INSTITUT NATIONAL DE LA STATISTIQUE ET DES ETUDES ECONOMIQUES
Série des Documents de Travail du CREST
(Centre de Recherche en Economie et Statistique)

n° 2010-08

A Term Structure Model with Level Factor cannot be Realistic and Arbitrage Free

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We thank Jonathan Ingersoll, Alain Monfort and Steve Ross for helpful comments. The second author gratefully acknowledges financial support of NSERC, Canada, and of the chair AXA/Risk Foundation : "Large Risks in Insurance". The views expressed in this paper are those of the authors and do not necessarily reflect those of the Banque de France.

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Abstract

A large part of the term structure literature interprets one of the underlying factors as a level factor. In this paper we consider a single factor model interpretable as a level factor model. We prove that this model is compatible with no-arbitrage restrictions and the positivity of rates under rather unrealistic conditions on the dynamic of the short term interest rate. This introduces some doubt on the relevance of the level interpretation of a factor in term structure models.

Keywords : Term Structure, Affine Model, No Arbitrage, Level Factor.

1 Introduction

The dynamic analysis of the term structure of interest rates reveals the existence of a limited number of underlying factors. It is usual to interpret sequentially these factors as a level factor, a slope (or steepness factor), a curvature (or butterfly factor), and so on, even if these notions have not been precisely defined in the literature [see e.g. Litterman, Scheinkman (1991), Jones (1991)].[3] This factor interpretation has also been extended to the field of option pricing [see Rogers, Tehranchi (2008) for a study of parallel shifts in the term structure of implied volatilities].

The aim of this note is to consider a single factor model, interpreted as a level factor. Loosely speaking, any shock on the factor X will imply a parallel shift in the whole term structure. From a modeling perspective, we suppose that there exists a process $(\xi_t)_{t\geq 0}$ such that for all t and all time-to-maturity h :

$$r(t,h) = r(0,h) + \xi_t$$

where $r(t,h)$ is the continuously compounded rate [4]. This means that the yield admits an additive decomposition :

$$r(t,h) = X_t + c(h), \tag{1.1}$$

(X_t) denotes the underlying stochastic factor ($X_t = X_0 + \xi_t$), and $c(.)$ is a baseline term structure. In the rest of the paper, we consider a discrete time term structure model, i.e. $t \in \mathcal{N}, h \in \mathcal{N} - \{0\}$.

In decomposition (1.1), the factor is defined up to an additive constant. Therefore, without loss of generality, we can always assume :

Assumption A.1 : $c(1) = 0$.

[3]"Level, slope and curvature factor loadings at the core of (term structure) models have their origin in the somewhat arbitrary and atheoretical field of yield curve fitting" [Krippner (2009)].

[4]Model (1.1) has been written for the continuously compounded rate. If we denote by $r^*(t,h)$ the rate which is not continuously compounded, we have : $\exp[-hr(t,h)] = [1 + r^*(t,h)]^{-h}$, or equivalently $r^*(t,h) = \exp[r(t,h)] - 1 = \exp[X_t + c(h)] - 1$. Thus the notion of level factor depends on the definition of the rate. We keep the continuously compounded definition in the rest of the paper, which is compatible with the existing literature.

Under Assumption A.1, the factor coincides with the short term interest rate : $X_t = r(t, 1)$.

We also assume that model (1.1) really is a single factor model :

Assumption A.2 : The support of the conditional distribution of (X_t) given X_0 is not reduced to a single point.

Finally, we assume nonnegative rates, with a short term interest rate, which can reach value zero.

Assumption A.3 :

i) The lower bound of the support of the distribution of X_t given X_0 is zero;

ii) $c(h) \geq 0, \forall h \in \mathcal{N} - \{0\}$.

In Section 2, we consider buy and hold strategies based on two zero coupon bonds and derive the necessary and sufficient conditions for no-arbitrage: the sequence $[c(h)]$ has to be a sequence of Cesaro means of a nonnegative increasing function. In Section 3, we discuss the implications of this result on the behavior of the long term interest rate. Section 4 exhibits all risk-neutral dynamics compatible with parallel shifts of the yield curve. We prove that they correspond to strong random walks and we explain how the behavior of the long term interest rate depends on the distribution of the innovation of this random walk [5]. Section 5 concludes. The history of parallel shift of the term structure in the financial literature is presented in appendix 1.

