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Nonparametric Identification and Estimation of a Common Value Auction Model

Ph. FÉVRIER [1]

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[1] ENSAE, 3 avenue Pierre Larousse, 92245 Malakoff Cedex, France. fevrier@ensae.fr
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Nonparametric Identification and Estimation of a common value auction model

Philippe Février *

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Abstract

Structural econometric studies on auctions have mainly focused on the independent private value paradigm. In this paper, we are interested in the "opposite" case known as the pure common value model. More precisely, we restrict our attention to a common value model defined by two functions : the density of the true value of the auctioned good and a unique function that appears in the definition of the conditional densities of the signals. We establish that this common value model is nonparametrically identified without any further restrictions. We then propose a one-step nonparametric estimation method and prove the uniform consistency of our estimators. We apply our method on simulated data and show that the technique we propose is adequate to recover the distribution functions of interest.

Keywords: Common Value; Auctions; Non Parametric Estimation.

***JEL* Classification:** C14;D44.

Résumé

Les études structurelle sur les enchères ont principalement porté sur le modèle à valeurs privées. Dans ce papier, nous nous intéressons au contraire au modèle à valeur commune. Plus précisément, nous étudions un modèle défini par deux fonctions: la densité de la valeur du bien mis aux enchères et une unique fonction qui apparaît dans la définition de la densité conditionnelle des signaux. Nous montrons que ce modèle est identifié non-paramétriquement sans restriction supplémentaire. Nous proposons alors une méthode d'estimation et prouvons la convergence uniforme de nos estimateurs. Finalement, nous utilisons cette méthode sur des données simulées et montrons que notre technique permet de retrouver convenablement les distributions d'intérêt.

Mots Clés: Valeur Commune; Enchère; Estimation Non-paramétrique.

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*ENSAE, 3 avenue Pierre Larousse, 92245 Malakoff Cedex, France, Email: fevrier@ensae.fr. I am very grateful to X. D'Haultfoeuille, J-P. Florens, P. Haile, I. Perrigne, P. Reny and Q. Vuong for helpful discussions and comments. The paper has been presented at the Cirano conference on auctions in Montreal and at the econometrics of auctions conference in Toulouse.

1 Introduction

Structural econometric approaches have been successfully applied during the last decade to study auction data. The aim of such analyzes is to recover the structural parameters of a theoretical model from the data using econometric methods. In the case of auctions, the econometrician is interested in estimating the distribution of the value of the good for each participant from the observed bids. It relies on the equilibrium that defines how bids depend on this distribution.

Previous studies mostly focused on the private value paradigm (PV) (Laffont et al. (1995), Donald and Paarsch (1996), Elyakime et al. (1994, 1997), Guerre et al. (2000)). In these models, each bidder knows his own private value for the auctioned good but does not know others' valuations.

The "opposite" case is known as the pure common value paradigm (CV). In this model, the value of the auctioned good is unknown but the same for each bidder. The participants receive a signal correlated with this value. It turns out that identification and estimation for CV models are more complicated than for PV models. The main reason behind these difficulties comes from the nonparametric identification of the CV model from observed bids (Laffont and Vuong (1996)). As a consequence, one has to impose some further restrictions to obtain identification results. Paarsch (1992) proposes a parametric approach, whereas Li et al. (2000) develop a nonparametric one to analyze the CV model. In their paper, the authors assume a multiplicative decomposition of the signals into a common component (the value of the good) and an indiosyncratic one (a specific signal) for each bidder. Adding some further restrictions, Li et al. show that the CV model is identifiable and propose a two-step nonparametric procedure to estimate the densities of both components.

In this paper, we analyze a CV model in which the knowledge of all densities of the signals conditionally on the value of the good reduces to the knowledge of a unique function. In this context, the CV model is defined by two functions only : the density of the true value and the unique function that enters in the conditional density of the signals. This model belongs to the class of CV models studied by Fevrier (2006) for which nonparametric identification is obtained without any further restrictions. Unfortunately, his proof does not allow us to derive a simple estimation method. Hence, we propose another way to prove identification on which our nonparametric estimation method will be based. Contrary to most of the studies, we show that it is possible to use directly the observed bids instead of using a two-step method as in Guerre et al. (2000). We prove the uniform consistency of our one-step nonparametric estimator and apply our method on simulated data. We show that this method is feasible and recovers correctly the distribution functions of interest.

The paper is organized as follows. Section 2 presents the CV model and our nonparametric identification results. Section 3 describes the estimation method we propose. Monte Carlo experiments are conducted in Section 4. Section 5 concludes.

2 The CV model and the Structural Approach

2.1 The Pure CV model

The general case

In the Pure Common Value model (Rothkopf (1969), Wilson (1977)), a single and indivisible good is auctioned to n bidders. The value V of the good, unknown to the bidders, is distributed following a distribution function $F_V(.)$ and a density function $f_V(.)$ on the support $[\underline{V}, \overline{V}]$ with $(\underline{V}, \overline{V}) \in \mathbb{R}^{+2}$.

Each bidder i receives a private signal S_i. The signals are conditionally independent given the common value V. We note $F_{S|V}(.|.)$ the distribution function of the signal given V and $f_{S|V}(.|.)$ the associated density function. We suppose that $f_{S|V}(.|.)$ satisfies the monotone likelihood ratio property.[1] Its support is denoted $[T_V, T^V]$ with $(T_V, T^V) \in \mathbb{R}^{+2}$.

Each player knows his private signal as well as the distribution functions. He does not know however the private signal of other bidders.

We study first price auctions. Each bidder submits a bid and the winner is the one who submits the highest bid. He obtains the object and pays his bid. Hence, a strategy for a player i is a function $b_i(.)$ that associates to each signal S_i the amount $b_i(S_i)$ that player i wants to bid. As shown by Milgrom and Weber (1982), a symmetric equilibrium exists in first price common value auctions. To describe this equilibrium, it is useful to introduce the following functions. We note $Y_i = \max_{j \neq i} S_j$ and $F_{Y_i|S_i}(.|.)$ (resp. $f_{Y_i|S_i}(.|.)$) the distribution function (resp. density function) of Y_i conditionally on the signal S_i of player i. We also introduce the function $V(s, y) = E[V|S_i = s, Y_i = y]$ that is the expected value of the good conditionally on the signal S_i of player i and the highest signal Y_i of the other players.

