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(Non) consistency of the Beta Kernel Estimator for Recovery Rate Distribution

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ABSTRACT

In this paper, we explain why a nonparametric approach based on a beta kernel [Renault, Scaillet (2004)] will lead to significant bias when applied to recovery rate distributions. This is due to a specific feature of these distributions, which admit strictly positive weights at 100 % corresponding to full recovery (and also at 0 % corresponding to total loss). Moreover, for distributions without point mass at 0% and 100%, the beta kernel approach features significant bias in finite sample. In large sample the method is consistent, but other competing approaches presented in the paper provide more accurate results.

Keywords : Loss-Given-Default, Recovery, Credit Risk, Kernel Estimation.

JEL number : C13, C14, G33.

1 Introduction

1.1 A feature of recovery rate distributions

A key building block of credit risk modelling is the recovery rate, equal to one minus the loss-given-default (LGD). By definition (see the discussion below) the observed recovery rates lie between 0 and 1. Generally, the recovery rate distribution shows strictly positive weight at 100 % (full recovery) and frequently at 0 % (total loss). This feature is observed for any type of loan and any way of measuring recovery.

For instance, Renault, Scaillet (2004) observe this feature for corporate bonds, when the recovery rate relies on the "price of defaulted security in the distressed debt market recorded shortly (in practice one month) after the default event". They explain that this "concentration of data at total recovery and total loss imply "multimodal distributions", which cannot be well fitted by the standard parametric beta distribution [see e.g. Gupton, Stein (2002), Appendix A, Taasche (2004), Kim and Kim (2006)], and advocate the use of nonparametric approach based on asymmetric kernel to account for the restricted domain.

Point mass at 100 % are also observed for other loans when loss-given-default is measured by appropriately discounting the recovery cash flows, that is, by considering carefully the workout process. For instance, the frequency of total recovery is about 20% for loans to cities and regions in France, but up to 40-60 % for segments of consumer retail credits.

The following reasons can explain the observed point mass at 100 %.

i) First, the definition of LGD depends on the selected definition of default. For instance, no reimbursement within 90 days is considered as a default by the regulator. This period length is not necessarily appropriate for cities or regions, which can solve their difficulties by increasing taxes. This process requires a longer period of time.

ii) Second, corporate default can arise when a corporate has transitory difficulties to reimburse, but is structurally in good health. The observed frequency at 100 % provides some idea on the magnitude of this phenomenon.

iii) Third, it is important to recall how loss-given-default is computed from the workout process. Indeed, it accounts for the debt amount (generally the Exposure-At-Default (EAD)), but also for penalties and for direct and indirect costs associated with collecting. Then, for given total payment by the borrower, the recovery rate is the ratio : (discounted total payment discounted cost)/EAD.

Since the total payment is nonnegative, smaller than the sum of the debt amount plus penalties, the recovery rate can be larger than 100 % (due to penalties), or negative (due to recovery costs). Moreover, these boundary effects are also very sensitive to the choice of discounting.

iv) Finally, in the case of large corporates, the debt is often traded on a secondary market, even after a failure. The EAD is generally equal to the face value. At default time, the value of the debt is generally smaller than the face value. However, the recovery rate is computed as the ratio of the market value of the debt one month later to the face value. It can be larger than 1, if the value of the debt is reevaluated by the market. This concerns mainly senior secured debt.

The boundary problem has been noted very early by the regulator, who imposes to truncate recovery rates to [0,1] to avoid negative values or values larger than one [3] [See Basel Committee on Banking Supervision (2005)].

1.2 The asymmetric beta kernel approach

Let us consider a continuous distribution on [0,1]. A nonparametric estimation method of the density based on a Gaussian kernel can lead to unconsistent results at the boundaries. This asymptotic bias can be eliminated by considering an asymmetric kernel. For instance, Chen (1999) [see also Bouezmarni, Rolin (2003)] proposed a beta kernel estimator of the density. This estimator is given by :

$$\hat{f}_n(x) = \frac{1}{n}\sum_{i=1}^{n} K(X_i, x/b+1, (1-x)/b+1), \tag{1}$$

where $K(u;\alpha,\beta) = \dfrac{1}{B(\alpha,\beta)}u^{\alpha-1}(1-u)^{\beta-1}, u \in [0,1]$,
with $B(.)$ denoting the beta function and b a smoothing parameter, which tends to zero when $n \to \infty$. If $X_1, \ldots, X_n$ is an iid sample from a continuous

[3] However, this truncation has not to be introduced, when a bank want to know its real risk, manage a portfolio of loans, or securitize a pool of credits.

distribution with density f_0, then the kernel density estimator $\hat{f}_n$ tends to f_0, when $n \to \infty$ [Bouezmarni, Scaillet (2005)].

However, this convergence is not uniform [4] on $[0, 1]$. This can be seen for instance from the upper bound on the variance of the beta kernel estimator derived by Chen [1999], Lemma 2. Indeed, this upper bound of the type : $ct \times [x(1-x)]^{-1/2}$ tends to infinity, when x tends to 0 or 1. This lack of uniform convergence has important practical consequences, since some integrals computed from the beta kernel estimator will not necessarily converge to their theoretical counterpart. This can arise for the total mass corresponding to the integral of the constant unitary function as well as for the computations of the expected LGD and its variability !!

1.3 Plan of the paper

In Section 2, we explain why the beta kernel approach provides a non consistent estimator of the distribution when there are point mass at 0% or 100%. This is due to the behaviour of the beta kernel at the boundaries. Moreover, the beta kernel has no unit mass. This lack of normalization implies that the fitted distribution has no unit mass and creates significant biases on the VaR in finite sample. We introduce in Section 3, two beta kernel approaches corrected for unit mass. They are called micro - and macro-beta kernel approaches, respectively. Their properties are compared by Monte-Carlo. Finally, in Section 3, the corrected micro-beta kernel method is compared with other approaches introduced in the nonparametric literature. Section 4 concludes.

2 The nonconsistency of the beta kernel approach for recovery rate distributions

2.1 The limit of the beta kernel estimator

It has been proposed to apply the beta kernel approach to recovery rate distributions [Renault, Scaillet (2004)]. For this type of application, it is necessary to check the properties of the nonparametric density estimator

[4] More precisely, it is uniform on every compact set included in $]0, 1[$ [see Bouezmarni, Scaillet (2005), Theorem 3.1].

(1), when the true distribution of the variable is a mixture of i) a point mass at 0, with probability p_0, ii) a point mass at 1, with probability p_1, iii) a continuous distribution f_0, with probability $1 - p_0 - p_1$.

Among the n observations, n_0(resp. n_1) are equal to 0 (resp. 1). Since $K(0, \alpha, \beta) = K(1, \alpha, \beta) = 0$, we have :

$$\begin{aligned}
\hat{f}_n(x) &= \frac{1}{n} \sum_{i;X_i \neq 0,1} K(X_i, x/b + 1, (1-x)/b + 1) \\
&= \frac{n - n_0 - n_1}{n} \frac{1}{n - n_0 - n_1} \sum_{i:X_i \neq 0,1} K(X_i, x/b + 1, (1-x)/b + 1) \\
&\simeq (1 - p_0 - p_1) f_0(x),
\end{aligned}$$

by applying the consistency result valid for continuous distributions. Thus, the beta kernel estimator approximates the continuous component of the mixture distribution. In particular, the associated cumulative distribution function is $(1 - p_0 - p_1)F_0(x), x \in (0, 1)$, and significantly differs from the true cdf :

$$\begin{aligned}
F(x) &= p_0 + (1 - p_0 - p_1)F_0(x), \text{ if } 0 \leq x < 1, \\
&= 1, \text{ if } x = 1.
\end{aligned}$$

This unconsistency has serious implications on the estimated CreditVaR, which corresponds to a quantile of the distribution, that is to the inverse of the cdf. Loosely speaking, if $p_0 = 6\%$, the true quantile at 5% correspond to a zero recovery rate, whereas the quantile computed from the beta kernel estimator will provide a strictly positive value of the recovery and a significant underestimation of the required capital.

