Estimation of the volatility persistence in a discretely observed diffusion model

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Abstract

We consider the stochastic volatility model

$$\mathrm{d}Y_t = \sigma_t \, \mathrm{d}B_t,$$

with B a Brownian motion and σ of the form

$$\sigma_t = \Phi\Big(\int_0^t a(u)\mathrm{d}W_u^H \Big),$$

where W^H is a fractional Brownian motion, independent of the driving Brownian motion B, with Hurst parameter $H \geq 1/2$. This model allows for persistence in the volatility σ. The parameter of interest is H and the functions Φ and a are treated as nuisance parameters. For a fixed objective time T, we construct from discrete data $Y_{i/n},\ i = 0, \ldots, nT$, a wavelet based estimator of H, inspired by adaptive estimation of quadratic functionals. We show that the accuracy of our estimator is $n^{-1/(4H+2)}$ and that this rate is optimal in a minimax sense.

Résumé

On considère le modèle à volatilité stochastique défini par les équations précédentes, où B est un mouvement brownien et W^H un mouvement brownien fractionnaire, indépendant de B, de paramètre de Hurst $H \geq 1/2$. Ce modèle permet de reproduire des propriétés de persistance dans la volatilité σ. Le paramètre d'intérêt est H et les fonctions Φ et a sont traitées comme des paramètres de nuisance. Pour un temps objectif fixé T, on construit à partir des données discrètes $Y_{i/n},\ i = 0, \ldots, nT$, un estimateur par ondelettes de H, inspiré de l'estimation adaptative des fonctionnelles quadratiques. On montre que la précision de notre estimateur est $n^{-1/(4H+2)}$ et que celle-ci est optimale au sens minimax.

Key words: Stochastic volatility models; High frequency data; Fractional Brownian motion; Adaptive estimation of quadratic functionals; Wavelet methods.
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1 Introduction

1.1 Stochastic volatility and volatility persistence

Since the celebrated model of Black and Scholes, the behaviour of financial assets is modeled by processes of type

$$\mathrm{d}S_t = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t,$$

where S is the price of the asset, B a Brownian motion and μ a drift process. The volatility coefficient σ represents the fluctuations of S and plays a crucial role in trading, option pricing and hedging. It is well known that stochastic volatility models, where the volatility is a random process, are a way to deal with the endemic time-varying volatility and to reproduce various stylised facts observed on the markets, see Shephard [30], Barndorff-Nielsen, Nicolato and Shephard [3]. Among these stylised facts, there are many arguings about volatility persistence. This presence of memory in the volatility has in particular consequences for option pricing, see Taylor [31], Comte, Coutin and Renault [8]. Hence continuous time dynamics have been introduced to capture this phenomenon, see Comte and Renault [9], Comte, Coutin and Renault [8] or Barndorff-Nielsen and Shephard [4]. Paradoxically, in statistical finance, the question of volatility persistence has been mostly treated with discrete time models, see among others Breidt, Crato and De Lima [6], Harvey [16], Andersen and Bollerslev [1], Robinson [29], Hurvich and Soulier [20], Teyssière [33]. Concurrently, statistical methods to detect this volatility persistence have been specifically developed for these models, see Hurvich, Moulines and Soulier [18], Deo, Hurvich and Lu [12], Hurvich and Ray [19], Lee [23], Jensen [22]. In this paper, our objective is to build for continuous time models a statistical program allowing to recover information about volatility persistence.

1.2 A diffusion model with fractional stochastic volatility

We consider a class of diffusion models whose volatility is a non-linear transformation of a stochastic integral with respect to fractional Brownian motion. Recall that a fractional Brownian motion $(W_t^H, t \geq 0)$, with Hurst parameter $H \in [0,1]$ is a self-similar centered Gaussian process with covariance function

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}).$$

As soon as $H > 1/2$, the use of fractional Brownian motion (fbm for short) is a way to allow for persistence. Indeed, its increments are positively correlated and the value of the Hurst parameter quantifies the presence of so-called long-memory in the dynamic, see Mandelbrot and Van Ness [25], Taqqu [13]. We define on a rich enough probability space $(\Omega, \mathcal{A}, \mathbb{P})$ a Brownian motion B and a fractional Brownian motion W^H, independent of B, with unknown Hurst parameter $H \in (1/2, 1)$. We fix an objective time $T > 0$ and we consider the 1-dimensional stochastic process Y defined by

$$Y_t = y_0 + \int_0^t \sigma_s \, \mathrm{d}B_s, \; y_0 \in \mathbb{R}, \quad t \in [0, T], \tag{1}$$

where σ is another 1-dimensional stochastic process of the form

$$\sigma_t = \Phi\Big(\int_0^t a(u) \mathrm{d}W_u^H \Big). \tag{2}$$

The functions Φ and a are deterministic and unknown. The stochastic integral with respect to fractional Brownian motion is defined as the limit in $\mathrm{L}^2(\mathbb{P})$ of the associated Riemann sums (for details and properties, we refer to Lin [24]).

This framework is an extension of the model introduced in mathematical finance by Comte and Renault [9]. Remark also that for $H = 1/2$, under smoothness assumptions on Φ, letting

$$a = 1, \; f = (\Phi^2)' \circ \Phi^{-1} \text{ and } g = (\Phi^2)'' \circ \Phi^{-1},$$

we equivalently have

$$\mathrm{d}\sigma_t^2 = g(\sigma_t^2)\mathrm{d}t + f(\sigma_t^2)\mathrm{d}W_t.$$

Thus, we (partially) retrieve the standard stochastic volatility diffusion framework, see for example Hull and White [17], Melino and Turnbull [26] or Musiela and Rutkowski [27] for a more exhaustive study.

Finally, the assumptions on a and Φ in the model (1)-(2) are the following:

Assumption A.

(i) $t \to a(t)$ is continuously differentiable,
(ii) There exist $0 \le \alpha < \beta \le T$ such that $\inf_{t \in [\alpha, \beta]} a^2(t) > 0$.

Assumption B. Let I denote the indicator function,

(i) $x \to \Phi(x)$ is twice continuously differentiable,
(ii) For some $c_1 > 0$, $c_2 > 0$ and $\gamma \ge 0$, $|(\Phi^2)'(x)| \ge c_1 |x|^\gamma I_{|x| \in [0,1]} + c_2 I_{|x|>1}$,
(iii) For some $c_3 > 0$ and $m \ge 0$, $|(\Phi^2)''(x)| \le c_3(1 + |x|^m)$.

We consider the preceding model. For technical reasons (see section 2.1.2), we take $T \geq 3$. We observe the diffusion at the sampling frequency n, that means we observe

$$Y^n = \{Y_{i/n},\ i = 0, \dots, nT\}.$$

For simplicity, we assume throughout the paper $n = 2^N$. We study the problem of the inference of H based on Y^n .

A rate $v_n \to 0$ is said to be *achievable* over $\mathcal{H} \subset (1/2, 1)$ if there exists an estimator $\widehat{H}_n = \widehat{H}_n(Y^n)$ such that the normalized error

$$\{v_n^{-1}(\widehat{H}_n - H)\}_{n \geq 1} \tag{3}$$

is bounded in probability, uniformly over $\mathcal{H}$. The rate v_n is moreover a *lower rate of convergence* on $\mathcal{H}$ if there exists $C > 0$ such that

$$\liminf_{n\to\infty} \inf_{F} \sup_{H\in\mathcal{H}} \mathbb{P}[v_n^{-1}|F - H| \geq C] > 0, \tag{4}$$

where the infimum is taken over all estimators $F = F(Y^n)$. We prove in this paper that the rate $v_n(H) = n^{-1/(4H+2)}$ is optimal in a minimax sense. This means that (3) and (4) agree with $v_n = v_n(H)$. We also exhibit an optimal estimator based on the behaviour of the wavelet coefficients of the process σ^2.

Theorem 1 *Grant assumptions A and B. The rate $v_n(H) = n^{-1/(4H+2)}$ is achievable over every compact set $\mathcal{H} \subset (1/2, 1)$. Moreover, the estimator $\widehat{H}_n$ explicitly constructed in section 2.2 achieves the rate $v_n(H)$.*

Our next result shows that, under an additional restriction on the non-degeneracy of the model, this result is indeed optimal.

Assumption C.
For some $c_4 > 0$, $c_5 > 0$, $c_4 \neq c_5$ and $c_6 > 0$, we have $c_4 \leq |\Phi(x)| \leq c_5$ and $|\Phi'(x)| \leq c_6$.

Theorem 2 *Grant assumptions A, B and C. The rate $v_n(H) = n^{-1/(4H+2)}$ is a lower rate over every compact set $\mathcal{H} \subset (1/2, 1)$ with non empty interior.*

• Contrary to other works, we do not consider intrinsically discrete data, but discretely observed data from a continuous underlying dynamic. Thus, as the objective time T is fixed, the dynamic between two data depends on the sampling frequency. This approach largely differs from those based on ergodic properties: in our context, the available information quantity does not increase because of longer observation period but because of higher sampling frequency. The estimation rates are naturally different according to the approaches. Compare our accuracy with the rate $n^{-(2/5-\varepsilon)}$ obtained by Hurvich, Moulines and Soulier in an ergodic context, see [18].

• Through this model, we aim at showing that we can recover the smoothness of the volatility from historical data. The following proposition, whose proof is given in appendix, shows that the Hurst parameter can be interpreted as a regularity parameter thanks to Besov smoothness spaces (see appendix).

Proposition 1 *(Smoothness of the volatility process). Under assumptions A and B,*

(i) Almost surely, the trajectory of $t \to \sigma_t^2$ belongs to the Besov space $\mathcal{B}^H_{2,\infty}$ but, for all $q < \infty$, a.s. it does not belong to $\mathcal{B}^H_{2,q}$.

(ii) For all $s < H$, almost surely, the trajectory of $t \to \sigma_t^2$ belongs to the Besov space $\mathcal{B}^s_{\infty,\infty}$ but, if moreover there exists $c > 0$ such that $|(\Phi^2)'(x)| > c$, then, a.s. it does not belong to $\mathcal{B}^H_{\infty,\infty}$.

