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On the Comparative Statics of the Optimal Reserve Price :
A Comment on "Reserve Price Signaling"

L. LAMY [1]

[1] Laboratoire d'Economie Industrielle, CREST-INSEE, 28 Rue des Saints-Pères, 75007 PARIS, France. laurent.lamy@ensae.fr

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Laurent Lamy*

*Laboratoire d'Economie Industrielle, CREST-INSEE, 28 rue des Saints-Pères 75007 Paris. e-mail: laurent.lamy@ensae.fr

Abstract

This comment finds an error in Cai, Riley and Ye [2] in presence of informational externalities between bidders, the correction of which gives a broader view on their comparative statics results with respect to n, the number of bidders. A linear specification of the informational externalities between bidders is analyzed. In contrast to their claim, the reserve price of the lowest type sellers is shown to be decreasing in n. We establish that the participation threshold is still increasing in n for high type sellers confirming their main insight. Nevertheless, it is the opposite comparative statics that holds for the reserve price.
Keywords: Auctions, Reserve price, Signaling
JEL classification: D44, D80, D82

Abstract

Ce commentaire souligne une erreur dans l'article de Cai, Riley and Ye [2] en présence d'externalités informationelles entre les enchérisseurs. La correction donne une perspective plus large sur les questions de statique comparative du prix de réserve par rapport au nombre d'enchérisseurs n. Une spécification linéaire est analysée. Contrairement à leurs résultats, le prix de réserve d'équilibre d'un vendeur d'un bien de basse qualité est décroissant avec n.
Mots-clés: Enchères, prix de réserve, Signaling
Classification: D44, D80, D82

1 Introduction

In the symmetric independent private value framework without externalities, it is well-known from Riley and Samuelson [13] and Myerson [11] that the optimal reserve price is independent of the number of bidders. Few papers attempt such comparative statics beyond this framework. Samuelson [14] and Levin and Smith [7] show, respectively in the first price auction with entry costs and in the second price auction with affiliation, that the optimal reserve price is decreasing in the number of bidders. Moreover, unless the number of bidders is small, the role of the reserve price policy perceived by the literature is limited: the extra profit from the reserve price and the probability that it sets the winning price goes to zero when the number of bidders goes to infinity. On the contrary, in a model where the seller has a private signal that both affects her reservation value and bidders' valuations, Cai, Riley and Ye [2] (henceforth CR&Y) show that the optimal reserve price is increasing in the number of bidders under fairly general specifications of bidders' valuations. In such a framework, the reserve price does not solely affect the winning price but also bidders' priors about the seller's private information. From this signaling channel, the extra profit captured by setting an adequate reserve price remains first order even if the number of bidders goes large.

However, the degree of validity of CR&Y's analysis of the unique separating equilibrium is much weaker than they have claimed: it is confined to the case without informational externalities between bidders as in their initial working paper [1] and in the closely related work of Jullien and Mariotti [5]. Those papers both exclude such informational externalities and are thus not subject to the errors we point out in the derivation of the equilibrium of the second price auction and in the proof of their comparative statics results. With such externalities, even if bidders' signals are assumed to be independent, the comparative statics analysis of the optimal reserve price with respect to the number of bidders is untractable without additional structure. Nevertheless, under a linear specification, we are able to separate the different channels for the comparative statics for low and high type sellers as it is done in Proposition [3.1], the main result of this comment, without imposing any additional structure on the underlying distributions of private signals. First, if the signal of the seller were publicly observable, i.e. exempt of any signaling incentives, then the optimal reserve price is shown to be nonincreasing in n, the number of bidders. At the unique separating equilibrium, the lowest type seller chooses this optimal reserve price which is thus nonincreasing in n. Second, if the seller's type is high enough, i.e. when the signaling effect is important, then we establish that CR&Y's comparative statics on the minimum type that bids still hold. Nevertheless, it does not imply that the reserve price is increasing in n for high type sellers. On the contrary we show that the reserve price is strictly decreasing in n for high type sellers.

Numerical simulations illustrate those asymptotic patterns. In a nutshell, they show that the signaling effect in CR&Y becomes rapidly prominent insofar as the minimum type that bids is decreasing in n except when the signal of the seller is extremely low. Nevertheless, their insight that the reserve price is increasing in n is not robust to informational externalities between bidders and it is rather the opposite result that seems to hold for intermediate values of the seller's signal.

The rest of this paper is organized as follows. In section 2, we summarize the model, point out its errors when there are informational externalities between bidders and show how the characterization of the unique separating equilibrium should be adapted. Section 3, the constructive part of this comment, discusses the comparative statics of the optimal reserve price with respect to the number of bidders under a linear specification for bidders' valuations. Section 4 concludes.

2 The separating equilibrium: errata

CR&Y considers the sale of a single indivisible item to n bidders in the general symmetric interdependent value model introduced by Milgrom and Weber [10] and analyses the second price auction. Each bidder receives a one-dimensional signal X_i such that $X_1, \ldots, X_n$ are affiliated and distributed according to a symmetric continuous density f which is assumed to be strictly positive on $\mathbb{X}^n$, where $\mathbb{X} = [\underline{x}, \overline{x}] \subset \mathbb{R}^+$. The seller also privately observes a one-dimensional signal S with support $\mathbb{S} = [\underline{s}, \overline{s}] \subset \mathbb{R}^+$ and which is drawn independently of $X_1, \ldots, X_n$. Bidder i's valuation for the item is given by $V_i = u(s, x_i, x_{-i})$: it depends on s, on his own signal's realization x_i and symmetrically on the other bidders' signals $X_{-i} = x_{-i}$. The function u is assumed to be non-negative, continuous and nondecreasing in all its arguments and strictly increasing in its two first arguments. Two functions are playing a central role in the analysis of the second price auction with informational externalities. Let us define the function $v : \mathbb{S} \times \mathbb{X} \to \mathbb{R}^+$ (respectively $w : \mathbb{S} \times \mathbb{X} \to \mathbb{R}^+$) by $v(s, x) = E[V_i | S = s, X_i = x, Y_i = x]$ (respectively $w(s, x) = E[V_i | S = s, X_i = x, Y_i \leq x]$), where Y_i denotes the first order statistic of the signals received by bidder i's opponents. From the monotonicity of u and affiliation, v and w are strictly increasing in both arguments. Differently from Milgrom and Weber [10], CR&Y also considers that the reservation value of the seller is given by $\xi(s)$, which is strictly increasing in s. It is this dependence that leaves some scope for signaling by means of a binding reserve price.