Proposition 2.1 *(Milgrom and Weber, 1982) In a common value first price auction, a symmetric equilibrium strategy exists and is given by:*

$$b(s) = V(s, s) - \int_{T_{\underline{V}}}^{s} L(\alpha|s) dV(\alpha, \alpha) \tag{2.1}$$

where $L(\alpha|s) = exp[-\int_\alpha^s f_{Y_i|S_i}(u|u)/F_{Y_i|S_i}(u|u)du]$.

Our model

We consider a pure CV model with n bidders ($n \geq 2$) and assume that the density of the signals conditionally on the value $V = v$ takes the form:

$$\forall s \in [T(\underline{V}), T(v)], \quad f_{S|V}(s|v) = \frac{h(s)}{H(T(v))}$$

[1] The density $f_{S|V}$ has the monotone likelihood ratio property if for all $s' > s$ and $v' > v$, $f_{S|V}(s|v)/f_{S|V}(s|v') \geq f_{S|V}(s'|v)/f_{S|V}(s'|v')$.

This function is defined on an interval $[T(\underline{V}), T(v)]$. $h(.)$ is the derivative of $H(.)$ and $H(T(\underline{V})) = 0$.

In this model, no restriction is imposed on the density of the value V whereas the distributions of the signals conditionally on the value V are supposed to be representable by a unique function $h(.)$. A natural example is a model in which the value of the good is distributed uniformly on $[\underline{V}, \overline{V}]$ and the signals are distributed uniformly on $[\underline{V}, 2v - \underline{V}]$ conditionally on $V = v$. This is the case if $h = 1$ and $T(v) = 2v - \underline{V}$. More generally, the function $h(.)$ and the interval $[T(\underline{V}), T(v)]$ define the amount of information that the signals carry over the value and play therefore a key role in the analysis.

We will need the following assumption.

Assumption 1 *1. for all $s \in [T(\underline{V}), T(\overline{V})]$, $h(s) > 0$ and for all $v \in [\underline{V}, \overline{V}]$, $f_V(v) > 0$;*

2. $h(.)$ is continuously differentiable.

3. $T(.)$ is strictly increasing and continuously differentiable.

Assumption 1 is a standard assumption in nonparametric literature. It imposes some smoothness on the densities and insures that these densities are bounded away from zero. Furthermore, $T(.)$ is supposed to be increasing for $f_{S|V}(.|.)$ to satisfy the monotone likelihood property.

2.2 Nonparametric Identification

A fundamental issue is to study if the model is identified nonparametrically i.e. to analyze if the observation of the bids determines uniquely the functions $F_V(.)$, $h(.)$ and $T(.)$. Of course, what is observed is important and we will suppose that, in every auction, all bids are available.

In the general case, Laffont and Vuong (1996) (see also Athey and Haile (2002)) have shown that the CV model is not identifiable. Fevrier (2006) proved however that the pure CV model is identified if there are some variations in the bounds of the conditional distribution functions of the signals. Under our specifications, proposition 2 of Fevrier (2006) applies and our model is nonparametrically identified.

Unfortunately, it is difficult to estimate the distribution functions based on Fevrier's identification proof. For this reason, we propose another way to prove identification upon which our estimation method will be based. We proceed in three steps.

- We first prove that for each $s' \in [T(\underline{V}), T(\overline{V})]$ and each $s \in [T(\underline{V}), s']$, $F_{S|V}(s|T^{-1}(s'))$ is identified.
- We then show that for each $s \in [T(\underline{V}), T(\overline{V})]$, $F_V(T^{-1}(s))$ is identified
- Finally, using the first order condition, we prove that $T^{-1}(.)$ is identified over its support $[T(\underline{V}), T(\overline{V})]$.

The identification of $F_V(.)$ and $F_{S|V}(.|.)$ is proved by combining these results for $n \geq 2$.

We will note, for each $k \leq n$, $f_S(s_1, ..., s_k)$ the joint density of $(s_1, ..., s_k)$, $F_S(s_1, ..., s_k)$ the joint distribution function, and $\frac{\partial F_S}{\partial s_1}(s_1, ..., s_k)$ and $\frac{\partial^2 F_S}{\partial s_1^2}(s_1, ..., s_k)$ the first and second partial derivatives of $F_S(s_1, ..., s_k)$ in s_1.

Identification of $F_{S|V}(.|T^{-1}(.))$

First, one has to remark that the model is defined up to a transformation of the signals. Indeed, observing $\phi(s)$ instead of s is equivalent in the model to replace $H(.)$ by $H \circ \phi^{-1}(.)$ defined over the segment $[\phi \circ T(\underline{V}), \phi \circ T(\overline{V})]$. We have to normalize the signals and a natural normalization is $b(s) = s$. Hence, we observe the signals and identify the bounds $T(\underline{V})$ and $T(\overline{V})$ by respectively the minimum and the maximum of the signals..

Given $s' \geq s$, we identify the distribution function

$$\begin{aligned}
F_S(s', s) &= P(S_1 \leq s', S_2 \leq s, V \leq T^{-1}(s)) + P(S_1 \leq s', S_2 \leq s, T^{-1}(s) \leq V \leq T^{-1}(s')) \\
&\quad + P(S_1 \leq s', S_2 \leq s, T^{-1}(s') \leq V) \\
&= F_V(T^{-1}(s)) + H(s) \int_{T^{-1}(s)}^{T^{-1}(s')} \frac{f_V(v)}{H(T(v))} dv + H(s')H(s) \int_{T^{-1}(s')}^{\overline{V}} \frac{f_V(v)}{H^2(T(v))} dv
\end{aligned}$$

and its partial derivative :

$$\frac{\partial F_S}{\partial s_1}(s', s) = h(s')H(s) \int_{T^{-1}(s')}^{\overline{V}} \frac{f_V(v)}{H^2(T(v))} dv$$

Hence, for all $s' \in [T(\underline{V}), T(\overline{V})]$ and all $s \in [T(\underline{V}), s']$, we identify

$$F_{S|V}(s|T^{-1}(s')) = \frac{H(s)}{H(s')} = \frac{\frac{\partial F_S}{\partial s_1}(s', s)}{\frac{\partial F_S}{\partial s_1}(s', s')} \tag{2.2}$$

