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Can One Really Estimate Nonstationary GARCH Models ?

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Can one really estimate nonstationary GARCH models?

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Abstract

Jensen and Rahbek (2004a) claim that consistency and asymptotic normality hold for the quasi-maximum likelihood estimator (QMLE) of (ω_0, α_0) in nonstationary ARCH(1) models. In fact their result only concerns a constrained QMLE, in which the intercept is fixed, and under a reinforced nonstationarity condition. Under this condition, we prove that the standard QMLE of α_0 is strongly consistent and asymptotically normal. Numerical experiments reveal that QMLE of ω_0 is likely to be inconsistent.

Keywords: ARCH, asymptotic normality, inconsistent estimator, nonstationarity, quasi-maximum likelihood estimation, strong consistency.

Résumé

Jensen et Rahbek (2004a) affirment que l'estimateur du quasi-maximum de vraisemblance (QMV) de (ω_0, α_0) est convergent et asymptotiquement normal, dans les modèles ARCH(1) non stationnaires. En fait, leur résultat concerne seulement un estimateur du QMV contraint, dans lequel le ω_0 est fixé, et sous une hypothèse de non stationnarité renforcée. Sous cette condition, nous montrons que l'estimateur du QMV standard de α_0 est fortement convergent et asymptotiquement normal. Des expériences numériques montrent que la convergence de l'estimateur du QMV de ω_0 est douteuse.

Mots-clés : ARCH, normalité asymptotique, estimateur non convergent, non stationnarité, quasi-maximum de vraisemblance, convergence forte.

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1 Introduction

In a recent paper, Jensen and Rahbek (henceforth JR) (2004a) claim that "consistency and asymptotic normality of the quasi-maximum likelihood estimator in the linear ARCH model" hold when the parameter is allowed "to be in the region where no stationary version of the process exists." The only model considered in their paper is in fact the two-parameters first-order ARCH. More importantly, the estimator studied in their paper is not the usual quasi-maximum likelihood estimator (QMLE). It is a constrained estimator of the ARCH parameters, where the first component is known. In a companion paper, JR (2004b) obtain a similar result for an estimator of the sub-vector (α_0, β_0) of the parameter vector, in the GARCH(1,1) framework. This estimator is also a constrained QMLE, in which the true intercept coefficient, ω_0, is replaced by an arbitrary fixed value ω. Precise definitions are given in the next section.

Apart from a minor point concerning the nonstationarity condition, our remarks do not concern the validity of the results established by JR, nor their proofs which are elegantly conducted. However we think that the initial claim is untrue, and can lead to severe misinterpretations of the role of stationarity in the implementation of GARCH models. There is a tendency among practitioners, and also theoreticians, to believe that the QMLE for GARCH is consistent and asymptotically normal without any stationarity constraint[1]. The aim of this Note is to draw attention on three points:

i) the estimator defined in the above-mentionned papers IS NOT the QMLE, which is the most widely used estimator of GARCH models,

ii) the asymptotic behavior of the QMLE of ω_0 is unknown and thus,

iii) despite their theoretical interest, those results have little, if any, consequence for the use of GARCH in practice.

This Note brings a complete answer to point i) in the ARCH(1) case, and gives highlights on points ii) and iii). More precisely, we show in Section 3 that the QMLE of α_0 is consistent

[1] See for instance Linton, Pan and Wang (2006): " Jensen and Rahbek (2004 a, 2004 b) were the first to consider the asymptotic theory of the QMLE for non-stationary ARCH/GARCH models. They showed that the likelihood-based estimator for the parameters in the first order ARCH/GARCH model is consistent and asymptotically Gaussian in the entire parameter region regardless of whether the process is strictly stationary or explosive." See also Caporale, Ntantamis, Pantelidis and Pittis (2005).

and asymptotically normal when the squared process almost surely converges to infinity.

We start be introducing the notation and main issues.