The first step of the analysis of the whole signaling game is the derivation of the equilibrium of the second price auction after the seller's move, i.e. the choice of the reserve price r, and given that bidders believe that the seller's signal is $\widehat{s}$. The symmetric Bayesian Nash Equilibrium has been derived by Milgrom and Weber [10]. It is characterized by the participation threshold $m(\widehat{s})$ such that $w(\widehat{s}, m(\widehat{s})) = r$, which is also referred to as the minimum type that bids. Then the equilibrium bidding strategy is $b(x) = v(\widehat{s}, x)$ for $x \geq m(\widehat{s})$ and not to participate (equivalently $b(x) < r$) for $x < m(\widehat{s})$. The proposed strategy in CR&Y does not correspond in general to this (unique) symmetric equilibrium. They wrongly state that "in equilibrium, the minimum bid must be equal to the reserve price" which corresponds to consider the above strategy with the participation threshold $m(\widehat{s})$ such that $v(\widehat{s}, m(\widehat{s})) = r$. Without informational externalities between bidders, i.e. if u does not depend on its third argument, the two functions v and w coincide and do not depend on the number of bidders and then CR&Y's analysis remains valid, e.g. for their linear independent valuation model which gives striking numerical results or for the two classes of value function (a) and (c). Nevertheless, in the general case with informational externalities between bidders, e.g. in the class of value function (b) listed in CR&Y, we have $v(x) > w(x)$ for $x > \underline{x}$ and the analysis is modified.[1]

Going from the model without informational externalities analyzed in the working paper version [1] to the full model, CRY made two additional errors in their analysis. Those errors are irrelevant when there are no informational externalities between bidders, more precisely when v and w do not depend on the number of bidders. Nevertheless, it matters when we want to adapt their results to a more general framework with informational externalities between bidders as it will be considered here. In their theorem 2, they claim that "In the separating equilibrium, the minimum type that bid $m(s)$, and *hence* the reserve price $r(s) = v(s, m(s))$ is higher for larger n". We claim that the argument is flawed. If the signal of the seller s is fixed, then $\frac{dr(s)}{dn} = \frac{\partial v}{\partial x} \cdot \frac{dm(s)}{dn} + \frac{\partial v}{\partial n}$. CR&Y implicitly argues that $\frac{dm(s)}{dn} \geq 0$ implies that $\frac{dr(s)}{dn} \geq 0$ as it would be if the function v were not to depend on n as in the case without informational externalities. Unfortunately, the natural assumption on the dependence with respect to n is that $\frac{\partial v}{\partial n} \leq 0$ as it is also assumed by CR&Y in their analysis. Indeed, the *hence* argument would be true in the above sentence only if "higher for larger n" is replaced by "lower for larger n", an argument that we use in the first part of Proposition [3.1]. In general, the comparative statics of the participation threshold and the reserve price may be opposite as it is the case for high type sellers in the second part of Proposition [3.1].

CR&Y drops the dependence with respect to n in their notation, which is a source of confusion in their comparative statics analysis. The second error

[1] The gap between the reserve price r and the lowest possible active bid $v(\widehat{s}, m(\widehat{s}))$ is a counterintuitive feature of the equilibrium when there are informational externalities between bidders. See Lamy [6] for a survey of auction environments where this gap is present and for an analysis of its consequences when the seller can bid strategically in the auction as any other bidder.

that appears in the proof of theorem 2 will be detailed after the presentation of the virtual surplus mapping. To avoid any confusion and at the expense of cumbersome notation, we always restore this dependence in the following notation, e.g. we note $w(s,x,n)$ and $m(s,n)$ instead of $w(s,x)$ and $m(s)$.

The rest of this section is devoted to the correction of the characterization of the unique separating equilibrium when there are informational externalities between bidders. General formulas that will be useful in the following section are given without proof. It is left to the reader to check that it corresponds to a straightforward adaption of CR&Y's analysis.

The expected payoff of a seller with type s, which is perceived to be type $\widehat{s}$ and that induces the participation threshold m by announcing a reserve price $w(\widehat{s},m,n)$, can be written as follows:

$$U(s,\widehat{s},m,n) = \xi(s)F_{(1:n)}(m)+w(\widehat{s},m,n)[F_{(2:n)}(m)-F_{(1:n)}(m)]+\int_m^{\overline{x}} v(\widehat{s},x,n)f_{(2:n)}(x)dx, \tag{1}$$

where $F_{(1:n)}$ and $F_{(2:n)}$ ($f_{(1:n)}$ and $f_{(2:n)}$) denote respectively the distribution (density) functions of the first and second highest signal statistics among the n potential buyers. The difference with the original expression in CR&Y is the second term, which corresponds to the event where only the highest bid is above the reserve price and hence the good is sold at the reserve price: in this term, $v(\widehat{s},m,n)$ has been replaced by $w(\widehat{s},m,n)$, the reserve price.

