INSTITUT NATIONAL DE LA STATISTIQUE ET DES ETUDES ECONOMIQUES
Série des Documents de Travail du CREST
(Centre de Recherche en Economie et Statistique)

n° 2008-05

Barlett's Formula for Non Linear Processes

C. FRANCQ[1]
J.-M. ZAKOÏAN[2]

[1] Université Lille III, GREMARS-EQUIPPE, BP 60149, 59653 Villeneuve d'Ascq Cédex, France.
Tél. : 33 (0) 3 20 41 64 87 (francq@univ-lille3.fr)
[2] CREST and GREMARS-EQUIPPE, 15 Boulevard Gabriel Péri, 92245 Malakoff Cédex, France.
Tél. : 33 (0) 1 41 17 78 25 (zakoian@ensae.fr).

Bartlett's formula for non linear processes

Christian Francq*and Jean-Michel Zakoïan†

Abstract

A Bartlett-type formula is proposed for the asymptotic distribution of the sample autocorrelations of nonlinear processes. The asymptotic covariances between sample autocorrelations are expressed as the sum of two terms. The first term corresponds to the standard Bartlett's formula for linear processes, involving only the autocorrelation function of the observed process. The second term, which is specific to nonlinear processes, involves the autocorrelation function of the observed process, the kurtosis of the linear innovation process and the autocorrelation function of its square. This formula is obtained under a symmetry assumption on the linear innovation process. An application to GARCH models is proposed.

Keywords : Bartlett formula, Empirical utocorrelations, GARCH model, Linear innovation, Nonlinear processes.

Résumé

Une formule de type Bartlett est proposée pour la loi asymptotique des autocorrélations empiriques de processus non linéaires. Les covariances asymptotiques des autocorrélations empiriques sont exprimées comme somme de deux termes. Le premier correspond à la formule de Bartlett standard pour processus linéaires et ne dépend que de la fonction d'autocorrélation du processus observé. Le second terme, spécifique aux processus non linéaires, dépend de la fonction d'autocorrélation du processus observé, de celle de son carré et du coefficient de kurtosis de l'innovation linéaire du carré. La formule est obtenue sous une hypothèse de symétrie du processus d'innovation linéaire. Une application aux modèles GARCH est proposée.

Keywords : Autocorrélations empiriques, formule de Bartlett, innovation linéaire, modèle GARCH, processus non linéaires.

*Université Lille III, GREMARS-EQUIPPE, BP 60149, 59653 Villeneuve d'Ascq cedex, France. E-mail: francq@univ-lille3.fr, tel: 33.3.20.41.64.87

†CREST and GREMARS-EQUIPPE, 15 Boulevard Gabriel Péri, 92245 Malakoff Cedex, France. E-mail: zakoian@ensae.fr, tel: 33.1.41.17.78.25

1 Introduction

In time series analysis, the estimation of the autocorrelation function plays a crucial role, in particular for identification problems (see *e.g.* Brockwell et Davis (1991)). Bartlett (1946) derived an explicit formula for the asymptotic covariance between sample autocorrelations. This formula is given in most time series textbooks, and most time series software packages plot the sample autocorrelation function with significance limits obtained from this formula[1]. Bartlett's formula was obtained for linear processes and it is well known (see *e.g.* Berlinet and Francq (1997), Diebold (1986), Romano and Thombs (1996)) that Bartlett's formula may be completely wrong for series exhibiting conditional heteroscedasticity or any other form of nonlinearity. The aim of this paper is to generalize Bartlett's formula to a wide class of nonlinear processes.

In order to give a precise definition of a linear process, first recall that the Wold decomposition (see Brockwell and Davis (1991), Section 5.7) states that any purely non deterministic stationary process can be written in the form

$$X_t = \sum_{\ell=-\infty}^{\infty} \phi_\ell \epsilon_{t-\ell}, \qquad (\epsilon_t) \sim \mathrm{WN}(0, \sigma^2) \tag{1.1}$$

where $\sum_\ell \phi_\ell^2 < \infty$. The process (ϵ_t) is called the linear innovation process of the process $X = (X_t)$, and the notation $(\epsilon_t) \sim \mathrm{WN}(0, \sigma^2)$ signifies that (ϵ_t) is a *weak white noise*, that is a stationary sequence of centered and uncorrelated random variables with common variance σ^2. An independent and identically distributed (iid) sequence of random variables with mean 0 and common variance σ^2 is sometimes called a *strong white noise*, and will be denoted by $\mathrm{IID}(0, \sigma^2)$. Obviously a strong white noise is also a weak white noise, because independence entails uncorrelatedness, but the reverse is not true. The process X is said to be *linear* when $(\epsilon_t) \sim \mathrm{IID}(0, \sigma^2)$, and is said to be *nonlinear* in the opposite case. The autoregressive moving average (ARMA) model with iid noise is the leading example of linear process (see *e.g.* Brockwell and Davis, 1991). Examples of nonlinear models include, among many others, the self-exciting threshold autoregressive (SETAR) model (see Tong, 1990), the smooth transition autoregression (STAR) model (see Teräsvirta (2004) and the references therein), the exponential autoregressive (EXPAR) model introduced by Haggan and Ozaki (1981), the bilinear model (see Granger and Andersen, 1978) and

[1]See *e.g.* the function `acf()` of the statistical software R, with its argument `ci.type = c("white", "ma")`.

the generalized autoregressive conditional heteroscedastic (GARCH) model introduced by Engle (1982) and Bollerslev (1986). Because numerous real time-series, in particular stock market returns, exhibit dynamics which can not be well mimicked by ARMA models with iid noises, nonlinear models are becoming more and more employed (see Tong (1990) and Fan and Yao (2003) for reference books on nonlinear time series analysis).

Before fitting any time series model to real data, it is common practice to draw the empirical autocovariances and analyze their significance. Because the standard Bartlett's formula can be unreliable when the underlying series is non linear, it is important to have an appropriate tool which could be used in very general settings. A question is therefore whether the standard Bartlett formula can be extended. More precisely, our aim in this paper is to derive a formula giving the asymptotic covariances between empirical autocovariances, in function of characteristics of the underlying processes. As we will see, the theoretical autocorrelations of the observed process will not suffice to characterize those asymptotic covariances, as is the case in the linear framework. It will also be of interest to know whether the standard Bartlett's formula can provide good approximations of the asymptotic autocovariances when the underlying process is non linear.

The plan of the paper is as follows. In Section 2 we begin by recalling the standard Bartlett's formula. Section 3 states a generalized Bartlett's formula which can be applied to both linear and nonlinear processes. Section 4 illustrates the generalized Bartlett's formula with GARCH models. Proofs are relegated to Section 5.

