

INSTITUT NATIONAL DE LA STATISTIQUE ET DES ETUDES ECONOMIQUES  
Série des Documents de Travail du CREST  
(Centre de Recherche en Economie et Statistique)

**n° 2005-49**

**A LARCH ( $\infty$ ) Vector Valued  
Process**

**P. DOUKHAN<sup>1</sup>**  
**G. TEYSSIERE<sup>2</sup>**  
**P. WINANT<sup>3</sup>**

Les documents de travail ne reflètent pas la position de l'INSEE et n'engagent que leurs auteurs.

Working papers do not reflect the position of INSEE but only the views of the authors.

---

<sup>1</sup> CREST, Laboratoire de Statistique, 3 avenue Pierre Larousse, 92245 Malakoff Cedex, France ([doukhan@ensae.fr](mailto:doukhan@ensae.fr))

<sup>2</sup> Statistique Appliquée et Modélisation Stochastique, Université Paris 1, Centre Pierre Mendès France, 90 rue de Tolbiac, 75634 Paris Cedex 13. France. ([stats@gillesteysiere.net](mailto:stats@gillesteysiere.net))

<sup>3</sup> ENS Lyon, 46 allée d'Italie, 69384 Lyon, France. ([pablo.winant@ens-lyon.org](mailto:pablo.winant@ens-lyon.org)).

---

# A LARCH( $\infty$ ) Vector Valued Process

Paul Doukhan<sup>1</sup>, Gilles Teyssière<sup>2</sup> and Pablo Winant<sup>3</sup>

<sup>1</sup> Laboratoire de Statistique, CREST. Timbre J340, 3, avenue Pierre Larousse, 92240 Malakoff, France. [doukhan@ensae.fr](mailto:doukhan@ensae.fr)

<sup>2</sup> Statistique Appliquée et MODélisation Stochastique, Université Paris 1, Centre Pierre Mendès France, 90 rue de Tolbiac, F-75634 Paris Cedex 13. France. [stats@gillesteysiere.net](mailto:stats@gillesteysiere.net)

<sup>3</sup> ENS Lyon, 46 allée d'Italie, 69384 Lyon, France. [pablo.winant@ens-lyon.org](mailto:pablo.winant@ens-lyon.org)

## Résumé

Nous introduisons une variante multivariée de l'équation LARCH( $\infty$ ) qui conduit à une approche simple de nombreux modèles comme le modèle bilinéaire ou les modèles GARCH. Pour y parvenir, nous donnons un développement chaotique de la solution de cette équation ARCH( $\infty$ ) et nous prouvons l'unicité de la solution sous des hypothèses raisonnables. Des approximations  $m$ -dépendantes ou  $N$ -Markoviennes de ce processus en permettent la simulation ; des bornes des coefficients de dépendance faible introduits par Doukhan et Louhichi (1999) sont aussi obtenues. Finalement les propriétés de dépendance à longue portée de tels modèles sont aussi considérées.

**Summary.** We introduce a vector version of the LARCH( $\infty$ ) equation yielding a simple approach to various models like bilinear or GARCH models. To this aim we provide an explicit chaotic expansion of a solution for this ARCH( $\infty$ ) equation, and show the uniqueness of this solution under reasonable conditions. Independent or  $N$ -Markov approximations of this process allow to simulate a trajectory or to derive bounds for their weak dependence coefficients as defined by Doukhan and Louhichi (1999). We finally consider a long range dependent version of this model; in this case we provide an existence and uniqueness result.

## 1 Introduction

The purpose of this chapter is to propose a unified framework for the study of ARCH( $\infty$ ) processes that are commonly used in the financial econometrics literature. We extend the study, based on Volterra expansions, of univariate ARCH( $\infty$ ) processes by Giraitis *et al.* [12] and Giraitis and Surgailis [11] to the multi-dimensional case.

Let  $\{\xi_t\}_{t \in \mathbb{Z}}$  be a sequence of real valued random independent and identically distributed matrices of size  $d \times m$ ,  $\{a_j\}_{j \in \mathbb{N}^*}$  be a sequence of real matrices  $m \times d$ , and  $a$  be a real vector of dimension  $m$ . The vector LARCH( $\infty$ ) process is defined as the solution to the recurrence equation:

$$X_t = \xi_t \left( a + \sum_{j=1}^{\infty} a_j X_{t-j} \right). \quad (1)$$

The following section 2 displays a chaotic expansion solution to this equation; we also consider a random field extension of this model. Some approximations of this solution are listed in the next section 3, where we consider approximations by  $m$ -dependent sequences, coupling results and approximations by Markov sequences. Section 4 details the weak dependence properties of the model and section 5 provides an existence and uniqueness condition for the solution of the previous equation; in that case, long range dependence may occur. The end of this section is dedicated to review examples of this vector valued model.

The vector LARCH( $\infty$ ) model nests a large variety of models, both first extensions being obvious:

1. The univariate linear LARCH( $\infty$ ) model, where the  $X_t$  and  $a_j$  are scalar,
2. The bilinear model, with

$$X_t = \zeta_t \left( \alpha + \sum_{j=1}^{\infty} \alpha_j X_{t-j} \right) + \beta + \sum_{j=1}^{\infty} \beta_j X_{t-j},$$

where all variables are scalar, and  $\zeta_t$  are iid centered innovations. We set

$$\xi_t = (\zeta_t, 1), \quad a = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad a_j = \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}.$$

In that case, the expansion (3) is the same as the one used by Giraitis and Surgailis [11].