At this step, it is useful to recall the main principles of nonparametric estimation of a distribution function. An appropriate estimation method has to satisfy the three following properties :

1) provide an estimator of the cdf in finite sample, that is, an increasing function satisfying the unit mass property;

2) then, provide a consistent estimator of the cdf;

3) then, provide a consistent estimator of the density function corresponding to the continuous part of the distribution.

For instance, the empirical cdf (i.e. the so-called historical simulation) satisfies 1), 2), but not 3). A Gaussian kernel estimation of the density satisfies 1), 2) and 3), except for the value of the density at the boundaries of the domain of the continuous part. Both approaches above provide consistent estimators of the cdf, and of the VaR. On the contrary, the beta-kernel estimator of the density does not satisfy 1) and 2), and can provide unconsistent estimator of the VaR. [5].

Of course the non consistency feature can be easily corrected by estimating separately the point mass at boundaries by their sample counterparts and the continuous part of the distribution by the beta kernel approach.

2.2 The normalization problem

Let us now discuss more precisely the normalization problem of the beta kernel. The general term of (1) can be explicited as :

$$g_b(x,u) = \frac{\Gamma(2+1/b)}{\Gamma(x/b+1)\Gamma((1-x)/b+1)} u^{x/b}(1-u)^{(1-x)/b}. \tag{2}$$

For any fixed value x, $g_b(x,.)$ defines a pdf on $[0,1]$, whereas for any fixed value u, $g(.,u)$ is not a pdf. Let us study both functions $g_b(x,.)$ and $g_b(.,u)$, and consider the total mass (integral) of $g_b(.,u)$ for various u.

Since $g_b(x,u)) = g_b(1-x, 1-u)$, the study of $g_b(x,.)$ [resp. $g_b(.,u)$] can be restricted to $x \in (0,0.5)$ (resp. $u \in [0,0.5]$). The functions $g_b(.,u)$ are plotted in Figure 1 for $u = 0.1, 0.2, 0.3, 0.4, 0.5$; the functions $g_b(x,.)$ are given in Figure 2 for $x = 0.1, 0.2, 0.3, 0.4, 0.5$. Note that Chen [Chen (1999)] considers functions $g_b(x,.)$, but not functions $g_b(.,u)$, which are more important since the kernel estimated density is a mixture of such functions.

[Insert Figure 1 : Kernel as a Function of x]

[Insert Figure 2 : Kernel as a Function of u]

The patterns of the two sets of curves are quite different. In particular, functions $g_b(u,.)$ are much more asymmetric around their maximum.

[5] The same type of remark applies, when the beta kernel estimator is used on data, which are preliminary transformed by the cdf of a beta distribution [see Hagman, Renault, Scaillet (2005)]

Since the kernel estimated density is a mixture of functions $g_b(., u)$, it is important to focus on their integrals and therefore on the lack of normalization of the beta kernel estimator.

As seen from Figure 3, these integrals can be either larger, or smaller than 1.

[Insert Figure 3 : Integral as Function of b and u]

They are much smaller than 1, when u is close to 0 or 1, especially for extreme risks. They are larger than 1, when u is close to 0.5. Moreover, the difference between the extreme values of the total mass increases with b.

In Figure 4, we have selected three sections of the surface given in Figure 3. These sections correspond to the "rule of thumb" procedure for selecting the bandwidth b proportional to $n^{-1/5}$, when $n = 100, 1000, 10000$, with proportionality coefficient 1. As already noted, things are worsening, when b increases and the function is very far from 1 in the case $n = 100$

[Insert Figure 4 : Integral as Function of u for $b = n^{-1/5}$]

3 Improved beta kernel methods

Let us now focus on the continuous part of the LGD distribution, and explain how the plain beta kernel approach can be improved. There exist at least two ways for correcting the lack of normalization of the standard beta kernel estimator.

In the "macro-beta" approach, the correction is performed globally. The beta kernel estimator $\hat{f}_n$ defined in (1) is replaced by :

$$\hat{f}_n^1(x) = \frac{\hat{f}_n(x)}{\int_0^1 \hat{f}_n(x)dx}. \tag{3}$$

In the "micro-beta" approach the correction is performed at the corporate (or loan) level. The estimator is defined by :

$$\hat{f}_n^{(1)}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{K(X_i, x/b+1, (1-x)/b+1)}{\int_0^1 K(X_i, x/b+1, (1-x)/b+1)dx}. \tag{4}$$

The finite sample properties of the three estimators $\hat{f}_n, \hat{f}_n^{(1)}$ and $\hat{f}_n^{(2)}$ have been compared by Monte-Carlo. We have considered three possible true distributions in the beta family. They correspond to different shapes, that are

the U-shape for $\alpha = 0.5, \beta = 0.5$,

the Skewed shape for $\alpha = 2.5, \beta = 0.5$,

the Bell-shape for $\alpha = 2.5, \beta = 2.5$.

These true underlying pdf are displayed on Figure 5.

[Insert Figure 5 : Beta Density Function]

We also provide in Figure 6 a zoom on their right tails, since the magnitude of the tail plays a key role in the Monte-Carlo experiment.

[Insert Figure 6 : Beta Density Function, Right Tails]

Note that this choice of distributions in the Monte-Carlo study is in favour of the standard beta kernel estimator, since we have not introduced point mass at the boundaries.

For each beta distribution, we realize 1000 replications of a drawing of size 100. For each drawing, we compute the beta, macro-beta and micro-beta kernel estimates at points $x = i/1000, i = 1, \ldots, 999$, with $b =$ standard error $\times n^{-2/5}$, that is the "rule of thumb". For each drawing and method, we computed the mean-squared errors (mse) between the estimations and the exact underlying pdf for the whole set of points, for the left tail ($i = 1, \ldots, 100$) and for the right tail ($i = 900, \ldots, 999$). Finally, we computed the average of mse over the 1000 replications. The results are given in Table 1.

Table 1 : Comparison of the three beta kernel methods.

Mean Squared Errors ;$n = 100; 1000$ replications.			
$\alpha = 0.5,\ \beta = 0.5$ (U-shape)	Global	Left Tail	Right Tail
Plain Beta	0.279	1.334	1.324
Macro-Beta	0.254	1.158	1.149
Micro-Beta	0.135	0.611	0.615
$\alpha = 2.5, \beta = 0.5$ (Skewed shape)	Global	Left Tail	Right Tail
Plain Beta	0.687	0.0013	6.71
Macro-Beta	0.561	0.0018	5.33
Micro-Beta	0.217	0.0013	2.02
$\alpha = 2.5, \beta = 2.5$ (Bell-shape)	Global	Left Tail	Right Tail
Plain Beta	0.031	0.028	0.029
Macro-Beta	0.031	0.025	0.027
Micro-Beta	0.031	0.028	0.029

Table 1 shows that the micro-beta method strongly dominates the macro-beta method, which, in turn, dominates the standard beta kernel in the U-shape and skewed shape cases. The three methods are almost equivalent in the Bell-shape case.

Moreover, as expected, the largest difference arises when considering the estimation of the tails. The estimation of the right tail in the skewed shape case, which contains more mass than the right tail of the U-shape, is especially difficult. Note also that the order of magnitude of the average mse in the U-shape and skewed shape cases is much larger than in Renault, Scaillet(2004). Indeed, the Gauss-Legendre quadrature approach that they use to compute the integrated squared errors has a smoothing effect, which implies an underestimation of the impact of heavy tails. This smoothing effect does not exist for thin tails (that is Bell-shape) and the values of Table 1 are similar to the values in Renault, Scaillet.