2 No-arbitrage condition for buy and hold strategies based on two zero-coupon bonds.

Let us consider at date t a portfolio of two zero-coupon bonds with time-to-maturity h_1 and $h_2, h_2 > h_1$, respectively. Its price at date t is :

$$\Pi_t(h_1, h_2, \alpha) = \alpha_1 B(t, h_1) + \alpha_2 B(t, h_2),$$

[5] The main result of this Section contradicts Theorem 4 in Ingersoll, Skelton, Weil (1978). We will see later on why their result is incomplete.

where $B(t,h) = \exp[-hr(t,h)]$ denotes the price of the zero-coupon bond, and (α_1, α_2) the allocations.

The value of this portfolio at date $t+k, k \leq h_1$, is :

$$\Pi_{t+k}(h_1 - k, h_2 - k, \alpha) = \alpha_1 B(t+k, h_1 - k) + \alpha_2 B(t+k, h_2 - k).$$

The no-arbitrage condition is the impossibility to ensure a positive future value with zero or negative initial endowment . This is equivalent to :

$$\begin{aligned} &\{\min_{t+k}[\Pi_{t+k}(h_1 - k, h_2 - k, \alpha] \geq 0\} \\ \Rightarrow &\{\min_t \Pi_t(h_1, h_2, \alpha)] \geq 0\}, \ \ \forall \alpha, k \leq h_1, h_2, \end{aligned} \tag{2.1}$$

where $\min_t$ is the minimum taken over the admissible values of the state variable of date t.

Condition (2.1) provides a restriction on zero-coupon prices, only if α_1 and α_2 have opposite sign. Thus, without loss of generality, we can choose $\alpha_1 = 1, \alpha_2 = -\alpha, \alpha > 0$, say.

Proposition 1 : Under model (1.1) and Assumptions A.1-A.3, the buy and hold strategies based on two zero-coupon bonds do not feature arbitrage opportunity if and only if the function $c^*(h) = hc(h)$ is such that :
$c^*(h+1) - c^*(h)$ is a nonnegative increasing function of h.

Proof : We have :

$$\begin{aligned} \Pi_{t+k}(h_1 - k, h_2 - k, \alpha) &= \exp[-(h_1 - k)X_{t+k} - c^*(h_1 - k)] \\ &- \alpha \exp[-(h_2 - k)X_{t+k} - c^*(h_2 - k)] \\ &= B(t+k, h_1 - k) \\ &\quad \{[1 - \alpha \exp[-(h_2 - h_1)X_{t+k}] \exp[-c^*(h_2 - k) + c^*(h_1 - k)]\}. \end{aligned}$$

Since $\alpha \geq 0, h_2 \geq h_1$, we deduce that :

$$\min_{t+k} \Pi_{t+k}(h_1 - k, h_2 - k, \alpha) \geq 0 \ \text{ if and only if } \ 1 - \alpha \exp[-c^*(h_2 - k) + c^*(h_1 - k)] \geq 0.$$

Therefore, $\min_{t+k} \Pi_{t+k}(h_1 - k, h_2 - k, \alpha)$ is nonnegative if and only if

$\alpha \leq \exp[c^*(h_2 - k) - c^*(h_1 - k)]$.

Similarly, $\min_t \Pi_t(h_1, h_2, \alpha)$ is strictly positive if and only if $\alpha \leq \exp[c^*(h_2) - c^*(h_1)]$.

The no-arbitrage condition is satisfied if and only if,

$$\{\alpha \leq \exp[c^*(h_2 - k) - c^*(h_1 - k)]\} \Rightarrow \{\alpha \leq \exp[c^*(h_2) - c^*(h_1)]\},$$

which is equivalent to :

$$c^*(h_2 - k) - c^*(h_1 - k) \leq c^*(h_2) - c^*(h_1), \forall k \leq h_1 \leq h_2. \qquad (2.2)$$

i) It is easily checked that condition (2.2) above is equivalent to the fact that the function $c^*(h_2 + k) - c^*(h_1 + k)$ is increasing in k for any $h_2 \geq h_1$.

ii) Finally, by noting that :

$$\begin{aligned} c^*(h_2 + k) - c^*(h_1 + k) &= [c^*(h_2 + k) - c^*(h_2 - 1 + k)] + [c^*(h_2 - 1 + k) - c^*(h_2 - 2 + k)] \\ &+ \ldots + [c^*(h_1 + 1 + k) - c^*(h_1 + k)], \end{aligned}$$

we get the increasingness condition.