3. With a suitable re-parameterization, this vector LARCH( $\infty$ ) includes the standard GARCH-type processes used in the financial econometrics literature for modeling the non-linear structure of the conditional second moments. The GARCH( $p, q$ ) model is defined as

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \sum_{j=1}^p \beta_j \sigma_{t-j}^2 + \gamma_0 + \sum_{j=1}^q \gamma_j r_{t-j}^2 \quad \gamma_0 > 0, \quad \gamma_j \geq 0, \quad \beta_i \geq 0, \end{aligned}$$

where the  $\varepsilon$  are centered and iid. This model is nested in the class of bilinear models with the following re-parameterization

$$\alpha_0 = \frac{\gamma_0}{1 - \sum \beta_i}, \quad \sum \alpha_i z^i = \frac{\sum \gamma_i z^i}{1 - \sum \beta_i z^i},$$

see Giraitis *et al.* [10]. The covariance function of the sequence  $\{r_t^2\}$  has an exponential decay, which is implied by the exponential decay of the sequence of weights  $\alpha_j$ ; see Giraitis *et al.* [12].

4. The ARCH( $\infty$ ) model, where the sequence of weights  $\beta_j$  might have either an exponential decay or a hyperbolic decay.

$$r_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \beta_0 + \sum_{j=1}^{\infty} \beta_j r_{t-j}^2,$$

with the following parameterization

$$X_t = r_t^2, \quad \xi_t = \left( \frac{\varepsilon_t^2 - \lambda_1}{\kappa}, 1 \right), \quad a = \begin{pmatrix} \kappa \beta_0 \\ \lambda_1 \beta_0 \end{pmatrix}, \quad a_j = \begin{pmatrix} \kappa \beta_j \\ \lambda_1 \beta_j \end{pmatrix},$$

where the  $\varepsilon$  are centered and iid,  $\lambda_1 = E(\varepsilon_0^2)$ , and  $\kappa^2 = \text{Var}(\varepsilon_0^2)$ . Note that the first coordinate of  $\xi_0$  is thus a centered random variable. Conditions for stationarity of the unidimensional ARCH( $\infty$ ) model have been derived using Volterra expansions by Giraitis *et al.* [12] and Giraitis and Surgailis [11]. The present paper is a multidimensional generalization of these previous works.

5. We can consider models with several innovations and variables like:

$$\begin{aligned} Z_t &= \zeta_{1,t} \left( \alpha + \sum_{j=1}^{\infty} \alpha_j^1 Z_{t-j} \right) + \mu_{1,t} \left( \beta + \sum_{j=1}^{\infty} \beta_j^1 Y_{t-j} \right) + \gamma + \sum_{j=1}^{\infty} \gamma_j^1 Z_{t-j} \\ Y_t &= \zeta_{2,t} \left( \alpha + \sum_{j=1}^{\infty} \alpha_j^2 Y_{t-j} \right) + \mu_{2,t} \left( \beta + \sum_{j=1}^{\infty} \beta_j^2 Z_{t-j} \right) + \gamma + \sum_{j=1}^{\infty} \gamma_j^2 Y_{t-j} \end{aligned}$$

This model is straightforwardly described through equation (1) with  $d = 2$

and  $m = 3$ . Here  $\xi_t = \begin{pmatrix} \zeta_{1,t} & \mu_{1,t} & 1 \\ \zeta_{2,t} & \mu_{2,t} & 1 \end{pmatrix}$  is a  $2 \times 3$  iid sequence,  $a_j = \begin{pmatrix} \alpha_j^1 & \alpha_j^2 \\ \beta_j^1 & \beta_j^2 \\ \gamma_j^1 & \gamma_j^2 \end{pmatrix}$

is a  $3 \times 2$  matrix and  $a = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$  is a vector in  $\mathbb{R}^3$  and the process

$X_t = \begin{pmatrix} Z_t \\ Y_t \end{pmatrix}$  is a vector of dimension 2. Dimensions  $m = 3$  and  $d = 2$  are only set here for simplicity. Replacing  $m = 3$  by  $m = 6$  would allow to consider different coefficients  $\alpha, \beta$  and  $\gamma$  for both lines in this system of two coupled equations.

This generalizes the class of multivariate ARCH( $\infty$ ) processes, defined in the  $p$ -dimensional case as:

$$R_t = \Sigma_t^{\frac{1}{2}} \varepsilon_t,$$

where  $R_t$  is a  $p$ -dimensional vector,  $\Sigma_t$  is a  $p \times p$  positive definite matrix, and  $\varepsilon_t$  is a  $p$ -dimensional vector. Those models are formally investigated by Farid Boussama in [2]; published references include [3] and [9]. This model is of interest in financial econometrics as the volatility of asset prices of linked markets, e.g., major currencies in the Foreign Exchange (FX) market are correlated, and in some cases display a common strong dependence structure; see [18]. This common dependence structure can be modeled with the assumption that the innovations  $\varepsilon_1, \dots, \varepsilon_p$  are correlated. An (empirically) interesting case for the bivariate model  $(X_t, Y_t)$  is obtained with the assumption that the  $(\zeta_{1,t}, \zeta_{2,t})$  are cross-correlated.

## 2 Existence and Uniqueness in $L^p$

In the sequel, we set  $A(x) = \sum_{j \geq x} \|a_j\|$ ,  $A = A(1)$ , where  $\|\cdot\|$  denotes the matrix norm.