Differentiating (1) we have

$$\frac{\partial U}{\partial m} = f_{(1:n)}(m)[\xi(s) - J(\widehat{s},m,n)] \tag{2a}$$

$$\frac{\partial U}{\partial \widehat{s}} = \frac{\partial w(\widehat{s},m,n)}{\partial \widehat{s}}[F_{(2:n)}(m) - F_{(1:n)}(m)] + \int_m^{\overline{x}} \frac{\partial v(\widehat{s},x,n)}{\partial \widehat{s}} f_{(2:n)}(x)dx, \tag{2b}$$

where the map $J(s,m,n)$, which corresponds to a generalization of the "virtual surplus" of Myerson [11], is given by the following expression:

$$J(s,x,n) = w(s,x,n)-\frac{\partial w(s,x,n)}{\partial x}\cdot\frac{F_{(2:n)}(x) - F_{(1:n)}(x)}{f_{(1:n)}(x)}+(v(s,x,n)-w(s,x,n))\cdot\frac{f_{(2:n)}(x)}{f_{(1:n)}(x)} \tag{3}$$

Compared to the related expression in CR&Y, a third term emerges with informational externalities: $(v(s,x,n) - w(s,x,n)) \cdot \frac{f_{(2:n)}(x)}{f_{(1:n)}(x)}$. As in CR&Y, a generalization of Myerson [11]'s regularity assumption is required.

Assumption 2.1 *For any s, the "generalized virtual surplus" $J(s,x,n)$ is strictly increasing in x.*

With this regularity assumption, we can compute the optimal reserve price or equivalently the optimal participation threshold $m^*(s,n)$ directly from equation (11a) in the case where the signal S is directly observable to the bidders. If the equation $J(s,m,n) = \xi(s)$ has a solution for $m \in [\underline{x}, \overline{x}]$, then it corresponds to the optimal participation threshold. On the other hand, if $J(s,m,n) > \xi(s)$ (respectively $J(s,m,n) < \xi(s)$) on the range $[\underline{x}, \overline{x}]$, then the optimal reserve price is non-binding and induces full participation, $m^*(s,n) = \underline{x}$ (resp. the seller prefers not to sell, i.e. $m^*(s,n) = \overline{x}$).

CR&Y's basic existence theorem is unaffected. It is generalized in the following proposition.

Proposition 2.1 *The differential equation (4),*

$$\frac{\partial s(m,n)}{\partial m} = \frac{f_{(1:n)}(m)[J(s,m,n) - \xi(s)]}{\frac{\partial w(s,m,n)}{\partial s}[F_{(2:n)}(m) - F_{(1:n)}(m)] + \int_m^{\overline{x}} \frac{\partial v(s,x,n)}{\partial s} f_{(2:n)}(x)dx}, \qquad (4)$$

with initial condition $(\underline{s}, m^*(\underline{s}, n))$ for the lowest type $\underline{s}$ characterizes the unique separating equilibrium for a given number of bidders n.

The main focus of CR&Y is then the comparative statics analysis of the participation threshold $m(s,n)$ and the corresponding reserve price $r(s,n)$ in this separating equilibrium. As it is clear from the differential equation (4) and the initial condition, an important step in order to prove that $m(s,n)$ is increasing in n (or equivalently that $s(m,n)$ is decreasing in n) is the condition that $J(s,m,n)$ is nonincreasing in n. If signals are independent and distributed according to the CDF F (the corresponding density is denoted by f), then the expression of J is:

$$J(s,x,n) = w(s,x,n) - \frac{\partial w(s,x,n)}{\partial x} \cdot \frac{1 - F(x)}{f(x)} + (n-1)(v(s,x,n) - w(s,x,n)) \cdot \frac{1 - F(x)}{F(x)} \qquad (5)$$

This step appears in CR&Y where the third term is absent and w should be replaced by v in the above expression. CR&Y states in the beginning of the proof of Theorem 2 that "$J(s,x)$ is nonincreasing in n as $v(s,x)$ is nonincreasing in n and $\frac{\partial v}{\partial s}$ is nondecreasing in n". Indeed, what would be right is that $J(s,x)$ is nonincreasing in n provided that $v(s,x)$ is nondecreasing in n and $\frac{\partial v}{\partial x}$ is nondecreasing in n. Thus, the extra assumption that $\frac{\partial v}{\partial x}$ is nondecreasing in n should have appeared in CR&Y's Theorem 2. Nevertheless, without informational externalities between bidders it is automatically satisfied. Indeed, more generally, CR&Y's whole analysis remains valid if $J(s,x,n)$ is nonincreasing in n. However, this assumption is unrealistic with externalities as emphasized in next section by means of a linear specification of the informational externalities.

Let us return to expression (5) to sketch the main forces that drive the comparative statics of the map $J(s,x,n)$ with respect to n and which thus plays a key role in the comparative statics of our variables of interest. It is natural to consider that the first term $w(s,x,n)$ is strictly decreasing in n for

$x > \underline{x}$ as it is assumed in CR&Y and satisfied in the linear independent model developed next section.[2] In particular, it means that if x is sufficiently close to $\overline{x}$ such that the second and third terms become negligible, then $J(s,x,n)$ is strictly decreasing in n. On the contrary, the natural assumptions for $-\frac{\partial w}{\partial x}$ and $(n-1)[v(s,x,n) - w(s,x,n)]$ is that those terms are nondecreasing in n for $x > \underline{x}$. Consequently, those terms exert an opposite force that may make the function $J(s,x,n)$ to be increasing in n. This broad pattern is illustrated in the left panel of Figure 1 where the map J in depicted for $n = 2, 3, 4, 10$ for a model that is specified in next section: we see here that $J(s,x,n)$ is increasing in n below a given threshold (around $x = 0.6$) and decreasing in n above.

3 Comparative Statics: the linear independent model

In this section, we consider a more specific valuation model referred to as the *linear independent model*. As in CR&Y, we first consider the case in which bidders' signals are independent and distributed according to the CDF F. Second, we consider the linear specification:

$$u(s, x_i, x_{-i}) = s + \alpha(n) \cdot x_i + (1 - \alpha(n)) \cdot \frac{\sum_{j \neq i} x_j}{n-1} \tag{6a}$$

$$\xi(s) = s, \tag{6b}$$

where $\alpha(1) = 1$ and $\alpha(n) \geq \frac{1}{n}$ is assumed to be a strictly decreasing function of n, which fits with the intuition that the relative weight of a bidder's signal in the estimation of his own valuation decreases with respect to the number of bidders. We conjecture that the different channels that matters for the comparative statics goes much beyond this specification, whose credit is tractability. In particular, thanks to the choices $\xi(s) = s$ and $u(s, x_i, x_{-i}) = s + \psi(x_i, x_{-i})$, the right term of the differential equation (4) does not depend on s and it is thus a standard linear first-order differential equation. Note also that our framework is then invariant to any translation of the seller's signal. Then the terminology 'low type sellers' refer to as the signals in the neighborhood of $\underline{s}$, independently of the specific value of $\underline{s}$. On the other hand, 'high type sellers' will refer to types that are sufficiently higher than $\underline{s}$. Note that if $\overline{s}$ is too close to $\underline{s}$, then 'high type sellers' may not exist.