2 Notation and discussion

JR (2004a) consider the ARCH(1) model, as given by

$$\begin{cases} \epsilon_t = \sqrt{h_t}\eta_t, \quad t = 1, 2, \dots \\ h_t = \omega_0 + \alpha_0 \epsilon_{t-1}^2 \end{cases} \tag{2.1}$$

under classical assumptions on the noise: the sequence (η_t) is assumed independent and identically distributed (iid) with zero mean and unit variance, and such that $\kappa_\eta = E\eta_1^4 < \infty$. In JR (2004a) the parameter $\omega_0 > 0$ **is assumed to be known** (for instance $\omega_0 = 1$), and only α_0 has to be estimated. They consider a **constrained QMLE** of α_0 defined by

$$\hat{\alpha}_n^c(\omega_0) = \arg \min_{\alpha \in [0,\infty)} \frac{1}{n} \sum_{t=1}^{n} \ell_t(\alpha), \quad \ell_t(\alpha) = \frac{\epsilon_t^2}{\sigma_t^2(\alpha)} + \log \sigma_t^2(\alpha), \tag{2.2}$$

where $\sigma_t^2(\alpha) = \omega_0 + \alpha\epsilon_{t-1}^2$, and an initial value is introduced for ϵ_0^2 (for instance $\epsilon_0^2 = 0$).

The necessary and sufficient condition for the existence of a strictly stationary solution to (2.1) is $E \log(\alpha_0 \eta_1^2) < 0$. When strict stationarity does not hold, i.e. under the assumption

$$\alpha_0 \geq \exp\left\{-E \log \eta_1^2\right\}, \tag{2.3}$$

JR (2004a) state that

$$\hat{\alpha}_n^c(\omega_0) \quad \text{is consistent} \tag{2.4}$$

and asymptotically normal:

$$\sqrt{n}\left(\hat{\alpha}_n^c(\omega_0) - \alpha_0\right) \xrightarrow{d} \mathcal{N}\left\{0, (\kappa_\eta - 1)\alpha_0^2\right\}, \quad \text{as } n \to \infty. \tag{2.5}$$

JR (2004a) use a result by Nelson (1990) stating that $h_t \to \infty$ almost surely as $t \to \infty$. This result is correct under the assumption

$$\alpha_0 > \exp\left\{-E \log \eta_1^2\right\}, \tag{2.6}$$

but Klüppelberg, Lindner and Maller (2004) note that the arguments given by Nelson are in failure when $\alpha_0 = \exp\left\{-E \log \eta_1^2\right\}$. These authors show that $h_t \to \infty$ in probability instead of almost surely. It follows that the results (2.4) and (2.5) are proven under the reinforced nonstationarity condition (2.6), but not under the general nonstationarity condition (2.3). We also note that JR (2004a) do not give a precise meaning to (2.4).

JR (2004b) consider a similar estimator for the sub-vector (α_0, β_0) of the parameter of a GARCH(1,1). The unknown parameter ω_0 is replaced by a fixed value ω, which is no longer assumed to be equal to ω_0. Thus (2.3) and (2.4) remain valid when $\hat{\alpha}_n^c(\omega_0)$ is replaced by $\hat{\alpha}_n^c(1)$, say. This point is important for practical purposes. From (2004b), it seems that (2.4) has to be understood as a convergence which holds in probability and locally, that is when $\hat{\alpha}_n^c(\omega_0)$ minimises $\sum_{t=1}^n \ell_t(\alpha)$ in a small neighborhood of α_0.

3 QMLE of a nonstationary ARCH(1) model

In this section we consider the QMLE of an ARCH(1), defined as a measurable solution of

$$(\hat{\omega}_n, \hat{\alpha}_n) = \arg\min_{\theta \in \Theta} \frac{1}{n} \sum_{t=1}^{n} \ell_t(\theta), \quad \ell_t(\theta) = \frac{\epsilon_t^2}{\sigma_t^2(\theta)} + \log \sigma_t^2(\theta), \tag{3.1}$$

where $\theta = (\omega, \alpha)$, Θ is a compact subset of $(0, \infty)^2$, and $\sigma_t^2(\theta) = \omega + \alpha\epsilon_{t-1}^2$ for $t = 1, \dots, n$ (with an initial value for ϵ_0^2). We will use the next result establishing the rate of the almost sure convergence of ϵ_t^2 to infinity under the reinforced nonstationarity condition (2.6).

Lemma 3.1 *Let the ARCH(1) defined by (2.1), with initial condition $\epsilon_0^2 \geq 0$. Then, if (2.6) holds,*

$$\frac{1}{h_n} = o(\rho^n) \qquad \text{and} \qquad \frac{1}{\epsilon_n^2} = o(\rho^n)$$

almost surely as $n \to \infty$ for any constant ρ such that

$$1 > \rho > \exp\left\{-E \log \eta_1^2\right\} / \alpha_0. \tag{3.2}$$

This lemma allows to obtain the strong consistency and asymptotic normality of the QMLE of α_0.

