theta_{2,1}' e_j}{1 - 2\sigma_{i,j,t}^2 \theta_{2,2}}, \frac{\sigma_{i,j,t}^2}{1 - 2\sigma_{i,j,t}^2 \theta_{2,2}}\right)
\end{aligned}
\tag{A.11}
$$

and Proposition 3 is proved.

Appendix 3
Deriving the no-arbitrage dynamics of the affine additive regime switching model

Let us assume the following Compound Autoregressive of order 1 [Car(1)] $\mathbb{Q}$-dynamics for $(y_{t+1}, z'_{t+1})'$:

$$
\begin{aligned}
& y_{t+1} = \mu_{0,1,t}^{\mathbb{Q}} + \mu_{0,2}^{\mathbb{Q}} y_t + \mu_1^{\mathbb{Q}}{}' z_t + \mu_2^{\mathbb{Q}}{}' z_{t+1} + [(\sigma_1^{\mathbb{Q}})' z_t + (\sigma_2^{\mathbb{Q}})' z_{t+1}]\xi_{t+1} \\
& \xi_{t+1} | z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{Q}}{\sim} N(0,1) \\
& \mathbb{Q}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{i,j}^{\mathbb{Q}} \,.
\end{aligned}
\tag{A.12}
$$

where $\mu_{0,1,t}^{\mathbb{Q}}$ denotes a deterministic function of t. It is easy to verify that the Laplace transform of $(y_{t+1}, z'_{t+1})'$, conditionally to $(\underline{y_t}, \underline{z_t})$, is:

$$
\begin{aligned}
\varphi_t^{\mathbb{Q}}(u,v) &= E_t[\exp(u y_{t+1} + v' z_{t+1})] \\
&= \exp[a^{\mathbb{Q}}(u,v) y_t + b^{\mathbb{Q}}(u,v)' z_t + c_t^{\mathbb{Q}}(u,v)]
\end{aligned}
\tag{A.13}
$$

with:

$$
\begin{aligned}
a^{\mathbb{Q}}(u,v) &= u\, \mu_{0,2}^{\mathbb{Q}}\, y_t \\
b^{\mathbb{Q}}(u,v)' &= \left[u \mu_1^{\mathbb{Q}}{}' + \tfrac{1}{2} u^2 (\sigma_1^{\mathbb{Q}})^2{}' + \Lambda'(u, v, \mu_2^{\mathbb{Q}}, \sigma_1^{\mathbb{Q}}, \sigma_2^{\mathbb{Q}}, \pi^{\mathbb{Q}}) \right] \\
c_t^{\mathbb{Q}}(u,v) &= u\, \mu_{0,1,t}^{\mathbb{Q}} \,,
\end{aligned}
\tag{A.14}
$$

and where the i^{th} component of $\Lambda(u, v, \mu_2^{\mathbb{Q}}, \sigma_1^{\mathbb{Q}}, \sigma_2^{\mathbb{Q}}, \pi^{\mathbb{Q}})$ is given by:

$$
\Lambda_i(u, v, \mu_2^{\mathbb{Q}}, \sigma_1^{\mathbb{Q}}, \sigma_2^{\mathbb{Q}}, \pi^{\mathbb{Q}}) = \log \sum_{j=1}^{J} \pi_{i,j}^{\mathbb{Q}} \exp\left[u \mu_{2,j}^{\mathbb{Q}} + v_j + \frac{1}{2} u^2 (\sigma_{2,j}^{\mathbb{Q}})^2 + u^2 \sigma_{1,i}^{\mathbb{Q}} \sigma_{2,j}^{\mathbb{Q}} \right] \,,\ \forall i \in \{1, \ldots, J\} \,.
\tag{A.15}
$$

Once we impose the ICC $\psi_t^{\mathbb{Q}}(1,0) = \log \varphi_t^{\mathbb{Q}}(1,0) = r_{t+1}$, the risk-neutral (pricing) affine dynamics takes the following form:

$$
\begin{aligned}
& y_{t+1} = r_{t+1} - \left[\lambda(\mu_2^{\mathbb{Q}}, \sigma_1^{\mathbb{Q}}, \sigma_2^{\mathbb{Q}}, \pi^{\mathbb{Q}}) + \tfrac{1}{2} (\sigma_1^{\mathbb{Q}})^2 \right]' z_t + \mu_2^{\mathbb{Q}}{}' z_{t+1} + [(\sigma_1^{\mathbb{Q}})' z_t + (\sigma_2^{\mathbb{Q}})' z_{t+1}]\xi_{t+1} \,, \\
& \xi_{t+1} | z_{t+1}, \underline{z_t}, \underline{y_t} \overset{\mathbb{Q}}{\sim} N(0,1) \,, \\
& \mathbb{Q}(z_{t+1} = e_j | z_t = e_i, \underline{z_{t-1}}, \underline{y_t}) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{i,j}^{\mathbb{Q}} \,,
\end{aligned}
\tag{A.16}
$$

where $\lambda_i(\mu_2^{\mathbb{Q}}, \sigma_1^{\mathbb{Q}}, \sigma_2^{\mathbb{Q}}, \pi^{\mathbb{Q}}) = \log \sum_{j=1}^{J} \pi_{i,j}^{\mathbb{Q}} \exp\left(\mu_{2,j}^{\mathbb{Q}} + \frac{1}{2} (\sigma_{2,j}^{\mathbb{Q}})^2 + \sigma_{1,i}^{\mathbb{Q}} \sigma_{2,j}^{\mathbb{Q}} \right)$, and the result is proved.

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