The impact of the tails on the average mse is also illustrated in Figures 7 and 8, showing typical graphs for estimated right tails for $n = 100$, in the U-shape and skewed shape case, respectively.

[Insert Figure 7 : Exact and Estimated Right Tails, $\alpha = 0.5, \beta = 0.5, n = 100$]

[Insert Figure 8 : Exact and Estimated Right Tails, $\alpha = 2.5, \beta = 0.5, n = 100$]

Figures 9 and 10 provide the similar graphs for sample size $n = 1000$.

[Insert Figure 9 : Exact and Estimated Right Tails, $\alpha = 0.5, \beta = 0.5, n = 1000$]

[Insert Figure 7 : Exact and Estimated Right Tails, $\alpha = 2.5, \beta = 0.5, n = 1000$]

To summarize, the Monte-Carlo study shows that the micro-beta method is the best one to estimate the continuous part of the LGD distribution.

4 Comparison of the micro-beta method with some alternative approaches

More generally the micro-beta approach can be compared with other estimation methods. We consider in this section the Gaussian kernel method, the truncated Gaussian kernel method and the transformed Gaussian kernel method.

i) The standard Gaussian kernel estimator is defined by :

$$\hat{f}_n^{(3)}(x) = \mathbb{1}_{[0,1]}(x) \frac{1}{nh} \sum_{i=1}^{n} \varphi\left(\frac{x - X_i}{h}\right), \tag{5}$$

where $\mathbb{1}$ denotes the indicator function and φ is the pdf of the standard normal.

ii) The truncated Gaussian kernel estimator is :

$$\hat{f}_n^{(4)}(x) = \mathbb{1}_{[0,1]}(x) \frac{1}{nh} \sum_{i=1}^{n} \left[\varphi\left(\frac{x - X_i}{h}\right) / \left[\Phi\left(\frac{1 - X_i}{h}\right) - \Phi\left(\frac{-X_i}{h}\right)\right]\right], \tag{6}$$

where Φ is the cdf of the standard normal, and corresponds to a mixture of Gaussian pdf truncated on $[0, 1]$.

iii) The transformed Gaussian kernel estimator is obtained by applying a Gaussian kernel to the data preliminary transformed by the logistic transformation : $x \to \log \frac{x}{1-x}$. We get :

$$\hat{f}_n^{(5)}(x) = \frac{\mathbb{1}_{[0,1]}(x)}{nhx(1-x)} \sum_{i=1}^{n} \varphi \left[\frac{\log\left(\frac{x}{1-x}\right) - \log\left(\frac{X_i}{1-X_i}\right)}{h} \right]. \tag{7}$$

The Monte-Carlo experiment is similar to the experiment performed in Section 5. The bandwidth is fixed to $h =$ standard error $\times n^{-1/5}$,.

Table 2 : Comparison of the Micro-beta Method with Gaussian Methods

Mean Squared Errors, $n = 100, 1000$ replications.			
$\alpha = 0.5, \beta = 0.5$ (U-shape)	Global	Left Tail	Right Tail
Plain Beta	0.279	1.334	1.324
Micro-Beta	0.135	0.611	0.615
Gaussian	0.598	2.936	2.939
Truncated Gaussian	0.493	2.219	2.225
Transformed Gaussian	0.151	0.739	0.687
$\alpha = 2.5, \beta = 0.5$ (Skewed shape)	Global	Left Tail	Right Tail
Plain Beta	0.687	0.0013	6.71
Micro-Beta	0.217	0.0013	2.02
Gaussian	1.729	0.0006	17.05
Truncated Gaussian	1.378	0.0008	13.06
Transformed Gaussian	0.226	0.0046	2.08
$\alpha = 2.5, \beta = 2.5$ (Bell-shape)	Global	Left Tail	Right Tail
Plain Beta	0.031	0.028	0.029
Micro-Beta	0.031	0.028	0.027
Gaussian	0.027	0.016	0.016
Truncated Gaussian	0.029	0.024	0.024
Transformed Gaussian	0.032	0.027	0.026

Table 2 shows that the standard Gaussian and the truncated Gaussian kernel methods provide poor fit in the U-shape and skewed shape case. In the Bell-shape case, all methods perform similarly, with a slight advantage to the standard Gaussian kernel.

The micro-beta kernel and the transformed Gaussian kernel perform well in all situations. They both always dominate the standard beta kernel estimator [6]. The micro-beta approach seems to be slightly better than the transformed Gaussian. However, even if a difference exists, the transformed Gaussian kernel estimator is much easier to compute, and, thus, appears as a serious competitor. Figures 11 and 12 show typical estimations of the right tails (without the Gaussian and truncated Gaussian methods), for the U-shape and skewed shape cases with $n = 100$.

[Insert Figure 11 : Exact and Estimated Right Tails, $\alpha = 0.5, \beta = 0.5, n = 100$]

[Insert Figure 12 : Exact and Estimated Right Tails, $\alpha = 2.5, \beta = 0.5, n = 100$]

Similar graphs are given for $n = 1000$ in Figures 13 and 14.

[Insert Figure 13 : Exact and Estimated Right Tails, $\alpha = 0.5, \beta = 0.5, n = 1000$]

[Insert Figure 14 : Exact and Estimated Right Tails, $\alpha = 2.5, \beta = 0.5, n = 1000$]

5 Conclusion

The message of the above paper is the following : to avoid important bias when computing a Credit VaR, both parametric and nonparametric estimation approaches of the recovery rate distribution have to account for the point mass appearing at 0 % (and 100%) explicitely. Moreover, the plain beta kernel method is not the most appropriate to estimate the continuous part of the LGD distribution.

[6] Renault and Scaillet concluded on the contrary that the standard beta method dominates the transformed Gaussian method. Again, these different conclusion are consequences of different treatments of the tails.

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FIGURE 1:Kernel as a function of x for u given (b=.2)
u=.1, u=.2, u=.3, u=. 4 , u=.5