Then we obtain the following expressions for w, v and $\frac{\partial s(m,n)}{\partial m}$:

$$w(s, x, n) = s + \alpha(n) \cdot x + (1 - \alpha(n)) \cdot E[X | X \leq x] \tag{7a}$$

[2] In Wilson [15]'s model, where bidders' signals X_i are identically and independently distributed conditional on an exogenous variable Z, Levin and Smith [7] show also that $w(s,x,n)$ and $v(s,x,n)$ are nonincreasing in n.

$$v(s,x,n) = s + (\alpha(n) + \frac{1-\alpha(n)}{n-1}) \cdot x + (1-\alpha(n)) \cdot \frac{n-2}{n-1} \cdot E[X|X \leq x] \quad (7b)$$

$$\frac{\partial s(m,n)}{\partial m} = \frac{n \cdot f(m) \cdot F^{n-1}(m)}{1-F^n(m)}[J(s,m,n) - \xi(s)] \quad (7c)$$

Added to the regularity assumption (2.1), we make two additional mild assumptions on the CDF F.

Assumption 3.1 *The reverse hazard rate $\frac{f(x)}{F(x)}$ is nonincreasing in x.*

Assumption 3.2 *Under public information of the seller's signal, the optimal participation threshold of the lowest type seller facing one bidder involves a probability of sale superior to one half: $F(m^*(\underline{s},1)) \leq \frac{1}{2}$.*

Assumption [A.3] is the only one that is not standard in the literature: it is always satisfied if $\underline{x}$ is high enough relative to $\underline{s}$.[3] We perform our numerical computations with the family $F(x) = x^\phi$ ($\phi > 0$) on the support $\mathbb{X} = [0,1]$ and with $\alpha(n) = \frac{1}{n}$.[4] The specification $\alpha(n) = \frac{1}{n}$ is not innocent: it is an upper bound for the informational externalities between bidders and thus we take the opposite case relative to CR&Y's computation where $\alpha(n) = 1$ for any n which corresponds to exclude any informational externalities between bidders.

Our following comparative statics results concern low and high type sellers. On the one hand, we obtain the exact opposite of CR&Y's results for low type sellers: the participation threshold and the reserve price are both nonincreasing in n. On the other hand, for high type sellers, we obtain CR&Y's results for the participation threshold but not for the reserve price: the participation threshold is actually strictly increasing in n but surprisingly the reserve price is strictly decreasing in n. The proof is relegated in the appendix.

Proposition 3.1 (Comparative Statics with respect to the number of bidders) ***[Low type sellers]***

In the separating equilibrium of the linear independent model, the minimum type that bids, $m^*(s,n)$, and hence the reserve price, $r(s,n) = w(s, m(s,n), n)$, is nonincreasing in n for the lowest type seller, i.e. for $s = \underline{s}$. By continuity, the monotonicity of the minimum type and hence the reserve price between n and $n' > n$ is strict for low type sellers if $m^*(\underline{s}, n) > \underline{x}$.

[High type sellers]

[3] In particular, it is satisfied if $\underline{x} \geq \frac{1}{f(\underline{x})}$ such that $m^*(\underline{s},1) = \underline{x}$. Anyway, even for $\underline{x} = \underline{s}$, it is satisfied for many standard distributions.

[4] Assumption (2.1) does not hold for $0 < \phi < 1$. Nevertheless, a weaker assumption is indeed sufficient to avoid bunching phenomena in our analysis: it is that $x \to (J(s,x,n) - \xi(s))$ is quasimonotone. This weaker assumption is satisfied as depicted in the left panels of Figure 1.

In the separating equilibrium of the linear independent model, the minimum type that bids, $m(s,n)$, is strictly increasing in the number of bidders for high-type sellers, i.e. if the signal of the seller is above a given threshold s^*. Nevertheless, the reserve price, $r(s,n) = w(s, m(s,n), n)$, is strictly decreasing in the number of bidders for sufficiently high types.

In the supplementary material of this comment, we show that Proposition [3.1] extends to the Conditionally Independent Private Information (CIPI) model studied by Li, Perrigne and Vuong [8] where bidders' private signals X_i are conditionally independent given a common component z and where the conditional distribution function $H(x|z)$ satisfies the Monotone Likelihood Ratio Property (MLRP) which implies that $(X_1, \cdots, X_n)$ are affiliated. The result for the lowest type sellers corresponds to an extension of Proposition 5 in Levin and Smith [7] with interdependent values. As the signal of the seller, s, becomes large, $\frac{\partial r(s,n)}{\partial n}$ is still approximately equal to $\frac{\partial w(s,\overline{x},n)}{\partial n}$ which is strictly positive. Therefore, the reserve price eventually decreases with n for high type sellers.

Under the assumption of independent signals, the Revenue Equivalence Theorem still applies with interdependent values (Theorem 3.5 in Milgrom [9]). Hence Proposition [3.1] still applies to any auction format, e.g. the first price auction, that puts the object in the hands of the highest type and induces the same participation threshold. Nevertheless, in the CIPI model, such an equivalence does not hold anymore. In first price auctions with affiliated values, Pinkse and Tan [12] derive a countervailing force to the competition effect that reduces the incentive to raise higher bids when the number of opponents increases. Therefore, we conjecture that a higher reserve price is more valuable in the first price auction and thus that the seller needs an even higher reserve price to signal her type.