• The accuracy $v_n(H)$ is slower by a polynomial order than the usual $n^{-1/2}$ of regular parametric models. This rate of convergence seems to be characteristic of high frequency parametric inference from noisy data in presence of fractional Brownian motion. Indeed, this rate is also found by Gloter and Hoffmann [14] in the high frequency inference of the finite dimensional parameter θ in the model

$$\mathrm{d}Y_t = \sigma_t \,\mathrm{d}B_t, \quad \sigma_t = \Phi(\theta, W_t^H).$$

Gloter and Hoffmann [15] also obtain this rate in the high frequency estimation of the Hurst parameter in the following model:

$$Y_i^n = \sigma W^H_{i/n} + a(W^H_{i/n})\xi_i^n, \tag{5}$$

where a is an unknown variance function and ξ_i^n a centered white noise.

1.5 Organisation of the paper

In section 2, we present our estimation method for the volatility Hurst parameter. Section 3 states the main propositions which lead to theorems 1 and 2. We prove in sections 4, 5 and 6 the results stated in section 3 concerning the upper bound whereas theorem 1 is proved in section 7. We end with the proof of theorem 2 in section 8. The proof of proposition 1 is given in appendix.

2 Estimation strategy

2.1 Estimation of the Hurst parameter: preliminaries

2.1.1 Estimation of H from direct observation of a fractional Brownian motion

Imagine we observe high frequency data at sampling rate n^{-1} of the trajectory of a fractional Brownian motion on the interval $[0,1]$. Then, we can recover the Hurst parameter at the parametric accuracy $n^{-1/2}$. Indeed, we can use as follows local properties of the trajectory of the fractional Brownian motion, see Istas and Lang [21], see also Berzin and Leon [5]. Let $s = (s_0, \dots, s_p) \in \mathbb{R}^{p+1}$ be such that for some positive integer $m(s)$:

$$\text{for } \; k = 0, \dots, m(s)-1: \; \sum_{i=0}^{p} s_i \, i^k = 0 \; \text{ and } \; \sum_{i=0}^{p} s_i \, i^{m(s)} \neq 0.$$

We define, for some sequence s and $i = 0, \dots, N-p-1$, the generalized difference

$$\Delta_{i,n} f = \sum_{j=0}^{m(s)} s_j f\Big(\frac{i+j}{n}\Big).$$

The integer $m(s)$ is called the order of the difference. For instance, the usual difference $s = (-1, 1)$ is of order 1 and $s = (1, -2, 1)$ is of order 2. Consider

$$V_n(H) = \sum_{i=1}^{n} (\Delta_{i,n} W^H)^2.$$

Istas and Lang [21] show that for $m(s) > 1$, there exists a constant $L > 0$ such that [1]

$$n^{2H-1} V_n(H) = L + \frac{1}{\sqrt{n}} \xi_n,$$

[1] The condition $m(s) > 1$ is necessary for $H > \frac{3}{4}$, if $H \leq \frac{3}{4}$, the result holds with $m(s) = 1$.

with ξ_n bounded in probability. Then, an estimator achieving the rate $n^{-1/2}$ is for example

$$\widehat{H} = \frac{1}{2}\Big(1 + \log_2 \frac{V_{\lfloor n/2 \rfloor}(H)}{V_n(H)}\Big).$$

2.1.2 *Estimation of H from noisy observation of a fractional Brownian motion*

Trying to recover the Hurst parameter from noisy data is more difficult. Indeed, Gloter and Hoffmann [15] show that the statistical structure of model (5) is significantly modified by the noise. They prove that the rate $n^{-1/(4H+2)}$ is optimal to estimate H in the minimax sense of (3) and (4). Their estimation strategy is based on the behaviour of the energy levels of the fractional Brownian motion that reflects the Besov properties of the trajectories. We adapt this strategy in this paper.

Pick a mother wavelet ψ with 2 vanishing moments. Hence, the wavelet support has a minimal length of 3, see Daubechies [10]. For j and k positive integers, let

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k),\ d_{jk} = \int \psi_{jk} W_s^H \mathrm{d}s \text{ and } Q_j = \sum_k d_{jk}^2.$$

The sequence of energy levels $(Q_j, j \geq 0)$ has the following scaling property [2] :

$$\frac{Q_{j+1}}{Q_j} = 2^{-2H} + o(1) \text{ as } j \to +\infty. \tag{6}$$

Gloter and Hoffmann [15] construct estimators $\widehat{d_{jk}^2}$ of the d_{jk}^2 up to a maximal resolution level $J_n = \lfloor \frac{1}{2}\log_2(n) \rfloor$. Setting

$$\widehat{Q}_j = \sum_k \widehat{d_{jk}^2},$$

one obtain a sequence of estimators:

$$\widehat{H}_{j,n} = -\frac{1}{2}\log_2 \frac{\widehat{Q}_{j+1,n}}{\widehat{Q}_{j,n}}, \quad j = 1, \dots, J_n. \tag{7}$$

Eventually, the estimator is $\widehat{H}_{J_n^*,n}$ where the optimal resolution level J_n^* is defined following the rules of adaptive estimation of quadratic functionals:

$$J_n^* = \max\Big\{ j = 1, \dots, J_n,\ \widehat{Q}_{j,n} \geq \frac{2^j}{n} \Big\}. \tag{8}$$

[2] For the moment, we do not specify the meaning of $o(\cdot)$.

2.2 Construction of an estimator

2.2.1 An Euler scheme-type transformation

By an Euler scheme-type transformation, we boil down the problem to a regression model. Indeed, we have

$$z_i^n = n(Y_{(i+1)/n} - Y_{i/n})^2 = \sigma_{i/n}^2 + \xi_i^n, \tag{9}$$

with

$$\sigma_{i/n}^2 = \Phi^2\Big(\int_0^{\frac{i}{n}} a(u)\mathrm{d}W_u^H\Big),$$

and

$$\xi_i^n = n\Big[\int_{\frac{i}{n}}^{\frac{i+1}{n}} (\sigma_t^2 - \sigma_{i/n}^2)\mathrm{d}t + \Big(\int_{\frac{i}{n}}^{\frac{i+1}{n}} \sigma_t\, \mathrm{d}B_t\Big)^2 - \int_{\frac{i}{n}}^{\frac{i+1}{n}} \sigma_t^2\mathrm{d}t\Big].$$

Conditional on the fbm W^H and up to negligible terms, the ξ_i^n are martingale increments with variance of order 1.

2.2.2 Estimation of the energy levels

Let ψ be a mother wavelet with 2 vanishing moments and support $[0,T]$. Let

$$d_{jk} = \int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}} \sigma_t^2\psi_{jk}(t)\mathrm{d}t \text{ and } Q_j = \sum_k d_{jk}^2.$$

By proving a scaling-type property on the energy levels analogous to (6), we can follow the strategy of section 2.1.2. The main difficulty lies here in the non-linearity introduced by the function Φ^2.

We now present the estimation of the energy levels. To get rid of boundary effects, without any loss of generality in our asymptotic framework, we do not take into account the wavelets ψ_{jk} whose support is not totally included in $[0,T]$. We have

$$d_{jk} = \sum_{l=0}^{T2^{N-j}-1} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2\psi_{jk}(t)dt.$$

A first natural estimator of d_{jk} is

$$\widetilde{d_{jk}} = \sum_{l=0}^{T2^{N-j}-1} z_{k2^{N-j}+l}^n \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t.$$

Let

$$M_{k,l,t} = \Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{t} \sigma_u\mathrm{d}B_u\Big)^2 - \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{t} \sigma_u^2\mathrm{d}u.$$

From (9), we have the following decomposition:

$$\widetilde{d_{jk}} - d_{jk} = b_{jk} + e_{jk} + f_{jk},$$

with

$$\begin{aligned}
b_{jk} &= \sum_{l=0}^{T2^{N-j}-1} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)(\sigma^2_{k2^{-j}+l2^{-N}} - \sigma_t^2)dt,\\
e_{jk} &= n \sum_{l=0}^{T2^{N-j}-1} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t \ M_{k,l,\frac{k}{2^j}+\frac{l+1}{2^N}},\\
f_{jk} &= n \sum_{l=0}^{T2^{N-j}-1} \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} (\sigma_t^2 - \sigma^2_{k2^{-j}+l2^{-N}})\mathrm{d}t.
\end{aligned}$$

In order to precisely estimate d_{jk}^2, we can not use $\widetilde{d_{jk}}^2$ because of the remaining term e_{jk}^2 that we have to compensate. The other terms are negligible.

Conditional on W^H, $(M_{k,l,t}, \quad t \geq 0)$ is a continuous local martingale. Let $\tilde{\mathbb{E}} = E[\quad |W^H]$ denote the expectation conditional on W^H. Then, by the independence of the Brownian increments,

$$\tilde{\mathbb{E}}[e_{jk}^2] = n^2 \sum_{l=0}^{T2^{N-j}-1} \Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t \Big)^2 \tilde{\mathbb{E}}[M^2_{k,l,\frac{k}{2^j}+\frac{l+1}{2^N}}].$$

Let

$$N_{k,l,t} = \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{t} \sigma_u \mathrm{d}B_u.$$

By Ito's formula,

$$M_{k,l,t} = 2\int_0^t \sigma_u N_u I_{\{u \geq \frac{k}{2^j}+\frac{l}{2^N}\}} \mathrm{d}B_u.$$

Let

$$a^2_{j,k,l} = \tilde{\mathbb{E}}[M^2_{k,l,\frac{k}{2^j}+\frac{l+1}{2^N}}] = 2\Big(\tilde{\mathbb{E}}\Big[(Y_{k2^{-j}+(l+1)2^{-N}} - Y_{k2^{-j}+l2^{-N}})^2\Big]\Big)^2.$$

We need to compensate $a^2_{j,k,l}$, so we estimate $a^2_{j,k,l}$ by

$$\widehat{a^2_{j,k,l}} = \left(\frac{\sqrt{2}}{h(n)} \sum_{p=1}^{h(n)} (Y_{k2^{-j}+(l+1+p)2^{-N}} - Y_{k2^{-j}+(l+p)2^{-N}})^2 \right)^2,$$

where $h(n) = \lfloor n^{1/2} \rfloor$. Let

$$\nu_{jk} = n^2 \sum_{l=0}^{T2^{N-j}-1} \Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t \Big)^2 a^2_{j,k,l},$$

$$\bar{\nu}_{jk} = n^2 \sum_{l=0}^{T2^{N-j}-1} \Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t \Big)^2 \widehat{a^2_{j,k,l}}.$$

Finally we define

$$\widehat{d^2_{jk}} = \widetilde{d_{jk}}^2 - \bar{\nu}_{jk} \text{ and } \hat{Q}_j = \sum_k \widehat{d^2_{jk}}.$$

We thus obtain our estimator $\widehat{H}_{J_n^*,n}$ of H with the same specifications as in (7) and (8).