**Theorem 2.1** *Let  $p > 0$ , we denote*

$$\varphi = \sum_{j \geq 1} \|a_j\|^{p \wedge 1} (E\|\xi_0\|^p)^{\frac{1}{p \wedge 1}}. \quad (2)$$

*If  $\varphi < 1$ , then a stationary solution in  $L^p$  to equation (1) is given by:*

$$X_t = \xi_t \left( a + \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_k \geq 1} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\dots-j_k} a \right). \quad (3)$$

*Proof.* The norm used for the matrices is any multiplicative norm. We have to show that expression (3) is well defined under the conditions stated above, converges absolutely in  $L^p$ , and that it satisfies equation (1).

*Step 1.* We first show that expression (3) is well defined (after the second line we omit to precise the norms). For  $p \geq 1$ , we have

$$\begin{aligned} & \sum_{j_1, \dots, j_k \geq 1} \|a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\dots-j_k}\|_{m \times m} \\ & \leq \sum_{j_1, \dots, j_k \geq 1} \|a_{j_1}\|_{m \times d} \cdots \|a_{j_k}\|_{m \times d} \|\xi_{t-j_1}\|_{d \times m} \cdots \|\xi_{t-j_1-\dots-j_k}\|_{d \times m} \end{aligned}$$

The series thus converges in norm  $L^p$  because

$$\begin{aligned}
 & \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_k \geq 1} (E \| a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\dots-j_k} \|^p)^{1/p} \\
 & \leq \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_k \geq 1} \| a_{j_1} \| \cdots \| a_{j_k} \| (E \| \xi_{t-j_1} \|^p)^{1/p} \cdots (E \| \xi_{t-j_1-\dots-j_k} \|^p)^{1/p} \\
 & \leq \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_k \geq 1} \| a_{j_1} \| \cdots \| a_{j_k} \| (E \| \xi_0 \|^p)^{\frac{k}{p}} \\
 & \leq \sum_{k=1}^{\infty} \varphi^k
 \end{aligned}$$

The series  $\sum_{k=1}^{\infty} \varphi^k$  is finite since  $\varphi < 1$ , hence the series (3) converges in  $L^p$ .

For  $p < 1$ , the convergence is defined through the metric  $d_p(U, V) = E \| U - V \|^p$  between vector valued  $L^p$  random variables  $U, V$  and we start from

$$\begin{aligned}
 & \left( \sum_{j_1, \dots, j_k \geq 1} \| a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\dots-j_k} \| \right)^p \\
 & \leq \sum_{j_1, \dots, j_k \geq 1} \| a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\dots-j_k} \|^p,
 \end{aligned}$$

and we use the same arguments as for  $p = 1$ .

*Step 2.* We now show that equation (3) is solution to equation (1):

$$\begin{aligned}
 X_t &= \xi_t \left( 1 + \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_k \geq 1} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\dots-j_k} \right) a \\
 &= \xi_t \left( a + \sum_{j_1 \geq 1} a_{j_1} \xi_{t-j_1} + \right. \\
 & \quad \left. + \sum_{k=2}^{\infty} \sum_{j_1 \geq 1} a_{j_1} \xi_{t-j_1} \sum_{j_2, \dots, j_k \geq 1} a_{j_2} \xi_{t-j_1-j_2} \cdots a_{j_k} \xi_{t-j_1-j_2-\dots-j_k} a \right) \\
 &= \xi_t \left( a + \sum_{j_1 \geq 1} a_{j_1} \xi_{t-j_1} \left( a + \right. \right. \\
 & \quad \left. \left. + \sum_{k=2}^{\infty} \sum_{j_2, \dots, j_k \geq 1} a_{j_2} \xi_{(t-j_1)-j_2} \cdots a_{j_k} \xi_{(t-j_1)-j_2-\dots-j_k} a \right) \right) \\
 &= \xi_t \left( a + \sum_{j \geq 1} a_j X_{t-j} \right).
 \end{aligned}$$

**Remark 2.1** *The uniqueness of this solution is not demonstrated without additional condition; see Theorem 2.2 and section 5 below.*

**Theorem 2.2** *Assume that  $p \geq 1$  then from (2),  $\varphi = \sum_j \|a_j\| \|\xi_0\|_p$ . Assume  $\varphi < 1$ . If a stationary solution  $(Y_t)_{t \in \mathbf{Z}}$  to equation (1) exists (a.s.), if  $Y_t$  is independent of the sigma-algebra generated by  $\{\xi_s; s > t\}$ , for each  $t \in \mathbf{Z}$ , then this solution is also in  $L^p$  and it is (a.s.) equal to the previous solution  $(X_t)_{t \in \mathbf{Z}}$  defined by equation (3).*

*Proof. Step 1.* We first prove that  $\|Y_0\|_p < \infty$ . From equation (1) and from  $\{Y_t\}_{t \in \mathbf{Z}}$ 's stationarity, we derive

$$\|Y_0\|_p \leq \|\xi_0\|_p \left( \|a\| + \sum_{j=1}^{\infty} \|a_j\| \|Y_0\|_p \right) < \infty,$$

hence, the first point in the theorem follows from:

$$\|Y_0\|_p \leq \frac{\|\xi_0\|_p \|a\|}{1 - \varphi} < \infty.$$

*Step 2.* As in [12] we write recursively  $Y_t = \xi_t \left( a + \sum_{j \geq 1} a_j Y_{t-j} \right) = X_t^m + S_t^m$ , with

$$\begin{aligned} X_t^m &= \xi_t \left( a + \sum_{k=1}^m \sum_{j_1, \dots, j_k \geq 1} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1 \cdots -j_k} a \right), \\ S_t^m &= \xi_t \left( \sum_{j_1, \dots, j_{m+1} \geq 1} a_{j_1} \xi_{t-j_1} \cdots a_{j_m} \xi_{t-j_1 \cdots -j_m} a_{j_{m+1}} Y_{t-j_1 \cdots -j_m} \right). \end{aligned}$$

