INSTITUT NATIONAL DE LA STATISTIQUE ET DES ETUDES ECONOMIQUES
Série des Documents de Travail du CREST
(Centre de Recherche en Economie et Statistique)

n° 2006-17

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C. GOURIEROUX[1]
W. LIU[2]

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[1] CREST and University of Toronto.
[2] Ph.D. Candidate, Department of Economics, University of Toronto.
Please send any comments to Wei Liu at : leo.liu@utoronto.ca

Efficient Portfolio Analysis Using Distortion Risk Measures

Christian Gourieroux* Wei Liu†

November 10, 2006

(Preliminary)

Abstract

We introduce nonparametric estimators of the sensitivity of distortion risk measure with respect to portfolio allocation. These estimators are used to derive the estimated efficient portfolio allocations when distortion risk measures define the constraints and the objectives, to study their asymptotic distributional properties, and to construct tests for the hypothesis of portfolio efficiency.

Keywords: Value-at-Risk, Tail-VaR, Loss-Given-Default, Distortion Risk Measures, Delta-DRM, Efficient Portfolio, Extended Sharpe Performance.
JEL Classification:

*CREST and University of Toronto.

†PhD Candidate, Department of Economics, University of Toronto. Please send any comments to Wei Liu at leo.liu@utoronto.ca.

1 Introduction

The second pillar of Basel II regulation concerns the control of internal models used by banks to measure and monitor the risk included in their asset portfolios. For a given portfolio with current allocation $\boldsymbol{a}_t$ and a given prediction horizon h, the practice consists in estimating the distribution of portfolio returns $y_{t+h}(\boldsymbol{a}_t)$ for the period $(t, t+h)$, and in deducing associated risk measures. These measures can be quantiles of the portfolio returns distribution (the so-called Value-at-Risk or VaR), other quantile based measures such as the expected shortfall (also called Tail-VaR, or conditional tail expectation), or more generally a distortion risk measure (DRM) (see Wang, 1996). However, the analysis of the return distribution of the current portfolio allocation can be insufficient. Indeed, it is important to look if this allocation can be improved, or if a small change of portfolio allocation has severe implications in terms of risk. In other words, we are interested in the ways the distribution of $y_{t+1}(\boldsymbol{a}_t)$ depends on allocation $\boldsymbol{a}_t$.

In Section 2, we introduce nonparametric estimators of the sensitivity of the distortion risk measures with respect to the portfolio allocation, called delta-DRM, and derive their asymptotic properties. Their finite sample properties are compared by a simulation study. In Section 3, we consider the portfolio allocation, which minimizes a distortion risk measure under budget constraint and restriction on another distortion risk measure. The framework includes the minimization of the VaR (or Tail-VaR) under expected return restriction as well as the minimization of Tail-VaR under VaR restriction. Then, we explain how to estimate the efficient portfolio allocation nonparametrically. The tests of efficiency of a given portfolio allocation are developed in Section 4. We discuss alternative test procedures based on a comparison of either risky allocations, or delta-DRM, or realized extended Sharpe (RES) performances. An analysis of finite sample properties of the estimators of the efficient allocation and of the extended Sharpe ratio is presented in Section 5. By considering Gaussian simulated returns, we have also the possibility to compare the empirical estimators introduced in our paper with the standard parametric estimators of the Gaussian framework. Section 6 concludes.

2 Sensitivity of distortion risk measures

In this section, we provide the expression of the sensitivity of distortion risk measures (or delta-DRM), which, loosely speaking, is a weighted expectation of the VaR sensitivity [called delta-VaR in the literature (see Garman, 1996)] with respect to the portfolio allocation. In the later part of this section, we introduce nonparametric estimators of delta-DRM and derive their asymptotic properties. For expository purpose, we drop the time index from the expression of returns and allocations in the definition of delta-DRM. The time is reintroduced to index the observations when the estimation problems are considered.

2.1 Expression of the delta-DRM

The DRMs are generally applied to portfolio loss (and profit) instead of profit (and loss). More precisely, we consider the opposite of the portfolio return $y(\boldsymbol{a}) = -\boldsymbol{a}'\boldsymbol{r}$, where $\boldsymbol{a}$ denotes the vector of allocations and $\boldsymbol{r}$ the $d \times 1$ primitive asset returns. For convenience, we denote $\boldsymbol{x} = -\boldsymbol{r}$ as the primitive asset losses, and thus, $y(\boldsymbol{a}) = \boldsymbol{a}'\boldsymbol{x}$. Throughout the paper, we use $(\boldsymbol{X}, Y)$ for the random variables and the lower cases for their realized values. A DRM for this portfolio is defined as:

$$\Pi(H, \boldsymbol{a}\,;\, F) = \int_0^1 Q(1-u; \boldsymbol{a})\, d\, H(u), \tag{2.1}$$

where F is the joint distribution of $\boldsymbol{X}$, $Q(\cdot; \boldsymbol{a})$ is the quantile function of $Y(\boldsymbol{a}) = \boldsymbol{a}'\boldsymbol{X}$, that is the inverse of the cdf $G(\cdot; \boldsymbol{a})$ of the (opposite) portfolio return. H is the cumulative distribution function of a positive measure, called distortion function (or Choquet capacity). As special cases, the VaR with risk level p is obtained when $H(u; p) = \mathbb{1}_{(u \geq p)}$ corresponds to a point mass at p, and the Tail-VaR with risk level p is obtained when $H(u; p) = (u/p) \wedge 1$ corresponds to the uniform distribution on $[0, p]$. In both examples above, the distortion function is parametrized by an additional parameter p and is used to define a family $\Pi(p; \boldsymbol{a}, F)$ of DRMs when p varies.

The delta-DRM provides the marginal risk contribution of a particular asset to the risk measure of the portfolio, and can be written as:

$$\begin{aligned} \frac{\partial \Pi}{\partial \boldsymbol{a}}(H, \boldsymbol{a}; F) &= \frac{\partial}{\partial \boldsymbol{a}} \int_0^1 Q(1-u; \boldsymbol{a}) d\, H(u) \\ &= \int_0^1 \frac{\partial Q}{\partial \boldsymbol{a}}(1-u; \boldsymbol{a}) d\, H(u). \end{aligned} \tag{2.2}$$

The explicit expression of VaR sensitivity with respect to portfolio allocation $\boldsymbol{a}$ has been derived in Gourieroux et al. (2000):

$$\frac{\partial Q}{\partial \boldsymbol{a}}(1-u; \boldsymbol{a}) \quad = \quad E[\boldsymbol{X} | \boldsymbol{a}'\boldsymbol{X} = Q(1-u; \boldsymbol{a})]. \tag{2.3}$$

Thus, we get:

$$\frac{\partial \Pi}{\partial \boldsymbol{a}}(H, \boldsymbol{a}; F) = \int_0^1 E[\boldsymbol{X} | \boldsymbol{a}'\boldsymbol{X} = Q(1-u; \boldsymbol{a})]\, d\, H(u). \tag{2.4}$$

Let us now discuss some features of the delta-DRM with useful implications for portfolio analysis.
i) The delta-DRM is homogeneous of degree zero in $\boldsymbol{a}$. Indeed, the DRM is homogeneous of degree one, since we have $\boldsymbol{a}' \frac{\partial Q}{\partial \boldsymbol{a}}(\cdot; \boldsymbol{a}) = E[\boldsymbol{a}'\boldsymbol{X} | \boldsymbol{a}'\boldsymbol{X} = Q(\cdot; \boldsymbol{a})] = Q(\cdot; \boldsymbol{a})$, which is the Euler characterization of homogeneous function of degree one.
ii) The delta-DRM is a weighted expectation of delta-VaRs at all risk levels. For instance, the delta-VaR(p) itself is a weighted expectation with weight equal to 1 at level p and equal to 0 otherwise. If H is differentiable, the expression (2.4) implies that the weighting function is the distortion density defined as $\partial H(u)/\partial u$. As shown in the next subsection, this facilitates the estimation of the delta-DRM once a delta-VaR estimator is available. Indeed, we can simply plug the delta-VaR estimator within the expectation expression (2.4).
iii) The delta-DRM can be alternatively interpreted as the expectation of the product between the distortion density and the primitive asset loss. More precisely, if function H is continuous and differ-

entiable,[1] we get the following expression (see Appendix B):

$$\frac{\partial \Pi}{\partial \boldsymbol{a}}(H, \boldsymbol{a}; F) = E\left[\boldsymbol{X}\, \frac{\partial H\Big(1 - G(\boldsymbol{a}'\boldsymbol{X}; \boldsymbol{a})\Big)}{\partial u}\right]. \tag{2.5}$$

The delta-DRM is equal to the expected co-movement between the primitive asset loss and the subjective perception, which is represented by the weights assigned to the losses. Indeed, the risk contribution of a particular asset is influenced jointly by the potential loss of the asset and how the risk measure is defined. The delta-DRM associated with the identity distortion function is simply the expected loss of the primitive asset.

2.2 Nonparametric estimators of the delta-DRM

We have seen in Section 2.1 that the delta-DRM can be written as a linear combination of delta-VaRs. In this section, we introduce nonparametric estimators of the delta-DRM. A first nonparametric estimator of the delta-DRM is obtained by substituting a kernel estimator of the delta-VaR into the expression (2.4). An alternative estimator of the delta-DRM is deduced from expression (2.5), under additional conditions on the distortion function.

2.2.1 Kernel estimator of the delta-DRM

Let us consider a sequence of observed (opposite) asset returns $\boldsymbol{x}_1, \dots, \boldsymbol{x}_T$. The delta-VaR admits the interpretation (2.3) as a conditional expectation. A kernel estimator of the delta-VaR is defined as follows:

$$\frac{\partial \widetilde{Q}_T}{\partial \boldsymbol{a}}(\,\cdot\,; \boldsymbol{a}) = \frac{\frac{1}{Th_T}\sum_{t=1}^{T} \boldsymbol{x}_t\, k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t - \widehat{Q}_T(\cdot;\boldsymbol{a})}{h_T}\right)}{\frac{1}{Th_T}\sum_{t=1}^{T} k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t - \widehat{Q}_T(\cdot;\boldsymbol{a})}{h_T}\right)}, \tag{2.6}$$

where $\widehat{Q}_T(u; \boldsymbol{a}) = \inf\left\{y : \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}_{(\boldsymbol{a}'\boldsymbol{x}_t \le y)} \ge u\right\}$ is the empirical quantile, k is a symmetric kernel function such that $\int k(u)d\,u = 1$ and $\int u\,k(u)d\,u = 0$, and h_T denotes the bandwidth. The estimator (2.6) is a Nadaraya-Watson estimator of the regression function $q \to E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X} = q]$ after substitution of the theoretical quantile $Q(\cdot; \boldsymbol{a})$ by its sample counterpart (see e.g. Yatchew, 2003, Chapter 3, for the definition and properties of the Nadaraya-Watson estimator).

The kernel estimator of the delta-DRM is defined as:

$$\frac{\partial \widetilde{\Pi}_T}{\partial \boldsymbol{a}}(H, \boldsymbol{a}) = \int_0^1 \frac{\partial \widetilde{Q}_T}{\partial \boldsymbol{a}}(1 - u; \boldsymbol{a})\, d\,H(u) = \int_0^1 \frac{\frac{1}{Th_T}\sum_{t=1}^{T} \boldsymbol{x}_t\, k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t - \widehat{Q}_T(1-u;\boldsymbol{a})}{h_T}\right)}{\frac{1}{Th_T}\sum_{t=1}^{T} k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t - \widehat{Q}_T(1-u;\boldsymbol{a})}{h_T}\right)}\, d\,H(u). \tag{2.7}$$

Since the empirical quantile function is a stepwise function, the integral (2.7) can be replaced by a

[1] The result still holds, if the distortion function H is continuous and almost everywhere differentiable. For instance, the distortion function associated with Tail-VaR is not differentiable when $u = p$, but the equality (2.5) is still valid with the right derivative of H. However, the relation is not valid for the application to the delta-VaR, in which H is not continuous.

summation. We get the equivalent expression:

$$\frac{\partial \widetilde{\Pi}_T}{\partial \boldsymbol{a}}(H, \boldsymbol{a}) = \sum_{i=1}^{T} \frac{\sum_{t=1}^{T} \boldsymbol{x}_t \, k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t - y_i^*(\boldsymbol{a})}{h_T}\right)}{\sum_{t=1}^{T} k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t - y_i^*(\boldsymbol{a})}{h_T}\right)} \left[H(1-\frac{i-1}{T}) - H(1-\frac{i}{T})\right], \tag{2.8}$$

where $y_1^*(\boldsymbol{a}) < y_2^*(\boldsymbol{a}) < \cdots < y_T^*(\boldsymbol{a})$ is the order statistics of the (opposite) observed portfolio returns associated with allocation $\boldsymbol{a}$.

The expression (2.7) is valid for any distortion function H, implying that the kernel estimator can be applied in the general framework, which includes delta-VaR as a special case. Although the rate of convergence of the kernel estimator of delta-VaR is $\sqrt{Th_T}$, the kernel estimator of a delta-DRM will converge at rate $\sqrt{T}$ due to its integral expression as shown in Section 2.3 (Similar results can be found in Ait-Sahalia (1993) and Gagliardini and Gourieroux (2006)).

2.2.2 Empirical estimator of delta-DRM

Alternatively, if the distortion function H is continuous and differentiable, the delta-DRM can be estimated by the sample analog of expression (2.5). The empirical estimator of the delta-DRM is defined as the sample average of the products between the observed (opposite) primitive asset returns and distortion densities,[2]

$$\frac{\partial \widehat{\Pi}_T}{\partial \boldsymbol{a}}(H; \boldsymbol{a}) = \frac{\partial \Pi}{\partial \boldsymbol{a}}(H, \boldsymbol{a}, \widehat{F}_T) = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}_t \frac{\partial H\Big(1 - \widehat{G}_T(\boldsymbol{a}'\boldsymbol{x}_t; \boldsymbol{a})\Big)}{\partial u}, \tag{2.9}$$

where $\widehat{G}_T(y; \boldsymbol{a}) = \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}_{(\boldsymbol{a}'\boldsymbol{x}_t \le y)}$ is the sample cdf of $Y(\boldsymbol{a})$. This estimator is also equal to:

$$\frac{\partial \widehat{\Pi}_T}{\partial \boldsymbol{a}}(H; \boldsymbol{a}) = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}_t^* \frac{\partial H\Big(1 - t/T\Big)}{\partial u}, \tag{2.10}$$

where $\boldsymbol{x}_t^*$, $t = 1, \ldots, T$, are the observations of (opposite) primitive asset returns reordered according to the order statistics of (opposite) portfolio returns $\big(y_t^*(\boldsymbol{a})\big)$.[3] For instance, the estimator for delta-TVaR can be calculated by:

$$\frac{\partial \widehat{TVaR_T}(p; \boldsymbol{a})}{\partial \boldsymbol{a}} = \frac{1}{Tp}\sum_{t=[T(1-p)]}^{T} \boldsymbol{x}_t^*,$$

where $[\cdot]$ denotes integer part. The empirical estimator is easier to compute and will also converge at the standard parametric rate $\sqrt{T}$.