3 The behaviour of the energy levels

We present here the steps that enable us to prove theorems 1 and 2.

3.1 Upper bound

Recall that

$$\sigma_t^2 = \Phi^2 \Big(\int_0^t a(u)\mathrm{d}W_u^H \Big), \ t \in [0,T].$$

Let

$$d_{jk} = \int \sigma_t^2 \psi_{jk}(t)\mathrm{d}t \text{ and } Q_j = \sum_k d_{jk}^2.$$

We write c for a constant depending on Φ, a, H, ψ and continuous in its arguments.

Proposition 2 *(Limit of the energy levels). Under assumptions A and B, there exists $c(\psi) > 0$ depending on ψ and H, continuous in its arguments and $c > 0$, such that*

$$\mathbb{E}\Big[\Big| 2^{2jH} Q_j - c(\psi) \int_0^T a^2(u) \Big\{ \Big(\Phi^2 \Big)' \Big(\int_0^u a(v)dW_v^H \Big) \Big\}^2 du \Big| \Big] \leq c 2^{-j/2}.$$

More precisely, proposition 2 enables us to obtain the following result:

Proposition 3 *(Scaling property). Under assumptions A and B, we have*

(i) For all $\varepsilon > 0$, there exist j_0 and $r > 0$ such that for all $j \geq j_0$,

$$\mathbb{P}\Big[2^{2jH} Q_j \geq r \Big] \geq 1 - \varepsilon,$$

(ii) For all $\varepsilon > 0$, there exist j_0 and $M > 0$ such that for all $j \geq j_0$,

$$\mathbb{P}\Big[2^{j/2} \sup_{l \geq j} \Big| \frac{Q_{l+1}}{Q_l} - 2^{-2H} \Big| \geq M \Big] \leq \varepsilon.$$

Finally, we have the following result for the estimator:

Proposition 4 *(Deviation of the estimator).*
Let $j_n(H) = \left\lfloor (2H+1)^{-1}\log_2(n)\right\rfloor$ *and* $\mathcal{H}$ *be a compact set included in* $(1/2, 1)$*. Under assumptions A and B, for all* $H \in \mathcal{H}$*,* $J_n \geq j_n(H)$ *and for any* $L > 0$*, the sequence*

$$\left\{ n 2^{j_n(H)/2} \sup_{J_n \geq j \geq j_n(H)-L} 2^{-j} \left| \widehat{Q}_{j,n} - Q_j \right| \right\}$$

is bounded in probability, uniformly over $\mathcal{H}$*.*

We then prove that propositions 3 and 4 together imply theorem 1.

3.2 Lower bound

Let $\mathbb{P}^n_f$ denote the law of the data $Y^n = \{Y_{i/n},\ i = 0, \ldots, nT\}$ conditional on $W^H = f$. The key point of the lower bound is the following:

Proposition 5 *(Distance in total variation). Under assumptions A, B and C, there exists* $c > 0$ *such that*

$$\|\mathbb{P}^n_f - \mathbb{P}^n_g\|^2_{TV} \leq cn\|f - g\|_2^2,$$

where $\|\cdot\|_{TV}$ *denotes the distance in total variation and* $\|\cdot\|_2$ *the usual* L^2 *norm of functions on* $[0, T]$ *with respect to the Lebesgue measure.*

4 Proof of proposition 2

4.1 Notation

In all the proofs, assumptions A and B are in force and we repeatedly use the notation c and $\tilde{c}$ for constants depending on Φ, a, H, ψ, continuous in their arguments and that may vary from line to line. Finally, for a function f, $\|f\|_\infty = \sup_t |f(t)|$.

4.2 Technical lemmas

We establish here several useful lemmas. We apply here ideas of Gloter and Hoffmann, see [14], initially developed for generalized differences.

Lemma 1 *Let f and g be two continuous deterministic bounded functions such that $|f| \leq g$. We have*

$$\mathbb{E}\Big[\Big(\int_0^T f(u)dW_u^H\Big)^2\Big] \leq \mathbb{E}\Big[\Big(\int_0^T |f(u)|dW_u^H\Big)^2\Big] \leq \mathbb{E}\Big[\Big(\int_0^T g(u)dW_u^H\Big)^2\Big].$$

Proof. Recall that (see Lin [24])

$$\mathbb{E}\Big[\Big(\int_0^T f(u)\mathrm{d}W_u^H\Big)^2\Big] \leq \|f\|_\infty T^{2H}.$$

Thus, the expectation of the square of the partial sums converges to the expectation of the square of the stochastic integral. We have

$$\int_0^T f(u)\mathrm{d}W_u^H = \lim_{n\to+\infty}\sum_{p=1}^n f(\{p-1\}Tn^{-1})[W^H_{pTn^{-1}} - W^H_{\{p-1\}Tn^{-1}}],$$

where the limit is taken in $\mathrm{L}^2(\mathbb{P})$. The results hold by stationarity arguments and because we easily check that

$$\mathbb{E}\Big[(W^H_{(p-\{q-1\})Tn^{-1}} - W^H_{(p-q)Tn^{-1}})W^H_{Tn^{-1}}\Big] \geq 0. \qquad \square$$

We now prove 2 lemmas on the expectation and covariance of the wavelet coefficients for the stochastic integral.

Lemma 2 *Let $\tilde{\sigma}_t = \int_0^t a(u)\,dW_u^H$, $\beta_{jk} = \int_0^T \tilde{\sigma}_t\psi_{jk}(t)\,dt$ and $F(t) = \int_0^t \psi(u)du$. For all positive integers j, k, there exists a bounded (independently from j and k) variable Z such that,*

$$\mathbb{E}[\beta_{jk}^2] = 2^{-j(1+2H)}(c(\psi)a(k2^{-j})^2 + 2^{-j}Z),$$

where $c(\psi)$ is positive and such that

$$c(\psi) = \mathbb{E}\Big[\Big(\int_0^T F(t)\,dW_t^H\Big)^2\Big].$$

Proof. The coefficient β_{jk} is equal to

$$2^{-j/2}\int_0^L \psi(v)\left(\int_0^{2^{-j}(k+v)} a(u)\mathrm{d}W_u^H\right)\mathrm{d}v.$$

By linearity and Fubini's theorem, β_{jk} is equal to

$$2^{-j/2}\int_0^{2^{-j}k}\int_0^T \psi(v)a(u)\mathrm{d}v\mathrm{d}W_u^H + 2^{-j/2}\int_{2^{-j}k}^{2^{-j}(k+T)}\int_{2^ju-k}^T \psi(v)a(u)\mathrm{d}v\mathrm{d}W_u^H.$$

Because $F(0) = F(T) = 0$, we have

$$\beta_{jk} = -2^{-j/2} \int_{2^{-j}k}^{2^{-j}(k+T)} F(2^j u - k) a(u) \mathrm{d}W_u^H.$$

We can already remark that β_{jk} is a Gaussian random variable. Let

$$F_p = F(2^j(2^{-j}k + \{p-1\}2^{-j}Tn^{-1}) - k) = F(\{p-1\}Tn^{-1}),$$

$$a_p = a(2^{-j}k + \{p-1\}2^{-j}Tn^{-1}).$$

By stationarity of the increments of the fbm, $2^j\mathbb{E}[\beta_{jk}^2]$ is the limit as n tends to infinity of the sum of the following two terms:

$$E_1^n = \sum_{p=1}^{n} F_p^2 a_p (2^{-j}Tn^{-1})^{4H},$$

$$E_2^n = 2 \sum_{1 \le p < q \le n} F_p F_q a_p a_q \mathbb{E}\Big[(W^H_{(p-\{q-1\})2^{-j}Tn^{-1}} - W^H_{(p-q)2^{-j}Tn^{-1}}) W^H_{2^{-j}Tn^{-1}}\Big].$$

By self-similarity property, we get

$$\begin{aligned} 2^j\mathbb{E}[\beta_{jk}^2] &= \lim_{n\to+\infty} 2^{-j2H}\mathbb{E}\Big[\Big(\sum_{p=1}^{n} F_p a_p [W^H_{pTn^{-1}} - W^H_{\{p-1\}Tn^{-1}}]\Big)^2\Big] \\ &= 2^{-j2H}\mathbb{E}\Big[\Big(\int_0^T F(t)\Big(a(k2^{-j}) + t2^{-j}a'(\theta_{k,j})\Big)\mathrm{d}W_t^H\Big)^2\Big], \end{aligned}$$

with $k2^{-j} \le \theta_{k,j} \le (k+T)2^{-j}$, by Taylor's formula. By lemma 1 and because a and a' are bounded, we get the result for $\mathbb{E}[\beta_{jk}^2]$. The positivity of $c(\psi)$ comes remarking that, by Ito's formula,

$$\int_0^T F(t)\mathrm{d}W_t^H = -\int_0^T \psi(t) W_t^H \mathrm{d}t.$$

This quantity is a non degenerate Gaussian variable, see Tewfik and Kim [32] or Delbeke and Abry [11]. The positivity of $c(\psi)$ follows. $\square$

Lemma 3 *(Decorrelation of the wavelet coefficients).*
Let $\tilde{\sigma}_t$ and β_{jk} be as in lemma 2. There exists c such that, for all j, k, k',

$$|\mathit{Cov}(\beta_{jk}\beta_{jk'})| \le 2^{-j(1+2H)} c(1 + |k - k'|)^{2(H-2)} + c2^{-j(2+2H)}.$$

Proof. We write θ for the value where the last term of a Taylor expansion is taken. Let

$$d_{jk} = \int_0^T \psi_{jk}(t) W_t^H \mathrm{d}t.$$

Let

$$F_p = F(\{p-1\}Tn^{-1}\Big), \quad a_{p,k} = a(2^{-j}k + \{p-1\}2^{-j}Tn^{-1}),$$

$$V_{j,p,k} = W^H_{2^{-j}k+p2^{-j}Tn^{-1}} - W^H_{2^{-j}k+\{p-1\}2^{-j}Tn^{-1}},\ r_{j,p,q,k,k'} = \mathbb{E}[V_{j,p,k}V_{j,q,k'}].$$