We have

$$\|S_t^m\|_p \leq \|\xi\|_p \sum_{j_1, \dots, j_{m+1} \geq 1} \|a_{j_1}\| \cdots \|a_{j_{m+1}}\| \|\xi\|_p^m \|Y_0\|_p = \|Y_0\|_p \varphi^{m+1}.$$

We recall the additive decomposition of the chaotic expansion  $X_t$  in equation (3) as a finite expansion plus a negligible remainder that can be controlled  $X_t = X_t^m + R_t^m$  where

$$R_t^m = \xi_t \left( \sum_{k > m} \sum_{j_1, \dots, j_k \geq 1} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1 \cdots -j_k} a \right),$$

satisfies

$$\|R_t^m\|_p \leq \|a\| \|\xi_0\|_p \sum_{k > m} \varphi^k \leq \|a\| \|\xi_0\|_p \frac{\varphi^m}{1 - \varphi} \rightarrow 0.$$

Then, the difference between those two solutions is controlled as a function of  $m$  with  $X_t - Y_t = R_t^m - S_t^m$ , hence

$$\begin{aligned} \|X_t - Y_t\|_p &\leq \|R_t^m\|_p + \|S_t^m\|_p \\ &\leq \frac{\varphi^m}{1 - \varphi} \|a\| \|\xi_0\|_p + \|Y_0\|_p \varphi^m \\ &\leq 2 \frac{\varphi^m}{1 - \varphi} \|a\| \|\xi_0\|_p \end{aligned}$$

thus,  $Y_t = X_t$  a.s.

We also consider the following extension of equation (1) to random fields  $\{X_t\}_{t \in \mathbb{Z}^D}$ :

**Lemma 2.1** *Assume that  $a_j$  are  $m \times d$ -matrices now defined for each  $j \in \mathbb{Z}^D \setminus \{0\}$ . Fix an arbitrary norm  $\|\cdot\|$  on  $\mathbb{Z}^D$ . We extend the previous function  $A$  to  $A(x) = \sum_{\|j\| \geq x} \|a_j\|$ ,  $A = A(1)$  and we suppose with  $p = \infty$  that  $\varphi = A \|\xi_0\|_\infty < 1$ . Then the random field*

$$X_t = \xi_t \left( a + \sum_{k=1}^{\infty} \sum_{j_1 \neq 0} \cdots \sum_{j_k \neq 0} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\dots-j_k} a \right) \quad (4)$$

is a solution to the recursive equation:

$$X_t = \xi_t \left( a + \sum_{j \neq 0} a_j X_{t-j} \right), \quad t \in \mathbb{Z}^D. \quad (5)$$

Moreover, each stationary solution to this equation is also bounded and equals  $X_t$ , a.s.

The proof is the same as before, we first prove that any solution is bounded and we expand it as the sum of the first terms in this chaotic expansion, up to a small remainder (wrt to sup norm); the only important modification follows from the fact that now  $j_1 + \dots + j_\ell$  may really vanish for nonzero  $j_i$ 's which entails that the bound with expectation has to be replaced by upper bounds.

**Remark 2.2** *In the previous lemma, the independence of the  $\xi$ 's does not play a role. We may have stated it for arbitrary random fields  $\{\xi_t\}$  such that  $\|\xi_t\|_\infty \leq M$  for each  $t \in \mathbb{Z}^D$ ; such models with dependent inputs are interesting but assumptions on the innovations are indeed very strong. This means that such models are heteroscedastic but with bounded innovations: according to [14], this restriction excludes extreme phenomena like crashes and bubbles. Mandelbrot school has shown from the seminal paper [15] that asset prices returns do not have a Gaussian distribution as the number of extreme deviations, the so-called "Noah effects", of asset returns is far greater than what is allowed by the Normal distribution, even with ARCH-type effects. It is the reason why this extension is not pursued in the present paper.*



### 3 Approximations

This section is aimed to approximate a sequence  $\{X_t\}$  given by (3), solution to eqn. (1) by a sequence  $\{\tilde{X}_t\}$ . We shall prove that we can control the approximation error  $E\|X_t - \tilde{X}_t\|$  within reasonable small bounds.

#### 3.1 Approximation by Independence

The purpose is to approximate  $X_t$  by a random variable independent of  $X_0$ . We set

$$\tilde{X}_t = \xi_t \left( a + \sum_{k=1}^{\infty} \sum_{j_1 + \dots + j_k < t} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\dots-j_k} a \right).$$

**Proposition 3.1** *Define  $\varphi$  from (2). A bound for the error is given by:*

$$E\|X_t - \tilde{X}_t\| \leq E\|\xi_0\| \left( E\|\xi_0\| \sum_{k=1}^{t-1} k \varphi^{k-1} A\left(\frac{t}{k}\right) + \frac{\varphi^t}{1-\varphi} \right) \|a\|.$$

Furthermore, we have as particular results that if  $b, C > 0$  and  $q \in [0, 1)$ , then for a suitable choice of constants  $K, K'$ :

$$E\|X_t - \tilde{X}_t\| \leq \begin{cases} K \frac{(\log(t))^{b \vee 1}}{t^b}, & \text{for Riemannian decays } A(x) \leq Cx^{-b}, \\ K'(q \vee \varphi)^{\sqrt{t}}, & \text{for geometric decays } A(x) \leq Cq^x. \end{cases}$$