Theorem 3.1 *Under the assumptions of Lemma 3.1, and if $\theta_0 = (\omega_0, \alpha_0) \in \Theta$, the QMLE defined in (3.1) satisfies*

$$\hat{\alpha}_n \to \alpha_0 \qquad a.s. \tag{3.3}$$

and, if θ_0 belongs to the interior of Θ,

$$\sqrt{n}\,(\hat{\alpha}_n - \alpha_0) \xrightarrow{d} \mathcal{N}\left\{0, (\kappa_\eta - 1)\alpha_0^2\right\} \tag{3.4}$$

as $n \to \infty$.

As already noted, this result as well as the results in JR papers do not give any insight on the asymptotic behavior of the QMLE of ω_0. However, a few remarks and numerical illustrations are in order concerning this issue.

In the proof of Theorem 3.1 it is shown that the score vector satisfies

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{\partial}{\partial\theta}\ell_t(\theta_0)\xrightarrow{d}\mathcal{N}\left\{0, J=(\kappa_\eta-1)\begin{pmatrix}0 & 0\\ 0 & \alpha_0^{-1}\end{pmatrix}\right\}.$$

The form of the asymptotic covariance matrix J of the score vector shows that, for n sufficiently large and almost surely, the variation of the log-likelihood $n^{-1/2}\sum_{t=1}^{n}\log\ell_t(\theta)$ is negligible when θ varies between (ω_0,α_0) and (ω_0+h,α_0) for small h. This leads to think that the QMLE of ω_0 is certainly inconsistent without the strict stationarity condition. Figure 3 presents some numerical evidence on the performance of the QMLE in finite samples through a simulation study. In all experiments, we use the sample size $n=200$ and $n=4000$ with 100 replications. The data of the top panel are generated from the second-order stationary ARCH(1) model (2.1) with the true parameter $\theta_0=(1,0.95)$. The data of the middle panel are generated from the strict stationary ARCH(1) model with $\theta_0=(1,1.5)$ and infinite variance. In those two panels the results are very similar, confirming that the second-order stationarity condition is not necessary for the use of the QMLE. The bottom panel, obtained for the explosive ARCH(1) model with $\theta_0=(1,4)$, confirms the asymptotic results for the QMLE of α_0. It also illustrates the impossibility to estimate parameter ω_0 with a reasonable accuracy under the nonstationarity condition (2.6). The results even worsen when the sample size increases.

4 Conclusion

To summarize, the results obtained by JR are interesting from a theoretical point of view, because they showed that strict stationarity is not compulsory for the estimation of ARCH coefficients. However, the scope of such results is much more limited than announced. More importantly, erroneous conclusions can be drawn from those results. To counterbalance the latter point, in this Note we showed that

i) the estimator used in JR (2004a, 2004b) is not the usual QMLE,

ii) the QMLE of α_0 is indeed strongly consistent and asymptotically normal but a stronger non-stationarity condition is required,