Let $k \geq k'$. We have

$$\mathbb{E}[\beta_{jk}\beta_{jk'}] = \lim_{n\to+\infty} 2^{-j}\sum_{p,q=1}^{n} F_pF_q a_{p,k}a_{q,k'}r_{j,p,q,k,k'}.$$

So, $\mathbb{E}[\beta_{jk}\beta_{jk'}]$ is the limit of the sum of the four following terms:

$$\begin{aligned}
T_1 &= a(k2^{-j})a(k'2^{-j})\mathbb{E}[d_{jk}d_{jk'}],\\
T_2 &= 2^{-2j}a(k2^{-j})\sum_{p,q=1}^{n} F_pF_q(\{q-1\}Tn^{-1})a'(\theta)r_{j,p,q,k,k'},\\
T_3 &= 2^{-2j}a(k'2^{-j})\sum_{p,q=1}^{n} F_pF_q(\{p-1\}Tn^{-1})a'(\theta)r_{j,p,q,k,k'},\\
T_4 &= 2^{-3j}\sum_{p,q=1}^{n} F_pF_q(\{p-1\}Tn^{-1})(\{q-1\}Tn^{-1})a'(\theta)a'(\theta)r_{j,p,q,k,k'}.
\end{aligned}$$

Using that

$$r_{j,p,q,k,k'} = 2^{-j2H}r_{0,p,q,k-k',0}$$

and Cauchy-Schwarz inequality, we easily prove that T_2, T_3 and T_4 are less than $c2^{-2j}2^{-2jH}$. Then, we use results on the decorrelation of the wavelet coefficients of the fbm which are easily obtained by fourth order Taylor expansions, see for example Tewfik and Kim [32] or Delbeke and Abry [11]. □

Lemma 4 *Let $\tilde{\sigma}_t$, β_{jk} and $c(\psi)$ be as in lemma 2. Let $\xi : [0,T] \to \mathbb{R}$ be a deterministic bounded function. Define*

$$\Sigma_j(\xi) = 2^j\sum_{k=0}^{T(2^j-1)}\{2^{j2H}\beta_{jk}^2 - c(\psi)2^{-j}a^2(k2^{-j})\}\xi_{k2^{-j}}. \tag{10}$$

Then,

$$\mathbb{E}[\Sigma_j(\xi)^2] \leq c\|\xi\|_\infty^2 2^j.$$

Proof. We have

$$\Sigma_j(\xi) = 2^j\sum_{k=0}^{T(2^j-1)}\{2^{j2H}\beta_{jk}^2 - 2^{j2H}\mathbb{E}[\beta_{jk}^2] + 2^{-2j}Z\}\xi_{k2^{-j}},$$

with Z a bounded variable, by lemma 2. Hence $\mathbb{E}[\Sigma_j(\xi)^2]$ is less than

$$2^{2j}\mathbb{E}\Big[\sum_{k,k'=0}^{T(2^j-1)} 2^{j4H}\{\beta_{jk}^2 - \mathbb{E}[\beta_{jk}^2]\}\{\beta_{jk'}^2 - \mathbb{E}[\beta_{jk'}^2]\}\xi_{k2^{-j}}\xi_{k'2^{-j}}\Big] + c\|\xi\|_\infty^2$$

$$\leq 2^{2j}2^{j4H}\sum_{k,k'=0}^{T(2^j-1)}\mathbb{E}[Y_kY_{k'}]\mathbb{E}[\beta_{jk}^2]\mathbb{E}[\beta_{jk'}^2]\xi_{k2^{-j}}\xi_{k'2^{-j}}+c\|\xi\|_\infty^2,$$

with $Y_k=\beta_{jk}^2/\mathbb{E}[\beta_{jk}^2]-1$. We apply Mehler's formula and we get that $\mathbb{E}[\Sigma_j(\xi)^2]$ is less than

$$2^{2j}2^{j4H}\|\xi\|_\infty^2 2\sum_{k,k'=0}^{T(2^j-1)}\mathrm{Cov}(\beta_{jk},\beta_{jk'})^2+c\|\xi\|_\infty^2$$

$$\leq 2^{2j}2^{j4H}\|\xi\|_\infty^2\tilde{c}\Big(\sum_{k,k'=0}^{T(2^j-1)}2^{-2j(1+2H)}(1+|k-k'|)^{4(H-2)}$$

$$+2^{-j}2^{-2j(1+2H)}(1+|k-k'|)^{2(H-2)}+2^{-2j}2^{-2j(1+2H)}\Big)+c\|\xi\|_\infty^2, \text{ by lemma 3,}$$

$$\leq \tilde{c}\|\xi\|_\infty^2\Big(\sum_{k=0}^{T(2^j-1)}\sum_{i=0}^{+\infty}(1+i)^{4(H-2)}+2^{-j}\sum_{k=0}^{T(2^j-1)}\sum_{i=0}^{+\infty}(1+i)^{2(H-2)}+1\Big)$$

$\leq \tilde{c}\|\xi\|_\infty^2 2^j$, because of the convergence of the infinite series for $H\in(1/2,1)$. □

Lemma 5 *Assume that $\xi:[0,T]\to\mathbb{R}$ is bounded and vanishes outside the interval $[k2^{-j_0},k'2^{-j_0}]\subset[0,T]$ for some $k,k',j_0\geq 1$, $k\neq k'$. Then, there exists $c>0$ such that for $j\geq j_0$, we have*

$$\mathbb{E}[\Sigma_j(\xi)^2]\leq c\|\xi\|_\infty^2|k'-k|2^{j-j_0}.$$

Proof. We have

$$\mathbb{E}[\Sigma_j(\xi)^2]\leq\tilde{c}\Big(\sum_{z,z'}^{T(2^j-1)}|\xi_{z2^{-j}}||\xi_{z'2^{-j}}|(1+|z-z'|)^{4(H-2)}$$

$$+2^{-j}|\xi_{z2^{-j}}||\xi_{z'2^{-j}}|(1+|z-z'|)^{2(H-2)}+2^{-2j}|\xi_{z2^{-j}}||\xi_{z'2^{-j}}|\Big)+c\|\xi\|_\infty^2.$$

By similar computations on the series as in proof of lemma 4, we get

$$\mathbb{E}[\Sigma_j(\xi)^2]\leq\tilde{c}\|\xi\|_\infty\sum_z|\xi_{z2^{-j}}|+c\|\xi\|_\infty^2.$$

As $\xi_{z2^{-j}}$ is different from zero only if $k2^{-j_0}\leq z2^{-j}\leq k'2^{-j_0}$, there are less than $|k-k'|2^{j-j_0}+1$ admissible values for z and so,

$$\mathbb{E}[\Sigma_j(\xi)^2]\leq\tilde{c}\|\xi\|_\infty^2|k'-k|2^{j-j_0}.\qquad\square$$

We now decompose the function $t\to((\Phi^2)')^2(\int_0^t a(u)\mathrm{d}W_u^H)$ in a wavelet basis with support $[0,T]$. Thus, we use the same wavelet as before but in another context.

Lemma 6 *(Decomposition in a wavelet basis). Let $h = ((\Phi^2)')^2$. Let ϕ be the scaling function associated to ψ. We write $\phi_{0k}(t) = \phi(t-k)$,*

$$c_k = \int h\Big(\int_0^t a(u)\,dW_u^H\Big)\phi_{0k}(t)\,dt \text{ and } c_{jk} = \int h\Big(\int_0^t a(u)\,dW_u^H\Big)\psi_{jk}(t)\,dt.$$

Then,

$$h\Big(\int_0^t a(u)\,dW_u^H\Big) = \sum_{k=0}^{r} c_k\phi_{0k}(t) + \sum_{j=0}^{+\infty}\sum_{k=0}^{T(2^j-1)} c_{jk}\psi_{jk}(t),$$

where r is a constant value depending on T and with

$$\mathbb{E}[c_0 + \cdots + c_r] \le \tilde{c}, \quad \mathbb{E}[c_{jk}^2] \le c2^{-j(1+2H)}.$$

Proof. From now on, we write $J(s,t)$ for $\int_s^t a(v)\mathrm{d}W_v^H$. Since the first moment of ψ vanishes, we have

$$\begin{aligned} c_{jk} =& \int \psi_{jk}(t)h[J(0,t)]\mathrm{d}t = 2^{-j/2}\int_0^T \psi(u)h[J(0,(k+u)2^{-j})]\mathrm{d}u \\ =& 2^{-j/2}\int_0^T \psi(u)\{J(0,(k+u)2^{-j}) - J(0,k2^{-j})\}h'(\eta)\mathrm{d}u. \end{aligned}$$

Here, η is a value between $J(0,k2^{-j})$ and $J(0,(k+u)2^{-j})$. By the continuity of the sample path of the stochastic integral, we know there exists a random point θ between $k2^{-j}$ and $(k+u)2^{-j}$ such that $\eta = J(0,\theta)$. Thus, we have

$$c_{jk}^2 \le c2^{-j}\int_0^T \psi^2(u)\{J(k2^{-j},(k+u)2^{-j})\}^2(h'[J(0,\theta)])^2\mathrm{d}u.$$

By lemma 1 and because $J(k2^{-j},(k+u)2^{-j})$ is a Gaussian variable,

$$\mathbb{E}[J(k2^{-j},(k+u)2^{-j})^4] \le c\|a\|_\infty^4 2^{-j4H}.$$

By assumption B [3], $\{h'[J(0,\theta)]\}^2 \le c(1+|J(0,\theta)|^m)^2$. To control $\mathbb{E}|J(0,\theta)|$, since $J(0,t)$ is a Gaussian process starting from 0 with continuous trajectories, we can use Dudley's entropy bound: there exists a universal constant c such that

$$\mathbb{E}[\sup_{t\in[0,T]}|J(0,t)|\,] \le c\int_0^{d(0,T)}\sqrt{\log N(T,d,\varepsilon)}\mathrm{d}\varepsilon,$$

where $d^2(s,t) = \mathbb{E}[\,|J(0,t)-J(0,s)|^2]$ and $N(T,d,\varepsilon)$ is the minimal number of balls of radius ε needed to recover $[0,T]$. Since

$$\mathbb{E}[\,|J(0,t)-J(0,s)|^2] \le c|t-s|^{2H},$$

[3] Remark that Assumption B implies there exist c and m positive such that $|(\Phi^2)'(x)| \le c(1+|x|^m)$ and consequently there exist c and m such that $|(((\Phi^2)')^2)'(x)| \le c(1+|x|^m)$.