2 Notation and Bartlett's formula for linear processes

The autocorrelation function of a real-valued stationary process $X = (X_t)$ is defined by

$$\rho_X(\cdot) = \frac{\gamma_X(\cdot)}{\gamma_X(0)}, \qquad \gamma_X(i) = \mathrm{Cov}(X_t, X_{t+i}) \quad \text{for all integers } t, i.$$

Assume that X is centered and that the observations are $X_1, \ldots, X_n$. The autocorrelation $\rho_X(i)$ and autocovariance $\gamma_X(i)$, for $0 \leq i < n$, are generally estimated by their sample versions

$$\hat{\rho}_X(i) = \hat{\rho}_X(-i) = \frac{\hat{\gamma}_X(i)}{\hat{\gamma}_X(0)}, \quad \hat{\gamma}_X(i) = \hat{\gamma}_X(-i) = \frac{1}{n}\sum_{t=1}^{n-i} X_t X_{t+i}.$$

For fixed $m \geq 1$, let us consider the following vectors of sample and theoretical autocovariances and autocorrelations

$$\gamma_m = (\gamma_X(0), \dots, \gamma_X(m)), \qquad \hat{\gamma}_m = (\hat{\gamma}_X(0), \dots, \hat{\gamma}_X(m)),$$

$$\rho_m = (\rho_X(1), \dots, \rho_X(m)) \quad \text{and} \quad \hat{\rho}_m = (\hat{\rho}_X(1), \dots, \hat{\rho}_X(m)).$$

The following theorem is standard (see Brockwell and Davis (1991), Chapter 7) and gives the asymptotic distribution of $\sqrt{n}\,(\hat{\gamma}_m - \gamma_m)$ and $\sqrt{n}\,(\hat{\rho}_m - \rho_m)$ in the case where X is a linear process.

Theorem 2.1 *Let $X = (X_t)$ be a linear process, that is a process satisfying (1.1) with $(\epsilon_t) \sim \mathrm{IID}(0, \sigma^2)$, $\sigma^2 > 0$. Assume also that $E(\epsilon_t^4) = \kappa\sigma^4 < \infty$ and $\sum_{\ell=-\infty}^{\infty} |\phi_\ell| < \infty$. The vectors $\sqrt{n}\,(\hat{\gamma}_m - \gamma_m)$ and $\sqrt{n}\,(\hat{\rho}_m - \rho_m)$ are asymptotically normally distributed with mean zero and variance given by Bartlett's formulas*

$$\lim_{n\to\infty} n\,Cov\{\hat{\gamma}_X(i), \hat{\gamma}_X(j)\} = v_{i,j}, \qquad \lim_{n\to\infty} n\,Cov\{\hat{\rho}_X(i), \hat{\rho}_X(j)\} = w_{i,j},$$

where for $i, j > 0$

$$v_{i,j} = (\kappa - 3)\gamma_X(i)\gamma_X(j) + \sum_{\ell=-\infty}^{\infty} \gamma_X(\ell)\left\{\gamma_X(\ell + j - i) + \gamma_X(\ell - j - i)\right\}, \tag{2.1}$$

$$\begin{aligned} w_{i,j} = \sum_{\ell=-\infty}^{\infty} \rho_X(\ell)\, &\{2\rho_X(i)\rho_X(j)\rho_X(\ell) - 2\rho_X(i)\rho_X(\ell + j) \\ &-2\rho_X(j)\rho_X(\ell + i) + \rho_X(\ell + j - i) + \rho_X(\ell - j - i)\}. \end{aligned} \tag{2.2}$$

It is important to note that the iid assumption on (ϵ_t) is very restrictive. Only linear models, essentially the ARMA models with iid noises, are covered by Theorem 2.1. In view of Wold's decomposition, if one can replace the assumption $(\epsilon_t) \sim \mathrm{IID}(0, \sigma^2)$ by the assumption $(\epsilon_t) \sim \mathrm{WN}(0, \sigma^2)$, then one can cover almost all the stationary nonlinear processes.

3 Bartlett's formula for non linear processes

Standard Bartlett's formula (2.2) only depends on the autocorrelation function of the process $X = (X_t)$, but is restricted to linear processes. The following theorem provides an extension of Bartlett's formula to nonlinear processes which, under a symmetry assumption, involves in addition the Kurtosis of the linear innovations ϵ_t of X and the autocorrelation function ρ_{ϵ^2} of (ϵ_t^2).

Theorem 3.1 *We consider the framework and assumptions of Theorem 2.1, but we relax the linearity assumption* $(\epsilon_t) \sim \text{IID}(0, \sigma^2)$ *and we make the following symmetry assumption*

$$E\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}\epsilon_{t_4} = 0 \qquad \textit{when} \qquad t_1 \neq t_2,\ t_1 \neq t_3 \ \textit{and}\ t_1 \neq t_4. \tag{3.1}$$

Then $\rho_{\epsilon^2} = \sum_{h=-\infty}^{+\infty} \rho_{\epsilon^2}(h)$ *exists, and we have the generalized Bartlett's formula for autocovariances*

$$\lim_{n\to\infty} n\,Cov\{\hat{\gamma}_X(i), \hat{\gamma}_X(j)\} \quad = \quad v_{i,j} + v_{i,j}^*, \tag{3.2}$$

where $v_{i,j}$ *is defined by (2.1) and*

$$\begin{aligned} v_{i,j}^* \quad &= \quad (\kappa - 1)\Big\{(\rho_{\epsilon^2} - 3)\gamma_X(i)\gamma_X(j) \\ &\quad + \sum_{\ell=-\infty}^{\infty} \gamma_X(\ell - i)\left\{\gamma_X(\ell - j) + \gamma_X(\ell + j)\right\}\rho_{\epsilon^2}(\ell)\Big\}. \end{aligned} \tag{3.3}$$

If

$$\sqrt{n}\left(\hat{\gamma}_{0,m} - \gamma_{0,m}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \Sigma_{\hat{\gamma}_{0,m}}\right) \qquad \textit{when } n \to \infty, \tag{3.4}$$

where the elements of $\Sigma_{\hat{\gamma}_{0,m}}$ *are given by (3.2), then*

$$\sqrt{n}\left(\hat{\rho}_m - \rho_m\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \Sigma_{\hat{\rho}_m}\right), \tag{3.5}$$

where the elements of $\Sigma_{\hat{\rho}_m}$, *for* $i, j > 0$, *are given by the generalized Bartlett's formula for autocorrelations*

$$\lim_{n\to\infty} n\,Cov\{\hat{\rho}_X(i), \hat{\rho}_X(j)\} = w_{i,j} + w_{i,j}^*, \tag{3.6}$$

where $w_{i,j}$ *is defined by (2.2) and*

$$\begin{aligned} w_{i,j}^* \quad &= \quad (\kappa - 1)\sum_{\ell=-\infty}^{\infty} \rho_{\epsilon^2}(\ell)\left[2\rho_X(i)\rho_X(j)\rho_X^2(\ell) - 2\rho_X(j)\rho_X(\ell)\rho_X(\ell + i)\right. \\ &\quad \left. -2\rho_X(i)\rho_X(\ell)\rho_X(\ell + j) + \rho_X(\ell + i)\left\{\rho_X(\ell + j) + \rho_X(\ell - j)\right\}\right]. \end{aligned} \tag{3.7}$$

We now give a series of remarks.