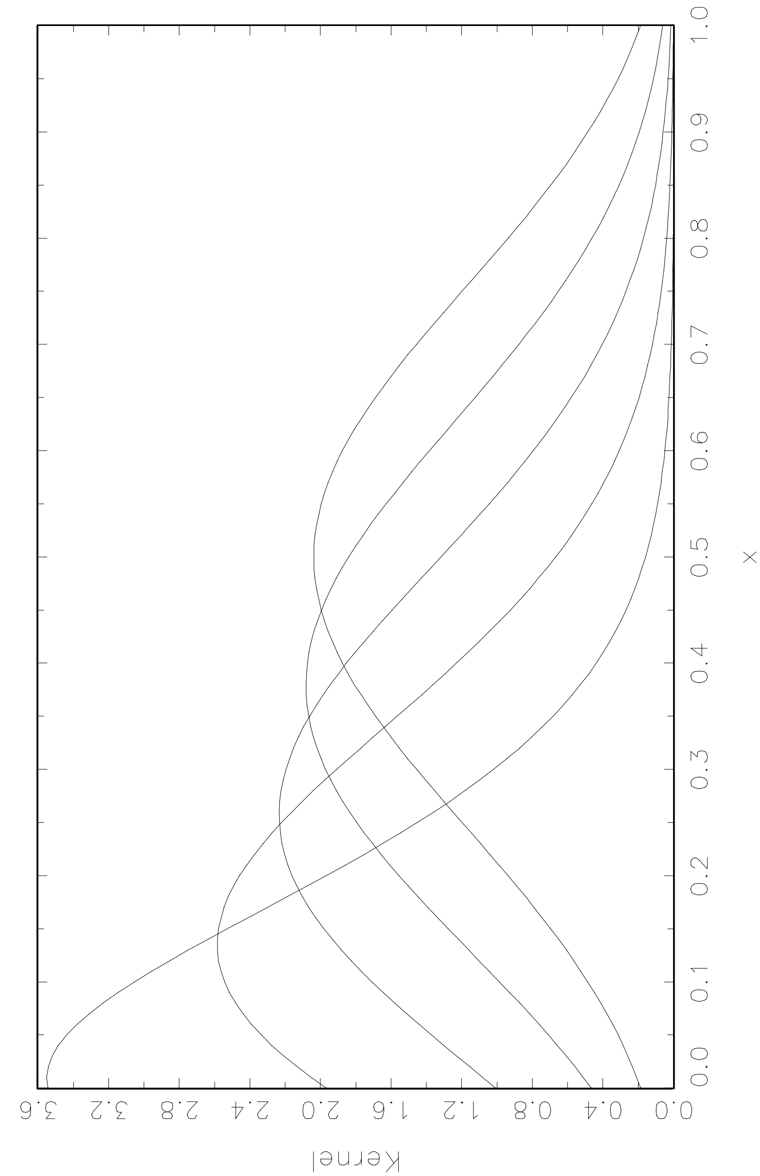
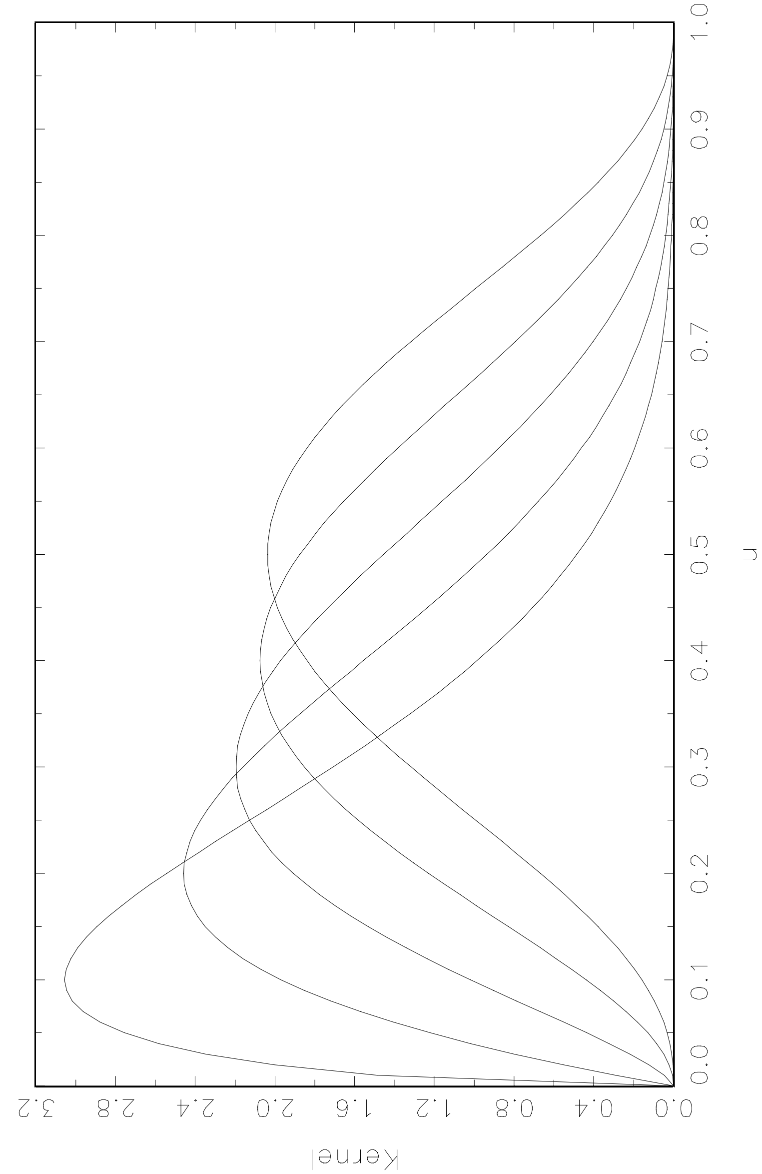


FIGURE 2:Kernel as a function of u for x given (b=.2)
x=.1, x=.2, x=.3, x=.4, x=.5