we easily get that $N(T,d,\varepsilon)$ is less than $cT\varepsilon^{-1/H}$. Using that H belongs to $(1/2,1)$, we easily bound $\mathbb{E}|J(0,\theta)|$ by a constant c. Then, since $J(0,t)$ is a Gaussian process, we also get $\mathbb{E}[\{h'[J(0,\theta)]\}^4] \le c$. Hence by Cauchy-Schwarz, $\mathbb{E}[c_{jk}^2] \le c2^{-j(1+2H)}$. By Taylor expansion of the coefficients, we also get $\mathbb{E}[c_0 + \cdots + c_r] \le \tilde{c}$. $\square$

Lemma 7 *Let h be as in lemma 6. We have*

$$\mathbb{E}\Big[\Big|2^{j(1+2H)}\sum_k\{\beta_{jk}^2 - c(\psi)2^{-j}a^2(k2^{-j})\}h\Big(\int_0^t a(u)dW_u^H\Big)\Big|\Big] \le c2^{\frac{j}{2}}$$

Proof. We know from lemma 6

$$h\Big(\int_0^t a(u)\mathrm{d}W_u^H\Big) = \sum_{k=0}^{r} c_k\phi_{0k}(t) + \sum_{j=0}^{+\infty}\sum_{k=0}^{T(2^j-1)} c_{jk}\psi_{jk}(t).$$

Let $S_j(h) = 2^{j(1+2H)}\sum_k\{\beta_{jk}^2 - c(\psi)2^{-j}a^2(k2^{-j})\}h\Big(\int_0^t a(u)\mathrm{d}W_u^H\Big)$. We can rewrite $S_j(h) = \sum_{k=0}^{r} c_k\Sigma_j(\phi_{0k}) + \sum_{j_1=0}^{+\infty} S_{j,j_1}$, with

$$S_{j,j_1} = \sum_{k=0}^{T(2^j-1)} c_{j_1k}\Sigma_j(\psi_{j_1k}).$$

For $i=0$ to k, $\mathbb{E}\Big[\,|c_i\Sigma_j(\phi_{0i})|\,\Big] \le c2^{j/2}$, by lemma 4. Now we prove that

$$\mathbb{E}\Big[\sum_{j_1=0}^{+\infty}|S_{j,j_1}|\Big] \le c2^{j/2}.$$

If $j_1 \le j$, by lemma 5,

$$\mathbb{E}\Big[\,|S_{j,j_1}|\,\Big] \le c\sum_{k=0}^{T(2^{j_1}-1)} 2^{-j_1(1+2H)/2}\Big(\mathbb{E}[\Sigma_j(\psi_{j_1k})^2]\Big)^{1/2} \le c2^{(1/2-H)j_1}2^{j/2}.$$

Because $H > 1/2$,

$$\sum_{j_1=0}^{j}\mathbb{E}\Big[\,|S_{j,j_1}|\,\Big] \le \tilde{c}2^{j/2}.$$

If $j < j_1$, ψ_{j_1k} has support $[k2^{-j_1},(k+T)2^{-j_1}]$, so $\Sigma_j(\psi_{j_1k}) = 0$ unless there exists $i \in [0,T(2^j-1)]$ such that $i2^{-j} \in [k2^{-j_1},(k+T)2^{-j_1}]$, that is:

$$k2^{j-j_1} \le i \le (k+T)2^{j-j_1}.$$

Thus, there are less than $c2^j$ possible values for i and moreover, for such i, the sum defining $\Sigma_j(\psi_{j_1k})$ is reduced to one single term, so, combining this result with lemma 2, we get

$$\mathbb{E}[\Sigma_j(\psi_{j_1k})^2] \leq c\|\psi_{j_1k}\|_\infty^2 \leq c2^{j_1},$$

and

$$\mathbb{E}\Big[\,|S_{j,j_1}|\,\Big] \leq c \sum_{1\leq k\leq T(2^{j_1}-1)} 2^{-j_1(1+2H)/2}2^{j_1/2} \leq c2^j2^{-j_1H}.$$

Finally

$$\sum_{j_1=0}^{+\infty}\mathbb{E}\Big[\,|S_{j,j_1}|\,\Big] = \sum_{j_1=0}^{j}\mathbb{E}\Big[\,|S_{j,j_1}|\,\Big] + \sum_{j_1=j+1}^{+\infty}\mathbb{E}\Big[\,|S_{j,j_1}|\,\Big] \leq c2^{j/2}. \qquad \square$$

Lemma 8 *(Riemann's approximation).*
Let $h = ((\Phi^2)')^2$ *and* $H(x,y,z) = a^2(x)h\Big(\int_y^z a(u)dW_u^H\Big)$. *Then,*

$$\mathbb{E}\Big[\Big|\frac{1}{n}\sum_{k=1}^{nT}H(k/n,0,k/n) - \int_0^T H(t,0,t)\,dt\Big|\Big] \leq cn^{-1/2}.$$

Proof. Let θ denote the value where the last term of a Taylor expansion is taken. Let

$$H_1'(x,y,z) = a^2(x)h'\Big(\int_y^z a(u)\mathrm{d}W_u^H\Big),$$
$$H_2'(x,y,z) = (a^2)'(x)h'\Big(\int_y^z a(u)\mathrm{d}W_u^H\Big).$$

We have that $\mathbb{E}\Big[\Big|\frac{1}{n}\sum_{k=1}^{nT}H(k/n,0,k/n) - \int_0^T H(t,0,t)\mathrm{d}t\Big|\Big]$ is less than

$$\sum_{k=1}^{nT}\int_{\frac{k-1}{n}}^{k/n}\mathbb{E}\Big[\,|(k/n-t)H_2'(\theta,0,k/n) - J(k/n,t)H_1'(t,0,\theta)|\,\Big]\mathrm{d}t.$$

By assumption B, it is less than

$$c\sum_{k=1}^{nT}\int_{\frac{k-1}{n}}^{k/n}\mathbb{E}\Big[(k/n-t)\|(a^2)'\|_\infty(|h(0)| + |J(0,k/n)||h'(\theta)|)$$
$$+\|a^2\|_\infty|J(k/n,t)|\{1+|J(0,\theta)|^m\}\Big]\mathrm{d}t,$$

which is less than

$$c\sum_{k=1}^{nT}\int_{\frac{k-1}{n}}^{k/n}\mathbb{E}\Big[(k/n-t)(1+|J(0,k/n)|\{1+|J(0,\theta)|^m\})$$
$$+|J(k/n,t)|\{1+|J(0,\theta)|^m\}\Big]\mathrm{d}t.$$

Finally, since $\mathbb{E}\Big[\,|J(k/n,t)|\,\Big] \leq cn^{-H}$ and $\mathbb{E}\Big[\,|J(0,\theta)|^m\,\Big] + \mathbb{E}\Big[\,|J(0,\theta)|\,\Big] \leq c$, we obtained the desired bound. $\square$

4.3 Proof of proposition 2

Proof. Recall that $J(s,t) = \int_s^t a(v)\mathrm{d}W_v^H$. Let θ denote the value where the last term of a Taylor expansion is taken. Let

$$\tilde{\beta}_{jk} = \int \psi_{jk}(t)\Phi^2\Big(\int_0^t a(u)\mathrm{d}W_u^H\Big)\mathrm{d}t,$$

which is equal to

$$2^{-\frac{j}{2}}(\Phi^2)'[J(0,k2^{-j})]\int_0^T \psi(u)J(k2^{-j},(k+u)2^{-j})\mathrm{d}u$$

$$+2^{-\frac{j}{2}}\int_0^T \psi(u)[J(k2^{-j},(k+u)2^{-j})]^2(\Phi^2)'(\theta)\mathrm{d}u.$$

This can be rewritten as

$$(\Phi^2)'[J(0,k2^{-j})]\beta_{jk} + 2^{-\frac{j}{2}}\int_0^T \psi(u)[J(k2^{-j},(k+u)2^{-j})]^2(\Phi^2)''(\theta)\mathrm{d}u.$$

So, $\tilde{\beta}_{jk}^2$ is equal to

$$h[J(0,k2^{-j})]\beta_{jk}^2 + 2^{-j}\Big(\int_0^T \psi(u)[J(k2^{-j},(k+u)2^{-j})]^2(\Phi^2)''(\theta)\mathrm{d}u\Big)^2$$

$$+2(\Phi^2)'[J(0,k2^{-j})]\beta_{jk}2^{-\frac{j}{2}}\int_0^T \psi(u)[J(k2^{-j},(k+u)2^{-j})]^2(\Phi^2)''(\theta)\mathrm{d}u.$$

Hence

$$\sum_k \Big\{2^{j2H}\tilde{\beta}_{jk}^2 - c(\psi)2^{-j}a^2(k2^{-j})h\Big(\int_0^{\frac{k}{2^j}} a(u)\mathrm{d}W_u^H\Big)\Big\} = T_1+T_2+T_3,$$

$$\begin{aligned}
\text{with}\quad T_1 &= \sum_k \{2^{j2H}\tilde{\beta}_{jk}^2 - c(\psi)2^{-j}a^2(k2^{-j})\}h[J(0,k2^{-j})],\\
T_2 &= 2^{j2H}\sum_k 2^{-j}\Big\{\int_0^T \psi(u)[J(k2^{-j},(k+u)2^{-j})]^2(\Phi^2)''(\theta)\mathrm{d}u\Big\}^2,\\
T_3 &= 2^{j2H}\sum_k T_{32}(k)T_{32}(k),\\
\text{and}\quad T_{31}(k) &= 2(\Phi^2)'[J(0,k2^{-j})]\beta_{jk},\\
T_{32}(k) &= 2^{-\frac{j}{2}}\int_0^T \psi(u)[J(k2^{-j},(k+u)2^{-j})]^2(\Phi^2)''(\theta)\mathrm{d}u.
\end{aligned}$$