[2] In some special cases (e.g. $H(u; p) = u^p$, for $p < 1$), the estimator can be unbounded since the empirical distortion density goes to infinity when $t = T$. This problem can be solved by replacing T by $T + 1$ in the distortion density.

[3] This estimator can be seen as a L-statistics constructed on two dependent series $\boldsymbol{x}_t$ and $y_t(\boldsymbol{a})$ (see e.g. Shorack and Wellner, 1986, Chapter 19).

2.3 Asymptotic properties of the delta-DRM estimators

The accuracy of the estimated risk measures and their sensitivities are often overlooked by practitioners. However, this potential estimation error can put investors in a very risky position. This issue has been categorized as the estimation risk in the risk management terminology of the Basle Committee. In this section, we derive the asymptotic distributions of both nonparametric estimators of the delta-DRM.

2.3.1 Asymptotic distributions of the estimators

The kernel estimator of the delta-DRM is defined as an integral of Nadaraya-Watson estimator of the delta-VaR, which is known to be asymptotically Gaussian. The empirical estimator of the delta-DRM is defined as a sample average. Thus, with i.i.d. observations $\boldsymbol{x}_t$, $t = 1, \dots, T$, the Central Limit Theorem applies directly. Thus, both estimators will converge to Gaussian processes. These results are summarized in the proposition below (see Appendices C.1–C.4).[4]

Proposition 1. *For an i.i.d. sequence $\boldsymbol{x}_t = (x_t^1, \dots, x_t^d)$, $t = 1, \dots, T$, following a joint distribution $F(\boldsymbol{X}) = C\left(F^1(X^1), \dots, F^d(X^d)\right)$ with copula function C and marginal cdf $F^i(X^i)$, if H is differentiable, we have:*

$$\sqrt{T}\left(\frac{\partial \widetilde{\Pi}_T}{\partial \boldsymbol{a}}(H, \boldsymbol{a}) - \frac{\partial \Pi}{\partial \boldsymbol{a}}(H, \boldsymbol{a}, F)\right)$$
$$\Rightarrow \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}} \frac{\partial}{\partial y}\big(E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X} = y]\big) \nabla H(y; \boldsymbol{a}) \mathbb{1}_{\boldsymbol{a}'\boldsymbol{x} \le y} dy \right\} d\,\mathcal{K}\big(F^1(x^1), \dots, F^d(x^d)\big)$$
$$- \int_{\mathbb{R}^d} \Big\{ \boldsymbol{x} - E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X} = \boldsymbol{a}'\boldsymbol{x}] \Big\} \nabla H(\boldsymbol{a}'\boldsymbol{x}; \boldsymbol{a})\, d\,\mathcal{K}\big(F^1(x^1), \dots, F^d(x^d)\big).$$

If H is twice differentiable, we have:

$$\sqrt{T}\left(\frac{\partial \widetilde{\Pi}_T}{\partial \boldsymbol{a}}(H, \boldsymbol{a}) - \frac{\partial \Pi}{\partial \boldsymbol{a}}(H, \boldsymbol{a}, F)\right)$$
$$\Rightarrow \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} \boldsymbol{z} \frac{\partial^2 H\big(1 - G(\boldsymbol{a}'\boldsymbol{z}; \boldsymbol{a})\big)}{\partial u^2} \mathbb{1}_{(\boldsymbol{a}'\boldsymbol{x} \le \boldsymbol{a}'\boldsymbol{z})} d\,F(\boldsymbol{z}) \right.$$
$$\left. - \Big\{ \boldsymbol{x} - E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X} = \boldsymbol{a}'\boldsymbol{x}] \Big\} \nabla H(\boldsymbol{a}'\boldsymbol{x}; \boldsymbol{a}) \right\} d\,\mathcal{K}\big(F^1(x^1), \dots, F^d(x^d)\big).$$

$$\sqrt{T}\left(\frac{\partial \widehat{\Pi}_T}{\partial \boldsymbol{a}}(H, \boldsymbol{a}) - \frac{\partial \Pi}{\partial \boldsymbol{a}}(H, \boldsymbol{a}, F)\right)$$
$$\Rightarrow \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} -\boldsymbol{z} \frac{\partial^2 H\big(1 - G(\boldsymbol{a}'\boldsymbol{z}; \boldsymbol{a})\big)}{\partial u^2} \mathbb{1}_{(\boldsymbol{a}'\boldsymbol{x} \le \boldsymbol{a}'\boldsymbol{z})} d\,F(\boldsymbol{z}) + \boldsymbol{x} \nabla H(\boldsymbol{a}'\boldsymbol{x}; \boldsymbol{a}) \right\} d\,\mathcal{K}\big(F^1(x^1), \dots, F^d(x^d)\big),$$

where $\Rightarrow$ denotes the weak convergence of processes,[5] $\nabla H(y; \boldsymbol{a}) = \frac{\partial H\left(1 - G(y; \boldsymbol{a})\right)}{\partial u}$, *and $\mathcal{K}$ is a multi-*

[4] We assume that the distortion function is continuous and differentiable. The asymptotic variances associated with the VaR and Tail-VaR are provided in Appendix C.5.

[5] The processes are indexed by allocation $\boldsymbol{a}$. When the distortion function depends on some pessimism parameter,

dimensional Brownian bridge on $[0,1]^d$.

The i.i.d. assumption is crucial for obtaining the proposition above. This assumption is implicitly used in the historical simulation method suggested by Basel committee. On the other hand, a similar result can be derived when $\boldsymbol{x}_t$ is stationary, in which case, Central Limit Theorem for stationary processes must be applied (see Gourieroux and Liu, 2006 Appendix A.3 for a brief discussion).

2.3.2 Estimation of the asymptotic variance

The variance-covariance matrices of both delta-DRM estimators corresponding to portfolio allocation $\boldsymbol{a}$ are given by (see Appendix C.5):

$$\Omega(\boldsymbol{a},\boldsymbol{a}) = \lim_{T\to\infty} V\left[\sqrt{T}\left(\frac{\partial\widetilde{\Pi}_T}{\partial \boldsymbol{a}}(H,\boldsymbol{a}) - \frac{\partial \Pi}{\partial \boldsymbol{a}}(H,\boldsymbol{a},F)\right)\right] = V\Big[A(\boldsymbol{a}'\boldsymbol{X},H) - \big(\boldsymbol{X} - E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}]\big)\nabla H(\boldsymbol{a}'\boldsymbol{X};\boldsymbol{a})\Big],$$

$$\Sigma(\boldsymbol{a},\boldsymbol{a}) = \lim_{T\to\infty} V\left[\sqrt{T}\left(\frac{\partial\widehat{\Pi}_T}{\partial \boldsymbol{a}}(H,\boldsymbol{a}) - \frac{\partial \Pi}{\partial \boldsymbol{a}}(H,\boldsymbol{a})\right)\right] = V\Big[\boldsymbol{X}\nabla H(\boldsymbol{a}'\boldsymbol{X};\boldsymbol{a}) - A(\boldsymbol{a}'\boldsymbol{X},H)\Big],$$

where,

$$\begin{aligned} A(y,H) &= \int \boldsymbol{z}\mathbb{1}_{(y\le \boldsymbol{a}'\boldsymbol{z})} \frac{\partial^2 H\big(1-G(\boldsymbol{a}'\boldsymbol{z};\boldsymbol{a})\big)}{\partial u^2} dF(\boldsymbol{z}) \\ &= E\left[\boldsymbol{X}^*\mathbb{1}_{(\boldsymbol{a}'\boldsymbol{X}^*\ge y)} \frac{\partial^2 H\big(1-G(\boldsymbol{a}'\boldsymbol{X}^*;\boldsymbol{a})\big)}{\partial u^2}\right], \end{aligned}$$

and $\boldsymbol{X}^*$ is independent of $\boldsymbol{X}$ with the same distribution. These variance-covariance matrices cannot be ordered and none of the estimator of delta-DRM is better than the other one.

Estimators of the variance-covariance matrices can be derived as follows: let us define the pseudo-observations of the components within the variance-covariance matrices as $\widehat{S}_{1t} = \boldsymbol{x}_t^* \frac{\partial H(1-t/T)}{\partial u}$, $\widehat{S}_{2t} = \left[\boldsymbol{x}_t^* - \frac{\partial \widetilde{Q}_T(1-t/T;\boldsymbol{a})}{\partial \boldsymbol{a}}\right]\frac{\partial H(1-t/T)}{\partial u}$ and $\widehat{S}_{3t} = \frac{1}{T}\sum_{i=t}^{T} \boldsymbol{x}_i^* \frac{\partial^2 H(1-i/T)}{\partial u^2}$. The variance-covariance matrices can be consistently estimated by their pseudo sample counterparts:

$$\begin{aligned} \widehat{\Omega}(\boldsymbol{a},\boldsymbol{a}) &= \frac{1}{T}\sum_{t=1}^{T}\left(\widehat{S}_{3t}-\widehat{S}_{2t}\right)\left(\widehat{S}_{3t}-\widehat{S}_{2t}\right)' - \frac{1}{T}\sum_{t=1}^{T}\left(\widehat{S}_{3t}-\widehat{S}_{2t}\right)\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{S}_{3t}-\widehat{S}_{2t}\right)', \\ \widehat{\Sigma}(\boldsymbol{a},\boldsymbol{a}) &= \frac{1}{T}\sum_{t=1}^{T}\left(\widehat{S}_{1t}-\widehat{S}_{3t}\right)\left(\widehat{S}_{1t}-\widehat{S}_{3t}\right)' - \frac{1}{T}\sum_{t=1}^{T}\left(\widehat{S}_{1t}-\widehat{S}_{3t}\right)\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{S}_{1t}-\widehat{S}_{3t}\right)', \end{aligned} \tag{2.11}$$

where $\boldsymbol{x}_t^*$ is defined as before [see e.g. Genest et al. (1995) for related results and regularity conditions, in the framework of copula estimation].

the results can be extended to processes doubly indexed by the allocation and the pessimism parameter (see Gourieroux and Liu, 2006 for the analysis of sensitivity w.r.t. pessimism parameter).

2.4 Finite sample properties of the estimators of the delta-DRM

The empirical estimator is less smooth than the kernel estimator with respect to allocation, especially if the distortion functions are discontinuous. Figure 1 illustrates this feature by considering simulated portfolios including two primitive assets. The two series of (opposite) returns are simulated from a bivariate Gaussian white noise. The covariance matrix is estimated from returns of the 1st and 5th deciles (in term of capitalization) of NYSE equity portfolios provided by CRSP. We use 250 observations corresponding to year 2000. The allocation (a^1) associated with asset 1 (resp. $a^2 = 1 - a^1$ associated with asset 2) is chosen to vary from 0.01 to 0.99 and the delta-DRM is estimated by both the kernel (solid lines) and empirical (dashed lines) approaches.[6] We consider two risk measures, that are the Tail-VaR (TVaR in short with $H(u;p) = (u/p) \wedge 1$) and the Proportional Hazard risk measure (PH in short with $H(u;p) = u^p$). TVaR is designed to measure the extreme losses (e.g. $p = 0.05$) with weighting function:

$$\nabla H(\boldsymbol{a}'\boldsymbol{X};\boldsymbol{a}) = \frac{1}{p}\mathbb{1}_{\left(\boldsymbol{a}'\boldsymbol{X} \geq Q(1-p;\boldsymbol{a})\right)},$$

which is not continuous in $\boldsymbol{a}$. In the left graph of Figure 1, we observe that the kernel estimator (solid line) of the delta-TVaR is much smoother than the empirical estimator (dashed line). In fact, the empirical estimator is not appropriate for estimating the sensitivity associated with the Tail-VaR (see also Scaillet, 2004). However, the right graph of Figure 1 shows that both estimators of the delta-PH are sufficiently smooth. Indeed, the PH risk measure assigns weights, which are continuous functions of $\boldsymbol{a}$, at all loss levels.[7]

Let us now study the accuracy of both estimators in finite sample. We replicate 1000 times the previous simulation and compute the associated simulated variances. The finite sample variances of the kernel estimators (solid lines) and of the empirical estimators (dashed lines) for both the delta-TVaR and delta-PH are provided in Figure 2. The finite sample accuracies of both estimators are of similar magnitudes. More precisely, the variance of the empirical estimator of the delta-TVaR is up to 20% higher than that of the corresponding kernel estimator. When considering delta-PH, the kernel estimator is less accurate, but the difference is only 5%.

3 Efficient portfolio

The mean-variance approach has been prevalent in portfolio management for the past decades (see e.g. Markowitz, 1952). In this framework, efficient portfolios are constructed by minimizing the variance while keeping a desired level of expected return and under a budget constraint. However, this approach can be misleading if the return distributions deviate from Gaussian. Indeed, the variance may not be a proper measure of risk. A few alternatives have been introduced to analyze the portfolio decision based on DRMs, such as VaR [Basak and Shapiro (2001)] and TVaR [Lemus (1999), Rockafellar and Uryasev (1999, 2000),Yamai and Yoshiba (2002)]. In a recent work, Gourieroux and

[6] Following Scaillet (2004), we choose the bandwidth parameter $h = 0.5 s_y T^{-1/5}$, where s_y is the estimated standard deviation of y_t.

[7] The weighting function associated with PH is $\nabla H(\boldsymbol{a}'\boldsymbol{X}) = p\big(1 - G(\boldsymbol{a}'\boldsymbol{X};\boldsymbol{a})\big)^{p-1}$ (see e.g. Gourieroux and Liu, 2006).

Monfort (2005) provide a complete analysis of expected utility based efficient portfolios, including the estimation of efficient portfolios and tests for portfolio efficiency. Their approach corresponds to the so-called direct approach of stochastic dominance. In this section, we revisit this issue of portfolio choices for the dual approach of risk comparison based on DRM (see e.g. Wang and Young, 1998 for a comparison of direct and dual approaches of stochastic dominance).

3.1 The efficiency frontier

i) The optimization problem

Let us consider a market with a risk-free asset and d primitive risky assets. Without loss of generality, the risk-free rate is set equal to zero; the current prices of risky assets are set equal to 1; we consider an agent facing a one-period investment decision on the allocation $\boldsymbol{a} = [a^1, ..., a^d]'$ of his/her initial wealth W_0. We impose no positivity restriction on the allocations, that is, short sales are allowed. Let us denote by $\boldsymbol{x} = [x^1, ..., x^d]'$ the vector of one-period (opposite) returns on the d risky assets. Similar to the mean-variance problem, the agent can solve the following DRM-DRM problem:

$$\min_{a^0, \boldsymbol{a}} \ \Pi(H_1, (a^0, \boldsymbol{a}), F_0), \quad \text{s.t.} \quad \Pi(H_0, (a^0, \boldsymbol{a}), F_0) = \Pi_0^* \text{ and } \ a^0 + \sum_{i=1}^{d} a^i = W_0, \tag{3.1}$$

where a^0 is the allocation in the risk-free asset and Π_0^* is a predetermined value. This includes several familiar optimization problems as special cases. For instance, by letting $H_0(u) = u$ and $H_1(u) = \mathbb{1}_{(u \geq p)}$ (resp. $H_1(u) = (u/p) \wedge 1$), we get the standard mean-VaR (resp. mean-TVaR) optimization problem. By defining $H_1(u) = (u/p) \wedge 1$ and $H_0(u) = \mathbb{1}_{(u \geq p)}$, we get the following optimization problem in the VaR-TVaR space:

$$\min_{a^0, \boldsymbol{a}} \ TVaR(p), \ \text{s.t.} \ \ VaR(p) = \Pi_0^* \text{ and } \ a^0 + \sum_{i=1}^{d} a^i = W_0. \tag{3.2}$$

The optimization problem (3.2) is coherent with the current regulation, in which the reserves are invested in a risk-free asset and used to bind a VaR(5%) constraint. In this framework, the constraint is an inequality constraint $VaR(5\%) \leq \Pi_0^*$, which will be binding at the optimum.