Informational externalities and affiliation between bidders exert a countervailing force to the signalling effect fingered by CR&Y. First, the participation threshold and the reserve price for the lowest type seller $\underline{s}$ correspond exactly to the ones prevailing if it were common knowledge that the seller is of type $\underline{s}$. We are not aware of any similar formalization of the insight that the reserve price is decreasing in the number of bidders when there are informational externalities. Nevertheless, in the second price auction with negative allocative externalities (whose analysis is very closed related to the second price auction with positive informational externalities), Jehiel and Moldovanu [4] emphasize that the optimal reserve price does, in general, depend on the number of bidders n. Moreover, in their simple illustration where the intrinsic value of the item is the private signal x and the payoff is $\alpha < 0$ if the item goes in the hands of another bidder, they derive a similar expression for what is here referred to as the map $J(s,x,n)$: it is equal to

$x - \frac{1-F(x)}{f(x)} - (n-1)\alpha \cdot \frac{1-F(x)}{F(x)}$.[5] If $(n-1)\alpha$ is increasing in the number of bidders, i.e. if the total externality suffered by non-purchasers is increasing in n, then we obtain that the optimal reserve price is decreasing in n.

Nevertheless, our numerical investigation suggests that this countervailing effect to CR&Y's signalling effect do matter only for extremely low signals. From our numerical simulations for $\phi = 1.5, 1, 0.5$ and for $n = 2, 3, 4, 10$, we see in the middle panel of Figure 1 that the participation threshold is higher for larger n provided that $s - \underline{s} > 0.04$, a small figure compared to the variation of bidders' private signal from 0 to 1. Note that the fact that, in our graphs, the curses $m(s, n)$ cross in the middle of the figure is a framing effect since we focus the graph on very low types for the seller. Thus our numerical investigation suggests that it is the signalling effect, CR&Y's main insight, that is quickly prominent insofar as the participation threshold is increasing in n.

Unlike the participation threshold, the comparative statics of the reserve price goes in the same sense at both extreme of the seller's range: it is decreasing in n, a result which inverts CR&Y's original result which is true only without externalities between bidders. Our numerical simulations, depicted in the right panel of Figure 1, show that this effect may be valid on most of the range of the seller's intermediate signals. For signals such that $s - \underline{s} > 0.1$, the reserve price is increasing in n in Figure 1.

4 Conclusion

Though we challenge CR&Y's results, our comment concerns mainly the comparative statics of the reserve price absent of any signalling incentives and in presence of informational externalities. We emphasize that CR&Y's signalling incentives in fixing the reserve price are still first order even with a large number of bidders. Similarly, reserve prices or other instruments could be used to signal other aspects of the seller's information, e.g. a better knowledge of the number of potential bidders. Signalling by an informed seller is a promising avenue of research. In this vain, Chakraborty et al. [3] analyses the strategic choice of the order of sale in sequential auctions for similar items of different quality.

Appendix

[5] See expression (40) in Jehiel and Moldovanu [4]. Indeed, when they went from equation (39) to equation (40), they assign the wrong sign to the additional term resulting from the allocative externalities and their final expression would imply the opposite comparative statics result. We correct here the error.

A Proof of Proposition [3.1]

In the linear independent model, the expression of the derivative of the map J with respect to the number of bidders is given by:

$$\frac{\partial J(s,x,n)}{\partial n} = \alpha'(n)\left[(2-\frac{1}{F(x)})\cdot[x-E[X|X\leq x]] - \frac{\partial(x-E[X|X\leq x])}{\partial x}\cdot\frac{1-F(x)}{f(x)}\right] \tag{8}$$

The derivative $\alpha'(n)$ is strictly negative as it has been assumed. Then, it remains to analyse the sign of the term in the brackets.

A preliminary lemma shows that the second term in the brackets is positive.

Lemma A.1 *If the reverse hazard rate $\frac{f(x)}{F(x)}$ is nonincreasing, then the function $x - E[X|X \leq x]$ is nondecreasing in x.*

Proof 1

$$x - E[X|X\leq x] = x - \int_{\underline{x}}^{x} u\frac{f(u)}{F(x)}du$$

Differentiating the above expression and after an integration by parts, we obtain:

$$\frac{\partial(x-E[X|X\leq x])}{\partial x} = 1 - \frac{f(x)\int_{\underline{x}}^{x}F(u)du}{(F(x))^2}$$

The assumption that the reverse hazard rate $\frac{f(x)}{F(x)}$ is nonincreasing implies that:

$$\frac{F(x)}{f(x)} \geq \frac{\int_{\underline{x}}^{x}F(u)du}{\int_{\underline{x}}^{x}f(u)du}$$

Finally, we have that $\frac{\partial(x-E[X|X\leq x])}{\partial x} \geq 0$

Let us return to expression (13). The first term in the brackets has the same sign as $(2-\frac{1}{F(x)})$. From assumption [A.3], we obtain that $J(\underline{s},x,n)$ is strictly increasing in n if x is in the range $[\underline{x}, m^*(\underline{s},1)]$. Thus we obtain that $m^*(\underline{s},n)$ and hence the corresponding reserve price are nonincreasing in n. It is strictly decreasing until $m^*(\underline{s},n) > \underline{x}$ and then is stuck to $m^*(\underline{s},n) = \underline{x}$. By continuity, if the monotonicity is strict for the lowest type seller, it is valid in the neighborhood of $\underline{s}$, i.e. for low type sellers. Thus we have proved the first part of proposition [3.1].