Remark 3.1 Following Remark 1 of Theorem 7.2.2 in Brockwell and Davis (1991), $w_{i,j}$ can also be written as

$$w_{i,j} = \sum_{\ell=1}^{\infty} w_i(\ell) w_j(\ell), \qquad \text{where} \qquad w_i(\ell) = \left\{2\rho_X(i)\rho_X(\ell) - \rho_X(\ell + i) - \rho_X(\ell - i)\right\}.$$

Similarly we have

$$w_{i,j}^* = (\kappa - 1) \sum_{\ell=1}^{\infty} \rho_{\epsilon^2}(\ell) w_i(\ell) w_j(\ell),$$

which shows that, whenever $\rho_X(\cdot)$, κ and $\rho_{\epsilon^2}(\cdot)$ are available, the standard and generalized Bartlett's formulas are computed very similarly.

Remark 3.2 Even for non linear processes, standard Bartlett's coefficients $v_{i,j}$ and $w_{i,j}$ provide good approximations of $\sqrt{n}\mathrm{Cov}(\hat{\gamma}_X(i), \hat{\gamma}_X(j))$ and $\sqrt{n}\mathrm{Cov}(\hat{\rho}_X(i), \hat{\rho}_X(j))$ when i or j is very large, because

$$v_{i,j}^* \to 0 \quad \text{and} \quad w_{i,j}^* \to 0 \qquad \text{when} \quad i \to \infty \quad \text{or} \quad j \to \infty.$$

Note however that, for fixed (i, j), it is easy to find examples of nonlinear processes such that $v_{i,j}^*/v_{i,j}$ and $w_{i,j}^*/w_{i,j}$ are arbitrarily large.

The following remark concerns the technical assumptions of the theorem.

Remark 3.3 The proof of the theorem reveals that the symmetry assumption (3.1) is only needed to obtain a tractable form for the asymptotic covariances, but is not required for their existence. Note also that (3.4) is not entailed by the assumptions made in Theorem 3.1, but general assumptions, such as mixing assumptions, are available in the literature in order to obtain a central limit theorem implying (3.4) and (3.5) (see *e.g.* Berlinet and Francq (1997) or Romano and Thombs (1996)).

The following remark shows that the validity of the standard Bartlett's formulas is actually not limited to the case where ϵ_t is a strong noise.

Remark 3.4 When the ϵ_t^2's are uncorrelated the standard Bartlett formulas apply because

$$v_{i,j}^* = -2(\kappa - 1)\gamma_X(i)\gamma_X(j) + (\kappa - 1)\gamma_X(i) \{\gamma_X(j) + \gamma_X(-j)\} = 0$$

and $w_{i,j}^* = 0$.

We now consider the particular case where X is a weak white noise.

Corollary 3.1 (Weak white noise) *If $X = (\epsilon_t)$, where (ϵ_t) satisfies the assumptions of Theorem 3.1, then for $i, j \geq 0$, the generalized Bartlett's formula for autocovariances (3.2) holds with*

$$\begin{cases} v_{i,j} = v_{i,j}^* = 0 & \text{if} \quad i \neq j \\ v_{i,i} = \gamma_\epsilon^2(0) \quad \text{and} \quad v_{i,i}^* = \rho_{\epsilon^2}(i)\gamma_{\epsilon^2}(0) & \text{if} \quad i > 0 \\ v_{0,0} = \gamma_{\epsilon^2}(0) \quad \text{and} \quad v_{0,0}^* = (\rho_{\epsilon^2} - 1)\gamma_{\epsilon^2}(0). & \end{cases}$$

Under the additional assumption (3.4), then for $i, j > 0$, the generalized Bartlett's formula for autocorrelations (3.6) holds with

$$\begin{cases} w_{i,j} = w^*_{i,j} = 0 & \textit{if} \quad i \neq j \\ w_{i,i} = 1 \quad \textit{and} \quad w^*_{i,i} = \frac{\gamma_{\epsilon^2}(i)}{\gamma^2_\epsilon(0)} & \textit{if} \quad i > 0. \end{cases}$$

It should be noted that the additional term $w^*_{i,i}$ can be arbitrarily large. Bartlett's formula is also particularly simple for the autocorrelations of MA(q) at lags $i > q$.

Corollary 3.2 (Moving average with non independent linear innovations) *If $X_t = \epsilon_t + \theta_1\epsilon_{t-1} + \cdots + \theta_q\epsilon_{t-q}$, where (ϵ_t) satisfies the assumptions of Theorem 3.1, then the asymptotic covariances $w_{i,j} + w^*_{i,j}$ defined in Theorem 3.1 are such that*

$$w_{i,i} = \sum_{\ell=-q}^{q} \rho^2_X(\ell) \quad \textit{and} \quad w^*_{i,i} = \frac{1}{\gamma^2_\epsilon(0)} \sum_{\ell=-q}^{q} \gamma_{\epsilon^2}(i-\ell)\rho^2_X(\ell)$$

for all $i > q$.

4 Application to GARCH models

The following lemma shows that the symmetry assumption (3.1) is satisfied for GARCH models with a symmetric innovation process.

Lemma 4.1 *Let (ϵ_t) be a GARCH(p, q) process defined by*

$$\begin{cases} \epsilon_t = \sqrt{h_t}\eta_t \\ h_t = \omega + \sum_{i=1}^q \alpha_i\epsilon^2_{t-i} + \sum_{j=1}^p \beta_j h_{t-j}, \end{cases} \tag{4.1}$$

where $\omega > 0$, $\alpha_i \geq 0$ ($i = 1, \ldots, q$), $\beta_j \geq 0$ ($j = 1, \ldots, p$), and where $(\eta_t) \sim \mathrm{IID}(0, 1)$, $E\eta_t^4 < \infty$, with η_t independent of $\{\epsilon_u, u < t\}$. Assume also that $E\epsilon_t^4 < \infty$. If the distribution of η_1 is symmetric then (3.1) holds true.

From Ling and McAleer (2002), there exists a solution to (4.1) such that $E\epsilon_t^4 < \infty$ if $\rho(A^{(2)}) < 1$, where $\rho(A^{(2)})$ denotes the spectral radius of $A^{(2)} = EA_t \otimes A_t$, the symbol $\otimes$ standing for the Kronecker product, and

$$A_t = \begin{pmatrix} \eta_t^2\boldsymbol{\alpha}'_{1:q-1} & \eta_t^2\alpha_q & \eta_t^2\boldsymbol{\beta}'_{1:p-1} & \eta_t^2\beta_p \\ I_{q-1} & 0_{q-1} & 0_{(q-1)\times(p-1)} & 0_{q-1} \\ \boldsymbol{\alpha}'_{1:q-1} & \alpha_q & \boldsymbol{\beta}'_{1:p-1} & \beta_p \\ 0_{(p-1)\times(q-1)} & 0_{p-1} & I_{p-1} & 0_{p-1} \end{pmatrix},$$

with $\boldsymbol{\alpha}_{1:q-1} = (\alpha_1, \ldots, \alpha_{q-1})'$, and $\boldsymbol{\beta}_{1:p-1} = (\beta_1, \ldots, \beta_{p-1})'$. Note that A_t is written for $p \geq 2$ and $q \geq 2$, but can be straightforwardly modified when $p < 2$ or $q < 2$. It is well known that the square of a GARCH process admits an ARMA representation of the form

$$\epsilon_t^2 - \sum_{i=1}^{p \wedge q} (\alpha_i + \beta_i)\epsilon_{t-i}^2 = \omega + \nu_t - \sum_{i=1}^{p} \beta_i \nu_{t-i},$$

where $\nu_t = \epsilon_t^2 - h_t = (\eta_t^2 - 1)h_t$ is a weak white noise. From this ARMA equation, the autocorrelation function $\rho_{\epsilon^2}(\cdot)$ can be easily computed (see *e.g.* Section 3.3 in Brockwell and Davis, 1991). It can be shown that $\rho_{\epsilon^2}(h) \geq 0$ for all h. Thus, in view of the form of $w_{i,j}^*$ given in Remark 3.1, the presence of GARCH effects makes the autocorrelations more difficult to estimate. More precisely, we have the following result.