Note also that convexity properties of the DRM with respect to portfolio allocation is useful to ensure a unique solution to the DRM-DRM optimization problem and sufficient first-order conditions. Such conditions are satisfied for the DRM when the distortion function is concave (see Denneberg, 1994).

Since a DRM is drift invariant,[8] the solution of the DRM-DRM problem depends on W_0 only by means of the budget constraint. Thus, for the allocations in risky assets, the optimization is equivalent to:

$$\min_{\boldsymbol{a}} \ \Pi(H_1, \boldsymbol{a}, F_0), \ \text{s.t.} \ \ \Pi(H_0, \boldsymbol{a}, F_0) = \Pi_0, \tag{3.3}$$

where $\Pi_0 = \Pi_0^* - W_0$ and F_0 is the joint distribution of excess returns.

[8] A function ρ is drift invariant if $\rho(X + c) = \rho(X) + c$, for any constant c.

ii) The first-order conditions

Let us denote λ the Lagrange multiplier associated with the DRM constraint. The optimization of the Lagrangian implies the following first-order conditions:

$$\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \boldsymbol{a}^*, F_0) - \lambda^* \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}^*, F_0) = 0, \tag{3.4a}$$

$$\Pi(H_0, \boldsymbol{a}^*, F_0) = \Pi_0, \tag{3.4b}$$

whose solution $\boldsymbol{a}^*, \lambda^*$ are the efficient portfolio and optimal Lagrange multiplier, respectively.[9] Since the DRM is homogeneous of degree one with respect to portfolio allocations, the system (3.4) implies:

$$\begin{aligned}
&(\boldsymbol{a}^*)' \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \boldsymbol{a}^*, F_0) - \lambda^* (\boldsymbol{a}^*)' \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}^*, F_0) = 0 \\
&\Leftrightarrow \Pi(H_1, \boldsymbol{a}^*, F_0) - \lambda^* \Pi(H_0, \boldsymbol{a}^*, F_0) = 0 \\
&\Leftrightarrow \lambda^* = \frac{\Pi(H_1, \boldsymbol{a}^*, F_0)}{\Pi(H_0, \boldsymbol{a}^*, F_0)} = \frac{\Pi(H_1, \boldsymbol{a}^*, F_0)}{\Pi_0}.
\end{aligned} \tag{3.5}$$

iii) The two funds separation theorem

The efficient allocation is characterized by:

$$\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \boldsymbol{a}^*, F_0) - \frac{\Pi(H_1, \boldsymbol{a}^*, F_0)}{\Pi_0} \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}^*, F_0) = 0, \tag{3.6a}$$

$$\Pi(H_0, \boldsymbol{a}^*, F_0) = \Pi_0. \tag{3.6b}$$

The efficient allocation $\boldsymbol{a}^*$ depends on the level Π_0 of the DRM constraint. Since the DRM are homogeneous function of degree one in the allocation, the equation (3.6a) is equivalent to:

$$\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \frac{\boldsymbol{a}^*}{\Pi_0}, F_0) - \Pi(H_1, \frac{\boldsymbol{a}^*}{\Pi_0}, F_0) \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \frac{\boldsymbol{a}^*}{\Pi_0}, F_0) = 0.$$

Thus, the optimal allocation associated with level Π_0 is Π_0 times the optimal allocation associated with level 1, and all efficient portfolios are proportional to a same portfolio when Π_0 varies. This is the well-known two-funds separation theorem (see e.g. Cass and Stiglitz, 1970, Ross, 1978), which is extended here to DRM measures of risk. The two-funds separation theorem is clearly a consequence of the homogeneity property. The efficiency frontier can be represented in the DRM-DRM space. The drift invariance and homogeneity properties imply that the efficiency frontier is a half-line starting at the point (W_0, W_0) corresponding to a full risk-free investment with a slope equal to λ^*.

iv) The optimization of the performance

The optimal Lagrange multiplier extends the standard notion of Sharpe ratio introduced in the mean-variance framework (see Sharpe, 1966) to the case of DRM. For instance if the DRM defining the constraint is the VaR(p) and the DRM to be optimized is TVaR(p), the Lagrange multiplier is simply the amplifying factor considered in Gourieroux and Liu (2006). Moreover, the extended Sharpe

[9] The solutions $\boldsymbol{a}^*, \lambda^*$ depend on H_0, H_1, Π_0, F_0. This dependence will be mentioned only when necessary.

ratio:

$$\lambda(H_0, H_1, \boldsymbol{a}, F_0) = \frac{\Pi(H_1, \boldsymbol{a}, F_0)}{\Pi(H_0, \boldsymbol{a}, F_0)} \tag{3.7}$$

is homogeneous of degree zero in the allocation. As a consequence the efficiency frontier can also be derived by minimizing the extended Sharpe ratio. Thus, an efficient allocation will also satisfy the first-order condition:

$$\frac{\partial \lambda}{\partial \boldsymbol{a}}(H_0, H_1, \boldsymbol{a}^*, F_0) = \frac{1}{\Pi(H_0, \boldsymbol{a}^*, F_0)} \left[\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \boldsymbol{a}^*, F_0) - \lambda(H_0, H_1, \boldsymbol{a}^*, F_0)\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}^*, F_0)\right] = 0. \tag{3.8}$$

The solution $\boldsymbol{a}^*$ of this first-order condition (3.8) is unique up to a multiplicative factor.

The solution becomes unique under the additional constraint $\Pi(H_0, \boldsymbol{a}^*, F_0) = \Pi_0$, or if we consider the modified first-order condition:

$$\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \boldsymbol{a}^*, F_0) - \frac{\Pi(H_1, \boldsymbol{a}^*, F_0)}{\Pi_0}\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}^*, F_0) = 0. \tag{3.9}$$

Indeed, by pre-multiplying (3.9) by $(\boldsymbol{a}^*)'$ and applying the homogeneity property, we get:

$$\begin{aligned} & \Pi(H_1, \boldsymbol{a}^*, F_0) - \frac{\Pi(H_1, \boldsymbol{a}^*, F_0)}{\Pi_0}\Pi(H_0, \boldsymbol{a}^*, F_0) = 0 \\ \Leftrightarrow \quad & \Pi(H_0, \boldsymbol{a}^*, F_0) = \Pi_0. \end{aligned}$$

Note that the theoretical results derived in Section 3.1 are also valid in a dynamic framework with serially dependent returns, considering the conditional distribution instead of the unconditional one.

3.2 Empirical estimator of the efficient allocation

Nonparametric estimators of the optimal allocation $\boldsymbol{a}^*$ and extended Sharpe ratio λ^* can be defined in two different ways according to the nonparametric estimator of the delta-DRM, which is used. For expository purpose, we consider below the empirical estimator, but a similar analysis can be done with the kernel estimator. The empirical estimators $\widehat{\boldsymbol{a}}_T$, $\widehat{\lambda}_T$ are solutions of the empirical counterparts of first-order conditions (3.4):

$$\begin{cases} \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \widehat{\boldsymbol{a}}_T, \widehat{F}_T) - \widehat{\lambda}_T \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \widehat{\boldsymbol{a}}_T, \widehat{F}_T) = 0, \\ \Pi(H_0, \widehat{\boldsymbol{a}}_T, \widehat{F}_T) = \Pi_0. \end{cases} \tag{3.10}$$

The system above provides jointly an estimator of the efficient allocation and of the extended Sharpe performance. This system is equivalent to:

$$\begin{cases} \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \widehat{\boldsymbol{a}}_T, \widehat{F}_T) - \frac{\Pi(H_1, \widehat{\boldsymbol{a}}_T, \widehat{F}_T)}{\Pi_0} \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \widehat{\boldsymbol{a}}_T, \widehat{F}_T) = 0, \\ \widehat{\lambda}_T = \frac{\Pi(H_1, \widehat{\boldsymbol{a}}_T, \widehat{F}_T)}{\Pi_0}, \end{cases} \tag{3.11}$$

which corresponds to the optimization of the extended Sharpe performance.

3.3 Asymptotic properties of the estimated efficient allocation and extended Sharpe performance

The estimators $\widehat{F}_T$, $\widehat{\boldsymbol{a}}_T$, and $\widehat{\lambda}_T$ converge to their theoretical counterparts, F_0, $\boldsymbol{a}^*$ and λ^*, when $T \to \infty$. Thus, the first-order conditions (3.11) can be expanded when $\widehat{F}_T$, $\widehat{\boldsymbol{a}}_T$ and $\widehat{\lambda}_T$ are close to F_0, $\boldsymbol{a}^*$, λ^*, respectively. The expansions are performed in Appendix D and summarized below.

Proposition 2.

$$
\begin{aligned}
i)\sqrt{T}(\widehat{\boldsymbol{a}}_T - \boldsymbol{a}^*) = &\left[-\frac{\partial^2 \Pi}{\partial \boldsymbol{a}\partial \boldsymbol{a}'}(H_1 - \lambda^* H_0, \boldsymbol{a}^*, F_0) + \frac{1}{\Pi_0}\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}^*, F_0)\frac{\partial \Pi}{\partial \boldsymbol{a}'}(H_1, \boldsymbol{a}^*, F_0)\right]^{-1} \\
&\sqrt{T}\left(\frac{\partial \widehat{\Pi}_T}{\partial \boldsymbol{a}}(H_1 - \lambda^* H_0, \boldsymbol{a}^*, F_0) - \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1 - \lambda^* H_0, \boldsymbol{a}^*, F_0)\right) \\
&- \left[-\frac{\partial^2 \Pi}{\partial \boldsymbol{a}\partial \boldsymbol{a}'}(H_1 - \lambda^* H_0, \boldsymbol{a}^*, F_0) + \frac{1}{\Pi_0}\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}^*, F_0)\frac{\partial \Pi}{\partial \boldsymbol{a}'}(H_1, \boldsymbol{a}^*, F_0)\right]^{-1} \\
&\frac{(\boldsymbol{a}^*)'}{\Pi_0}\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}^*, F_0)\sqrt{T}\left[\frac{\partial \widehat{\Pi}_T}{\partial \boldsymbol{a}}(H_1, \boldsymbol{a}^*, F_0) - \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \boldsymbol{a}^*, F_0)\right] + o_p(1);
\end{aligned}
$$

$$
ii)\sqrt{T}(\widehat{\lambda}_T - \lambda^*) = \frac{1}{\Pi_0}\frac{\partial \Pi}{\partial \boldsymbol{a}'}(H_1, \boldsymbol{a}^*, F_0)\sqrt{T}(\widehat{\boldsymbol{a}}_T - \boldsymbol{a}^*) + \frac{1}{\Pi_0}(\boldsymbol{a}^*)'\sqrt{T}\left[\frac{\partial \widehat{\Pi}_T}{\partial \boldsymbol{a}}(H_1, \boldsymbol{a}^*, F_0) - \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \boldsymbol{a}^*, F_0)\right] + o_p(1).
$$

Thus, the asymptotic properties of $\widehat{\boldsymbol{a}}_T, \widehat{\lambda}_T$ can be derived from the asymptotic properties of the estimated DRM sensitivities (see Proposition 1).

If the DRM is twice differentiable, we can explicit its second-order derivative by differentiating (2.5) with respect to $\boldsymbol{a}$. More precisely, we have:

$$
\begin{aligned}
\frac{\partial^2 \Pi}{\partial \boldsymbol{a}\partial \boldsymbol{a}'}(H, \boldsymbol{a}, F) &= \frac{\partial}{\partial \boldsymbol{a}'}E\left[\boldsymbol{X}\,\frac{\partial H\big(1 - G(\boldsymbol{a}'\boldsymbol{X}; \boldsymbol{a})\big)}{\partial u}\right] \\
&= -E\left[\boldsymbol{X}\frac{\partial^2 H(1 - G(\boldsymbol{a}'\boldsymbol{X}; \boldsymbol{a}))}{\partial u^2}\left\{g(\boldsymbol{a}'\boldsymbol{X}; \boldsymbol{a})\boldsymbol{X}' + \frac{\partial G(\boldsymbol{a}'\boldsymbol{X}; \boldsymbol{a})}{\partial \boldsymbol{a}'}\right\}\right] \\
&= -E\left[\boldsymbol{X}\boldsymbol{X}'\frac{\partial^2 H\big(1 - G(\boldsymbol{a}'\boldsymbol{X}; \boldsymbol{a})\big)}{\partial u^2}g(\boldsymbol{a}'\boldsymbol{X}; \boldsymbol{a})\right] + E\left[\boldsymbol{X}E\big[\boldsymbol{X}'|\boldsymbol{a}'\boldsymbol{X}\big]\frac{\partial^2 H\big(1 - G(\boldsymbol{a}'\boldsymbol{X}; \boldsymbol{a})\big)}{\partial u^2}g(\boldsymbol{a}'\boldsymbol{X}; \boldsymbol{a})\right],
\end{aligned}
$$

since $\partial G(Y; \boldsymbol{a})/\partial \boldsymbol{a} = -E[\boldsymbol{X}|Y]g(Y; \boldsymbol{a})$. Then, by applying the iterated expectation theorem, we get,

$$
\frac{\partial^2 \Pi}{\partial \boldsymbol{a}\partial \boldsymbol{a}'}(H, \boldsymbol{a}, F) = -E\left[V(\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X})\frac{\partial^2 H\big(1 - G(\boldsymbol{a}'\boldsymbol{X}; \boldsymbol{a})\big)}{\partial u^2}g(\boldsymbol{a}'\boldsymbol{X}; \boldsymbol{a})\right]. \tag{3.12}
$$

Since $V(\boldsymbol{a}'\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}) = 0$, we see that $\boldsymbol{a}'\frac{\partial^2 \Pi}{\partial \boldsymbol{a}\partial \boldsymbol{a}'}(H, \boldsymbol{a}, F)\boldsymbol{a} = 0$, and the Hessian of the DRM has a diminished rank $d-1$. This is a consequence of the homogeneity property of the DRM. If the distortion function is concave, $\frac{\partial^2 H}{\partial u^2}$ is negative and the matrix $\frac{\partial^2 \Pi}{\partial \boldsymbol{a}\partial \boldsymbol{a}'}$ is positive semi-definite. This provides a proof of the result by Denneberg (1994) mentioned above.

4 Efficiency test of a given portfolio

The asymptotic expressions of the delta-DRM estimators and the estimated efficient allocations can be used to construct different tests of the efficiency of a given portfolio $\boldsymbol{a}_0$, say. As usual three types of test statistics can be considered by analogy with the Wald, Lagrange Multiplier and Likelihood Ratio tests introduced in the statistical literature. For expository purpose, we provide the results associated with the empirical estimators of the delta-DRMs. A similar analysis can be performed for the kernel estimator.