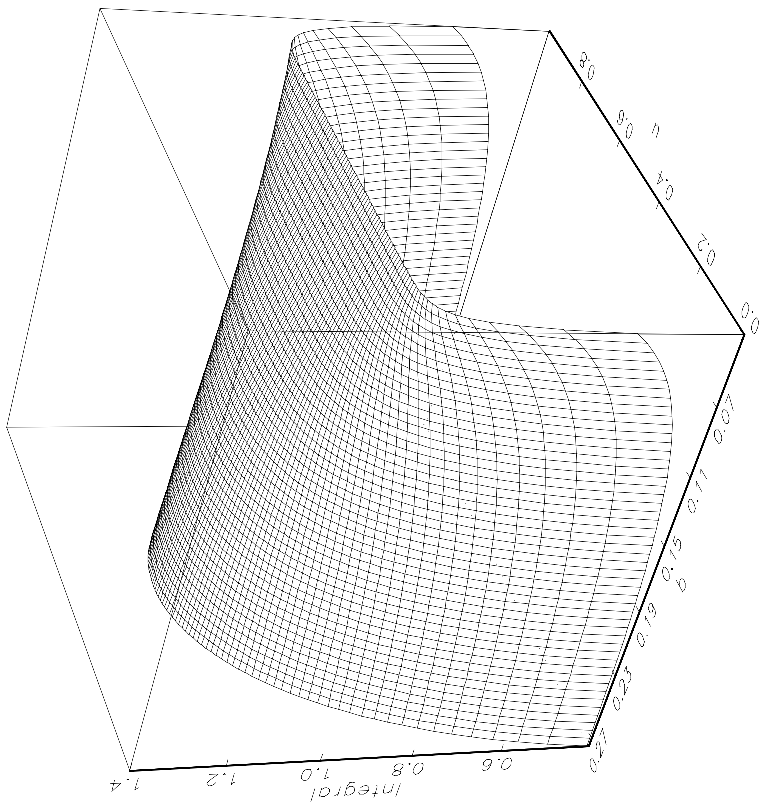


FIGURE 3:Integral as a function of b and u

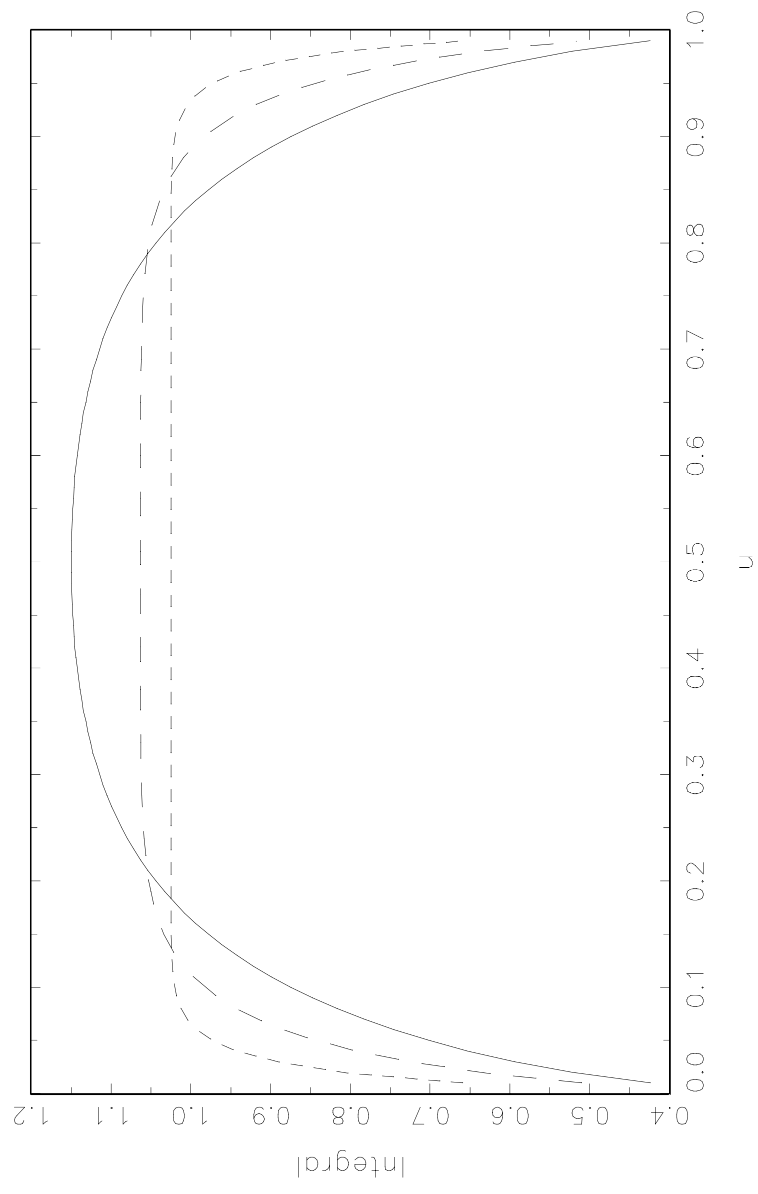


FIGURE 4:Integral as a function of u for b=n^(-2/5)
n=100(solid),n=1000(dashes),n=10000(short dashes)

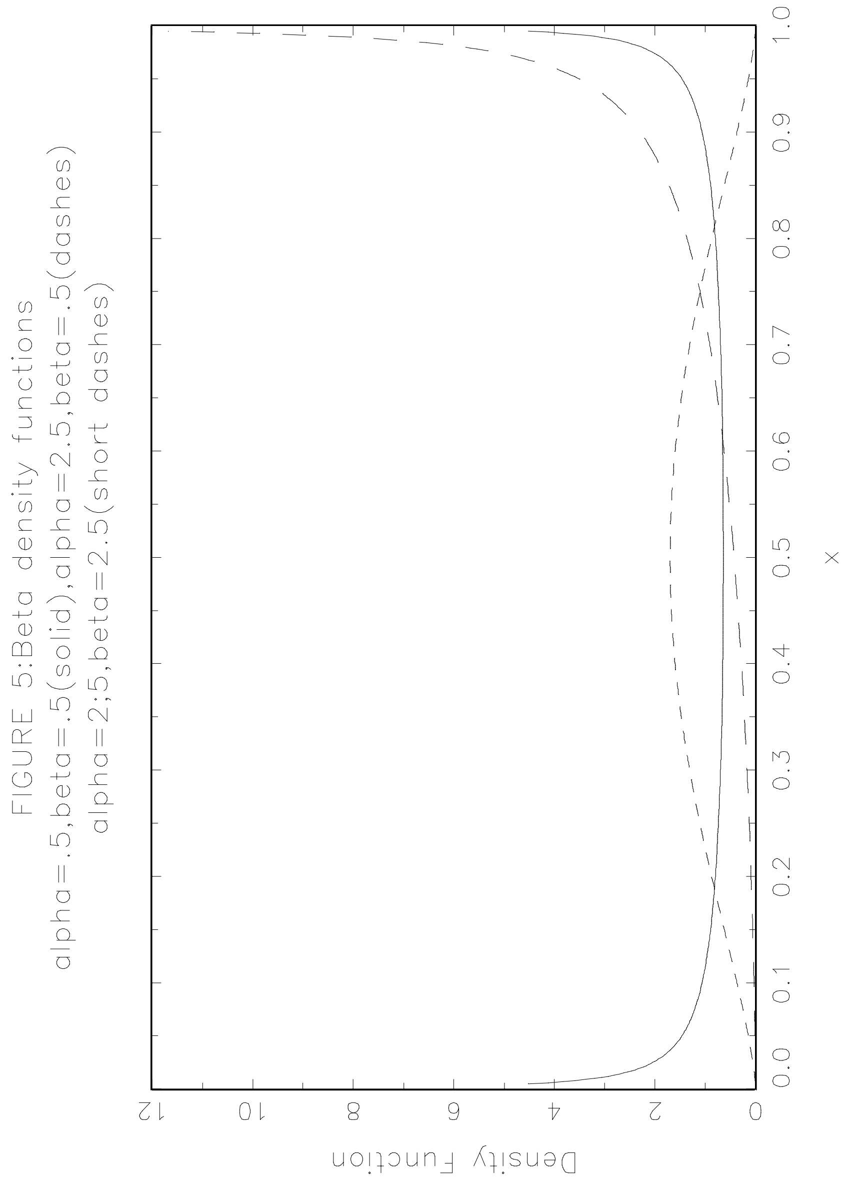


FIGURE 5:Beta density functions
alpha=.5,beta=.5(solid),alpha=2.5,beta=.5(dashes)
alpha=2;5,beta=2.5(short dashes)
Density Function
x
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
0 2 4 6 8 10 12