Remark 4.1 Consider the general framework of Theorem 3.1. If the linear innovation process (ϵ_t) is a GARCH process satisfying the assumptions of Lemma 4.1 then

$$w_{i,i}^* \geq 0 \quad \text{for all} \quad i > 0.$$

To compute the generalized Bartlett's formula, we also need $\kappa - 1 = \gamma_{\epsilon^2}(0)/\gamma_\epsilon^2(0)$, where $\gamma_\epsilon(0) = \omega \left\{1 - \sum_{i=1}^{p \wedge q} (\alpha_i + \beta_i)\right\}^{-1}$ and $\gamma_{\epsilon^2}(0) = E\epsilon_t^4 - \gamma_\epsilon^2(0)$. It can be shown that

$$E\epsilon_t^4 = \mathbf{e}_1 \left(I_{(p+q)^2} - A^{(2)}\right)^{-1} \left\{\underline{b}^{(2)} + \gamma_\epsilon(0) \left(EA_t \otimes \underline{b}_t + E\underline{b}_t \otimes A_t\right) \mathbf{1}_{p+q}\right\} \qquad (4.2)$$

where $\mathbf{e}_1 = (1, 0'_{p+q-1})'$, $\underline{b}_t = (\omega\eta_t, 0'_{q-1}, \omega, 0'_{p-1})'$, $\underline{b}^{(2)} = E\underline{b}_t \otimes \underline{b}_t$ and $\mathbf{1}_{p+q} = (1, \ldots, 1)' \in \mathbb{R}^{p+q}$.

It is then easy to compute Bartlett's coefficients $v_{i,j} + v_{i,j}^*$ and $w_{i,j} + w_{i,j}^*$. For instance in the case of an observed ARCH(1), $X = (\epsilon_t)$, letting $\mu_4 = E\eta_t^4$ we get, when $\mu_4\alpha_1^2 < 1$,

$$\lim_{n\to\infty} n\mathrm{Var}\left\{\hat{\gamma}_X(i)\right\} = \left(\frac{\omega}{1-\alpha_1}\right)^2 \left(1 + \frac{(\mu_4 - 1)\alpha_1^i}{1 - \mu_4\alpha_1^2}\right).$$

It is seen that in the presence of an ARCH(1) effect, the asymptotic variances of the $\hat{\gamma}_X(i)$ can be arbitrarily large (when $\mu_4\alpha_1^2$ is close to 1), increase with α_1 (and thus are always larger than for the iid noise, obtained for $\alpha_1 = 0$), and decrease to the squared unconditional variance of X_t when i increases.

An approximation of the standard deviation of $\hat{\rho}_X(i)$ is then given by $\sigma_{\hat{\rho}_X(i)} = \sqrt{(w_{i,i} + w_{i,i}^*)/n}$. Using the delta method (see *e.g.* Proposition 6.4.3 in Brockwell and Davis, 1991), one can also obtain asymptotic standard deviations for the sample partial autocorrelations $\hat{r}_X(i)$, or for any other statistic depending on a finite number of sample autocovariances/autocorrelations. Statistical issues are not considered in the present paper, but it is clear that $\sigma_{\hat{\rho}_X(i)}$ and all the other theoretical moments must be replaced by estimates in statistical applications.

As an illustration, consider the following ARMA(2,1)-GARCH(1,1) model

$$\begin{cases} X_t - 0.8X_{t-1} + 0.8X_{t-2} = \epsilon_t - 0.8\epsilon_{t-1} \\ \epsilon_t = \sigma_t \eta_t, \quad \eta_t \text{ iid } \mathcal{N}(0,1) \\ \sigma_t^2 = 1 + 0.2\epsilon_{t-1}^2 + 0.6\sigma_{t-1}^2. \end{cases} \tag{4.3}$$

Figure 1 displays the autocorrelation and partial autocorrelation functions, as well as bands in which the sample autocorrelations and sample partial autocorrelations should be included with a probability approximately equal to 95%, when $n = 1,000$.

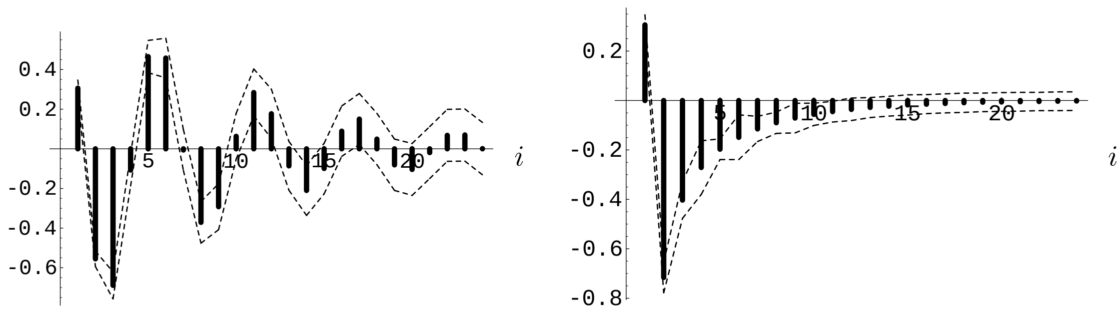

Figure 1: The left panel displays the autocorrelations $\rho_X(i)$ of Model (4.3) and the band $\rho_X(i) \pm 1.96\sigma_{\hat{\rho}_X(i)}$ in dotted lines, for $n = 1,000$. The right panel is similar for the partial autocorrelations $r_X(i)$.

5 Proofs

This section is devoted to the proof of Theorem 3.1, which is the main result of this paper. The proof of the other results, in particular Lemma 4.1 and Corollary 3.1, are not given here but are available from the authors on request.