4.1 The null hypothesis

Let us consider a given allocation in risky assets $\boldsymbol{a}_0$, say. The hypothesis to be tested is the DRM-DRM efficiency of $\boldsymbol{a}_0$ for given risk measures, that are given distortion functions H_0, H_1. If $\boldsymbol{a}^*(H_0, H_1, \Pi_0, F_0)$ denotes the efficient allocation associated with constraint level Π_0, the null hypothesis can be written as:

$$\begin{aligned}\mathcal{H}_0 &= \Big\{\exists \Pi_0 \text{ s.t.: } \boldsymbol{a}^*(H_0, H_1, \Pi_0, F_0) = \boldsymbol{a}_0\Big\} \\ &= \Big\{\boldsymbol{a}^*(H_0, H_1, \Pi_0, F_0) \text{ and } \boldsymbol{a}_0 \text{ are proportional}\Big\}.\end{aligned} \tag{4.1}$$

Alternatively, the efficiency hypothesis can also be defined from the extended Sharpe ratio by:

$$\mathcal{H}_0 = \Big\{\boldsymbol{a}_0 \;\text{ optimizes } \lambda(H_0, H_1, \boldsymbol{a}, F_0)\Big\}. \tag{4.2}$$

The null hypothesis involves $d-1$ independent restrictions on the true distribution F_0, and we can expect test procedures with $d-1$ degree of freedom.

4.2 The constrained and unconstrained estimators

The general results of Section 3.2, 3.3 cannot be used directly. Indeed, the estimator $\widehat{\boldsymbol{a}}_T = \widehat{\boldsymbol{a}}_T(H_0, H_1, \Pi_0)$ assumes a known constraint level Π_0. This is not the case when considering the tests for efficiency. However, we can define the realized constrained level under the null:

$$\widehat{\Pi}_{0T} = \Pi(H_0, \boldsymbol{a}_0, \widehat{F}_T), \tag{4.3}$$

and look for the efficient allocation and Lagrange multiplier corresponding to this level. The unconstrained estimators are denoted by $\widehat{\widehat{\boldsymbol{a}}}_T, \widehat{\widehat{\lambda}}_T$ and satisfy the first-order conditions:

$$\left\{\begin{aligned} &\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \widehat{\widehat{\boldsymbol{a}}}_T, \widehat{F}_T) - \frac{\Pi(H_1, \widehat{\widehat{\boldsymbol{a}}}_T, \widehat{F}_T)}{\widehat{\Pi}_{0T}} \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \widehat{\widehat{\boldsymbol{a}}}_T, \widehat{F}_T) = 0, \\ &\qquad\qquad\qquad\qquad \widehat{\widehat{\lambda}}_T = \frac{\Pi(H_1, \widehat{\widehat{\boldsymbol{a}}}_T, \widehat{F}_T)}{\widehat{\Pi}_{0T}}. \end{aligned}\right. \tag{4.4}$$

The estimators constrained by the null hypothesis are $\widehat{\widehat{\boldsymbol{a}}}_{0T} = \widehat{\mu}_T \boldsymbol{a}_0$ and $\widehat{\widehat{\lambda}}_{0T}$, say. The estimator of the constrained allocation satisfies the first-order condition:

$$\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \widehat{\mu}_T \boldsymbol{a}_0, \widehat{F}_T) - \frac{\Pi(H_1, \widehat{\mu}_T \boldsymbol{a}_0, \widehat{F}_T)}{\Pi(H_0, \boldsymbol{a}_0, \widehat{F}_T)} \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \widehat{\mu}_T \boldsymbol{a}_0, \widehat{F}_T) = 0.$$

By using the homogeneity property of the DRM, we get:

$$\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1, \boldsymbol{a}_0, \widehat{F}_T) - \widehat{\mu}_T \frac{\Pi(H_1, \boldsymbol{a}_0, \widehat{F}_T)}{\Pi(H_0, \boldsymbol{a}_0, \widehat{F}_T)} \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}_0, \widehat{F}_T) = 0.$$

Then, pre-multiplying the system by $\boldsymbol{a}_0'$ and using the Euler condition, we get: $\widehat{\mu}_T = 1$. We deduce that the constrained estimators are:

$$\widehat{\widehat{\boldsymbol{a}}}_{0T} = \boldsymbol{a}_0, \quad \widehat{\widehat{\lambda}}_{0T} = \frac{\Pi(H_1, \boldsymbol{a}_0, \widehat{F}_T)}{\Pi(H_0, \boldsymbol{a}_0, \widehat{F}_T)},$$

where $\widehat{\widehat{\lambda}}_{0T}$ is the realized extended Sharpe (RES) ratio of portfolio $\boldsymbol{a}_0$.

4.3 Asymptotic expansion of the unconstrained estimated allocation

The asymptotic expansion of the difference between the unconstrained and constrained efficient allocations $\widehat{\widehat{\boldsymbol{a}}}_T - \boldsymbol{a}_0$ is derived in Appendix E.

Proposition 3. *We get:*

$$\begin{aligned}
\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T - \boldsymbol{a}_0) &= B^{-1} A \sqrt{T} \left[\frac{\partial \widehat{\Pi}_T}{\partial \boldsymbol{a}}(H_1 - \lambda_0 H_0, \boldsymbol{a}_0, F_0) - \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1 - \lambda_0 H_0, \boldsymbol{a}_0, F_0) \right] + o_p(1), \\
&= B^{-1} \sqrt{T} \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1 - \widehat{\widehat{\lambda}}_{0T} H_0, \boldsymbol{a}_0, \widehat{F}_T) + o_p(1),
\end{aligned}$$

where

$$\begin{aligned}
A &= Id - \frac{1}{\Pi(H_0, \boldsymbol{a}_0, F_0)} \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}_0, F_0) \boldsymbol{a}_0', \\
B &= -\frac{\partial^2 \Pi}{\partial \boldsymbol{a} \partial \boldsymbol{a}'}(H_1 - \lambda_0 H_0, \boldsymbol{a}_0, F_0) + \frac{1}{\Pi(H_0, \boldsymbol{a}_0, F_0)} \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}_0, F_0) \frac{\partial \Pi}{\partial \boldsymbol{a}'}(H_1, \boldsymbol{a}_0, F_0).
\end{aligned}$$

The matrix A is such that:

$$\begin{aligned}
\boldsymbol{a}_0' A &= \boldsymbol{a}_0' \left(Id - \frac{1}{\Pi(H_0, \boldsymbol{a}_0, F_0)} \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}_0, F_0) \boldsymbol{a}_0' \right) \\
&= \boldsymbol{a}_0' - \frac{\boldsymbol{a}_0' \frac{\partial \Pi}{\partial \boldsymbol{a}}(H_0, \boldsymbol{a}_0, F_0)}{\Pi(H_0, \boldsymbol{a}_0, F_0)} \boldsymbol{a}_0' \\
&= \boldsymbol{a}_0' - \boldsymbol{a}_0' = 0, \quad \text{by the Euler condition.}
\end{aligned}$$

Thus, matrix A has rank $d - 1$. In particular, if Σ denotes the asymptotic variance-covariance matrix of $\sqrt{T}\left[\frac{\partial\widehat{\Pi}_T}{\partial \boldsymbol{a}}(H_1-\lambda_0 H_0,\boldsymbol{a}_0,F_0)-\frac{\partial \Pi}{\partial \boldsymbol{a}}(H_1-\lambda_0 H_0,\boldsymbol{a}_0,F_0)\right]$, we have asymptotically

$$\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T-\boldsymbol{a}_0)\xrightarrow{d} N\big(0,B^{-1}A\Sigma A'(B')^{-1}\big). \tag{4.5}$$

The variance-covariance matrix of $\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T-\boldsymbol{a}_0)$ has also a diminished rank $d-1$, since the difference between estimated efficient allocations satisfies the restriction $\frac{\partial\Pi}{\partial \boldsymbol{a}'}(H_0,\boldsymbol{a}_0,F_0)\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T-\boldsymbol{a}_0)=o_p(1)$ due to the DRM constraint.

4.4 The test statistics

The test statistics for the efficiency hypothesis are the following.

i) **The Wald statistic**

The Wald statistic is based on a comparison of the estimated efficient allocation with the given allocation $\boldsymbol{a}_0$. It is defined as:

$$\xi_W=\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T-\boldsymbol{a}_0)'\left[\widehat{V}(\widehat{\widehat{\boldsymbol{a}}}_T)\right]^{-}\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T-\boldsymbol{a}_0), \tag{4.6}$$

where $\big[\widehat{V}(\widehat{\widehat{\boldsymbol{a}}}_T)\big]^{-}=\widehat{B}'\big(\widehat{A}\widehat{\Sigma}\widehat{A}'\big)^{-}\widehat{B}$ is a consistent estimator of the asymptotic variance-covariance matrix of $\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T-\boldsymbol{a}_0)$, and $[\cdot]^{-}$ denotes a generalized inverse. Indeed, we know that $V(\widehat{\widehat{\boldsymbol{a}}}_T)$ has the reduced rank $d-1$.

ii) **The delta-DRM statistic**

The delta-DRM statistics is given by:

$$\xi_{delta-DRM}=T\frac{\partial\Pi}{\partial \boldsymbol{a}'}\left(H_1-\widehat{\widehat{\lambda}}_{0T}H_0,\boldsymbol{a}_0,\widehat{F}_T\right)\left(\widehat{A}\widehat{\Sigma}\widehat{A}'\right)^{-}\frac{\partial\Pi}{\partial \boldsymbol{a}}\left(H_1-\widehat{\widehat{\lambda}}_{0T}H_0,\boldsymbol{a}_0,\widehat{F}_T\right). \tag{4.7}$$

From Proposition 3, we deduce the result below.

Proposition 4. *Under the null hypothesis of efficiency, the Wald and delta-DRM statistics are asymptotically equivalent and follow asymptotically a chi-square distribution with $d-1$ degree of freedom.*

iii) **The RES based statistic**

Finally, let us consider the statistic

$$\xi_\lambda=T\left(\widehat{\widehat{\lambda}}_T-\widehat{\widehat{\lambda}}_{0,T}\right), \tag{4.8}$$

based on the comparison of RES ratios. This is the analogue of the standard likelihood ratio test encountered in maximum likelihood theory. Under the null hypothesis, we have:

$$\begin{aligned}
T\left(\widehat{\widehat{\lambda}}_T - \widehat{\widehat{\lambda}}_{0T}\right) &= T\left[\lambda(H_0, H_1, \widehat{\widehat{\boldsymbol{a}}}_T, \widehat{F}_T) - \lambda(H_0, H_1, \boldsymbol{a}_0, \widehat{F}_T)\right] \\
&= \sqrt{T}\left[\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T - \boldsymbol{a}_0)\right]' \frac{\partial \lambda}{\partial \boldsymbol{a}}(H_0, H_1, \boldsymbol{a}_0, \widehat{F}_T) + o_p(1) \\
&= \sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T - \boldsymbol{a}_0)' \frac{\partial^2 \lambda}{\partial \boldsymbol{a} \partial F'}(H_0, H_1, \boldsymbol{a}_0, F_0)\sqrt{T}(\widehat{F}_T - F_0) + o_p(1),
\end{aligned} \tag{4.9}$$

since under the null $\frac{\partial \lambda}{\partial \boldsymbol{a}}(H_0, H_1, \boldsymbol{a}_0, F_0) = 0$.

Moreover, the unconstrained efficient allocation $\widehat{\widehat{\boldsymbol{a}}}_T$ satisfies the first-order conditions:

$$\frac{\partial \lambda}{\partial \boldsymbol{a}}(H_0, H_1, \widehat{\widehat{\boldsymbol{a}}}_T, \widehat{F}_T) = 0;$$

these conditions can be expanded to get:

$$\frac{\partial^2 \lambda}{\partial \boldsymbol{a} \partial \boldsymbol{a}'}(H_0, H_1, \boldsymbol{a}_0, F_0)\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T - \boldsymbol{a}_0) + \frac{\partial^2 \lambda}{\partial \boldsymbol{a} \partial F'}(H_0, H_1, \boldsymbol{a}_0, F_0)\sqrt{T}(\widehat{F}_T - F_0) + o_p(1) = 0.$$

By substituting in (4.9), we get the following proposition.

Proposition 5. *Under the null hypothesis of efficiency,*

$$\begin{aligned}
T\left(\widehat{\widehat{\lambda}}_T - \widehat{\widehat{\lambda}}_{0T}\right) &= \sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T - \boldsymbol{a}_0)'\left[-\frac{\partial^2 \lambda}{\partial \boldsymbol{a} \partial \boldsymbol{a}'}(H_0, H_1, \boldsymbol{a}_0, F_0)\right]\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T - \boldsymbol{a}_0) + o_p(1) \\
&= \sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T - \boldsymbol{a}_0)'\left[-\frac{\partial^2 \Pi}{\partial \boldsymbol{a} \partial \boldsymbol{a}'}(H_1 - \lambda_0 H_0, \boldsymbol{a}_0, F_0)\right]\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T - \boldsymbol{a}_0) + o_p(1).
\end{aligned}$$

In general $-\frac{\partial^2 \Pi}{\partial \boldsymbol{a} \partial \boldsymbol{a}'}(H_1 - \lambda_0 H_0, \boldsymbol{a}_0, F_0)$ is not a generalized inverse of $V(\widehat{\widehat{\boldsymbol{a}}}_T)$, and the RES based statistic is not equivalent to the two other statistics. This result is compatible with the general theory of asymptotic tests when the objective function cannot be interpreted as a log-likelihood function (see Gourieroux and Monfort, 1995, Vol. 2, Chapter 18).

5 Finite sample properties

We will now discuss the finite sample properties of the empirical estimators of the efficient allocation and extended Sharpe ratio for a TVaR-VaR optimization problem. By considering Gaussian returns in the Monte-Carlo, we will also compare the nonparametric estimators with the parametric estimators computed using the Gaussian assumption. Let us first describe the DRM-DRM problem in a Gaussian framework.

5.1 The DRM-DRM problem in a Gaussian framework

Let us assume Gaussian (opposite) excess returns: $\boldsymbol{X} \sim N(-\boldsymbol{m}, \Omega)$. The quantile function associated with the (negative) portfolio return $y(\boldsymbol{a}) = \boldsymbol{a}'\boldsymbol{x}$ is given by:

$$Q(u; \boldsymbol{a}) = -\boldsymbol{a}'\boldsymbol{m} + \Phi^{-1}(u)(\boldsymbol{a}'\Omega\boldsymbol{a})^{1/2}. \tag{5.1}$$

We deduce the expression of a DRM in the Gaussian framework:

$$\begin{aligned}\Pi(H, \boldsymbol{a}, \Phi) &= \int_0^1 \left[-\boldsymbol{a}'\boldsymbol{m} + \Phi^{-1}(u)(\boldsymbol{a}'\Omega\boldsymbol{a})^{1/2}\right] dH(u) \\ &= -\boldsymbol{a}'\boldsymbol{m} + (\boldsymbol{a}'\Omega\boldsymbol{a})^{1/2}\beta(H),\end{aligned} \tag{5.2}$$

where $\beta(H) = \int_0^1 \Phi^{-1}(1-u)dH(u)$. Thus, the DRM is an affine combination of the expected (negative) portfolio return and its standard error.