Identification of $F_V(T^{-1}(.))$

The distribution function and the density of a signal $s \in [T(\underline{V}), T(\overline{V})]$ are given, respectively, by

$$F_S(s) = F_V(T^{-1}(s)) + H(s) \int_{T^{-1}(s)}^{\overline{V}} \frac{f_V(v)}{H(T(v))} dv$$

and

$$f_S(s) = h(s) \int_{T^{-1}(s)}^{\overline{V}} \frac{f_V(v)}{H(T(v))} dv$$

Rewriting the first equation and using the second one, we have

$$F_V(T^{-1}(s)) = F_S(s) - \frac{H(s)}{h(s)} f_S(s)$$

that is, using $f_S(s,s) = h(s)^2 \int_{T^{-1}(s)}^{\overline{V}} \frac{f_V(v)}{H^2(T(v))} dv = \frac{h(s)}{H(s)} \frac{\partial F_S}{\partial s_1}(s,s)$

$$F_V(T^{-1}(s)) = F_S(s) - \frac{\frac{\partial F_S}{\partial s_1}(s,s)}{f_S(s,s)} f_S(s) \tag{2.3}$$

The right hand side of equation (2.3) is identified. Hence $F_V(T^{-1}(.))$ also is.

Identification of $T^{-1}(.)$

We prove in appendix A that the first order condition can be rewritten for all $s \in [T(\underline{V}), T(\overline{V})]$ as

$$\begin{aligned} T^{-1}(s) = & s + \frac{\frac{\partial F_S}{\partial s_1}(s,s)}{(n-1) f_S(s,s)} + \left[\frac{F_{S_{\max}}(s) - F_V(T^{-1}(s))}{n-1} \right] \times \\ & \frac{\frac{\partial^2 F_S}{\partial s_1^2}(s,s) - f_S'(s) - n f_S(s) + n \frac{f_S(s) f_S(s,s)}{\frac{\partial F_S}{\partial s_1}(s,s)}}{f_S(s) \frac{\partial^2 F_S}{\partial s_1^2}(s,s) - f_S'(s) \frac{\partial F_S}{\partial s_1}(s,s)} \end{aligned} \tag{2.4}$$

where $F_{S_{\max})}(.)$ is the distribution of the highest signal.

The functions that appear in the right hand side of equation (2.4) are identified. Hence, $T^{-1}(.)$, and $T(.)$, also are.

Identification of $F_V(.)$ *and* $F_{S|V}(.|.)$

Combining the previous results, we conclude that the distribution function of the value $F_V(.) = F_V(T^{-1}(T(.)))$ and the conditional distributions of the signals $F_{S|V}(.|.) = F_{S|V}(.|T^{-1}(T(.)))$ are identified.

Proposition 2.2 *The model is nonparametrically identified.*

This identification result is important and gives with Li et al. (2000) another nonparametric identification result for pure CV auctions upon which an estimation method can be based. The model estimated by Li et al. may be seen as more natural, but our model has some nice properties. First, the knowledge of the joint density function of the signals and the value is reduced to the knowledge of three functions of one variable $h(.)$, $T(.)$ and $F_V(.)$. No other restriction is needed and full nonparametric identification is achieved. Moreover, we will show in the next part that it can be easily estimated. Finally, the model is overidentified. It imposes several restrictions on the distributions of the bids that can be tested.

3 Estimation

The estimation method is based on our identification result and will follow the same logic. It consists in estimating some distribution functions of the bids and to use them to construct estimates for $F_{S|V}(.|T^{-1}(.))$, $F_V(T^{-1}(.))$ and $T^{-1}(.)$.

Let $n \geq 2$ be a given number of bidders.[2] Let L be the number of auctions with n bidders indexed by l. We note $\{s_{il};\ i = 1, ..., n;\ l = 1, ..., L\}$ the observed signals.

3.1 Estimation of $F_{S|V}(.|T^{-1}(.))$

Given $s_1 \geq s_2$, we first apply kernel techniques to estimate nonparametrically $\frac{\partial F_S}{\partial s_1}(s_1, s_2)$ from the observations s_{il}:

$$\widehat{\frac{\partial F_S}{\partial s_1}}(s_1, s_2) = \frac{1}{Lh_1} \sum_{l=1}^{L} \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} K\left(\frac{s_1 - s_{il}}{h_1}\right) 1\left(s_2 \geq s_{jl}\right)$$

where h_1 is some bandwidth and $K(.)$ a kernel.

We note $S_{\min} = \min_{i,l} s_{il}$ and $S_{\max} = \max_{i,l} s_{il}$. Then, using equation (2.2), for all $s' \in [S_{\min}, S_{\max}]$ and all $s \in [S_{\min}, s']$, we estimate $\Phi(.|.) = F_{S|V}(.|T^{-1}(.))$ as the ratio:

$$\widehat{\Phi}(s|s') = \frac{\widehat{\frac{\partial F_S}{\partial s_1}}(s', s)}{\widehat{\frac{\partial F_S}{\partial s_1}}(s', s')}$$

3.2 Estimation of $F_V(T^{-1}(.))$

In a first step, we estimate nonparametrically $f_S(.)$, $f_S(., .)$ and $F_S(.)$ by the kernel density estimators and the empirical distribution, i.e., by

$$\widehat{f}_S(s_1) = \frac{1}{Lh_1} \sum_{l=1}^{L} \frac{1}{n} \sum_{1 \leq i \leq n} K\left(\frac{s_1 - s_{il}}{h_1}\right)$$

$$\widehat{f}_S(s_1, s_2) = \frac{1}{Lh_2^2} \sum_{l=1}^{L} \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} K\left(\frac{s_1 - s_{il}}{h_2}\right) K\left(\frac{s_2 - s_{jl}}{h_2}\right)$$

$$\widehat{F}_S(s_1) = \frac{1}{L} \sum_{l=1}^{L} \frac{1}{n} \sum_{1 \leq i \leq n} 1\left(s_1 \geq s_{il}\right)$$

Then, using equation (2.3), we estimate nonparametrically the function $\Psi(.) = F_V(T^{-1}(.))$ over the support $[S_{\min}, S_{\max}]$ by

[2] We suppose that the number of bidders is constant in all auctions. If the number of bidders varies in the dataset, it is possible to apply kernel techniques to the discrete variable n (see Guerre et al. (2000)).

$$\widehat{\Psi}(s) = \widehat{F}_S(s) - \frac{\widehat{\frac{\partial F_S}{\partial s_1}}(s,s)}{\widehat{f}_S(s,s)} \widehat{f}_S(s)$$

Several kernels and bandwidths can be used. However, to simplify, we will always refer to the same kernel $K(.)$ in each estimation. h_1 and h_2 are some bandwidths used for univariate and bivariate densities.