Using (3.1) and setting $\phi_{\ell_1,\ell_2,\ell_3,\ell_4} = \phi_{\ell_1}\phi_{\ell_2}\phi_{\ell_3}\phi_{\ell_4}$, we obtain

$$\begin{aligned} EX_tX_{t+i}X_{t+h}X_{t+j+h} &= \sum_{\ell_1,\ell_2,\ell_3,\ell_4} \phi_{\ell_1,\ell_2,\ell_3,\ell_4} E\epsilon_{t-\ell_1}\epsilon_{t+i-\ell_2}\epsilon_{t+h-\ell_3}\epsilon_{t+j+h-\ell_4} \\ &= \sum_{\ell_1,\ell_3} \phi_{\ell_1,\ell_1+i,\ell_3,\ell_3+j} E\epsilon_{t-\ell_1}^2\epsilon_{t+h-\ell_3}^2 + \sum_{\ell_1,\ell_2} \phi_{\ell_1,\ell_1+h,\ell_2,\ell_2+h+j-i} E\epsilon_{t-\ell_1}^2\epsilon_{t+i-\ell_2}^2 \\ &+ \sum_{\ell_1,\ell_2} \phi_{\ell_1,\ell_1+h+j,\ell_2,\ell_2+h-i} E\epsilon_{t-\ell_1}^2\epsilon_{t+i-\ell_2}^2 - 2E\epsilon_t^4 \sum_{\ell_1} \phi_{\ell_1,\ell_1+i,\ell_1+h,\ell_1+h+j}. \end{aligned} \tag{5.1}$$

The last equality is obtained by summing over $\ell_1, \ell_2, \ell_3, \ell_4$ such that the indices of $\{\epsilon_{t-\ell_1}, \epsilon_{t+i-\ell_2}, \epsilon_{t+h-\ell_3}, \epsilon_{t+j+h-\ell_4}\}$ are equal two-by-two, which corresponds to the first three sums, and then removing two times the sum in which the four indices are equal. We have also

$$\gamma_X(i) = \sum_{\ell_1,\ell_2} \phi_{\ell_1}\phi_{\ell_2} E\epsilon_{t-\ell_1}\epsilon_{t+i-\ell_2} = \gamma_\epsilon(0)\sum_{\ell_1}\phi_{\ell_1}\phi_{\ell_1+i}. \tag{5.2}$$

By stationarity,

$$\lim_{n\to\infty} n\mathrm{Cov}\{\hat\gamma_X(i), \hat\gamma_X(j)\} = \sum_{h=-\infty}^{\infty} \mathrm{Cov}\{X_t X_{t+i}, X_{t+h}X_{t+j+h}\}.$$

In view of (5.1) and (5.2), the existence of the last sum is guaranteed by the conditions $\sum|\phi_{\ell_1}| < \infty$ and $\sum|\rho_{\epsilon^2}(h)| < \infty$, and this sum is equal to

$$\begin{aligned}
&\sum_{\ell_1,\ell_3}\phi_{\ell_1,\ell_1+i,\ell_3,\ell_3+j}\sum_h\left\{E\epsilon^2_{t-\ell_1}\epsilon^2_{t+h-\ell_3} - \gamma^2_\epsilon(0)\right\} + \sum_{h,\ell_1,\ell_2}\phi_{\ell_1,\ell_1+h,\ell_2,\ell_2+h+j-i}E\epsilon^2_{t-\ell_1}\epsilon^2_{t+i-\ell_2}\\
&+\sum_{h,\ell_1,\ell_2}\phi_{\ell_1,\ell_1+h+j,\ell_2,\ell_2+h-i}E\epsilon^2_{t-\ell_1}\epsilon^2_{t+i-\ell_2} - 2E\epsilon_t^4\sum_{h,\ell_1}\phi_{\ell_1,\ell_1+i,\ell_1+h,\ell_1+h+j}\\
=\;&\gamma_{\epsilon^2}(0)\rho_{\epsilon^2}\sum_{\ell_1}\phi_{\ell_1}\phi_{\ell_1+i}\sum_{\ell_3}\phi_{\ell_3}\phi_{\ell_3+j} + \sum_{\ell_1,\ell_2}\phi_{\ell_1}\phi_{\ell_2}E\epsilon^2_{t-\ell_1}\epsilon^2_{t+i-\ell_2}\sum_h\phi_{\ell_1+h}\phi_{\ell_2+h+j-i}\\
&+\sum_{\ell_1,\ell_2}\phi_{\ell_1}\phi_{\ell_2}E\epsilon^2_{t-\ell_1}\epsilon^2_{t+i-\ell_2}\sum_h\phi_{\ell_1+h+j}\phi_{\ell_2+h-i} - 2E\epsilon_t^4\sum_{\ell_1}\phi_{\ell_1}\phi_{\ell_1+i}\sum_h\phi_{\ell_1+h}\phi_{\ell_1+h+j},
\end{aligned}$$

using Fubini's theorem for the permutation of summation symbols. Using again (5.2) and $\gamma_{\epsilon^2}(0) = (\kappa-1)\gamma^2_\epsilon(0)$ we obtain

$$\begin{aligned}
&\lim_{n\to\infty} n\mathrm{Cov}\{\hat\gamma_X(i), \hat\gamma_X(j)\}\\
=\;&\gamma_{\epsilon^2}(0)\rho_{\epsilon^2}\gamma_\epsilon^{-2}(0)\gamma_X(i)\gamma_X(j) + \sum_{\ell_1,\ell_2}\phi_{\ell_1}\phi_{\ell_2}E\epsilon^2_{t-\ell_1}\epsilon^2_{t+i-\ell_2}\gamma_\epsilon^{-1}(0)\gamma_X(\ell_2+j-i-\ell_1)\\
&+\sum_{\ell_1,\ell_2}\phi_{\ell_1}\phi_{\ell_2}E\epsilon^2_{t-\ell_1}\epsilon^2_{t+i-\ell_2}\gamma_\epsilon^{-1}(0)\gamma_X(\ell_2-j-i-\ell_1) - 2E\epsilon_t^4\gamma_\epsilon^{-2}(0)\gamma_X(i)\gamma_X(j)\\
=\;&\{(\kappa-1)\rho_{\epsilon^2} - 2\kappa\}\gamma_X(i)\gamma_X(j)\\
&+\gamma_\epsilon^{-1}(0)\sum_{\ell_1,\ell_2}\phi_{\ell_1}\phi_{\ell_2}\{\gamma_X(\ell_2+j-i-\ell_1) + \gamma_X(\ell_2-j-i-\ell_1)\}\left\{\gamma_{\epsilon^2}(i-\ell_2+\ell_1) + \gamma^2_\epsilon(0)\right\}.
\end{aligned}$$

Setting $\ell = \ell_2 - \ell_1$, we finally obtain

$$\begin{aligned}
&\lim_{n\to\infty} n\mathrm{Cov}\{\hat\gamma_X(i), \hat\gamma_X(j)\} = \{(\kappa-1)\rho_{\epsilon^2} - 2\kappa\}\gamma_X(i)\gamma_X(j)\\
&+\gamma_\epsilon^{-2}(0)\sum_{\ell=-\infty}^{\infty}\gamma_X(\ell)\{\gamma_X(\ell+j-i) + \gamma_X(\ell-j-i)\}\left\{\gamma_{\epsilon^2}(i-\ell) + \gamma^2_\epsilon(0)\right\},
\end{aligned} \tag{5.3}$$

which, after simple algebra, can be written as $v_{i,j} + v^*_{i,j}$, where $v_{i,j}$ and $v^*_{i,j}$ are given by (2.1) and (3.3).