To prove the nonnegativity, we have to check that $c^*(2) - c^*(1) = 2c(2)$ is nonnegative (since $c^*(1) - c^*(0) = 0$). This is a direct consequence of Assumption A.3.

QED

Corollary 1 : The no-arbitrage condition of Proposition 1 is satisfied if and only if the sequence $[c(h)]$ is a sequence of Cesaro means of a nonnegative increasing function.

Proof : We have :

$$c(h) = c^*(h)/h = \frac{1}{h} \sum_{l=1}^{h} \Delta c^*(l),$$

with $\Delta c^*(l) = c^*(l) - c^*(l-1)$.

The result follows from Proposition 1.

QED

Corollary 2 : Under model (1,1), Assumption A1-A3 and no-arbitrage, function c^* is superadditive, that is,

$$c^*(h_1) + c^*(h_2) \leq c^*(h_1 + h_2), \forall h_1, h_2 \in \mathcal{N} - \{0\}.$$

Proof : Indeed, let us consider the special case of inequality (2.2) for $k = h_1$. We get :

$$c^*(h_2 - h_1) \leq c^*(h_2) - c^*(h_1), \quad \forall h_1 \leq h_2.$$

QED

This condition was expected. Indeed, under Assumption A.3 the lower bound of the support of $r(t,h)$ is equal to $c(h)$. It has been proved in Gourieroux-Monfort (2010) that h times this lower bound, that is, $c^*(h) = hc(h)$ is necessarily superadditive under no-arbitrage condition.

3 Behavior of the long term interest rate

Proposition 2 : Under model (1.1) and Assumptions A.1-A.3, we get one of the two following cases :

i) $r(t,\infty) = +\infty$:

ii) $r(t,\infty) = X_t + c_\infty$, where c_∞ is a given positive constant.

Proof : Since $\Delta c^*(h)$ is nonnegative increasing, we have either $\lim_{h\to\infty} \Delta c^*(h) = \infty$, or $\lim_{h\to\infty} \Delta c^*(h) = c_\infty < \infty$ say. Since $\Delta c^*(h)$ is a nonnegative increasing function, we deduce that the Cesaro mean $[c(h)]$ is such that :

$$\begin{aligned} c(h) &= \frac{1}{h}\sum_{l=1}^{h} \Delta c^*(l) \leq \Delta c^*(h), \forall h, \\ \text{and} \quad c(h) &\geq \frac{1}{h}\sum_{l=k+1}^{h} \Delta c^*(l) \geq \frac{h-k}{h}\Delta c^*(k), \forall k \leq h. \end{aligned}$$

These two inequalities explain why the sequences $[c(h)]$ and $[\Delta c^*(h)]$ have the same asymptotic behavior. For instance, let us assume that $\lim_{h\to\infty} \Delta c^*(h) = +\infty$. Then, from the second inequality, we get :

$$\lim_{h\to\infty} \inf c(h) \geq \Delta c^*(k), \forall k,$$

which implies $\lim_{h\to\infty} \inf c(h) \geq +\infty$. We deduce that $\lim_{h\to\infty} c(h) = +\infty$. When $\lim_{h\to\infty} \Delta c^*(h) = c_\infty$, the joint use of the two inequalities shows that $\lim_{h\to\infty} \inf c(h)$ and $\lim_{h\to\infty} \sup c(h)$ exist and are equal to c_∞.

QED

Proposition 2 shows that the case, where the long term interest rate does not exist due, for instance, to a periodic asymptotic behavior of function c has been eliminated.

