INSTITUT NATIONAL DE LA STATISTIQUE ET DES ETUDES ECONOMIQUES
Série des Documents de Travail du CREST
(Centre de Recherche en Economie et Statistique)

n° 2005-46

Uniform Limit Theorems for the Integrated Periodogram of Weakly Dependent Time Series and their Applications to Whittle's Estimate

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Uniform limit theorems for the integrated periodogram of weakly dependent time series and their applications to Whittle's estimate

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Nous montrons un théorème de limite central pour des fonctionnelles générales de séries temporelles. Notre énoncé valide pour de nombreuses classes de séries temporelles vaut en particulier pour des modèles ARCH(∞), bilinéaires, linéaires non causaux ou and ARCH(∞)
Nous prouvons des énoncés uniformes comme une loi des grands nombres ou un théorème de limite central pour le périodogramme d'une série temporelle, intégré par une classe de Sobolev. Ces énoncés probabilistes sont utilisés pour déduire les propriétés asymptotiques de l'estimateur de Whittle. Sous des conditions générales de dépendance faibles nous prouvons ainsi la consistance et la normalité a Ainsi la notion causale de θ-dépendance faible prouve ces énoncés de manière unifiée pour des modèles LARCH(∞) ou bilinéaires processes, pendant que la notion non causale de η-dépendance faible prouve (pour la première fois) de tels énoncés des séries linéaires non causales, des modèles de Volterra, ou des modèles LARCH(∞) non causaux.

Abstract

We prove uniform convergence results like a law of large numbers and a central limit theorem for the integrated periodogram of a weak dependent time series. Those probabilistic results are used for Whittle's parametric estimation. Using a general weakly dependent frame, we derive new results, *i.e.* uniform limit theorems and asymptotic normality of the Whittle estimate, for a large variety of models. For instance, the causal θ-weak dependence property allows a new and unique proof of those results for LARCH(∞) and bilinear processes, whereas the non causal η-weak dependence property provides for the first time those limit theorems for two sided linear, Volterra, bilinear and LARCH(∞) processes.

Key words: Periodogram, Weak dependence, Whittle estimate.
AMS Subject Classification: 60F17, 60F25, 62M09, 62M10, 62M15.

Running title: Uniform limit theorems for the integrated periodogram and Whittle's estimate.

*This author aknowledges the program ECOS-NORD of Fonacit, Venezuela, for its support.
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1 Introduction

The parametric estimation from a sample of a stationary time series is an important statistic domain both for the theoretical research and for its practical applications on (biometric, hydrologic, econometric, financial,...) data. The Whittle's approximation likelihood estimate is particularly attractive for numerous models like ARMA, linear,..., processes for mainly two reasons: firstly, the Whittle's contrast is not depending on the marginal law of the time series but only on its spectral density, secondly, its computation time is smaller than other parametric estimates like exact likelihood ones. Numerous papers have been written on this estimate after the seminal paper of Whittle and in particular Hannan (1973), Rosenblatt (1985) or Giraitis and Robinson (2001), who have established respectively the asymptotic normality for Gaussian and causal linear, strong mixing and LARCH(∞) processes. The main goal of the present paper is to provide a unified treatment of this asymptotic normality for a very rich class of weakly dependent time processes, including those previously quoted, but also some never studied non causal or non linear processes (the case of long range dependent processes was studied by Fox and Taqqu, 1986, and Giraitis and Surgailis, 1990).

More precisely, let $X = (X_k)_{k\in\mathbb{Z}}$ be a zero mean fourth-order stationary time series with real values. Denote $(R(s))_s$ the covariogram of X, and $(\kappa_4(i,j,k))_{i,j,k}$ the fourth cumulants of X, such that:

$$\begin{aligned} R(s) &= \operatorname{Cov}(X_0, X_s) = \mathbb{E}(X_0X_s), \quad \text{for } s \in \mathbb{Z}, \\ \kappa_4(i,j,k) &= \mathbb{E}X_0X_iX_jX_k - \mathbb{E}X_0X_i\mathbb{E}X_jX_k - \mathbb{E}X_0X_j\mathbb{E}X_iX_k - \mathbb{E}X_0X_k\mathbb{E}X_iX_j, \quad \text{for } (i,j,k) \in \mathbb{Z}^3. \end{aligned}$$

We will use the following assumption on X:

Assumption M: X is such that:

$$\gamma = \sum_{\ell\in\mathbb{Z}} R(\ell)^2 < \infty \quad \text{and} \quad \kappa_4 = \sum_{i,j,k} |\kappa_4(i,j,k)| < \infty. \tag{1}$$

For X satisfying assumption M, the periodogram of X is:

$$I_n(\lambda) = \frac{1}{2\pi \cdot n}\left|\sum_{k=1}^{n} X_k e^{-ik\lambda}\right|^2, \quad \text{for } \lambda \in [-\pi, \pi[.$$