3.3 Estimation of $T^{-1}(.)$

The estimation of $T^{-1}(.)$ requires to first estimate the distribution function $F_{S_{\max}}(.)$, as well as the derivatives $f'_S(.)$ and $\frac{\partial^2 F}{\partial s_1^2}(.,.)$. These functions are nonparametrically estimated by the empirical distribution of $S_{\max}$ and by the derivatives of the kernel estimators of $f_S(.)$ and $F_{1,S}(.,.)$, respectively.

$$\widehat{F}_{S_{\max}}(s_1) = \frac{1}{L} \sum_{l=1}^{L} 1\left(s_1 \geq \max(s_{il})\right)$$

$$\widehat{f'}_S(s_1) = \frac{1}{Lh_1'^2} \sum_{l=1}^{L} \frac{1}{n} \sum_{1 \leq i \leq n} k\left(\frac{s_1 - s_{il}}{h_1'}\right)$$

$$\widehat{\frac{\partial^2 F_S}{\partial s_1^2}}(s_1, s_2) = \frac{1}{Lh_1'^2} \sum_{l=1}^{L} \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} k\left(\frac{s_1 - s_{il}}{h_1'}\right) 1\left(s_2 \geq s_{jl}\right)$$

where $k(.)$ is the derivative of the kernel $K(.)$ and h_1' a bandwidth.

Hence, it is possible to estimate $T^{-1}(.)$ over the support $[S_{\min}, S_{\max}]$ using equation (2.4):

$$\begin{aligned}\widehat{T^{-1}}(s) =& s + \frac{\widehat{\frac{\partial F_S}{\partial s_1}}(s,s)}{(n-1)\widehat{f}_S(s,s)} + \left[\frac{\widehat{F}_{S_{\max}}(s) - \widehat{\Psi}(s)}{n-1}\right] \times \\ & \frac{\widehat{\frac{\partial^2 F_S}{\partial s_1^2}}(s,s) - \widehat{f'}_S(s) - n\widehat{f}_S(s) + n\frac{\widehat{f}_S(s)\widehat{f}_S(s,s)}{\widehat{\frac{\partial F_S}{\partial s_1}}(s,s)}}{\widehat{f}_S(s)\widehat{\frac{\partial^2 F_S}{\partial s_1^2}}(s,s) - \widehat{f'}_S(s)\widehat{\frac{\partial F_S}{\partial s_1}}(s,s)}\end{aligned}$$

3.4 Estimation of $F_V(.)$ and $F_{S|V}(.|.)$

To estimate the distribution function of the value and the conditional distribution function of the signals, we first have to estimate $T(.)$. The main difficulty arises because $\widehat{T^{-1}}(.)$ may not be invertible. However, one can always define:

$$\widehat{T}(v) = \inf\{s/\widehat{T^{-1}}(s) = v\}$$

over the support $\left[\min_{s\in[S_{\min},S_{\max}]} \widehat{T^{-1}}(s), \max_{s\in[S_{\min},S_{\max}]} \widehat{T^{-1}}(s)\right]$.

We estimate nonparametrically the distribution function of the values $F_V(.)$ by

$$\widehat{F}_V(v) = \widehat{\Psi}\left(\widehat{T}(v)\right)$$

and the conditional distribution function of the signals $F_{S|V}(.|.)$ by

$$\widehat{F}_{S|V}(s|v) = \widehat{\Phi}\left(s|\widehat{T}(v)\right)$$

3.5 Consistency of the estimators

As in Guerre et al. (2000), our estimation method relies heavily on the distributions of the bids that are observed in the data. However, because we observe the signals, our method is a one step nonparametric estimation method. In that sense, it is easier than the estimation procedure proposed by Li et al. (2000).

A consequence is that the asymptotic properties and in particular the uniform consistency of our estimators are easily obtained under standard assumptions.

Assumption 2 *The vectors $(s_{1l}, ..., s_{nl})$, $l = 1, ..., L$, are independently and identically distributed.*

This hypothesis insures that the signals are independent across auctions. If this was not the case, the equilibrium strategy derived by Milgrom and Weber (1982) would not longer apply and dynamic considerations should be taken into account.

To prove the consistency, some hypotheses have to be made on the smoothness of the kernel and on the bandwidths used in the estimation procedure.

Assumption 3 $K(.)$ *is a symmetric kernel with a bounded support and twice continuous derivatives.*

Assumption 4 *The bandwidths h_1, h'_1 and h_2 are of the form :*
$h_1 = \lambda_1 \left(\frac{\log L}{L}\right)^{1/5}$, $h'_1 = \lambda'_1 \left(\frac{\log L}{L}\right)^{1/7}$, $h_2 = \lambda_2 \left(\frac{\log L}{L}\right)^{1/6}$
where the λ's are strictly positive constants.

Assumption 3 is standard. Assumption 4 deals with the choice of the bandwidths and requires more attention. h_1, h'_1 and h_2 are chosen such that our kernel estimators $\widehat{\frac{\partial F_S}{\partial s_1}}(.,.)$, $\widehat{f}_S(.)$, $\widehat{f}_S(.,.)$, $\widehat{f}'_S(.)$ and $\widehat{\frac{\partial^2 F_S}{\partial s_1^2}}(.,.)$ converge uniformly at the best possible rate (see Scott, 1992). The $\log L$ appears because we deal with uniform consistency. A larger bandwidth is required for multivariate densities ($h_2 > h_1$). A larger smoothing parameter h'_1 is also required as the derivative of a function is noisier than the function itself.