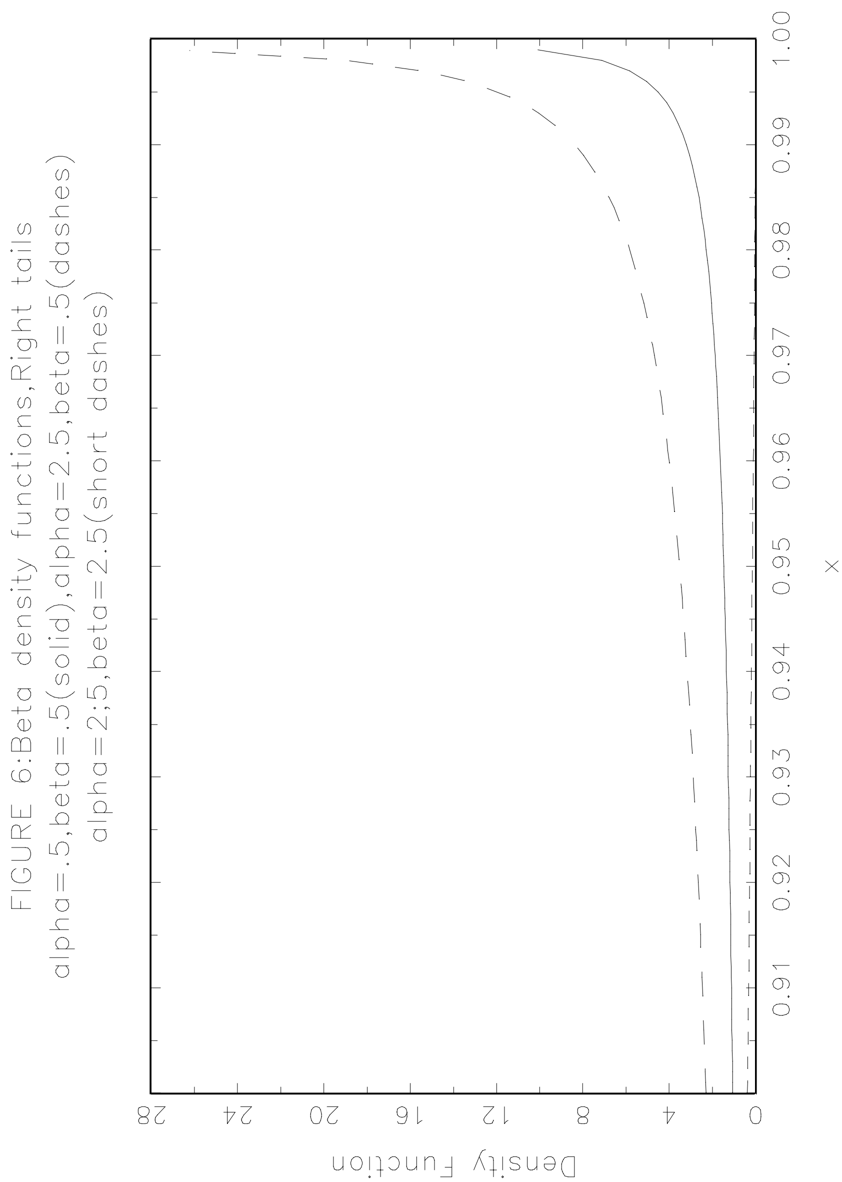


FIGURE 6:Beta density functions,Right tails
alpha=.5,beta=.5(solid),alpha=2.5,beta=.5(dashes)
alpha=2;5,beta=2.5(short dashes)
Density Function
x
0.91
0.92
0.93
0.94
0.95
0.96
0.97
0.98
0.99
1.00
0
4
8
12
16
20
24
28

FIGURE 7 :Exact and Estimated Right Tail ,alpha=.5,beta=.5 ,n=100
True:Solid, Plain Beta:Dashes,Macro−Normalized Beta:Short Dashes,
Micro−Normalized Beta:Dots and Dashes

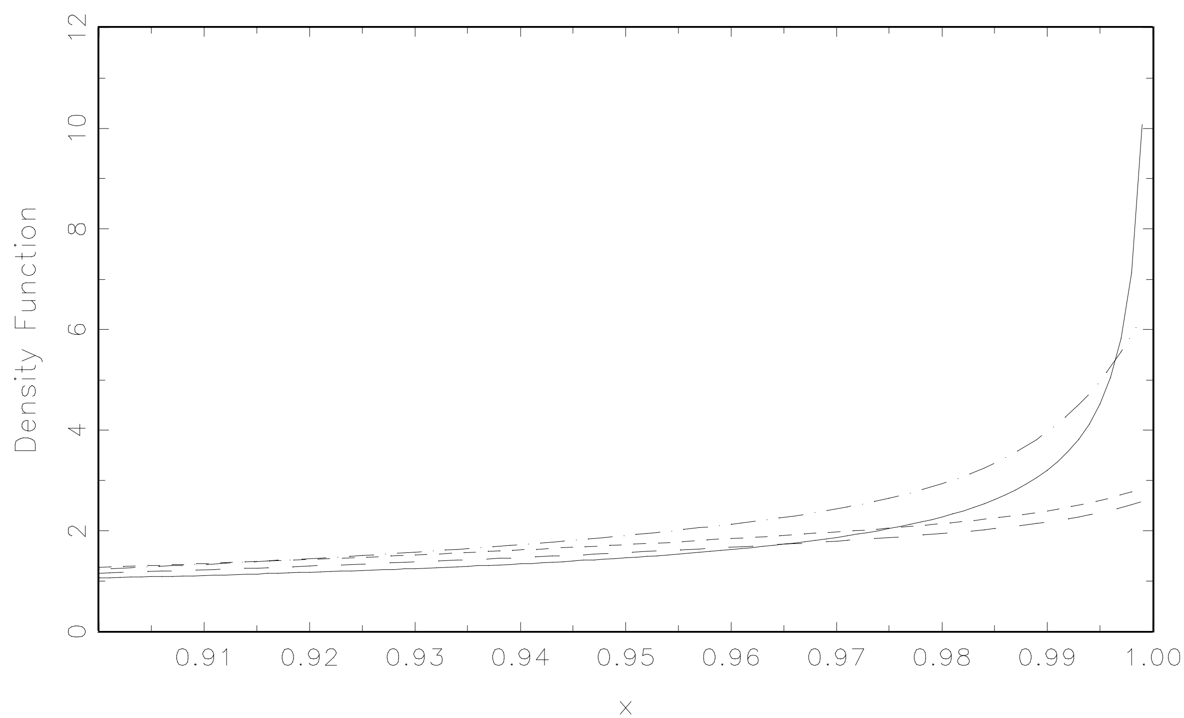


FIGURE 8 :Exact and Estimated Right Tail ,alpha=2.5,beta=.5 ,n=100
True:Solid, Plain Beta:Dashes,Macro−Normalized Beta:Short Dashes,
Micro−Normalized Beta:Dots and Dashes

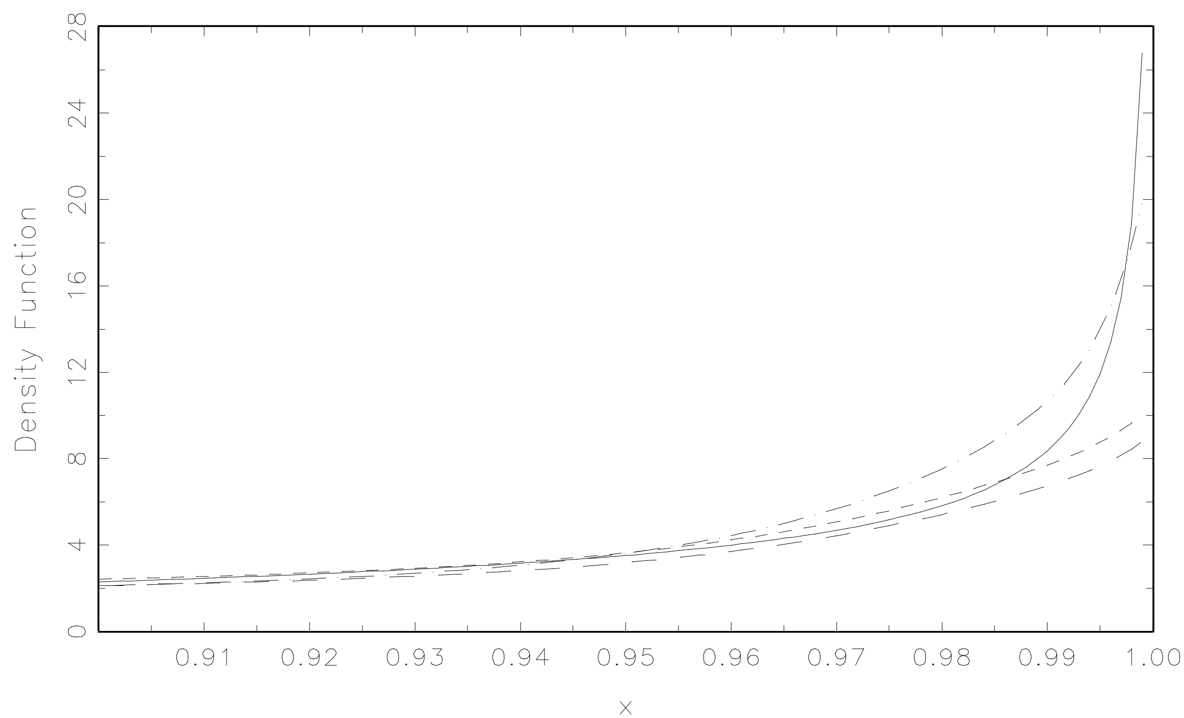


FIGURE 9 :Exact and Estimated Right Tail ,alpha=.5,beta=.5 ,n=1000
True:Solid, Plain Beta:Dashes,Macro−Normalized Beta:Short Dashes,
Micro−Normalized Beta:Dots and Dashes