Now consider the case of high type sellers. Differentiating the right hand of equation (7c), we have:

$$\frac{\partial^2 s(m,n)}{\partial m \partial n} = \frac{nf(m)F^{2n-1}(m)\log(F(m))}{(1-F^n(m))^2}\cdot[J(s,m,n)-\xi(s)]$$

$$+[1+n\log(F(m))]\cdot\frac{f(m)F^{n-1}(m)}{(1-F^n(m))}\cdot[J(s,m,n)-\xi(s)]+\frac{nf(m)F^{n-1}(m)}{(1-F^n(m))}\cdot\frac{\partial J(s,m,n)}{\partial n}$$

Then note that $1-F(x)=(\overline{x}-x)\cdot f(\overline{x})+o(\overline{x}-x)$ where $f(\overline{x})>0$ and $\frac{o(\overline{x}-x)}{(\overline{x}-x)}$ is a bounded function, $\lim_{x\to\overline{x}}\frac{\partial J(s,x,n)}{\partial n}=\alpha'(n)\cdot[\overline{x}-E[X|X\leq\overline{x}]]<0$ and $\lim_{x\to\overline{x}}J(s,x,n)=\alpha(n)\cdot\overline{x}+(1-\alpha(n))E[X|X\leq\overline{x}]>0$. Then we make the asymptotic development of each term of the above expression according to the powers of $(\overline{x}-x)$. The first term has the form $\frac{1}{(\overline{x}-x)}\cdot\frac{\alpha(n)\cdot\overline{x}+(1-\alpha(n))E[X|X\leq\overline{x}]}{n}+O(1)$, where $O(1)$ represents a bounded residual. The second term has the form $-\frac{1}{(\overline{x}-x)}\cdot\frac{\alpha(n)\cdot\overline{x}+(1-\alpha(n))E[X|X\leq\overline{x}]}{n}+O(1)$. The third term is equivalent to $\frac{\alpha'(n)\cdot[\overline{x}-E[X|X\leq\overline{x}]]}{(\overline{x}-x)}$. Finally we obtain that $\frac{\partial^2 s(x,n)}{\partial m\partial n}$ is equivalent to $\frac{K}{(\overline{x}-x)}$, where the strictly negative constant K is equal to $\alpha'(n)\cdot[\overline{x}-E[X|X\leq\overline{x}]]$, as x goes to $\overline{x}$.

Take $\widetilde{x}\in(m^*(\underline{s},n),1)$ and $x>\widetilde{x}$. We have:

$$\frac{\partial s(x,n)}{\partial n}=\frac{\partial s(\widetilde{x},n)}{\partial n}+\int_x^{\widetilde{x}}\frac{\partial^2 s(u,n)}{\partial m\partial n}du\xrightarrow[x\to\overline{x}]{}-\infty$$

The first term $\frac{\partial s(\widetilde{x},n)}{\partial n}$ is bounded since $\frac{\partial^2 s(m,n)}{\partial m\partial n}$ is bounded on the range $[\underline{x},\widetilde{x}]$. On the other hand the second term goes to minus infinity as x goes to $\overline{x}$. Thus we are done with the comparative statics of the minimum type that bids for high type sellers: $m(s,n)$ is strictly increasing in n if s is high enough.

Now consider the reserve price: $r(s,n)=w(s,m(s,n),n)$. We have:

$$\frac{\partial r(s,n)}{\partial n}=\frac{\partial w}{\partial x}\cdot\frac{\partial m}{\partial n}+\frac{\partial w}{\partial n}$$

We have seen that $\frac{\partial s(m,n)}{\partial n}$ goes to minus infinity when the minimum type that bids goes to $\overline{x}$ or equivalently $\frac{\partial m(s,n)}{\partial n}$ goes to zero for high type sellers. Finally, $\frac{\partial r(s,n)}{\partial n}$ is equivalent to $\frac{\partial w}{\partial n}$ as s goes large which proves the second part of proposition [3.1] since w is strictly decreasing in n.

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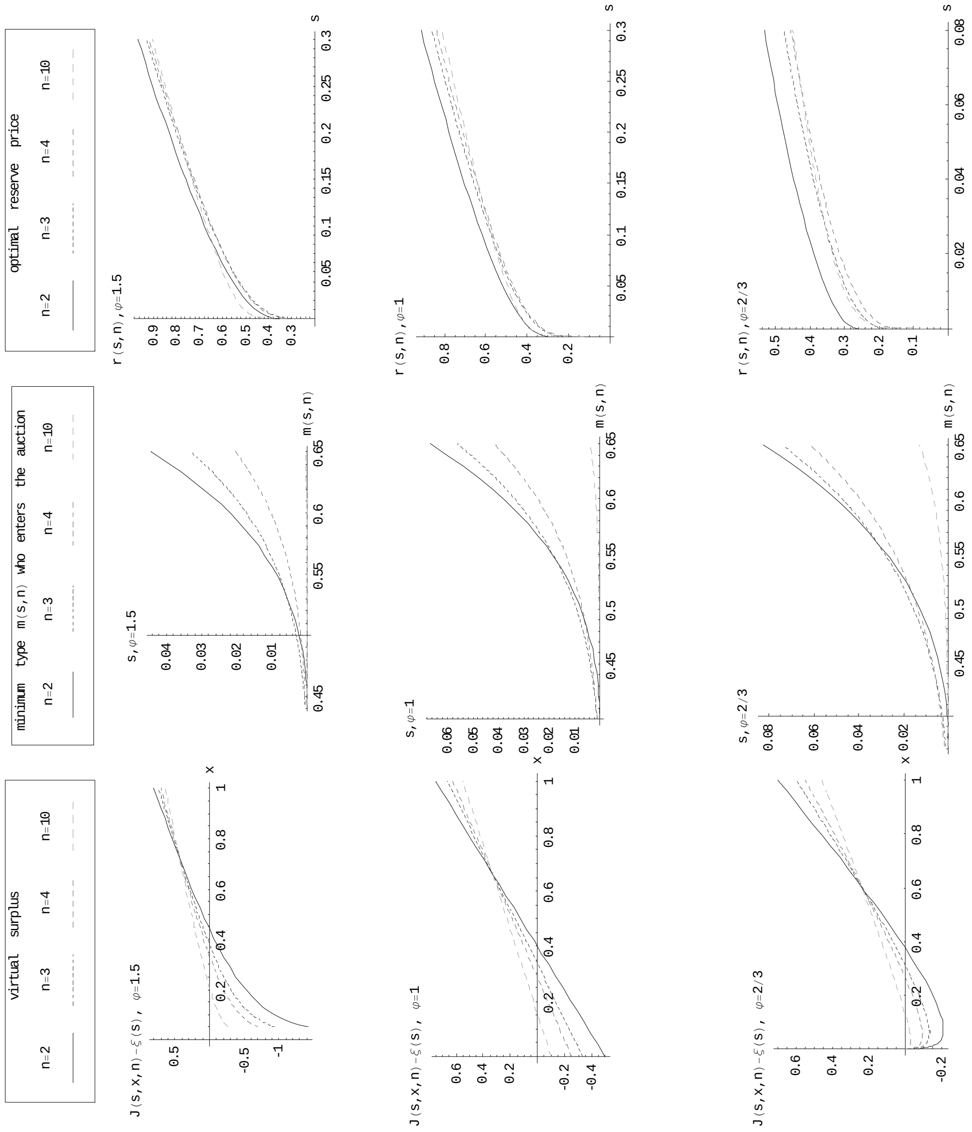