Clearly, $\mathbb{E}\Big[|T_2|\Big] \le c2^{-j2H}$ and $\mathbb{E}\Big[|T_3|\Big] \le c2^{-jH}$. By lemma 7, we have $\mathbb{E}\Big[|T_1|\Big] \le c2^{-j/2}$ and we finally get the result by lemma 8. $\square$

5 Proof of proposition 3

Proof. We begin by the proof of (i). With the notation of assumption A, there exists $\eta > 0$ such that

$$c(\psi)\int_0^T a^2(u)\{(\Phi^2)'[J(0,u)]\}^2 \mathrm{d}u \geq \eta \int_\alpha^\beta \{(\Phi^2)'[J(0,u)]\}^2 \mathrm{d}u.$$

Let

$$Z = \eta \int_\alpha^\beta \{(\Phi^2)'[J(0,u)]\}^2 \mathrm{d}u.$$

Suppose there exists $\varepsilon > 0$ such that for all $r > 0$, $\mathbb{P}[Z \leq r] \geq \varepsilon$. Since $Z \geq 0$, $\mathbb{P}[Z = 0] \geq \varepsilon$. By assumption B, this implies $J(0,u) = 0$ on (α, β) with positive probability which is absurd by assumption A and because $J(0,u)$ is a continuous Gaussian process. Then, for $\varepsilon > 0$, there exists $r > 0$ such that

$$\mathbb{P}\Big[Z \geq 2r\Big] \geq 1 - \varepsilon.$$

By Markov's inequality, we have

$$\mathbb{P}\Big[2^{2jH}Q_j \notin [Z - r, Z + r]\Big] = \mathbb{P}\Big[\,|2^{2jH}Q_j - Z| > r\Big] \leq c\frac{2^{-j/2}}{r},$$

thus,

$$\sum_{j\geq 0} \sup_H \mathbb{P}\Big[2^{2jH}Q_j \notin [Z - r, Z + r]\Big] < +\infty.$$

Then, by Borel Cantelli's lemma, for large enough j a.s.

$$2^{2jH}Q_j \geq Z - r.$$

We now prove (ii). Let $\varepsilon > 0$, r and j_0 associated by proposition 3 (i) and $j \geq j_0$. We have

$$\begin{aligned}
\mathbb{P}\Big[2^{j/2}\sup_{l\geq j}\Big|\frac{Q_{l+1}}{Q_l} - 2^{-2H}\Big| \geq M\Big] &= \mathbb{P}[\sup_{l\geq j}|Q_{l+1} - 2^{-2H}Q_l| \geq MQ_l 2^{-j/2}] \\
&\leq \varepsilon + \mathbb{P}\Big[\sup_{l\geq j}|Q_{l+1} - 2^{-2H}Q_l| \geq M2^{-j/2}2^{-2lH}r\Big] \\
&\leq \varepsilon + \sum_{l\geq j\geq j_0}\mathbb{E}\Big[\,|Q_{l+1} - 2^{-2H}Q_l|\,\Big]2^{2lH}2^{j/2}\{Mr\}^{-1}.
\end{aligned}$$

But, $\mathbb{E}\Big[\,|Q_{l+1} - 2^{-2H}Q_l|\,\Big]$ is equal to

$$\mathbb{E}\Big[\,|Q_{l+1} - 2^{-2(l+1)H}Z + 2^{-2(l+1)H}Z - 2^{-2H}Q_l|\,\Big] \leq c2^{-l(2H+1/2)},$$

thus,

$$\mathbb{P}\Big[2^{j/2}\sup_{l\geq j}\Big|\frac{Q_{l+1}}{Q_l}-2^{-2H}\Big|\geq M\Big]\leq \varepsilon + c\sum_{l\geq j\geq j_0}2^{-l/2}2^{j/2}\{Mr\}^{-1}.$$

For large enough M, this can be made arbitrarily small. $\square$

6 Proof of proposition 4

Proof. With the notation of section 2.2.2, we have

$$\widehat{Q_j}-Q_j=\sum_k b_{jk}^2+\sum_k f_{jk}^2+\sum_k b_{jk}f_{jk}+\sum_k d_{jk}b_{jk}+\sum_k d_{jk}f_{jk}$$

$$+\sum_k(e_{jk}^2-\bar{\nu}_{jk})+\sum_k e_{jk}f_{jk}+\sum_k b_{jk}e_{jk}+\sum_k d_{jk}e_{jk}+\sum_k \nu_{jk}-\bar{\nu}_{jk}.$$

Following Gloter and Hoffmann [15], it is enough to prove

$$\sup_{J_n\geq j\geq j_n(H)-L}\ \sup_{H\in[H_-,H_+]}2^{-j/2}\mathbb{E}\Big[\,|\widehat{Q}_{j,n}-Q_j|\,\Big]\leq cn^{-1}.$$

Now we bound the 10 terms one by one.

- Term 1: let $V_{tl}=\sigma_t^2-\sigma_{k2^{-j}+l2^{-N}}^2$. We have

$$\begin{aligned}\mathbb{E}[b_{jk}^2]=&\sum_{l=0}^{T(2^{N-j}-1)}\sum_{l'=0}^{T(2^{N-j}-1)}\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\int_{\frac{k}{2^j}+\frac{l'}{2^N}}^{\frac{k}{2^j}+\frac{l'+1}{2^N}}\psi_{jk}(t)\psi_{jk}(t')\mathbb{E}[V_{tl}V_{tl'}]\mathrm{d}t\mathrm{d}t'\\ &\leq c2^j\sum_l\sum_{l'}\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\int_{\frac{k}{2^j}+\frac{l'}{2^N}}^{\frac{k}{2^j}+\frac{l'+1}{2^N}}\Big(\mathbb{E}[V_{tl}^2]\mathbb{E}[V_{tl'}^2]\Big)^{1/2}\mathrm{d}t\mathrm{d}t'.\end{aligned}$$

Moreover,

$$V_{tl}=(\Phi^2)'\Big(\int_0^{t-\frac{k}{2^j}+\frac{l}{2^N}}a(u)\mathrm{d}W_u^H\Big)\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+v}{2^N}}a(u)\mathrm{d}W_u^H,\text{ with v}\in[0,1].$$

By assumption B and the same arguments as previously, $\mathbb{E}[V_{tl}^2]\leq c2^{-2NH}$. Hence $\mathbb{E}[b_{jk}^2]\leq c2^{-j}n^{-1}$.

- Term 2 and term 3 follow easily with the same order.

• Term 4: we easily prove as in lemma 6 that $\mathbb{E}[d_{jk}^2] \leq c2^{-j(1+2H)}$ and then, because $j \geq \frac{1}{2H+1}\log_2(n)$, $\mathbb{E}\Big[\,|d_{jk}b_{jk}|\,\Big] \leq c2^{-j/2}n^{-1}$.

• Term 5 follows as term 4 with the same order.

• Term 6: we argue first conditional on W^H. We write $\tilde{\mathbb{E}}$ for the expectation conditional on W^H. Because of the independence of the Brownian increments and because the variables are centered, we have

$$\tilde{\mathbb{E}}\Big[\Big(\sum_k e_{jk}^2 - \nu_{jk}\Big)^2\Big] = \sum_k \tilde{\mathbb{E}}\Big[(e_{jk}^2 - \nu_{jk})^2\Big],$$

$$\tilde{\mathbb{E}}\Big[(e_{jk}^2 - \nu_{jk})^2\Big] = \mathrm{Var}[e_{jk}^2] \leq \tilde{\mathbb{E}}[e_{jk}^4 + \nu_{jk}^2].$$

Let

$$M_l = \Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t \,\mathrm{d}B_t\Big)^2 - \int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2 \mathrm{d}t.$$

Because the variables $M_l, \quad l = 0, \dots, T(2^{N-j}-1)$ are centered and independent, we get that $\tilde{\mathbb{E}}[e_{jk}^4]$ is equal to

$$\sum_{l=0}^{T(2^{N-j}-1)} \sum_{l'=0}^{T(2^{N-j}-1)} n^4\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t \int_{\frac{k}{2^j}+\frac{l'}{2^N}}^{\frac{k}{2^j}+\frac{l'+1}{2^N}} \psi_{jk}(t)\mathrm{d}t\Big)^2 \tilde{\mathbb{E}}[M_l^2 M_{l'}^2].$$

Indeed the product of terms of power 3 with terms of power 1 are equal to zero. But, we have the following equality in law

$$M_l^2 \overset{\mathcal{L}}{=} \Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2 \mathrm{d}t\Big)^2 (Z^2-1)^2,$$

with Z a standard Gaussian variable. Hence,

$$\tilde{\mathbb{E}}[M_l^4] = c\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2 \mathrm{d}t\Big)^4.$$

Now, we have

$$\mathbb{E}\Big[\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2 \mathrm{d}t\Big)^4\Big] \leq \iiiint \Big(\mathbb{E}[\sigma_{t_1}^8]\mathbb{E}[\sigma_{t_2}^8]\mathbb{E}[\sigma_{t_3}^8]\mathbb{E}[\sigma_{t_4}^8]\Big)^{1/4} \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4.$$

Moreover, there exists $\theta \in [0, T]$ such that,

$$\sigma_t^2 = \Phi^2\Big(\int_0^t a(u)\mathrm{d}W_u^H\Big) = \Phi^2(0) + \Phi^{2'}\Big(\int_0^\theta a(u)\mathrm{d}W_u^H\Big)\int_0^t a(u)\mathrm{d}W_u^H.$$

Since the stochastic integral is a Gaussian variable with finite moments, together with assumption B, we get $\mathbb{E}[\sigma_t^8] \leq c$. Hence $\mathbb{E}[e_{jk}^4] \leq cn^{-2}$. We have

$$\mathbb{E}[\nu_{jk}^2] = 4n^4 \sum_{l=0}^{T(2^{N-j}-1)} \sum_{l^4=0}^{T(2^{N-j}-1)} \Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \psi_{jk}(t)\mathrm{d}t \int_{\frac{k}{2^j}+\frac{l'}{2^N}}^{\frac{k}{2^j}+\frac{l'+1}{2^N}} \psi_{jk}(t)\mathrm{d}t \Big)^2$$
$$\mathbb{E}\Big[\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}} \sigma_t^2 \mathrm{d}t \Big)^2 \Big(\int_{\frac{k}{2^j}+\frac{l'}{2^N}}^{\frac{k}{2^j}+\frac{l'+1}{2^N}} \sigma_t^2 \mathrm{d}t \Big)^2 \Big].$$

In the same way as for $\mathbb{E}[e_{jk}^4]$, we get $\mathbb{E}[\nu_{jk}^2] \leq cn^{-2}$.