Under our assumptions, we prove that $\widehat{F}_V(.)$ and $\widehat{F}_{S|V}(.|.)$ are uniformly consistent estimators for $F_V(.)$ and $F_{S|V}(.|.)$.

Proposition 3.1 *Under assumptions 1-4, for any closed inner subset $\mathcal{C}(V)$ of $[\underline{V}, \overline{V}]$),*

$$\lim_{n\to+\infty} \sup_{v\in\mathcal{C}(V)} |\widehat{F}_V(v) - F_V(v)| = 0$$

with probability one.

Under assumptions 1-4, for any $v \in [\underline{V}, \overline{V}]$ and any closed inner subset $\mathcal{C}_v(S)$ of $[T(\underline{V}), T(v)]$),

$$\lim_{n\to+\infty} \sup_{s\in\mathcal{C}_v(S)} |\widehat{F}_{S|V}(s|v) - F_{S|V}(s|v)| = 0$$

with probability one.

Proof See appendix A.

4 Monte Carlo experiment

In this section, we conduct a Monte Carlo experiment to evaluate our nonparametric procedure. To illustrate our method, we suppose that the value of the good is uniformly distributed on $[0, 2]$ and that the signals are uniformly distributed on $[0, 2v]$ conditionally on v. The equilibrium strategy is given by proposition (2.1) and can be rewritten in our case as (see Appendix A):

$$b(s) = 4 - \frac{32}{s} + \frac{128}{s^2} \ln\left(1 + \frac{s}{4}\right) \tag{4.1}$$

However, normalizing the bids by $b(s) = s$ is equivalent to consider a model in which V is distributed uniformly on $[0, 2]$ and the density of the signals conditionally on $V = v$ is $f_{S|V}(s|v) = \frac{1}{2v}(b^{-1})'(s)$ defined on the interval $[0, b(2v)]$.

Our Monte Carlo study consists in $R = 400$ replications. For each replication, we simulate $L = 500$ auctions with $n = 3$ bidders.[3] The bids $b^r_{il} = s^r_{il}$ are computed numerically for $l = 1, ..., 500$, $i = 1, 2, 3$ and $r = 1, ..., 500$. For each replication r, we estimate $\widehat{\Psi}^r(.)$, $\widehat{\Phi}^r(.|b(3.6))$, $\widehat{T^{-1}}^r(.)$, $\widehat{T}^r(.)$, $\widehat{F}^r_V(.)$ and $\widehat{F}^r_{S|V}(.|1.8)$.

To apply our nonparametric procedure, we need to address the choice of the kernel function and of the bandwidths. We choose a kernel that satisfies assumption 3. This is the case for the Epanechnikov kernel defined as $K(u) = \frac{3}{4}(1 - u)^2 1(|u| \leq 1)$. We define the bandwidths according to assumption 4 and use the so-called rule of thumb

[3]Such a dataset is similar to real data used to estimate auction models.

(Scott, 1992) to define the λs. Hence, $h_1 = 1.06\widehat{\sigma_S}(\frac{\log(L)}{L})^{1/5}$, $h_1' = 1.06\widehat{\sigma_S}(\frac{\log(L)}{L})^{1/7}$ and $h_2 = 1.06\widehat{\sigma_S}(\frac{\log(L)}{L})^{1/6}$ where $\widehat{\sigma_S}$ is the standard deviation of the signals.

The results are summarized in Figures 1-6 in Appendix B. Each figure displays the true function in plain line and the mean of the 400 estimates in dotted line. We also display in each figure the pointwise 90% confidence interval and represent in dashed lines the 5% and 95% percentiles of the estimates when $L = 500$.

Figure 1 depicts the estimation of $\Phi(.|b(3.6))$. This function is well estimated as the estimated function confounds with the true function. Figures 2 and 3 show, however, that the finite sample bias is not negligible in the estimation of $\Psi(.)$ and $T^{-1}(.)$. In nonparametric estimation, the curse of dimensionality is a problem and this bias is mainly due to the estimation of bivariate densities that appear in the estimation of these functions. Asymptotically, when L increases, the bias disappears. One can also remark that the bias is more important near the bounds of the interval. This effect corresponds to the boundary effect of kernel estimators near the extremities of the density's supports.

As depicted in figure 4, the inverse procedure we use allows us to recover correctly the function $T(.)$. Finally, figures 5 and 6 prove that our estimation method gives rather good estimators of the distribution functions of interest $F_V(.)$ and $F_{S|V}(.|1.8)$. Because we estimate distribution functions, it would be possible to improve our estimates by imposing that $\widehat{F}_V(.)$ and $\widehat{F}_{S|V}(.|1.8)$ are increasing functions that satisfy $\widehat{F}_V(\widehat{T^{-1}}(\max s_{il})) = 1$ and $\widehat{F}_{S|V}(\widehat{T}(1.8)|1.8) = 1$.

5 Conclusion

In this paper, we studied a pure common value model defined by two functions: the distribution function of the value of the good and a unique function that enters in the definition of the conditional densities of the signals. We proved that this model is nonparametrically identified without any further restriction. We then proposed a one-step nonparametric estimation method and prove that the estimators are uniformly convergent. We finally apply our estimation procedure to simulated data. Our method is easy to implement and our estimators predict correctly the true densities.

References

Athey, S., Haile, P., 2002. Identification in Standard Auction Models. *Econometrica*, 70, 2107-2140.

Donald, S., Paarsch, H., 1996. Identification, estimation and testing in parametric empirical models of auctions within the independent private values paradigm. *Econometric Theory*, 12, 517-567.

Elyakime, B., Laffont, J-J., Loisel, P., Vuong, Q., 1997. First-Price Sealed-Bid Auctions With Secret Reservation Prices. *Annales d'Economie et de Statistique*, 34, 115-141.

Elyakime, B., Laffont, J-J., Loisel, P., Vuong, Q., 1997. Auctioning and Bargaining: An Econometric Study of Timber Auctions With Secret Reservation Prices. *Journal of Business and Economic Statistics*, 15, 209-220.

Fevrier, P., 2006, The mineral rights model is nonparametrically identified. Mimeo Crest.

Guerre, E., Perrigne, I., Vuong, Q., 2000. Optimal Nonparametric Estimation of First Price Auctions. *Econometrica*, 68, 525-574.