Proposition 2 concerns the limiting behavior of the long run spot interest rate when the whole term structure moves by parallel shifts. The instantaneous forward interest rate is given by :

$$f(t, h) = hr(t, h) - (h-1)r(t, h-1).$$

Under model (1.1), the instantaneous forward rate is equal to :

$$f(t, h) = X_t + c^*(h) - c^*(h-1), \forall t, h.$$

It is not a constant function of time. In particular, if $\lim_{h\to\infty} \Delta c^*(h)$ exists, the long run instantaneous forward interest rate also exists and is stochastic.

4 Risk-neutral factor dynamic

Proposition 3 : Under model (1.1) and Assumptions A.1-A.3, the factor process is a Markov process under the risk-neutral probability Q and we have :

$$\overset{Q}{E}_t \left[\exp(-hX_{t+1})\right] = \exp[-hX_t + c^*(h) - c^*(h+1)].$$

Proof : Under no-arbitrage, we have :

$$B(t,h+1) = \overset{Q}{E_t} \{\exp[-r(t,1)]B(t+1,h)\}, \forall h,$$

or, equivalently :

$$\exp[-(h+1)r(t,h+1)] = \exp[-r(t,1)] \overset{Q}{E_t} \{\exp[-hr(t+1,h)]\}, \forall h.$$

By decomposition (1.1), we deduce :

$$\exp[-(h+1)X_t - c^*(h+1)] = \exp(-X_t) \overset{Q}{E_t} \{\exp[-hX_{t+1} - c^*(h)]\},$$

or :

$$\overset{Q}{E_t} [\exp(-hX_{t+1})] = \exp[-hX_t + c^*(h) - c^*(h+1)], \forall h.$$

For a nonnegative variable, the knowledge of the Laplace transform for negative integer characterizes the distribution [6]. We deduce that the conditional distribution of X_{t+1} given its past depends on the past by means of the most recent observation. This is the Markov property and Proposition 3 follows.

QED

The conditional log-Laplace transform is an affine function of the current value of the process. This is exactly the definition of a Compound Autoregressive (CaR) process [see Darolles, Gourieroux, Jasiak (2006)], also called Affine process in continuous time [Duffie, Kan (1996), Duffie, Filipovic, Schachermayer (2003)].

Proposition 4 [7] : Under model (1.1) and Assumptions A.1-A.3, the level factor process is a strong random walk under Q :

[6] Indeed, let us denote $Z = \exp(-X)$. Variable Z takes values in $(0,1)$. Thus, for any argument u, the series $\Sigma_{h=0}^{\infty} \frac{E(Z^h)(iu)^h}{h!}$ is uniformly absolutely convergent. We deduce that the characteristic function $\psi(u) = E[\exp(iuZ)]$ exists [see Feller (1971), Vol2, p430].

[7] Proposition 4 contradicts Theorem 4 in Ingersoll, Skelton, Weil (1978), where it is said that any parallel shift in a term structure is not arbitrage free. The random walk models in Proposition 4 are both compatible with parallel shift and no-arbitrage. This contradiction is due to a misleading proof in ISW (1978), p635, l3, where the effect of diminishing time-to-maturity is omitted when computing the future value of the portfolio of zero-coupon bonds. In some sense, they have implicitly assumed a flat term structure $c(h) = 0$ [see the discussion in Appendix 1].

$$X_{t+1} = X_t + \varepsilon_{t+1},$$

where (ε_t) is under Q a sequence of nonnegative i.i.d. variables with Laplace transform :

$$\psi_\varepsilon(h) = \overset{Q}{E} [\exp(-h\varepsilon_t)] = \exp[c^*(h) - c^*(h+1)].$$

Proof : Let us denote $\varepsilon_{t+1} = X_{t+1} - X_t$. From Proposition 3, we deduce that : $\overset{Q}{E_t} [\exp(-h\varepsilon_{t+1})] = \exp[c^*(h) - c^*(h+1)]$. This shows that the conditional distribution of ε_{t+1} is independent of the past and provides the form of its Laplace transform. Moreover, ε is nonnegative, since by Assumption A.3, X_t can be arbitrary close to zero. In this case $\varepsilon_{t+1} = X_{t+1}$, which is nonnegative.