• Term 7: in the preceding proof, we have shown $\mathbb{E}[e_{jk}^2] \leq cn^{-1}$ and so we obtain $\mathbb{E}\Big[|f_{jk}e_{jk}| \Big] \leq c2^{-j/2}n^{-1}$.

• Term 8 follows exactly as term 7.

• Term 9: we argue first conditional on W^H. Because of the independence of the Brownian increments and because the variables are centered, we have

$$\tilde{\mathbb{E}}\Big[\Big(\sum_k e_{jk}d_{jk} \Big)^2 \Big] = \sum_k d_{jk}^2 \tilde{\mathbb{E}}[e_{jk}^2].$$

Again because of the independence of the Brownian increments and because the variables are centered, we have

$$d_{jk}^2\tilde{\mathbb{E}}[e_{jk}^2] = c \sum_{l_1}\sum_{l_2} \int_{\frac{k}{2^j}+\frac{l_1}{2^N}}^{\frac{k}{2^j}+\frac{l_1+1}{2^N}} \psi_{jk}(t_1)\sigma_{t_1}^2 \mathrm{d}t_1 \int_{\frac{k}{2^j}+\frac{l_2}{2^N}}^{\frac{k}{2^j}+\frac{l_2+1}{2^N}} \psi_{jk}(t_2)\sigma_{t_2}^2 \mathrm{d}t_2$$
$$\sum_{l_3} n^2 \Big(\int_{\frac{k}{2^j}+\frac{l_3}{2^N}}^{\frac{k}{2^j}+\frac{l_3+1}{2^N}} \psi_{jk}(t_3)\mathrm{d}t_3 \Big)^2 \Big(\int_{\frac{k}{2^j}+\frac{l_3}{2^N}}^{\frac{k}{2^j}+\frac{l_3+1}{2^N}} \sigma_{t_3}^2 \mathrm{d}t_3 \Big)^2.$$

So, we get

$$\mathbb{E}[d_{jk}^2 e_{jk}^2] = n^2 \sum_{l_3} \Big(\int_{\frac{k}{2^j}+\frac{l_3}{2^N}}^{\frac{k}{2^j}+\frac{l_3+1}{2^N}} \psi_{jk}(t_3)\mathrm{d}t_3 \Big)^2 \int_{\frac{k}{2^j}+\frac{l_3}{2^N}}^{\frac{k}{2^j}+\frac{l_3+1}{2^N}} \int_{\frac{k}{2^j}+\frac{l_3}{2^N}}^{\frac{k}{2^j}+\frac{l_3+1}{2^N}}$$

$$\mathbb{E}\Big[\int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}} \psi_{jk}(t_1)\sigma_{t_1}^2 \mathrm{d}t_1 \int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}} \psi_{jk}(t_2)\sigma_{t_2}^2\sigma_{t_3}^2\sigma_{t_4}^2 \mathrm{d}t_2 \Big] \mathrm{d}t_3 \mathrm{d}t_4.$$

Because of the vanishing moments of the wavelet, we have

$$
\begin{aligned}
&\mathbb{E}\Big[\int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}}\psi_{jk}(t_1)\sigma_{t_1}^2\mathrm{d}t_1\int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}}\psi_{jk}(t_2)\sigma_{t_2}^2\sigma_{t_3}^2\sigma_{t_4}^2\mathrm{d}t_2\Big]\\
&=\mathbb{E}\Big[\int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}}\psi_{jk}(t_1)\int_{\frac{k}{2^j}}^{\frac{k+T}{2^j}}\psi_{jk}(t_2)V_{t_10}V_{t_20}\sigma_{t_3}^2\sigma_{t_4}^2\mathrm{d}t_2\mathrm{d}t_1\Big]\\
&\leq c2^j2^{-2j}\Big(\mathbb{E}[V_{t_10}^4]\mathbb{E}[V_{t_20}^4]\Big)^{1/4}\leq c2^{-j}2^{-j2H}.
\end{aligned}
$$

Consequently, $\mathbb{E}[d_{jk}^2e_{jk}^2]\leq cn^{-1}2^{-3j}$, but, as $j\geq\dfrac{\log_2 n}{3}$, $\mathbb{E}[d_{jk}^2e_{jk}^2]\leq cn^{-2}$.

- Term 10: let $X=\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\sigma_t\,\mathrm{d}B_t\Big)^2$ and $X_i=\Big(\int_{\frac{k}{2^j}+\frac{l+i}{2^N}}^{\frac{k}{2^j}+\frac{l+i+1}{2^N}}\sigma_t\,\mathrm{d}B_t\Big)^2$. Then,

$$
\begin{aligned}
\nu_{jk}&=2\sum_{l=0}^{2^{N-j}-1}n^2\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\psi_{jk}(t)\mathrm{d}t\Big)^2\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\sigma_u^2\mathrm{d}u\Big)^2,\\
\bar{\nu}_{jk}&=2\sum_{l=0}^{2^{N-j}-1}n^2\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\psi_{jk}(t)\mathrm{d}t\Big)^2\Big(\frac{1}{h}\sum_{i=0}^{h}X_i\Big)^2,
\end{aligned}
$$

where $h=h(n)=\lfloor n^{1/2}\rfloor$. The term $\nu_{jk}-\bar{\nu}_{jk}$ is equal to

$$
\begin{aligned}
2\sum_{l=0}^{2^{N-j}-1}n^2\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\psi_{jk}(t)\mathrm{d}t\Big)^2\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\sigma_u^2\mathrm{d}u+\frac{1}{h}\sum_{i=0}^{h}X_i\Big)&\\
\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\sigma_u^2\mathrm{d}u-\frac{1}{h}\sum_{i=0}^{h}X_i\Big)&.
\end{aligned}
$$

We argue first conditional on W^H. We have

$$
\begin{aligned}
&\tilde{\mathbb{E}}\Big[\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\sigma_u^2\mathrm{d}u-\frac{1}{h}\sum_{i=0}^{h}X_i\Big)^2\Big]\\
&\leq c\tilde{\mathbb{E}}\Big[\Big(\frac{1}{h}\sum_{i=0}^{h}\{X_i-\tilde{\mathbb{E}}[X_i]\}\Big)^2\Big]+c\tilde{\mathbb{E}}\Big[\Big(\frac{1}{h}\sum_{i=0}^{h}\tilde{\mathbb{E}}[X_i]-\tilde{\mathbb{E}}[X]\Big)^2\Big],
\end{aligned}
$$

with the following equality in law

$$
X_i-\tilde{\mathbb{E}}[X_i]\overset{\mathcal{L}}{=}\Big(\int_{\frac{k}{2^j}+\frac{l+i}{2^N}}^{\frac{k}{2^j}+\frac{l+i+1}{2^N}}\sigma_t^2\mathrm{d}t\Big)(Z^2-1),
$$

with Z a standard Gaussian variable. Now,

$$
\mathbb{E}\Big[\Big(\int_{\frac{k}{2^j}+\frac{l+i}{2^N}}^{\frac{k}{2^j}+\frac{l+i+1}{2^N}}\sigma_t^2\mathrm{d}t\Big)^2\Big]\leq c2^{-2N}.
$$

Then, by independence of the Brownian increments and because the variables are centered,

$$\tilde{\mathbb{E}}\Big[\Big(\frac{1}{h}\sum_{i=0}^{h}\{X_i-\tilde{\mathbb{E}}[X_i]\}\Big)^2\Big]=\frac{1}{h^2}\sum_{i=0}^{h}\tilde{\mathbb{E}}\Big[(X_i-\tilde{\mathbb{E}}X_i)^2\Big]\le\frac{c}{h}2^{-2N}.$$

For the other term, we have

$$\mathbb{E}\Big[\Big(\frac{1}{h}\sum_{i=0}^{h}\tilde{\mathbb{E}}X_i-\tilde{\mathbb{E}}X\Big)^2\Big]=\mathbb{E}\Big[\Big(\frac{1}{h}\sum_{i=0}^{h}\int_{\frac{k}{2^j}+\frac{l+i}{2^N}}^{\frac{k}{2^j}+\frac{l+i+1}{2^N}}\sigma_t^2\mathrm{d}t-\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\sigma_t^2\mathrm{d}t\Big)^2\Big]$$
$$=\frac{1}{h^2}\sum_{i=0}^{h}\sum_{g=0}^{h}\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\mathbb{E}\Big[(\sigma^2_{u+i2^{-N}}-\sigma^2_u)(\sigma^2_{v+g2^{-N}}-\sigma^2_v)\Big]\mathrm{d}u\mathrm{d}v$$
$$\le\frac{c}{h^2}\sum_{i=0}^{h}\sum_{g=0}^{h}\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}(ig)^H2^{-2NH}\mathrm{d}u\mathrm{d}v\le c2^{-2N}n^{-1}h^2.$$

Eventually,

$$\mathbb{E}\Big[\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\sigma_u^2\mathrm{d}u-\frac{1}{h}\sum_{i=0}^{h}X_i\Big)^2\Big]\le c\frac{n^{-2}}{\sqrt{n}}.$$

We easily check that the term

$$\mathbb{E}\Big[\Big(\int_{\frac{k}{2^j}+\frac{l}{2^N}}^{\frac{k}{2^j}+\frac{l+1}{2^N}}\sigma_u^2\mathrm{d}u+\frac{1}{h}\sum_{i=0}^{h}X_i\Big)^2\Big]$$

is less than cn^{-2} and finally $\mathbb{E}\Big[|\nu_{jk}-\bar{\nu}_{jk}|\Big]\le cn^{-1}2^{-j/2}$, because $j\le\dfrac{\log_2 n}{n}$. □

7 Proof of theorem 1

We now prove that proposition 3 and 4 together imply theorem 1.