Let us now consider a DRM-DRM problem such as:

$$\min_{\boldsymbol{a}} \ -\boldsymbol{a}'\boldsymbol{m} + (\boldsymbol{a}'\Omega\boldsymbol{a})^{1/2}\beta(H_1) \text{ s.t. } \ -\boldsymbol{a}'\boldsymbol{m} + (\boldsymbol{a}'\Omega\boldsymbol{a})^{1/2}\beta(H_0) = \Pi_0. \tag{5.3}$$

The first-order conditions imply that the efficient allocation is proportional to the efficient mean-variance allocation $\boldsymbol{a}_0^* = \Omega^{-1}\boldsymbol{m}$. In particular, it is easily checked that the set of efficient allocations is given by:

$$\boldsymbol{a}^* = \Omega^{-1}\boldsymbol{m}\frac{\Pi_0}{\boldsymbol{m}'\Omega^{-1}\boldsymbol{m}}\left[\frac{\left(\boldsymbol{m}'\Omega^{-1}\boldsymbol{m}\right)^{1/2}}{-\left(\boldsymbol{m}'\Omega^{-1}\boldsymbol{m}\right)^{1/2} + \beta(H_0)}\right],$$

which is generated by the vector $\boldsymbol{a}_0^*$ and only depend on the choice of (Π_0, H_0).[10] We can note that $\boldsymbol{a}^*$ is equivalent to mean-variance efficient allocation when the expected return is equal to $\Pi_0 \left(\boldsymbol{m}'\Omega^{-1}\boldsymbol{m}\right)^{1/2} / \left[-\left(\boldsymbol{m}'\Omega^{-1}\boldsymbol{m}\right)^{1/2} + \beta(H_0)\right]$. The associated extended Sharpe ratio is:

$$\lambda^* = \frac{-\boldsymbol{a}^{*'}\boldsymbol{m} + (\boldsymbol{a}^{*'}\Omega\boldsymbol{a}^*)^{1/2}\beta(H_1)}{-\boldsymbol{a}^{*'}\boldsymbol{m} + (\boldsymbol{a}^{*'}\Omega\boldsymbol{a}^*)^{1/2}\beta(H_0)} = \frac{-\left(\boldsymbol{m}'\Omega^{-1}\boldsymbol{m}\right)^{1/2} + \beta(H_1)}{-\left(\boldsymbol{m}'\Omega^{-1}\boldsymbol{m}\right)^{1/2} + \beta(H_0)}. \tag{5.4}$$

The extended Sharpe ratio is in a one-to-one relationship with the standard Sharpe ratio, that is $\boldsymbol{m}'\Omega^{-1}\boldsymbol{m}$, but depends on both selected distortion measures.

5.2 The Monte-Carlo study

Let us consider two risky assets with excess returns independently drawn in a Gaussian distribution with parameters $\boldsymbol{m} = (0.00044, 0.007)'$, $\Omega = \begin{pmatrix} 0.0031 & 0.0028 \\ 0.0028 & 0.0064 \end{pmatrix}$. These parameters are estimated using the same data set as in Section 2.4.[11] The corresponding mean-variance efficient portfolio is

[10] As long as the convexity of the Lagrangian objective function is ensured.

[11] The returns are multiplied by 10 before calculating these parameters. This is to ensure the numerical invertibility of the variance-covariance matrix from the simulated data.

$\boldsymbol{a}_0^* = (-1.415, 1.722)'$, and can be normalized as $\widetilde{\boldsymbol{a}}_0 = (-4.6, 5.6)'$ to get a sum of the components equal to 1.

Then, we introduce a TVaR-VaR problem for risk level $p = 5\%, 10\%$, respectively. For a given risk level and a given set of $T = 501$ return observations, we compute:

i) the empirical estimate of the efficient allocation normalized by $\widehat{\widehat{a}}_T^1 + \widehat{\widehat{a}}_T^2 = 1$;

ii) the empirical estimate of the extended Sharpe ratio;

iii) the parametric estimate of the normalized efficient allocation $\widehat{\Omega}_T^{-1}\widehat{\boldsymbol{m}}_T / \left(\iota'\widehat{\Omega}_T^{-1}\widehat{\boldsymbol{m}}_T\right)$, where $\widehat{\boldsymbol{m}}_T$ and $\widehat{\Omega}_T$ are the realized mean and volatility, ι is a $d \times 1$ vector of ones;

iv) the parametric estimate of the standard Sharpe ratio $\widehat{\boldsymbol{m}}_T'\widehat{\Omega}_T^{-1}\widehat{\boldsymbol{m}}_T'$;

v) the implied Sharpe ratio from the empirical estimate of extended Sharpe ratio

$$\widetilde{\lambda}_T = \left(\frac{\widehat{\widehat{\lambda}}_T\beta(H_0) - \beta(H_1)}{\widehat{\widehat{\lambda}}_T - 1}\right)^2.$$

These computations are replicated $S = 400$ times to get the finite sample distributions.

The finite sample distributions of the normalized estimated allocation in asset 1 $(a^1)^*$ are provided in Figure 3. The solid line corresponds to the parametric estimator, the dashed line (resp. shorter dashed line) to the empirical estimator when $p = 0.05$ (resp. $p = 0.1$) and the vertical (thick) line indicates the true efficient allocation $\widetilde{a}_0^1$. These curves are produced for the range $[-100, 100]$, but it is important to note that more simulated values for the parametric estimator are outside this range than those for the empirical estimators. This is a consequence of the lack of robustness of the parametric mean-variance approach to extreme returns. It is observed that all estimators feature bias in finite sample. The bias is larger for the maximum likelihood parametric estimator than for the empirical estimator when $p = 0.05$, and that for the empirical estimator when $p = 0.1$. The average biases are 7, 6.26 and 5.46, respectively. The order of their finite sample variances are the same as that of their biases. The parametric estimator can be very noisy (1656.32). Between the empirical estimators, the one with $p = 0.05$ has slightly larger variance than that with $p = 0.1$ (139.02 v.s. 117.76).

The distributions of implied Sharpe ratios are plotted in Figure 4. The vertical (thicker) line represents the true value (i.e. 0.107). The parametric estimator (solid line) exhibits the smallest bias (the bias is 0.013) and finite sample variance (0.002). As expected, due the shortage of observations, the empirical estimator with $p = 0.05$ (dashed line) is less accurate as compared to the one with $p = 0.1$ (shorter dashed line), in the sense that it has a larger bias (the average biases of the implied Sharpe ratio are 0.219 and 0.13, respectively) and a larger variance (0.094 v.s. 0.038).

6 Conclusion

In this paper, we first introduce two nonparametric estimators of the delta-DRM, defined as the sensitivity of DRM with respect to portfolio allocations, and derive their asymptotic distributions. Then, these estimators are used to calculate the efficient portfolio allocations assuming a DRM objective and a DRM constraint. We show that the limiting behaviors of the estimators of efficient allocations only depend on the asymptotic properties of the delta-DRM. Three test statistics are proposed to test the efficiency hypothesis for a given portfolio. These test statistics are analogous to the standard Wald, LM and LR test statistics in the Maximum Likelihood context. Finally, a Monte-Carlo study is implemented to compare the finite sample properties of the nonparametric estimators and the parametric estimators in a Gaussian framework. We find that, when estimating the efficient allocation in finite sample, the empirical estimator may be preferred to its parametric counterpart even though the parametric model is well specified. While considering the implied Sharpe ratio, the parametric estimator dominates. Of course, the advantage of the empirical estimator is its consistency even if the parametric model is misspecified.

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Appendices

Appendix A Preliminary lemma

Lemma 1 (Scaillet (2004), Proof of Proposition 3.1)**.** *Let us consider a d-dimensional random vector $\boldsymbol{X}$ with a continuous joint distribution F. For any given $d \times 1$ real vector $\boldsymbol{a}$, any continuous function $\Psi : \mathbb{R} \to \mathbb{R}$, any continuous function φ: $\mathbb{R}^d \to \mathbb{R}$, any value ξ and any kernel k such that $\int k(u)d\,u = 1$ and $\int u\,k(u)d\,u = 0$, we have, as $T \to \infty$:*

(i).

$$\begin{aligned}\frac{1}{h_T}\int_{\mathbb{R}} \boldsymbol{x}k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-y}{h_T}\right)\Psi(y)d\,y &= -\boldsymbol{x}\Psi(\boldsymbol{a}'\boldsymbol{x}) + O(h_T^2),\\ \frac{1}{h_T}\int_{\mathbb{R}} E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=y]k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-y}{h_T}\right)\Psi(y)d\,y &= -E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=\boldsymbol{a}'\boldsymbol{x}]\Psi(\boldsymbol{a}'\boldsymbol{x}) + O(h_T^2);\end{aligned}$$

(ii).

$$\begin{aligned}\frac{1}{h_T}E\left[\varphi(\boldsymbol{X})k\left(\frac{\boldsymbol{a}'\boldsymbol{X}-\xi}{h_T}\right)\right] &= E\left[\varphi(\boldsymbol{X})|\boldsymbol{a}'\boldsymbol{X}=\xi\right]g(\xi;\boldsymbol{a}) + O(h_T^2),\\ \frac{1}{h_T}E\left[\left(\varphi(\boldsymbol{X})k\left(\frac{\boldsymbol{a}'\boldsymbol{X}-\xi}{h_T}\right)\right)^2\right] &= E\left[\varphi(\boldsymbol{X})\varphi(\boldsymbol{X})'|\boldsymbol{a}'\boldsymbol{X}=\xi\right]g(\xi;\boldsymbol{a})\int k(u)^2 d\,u + O(h_T),\end{aligned}$$

where h_T is the bandwidth satisfying $h_T \to 0$ and $T\,h_T \to \infty$ as $T \to \infty$, and $g(\cdot;\boldsymbol{a})$ is the pdf of $\boldsymbol{a}'\boldsymbol{X}$.

Appendix B Comovement interpretation of the delta-DRM

Proof. If H is continuous and differentiable in u, the ith right hand side of equation (2.4) can be written as:

$$RHS^i = \int_0^1 \frac{\int_{\mathbb{R}^{d-1}} \frac{x^i}{a^j} f\left(x^1,\ldots,\frac{Q(1-u;\boldsymbol{a})-\sum_{l\neq j}a^l x^l}{a^j},\ldots,x^d\right)\prod_{l\neq j}dx^l}{g(Q(1-u;\boldsymbol{a});\boldsymbol{a})}\frac{\partial H(u)}{\partial u}d\,u.$$

By applying the change of variables $y = Q(1-u;\boldsymbol{a})$, this expression becomes:

$$\begin{aligned}RHS^i &= \int_{\mathbb{R}} \frac{\int_{\mathbb{R}^{d-1}} \frac{x^i}{a^j} f\left(x^1,\ldots,\frac{y-\sum_{l\neq j}a^l x^l}{a^j},\ldots,x^d\right)\prod_{l\neq j}dx^l}{g(y;\boldsymbol{a})}\frac{\partial H(1-(G(y;\boldsymbol{a}))}{\partial u}d\left(1-G(y;\boldsymbol{a})\right)\\ &= \int_{\mathbb{R}^d} \frac{x^i}{a^j} f\left(x^1,\ldots,\frac{y-\sum_{l\neq j}a^l x^l}{a^j},\ldots,x^d\right)\frac{\partial H(1-(G(y;\boldsymbol{a}))}{\partial u}\prod_{l\neq j}dx^l d\,y.\end{aligned}$$

Since $\frac{1}{a^j} f\left(x^1, \ldots, \frac{y-\sum_{l\neq j} a^l x^l}{a^j}, \ldots, x^d\right)$ is the joint density of distribution of $(X^l,\, l \neq j)$ and $\boldsymbol{a}'\boldsymbol{X}$, we deduce the result. $\square$

Appendix C Asymptotic expansions of the estimators

C.1 Regularity conditions

A sufficient set of regularity conditions for deriving the asymptotic expansions of the random estimators and applying functional limit theorems is given below.

1. Conditions on the return process and portfolio allocation:

   Assumption A. 1. *The process $(\boldsymbol{x}_t)$ is a strong white noise, that is the $\boldsymbol{x}_t$'s are i.i.d. random vectors.*

   Assumption A. 2. *The distribution of $(\boldsymbol{X})$ is continuous, with a continuous strictly positive density.*

   Assumption A. 3. $E\left[||\boldsymbol{X}||^{\gamma}\right] < \infty$ *for some* $\gamma > 0$.

   Assumption A. 4. *The true portfolio allocation belongs to a bounded set* $A = \prod_{i=1}^{d}\left(\underline{a}^i, \overline{a}^i\right)$, *say.*

   Assumption A.2 avoids nonstandard behavior, when the distribution features some point masses (see e.g. Laurent, 2003 for a discussion of this problem), and Assumption A.3 imposes a uniform tail behavior for the portfolio returns. The value of γ has to be sufficiently large to give a meaning to the DRM of interest.

   Assumption A. 5. *The distribution of $\boldsymbol{X}$ given $\boldsymbol{a}'\boldsymbol{X} = y$ is continuous on $\{\boldsymbol{a}'\boldsymbol{X} = y\}$ with a continuous strictly positive density. The conditional moments $E\left[||\boldsymbol{X}||^{\gamma} \middle| \boldsymbol{a}'\boldsymbol{X} = y\right]$ exist for any y and any $\boldsymbol{a} \in A$.*

2. Conditions on the distortion function:

   Assumption A. 6 (Smoothness)**.** *The distortion function H is increasing on* $(0, 1]$, *twice continuously differentiable.*

   Assumption A. 7 (Boundedness)**.** *[see Shorack (1972), or Shorack and Wellner (1986), Chapter 19] We have* $\left|\frac{\partial H(u)}{\partial u}\right| \leq c u^{-\alpha}(1-u)^{-\beta}$, *for* $\alpha < 1/2 - 1/\gamma$, $\beta < 1/2 - 1/\gamma$, $\gamma > 0$ *and* $c < \infty$.

   Assumption A.7 ensures that the weighting function do not attribute too much weight on extreme risks. It is used jointly with Assumption A.3 on the tail behavior of the return distribution.

Assumption A. 8. *The discretized distortion function (resp. first, second order derivative of the distortion function) tends uniformly to the true distortion function (resp first, second order derivative of the distortion function).*

This assumption is introduced to get a negligible discretization error.

3. Conditions on the kernel:

Assumption A. 9. *[see Fermanian and Scaillet (2005)] k is a strictly symmetric Parzen kernel of order 2 on $\mathbb{R}$ such that $\lim_{u\to\infty} uk(u) = 0$, $\int |u|k(u)du < \infty$, $\int k(u)du = 1$ and $\int uk(u)du = 0$. k is three times differentiable, k' and k'' are integrable and k''' is bounded.*

Assumption A.9 is satisfied by standard kernels such as the Gaussian kernel. It is not satisfied by some optimal kernels, such as the Epanechnikov kernel.

4. Conditions on the bandwidth:

Assumption A. 10. *$Th_T^5 \to 0$ and $Th_T^{7/2} \to \infty$ when T tends to infinity.*

Assumption A.10 removes the bias from the kernel estimator (see Yatchew (2003)).

Under this set of conditions, we can apply the multivariate Functional Central Limit Theorem to the sample cdf.