**Remark 3.1** *Note that in the first case this decay is essentially the same Riemannian one while it is sub-geometric (like  $t \mapsto e^{-c\sqrt{t}}$ ) when the decay of the coefficients is geometric.*

**Remark 3.2** *In the paper Riemannian or Geometric decays always refer to the previous relations.*

*Idea of the Proof.* A careful study of the terms in  $X_t$ 's expansion which do not appear in  $\tilde{X}_t$  entails the following bound with the triangular inequality. For this, quote that if  $j_1 + \dots + j_k \geq t$  for some  $k \geq 1$  then, at least, one of the indices  $j_1, \dots, j_k$  is larger than  $t/k$ . The additional term corresponds to those terms with indices  $k > t$  in the expansion (3).

The following extension to the case of the random fields determined in lemma 2.1 is immediate by setting

$$\tilde{X}_t = \xi_t \left( a + \sum_{k=1}^{\infty} \sum_{\substack{j_1, \dots, j_k \neq 0 \\ \|j_1\| + \dots + \|j_k\| < \|t\|}} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\dots-j_k} a \right).$$

**Proposition 3.2** *The random field  $(X_t)_{t \in \mathbf{Z}^D}$  defined in lemma 2.1 satisfies:*

$$E\|X_t - \tilde{X}_t\| \leq E\|\xi_0\| \left( \|\xi_0\|_\infty \sum_{1 \leq k < \|t\|} k\varphi^{k-1} A\left(\frac{\|t\|}{k}\right) + \frac{\varphi^{\|t\|}}{1-\varphi} \right) \|a\|.$$

### 3.2 Coupling

First note that the variable  $\tilde{X}_t$  which approximates  $X_t$  does not follow the same distribution. For dealing with this issue, it is sufficient to construct a sequence of iid random variables  $\xi'_i$  which follow the same distribution as the one of the  $\xi_i$ , each term of the sequence being independent of all the  $\xi_i$ . We then set

$$\xi_t^* = \begin{cases} \xi_t & \text{if } t > 0 \\ \xi'_t & \text{if } t \leq 0 \end{cases}, \text{ and } X_t^* = \xi_t \left( a + \sum_{k=1}^{\infty} \sum_{j_1, \dots, j_k} a_{j_1} \xi_{t-j_1}^* \cdots a_{j_k} \xi_{t-j_1-\dots-j_k}^* a \right).$$

Coefficients  $\tau_t$  for the  $\tau$ -dependence introduced by Dedecker and Prieur [6] are easily computed. In this case, we find the upper bounds from above, up to a factor 2:

$$\tau_t = E\|X_t - X_t^*\| \leq 2E\|\xi_0\| \left( E\|\xi_0\| \sum_{k=1}^{t-1} k\varphi^{k-1} A\left(\frac{t}{k}\right) + \frac{\varphi^t}{1-\varphi} \right) \|a\|;$$

see also Rüschendorf [17], Prieur [16]. These coefficients  $\tau_k$  are defined as  $\tau_k = \tau(\sigma(X_i, i \leq 0), X_k)$  where for each random variable  $X$  and each  $\sigma$ -algebra  $\mathcal{M}$  one sets

$$\tau(\mathcal{M}, X) = E \left\{ \sup_{\text{Lip } f \leq 1} \left| \int f(x) \mathbb{P}_{X|\mathcal{M}}(dx) - \int f(x) \mathbb{P}_X(dx) \right| \right\}$$

where  $\mathbb{P}_X$  and  $\mathbb{P}_{X|\mathcal{M}}$  denotes the distribution and the conditional distribution of  $X$  on the  $\sigma$ -algebra  $\mathcal{M}$  and  $\text{Lip } f = \sup_{x \neq y} |f(x) - f(y)| / \|x - y\|$ .

### 3.3 Markovian Approximation

We consider equation (1) truncated at the order  $N$ :  $Y_t = \xi_t(a + \sum_{j=1}^N a_j Y_{t-j})$ . The solution considered above can be rewritten as

$$X_t^N = \xi_t \left( a + \sum_{k=1}^{\infty} \sum_{N \geq j_1, \dots, j_k \geq 1} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1-\dots-j_k} a \right).$$

We can easily find an upper bound of the error:  $E\|X_t - X_t^N\| \leq \sum_{k=1}^{\infty} A(N)^k$ . As in proposition 3.1, in the Riemannian case, this bound of the error writes as  $C \sum_{k=1}^{\infty} N^{-bk} \leq C/(N^b - 1)$  with  $b > 1$ , while in the geometric case, this writes as  $Cq^N/(1 - q^N) \leq Cq^N/(1 - q)$ ,  $0 < q < 1$ .