QED

Since ε is nonnegative, $\psi_\varepsilon(h)$ is smaller than 1 and a decreasing function of h. We deduce that $c^*(h+1) - c^*(h)$ is a nonnegative increasing function of h (which is Proposition 1). We also get the following Corollaries :

Corollary 2 : Model (1.1) is compatible with the no-arbitrage condition if and only if function c^* is such that : $\exp[c^*(h) - c^*(h+1)]$ is the Laplace transform of a positive variable.

Corollary 3 : Under model (1.1) and Assumptions A.1-A.3, the factor process is a non decreasing function of time: the term structure cannot make uniform downward move.

Let us now come back to the behavior of the long term interest rate. We have the following proposition :

Proposition 5 : For a strong random walk under Q, the long term interest rate exists, if and only if :

$$\lim_{h\to\infty} \{-\log E[\exp(-h\varepsilon)]\} = -\log P[\varepsilon = 0] = c_\infty < \infty;$$

then the long run interest rate is equal to :

$$r(t, \infty) = X_t + c_\infty.$$

Proof : The first condition concerning $\lim_{h\to\infty}\{-\log E[\exp(-h\varepsilon)]\} = c_\infty < \infty$ is a direct consequence of Proposition 4 and the proof of Proposition 2.

Moreover, we have

$$E[\exp(-h\varepsilon)] = P[\varepsilon = 0] + \int \mathbb{1}_{x>0} \exp(-xh) dF(x).$$

But $\lim_{h\to\infty} \exp(-hx) = 0, \forall x > 0$, and since $\exp(-hx) \in (0,1)$, we deduce by Beppo-Levi theorem that $\lim_{h\to\infty} \int \mathbb{1}_{x>0} \exp(-hx) dF(x) = 0$. The result follows.

QED

In this framework, the long term rate exists, is stochastic and provides the same information than the underlying factor. This contradicts Lemma 3 in El Karoui, Frachot, Geman (1998) which asserts that the long term yield (if it exists) cannot be stochastic in a one-factor model.

The need for an innovation with point mass at zero explains the strange behavior of the long run interest rate, in affine models with a level factor following a Gaussian random walk, even if this factor is not positive [see e.g. Christensen, Diebold, Rudebusch (2008)]. In this case, it is seen that the long run interest rate is equal to $-\infty$.

To illustrate Corollary 2, let us consider a random walk with a Poisson innovation $\varepsilon_t \sim \mathcal{P}(\lambda)$. We have :

$$\begin{aligned} \psi_\varepsilon(h) &= \exp\{-\lambda[1-\exp(-h)]\}, \\ -\log P[\varepsilon = 0] &= \lambda, \end{aligned}$$

and the interest rate with time-to-maturity h is :

$$r(t,h) = X_t + \lambda\{\frac{h-1}{h} + \frac{1}{h}[1 - \frac{1-\exp(-h)}{1-\exp(-1)}]\}.$$

We check that : $\lim_{h\to\infty} \psi_\varepsilon(h) = r(t,\infty) - X_t = -\log P[\varepsilon = 0] = \lambda$.

The results above concern the risk-neutral dynamics. It is known that the historical and risk-neutral dynamics are weakly linked [see Rogers (1977)]. For instance, the historical dynamic of (X_t) is not necessarily affine, and does not necessarily feature a unit root. Nevertheless, the historical and risk-neutral distributions have a same support : in particular the process (X_t) is also increasing under the historical probability and the probability that $X_{t+1} = X_t$ is nonzero if the long run interest rate exists. Similarly, when it exists, the long run interest rate is also an increasing function of time. Therefore under model (1.1), either the long term spot interest rate does not exist, or if it exists it can never fall. [8]

5 Concluding remarks

A large part of the term structure literature interprets one of underlying factors as a level factor model. In this paper, we have considered a single factor model, interpretable as a level factor. We have seen that this model is compatible with the positivity of rate and the no-arbitrage restrictions under rather unrealistic rate dynamics. The short term interest rate is stochastic, but an increasing function of time. Moreover, if the long run interest rate exists, the short term interest rate has a nonzero probability to coincide with the previous rate. Both facts do not correspond to observed evolutions of short term rates. [9]

This introduces some doubt on the relevance of the level interpretation of the underlying factor, but also on the practice which consists in considering parallel drift as basic shocks on a term structure, without checking if these drifts are compatible with the existing term structure pattern and no arbitrage [see the discussion in appendix 1].