Multivariate Functional Limit Theorem. *Let $\boldsymbol{x}_1, ..., \boldsymbol{x}_T$ be i.i.d. random observations in $\mathbb{R}^d$ with continuous marginal cdfs $F^j(x^j)$ with $j = 1, ..., d$, $\boldsymbol{u} = (u^1, \ldots, u^d) = \big(F^1(x^1), ..., F^d(x^d)\big)$ a $1 \times d$ vector of uniformly distributed ranks and $\widetilde{F}_T(\boldsymbol{x})$ its joint sample cdf, we have:*

$$\sqrt{T}\big(\widetilde{F}_T(\boldsymbol{x}) - F(\boldsymbol{x})\big) \Rightarrow \mathcal{K}(\boldsymbol{u}), \tag{C-1}$$

where $\mathcal{K}$ is a multivariate Brownian bridge on $[0,1]$, which is Gaussian with zero mean and covariance,

$$COV\big(\mathcal{K}(\boldsymbol{u}), \mathcal{K}(\boldsymbol{u}')\big) = C(\boldsymbol{u} \wedge \boldsymbol{u}') - C\big(\boldsymbol{u}\big)C\big(\boldsymbol{u}'\big), \tag{C-2}$$

where $\boldsymbol{u} \wedge \boldsymbol{u}' = \{\min(u^1, (u')^1), \ldots, \min(u^d, (u')^d)\}$.

Note that, for any real function Ψ such that $E[\Psi(\boldsymbol{X})^2] < \infty$, we have:

$$E\left[\int \Psi(\boldsymbol{x}) d\mathcal{K}\Big(F^1(x^1), \ldots, F^d(x^d)\Big)\right] = 0,$$
$$E\left[\left(\int \Psi(\boldsymbol{x}) d\mathcal{K}\Big(F^1(x^1), \ldots, F^d(x^d)\Big)\right)^2\right] = V\big(\Psi(\boldsymbol{x})\big).$$

C.2 Kernel estimator

Let us consider the expansion of the kernel estimator of the delta-VaR (2.6). We have:

$$
\begin{aligned}
&\sqrt{Th_T}\left(\frac{\partial\widetilde{Q}_T}{\partial \boldsymbol{a}}(\,\cdot\,;\boldsymbol{a})-\frac{\partial Q}{\partial \boldsymbol{a}}(\,\cdot\,;\boldsymbol{a})\right)\\
&=\sqrt{Th_T}\left(\frac{\partial\widetilde{Q}_T}{\partial \boldsymbol{a}}(\,\cdot\,;\boldsymbol{a})-E\big[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=Q(\cdot\,;\boldsymbol{a})\big]\right)\\
&=\sqrt{Th_T}\left(\frac{\frac{1}{Th_T}\sum_{t=1}^T \boldsymbol{x}_t\,k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t-\widehat{Q}_T(\cdot;\boldsymbol{a})}{h_T}\right)}{\frac{1}{Th_T}\sum_{t=1}^T k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t-\widehat{Q}_T(\cdot;\boldsymbol{a})}{h_T}\right)}-\frac{\frac{1}{Th_T}\sum_{t=1}^T \boldsymbol{x}_t\,k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t-Q(\cdot;\boldsymbol{a})}{h_T}\right)}{\frac{1}{Th_T}\sum_{t=1}^T k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t-Q(\cdot;\boldsymbol{a})}{h_T}\right)}\right)\\
&\quad+\sqrt{Th_T}\left(\frac{\frac{1}{Th_T}\sum_{t=1}^T \boldsymbol{x}_t\,k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t-Q(\cdot;\boldsymbol{a})}{h_T}\right)}{\frac{1}{Th_T}\sum_{t=1}^T k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t-Q(\cdot;\boldsymbol{a})}{h_T}\right)}-\frac{E\left[\boldsymbol{X}\mathbb{1}_{\left(\boldsymbol{a}'\boldsymbol{X}=Q(\cdot\,;\boldsymbol{a})\right)}\right]}{g\big(Q(\cdot;\boldsymbol{a});\boldsymbol{a}\big)}\right)\\
&=\nabla\mathcal{K}_T\big(Q(\cdot;\boldsymbol{a});\boldsymbol{a}\big)\sqrt{Th_T}\big(\widehat{Q}_T(\cdot;\boldsymbol{a})-Q(\cdot;\boldsymbol{a})\big)+\frac{1}{g\big(Q(\cdot;\boldsymbol{a});\boldsymbol{a}\big)}\frac{1}{\sqrt{h_T}}\int_{\mathbb{R}^d}\boldsymbol{x}k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-Q(\cdot\,;\boldsymbol{a})}{h_T}\right)\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x})-F(\boldsymbol{x})\right]\\
&\quad-\frac{E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=Q(\cdot\,;\boldsymbol{a})]}{g\big(Q(\cdot\,;\boldsymbol{a});\boldsymbol{a}\big)}\frac{1}{\sqrt{h_T}}\int_{\mathbb{R}^d}k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-Q(\cdot\,;\boldsymbol{a})}{h_T}\right)\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x})-F(\boldsymbol{x})\right]+o_p(\sqrt{h_T})
\end{aligned}
$$

where:

$$
\nabla\mathcal{K}_T\big(Q(\cdot;\boldsymbol{a});\boldsymbol{a}\big)=\frac{\partial}{\partial Q(\cdot;\boldsymbol{a})}\left(\frac{\frac{1}{Th_T}\sum_{t=1}^T \boldsymbol{x}_t\,k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t-Q(\cdot;\boldsymbol{a})}{h_T}\right)}{\frac{1}{Th_T}\sum_{t=1}^T k\left(\frac{\boldsymbol{a}'\boldsymbol{x}_t-Q(\cdot;\boldsymbol{a})}{h_T}\right)}\right),
$$

and $\widehat{F}_T(\boldsymbol{x})=\frac{1}{T}\sum_{t=1}^T\mathbb{1}_{(\boldsymbol{x}_t\leq\boldsymbol{x})}$ denotes the sample multivariate cdf and we use the standard expansion for the Nadaraya-Watson estimator. Thus, we get:

$$
\begin{aligned}
\sqrt{Th_T}\left(\frac{\partial\widetilde{Q}_T}{\partial \boldsymbol{a}}(\,\cdot\,;\boldsymbol{a})-\frac{\partial Q}{\partial \boldsymbol{a}}(\,\cdot\,;\boldsymbol{a})\right)&=\frac{1}{g\big(Q(\cdot\,;\boldsymbol{a});\boldsymbol{a}\big)}\frac{1}{\sqrt{h_T}}\int_{\mathbb{R}^d}\boldsymbol{x}k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-Q(\cdot\,;\boldsymbol{a})}{h_T}\right)\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x})-F(\boldsymbol{x})\right]\\
&-\frac{E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=Q(\cdot\,;\boldsymbol{a})]}{g\big(Q(\cdot\,;\boldsymbol{a});\boldsymbol{a}\big)}\frac{1}{\sqrt{h_T}}\int_{\mathbb{R}^d}k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-Q(\cdot\,;\boldsymbol{a})}{h_T}\right)\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x})-F(\boldsymbol{x})\right]+o_p(\sqrt{h_T}).
\end{aligned}
$$

Due to its integral expression, the kernel estimator of the delta-DRM will converge at the standard parametric rate $\sqrt{T}$. Indeed, we have:

$$
\begin{aligned}
&\sqrt{T}\left(\frac{\partial\widetilde{\Pi}_T}{\partial \boldsymbol{a}}(H,\boldsymbol{a})-\frac{\partial \Pi}{\partial \boldsymbol{a}}(H,\boldsymbol{a})\right)\\
&=\int_0^1\nabla\mathcal{K}_T\big(Q(1-u;\boldsymbol{a});\boldsymbol{a}\big)\sqrt{T}\big(\widehat{Q}_T(1-u;\boldsymbol{a})-Q(1-u;\boldsymbol{a})\big)d\,H(u)\\
&\quad+\int_0^1\frac{1}{g\big(Q(1-u\,;\boldsymbol{a});\boldsymbol{a}\big)}\left\{\frac{1}{h_T}\int_{\mathbb{R}^d}\boldsymbol{x}k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-Q(1-u\,;\boldsymbol{a})}{h_T}\right)\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x})-F(\boldsymbol{x})\right]\right\}d\,H(u)\\
&\quad-\int_0^1\frac{E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=Q(1-u\,;\boldsymbol{a})]}{g\big(Q(1-u\,;\boldsymbol{a});\boldsymbol{a}\big)}\left\{\frac{1}{h_T}\int_{\mathbb{R}^d}k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-Q(1-u\,;\boldsymbol{a})}{h_T}\right)\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x})-F(\boldsymbol{x})\right]\right\}d\,H(u)+o_p(1).
\end{aligned}
$$

By a change of variables, $y = Q(1-u\,;\boldsymbol{a}) \Leftrightarrow u = 1 - G(y;\boldsymbol{a})$, we obtain:

$$
\begin{aligned}
&\sqrt{T}\left(\frac{\partial \widetilde{\Pi}_T}{\partial \boldsymbol{a}}(H,\boldsymbol{a}) - \frac{\partial \Pi}{\partial \boldsymbol{a}}(H,\boldsymbol{a})\right)\\
&= -\int_{\mathbb{R}} \nabla \mathcal{K}_T\big(y;\boldsymbol{a}\big)\sqrt{T}\big[\widehat{Q}_T(1-G(y;\boldsymbol{a});\boldsymbol{a}) - y\big]\, dH\big(1-G(y;\boldsymbol{a})\big)\\
&\quad -\int_{\mathbb{R}} \frac{1}{g\big(y;\boldsymbol{a}\big)}\left\{\frac{1}{h_T}\int_{\mathbb{R}^d} \boldsymbol{x}k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-y}{h_T}\right)\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x}) - F(\boldsymbol{x})\right]\right\} dH\big(1-G(y;\boldsymbol{a})\big)\\
&\quad +\int_{\mathbb{R}} \frac{E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=y]}{g\big(y;\boldsymbol{a}\big)}\left\{\frac{1}{h_T}\int_{\mathbb{R}^d} k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-y}{h_T}\right)\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x}) - F(\boldsymbol{x})\right]\right\} dH\big(1-G(y;\boldsymbol{a})\big) + o_p(1).
\end{aligned}
$$

When H is first-order differentiable, we get:

$$
\begin{aligned}
&\sqrt{T}\left(\frac{\partial \widetilde{\Pi}_T}{\partial \boldsymbol{a}}(H,\boldsymbol{a}) - \frac{\partial \Pi}{\partial \boldsymbol{a}}(H,\boldsymbol{a})\right)\\
&= \int_{\mathbb{R}} \nabla \mathcal{K}_T\big(y;\boldsymbol{a}\big)\sqrt{T}\big[\widehat{Q}_T(G(y;\boldsymbol{a});\boldsymbol{a}) - y\big]\frac{\partial H(1-G(y;\boldsymbol{a}))}{\partial u} g(y;\boldsymbol{a})dy\\
&\quad +\int_{\mathbb{R}} \left\{\frac{1}{h_T}\int_{\mathbb{R}^d} \boldsymbol{x}k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-y}{h_T}\right)\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x}) - F(\boldsymbol{x})\right]\right\}\frac{\partial H(1-G(y;\boldsymbol{a}))}{\partial u}dy\\
&\quad -\int_{\mathbb{R}} E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=y]\left\{\frac{1}{h_T}\int_{\mathbb{R}^d} k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-y}{h_T}\right)\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x}) - F(\boldsymbol{x})\right]\right\}\frac{\partial H(1-G(y;\boldsymbol{a}))}{\partial u}dy + o_p(1)\\
&= -\int_{\mathbb{R}} \nabla \mathcal{K}_T\big(y;\boldsymbol{a}\big)\left\{\int_{\mathbb{R}^d} \mathbb{1}_{\boldsymbol{a}'\boldsymbol{x}\le y}\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x}) - F(\boldsymbol{x})\right]\right\}\frac{\partial H(1-G(y;\boldsymbol{a}))}{\partial u}dy\\
&\quad +\int_{\mathbb{R}} \left\{\frac{1}{h_T}\int_{\mathbb{R}^d} \boldsymbol{x}k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-y}{h_T}\right)\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x}) - F(\boldsymbol{x})\right]\right\}\frac{\partial H(1-G(y;\boldsymbol{a}))}{\partial u}dy\\
&\quad -\int_{\mathbb{R}} E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=y]\left\{\frac{1}{h_T}\int_{\mathbb{R}^d} k\left(\frac{\boldsymbol{a}'\boldsymbol{x}-y}{h_T}\right)\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x}) - F(\boldsymbol{x})\right]\right\}\frac{\partial H(1-G(y;\boldsymbol{a}))}{\partial u}dy + o_p(1),
\end{aligned}
$$

where the second equality is a consequence of the Bahadur representation of the nonparametric quantile estimator:

$$
\sqrt{T}\big(\widehat{Q}_T(G(y;\boldsymbol{a});\boldsymbol{a}) - y\big) = -\frac{1}{g(y;\boldsymbol{a})}\int_{\mathbb{R}^d} \mathbb{1}_{\boldsymbol{a}'\boldsymbol{x}\le y}\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x}) - F(\boldsymbol{x})\right] + o_p(1).
$$

By commuting the integrations with respect to $\boldsymbol{x}$ and y and applying Lemma 1 i) in Appendix A, we deduce the limiting behavior of the kernel estimator:

$$
\begin{aligned}
&\sqrt{T}\left(\frac{\partial\widetilde{\Pi}_T}{\partial \boldsymbol{a}}(H,\boldsymbol{a})-\frac{\partial\Pi}{\partial \boldsymbol{a}}(H,\boldsymbol{a})\right)\\
&\Rightarrow \quad \int_{\mathbb{R}^d}\left\{\int_{\mathbb{R}}\frac{\partial}{\partial y}\big(E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=y]\big)\Psi(y;\boldsymbol{a})\mathbb{1}_{\boldsymbol{a}'\boldsymbol{x}\le y}dy\right\}d\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big)\\
&\quad -\int_{\mathbb{R}^d}\boldsymbol{x}\Psi(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\,d\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big)+\int_{\mathbb{R}^d}E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=\boldsymbol{a}'\boldsymbol{x}]\Psi(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\,d\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big)\\
&=\int_{\mathbb{R}^d}\int_{\mathbb{R}}\frac{\partial}{\partial y}\big(E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=y]\big)\Psi(y;\boldsymbol{a})\mathbb{1}_{\boldsymbol{a}'\boldsymbol{x}\le y}dyd\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big)\\
&\qquad -\int_{\mathbb{R}^d}\Big\{\boldsymbol{x}-E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=\boldsymbol{a}'\boldsymbol{x}]\Big\}\Psi(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\,d\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big).
\end{aligned}
\tag{C-3}
$$