Hardle, W., 1991. Smoothing techniques with implementation in S. New-York: Springer Verlag.

Laffont, J-J., Ossard, H., Vuong Q., 1995. Econometrics of First-Price Auctions. *Econometrica*, 63, 953-980.

Laffont, J-J., Vuong Q., 1996. Structural Analysis of Auction Data. *American Economic Review, Proceedings*, 36, 414-420.

Li T., Perrigne, I., Vuong, Q., 2000. Conditionally Independent Private Information in OCS Wildcat Auctions. *Journal of Econometrics*, 98, 129-161.

Milgrom, P.R., Weber, R.J., 1982. A theory of auctions and competitive bidding. *Econometrica*, 50, 1089-1122.

Paarsch, H., 1992. Deciding between the common and private value paradigms in empirical models of auctions. *Journal of Econometrics*, 51, 191-215.

Porter, R.H., 1995. The role of information in U.S. offshore oil and gas lease auctions. *Econometrica*, 63, 1-27.

Rothkopf, M., 1969. A model of rational competitive bidding. *Management Science*, 15, 774-777.

Scott, D., 1992. Multivariate density estimation. John Wiley & Sons, Inc.

Schuster, E.F., 1969. Estimation of a probability density function and its derivatives. *The Annals of Mathematical Statistics*, 40, 1187-1195.

Wilson, R., 1977. A bidding model of perfect competition. *Review of Economic Studies*, 44, 511-518.

Appendix A

Derivation of equation (2.4)

Deriving the first order condition given in equation (2.1) and using $b(s) = s$, we have

$$1 = \frac{f_{Y|S}}{F_{Y|S}}(s,s)\int_{T(\underline{V})}^{s} L(\alpha|s)dV(\alpha,\alpha) = \frac{f_{Y|S}}{F_{Y|S}}(s,s)(V(s,s)-s)$$

Hence, with obvious notations,

$$f_{Y,S}(s,s)V(s,s) = sf_{Y,S}(s,s) + F_{Y|S}(s,s)$$

i.e.

$$\begin{aligned}(n-1)h^2(s)H^{n-2}(s)\int_{T^{-1}(s)}^{\overline{V}} v\frac{f_V(v)}{H^n(T(v))}dv =&(n-1)sh^2(s)H^{n-2}(s)\int_{T^{-1}(s)}^{\overline{V}} \frac{f_V(v)}{H^n(T(v))}dv\\ &+ h(s)H^{n-1}(s)\int_{T^{-1}(s)}^{\overline{V}} \frac{f_V(v)}{H^n(T(v))}dv\end{aligned}$$

or equivalently

$$\left[s + \frac{H(s)}{(n-1)h(s)}\right]\int_{T^{-1}(s)}^{\overline{V}} \frac{f_V(v)}{H^n(T(v))}dv = \int_{T^{-1}(s)}^{\overline{V}} \frac{vf_V(v)}{H^n(T(v))}dv$$

Deriving this equation in s and rearranging the terms allow us to find $T^{-1}(.)$:

$$\begin{aligned}T^{-1}(s) =&s + \frac{H(s)}{(n-1)h(s)}\\ &-\left[\frac{n}{n-1} - \frac{H(s)h'(s)}{(n-1)h^2(s)}\right]\frac{H^n(s)\int_{T^{-1}(s)}^{\overline{V}} \frac{f_V(v)}{H^n(T(v))}dv}{(T^{-1})'(s)f_V(T^{-1}(s))}\end{aligned}$$

Replacing in this equation $(T^{-1})'(s)f_V(T^{-1}(s))$ by $\frac{H(s)h'(s)}{h^2(s)}f_S(s) - \frac{H(s)}{h(s)}f_S'(s)$ and $\frac{h'(s)}{h(s)}$ by $\frac{\frac{\partial^2 F_S}{\partial s_1^2}(s,s)-f_S'(s)}{\frac{\partial F_S}{\partial s_1}(s,s)-f_S(s)}$, we obtain after some straightforward calculus:

$$\begin{aligned}T^{-1}(s) =&s + \frac{\frac{\partial F_S}{\partial s_1}(s,s)}{(n-1)f_S(s,s)} + \left[\frac{F_{S_{\max}}(s) - F_V(T^{-1}(s))}{n-1}\right]\times\\ &\frac{\frac{\partial^2 F_S}{\partial s_1^2}(s,s) - f_S'(s) - nf_S(s) + n\frac{f_S(s)f_S(s,s)}{\frac{\partial F_S}{\partial s_1}(s,s)}}{f_S(s)\frac{\partial^2 F_S}{\partial s_1^2}(s,s) - f_S'(s)\frac{\partial F_S}{\partial s_1}(s,s)}\end{aligned} \tag{5.1}$$

Proof of proposition 3.1

Under our assumptions (Hardle 1991, Schuster 1969), the nonparametric estimators $\widehat{\frac{\partial F_S}{\partial s_1}}(.,.)$, $\widehat{f}_S(.)$, $\widehat{f}_S(.,.)$, $\widehat{F}_S(.)$, $\widehat{F}_{S_{\max}}(.)$, $\widehat{f}'_S(.)$ and $\widehat{\frac{\partial^2 F_S}{\partial s_1^2}}(.,.)$ are uniformly consistent estimators for $\frac{\partial F_S}{\partial s_1}(.,.)$, $f_S(.)$, $f_S(.,.)$, $F_S(.)$, $F_{S_{\max}}(.)$ $f'_S(.)$ and $\frac{\partial^2 F_S}{\partial s_1^2}(.,.)$.

Hence, because $\widehat{\Psi}(.)$ and $\widehat{\Phi}(.|.)$ and $\widehat{T^{-1}}(.)$ are simple transformations of the previous functions (see equations (2.2), (2.3) and (2.4)), we conclude that these estimators are uniformly consistent estimators for $F_V(T^{-1}(.))$, $F_{S|V}(.|T^{-1}(.))$ and $T^{-1}(.)$.