Figure 1: Numerical Example

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A On the Comparative Statics of the Optimal Reserve Price: a comment on "Reserve price signaling": Supplementary Material

A.1 Introduction and notation

The aim of this supplementary material is to extend Proposition 3.1 under a quite general form of affiliation.[6] We consider the Conditionally Independent Private Information (CIPI) model studied by Li, Perrigne and Vuong [8] where bidders' private signals X_i are conditionally independent given a common component z distributed according to the density g with support $[\underline{z}, \overline{z}]$. Denote the conditional distribution and density functions by $H(x|z)$ and $h(x|z)$ with support $[\underline{x}, \overline{x}]$. Assume that $h(x|z)$ satisfies the Monotone Likelihood Ratio Property (MLRP), so that $(X_1, \cdots, X_n)$ are affiliated. We maintain the assumption that the signal observed by the seller is independent of $(Z, X_1, \cdots, X_n)$. The following section generalizes Proposition 3.1 to the *linear affiliated model.*

$$u(s, x_i, x_{-i}) = s + \alpha(n) \cdot x_i + (1 - \alpha(n)) \cdot \frac{\sum_{j \neq i} x_j}{n-1} \tag{9a}$$

$$\xi(s) = s, \tag{9b}$$

Additional to the notation in the paper which immediately adapt to the present framework, we denote by $f_{(1:n)}(.|z)$ and $F_{(2:n)}(.|z)$ ($f_{(1:n)}(.|z)$ and $f_{(2:n)}(.|z)$) respectively the distribution (density) functions of the first and second highest signal statistics among all the signals of the n buyers conditional on the common component z.

A.2 Comparative Statics in the linear affiliated model

The expected payoff of a seller with type s, which is perceived to be type $\widehat{s}$ and that induces the participation threshold m by announcing a reserve price $w(\widehat{s}, m, n)$, can be written as follows:

$$U(s, \widehat{s}, m, n) = \int_{\underline{z}}^{\overline{z}} \left(\xi(s) f_{(1:n)}(m|z) + w(\widehat{s}, m, n)[F_{(2:n)}(m|z) - F_{(1:n)}(m|z)] + \int_m^{\overline{x}} v(\widehat{s}, x, n) dF_{(2:n)}(x|z) \right) \cdot g(z)dz, \tag{10}$$

Differentiating (10) we have

$$\frac{\partial U}{\partial m} = \int_{\underline{z}}^{\overline{z}} \left(f_{(1:n)}(m|z)[\xi(s) - J(\widehat{s}, m, n|z)] \right) \cdot g(z)dz \tag{11a}$$

[6] By the de Finetti's Theorem, any affiliated model is a CIPI model under the assumption that $(X_1, \cdots, X_m)$ is exchangeable and identically distributed for any number of bidders $n \geq m$.

$$\frac{\partial U}{\partial \widehat{s}} = \int_{\underline{z}}^{\overline{z}} \left(\frac{\partial w(\widehat{s}, m, n)}{\partial \widehat{s}} [F_{(2:n)}(m|z) - f_{(1:n)}(m|z)] + \int_{m}^{\overline{x}} \frac{\partial v(\widehat{s}, x, n)}{\partial \widehat{s}} dF_{(2:n)}(x|z) \right) \cdot g(z) dz, \tag{11b}$$

where the map $J(s, m, n|z)$, which corresponds to a generalization of the "virtual surplus" of Myerson [11] conditional on the common component z, is given by the following expression:

$$J(s, x, n|z) = w(s, x, n) - \frac{\partial w(s, x, n)}{\partial x} \cdot \frac{F_{(2:n)}(x|z) - F_{(1:n)}(x|z)}{f_{(1:n)}(x|z)} + (v(s, x, n) - w(s, x, n)) \cdot \frac{f_{(2:n)}(x|z)}{f_{(1:n)}(x|z)} \tag{12}$$

As in CRY and our comment, a generalization of Myerson [11]'s regularity assumption is required.

Assumption A.1 *For any s, the map $\int_{\underline{z}}^{\overline{z}} \left(f_{(1:n)}(m|z)[\xi(s) - J(s, x, n|z)] \right) \cdot g(z) dz$ is strictly decreasing in x.*

Similarly to assumptions 3.1 and 3.2, we make two additional mild assumptions on the distribution of bidders' signal conditional on z.