## 4 Weak Dependence

Consider integers  $u, v \geq 1$ . Let  $i_1 < \dots < i_u, j_1 < \dots < j_v$  be integers with  $j_1 - i_u \geq r$ , we set  $U$  and  $V$  for the two random vectors  $U = (X_{i_1}, X_{i_2}, \dots, X_{i_u})$  and  $V = (X_{j_1}, X_{j_2}, \dots, X_{j_v})$ . We fix a norm  $\|\cdot\|$  on  $\mathbf{R}^d$ . For a function  $h : (\mathbf{R}^d)^w \rightarrow \mathbf{R}$  we set

$$\text{Lip}(h) = \sup_{x_1, y_1, \dots, x_w, y_w \in \mathbf{R}^d} \frac{|h(x_1, \dots, x_w) - h(y_1, \dots, y_w)|}{\sum_{i=1}^w \|x_i - y_i\|}.$$

**Theorem 4.1** *Assume that the coefficient defined by (2) satisfies  $\varphi < 1$ . The solution (3) to the equation (1) is  $\theta$ -weakly dependent, see [4]. This means that:*

$$|\text{Cov}(f(U), g(V))| \leq 2v\|f\|_\infty \text{Lip}(g)\theta_r,$$

for any integers  $u, v \geq 1, i_1 < \dots < i_u, j_1 < \dots < j_v$  such that  $j_1 - i_u \geq r$ ; with

$$\theta_r = E\|\xi_0\| \left( E\|\xi_0\| \sum_{k=1}^{r-1} k\varphi^{k-1} A\left(\frac{r}{k}\right) + \frac{\varphi^r}{1-\varphi} \right) \|a\|.$$

*Proof.* For calculating a weak dependence bound, we approximate the vector  $V$  by the vector  $\hat{V} = (\hat{X}_{j_1}, \hat{X}_{j_2}, \dots, \hat{X}_{j_v})$ , where we set

$$\hat{X}_t = \xi_t \left( a + \sum_{k=1}^{\infty} \sum_{j_1 + \dots + j_k < s} a_{j_1} \xi_{t-j_1} \cdots a_{j_k} \xi_{t-j_1 - \dots - j_k} a \right).$$

Note that for each index  $j \in \mathbb{Z}$ ,  $\hat{X}_j$  is independent of  $(X_{j-s})_{s \geq r}$ . Note that for  $1 \leq k \leq v$ ,  $E\|X_{j_k} - \hat{X}_{j_k}\| \leq \theta_r$  defined in theorem 4.1. Then

$$\begin{aligned} |\text{Cov}(f(U), g(V))| &\leq \left| E\left(f(U)(g(V) - g(\hat{V}))\right) - E(f(U))E(g(V) - g(\hat{V})) \right| \\ &\leq 2\|f\|_\infty E\left|g(V) - g(\hat{V})\right| \\ &\leq 2\|f\|_\infty \text{Lip}(g) \sum_{k=1}^v E\|X_{j_k} - \hat{X}_{j_k}\| \\ &\leq 2v\|f\|_\infty \text{Lip}(g)\theta_r. \end{aligned}$$

**Remark 4.1** *We obtain explicit expressions for this bound in proposition 3.1 for the Riemannian and geometric decay rates.*

**Remark 4.2** *In the case of random fields the  $\eta$ -weak dependence condition in [8] or [7] holds in a similar way with*

$$\eta_r = 2E\|\xi_0\| \left( \|\xi_0\|_\infty \sum_{k < r/2} k\varphi^{k-1} A\left(\frac{r}{k}\right) + \frac{\varphi^{\lceil r/2 \rceil}}{1-\varphi} \right) \|a\|,$$

which means that the previous bound now writes as

$$|\text{Cov}(f(U), g(V))| \leq (u\|g\|_\infty \text{Lip}(f) + v\|f\|_\infty \text{Lip}(g))\eta_r.$$

The argument is the same except for the fact that now  $\hat{U}$  and  $\hat{V}$  are independent vectors with truncations at a level  $s = \lceil r/2 \rceil$  but  $\hat{V}$  and  $U$  are not necessarily independent (recall that independence of  $U$  and  $\hat{V}$  follows from  $s \geq r$  in the proof for the causal case). This point makes the previous bound a bit more complicated than the one in theorem 4.1 and it explains the appearance of the factor 2 in the expression of  $\eta_r$ .

**Remark 4.3** Those weak dependence conditions imply various limit theorems both for partial sums processes and for the empirical process (see [8], [4] and [7]).

## 5 $L^2$ Properties

For the univariate case, the uniqueness of a stationary solution to equation (1) has been demonstrated by Giraitis *et al.* [12].

We first present an existence and uniqueness condition for the model in  $L^2$ . The situation is then no longer necessarily weakly dependent.

**Theorem 5.1** Assume that the iid sequence  $\{\xi_t\}$  satisfies  $E(\xi_k) = 0$ .

Assume that the matrix  $S = \sum_{k=1}^{\infty} a'_k E(\xi'_k \xi_k) a_k$  has a spectral radius which satisfies  $\rho(S) < 1$ .

Then there exists a stationary solution in  $L^2$  to equation (1) given by (3). Moreover the solution in  $L^2$  to equation (1) is unique.

**Remark 5.1** • The assumption  $\rho(S) < 1$  implies  $\xi_t \in L^2$  for  $t \in \mathbb{Z}$ .

- In [11], the example 2 of the bilinear model displays the double long memory property when the corresponding series  $\alpha_j$  and  $\beta_j$  are not summable but

$$\sum_{j=1}^{\infty} (\alpha_j^2 E \zeta_0^2 + \beta_j^2) < 1.$$

As a particular case, the squares of the LARCH( $\infty$ ) process, example 1, display long-range dependence as well. Those authors prove that the corresponding partial sums process converges to the fractional Brownian Motion, appropriately normalized (with normalization  $\gg \sqrt{n}$ ).

- Models GARCH( $p, q$ ), in example 3, are always weakly dependent, in the sense of [8].
- Note that [12] and [11] prove that the stationary ARCH( $\infty$ ) model, described as example 4 in section 1, is not long range dependent in the previous sense; more precisely the sequence of partial sums processes, normalized with  $\sqrt{n}$ , converges to the Brownian Motion.