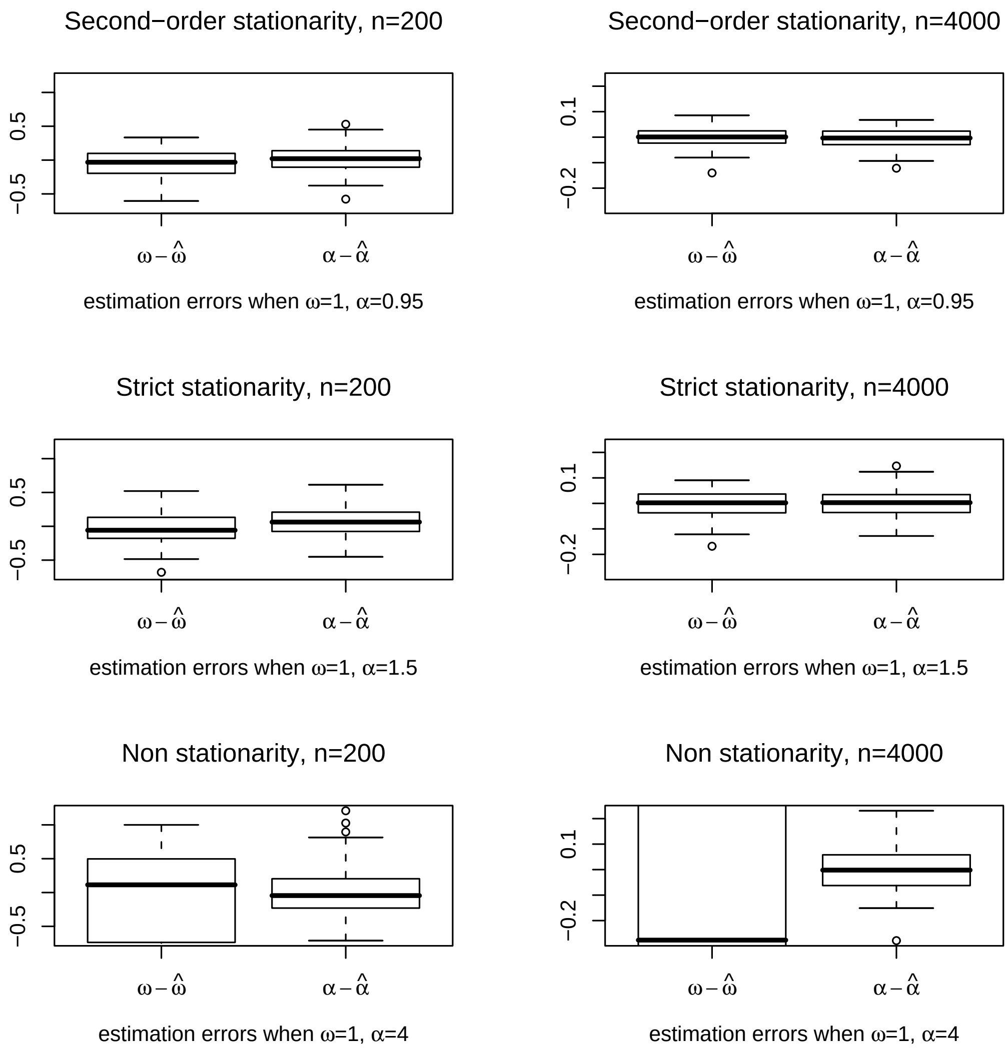

Figure 1: Boxplots of estimation errors for the QMLE of the parameters ω_0 and α_0 of an ARCH(1), with $\eta_t \sim \mathcal{N}(0,1)$.

iii) no asymptotic result holds for the parameter ω_0. Numerical experiments lead to think that it is inconsistent.

We conclude by recalling that a nonstationary GARCH generates explosive trajectories which have little compatibility with real financial series. The study of the behavior of the QMLE in this framework thus has little practical significance.

5 Proofs

Proof of Lemma 3.1. We have

$$\begin{aligned}\rho^n h_n &= \rho^n \omega_0 \left\{1 + \sum_{t=1}^{n-1} \alpha_0^t \eta_{n-1}^2 \dots \eta_{n-t}^2\right\} + \rho^n \alpha_0^n \eta_{n-1}^2 \dots \eta_1^2 \epsilon_0^2 \\ &\geq \rho^n \omega_0 \prod_{t=1}^{n-1} \alpha_0 \eta_t^2. \end{aligned} \tag{5.1}$$

Thus

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \rho^n h_n &\geq \lim_{n\to\infty} \frac{1}{n} \left\{\log \rho\omega_0 + \sum_{t=1}^{n-1} \log \rho\alpha_0\eta_t^2\right\} \\ &= E \log \rho\alpha_0\eta_1^2 > 0,\end{aligned}$$

using (3.2) for the last inequality. It follows that $\log \rho^n h_n$, and hence $\rho^n h_n$, tends to $+\infty$ almost surely as $n \to \infty$. For any real-valued function f, let $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$, so that $f(x) = f^+(x) - f^-(x)$. Since $E \log^+ \eta_1^2 \leq E\eta_1^2 = 1$, we have $E|\log \eta_1^2| = \infty$ if and only if $E \log \eta_1^2 = -\infty$. Thus (2.6) implies $E|\log \eta_1^2| < \infty$, which entails that $\log \eta_n^2/n \to 0$ almost surely as $n \to \infty$. Therefore, using (5.1), $\liminf_{n\to\infty} n^{-1} \log \rho^n \eta_n^2 h_n \geq E \log \rho\alpha_0\eta_1^2 > 0$, and $\rho^n \epsilon_n^2 = \rho^n \eta_n^2 h_n \to +\infty$ almost surely by already given arguments.