The vector $(\hat\rho_X(i), \hat\rho_X(j))$ is a function of $(\hat\gamma_X(0), \hat\gamma_X(i), \hat\gamma_X(j))$. The Jacobian of this transformation is

$$J = \begin{pmatrix} -\frac{\gamma_X(i)}{\gamma_X^2(0)} & \frac{1}{\gamma_X(0)} & 0 \\ -\frac{\gamma_X(j)}{\gamma_X^2(0)} & 0 & \frac{1}{\gamma_X(0)} \end{pmatrix}.$$

Let Σ be the variance matrix of $(\hat\gamma_X(0), \hat\gamma_X(i), \hat\gamma_X(j))$. By the delta method, we obtain

$$\begin{aligned} & \lim_{n\to\infty} n\mathrm{Cov}\left\{\hat\rho(i), \hat\rho(j)\right\} = J\Sigma J'(1,2) \\ = \; & \frac{\gamma_X(i)\gamma_X(j)}{\gamma_X^4(0)}\Sigma(1,1) - \frac{\gamma_X(i)}{\gamma_X^3(0)}\Sigma(1,3) - \frac{\gamma_X(j)}{\gamma_X^3(0)}\Sigma(2,1) + \frac{1}{\gamma_X^2(0)}\Sigma(2,3). \end{aligned}$$

Using (5.3) to determine the elements of Σ, this asymptotic covariance is

$$\begin{aligned} & \{(\kappa-1)\rho_{\epsilon^2} - 2\kappa\}\left\{\frac{\gamma_X(i)\gamma_X(j)}{\gamma_X^4(0)}\gamma_X^2(0) - \frac{\gamma_X(i)}{\gamma_X^3(0)}\gamma_X(0)\gamma_X(j)\right. \\ & \left. -\frac{\gamma_X(j)}{\gamma_X^3(0)}\gamma_X(i)\gamma_X(0) + \frac{1}{\gamma_X^2(0)}\gamma_X(i)\gamma_X(j)\right\} \\ & +\gamma_\epsilon^{-2}(0)\sum_{\ell=-\infty}^{\infty}\left[\frac{\gamma_X(i)\gamma_X(j)}{\gamma_X^4(0)}2\gamma_X^2(\ell)\left\{\gamma_{\epsilon^2}(-\ell) + \gamma_\epsilon^2(0)\right\}\right. \\ & -\frac{\gamma_X(i)}{\gamma_X^3(0)}\gamma_X(\ell)\left\{\gamma_X(\ell+j) + \gamma_X(\ell-j)\right\}\left\{\gamma_{\epsilon^2}(-\ell) + \gamma_\epsilon^2(0)\right\} \\ & -\frac{\gamma_X(j)}{\gamma_X^3(0)}\gamma_X(\ell)\left\{\gamma_X(\ell-i) + \gamma_X(\ell-i)\right\}\left\{\gamma_{\epsilon^2}(i-\ell) + \gamma_\epsilon^2(0)\right\} \\ & \left. +\frac{1}{\gamma_X^2(0)}\gamma_X(\ell)\left\{\gamma_X(\ell+j-i) + \gamma_X(\ell-j-i)\right\}\left\{\gamma_{\epsilon^2}(i-\ell) + \gamma_\epsilon^2(0)\right\}\right]. \end{aligned}$$

As function of the autocorrelations, the previous quantity is written as

$$\begin{aligned} & \sum_{\ell=-\infty}^{\infty}\left[2\rho_X(i)\rho_X(j)\rho_X^2(\ell) - \rho_X(i)\rho_X(\ell)\left\{\rho_X(\ell+j) + \rho_X(\ell-j)\right\}\right. \\ & \left. -\rho_X(j)\rho_X(\ell)\left\{\rho_X(\ell-i) + \rho_X(\ell-i)\right\} + \rho_X(\ell)\left\{\rho_X(\ell+j-i) + \rho_X(\ell-j-i)\right\}\right] \\ & +(\kappa-1)\sum_{\ell=-\infty}^{\infty}\rho_{\epsilon^2}(\ell)\left[2\rho_X(i)\rho_X(j)\rho_X^2(\ell) - \rho_X(i)\rho_X(\ell)\left\{\rho_X(\ell+j) + \rho_X(\ell-j)\right\}\right. \\ & \left. -\rho_X(j)\rho_X(\ell-i)\left\{\rho_X(\ell) + \rho_X(\ell)\right\} + \rho_X(i-\ell)\left\{\rho_X(-\ell+j) + \rho_X(-\ell-j)\right\}\right]. \end{aligned}$$

Noting that

$$\sum_\ell \rho_X(\ell)\rho_X(\ell+j) \quad = \quad \sum_\ell \rho_X(\ell)\rho_X(\ell-j),$$

we obtain

$$\lim_{n\to\infty} n\mathrm{Cov}\left\{\hat\rho(i), \hat\rho(j)\right\} \quad = \quad w_{i,j} + w^*_{i,j},$$

where $w_{i,j}$ is given by (2.2) and $w_{i,j}^*$ is given by (3.7)

□

Appendix

A Proof of technical results

Proof of Lemma 4.1. It is shown in Francq and Zakoïan (2004) that, if the distribution of η_t is symmetric then

$$\forall j, \quad E\left\{g(\epsilon_t^2, \epsilon_{t-1}^2, \dots)\epsilon_{t-j} f(\epsilon_{t-j-1}, \epsilon_{t-j-2}, \dots)\right\} = 0, \tag{A.1}$$

for any functions f and g such that the expectation exists. Let four indices t_i, $i = 1, \dots, 4$, such that $t_1 \leq t_2 \leq t_3 \leq t_4$. We have to show that $E\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}\epsilon_{t_4} = 0$ when one of the indices is different from the three others.

If $t_3 < t_4$, then

$$\begin{aligned} E\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}\epsilon_{t_4} &= E\left[E\left(\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}\epsilon_{t_4} \mid \{\epsilon_u, u < t_4\}\right)\right] \\ &= E\left[\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3} h_{t_4} E\left(\eta_{t_4} \mid \{\epsilon_u, u < t_4\}\right)\right] = 0, \end{aligned}$$

because h_{t_4} is measurable with respect to the σ-field generated by $\{\epsilon_u, u < t_4\}$ and because η_{t_4} is centered and independent of $\{\epsilon_u, u < t_4\}$. The result can also be obtained from (A.1) with $g = 1$, $t - j = t_4$ and $f(\epsilon_{t_4-1}, \epsilon_{t_4-2}, \dots) = \epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}$.

Assume therefore that $t_1 < t_2 \leq t_3 = t_4$. Applying (A.1) with $g(x) = f(x) = x$, we have

$$E\epsilon_{t_1}\epsilon_{t_2}\epsilon_{t_3}\epsilon_{t_4} = E\left\{g(\epsilon_{t_3}^2)\epsilon_{t_2} f(\epsilon_{t_1})\right\} = 0$$

and the conclusion follows.