Proof. Following lemma 1 of Gloter and Hoffmann [15], we easily obtain that for all ε positive, there exist n_0 and $M>0$, such that for all $n\ge n_0$,

$$\mathbb{P}[n^{1/(4H+2)}|\widehat{H}_n-H|\ge M]\le\varepsilon. \tag{11}$$

With no loss of generality, we may demand $\widehat{H}\le C$, with $C>1$ a constant value, by considering $\widetilde{H}=\widehat{H}I_{|\widehat{H}|\le C}$. Let $\varepsilon>0$, n_0, M associated by (11). For $n\ge n_0$, if $(C-1)n^{1/(4H+2)}>M$, we have

$$\mathbb{P}[\widehat{H}_n\ge C]\le\mathbb{P}[n^{1/(4H+2)}|\widehat{H}_n-H|\ge(C-1)n^{1/(4H+2)}]\le\varepsilon.$$

Let $n_0^* \geq n_0$ such that $(C-1)n_0^* \geq M$. For all $n \leq n_0^*$,

$$n^{1/(4H+2)}|\widetilde{H}_n - H| \leq (C+1)(n_0^*)^{1/(4H+2)}.$$

Let $M_1 = \max\{M, (C+1)(n_0^*)^{1/(4H+2)}\}$. For all n,

$$\mathbb{P}[n^{1/(4H+2)}|\widetilde{H}_n - H| \geq M_1] \leq \varepsilon. \qquad \square$$

8 Proof of theorem 2

8.1 Proof of proposition 5

Proof. We observe

$$\Big\{Y_{i/n} = y_0 + \int_0^{i/n} \Phi\Big(\int_0^s a(u)\mathrm{d}W_u^H\Big)\mathrm{d}B_s, \quad i = 1, \dots, nT\Big\}.$$

Consider the equivalent sample

$$\Big\{Z_{i/n} = Y_{i/n} - Y_{(i-1)/n}, \quad i = 1, \dots, nT\Big\}.$$

Conditional on $W^H = f$, $Z_{i/n}$ is a centered Gaussian variable with variance σ_i where

$$\sigma_i = \int_{(i-1)/n}^{i/n} \Phi^2\Big(\int_0^s a(u)\mathrm{d}f_u\Big)\mathrm{d}s.$$

Moreover, conditional on W^H, the observations are independent. We define by $K(\mu,\nu) = \int(\log\frac{\mathrm{d}\mu}{\mathrm{d}\nu})\mathrm{d}\mu \leq +\infty$ the Kullback-Leibler divergence between two probability measures μ and ν. We recall the classical Pinsker's inequality $\|\mu-\nu\|_{TV} \leq \sqrt{2}K(\mu,\nu)^{1/2}$. Let $\mathbb{P}_f^n$ be the law of the sample conditional on $W^H = f$, let

$$\beta_i = \int_{(i-1)/n}^{i/n} \Phi^2\Big(\int_0^s a(u)\mathrm{d}g_u\Big)\mathrm{d}s.$$

We have

$$\|\mathbb{P}_f^n - \mathbb{P}_g^n\|_{TV} \leq \sqrt{2}K(\mathbb{P}_f^n, \mathbb{P}_g^n)^{1/2}.$$

By classical computations, we get

$$K(\mathbb{P}_f^n, \mathbb{P}_g^n) = \frac{1}{2}\sum_{i=1}^{nT}\Big(-\log\frac{\sigma_i}{\beta_i} - 1 + \frac{\sigma_i}{\beta_i}\Big).$$

By assumption C, we have $(c_4/c_5)^2 \leq \sigma_i/\beta_i \leq (c_5/c_4)^2$. Let $a = (c_4/c_5)^2$, $b = (c_5/c_4)^2$ and $c \geq 1/2$. Consider

$$z(x) = \log x - 1 + 1/x - c(x-1)^2, \ x \in [a,b].$$

We have $z(a) = \log a - 1 + 1/a - c(a-1)^2$, so, if $c \geq \frac{\log a - 1 + 1/a}{(a-1)^2}$, we have $z(a) \leq 0$. Take

$$c = c^* = \max\Big(\frac{1}{2}, \frac{\log a - 1 + 1/a}{(a-1)^2}\Big).$$

Hence z is negative on $[a, b]$, consequently, $K(\mathbb{P}^n_f, \mathbb{P}^n_g)$ is less than

$$\begin{aligned}
&\frac{\sqrt{2}}{2} c^* \sum_{i=1}^{nT} \left(\frac{\beta_i}{\sigma_i} - 1\right)^2 \\
&\leq \tilde{c} n^2 \sum_{i=1}^{nT} \left(\int_{(i-1)/n}^{i/n} \Big| \Phi\Big(\int_0^s a(u) \mathrm{d} f_u \Big) - \Phi\Big(\int_0^s a(u) \mathrm{d} g_u \Big) \Big| \mathrm{d}s \right)^2 \\
&\leq \tilde{c} n \int_0^T \left| \int_0^s a(u) \mathrm{d} f_u - \int_0^s a(u) \mathrm{d} g_u \right|^2 \mathrm{d}s \\
&\leq \tilde{c} n \int_0^T \left| a(s)(f(s) - g(s)) + \int_0^s a'(u)(g(u) - f(u)) \mathrm{d}u \right|^2 \mathrm{d}s \\
&\leq \tilde{c} n \|f - g\|_2^2. \qquad \square
\end{aligned}$$

8.2 Proof of theorem 2

Proposition 5 together with proposition 5 of Gloter and Hoffmann [15] imply the lower bound.

9 Appendix: proof of proposition 1

The link between Besov spaces and Gaussian processes has been largely studied, see in particular Ciesielski, Kerkyacharian and Roynette [7]. Nualart and Ouknine [28] study in particular the case of the stochastic integral driven by a fractional Brownian motion. We give here some simple proofs for our case. Let (ϕ, ψ) be a wavelet basis,

$$\alpha_{0k} = \int f(x) \phi_{0k}(x) \mathrm{d}x, \qquad \beta_{jk} = \int f(x) \psi_{jk}(x) \mathrm{d}x.$$

Recall that in term of wavelets coefficients, the Besov spaces $\mathcal{B}^s_{p,q}$, with $s \in [0,1]$, $1 \leq p, q < \infty$ are Banach spaces on $[0, T]$ equipped with the norm

$$\|f\|_{\mathcal{B}^s_{p,q}} = \|\alpha_{0.}\|_{l_p} + \left(\sum_j \left(2^{j(s-1/p+1/2)} \|\beta_{j.}\|_{l_p} \right)^q \right)^{1/q},$$

where

$$\|\beta_{j.}\|_{l_p} = \Big(\sum_k |\beta_{jk}|^p\Big)^{1/p}.$$

If p or q is equal to ∞, then the corresponding norm in p or q is replaced by the sup norm. For details, we refer to Ciesielski, Kerkyacharian and Roynette [7].

First, we show that the trajectory of $t \to \sigma_t^2$ belongs a.s. to $\mathcal{B}^H_{2,\infty}$. It is enough to prove that $\sup_j 2^{2jH} Q_j < \infty$. We know that for all positive ε, there exist j_0 and $M > 0$ such that

$$\mathbb{P}\Big[2^{j/2} \sup_{l \geq j \geq j_0} \Big|\frac{Q_{l+1}}{Q_l} - 2^{-2H}\Big| \geq M\Big] \geq \varepsilon.$$

This implies that

$$\mathbb{P}\Big[\exists j_0, \exists M, 2^{j/2} \sup_{l \geq j \geq j_0} \Big|\frac{Q_{l+1}}{Q_l} - 2^{-2H}\Big| \leq M\Big] = 1.$$

Let $u_j = 2^{2jH} Q_j$. For such j_0, for all $j \geq j_0$, $|u_{j+1}/u_j| \leq 1 + \tilde{M} 2^{-j/2}$. Thus, $\log u_{j+1} - \log u_j \leq \log(1 + \tilde{M} 2^{-j/2}) \leq \tilde{M} 2^{-j/2}$ and $\log u_n \leq c$. Hence the trajectory belongs a.s. to $\mathcal{B}^H_{2,\infty}$. Nevertheless, it does not belong to $\mathcal{B}^H_{2,q}$, $q < \infty$, as a matter of fact, for all ε positive, there exist j_0 and $r > 0$ such that for all $j \geq j_0$, $\mathbb{P}[2^{2jH} Q_j \geq r] \geq 1 - \varepsilon$. So, almost surely,

$$\sum_{j=0}^{+\infty} (2^{2jH} Q_j)^q = +\infty.$$

The fact that for $s < H$, the trajectory belongs almost surely to $\mathcal{B}^s_{\infty,\infty}$ is clear by Kolmogorov's criterion and preceding calculations on the expectations. We now prove that it does not belong to $\mathcal{B}^H_{\infty,\infty}$. We take $[s,t]$ on which a is positive. Suppose that almost surely, there exists $\tilde{c}$ such that for all (s,t),

$$\Big|\Phi^2\Big(\int_0^t a(u)\mathrm{d}W_u^H\Big) - \Phi^2\Big(\int_0^s a(u)\mathrm{d}W_u^H\Big)\Big| \leq \tilde{c}|t-s|^H.$$

Because there exists $c > 0$ such that for all x, $|(\Phi^2)'(x)| > c$, this implies

$$\Big|\int_s^t a(u)\mathrm{d}W_u^H\Big| \leq c|t-s|^H.$$

Ito's formula gives: $\Big|\dfrac{W_t^H a(t) - W_s^H a(s)}{(t-s)^H} - \dfrac{(t-s)^{1-H}\int_s^t a'(u) W_u^H \mathrm{d}u}{(t-s)}\Big| \leq c$. If s tends to t, $\dfrac{\int_s^t a'(u) W_u^H \mathrm{d}u}{t-s}$ tends to $a'(t)W_t^H$ and so for fixed $\varepsilon > 0$, for $|t-s|$

small enough,

$$\left|\frac{(t-s)^{1-H}\int_s^t a'(u)W_u^H \mathrm{d}u}{t-s}\right| \leq \varepsilon$$

and consequently,

$$\left|\frac{(W_t^H - W_s^H)a(t)}{(t-s)^H} - \frac{W_s^H(a(s)-a(t))}{(t-s)^H}\right| \leq c+\varepsilon.$$

Eventually, because a is positive, we get for $|t-s|$ small enough:

$$\left|\frac{W_t^H - W_s^H}{(t-s)^H}\right| \leq \frac{c+2\varepsilon}{\min_x a(x)},$$

which is absurd because the fbm is H Hölderian on no interval (this is a consequence of a law of the iterated logarithm shown by Arcones, see [2]).

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