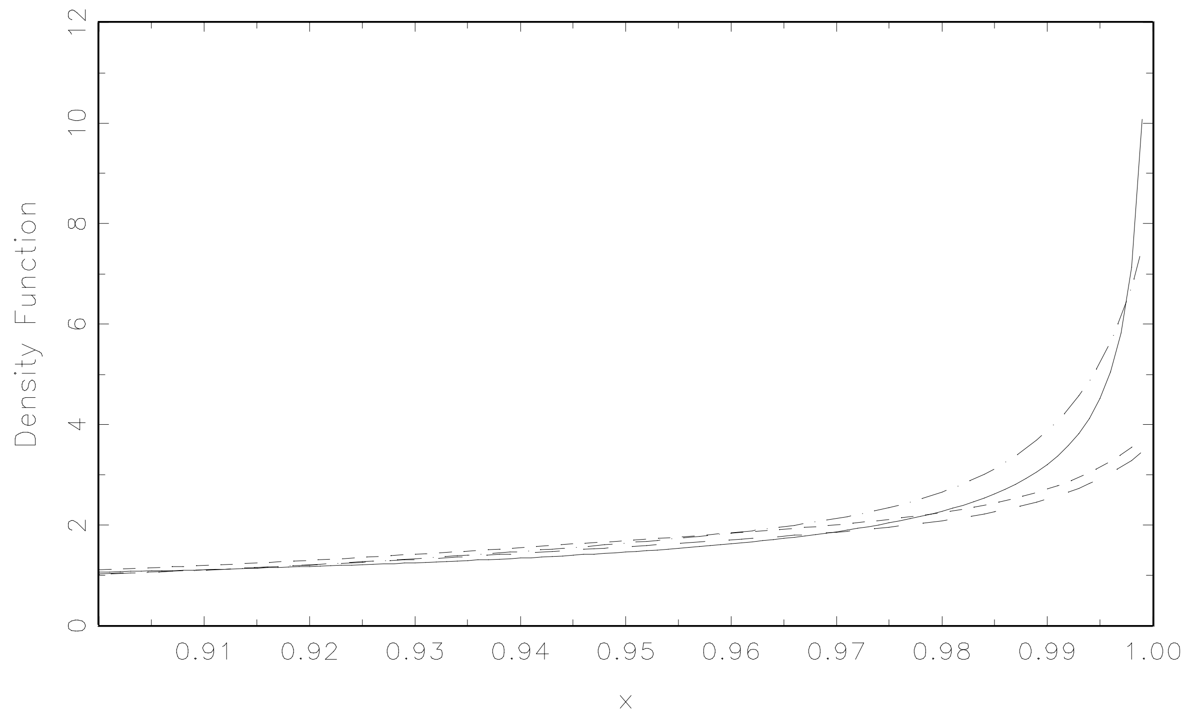


FIGURE 10 :Exact and Estimated Right Tail ,alpha=2.5,beta=.5 ,n=1000
True:Solid, Plain Beta:Dashes,Macro−Normalized Beta:Short Dashes,
Micro−Normalized Beta:Dots and Dashes

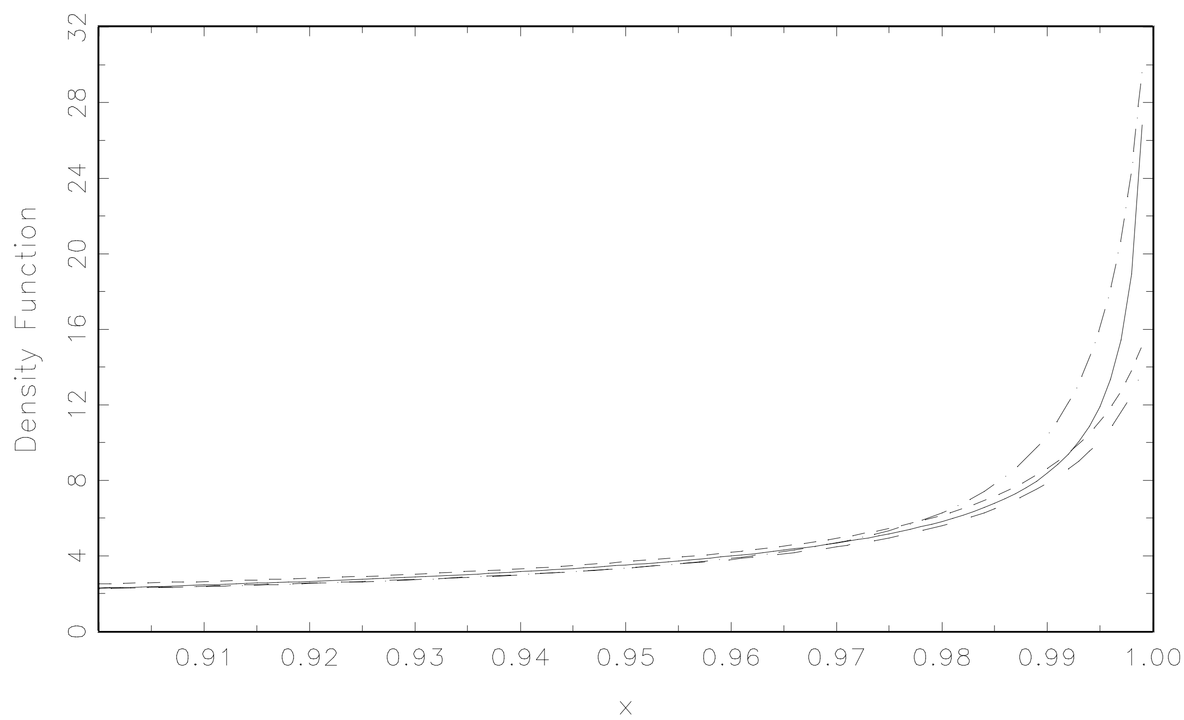


FIGURE 11:Exact and Estimated Right Tail,alpha=.5,beta=.5 ,n=100
True:Solid, Plain Beta:Dashes,Transformed Gaussian:Short Dashes,
Micro-Normalized Beta:Dots and Dashes