$\square$

Proof of (3.3). Note that $(\hat{\omega}_n, \hat{\alpha}_n) = \arg\min_{\theta\in\Theta} Q_n(\theta)$, where

$$Q_n(\theta) = \frac{1}{n} \sum_{t=1}^{n} \{\ell_t(\theta) - \ell_t(\theta_0)\}.$$

We have

$$\begin{aligned}Q_n(\theta) &= \frac{1}{n} \sum_{t=1}^{n} \eta_t^2 \left\{\frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)} - 1\right\} + \log \frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} \\ &= \frac{1}{n} \sum_{t=1}^{n} \eta_t^2 \frac{(\omega_0 - \omega) + (\alpha_0 - \alpha)\epsilon_{t-1}^2}{\omega + \alpha\epsilon_{t-1}^2} + \log \frac{\omega + \alpha\epsilon_{t-1}^2}{\omega_0 + \alpha_0\epsilon_{t-1}^2}.\end{aligned}$$

For any $\theta \in \Theta$, we have $\alpha \neq 0$. Letting

$$O_n(\alpha) = \frac{1}{n}\sum_{t=1}^{n} \eta_t^2 \frac{(\alpha_0 - \alpha)}{\alpha} + \log \frac{\alpha}{\alpha_0}$$

and

$$d_t = \frac{\alpha(\omega_0 - \omega) - \omega(\alpha_0 - \alpha)}{\alpha(\omega + \alpha\epsilon_{t-1}^2)},$$

we have

$$Q_n(\theta) - O_n(\alpha) \quad = \quad \frac{1}{n}\sum_{t=1}^{n} \eta_t^2 d_{t-1} + \frac{1}{n}\sum_{t=1}^{n} \log \frac{(\omega + \alpha\epsilon_{t-1}^2)\alpha_0}{(\omega_0 + \alpha_0\epsilon_{t-1}^2)\alpha} \to 0 \quad \text{a.s.}$$

since, by Lemma 3.1, $\epsilon_t^2 \to \infty$ almost surely as $t \to \infty$. Moreover this convergence is uniform on the compact set Θ:

$$\lim_{n\to\infty} \sup_{\theta\in\Theta} |Q_n(\theta) - O_n(\alpha)| = 0 \quad \text{a.s.} \tag{5.2}$$

Let α_0^- and α_0^+ denote two constants such that $0 < \alpha_0^- < \alpha_0 < \alpha_0^+$. Introducing $\hat{\sigma}_\eta^2 = n^{-1}\sum_{t=1}^{n} \eta_t^2$, the solution of

$$\alpha_n^* = \arg\min_{\alpha} O_n(\alpha)$$

is $\alpha_n^* = \alpha_0 \hat{\sigma}_\eta^2$. This solution belongs to the interval (α_0^-, α_0^+) for sufficiently large n. Thus

$$\alpha_n^{**} = \arg \min_{\alpha \notin (\alpha_0^-, \alpha_0^+)} O_n(\alpha) \in \{\alpha_0^-, \alpha_0^+\}$$

and

$$\lim_{n\to\infty} O_n(\alpha_n^{**}) = \min\left\{\lim_{n\to\infty} O_n(\alpha_0^-), \lim_{n\to\infty} O_n(\alpha_0^+)\right\} > 0.$$

This result and (5.2) show that almost surely

$$\lim_{n\to\infty} \min_{\theta\in\Theta,\, \alpha\notin(\alpha_0^-, \alpha_0^+)} Q_n(\theta) > 0.$$

Since $\min_\theta Q_n(\theta) \leq Q_n(\theta_0) = 0$, it follows that

$$\lim_{n\to\infty} \arg\min_{\theta\in\Theta} Q_n(\theta) \in (0, \infty) \times (\alpha_0^-, \alpha_0^+).$$

Because the interval (α_0^-, α_0^+) containing α_0 can be chosen arbitrarily small, we get the convergence in (3.3).