C.3 Empirical estimator

Let us now consider the expansion of the empirical estimator (2.9). When H is twice differentiable, we have:

$$
\begin{aligned}
&\sqrt{T}\left(\frac{\partial\widehat{\Pi}_T}{\partial \boldsymbol{a}}(H,\boldsymbol{a})-\frac{\partial\Pi}{\partial \boldsymbol{a}}(H,\boldsymbol{a})\right)\\
&=\sqrt{T}\left(\int_{\mathbb{R}^d}\boldsymbol{x}\frac{\partial H\big(1-\widehat{G}_T(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\big)}{\partial u}d\,\widehat{F}_T(\boldsymbol{x})-\int_{\mathbb{R}^d}\boldsymbol{x}\frac{\partial H\big(1-G(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\big)}{\partial u}d\,F(\boldsymbol{x})\right)\\
&=\sqrt{T}\left(\int_{\mathbb{R}^d}\boldsymbol{x}\frac{\partial H\big(1-\widehat{G}_T(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\big)}{\partial u}d\,\widehat{F}_T(\boldsymbol{x})-\int_{\mathbb{R}^d}\boldsymbol{x}\frac{\partial H\big(1-G(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\big)}{\partial u}d\,\widehat{F}_T(\boldsymbol{x})\right)\\
&\quad+\sqrt{T}\left(\int_{\mathbb{R}^d}\boldsymbol{x}\frac{\partial H\big(1-G(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\big)}{\partial u}d\,\widehat{F}_T(\boldsymbol{x})-\int_{\mathbb{R}^d}\boldsymbol{x}\frac{\partial H\big(1-G(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\big)}{\partial u}d\,F(\boldsymbol{x})\right)\\
&=-\int_{\mathbb{R}^d}\boldsymbol{x}\frac{\partial^2 H\big(1-G(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\big)}{\partial u^2}\sqrt{T}\big[\widehat{G}_T(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})-G(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\big]d\,F(\boldsymbol{x})\\
&\quad+\int_{\mathbb{R}^d}\boldsymbol{x}\frac{\partial H\big(1-G(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\big)}{\partial u}\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x})-F(\boldsymbol{x})\right]+o_p(1).
\end{aligned}
$$

Since $G(\boldsymbol{a}'\boldsymbol{z};\boldsymbol{a})=P[\boldsymbol{a}'\boldsymbol{X}\le\boldsymbol{a}'\boldsymbol{z}]=\int_{\mathbb{R}^d}\mathbb{1}_{(\boldsymbol{a}'\boldsymbol{x}\le\boldsymbol{a}'\boldsymbol{z})}d\,F(\boldsymbol{x})$, and $\widehat{G}_T(\boldsymbol{a}'\boldsymbol{z};\boldsymbol{a})=\int_{\mathbb{R}^d}\mathbb{1}_{(\boldsymbol{a}'\boldsymbol{x}\le\boldsymbol{a}'\boldsymbol{z})}d\widehat{F}_T(\boldsymbol{x})$, the right hand side of the expansion can be rewritten as:

$$
\int_{\mathbb{R}^d}\left\{\int_{\mathbb{R}^d}-\boldsymbol{z}\frac{\partial^2 H\big(1-G(\boldsymbol{a}'\boldsymbol{z};\boldsymbol{a})\big)}{\partial u^2}\mathbb{1}_{(\boldsymbol{a}'\boldsymbol{x}\le\boldsymbol{a}'\boldsymbol{z})}d\,F(\boldsymbol{z})+\boldsymbol{x}\frac{\partial H\big(1-G(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\big)}{\partial u}\right\}\sqrt{T}d\left[\widehat{F}_T(\boldsymbol{x})-F(\boldsymbol{x})\right]+o_p(1),
$$

which weakly converges to:

$$
\int_{\mathbb{R}^d}\left\{\int_{\mathbb{R}^d}-\boldsymbol{z}\frac{\partial^2 H\big(1-G(\boldsymbol{a}'\boldsymbol{z};\boldsymbol{a})\big)}{\partial u^2}\mathbb{1}_{(\boldsymbol{a}'\boldsymbol{x}\le\boldsymbol{a}'\boldsymbol{z})}d\,F(\boldsymbol{z})+\boldsymbol{x}\frac{\partial H\big(1-G(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\big)}{\partial u}\right\}d\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big).
$$

C.4 Alternative expansion of the kernel estimator

If the distortion function H is twice differentiable, we can integrate by part the first component in the asymptotic expression of the kernel estimator given in (C-3). Let us denote by $\boldsymbol{Z}$ a variable with the same distribution as $\boldsymbol{X}$. We have:

$$\int_{\mathbb{R}} \frac{\partial}{\partial y}\big(E[Z^i|\boldsymbol{a}'\boldsymbol{Z}=y]\big)\nabla H(y;\boldsymbol{a})\mathbb{1}_{\boldsymbol{a}'\boldsymbol{x}\leq y}dy$$
$$=\mathbb{1}_{\boldsymbol{a}'\boldsymbol{x}\leq y}\nabla H(y;\boldsymbol{a})E[Z^i|\boldsymbol{a}'\boldsymbol{Z}=y]\Big]_{-\infty}^{\infty} - \int_{\mathbb{R}} E[Z^i|\boldsymbol{a}'\boldsymbol{Z}=y]\mathbb{1}_{\boldsymbol{a}'\boldsymbol{x}\leq y}d\nabla H(y)$$
$$=\nabla H(\infty;\boldsymbol{a})E[Z^i|\boldsymbol{a}'\boldsymbol{Z}=\infty] + \int_{\mathbb{R}} \mathbb{1}_{\boldsymbol{a}'\boldsymbol{x}\leq y}\frac{\partial^2 H\big(1-G(y;\boldsymbol{a});\boldsymbol{a}\big)}{\partial u^2}\frac{\int_{\mathbb{R}^{d-1}}\frac{z^i}{a^j}f\left(z^1,\ldots,\frac{y-\sum_{l\neq j}a^l z^l}{a^j},\ldots,z^d\right)\prod_{l\neq j}d\,z^l}{\int_{\mathbb{R}^{d-1}}\frac{1}{a^j}f\left(z^1,\ldots,\frac{y-\sum_{l\neq j}a^l z^l}{a^j},\ldots,z^d\right)\prod_{l\neq j}d\,z^l}g(y;\boldsymbol{a})dy.$$

Since $g(y;\boldsymbol{a})=\int_{\mathbb{R}^{d-1}}\frac{1}{a^j}f\left(z^1,\ldots,\frac{y-\sum_{l\neq j}a^l z^l}{a^j},\ldots,z^d\right)\prod_{l\neq j}d\,z^l$ and $\int_{\mathbb{R}^d}d\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big)=0$, we get:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}}\frac{\partial}{\partial y}\big(E[Z^i|\boldsymbol{a}'\boldsymbol{Z}=y]\big)\nabla H(y;\boldsymbol{a})\mathbb{1}_{\boldsymbol{a}'\boldsymbol{x}\leq y}dy\,d\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big)\\
&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{1}_{\boldsymbol{a}'\boldsymbol{x}\leq\boldsymbol{a}'\boldsymbol{z}}\frac{\partial^2 H\big(1-G(\boldsymbol{a}'\boldsymbol{z};\boldsymbol{a});\boldsymbol{a}\big)}{\partial u^2}z^i f(z^1,\ldots,z^d)dz^1\cdots dz^d d\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big)\\
&\quad+\nabla H(\infty)E[Z^i|\boldsymbol{a}'\boldsymbol{Z}=\infty]\int_{\mathbb{R}^d}d\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big)\\
&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{1}_{\boldsymbol{a}'\boldsymbol{x}\leq\boldsymbol{a}'\boldsymbol{z}}\frac{\partial^2 H\big(1-G(\boldsymbol{a}'\boldsymbol{z};\boldsymbol{a});\boldsymbol{a}\big)}{\partial u^2}z^i f(z^1,\ldots,z^d)dz^1\cdots dz^d d\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big).
\end{aligned}$$

The above equation holds for any asset $i=1,\ldots,d$. Thus, we deduce that:

$$\begin{aligned}
&\sqrt{T}\left(\frac{\partial\widetilde{\Pi}_T}{\partial\boldsymbol{a}}(H,\boldsymbol{a})-\frac{\partial\Pi}{\partial\boldsymbol{a}}(H,\boldsymbol{a})\right)\Rightarrow\\
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\boldsymbol{z}\,\mathbb{1}_{\boldsymbol{a}'\boldsymbol{x}\leq\boldsymbol{a}'\boldsymbol{z}}\frac{\partial^2 H\big[1-G(\boldsymbol{a}'\boldsymbol{z};\boldsymbol{a});\boldsymbol{a}\big]}{\partial u^2}d\,F(\boldsymbol{z})\,d\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big)\\
&\qquad-\int_{\mathbb{R}^d}\Big[\boldsymbol{x}-E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=\boldsymbol{a}'\boldsymbol{x}]\Big]\nabla H(\boldsymbol{a}'\boldsymbol{x};\boldsymbol{a})\,d\,\mathcal{K}\big(F^1(x^1),\ldots,F^d(x^d)\big).
\end{aligned}\tag{C-4}$$

C.5 Asymptotic variances

The asymptotic variance-covariances of both estimators are obtained by using (see C.1):

$$V\left(\int_{\mathbb{R}^d}\psi(\boldsymbol{x})d\mathcal{K}\left(F^1(x^1),\ldots,F^d(x^d)\right)\right)=V\big(\psi(\boldsymbol{x})\big).$$

a) If the distortion function H is twice differentiable, the variance-covariance matrices of the limiting processes are:

$$\Omega(\boldsymbol{a},\boldsymbol{a}) = V\left[\int \boldsymbol{z}\mathbb{1}_{(\boldsymbol{a}'\boldsymbol{X}\le\boldsymbol{a}'\boldsymbol{z})}\frac{\partial^2 H\big(1-G(\boldsymbol{a}'\boldsymbol{z};\boldsymbol{a});\boldsymbol{a}\big)}{\partial u^2}dF(\boldsymbol{z}) - \big(\boldsymbol{X} - E[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}]\big)\nabla H(\boldsymbol{a}'\boldsymbol{X};\boldsymbol{a})\right],$$

$$\Sigma(\boldsymbol{a},\boldsymbol{a}) = V\left[\boldsymbol{X}\nabla H(\boldsymbol{a}'\boldsymbol{X};\boldsymbol{a}) - \int \boldsymbol{z}\mathbb{1}_{(\boldsymbol{a}'\boldsymbol{X}\le\boldsymbol{a}'\boldsymbol{z})}\frac{\partial^2 H\big(1-G(\boldsymbol{a}'\boldsymbol{z};\boldsymbol{a});\boldsymbol{a}\big)}{\partial u^2}dF(\boldsymbol{z})\right].$$

b) However, the distortion functions H associated with the VaR and Tail-VaR are not twice differentiable. The variance-covariance matrices of their kernel estimators are given below.
b.i) For the delta-VaR, we have:

$$\lim_{T\to\infty} V\left(\sqrt{T\,h_T}\left[\frac{\partial\widetilde{VaR}_{T,k}(p,\boldsymbol{a})}{\partial\boldsymbol{a}} - \frac{\partial VaR(p,\boldsymbol{a})}{\partial\boldsymbol{a}}\right]\right)$$
$$= \frac{1}{g\big(Q(1-p\,;\boldsymbol{a})\,;\,\boldsymbol{a}\big)}\int k^2(u)du\left[E\big[\boldsymbol{x}^2|\boldsymbol{a}'\boldsymbol{x}=Q(1-p;\boldsymbol{a})\big] - \Big(E\big[\boldsymbol{x}|\boldsymbol{a}'\boldsymbol{x}=Q(1-p;\boldsymbol{a})\big]\Big)^2\right].$$

Proof. Let us denote $g = g\big(Q(1-p\,;\boldsymbol{a})\,;\,\boldsymbol{a}\big)$; we get:

$$V\left(\sqrt{T\,h_T}\left[\frac{\partial\widetilde{Q}_T(\cdot,\boldsymbol{a})}{\partial\boldsymbol{a}} - \frac{\partial Q(\cdot,\boldsymbol{a})}{\partial\boldsymbol{a}}\right]\right)$$
$$= \frac{1}{g^2}\frac{1}{h_T}\left\{E\left[\boldsymbol{X}^2k^2\left(\frac{\boldsymbol{a}'\boldsymbol{X}-Q(\cdot;\boldsymbol{a})}{h_T}\right)\right] - \left[E\,\boldsymbol{X}\,k\left(\frac{\boldsymbol{a}'\boldsymbol{X}-Q(\cdot;\boldsymbol{a})}{h_T}\right)\right]^2\right\}$$
$$+\frac{\left(E\left[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=Q(\cdot;\boldsymbol{a})\right]\right)^2}{g^2}\frac{1}{h_T}\left[E\,k^2\left(\frac{\boldsymbol{a}'\boldsymbol{X}-Q(\cdot;\boldsymbol{a})}{h_T}\right) - \left(E\,k\left(\frac{\boldsymbol{a}'\boldsymbol{X}-Q(\cdot;\boldsymbol{a})}{h_T}\right)\right)^2\right]$$
$$-\frac{2\left(E\left[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=Q(\cdot;\boldsymbol{a})\right]\right)}{g^2}\frac{1}{h_T}\left\{E\left[\boldsymbol{X}\,k^2\left(\frac{\boldsymbol{a}'\boldsymbol{X}-Q(\cdot;\boldsymbol{a})}{h_T}\right)\right]\right.$$
$$\left.-E\left[\boldsymbol{X}\,k\left(\frac{\boldsymbol{a}'\boldsymbol{X}-Q(\cdot;\boldsymbol{a})}{h_T}\right)\right]\;E\left[k\left(\frac{\boldsymbol{a}'\boldsymbol{X}-Q(\cdot;\boldsymbol{a})}{h_T}\right)\right]\right\}.$$

By applying Lemma **1** ii) in Appendix A for $\varphi(\boldsymbol{x}) = 1, \boldsymbol{x}, \boldsymbol{x}^2$, respectively, we deduce:

$$\lim_{T\to\infty} V\left(\sqrt{T\,h_T}\frac{\partial\widetilde{VaR}_T(p,\boldsymbol{a})}{\partial\boldsymbol{a}} - \frac{\partial VaR(p,\boldsymbol{a})}{\partial\boldsymbol{a}}\right)$$
$$= \frac{1}{g}\int k^2(u)du\left[E\big[\boldsymbol{X}^2|\boldsymbol{a}'\boldsymbol{X}=Q(1-p;\boldsymbol{a})\big] - \Big(E\big[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=Q(1-p;\boldsymbol{a})\big]\Big)^2\right] + o(1).$$

□

ii) For the delta-TVaR, we have:

$$\Omega_{TVaR(p)} = V\left(-\frac{1}{p}E\big[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=Q(1-p;\boldsymbol{a})\big]\,\mathbb{1}_{\boldsymbol{a}'\boldsymbol{X}\le Q(1-p;\boldsymbol{a})} - \frac{1}{p}\boldsymbol{X}\mathbb{1}_{\boldsymbol{a}'\boldsymbol{X}\ge Q(1-p;\boldsymbol{a})}\right.$$

$$+\frac{1}{p}E\left[\boldsymbol{X}|\boldsymbol{a}'\boldsymbol{X}=Q(1-p;\boldsymbol{a})\right]\mathbb{1}_{\boldsymbol{a}'\boldsymbol{X}\geq Q(1-p;\boldsymbol{a})}\Big).$$