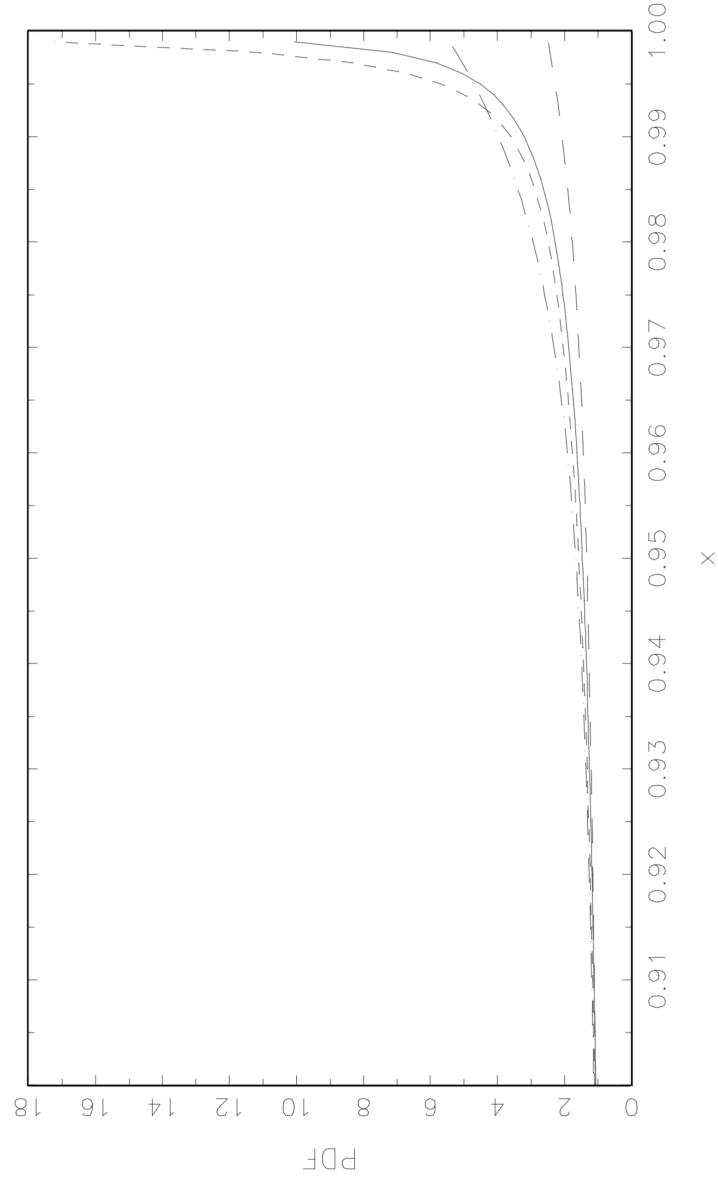


FIGURE 12:Exact and Estimated Right Tail,alpha=2.5,beta=.5 ,n=100
True:Solid, Plain Beta:Dashes,Transformed Gaussian:Short Dashes,
Micro−Normalized Beta:Dots and Dashes

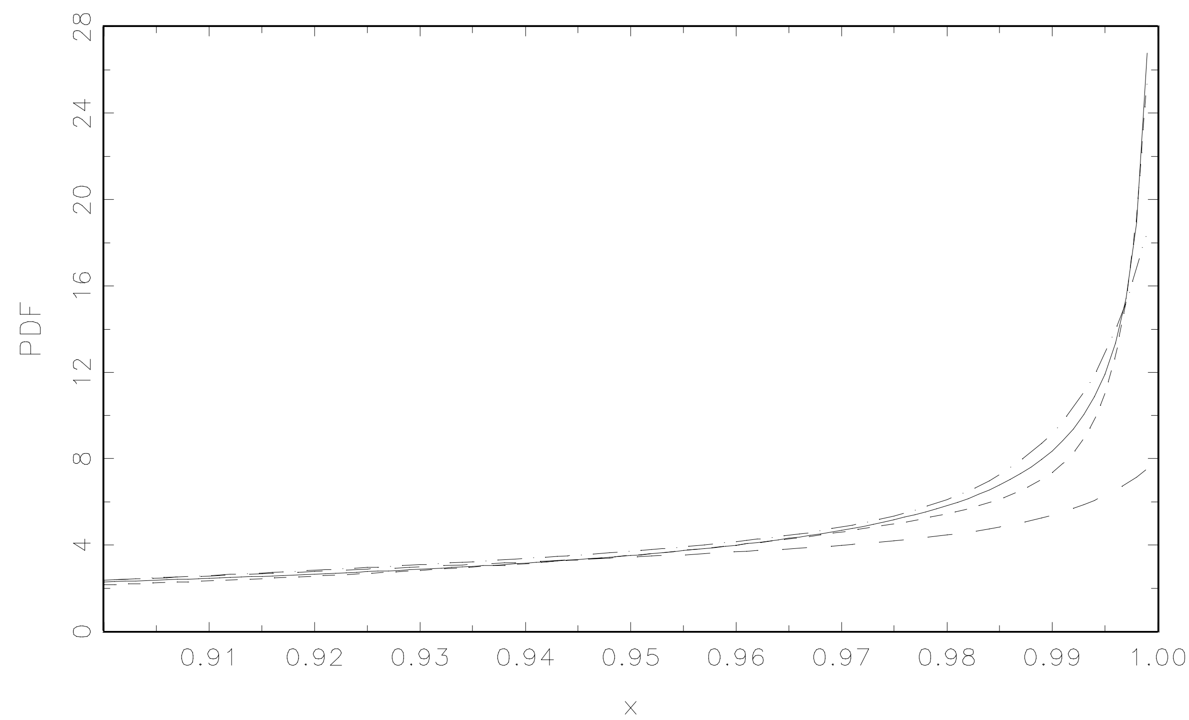


FIGURE 13:Exact and Estimated Right Tail,alpha=.5,beta=.5 ,n=1000
True:Solid, Plain Beta:Dashes,Transformed Gaussian:Short Dashes,
Micro−Normalized Beta:Dots and Dashes

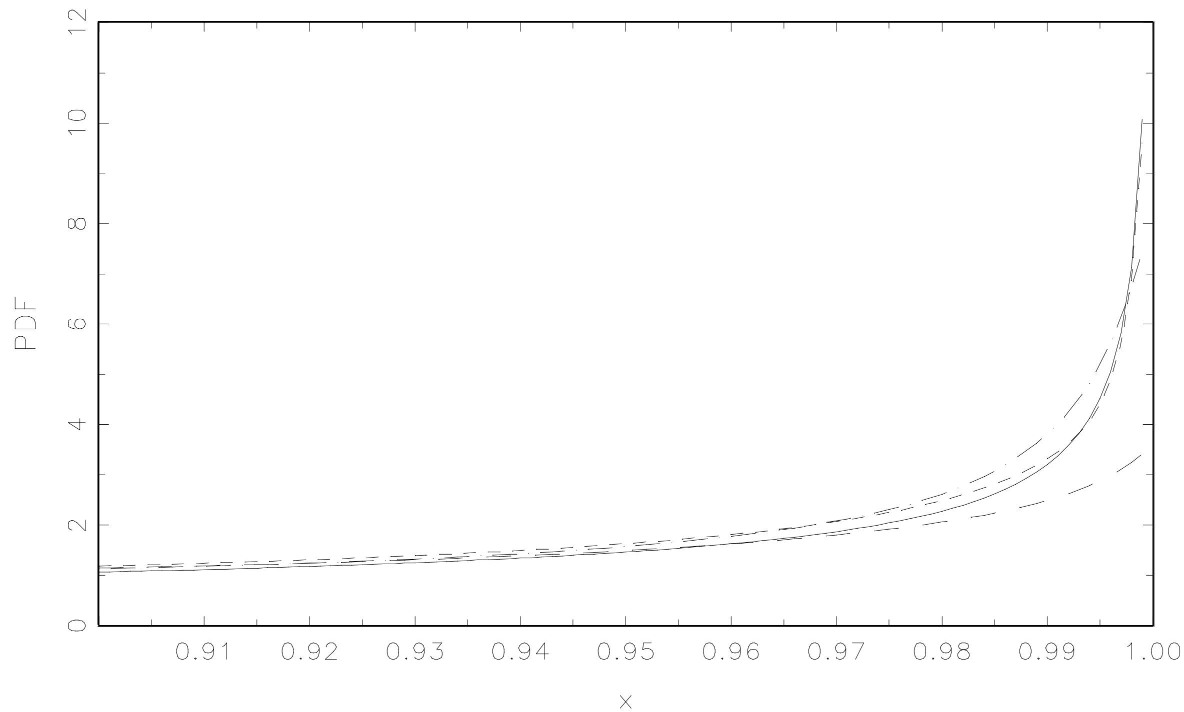


FIGURE 14:Exact and Estimated Right Tail,alpha=2.5,beta=.5 ,n=1000
True:Solid, Plain Beta:Dashes,Transformed Gaussian:Short Dashes,
Micro-Normalized Beta:Dots and Dashes