The main difficulty comes from the inversion of $\widehat{T^{-1}}(.)$ and we have to establish the uniform strong convergence of $\widehat{T}(.)$.
Let $\mathcal{C}(V)$ be an inner subset of $[\underline{V}, \overline{V}]$. Because $\widehat{T^{-1}}(.)$ converges uniformly, $\mathcal{C}(V)$ is included in $\left[\min_{s\in[S_{\min},S_{\max}]} \widehat{T^{-1}}(s), \max_{s\in[S_{\min},S_{\max}]} \widehat{T^{-1}}(s)\right]$ almost surely when L is large enough. Hence, $\widehat{T}(.)$ is well defined on $\mathcal{C}(V)$.
By definition, for any $v \in \mathcal{C}(V)$, it exists $s_v = \widehat{T}(v)$ such that

$$
\begin{aligned}
|T(v) - \widehat{T}(v)| &= |T(\widehat{T^{-1}}(s_v)) - s_v| \\
&= |T(\widehat{T^{-1}}(s_v)) - T(T^{-1}(s_v))| \\
&= |T'(\tilde{v})(\widehat{T^{-1}}(s_v) - T^{-1}(s_v))| \quad \text{where } \tilde{v} \in [\min(v, T^{-1}(s_v)), \max(v, T^{-1}(s_v))] \\
&\leq C \sup_{s\in\mathcal{C}(S)} |\widehat{T^{-1}}(s) - T^{-1}(s)|
\end{aligned}
$$

where C is an upper bound of the derivative $T'(.)$ (which exists by assumption 1) and $\mathcal{C}(S)$ is an inner subset of $[T(\underline{V}), T(\overline{V})]$ that contains $\{s_v, v \in \mathcal{C}(V)\}$ with probability one.
Hence,

$$
\sup_{v\in\mathcal{C}(V)} |T(v) - \widehat{T}(v)| \leq C \sup_{s\in\mathcal{C}(S)} |\widehat{T^{-1}}(s) - T^{-1}(s)|
$$

and the uniform strong convergence of $\widehat{T}(.)$ is proved.

The uniform consistency of $\widehat{F}_V(v) = \widehat{\Psi}\left(\widehat{T}(v)\right)$ and $\widehat{F}_{S|V}(s|v) = \widehat{\Phi}\left(s|\widehat{T}(v)\right)$ falls out naturally from the previous results.

Derivation of equation (4.1)

By proposition (2.1),

$$b(s) = V(s,s) - \int_{S_{\underline{V}}}^{s} L(\alpha|s) dV(\alpha,\alpha)$$

When V is uniformly distributed on $[0,2]$ and when the signals conditionally on $V = v$ are uniformly distributed on $[0,2v]$, straightforward calculus lead to

$$V(s,s) = \frac{\int_{s/2}^{2} 2v(\frac{1}{2v})^2 \frac{s}{2v} \frac{1}{2} dv}{\int_{s/2}^{2} 2(\frac{1}{2v})^2 \frac{s}{2v} \frac{1}{2} dv} = \frac{4s}{4+s}$$

Similarly,

$$\begin{aligned} L(\alpha,s) &= \exp\left[-\int_{\alpha}^{s} \frac{\int_{s/2}^{2} 2(\frac{1}{2v})^2 \frac{s}{2v} \frac{1}{2} dv}{\int_{s/2}^{2} \frac{1}{2v} \left(\frac{s}{2v}\right)^2 \frac{1}{2} dv} ds \right] \\ &= \exp\left[-\int_{\alpha}^{s} \frac{2}{s} ds \right] = \left(\frac{\alpha}{s}\right)^2 \end{aligned}$$

Combining both results, we have

$$\begin{aligned} b(s) &= \int_{0}^{s} \frac{4\alpha}{4+\alpha} \frac{2\alpha}{s^2} d\alpha \\ &= 4 - \frac{32}{s} + \frac{128}{s^2} \ln\left(1 + \frac{s}{4}\right) \end{aligned}$$