Appendix D Expansions of $\widehat{\boldsymbol{a}}_T$ and $\widehat{\lambda}_T$

i) Let us consider the first equation of system (3.10). We get:

$$\begin{aligned}&\left[\frac{\partial^2\Pi}{\partial\boldsymbol{a}\partial\boldsymbol{a}'}(H_1-\lambda^*H_0,\boldsymbol{a}^*,F_0)-\frac{1}{\Pi_0}\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_0,\boldsymbol{a}^*,F_0)\frac{\partial\Pi}{\partial\boldsymbol{a}'}(H_1,\boldsymbol{a}^*,F_0)\right]\sqrt{T}(\widehat{\boldsymbol{a}}_T-\boldsymbol{a}^*)\\&+\left[\frac{\partial^2\Pi}{\partial\boldsymbol{a}\partial F'}(H_1-\lambda^*H_0,\boldsymbol{a}^*,F_0)-\frac{1}{\Pi_0}\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_0,\boldsymbol{a}^*,F_0)\frac{\partial\Pi}{\partial F'}(H_1,\boldsymbol{a}^*,F_0)\right]\sqrt{T}(\widehat{F}_T-F_0)=o_p(1),\end{aligned}\tag{D-1}$$

where $\frac{\partial\Pi}{\partial F'}$ stands for Hadamard derivative. Equivalently, we have:

$$\begin{aligned}&\left[-\frac{\partial^2\Pi}{\partial\boldsymbol{a}\partial\boldsymbol{a}'}(H_1-\lambda^*H_0,\boldsymbol{a}^*,F_0)+\frac{1}{\Pi_0}\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_0,\boldsymbol{a}^*,F_0)\frac{\partial\Pi}{\partial\boldsymbol{a}'}(H_1,\boldsymbol{a}^*,F_0)\right]\sqrt{T}(\widehat{\boldsymbol{a}}_T-\boldsymbol{a}^*)\\&=\sqrt{T}\left[\frac{\partial\widehat{\Pi}_T}{\partial\boldsymbol{a}}(H_1-\lambda^*H_0,\boldsymbol{a}^*,F_0)-\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_1-\lambda^*H_0,\boldsymbol{a}^*,F_0)\right]\\&\quad-\frac{1}{\Pi_0}\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_0,\boldsymbol{a}^*,F_0)\sqrt{T}\left[\widehat{\Pi}_T(H_1,\boldsymbol{a}^*,F_0)-\Pi(H_1,\boldsymbol{a}^*,F_0)\right]+o_p(1).\end{aligned}$$

The result of Proposition 2 i) follows by noting that $\widehat{\Pi}_T(H_1,\boldsymbol{a}^*,F_0)=(\boldsymbol{a}^*)'\frac{\partial\widehat{\Pi}_T}{\partial\boldsymbol{a}}(H_1,\boldsymbol{a}^*,F_0)$.

ii) Let us now consider the expansion of $\widehat{\lambda}_T$, we get:

$$\sqrt{T}(\widehat{\lambda}_T-\lambda^*)=\frac{1}{\Pi_0}\frac{\partial\Pi}{\partial\boldsymbol{a}'}(H_1,\boldsymbol{a}^*,F_0)\sqrt{T}(\widehat{\boldsymbol{a}}_T-\boldsymbol{a}^*)+\frac{1}{\Pi_0}\sqrt{T}\left[\widehat{\Pi}_T(H_1,\boldsymbol{a}^*,F_0)-\Pi(H_1,\boldsymbol{a}^*,F_0)\right]+o_p(1).$$

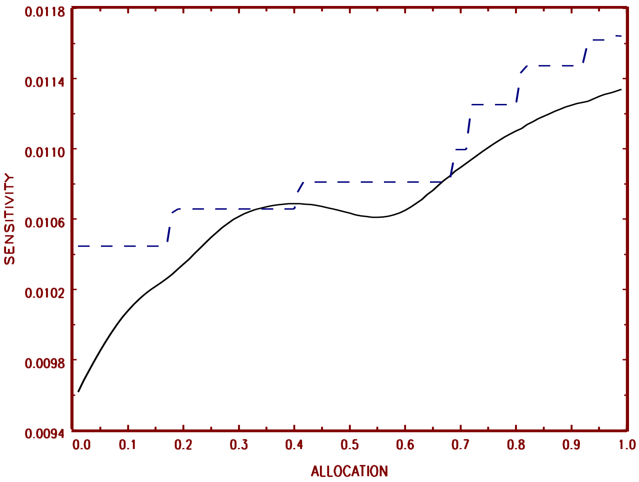
Appendix E Expansion of $\widehat{\widehat{\boldsymbol{a}}}_T$

The expansion of the first equation of (4.4) provides:

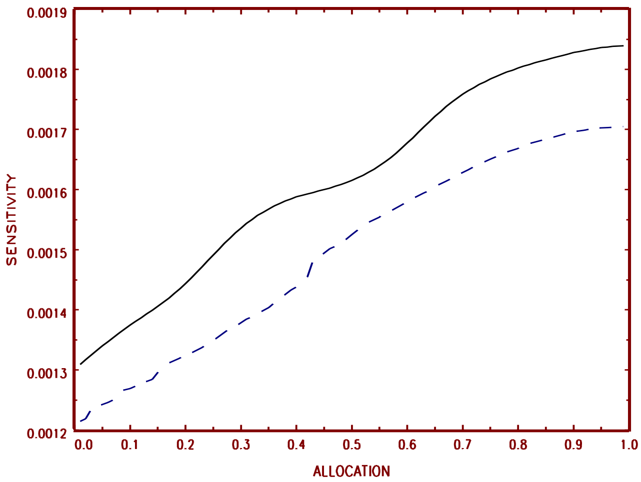
$$\begin{aligned}&B\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T-\boldsymbol{a}_0)\\&=\sqrt{T}\left[\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_1,\boldsymbol{a}_0,\widehat{F}_T)-\frac{\Pi(H_1,\boldsymbol{a}_0,\widehat{F}_T)}{\Pi(H_0,\boldsymbol{a}_0,\widehat{F}_T)}\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_0,\boldsymbol{a}_0,\widehat{F}_T)\right]+o_p(1)\\&=\sqrt{T}\left[\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_1,\boldsymbol{a}_0,\widehat{F}_T)-\frac{\boldsymbol{a}_0'\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_1,\boldsymbol{a}_0,\widehat{F}_T)}{\boldsymbol{a}_0'\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_0,\boldsymbol{a}_0,\widehat{F}_T)}\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_0,\boldsymbol{a}_0,\widehat{F}_T)\right]+o_p(1)\\&=\left[Id-\frac{1}{\Pi(H_0,\boldsymbol{a}_0,F_0)}\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_0,\boldsymbol{a}_0,F_0)\boldsymbol{a}_0'\right]\sqrt{T}\left[\frac{\partial\widehat{\Pi}_T}{\partial\boldsymbol{a}}(H_1-\lambda_0H_0,\boldsymbol{a}_0,F_0)-\frac{\partial\Pi}{\partial\boldsymbol{a}}(H_1-\lambda_0H_0,\boldsymbol{a}_0,F_0)\right]+o_p(1).\end{aligned}$$

Note finally that the first equality can be written as:

$$B\sqrt{T}(\widehat{\widehat{\boldsymbol{a}}}_T-\boldsymbol{a}_0)=\sqrt{T}\frac{\partial\Pi}{\partial\boldsymbol{a}}\left(H_1-\widehat{\widehat{\lambda}}_{0T}H_0,\boldsymbol{a}_0,\widehat{F}_T\right)+o_p(1).$$



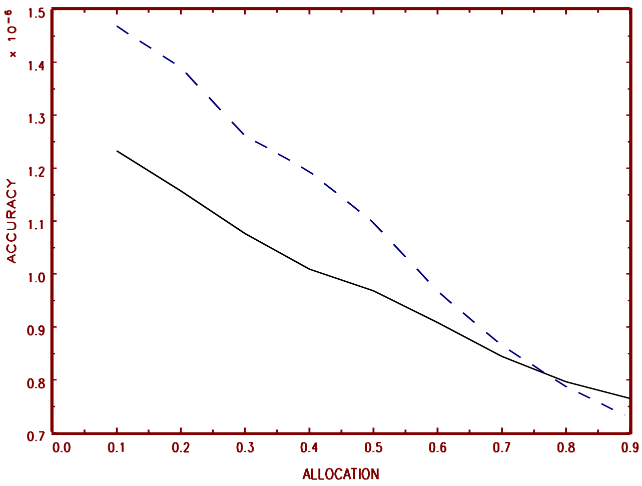
delta-TVaR(0.05)



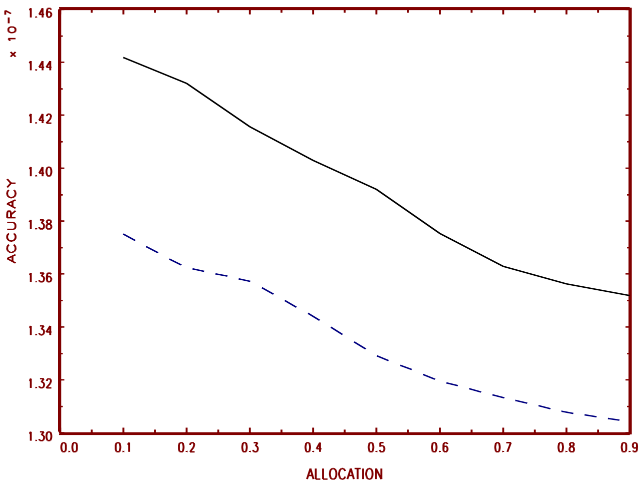
delta-PH(0.7)

Figure 1: Estimators of delta-TVaR and delta-PH

The average of estimates are obtained from a sample size 250 and a simulation size 1000. The delta-TVaR and delta-PH are estimated for a^1 varying from 0.1 to 0.9 and the risk/pessimistic parameters are equal to 0.05 and 0.7, respectively. The kernel estimators are represented by solid line and the empirical estimators by dashed line.



delta-TVaR(0.05)



delta-PH(0.7)

Figure 2: Finite sample variances of estimators of delta-TVaR and delta-PH

The variances are calculated from a sample size 250 and a simulation size 1000. The delta-TVaR and delta-PH are estimated for a^1 varying from 0.1 to 0.9 and the risk/pessimistic parameters are equal to 0.05 and 0.7, respectively. The kernel estimators are represented by solid line and the empirical estimators by dashed line.

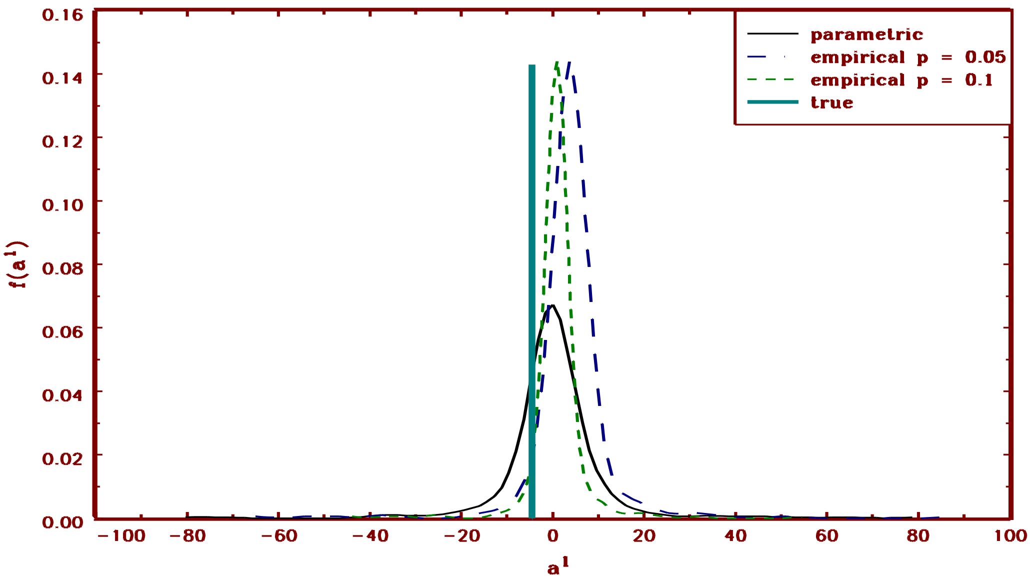


Figure 3: Finite sample distribution of estimators of $(a^1)^*$

The densities are estimated from 400 simulated values of the normalized allocation on asset one using kernel method. The curves are provided for the range $[-100, 100]$. The true value (vertical line) is -4.6. The densities plotted are for i) the parametric estimator (solid line); ii) the empirical estimator when $p = 0.05$ (dashed line); and iii) the empirical estimator when $p = 0.1$ (shorter dashed line). The averages calculated from the simulated estimates are 2.4, 1.66 and 0.86, respectively. The finite sample variances are 1656.32 (parametric), 139.02 (empirical when $p = 0.05$) and 117.76 (empirical when $p = 0.1$).

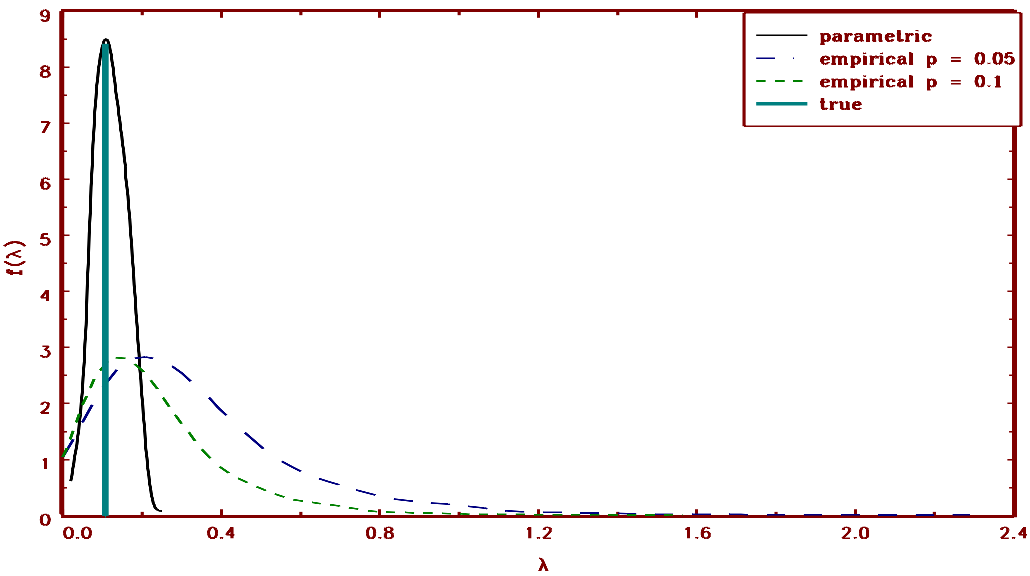


Figure 4: Finite sample distribution of estimators of standard Sharpe ratio
The densities are estimated from 400 simulated values of the standard Sharpe ratio using kernel method. The true value (vertical line) is 0.107. The densities plotted are for i) the parametric estimator (solid line); ii) the empirical estimator when $p = 0.05$ (dashed line); and iii) the empirical estimator when $p = 0.1$ (shorter dashed line). The averages calculated from the simulated estimates are 0.12, 0.326 and 0.237, respectively. The finite sample variances are 0.002 (parametric), 0.094 (empirical when $p = 0.05$) and 0.038 (empirical when $p = 0.1$).