Assumption A.2 *For any common component z, the reverse hazard rate $\frac{h(x|z)}{H(x|z)}$ is non-increasing in x.*

Assumption A.3 *Under public information of the seller's signal and for any common component z, the optimal participation threshold of the lowest type seller facing one bidder involves a probability of sale superior to one half for any z: $H(m^*(\underline{s}, 1)|z) \leq \frac{1}{2}$.*

As in the proof of proposition 3.1 in the comment, those assumptions guarantee that $J(\underline{s}, x, n|z)$ is strictly increasing in n in the range $[\underline{x}, m^*(\underline{s}, 1)]$ for the linear affiliated model where the expression of $J(\underline{s}, x, n|z)$ is given by:

$$\frac{\partial J(s, x, n|z)}{\partial n} = \alpha'(n) \left[(2 - \frac{1}{H(x|z)}) \cdot [x - E[X|X \leq x, X_1 = x]] - \frac{\partial (x - E[X|X \leq x, X_1 = x])}{\partial x} \cdot \frac{1 - H(x|z)}{h(x|z)} \right] \tag{13}$$

In particular, assumption A.2 guarantees that $(x - E[X|X \leq x, X_1 = x]) = E_{Z|X_1 = x}[x - E[X|X \leq x, Z = z]]$ is increasing in x after the inversion of the expectation and the derivation.

In the separating equilibrium, the lowest type seller equilibrium reserve price $r^*(\underline{s}, n)$ (or equivalently the optimal threshold $m^*(\underline{s}, n)$) corresponds to the one that would be optimal were her signal publicly observable. Finally, $m^*(\underline{s}, n)$ is characterized by the equation $\frac{\partial U}{\partial m}(\underline{s}, \underline{s}, m^*(\underline{s}, n), n) = 0$.

Consider expression (11a) for $s = \widehat{s} = \underline{s}$:

$$\frac{\partial U}{\partial m}(\underline{s}, \underline{s}, m, n) = \int_{\underline{z}}^{\overline{z}} \left(n \cdot H^{n-1}(m|z) \cdot h(m|z)[\xi(\underline{s}) - J(\underline{s}, m, n|z)] \right) \cdot g(z) dz \tag{14}$$

Lemma A.1 $\frac{\partial U}{\partial m}(\underline{s}, \underline{s}, m^*(s,n), n+1) < 0$

Proof 2

$$\frac{\partial U}{\partial m}(\underline{s}, \underline{s}, m^*(\underline{s},n), n+1) = \int_{\underline{z}}^{\overline{z}} ((n+1) \cdot H^n(m|z) \cdot h(m|z)[J(\underline{s}, m, n|z) - J(\underline{s}, m, n+1|z)]) \cdot g(z)dz \quad (15)$$
$$+ \int_{\underline{z}}^{\overline{z}} ((n+1) \cdot H^n(m|z) \cdot h(m|z)[\xi(\underline{s}) - J(\underline{s}, m, n|z)]) \cdot g(z)dz \quad (16)$$

The first term is negative since $J(\underline{s}, x, n|z)$ is strictly increasing in n in the range $[\underline{x}, m^*(\underline{s}, 1)]$. To prove that the second term is non-positive, we use the following lemma whose proof is left to the reader.

Lemma A.2 *Let $z \to A(z)$ be a quasimonotone function such that $\int_{\underline{z}}^{\overline{z}} A(z)dz = 0$ and $z \to B(z)$ be a nonincreasing function, then $\int_{\underline{z}}^{\overline{z}} B(z) \cdot A(z)dz \leq 0$*

Note first that $z \to J(\underline{s}, x, n|z)$ is nonincreasing in z since $H(x|z)$ and $\frac{H(x|z)}{h(x|z)}$ are both nonincreasing in z from MLRP. Consequently $A(z) = \cdot H^{n-1}(m|z) \cdot h(m|z)[\xi(\underline{s}) - J(\underline{s}, m, n|z)] \cdot g(z)$ is quasimonotone. Let $B(z) = H(m|z)$ which is nonincreasing and apply lemma A.2 to conclude.

We conclude with assumption (A.1) that $m^*(\underline{s}, n)$ is nonincreasing in n.

Corollary A.1 *The optimal threshold $m^*(\underline{s}, n)$ is nonincreasing.*

The differential equation that characterizes the unique separating equilibrium in the linear affiliated model is:

$$\frac{\partial s(m,n)}{\partial m} = \int_{\underline{z}}^{\overline{z}} \frac{n \cdot h(m|z) \cdot H^{n-1}(m|z)}{1 - H^n(m|z)} [J(s, m, n|z) - \xi(s)] \cdot g(z)dz \quad (17)$$

As in the proof of Proposition 3.1 in our comment, the same kind of analysis with a Taylor development at the upper bound of the bidders' support can be lead after inverting the order of integration and differentiation.

Finally we obtain that $\frac{\partial^2 s(x,n)}{\partial m \partial n}$ is equivalent to $\frac{K}{(\overline{x}-x)}$, where the strictly negative constant K is equal to $\int_{\underline{z}}^{\overline{z}} \alpha'(n) \cdot [\overline{x} - E[X|X \leq \overline{x}]]g(z)dz$, as x goes to $\overline{x}$.

Then we have:

$$\frac{\partial s(x,n)}{\partial n} \longrightarrow_{x \to \overline{x}} -\infty.$$

We conclude in the same way as in the proof of Proposition 3.1 that the minimum type that bids (resp. the reserve price) is increasing (decreasing) in the number of bidders for high type sellers.

Finally, we have proved that Proposition 3.1 remains true in the linear affiliated model and we obtain the more general following proposition.

Proposition A.2 (Comparative Statics with respect to the number of bidders) ***[Low type sellers]***

In the separating equilibrium of the linear ***affiliated*** *model, the minimum type that bids,* $m^*(s,n)$*, and hence the reserve price,* $r(s,n) = w(s, m(s,n), n)$*, is non-increasing in* n *for the lowest type seller, i.e. for* $s = \underline{s}$*. The monotonicity of the minimum type and hence the reserve price between* n *and* $n' > n$ *is strict for low type sellers if* $m^*(\underline{s}, n) > \underline{x}$*.*

[High type sellers]

In the separating equilibrium of the linear ***affiliated*** *model, the minimum type that bids,* $m(s,n)$*, is strictly increasing in the number of bidders for high-type sellers, i.e. if the signal of the seller is above a given threshold* s^**. Nevertheless, the reserve price,*$r(s,n) = w(s, m(s,n), n)$*, is strictly decreasing in the number of bidders for sufficiently high types.*