*Proof. Step 1: existence.* Define  $T = E(\xi'_k \xi_k)$ . Considering the chaotic solution (3) and setting

$$C_t(k_2, \dots, k_\ell) = \xi_t a_{k_2} \xi_{t-k_2} \cdots a_{k_\ell} \xi_{t-k_2 \cdots k_\ell} a$$

we write  $E(X'_t X_t) = a' E \xi'_t \xi_t a + B = a' T a + B$  where

$$\begin{aligned} B &= \sum_{\ell, k_1, \dots, k_\ell \geq 1} EC'_{t-k_1}(k_2, \dots, k_\ell) a'_{k_1} T a_{k_1} C_{t-k_1}(k_2, \dots, k_\ell) \\ &= \sum_{\ell, k_1, \dots, k_\ell} EC'_{t-k_1}(k_2, \dots, k_\ell) a'_{k_1} E \xi'_{t-k_1} \xi_{t-k_1} a_{k_1} C_{t-k_1}(k_2, \dots, k_\ell) \\ &= \sum_{\ell, k_1, \dots, k_\ell} EC'_{t-k_1}(k_2, \dots, k_\ell) (E a'_{k_1} \xi'_{t-k_1} \xi_{t-k_1} a_{k_1}) C_{t-k_1}(k_2, \dots, k_\ell) \\ &= \sum_{\ell, k_1, \dots, k_\ell} EC'_t(k_2, \dots, k_\ell) (E a'_{k_1} \xi'_{t-k_1} \xi_{t-k_1} a_{k_1}) C_t(k_2, \dots, k_\ell) \\ &= \sum_{\ell, k_2, \dots, k_\ell} EC'_t(k_2, \dots, k_\ell) S C_t(k_2, \dots, k_\ell) \\ &\leq \rho(S) \sum_{\ell, k_2, \dots, k_\ell} EC'_t(k_2, \dots, k_\ell) C_t(k_2, \dots, k_\ell) \\ &\leq E(\xi_0 a)' (\xi_0 a) \sum_{\ell=1}^{\infty} \rho(S)^\ell \quad (\text{recursively}) \\ &\leq a' a \rho(T) \sum_{\ell=1}^{\infty} \rho(S)^\ell, \end{aligned}$$

hence,

$$E(X'_t X_t) \leq a' T a + a' a \frac{\rho(T)}{1 - \rho(S)} < \infty \quad (6)$$

In the previous relations we both use the fact that the  $\xi_t$  are centered and iid and the relation  $v' A v \leq v' v \rho(A)$  which holds if  $A$  denotes a non-negative  $d \times d$  matrix and  $v \in \mathbb{R}^d$ . This concludes the proof of the existence of a solution in  $L^2$ .

*Step 2:  $L^2$  uniqueness.* Let now  $X_t^1$  and  $X_t^2$  be two solutions to equation (1) in  $L^2$ . Define  $\tilde{X}_t = X_t^1 - X_t^2$ , then  $\tilde{X}_t$  is solution to

$$\tilde{X}_t = \xi_t \tilde{A}_t, \quad \tilde{A}_t = \sum_{k=1}^{\infty} a_k \tilde{X}_{t-k}. \quad (7)$$

Now we use (7) and the fact that  $\tilde{X}_t$  is centered and thus  $E \tilde{X}_s \tilde{X}_t = 0$  for  $s \neq t$  to derive

$$\begin{aligned}
 E \left( (\tilde{X}_t g)' (\tilde{X}_t g) \right) &= \sum_{k=1}^{\infty} g' E \left( \tilde{X}'_{t-k} a'_{t-k} T a_{t-k} \tilde{X}_{t-k} \right) g \\
 &= \sum_{k=1}^{\infty} g' E \left( \tilde{X}'_t a'_{t-k} T a_{t-k} \tilde{X}_t \right) g \\
 &= g' E \left( \tilde{X}'_t S \tilde{X}_t \right) g \\
 &= E \left( (\tilde{X}_t g)' S (\tilde{X}_t g) \right) \\
 &\leq \rho(S) E \left( (\tilde{X}_t g)' (\tilde{X}_t g) \right)
 \end{aligned}$$

From equation (6), this expression is finite and thus the assumption  $\rho(S) < 1$  concludes the proof.

**Remark 5.2** *The proof does not extend to the case of random fields because in this case the previous arguments of independence cannot be used. In that case we cannot address the question of uniqueness.*

The previous  $L^2$  existence and uniqueness assumptions do not imply that  $\sum_{j \geq 1} \|a_j\| < \infty$ , thus this situation is perhaps not a weakly dependent one. Giraitis and Surgailis [11], prove results both for the partial sums processes of  $X_t$  and  $X_t^2 - EX_t^2$ . In our vector case the second problem is difficult and will be addressed in a forthcoming work. However  $X_t$  is now the increment of a (vector valued-)martingale and thus we partially extend Theorem 6.2 in [11], providing a version of Donsker's theorem for partial sums processes of  $\{X_t\}$ .