[8] Several authors argue that this property of the long term spot rate is a consequence of no-arbitrage [Dybvig, Ingersoll, Ross (1996), El Karoui, Frachot, Geman (1998)], but prove this property under additional assumptions. These assumptions can be a predetermined long interest rate [DIR (1996)], or a long rate satisfying a diffusion equation [EFG (1998)].

[9] As noted in Andersen, Lund (1997), "We simply do not know of any theoretical rationale for explosive interest rate series".

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Appendix 1
Parallel Shift in the Term Structure Literature

i) **From a coupon bond to a zero-coupon bond.**

Very early in the literature [Macaulay (1931), (1938), p48] is the idea to replace a coupon bond by an "equivalent" zero-coupon bond, to facilitate the comparison of bonds with varying maturities and seasoning. More precisely, let us consider at time t a coupon bond with nonnegative coupons $A_h, h = 0, 1 \ldots$ at the different times-to-maturity, and a current price $\Pi_t(A)$. To create the "equivalent" zero-coupon bond, we have to define the corresponding rate and time-to-maturity. They are usually defined as follows : the equivalent rate, or yield, is the solution $r_t^I(A)$ of the equation :

$$\Pi_t(A) = \sum_{h=0}^{\infty} A_h \exp[-h r_t^I(A)].$$

The equivalent time-to-maturity is the so-called Macaulay's duration [10] defined by :

$$D_t^I(A) = \sum_{h=0}^{\infty} h A_h \exp[-h r_t^I(A)] / \sum_{h=0}^{\infty} A_h \exp[-h r_t^I(A)].$$

It is equal to the average time-to-maturity of the coupons weighted by the discounted coupons.

In a modern terminology, these two notions are an implied rate and an implied time-to-maturity, since they are computed from a misspecified term structure model, which assumes a flat term structure, possibly varying in time :

$$r(t, h) = X_t, \forall h, \text{ say.} \tag{A.1}$$

ii) **The duration as a sensitivity parameter.**

It is also well-known that the duration is a measure of the sensitivity of the bond price with respect to shock on the level of interest rate, which does not depend on time-to-maturity due to the assumption of a flat term

[10] The eponym "Macaulay's duration" has been introduced in Fisher, Weil (1971), p416.

structure [see e.g. Hicks (1939), Redington (1952), Fisher (1966), Hopewell, Kaufman (1973)] :

$$D_t^I(A) = \frac{\partial \log \Pi_t(A)}{\partial r_t^I(A)}. \tag{A.2}$$

iii) **Consistency with no-arbitrage**

The flat term structure model underlying the derivation and interpretation of the yield and duration hardly coincides with the true term structure. Nevertheless, this misspecified model should be consistent with no-arbitrage restrictions.

From (A.1), we note that the underlying model is a special case of model (1.1) with $c(h) = 0, \forall h$. By arguments similar to the arguments in Section 4, we deduce that, under no-arbitrage, the dynamic of (X_t) is such that :

$$X_{t+1} = X_t, \forall t. \tag{A.3}$$

Therefore, under no-arbitrage, the term structure is flat at all dates if and only if it is also time independent :

$$\begin{aligned} & X_t = X_0, \forall t, \\ \Leftrightarrow \quad & r(t,h) = X_0, \forall t, h \text{ say.} \end{aligned} \tag{A.4}$$

Thus the no-arbitrage restriction induces strong links between the pattern of the term structure (i.e. flat) and its evolution (i.e. constant in time).

The underlying model can be stochastic if the initial value is stochastic, but the associated notion of shock is very special. Indeed, a shock on X_0 can be introduced : this shock will have a drift effect not only on the term structure at date t, but on the term structures of all dates jointly. Under no-arbitrage on the underlying misspecified model, a transitory shock on a term structure, that is a shock specific to date t, cannot be defined. This shock is systematically permanent.[11]

Note finally that the aim of Assumption A.2 was to eliminate this very special limiting case.

[11] Theorem 4 in Ingersoll, Skelton, Weil (1978) provides an alternative proof of the result. They show that a transitory shift in a flat term structure is not arbitrage free.