□

Proof of Corollary 3.1. When $X = (\epsilon_t)$,

$$\begin{aligned}
v_{i,j} &= (\kappa-3)\gamma_\epsilon(i)\gamma_\epsilon(j) + \gamma_\epsilon(0)\left\{\gamma_\epsilon(j-i)+\gamma_\epsilon(-j-i)\right\} \\
&= \begin{cases} 0 & \text{if} \quad i \neq j \\ \gamma_\epsilon^2(0) & \text{if} \quad i = j > 0 \\ (\kappa-1)\gamma_\epsilon^2(0) & \text{if} \quad i = j = 0, \end{cases} \\
w_{i,j} &= -2\rho_\epsilon(j)\rho_\epsilon(i) + \rho_\epsilon(j-i) + \rho_\epsilon(-j-i) \\
&= \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j > 0, \end{cases} \\
v^*_{i,j} &= (\rho_{\epsilon^2}-3)(\kappa-1)\gamma_\epsilon(i)\gamma_\epsilon(j) + (\kappa-1)\gamma_\epsilon(0)\left\{\gamma_\epsilon(i-j)+\gamma_\epsilon(i+j)\right\}\rho_{\epsilon^2}(i) \\
&= \begin{cases} 0 & \text{if} \quad i \neq j \\ (\kappa-1)\gamma_\epsilon^2(0)\rho_{\epsilon^2}(i) & \text{if} \quad i = j > 0 \\ (\rho_{\epsilon^2}-1)(\kappa-1)\gamma_\epsilon^2(0) & \text{if} \quad i = j = 0, \end{cases} \\
w^*_{i,j} &= (\kappa-1)\left[-2\rho_\epsilon(i)\rho_\epsilon(j) + \rho_{\epsilon^2}(i)\left\{\rho_\epsilon(i+j)+\rho_\epsilon(i-j)\right\}\right] \\
&= \begin{cases} 0 & \text{if} \quad i \neq j \\ (\kappa-1)\rho_{\epsilon^2}(i) & \text{if} \quad i = j > 0. \end{cases}
\end{aligned}$$

The conclusion then follows from $(\kappa-1) = \gamma_{\epsilon^2}(0)/\gamma_\epsilon^2(0)$.

□

Proof of Corollary 3.2. Because $\rho_X(\ell) = 0$ for $|\ell| > q$, we have

$$\begin{aligned}
w_{i,j} &= \sum_{\ell=-q}^{q} \rho_X(\ell)\left[2\rho_X(i)\rho_X(j)\rho_X(\ell) - 2\rho_X(i)\rho_X(\ell+j)\right. \\
&\quad \left. -2\rho_X(j)\rho_X(\ell+i) + \rho_X(\ell+j-i) + \rho_X(\ell-j-i)\right]
\end{aligned}$$

and for $i, j > q$

$$w_{i,j} = \sum_{\ell=-q}^{q} \rho_X(\ell)\rho_X(\ell+j-i).$$

The expression of $w_{i,i}$ follows. Similarly, for $i > q$

$$\begin{aligned}
w^*_{i,i} &= (\kappa-1)\sum_{\ell=-\infty}^{\infty} \rho_{\epsilon^2}(\ell)\rho_X(\ell+i)\left\{\rho_X(\ell+i)+\rho_X(\ell-i)\right\} \\
&= (\kappa-1)\sum_{\ell=-q}^{q} \rho_{\epsilon^2}(i-\ell)\rho_X^2(\ell).
\end{aligned}$$

□

Proof of (4.2). Model (4.1) can be written in vector form as

$$\underline{z}_t = \underline{b}_t + A_t \underline{z}_{t-1},$$

where $\underline{z}_t = \left(\epsilon_t^2, \ldots, \epsilon_{t-q+1}^2, \sigma_t^2, \ldots, \sigma_{t-p+1}^2\right)'$. Using the independence between $\underline{z}_t$ and $(\underline{b}_t, A_t)$, together with elementary properties of the Kronecker product, we obtain

$$\begin{aligned}
E\underline{z}_t^{\otimes 2} &= E(\underline{b}_t + A_t\underline{z}_{t-1}) \otimes (\underline{b}_t + A_t\underline{z}_{t-1}) \\
&= E\underline{b}_t \otimes \underline{b}_t + EA_t\underline{z}_{t-1} \otimes \underline{b}_t + E\underline{b}_t \otimes A_t\underline{z}_{t-1} + EA_t\underline{z}_{t-1} \otimes A_t\underline{z}_{t-1} \\
&= E\underline{b}_t^{\otimes 2} + EA_t \otimes \underline{b}_t E\underline{z}_{t-1} + E\underline{b}_t \otimes A_t E\underline{z}_{t-1} + EA_t^{\otimes 2} E\underline{z}_{t-1}^{\otimes 2}.
\end{aligned}$$

Thus

$$E\underline{z}_t^{\otimes 2} = \left(\mathbb{I}_{(p+q)^2} - A^{(2)}\right)^{-1} \left\{\underline{b}^{(2)} + \left(EA_t \otimes \underline{b}_t + E\underline{b}_t \otimes A_t\right) E\underline{z}_t\right\}$$

and (4.2) follows. To compute $A^{(2)}$, one can use that $A_t = \eta_t^2 B + C$, where B and C are deterministic matrices. Thus we have

$$A^{(2)} = E(\eta_t^2 B + C) \otimes (\eta_t^2 B + C) = B \otimes BE\eta_t^4 + B \otimes C + C \otimes B + C \otimes C.$$

Similarly we obtain $EA_t \otimes \underline{b}_t$ and $E\underline{b}_t \otimes A_t$.

□

Proof of Remark 4.1. Because

$$w_{i,i}^* = (\kappa - 1)\sum_{\ell=1}^{\infty} \rho_{\epsilon^2}(\ell) w_i^2(\ell),$$

the result comes from Proposition 1 below.

□

Proposition 1 *If (ϵ_t) is a GARCH process and $E\epsilon_t^4 < \infty$ then*

$$\gamma_{\epsilon^2}(h) = Cov(\epsilon_t^2, \epsilon_{t-h}^2) > 0 \qquad \forall h.$$

Proof. It suffices to show that we have a MA(∞) of the form

$$\epsilon_t^2 = c + \nu_t + \sum_{\ell=1}^{\infty} \phi_\ell \nu_{t-\ell}, \quad \text{with} \quad \phi_\ell \geq 0 \quad \forall \ell.$$

Indeed, $\nu_t := \epsilon_t^2 - h_t = (\eta_t^2 - 1)h_t$ being a weak white noise, we have

$$\gamma_{\epsilon^2}(h) = E\nu_1^2 \sum_{\ell=0}^{\infty} \phi_\ell \phi_{\ell+|h|}, \qquad \text{with the notation } \phi_0 = 1.$$

Denoting by B the backshift operator, and introducing the notation $\alpha(z) = \sum_{i=1}^{q} \alpha_i z^i$, $\beta(z) = \sum_{j=1}^{p} \beta_j z^j$ and $\phi(z) = \sum_{\ell=1}^{\infty} \phi_\ell z^\ell$, we obtain

$$\epsilon_t^2 = \{1 - (\alpha + \beta)(1)\}^{-1}\omega + \{1 - (\alpha + \beta)(B)\}^{-1}(1 - \beta(B))\nu_t = c + \phi(B)\nu_t.$$

Since $1-\beta(B) = 1-(\alpha+\beta)(B)+\alpha(B)$, we obtain ϕ_ℓ as the coefficient of z^ℓ in the division of $\alpha(z)$ by $1-(\alpha+\beta)(z)$ according to the increasing powers of z. By recurrence on ℓ, it is easy to see that these coefficients are positive because the polynomials $\alpha(z)$ and $(\alpha+\beta)(z)$ have positive coefficients.