$\square$

The following result will be used to establish the asymptotic normality of the QMLE of α_0.

Lemma 5.1 *Under the assumptions of Theorem 3.1, we have*

$$\sum_{t=1}^{\infty} \sup_{\theta\in\Theta} \left| \frac{\partial}{\partial\omega} \ell_t(\theta) \right| \quad < \quad \infty \quad a.s., \tag{5.3}$$

$$\sum_{t=1}^{\infty} \sup_{\theta\in\Theta} \left\| \frac{\partial^2}{\partial\omega\partial\theta} \ell_t(\theta) \right\| \quad < \quad \infty \quad a.s., \tag{5.4}$$

$$\sup_{\theta\in\Theta} \left| \frac{1}{n} \sum_{t=1}^{n} \frac{\partial^2}{\partial\alpha^2} \ell_t(\omega,\alpha_0) - \frac{1}{\alpha_0^2} \right| \quad = \quad o(1) \quad a.s., \tag{5.5}$$

$$\frac{1}{n} \sum_{t=1}^{n} \sup_{\theta\in\Theta} \left| \frac{\partial^3}{\partial\alpha^3} \ell_t(\theta) \right| \quad = \quad O(1) \quad a.s., \tag{5.6}$$

Proof. Using Lemma 3.1, there exist a real random variable K and a constant $\rho \in (0,1)$ independent of θ and t such that

$$\begin{aligned} \left| \frac{\partial}{\partial\omega} \ell_t(\theta) \right| &= \left| \frac{1}{\sigma_t^2(\theta)} \frac{\partial\sigma_t^2(\theta)}{\partial\omega} \left(1 - \frac{\epsilon_t^2}{\sigma_t^2(\theta)} \right) \right| \\ &= \left| \frac{-(\omega_0 + \alpha_0\epsilon_{t-1}^2)\eta_t^2}{(\omega + \alpha\epsilon_{t-1}^2)^2} + \frac{1}{\omega + \alpha\epsilon_{t-1}^2} \right| \leq K\rho^t(\eta_t^2 + 1). \end{aligned}$$

Since $\sum_{t=1}^{\infty} K\rho^t(\eta_t^2+1)$ has a finite expectation, it is almost surely finite. Thus (5.3) is proved, and (5.4) can be obtained by the same arguments. We have

$$\begin{aligned} \frac{\partial^2 \ell_t(\omega,\alpha_0)}{\partial\alpha^2} - \frac{1}{\alpha_0^2} &= \left\{ 2\frac{(\omega_0 + \alpha_0\epsilon_{t-1}^2)\eta_t^2}{\omega + \alpha_0\epsilon_{t-1}^2} - 1 \right\} \frac{\epsilon_{t-1}^4}{(\omega + \alpha_0\epsilon_{t-1}^2)^2} - \frac{1}{\alpha_0^2} \\ &= \left(2\eta_t^2 - 1\right) \frac{\epsilon_{t-1}^4}{(\omega + \alpha_0\epsilon_{t-1}^2)^2} - \frac{1}{\alpha_0^2} + r_{1,t} \\ &= 2\left(\eta_t^2 - 1\right) \frac{1}{\alpha_0^2} + r_{1,t} + r_{2,t} \end{aligned}$$

where

$$\sup_{\theta\in\Theta} |r_{1,t}| = \sup_{\theta\in\Theta} \left| \frac{2(\omega_0 - \omega)\eta_t^2}{(\omega + \alpha_0\epsilon_{t-1}^2)} \frac{\epsilon_{t-1}^4}{(\omega + \alpha_0\epsilon_{t-1}^2)^2} \right| = o(1) \quad \text{a.s.}$$

and

$$\begin{aligned} \sup_{\theta\in\Theta} |r_{2,t}| &= \sup_{\theta\in\Theta} \left| (2\eta_t^2 - 1) \left\{ \frac{\epsilon_{t-1}^4}{(\omega + \alpha_0\epsilon_{t-1}^2)^2} - \frac{1}{\alpha_0^2} \right\} \right| \\ &= \sup_{\theta\in\Theta} \left| (2\eta_t^2 - 1) \left\{ \frac{\omega^2 + 2\alpha_0\epsilon_{t-1}^2}{\alpha_0^2(\omega + \alpha_0\epsilon_{t-1}^2)^2} \right\} \right| = o(1) \quad \text{a.s.} \end{aligned}$$