**Proposition 5.1** *Assume that the assumptions in theorem 5.1 hold. Then  $S_n(t)/\sqrt{\text{Var}S_n(t)}$  converges to  $\Sigma W(t)$ , if  $S_n(t) = \sum_{1 \leq i \leq nt} X_i$  for  $0 \leq t \leq 1$  and where  $W(t)$  is a  $\mathbb{R}^d$  valued Brownian motion and  $\Sigma$  is a symmetric non negative matrix such that  $\Sigma^2$  is the covariance matrix of  $X_0$ . The convergence holds for finite dimensional distributions.*

**Remark 5.3** • *The convergence only holds for any  $k$ -tuples  $(t_1, \dots, t_k) \in [0, 1]^k$  and since the section is related to  $L^2$  properties we cannot use the tightness arguments in [11] to obtain the Donsker theorem; indeed tightness is obtained through moment inequalities of order  $p > 2$ .  $L^p$  existence conditions are obtained in [11] for the bilinear case if  $p = 4$ ; the method is based on the diagram formula and does not extend simply to this vector valued case. A bound for the moments of order  $p > 2$  of the partial sum process  $S_n(t)$  can be obtained using Rosenthal inequality, Theorem 2.11 in [13], if  $E\|X_t\|^p < \infty$ . This inequality would imply the functional convergence in the Skohorod space  $D[0, 1]$  if  $p > 2$ .*

- *If  $E\xi_0 \neq 0$  (as for the case of the bilinear model in [11]), we may write  $X_t = \Delta M_t + E\xi_0 \left( a + \sum_{j=1}^{\infty} a_j X_{t-j} \right)$  where*

$$\Delta M_t = (\xi_t - E\xi_t) \left( a + \sum_{j=1}^{\infty} a_j X_{t-j} \right)$$

is a martingale increment. This martingale also obeys a central limit theorem. then,

$$n^{-1/2} S_n(t) \rightarrow \bar{\Sigma} W(t),$$

where  $W(t)$  is a vector Brownian motion, where  $\bar{\Sigma}' \bar{\Sigma} = \Sigma$ . If  $E\xi_0 = 0$  this is a way to prove proposition 5.1, which is a multi-dimensional extension of the proof in [11].

For the case of the bilinear model, Giraitis and Surgailis also prove the (functional) convergence of the previous sequence of process to a Fractional Brownian Motion in [11]. For this, Riemannian decays of the coefficients are assumed. The covariance function of the process is also completely determined to prove such results; this is a quite difficult point to extend to our vector valued frame.

- A final comment concerns the analogue for powers of  $X_t$  which, if suitably normalized, are proved to converge to some higher order Rosenblatt process in [11] for the bilinear case. We have a structural difficulty to extend it; the only case which may reasonably be addressed is the real valued one ( $d = 1$ ), but it also presents very heavy combinatorial difficulties. Computations for the covariances of the processes  $(X_t^k)_{t \in \mathbf{Z}}$  will be addressed in a forthcoming work in order to extend those results.

**Acknowledgements.** The authors are grateful to the referees for their valuable comments.

## References

1. Bougerol, P., Picard, N. (1992). Stationarity of GARCH processes and of some nonnegative time series. *J. of Econometrics*, 52, 115–127.
2. Boussama, F. (1998). Ergodicité, mélange, et estimation dans les modèles GARCH. PhD Thesis, University Paris 7.
3. Boussama, F. (2000). Normalité asymptotique de l'estimateur du pseudo-maximum de vraisemblance d'un modèle GARCH. *Comptes Rendus de l'Académie des Sciences, Série I*, Volume 331-1, 81–84.
4. Dedecker, J., Doukhan, P. (2003). A new covariance inequality and applications, *Stoch. Proc. Appl.*, 106, 63–80.
5. Dedecker, J., Prieur, C. (2004). New dependence coefficients. Examples and applications to statistics. *Prob. Theory Relat. Fields*, In press.
6. Dedecker, J., Prieur, C. (2003). Coupling for  $\tau$ -dependent sequences and applications. *J. of Theoret. Prob.*, forthcoming.
7. Doukhan, P., Lang, G. (2002). Rates of convergence in the weak invariance principle for the empirical repartition process of weakly dependent sequences. *Stat. Inf. for Stoch. Proc.*, 5, 199–228.

8. Doukhan, P., Louhichi, S. (1999). A new weak dependence condition and applications to moment inequalities. *Stoch. Proc. Appl.*, 84, 313–342.
9. Engle, R.F. Kroner, K.F. (1995). Multivariate simultaneous generalized ARCH. *Econometric Theory*, 11, 122–150.
10. Giraitis, L., Leipus, R., Surgailis, D. (2005). Recent advances in ARCH modelling, in: Teyssière, G. and Kirman, A., (Eds.), *Long-Memory in Economics*. Springer Verlag, Berlin. pp 3–38.
11. Giraitis, L., Surgailis, D. (2002). ARCH-type bilinear models with double long memory, *Stoch. Proc. Appl.*, 100, 275–300
12. Giraitis, L., Kokoszka, P., Leipus, R. (2000). Stationary ARCH models: dependence structure and central limit theorems. *Econometric Theory*, 16, 3–22.
13. Hall, P., Heyde, C.C. (1980). *Martingale Limit Theory and Its Applications*. Academic Press.
14. Mandelbrot, B.B., Hudson R.L. (2004). *The Misbehavior of Markets: A Fractal View of Risk, Ruin, and Reward*. Basic Books, New-York.
15. Mandelbrot, B.B. (1963). The variation of certain speculative prices. *Journal of Business*, 36, 394–419.
16. Prieur, C. (2004). Recent results on weak dependence. Statistical applications. Dynamical system’s example. This volume.
17. Rüschemdorf, L. (1985). The Wasserstein distance and approximation theorems. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 70, 117–129.
18. Teyssière, G. (1997). Modelling Exchange rates volatility with multivariate long-memory ARCH processes. *Preprint*.