□

Proof that (5.3) can be written as (3.2) Because $\gamma_{\epsilon^2}(0) = (\kappa-1)\gamma_\epsilon^2(0)$, (5.3) can be written as

$$\begin{aligned}
&\lim_{n\to\infty} n\mathrm{Cov}\left\{\hat\gamma_X(i), \hat\gamma_X(j)\right\} = \left\{(\kappa-1)\rho_{\epsilon^2} - 2\kappa\right\}\gamma_X(i)\gamma_X(j) \\
&+\gamma_\epsilon^{-2}(0)\sum_{\ell=-\infty}^{\infty}\gamma_X(\ell)\left\{\gamma_X(\ell+j-i)+\gamma_X(\ell-j-i)\right\}\left\{\gamma_{\epsilon^2}(i-\ell)+\gamma_\epsilon^2(0)\right\} \\
&= (\rho_{\epsilon^2}-3)(\kappa-1)\gamma_X(i)\gamma_X(j) + (\kappa-3)\gamma_X(i)\gamma_X(j) \\
&+\sum_{\ell=-\infty}^{\infty}\gamma_X(\ell)\left\{\gamma_X(\ell+j-i)+\gamma_X(\ell-j-i)\right\} \\
&+(\kappa-1)\sum_{\ell=-\infty}^{\infty}\gamma_X(\ell)\left\{\gamma_X(\ell+j-i)+\gamma_X(\ell-j-i)\right\}\rho_{\epsilon^2}(i-\ell).
\end{aligned}$$

Setting $h = i-\ell$ and using the parity of the autocorrelation functions, we obtain

$$\begin{aligned}
&\sum_{\ell=-\infty}^{\infty}\gamma_X(\ell)\left\{\gamma_X(\ell+j-i)+\gamma_X(\ell-j-i)\right\}\rho_{\epsilon^2}(i-\ell) \\
=&\sum_{h=-\infty}^{\infty}\gamma_X(-h+i)\left\{\gamma_X(-h+j)+\gamma_X(-h-j)\right\}\rho_{\epsilon^2}(h),
\end{aligned}$$

which gives (3.2), where $v_{i,j}$ and $v_{i,j}^*$ are given by (2.1) and (3.3).

□

B Additional examples

Romano and Thombs (1996) considered weak white noises of the form $\epsilon_t = \eta_t\eta_{t-1}\cdots\eta_{t-k+1}$ where $(\eta_t) \sim \mathrm{IID}(0,\sigma^2)$, with $\sigma^2 > 0$, $E\eta_1^4 = \mu_4 < \infty$ and $k \geq 1$. It is clear that (3.1) is satisfied for such noises. It is obvious to check that ϵ_t is a weak white noise, such that $\gamma_\epsilon(0) = \sigma^{2k}$ and we have

$$\gamma_{\epsilon^2}(i) \quad = \quad \begin{cases} \sigma^{4i}(\mu_4 - \sigma^4)^{k-i} & \text{for } i = 0, \dots, k-1 \\ 0 & \text{for } i \geq k \end{cases}$$

which shows that the ϵ_t's are not independent when $k > 1$ and η_t^2 is not almost surely constant. Note that Corollary 3.1 holds with

$$w_{i,i}^* = \frac{\gamma_{\epsilon^2}(i)}{\gamma_\epsilon^2(0)} = \left(\frac{\mu_4}{\sigma^4} - 1\right)^{k-i} \geq 0$$

when $k > i$, and that $w_{i,i}^*$ can be made arbitrarily large.

Romano and Thombs (1996) also considered weak white noises of the form $\epsilon_t = \eta_t/\eta_{t-1}$ where $(\eta_t) \sim \text{IID}(0, \sigma^2)$ and$E\eta_1^{-4} < \infty$. It is interesting to note that (3.1) may not hold because

$$E\epsilon_t^2\epsilon_{t-1}\epsilon_{t-2} = E\frac{\eta_t^2}{\eta_{t-1}^2}\frac{\eta_{t-1}}{\eta_{t-2}}\frac{\eta_{t-2}}{\eta_{t-3}} = \left\{E\left(\frac{1}{\eta_1}\right)\right\}^2.$$

When the marginal distribution of η_1 is symmetric (3.1) is however satisfied. In this case we have

$$\gamma_{\epsilon^2}(i) \quad = \quad \begin{cases} \mu_4 E\left(\frac{1}{\eta_1^4}\right) - \left\{\sigma^2 E\left(\frac{1}{\eta_1^2}\right)\right\}^2 & \text{for } i = 0 \\ \sigma^2 E\left(\frac{1}{\eta_1^2}\right) - \left\{\sigma^2 E\left(\frac{1}{\eta_1^2}\right)\right\}^2 & \text{for } i = 1 \\ 0 & \text{for } i \geq 2 \end{cases}$$

which shows that the ϵ_t's are not independent. Note that Corollary 3.1 holds with

$$w_{1,1}^* = \frac{1}{\sigma^2 E\left(\frac{1}{\eta_1^2}\right)} - 1 < 0,$$

by Jensen's inequality.

References

Bartlett, M.S. (1946) On the theoretical specification and sampling properties of autocorrelated time series. *Supplement to the Journal of the Royal Statistical Society* 8 27-41.

Berlinet A., Francq C. (1997) On Bartlett's formula for nonlinear processes. *Journal of Time Series Analysis* 18 535–552.

Bollerslev T. (1986) Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* 31 307–327.

Brockwell P.J., Davis R.A. (1991) *Time Series: Theory and Methods.* Springer-Verlag, New-York.

Diebold F.X. (1986) Testing for Serial Correlation in the Presence of ARCH. *Proceedings of the Business and Economics Statistics Section*, American Statistical Association, 323–328.

Engle R.F. (1982) Autoregressive conditional heteroskedasticity with estimates of the variance of the United Kingdom inflation. *Econometrica* 50 987–1007.

Fan J., Yao Q. (2003) Nonlinear Time Series : Nonparametric and Parametric Methods. Springer-Verlag, New York.

Francq, C., and J. M. Zakoïan (2004) Maximum Likelihood Estimation of Pure GARCH and ARMA-GARCH Processes. *Bernoulli* 10, 605–637.

Granger C.W.J., Andersen A.P. (1978) *An introduction to bilinear time series models.* Vandenhoek and Ruprecht, Gottingen.

Haggan V., Ozaki T. (1981) Modelling Nonlinear Random Vibrations Using an Amplitude-Dependent Autoregressive Time Series Model. *Biometrika* 68 189–196.

Ling S., McAleer M. (2002) Necessary and Sufficient Moment Conditions for the GARCH(r, s) and Asymmetric Power GARCH(r, s) Models. *Econometric Theory* 18 722–729.

Romano J.P., Thombs L.A. (1996) Inference for Autocorrelations under Weak Assumptions. *Journal of the American Statistical Association* 91 590–600.

Teräsvirta T. (2004) Smooth Transition Regression Modeling, in *Applied Time Series Econometrics* Eds. H. Lütkepohl and M. Krätzig, Cambridge University Press, Cambridge 222–242.

Tong H. (1990) *Non-Linear Time Series. A dynamical System Approach.* Oxford University Press, Oxford .