Appendix B

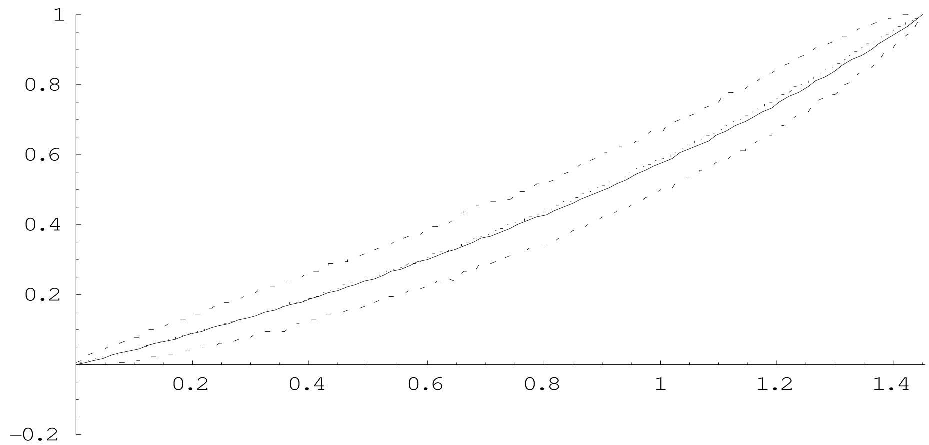

Figure 1: The true function $\Phi(.|b(3.6))$ and its estimation

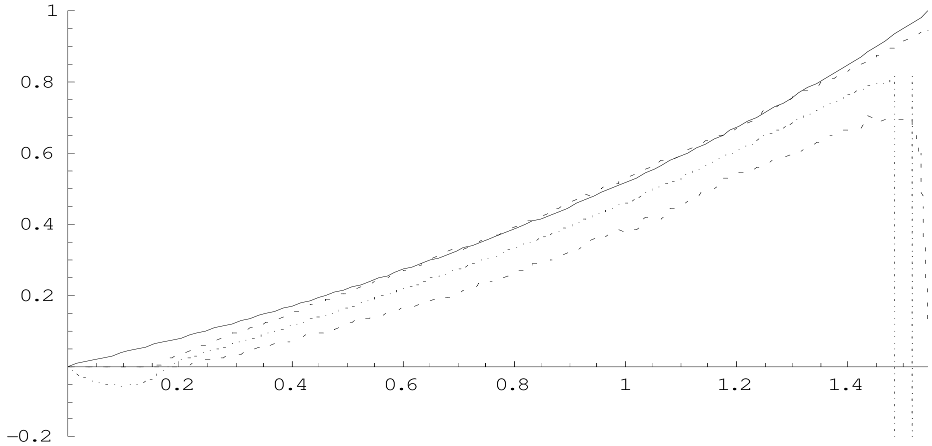

Figure 2: The true function $\Psi(.)$ and its estimation

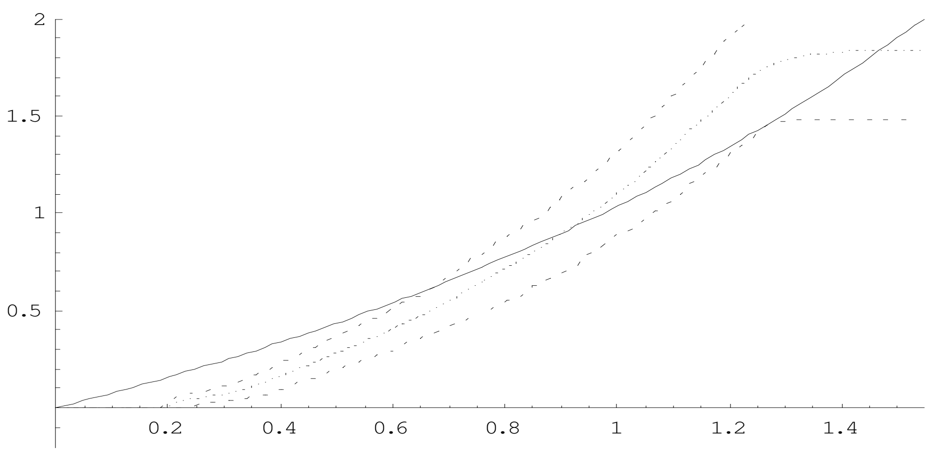

Figure 3: The true function $T^{-1}(.)$ and its estimation

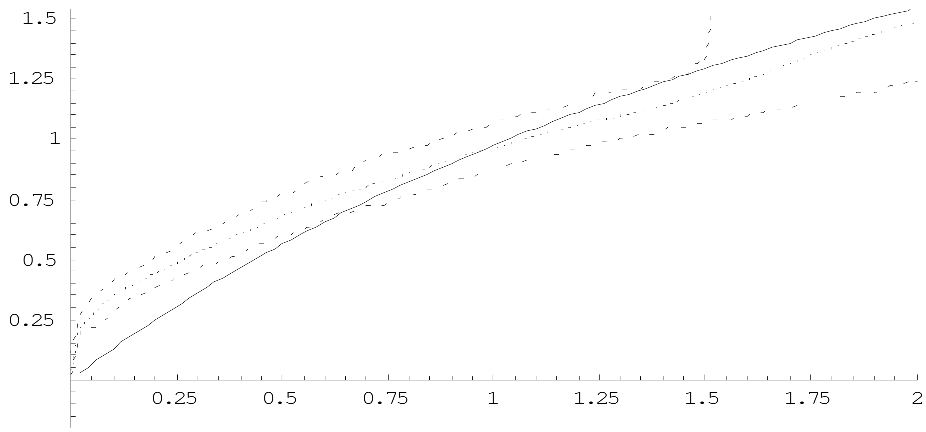

Figure 4: The true function $T(.)$ and its estimation

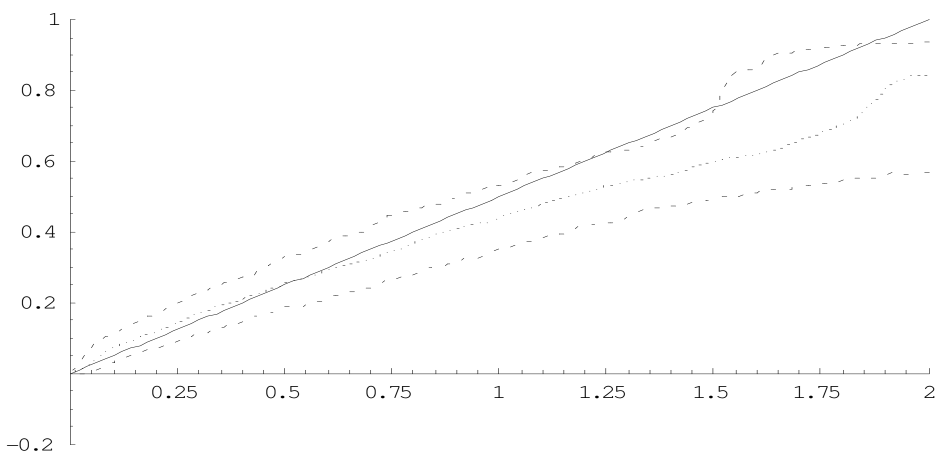

Figure 5: The true function $F_V(.)$ and its estimation

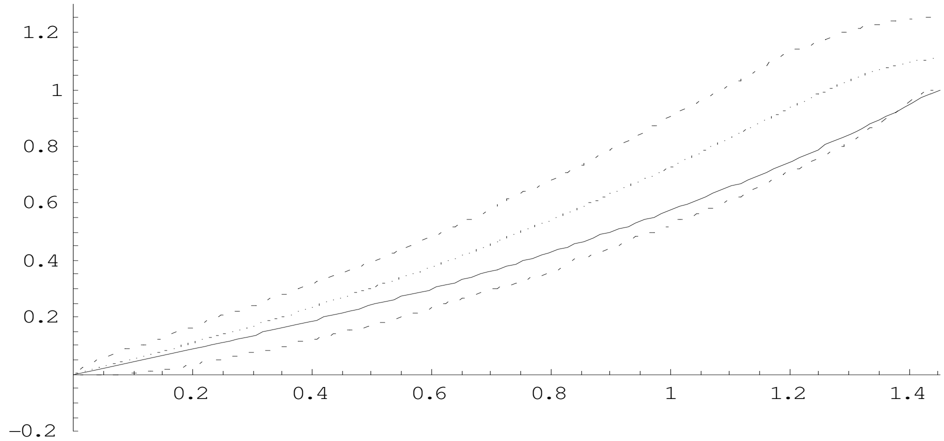

Figure 6: The true function $F_{S|V}(.|1.8)$ and its estimation