as $t \to \infty$. Therefore (5.5) is established. To prove (5.6), it suffices to remark that

$$\begin{aligned} \left| \frac{\partial^3}{\partial\alpha^3} \ell_t(\theta) \right| &= \left| \left\{ 2 - 6\frac{(\omega_0 + \alpha_0\epsilon_{t-1}^2)\eta_t^2}{\omega + \alpha\epsilon_{t-1}^2} \right\} \left(\frac{\epsilon_{t-1}^2}{\omega + \alpha\epsilon_{t-1}^2} \right)^3 \right| \\ &\leq \left\{ 2 + 6\left(\frac{\omega_0}{\omega} + \frac{\alpha_0}{\alpha} \right) \eta_t^2 \right\} \frac{1}{\alpha^3}. \end{aligned}$$

$\square$

Proof of (3.4). Notice that we cannot use that fact that the derivative of the criterion cancels at $\hat{\theta}_n = (\hat{\omega}_n, \hat{\alpha}_n)$ since we only have the convergence of $\hat{\alpha}_n$ to α_0. Thus the minimum could lie on the boundary of Θ, even asymptotically. However, the partial derivative with respect to α is asymptotically equal to zero at the minimum since $\hat{\alpha}_n \to \alpha_0$ and $(\omega_0, \alpha_0) \in \overset{\circ}{\Theta}$. Hence, an expansion of the criterion derivative gives

$$\begin{pmatrix} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \frac{\partial}{\partial \omega} \ell_t(\hat{\theta}_n) \\ 0 \end{pmatrix} = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \frac{\partial}{\partial \theta} \ell_t(\theta_0) + J_n \sqrt{n}(\hat{\theta}_n - \theta_0) \tag{5.7}$$

where J_n is a 2×2 matrix whose elements have the form

$$J_n(i,j) = \frac{1}{n} \sum_{t=1}^{n} \frac{\partial^2}{\partial \theta_j \partial \theta_j} \ell_t(\theta_{i,j}^*)$$

where $\theta_{i,j}^* = (\omega_{i,j}^*, \alpha_{i,j}^*)$ is between $\hat{\theta}_n$ and θ_0. By Lemma 3.1 and from the Lindeberg central limit theorem for martingale differences we have

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \frac{\partial}{\partial \alpha} \ell_t(\theta_0) &= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (1 - \eta_t^2) \frac{\epsilon_{t-1}^2}{\omega_0 + \alpha_0 \epsilon_{t-1}^2} \\ &= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (1 - \eta_t^2) \frac{1}{\alpha_0} + o_P(1) \\ &\xrightarrow{d} \mathcal{N}\left(0, \frac{\kappa_\eta - 1}{\alpha_0^2}\right). \end{aligned} \tag{5.8}$$

By (5.4), in Lemma 5.1, and the compactness of Θ we have

$$J_n(2,1)\sqrt{n}(\hat{\omega}_n - \omega_0) \leq \sum_{t=1}^{\infty} \sup_{\theta \in \Theta} \left\| \frac{\partial^2}{\partial \omega \partial \theta} \ell_t(\theta) \right\| \frac{1}{\sqrt{n}} (\hat{\omega}_n - \omega_0) \to 0 \quad \text{a.s.} \tag{5.9}$$

An expansion of the function

$$\alpha \mapsto \frac{1}{n} \sum_{t=1}^{n} \frac{\partial^2}{\partial \alpha^2} \ell_t(\omega_{2,2}^*, \alpha)$$

gives

$$J_n(2,2) = \frac{1}{n} \sum_{t=1}^{n} \frac{\partial^2}{\partial \alpha^2} \ell_t(\omega_{2,2}^*, \alpha_0) + \frac{1}{n} \sum_{t=1}^{n} \frac{\partial^3}{\partial \alpha^3} \ell_t(\omega_{2,2}^*, \alpha^*)(\alpha_{2,2}^* - \alpha_0)$$

where α^* is between $\alpha_{2,2}^*$ and α_0. Using (5.5), (5.6) and (3.3) we get

$$J_n(2,2) \to \frac{1}{\alpha_0^2} \quad \text{a.s.} \tag{5.10}$$

The conclusion follows, by considering the second component in (5.7) and from (5.8), (5.9) and (5.10).

□

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