Now, let $g : [-\pi,\pi[\to \mathbb{R}$ a 2π-periodic function such that $g \in \mathbb{L}^2([-\pi,\pi[)$ and define:

$$\begin{aligned} J_n(g) &= \int_{-\pi}^{\pi} g(\lambda) I_n(\lambda)\, d\lambda, \quad \text{the integrated periodogram of } X \\ \text{and} \quad J(g) &= \int_{-\pi}^{\pi} g(\lambda) f(\lambda)\, d\lambda, \end{aligned}$$

with f the spectral density of X (that exists and is in $\mathbb{L}^2([-\pi,\pi[)$ from Assumption M) defined by:

$$f(\lambda) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} R(k)\, e^{ik\lambda} \quad \text{for } \lambda \in [-\pi,\pi[.$$

Recall that $I_n(\lambda) = \dfrac{1}{2\pi}\displaystyle\sum_{|k|<n} \widehat{R}_n(k) e^{-ik\lambda}$ with $\widehat{R}_n(k) = \dfrac{1}{n}\displaystyle\sum_{j=1\vee(1-k)}^{(n-k)\wedge n} X_jX_{j+k}$, a biased estimate of $R(k)$. A special case of the integrated periodogram is the Whittle's contrast, defined as a function $\beta \to J_n(h_\beta)$, where h_β is included in a class of functions depending on the vector of parameters β.

The Whittle's estimate is a minimization of this contrast. As a consequence, uniform limit theorems of integrated periodogram $J_n(.)$ are the appropriate tools for obtaining uniform limit theorems of the Whittle's contrast, that imply, under supplementary conditions concerning the regularity of the spectral density, limit theorems for Whittle's estimators.

A uniform strong law of large numbers of integrated periodograms on a Sobolev-type space (included in the space of 2π-periodic $\mathbb{L}^2$-functions) is first established only under assumption M. Other assumptions on the dependence properties of the time series have to be specified for establishing central limit theorems. Our choice has been to consider time series satisfying weak dependence properties introduced and developed in Doukhan and Louhichi (1999). Numerous reasons may explain this choice. First, this frame of dependence includes a lot of models like causal or non causal linear, bilinear, strong mixing processes or also dynamical systems. Secondly, these properties of dependence are independent of the marginal distribution of the time series, that can be as well a discrete one, Lebesgue measurable one or anything else. Finally, these definitions of dependence can be easily used in various statistic contexts, in particular in the case of the integrated periodogram that is a quadratic form of the time series.

Two frames of weak dependence are considered here. The first one exploits a causal property of dependence, the θ-weak dependence property (see Dedecker and Doukhan, 2003). Under certain conditions, the uniform limit theorems for integrated periodogram and asymptotic normality of the Whittle's estimate are established. These general results are new and extend Hannan (1973) and Rosenblatt (1985) classical results for causal linear or strong mixing processes. For example, parametric and causal ARCH(∞) or bilinear (a very general class of models introduced by Giraitis and Surgailis 2002, see definition (16)) processes are considered; under certain conditions, the asymptotic normality of Whittle estimators of those two classes of models is established with the same method (the case of causal ARCH(∞), and therefore of GARCH(p,q), was already treated by Giraitis and Robinson, 2001, under less restrictive conditions. However, their proof is *ad hoc* and cannot be used in a more general frame).

The second type of weak dependence concerns η-weakly dependent processes. The important point is that the definition of the η-weak dependent property allows to obtain central limit theorems for non causal processes. These results are new and can be applied for instance, to two-sided linear, Volterra, bilinear or LARCH(∞) processes (see their definition below in Section 3). Let us remark that usual proofs of central limit theorems for integrated periodogram are established by considering increments of martingales or asymptotic results for strong mixing processes, this is not at all adapted for non causal processes, even in the simple case of two-sided linear models. The proof of our results is a corollary of a general functional central limit theorem for η-weakly dependent processes, established by using a Bernstein's blocks method. Even if our results may be sub optimal in terms of the conditions linking the moment assumption with the decay rate of weak dependence of the time series, they however cover numerous models and open new perspectives of treatments for non causal processes.

The paper is organized as follows. In Section 2, uniform limit theorems are presented with some applications to time series. Section 3 is devoted to limit theorems satisfied by Whittle's estimators, that are applied to several examples of causal and non causal processes. Section 4 contains the main proofs, and an useful lemma is presented in an appendix (Section 5).

2 Uniform limit theorems

2.1 Notations and assumptions

Afterward, we shall use the zero mean random variables $(Y_{j,k})_{j,k\in\mathbb{Z}}$ such that:

$$Y_{j,k} = X_j X_{j+k} - R(k), \quad \text{for all} \quad (j,k) \in \mathbb{Z}^2.$$

We intend to work in a Sobolev space $\mathcal{H}_s$ of locally $\mathbb{L}^2$ and $2\pi-$periodic functions, defined by:

$$\mathcal{H}_s = \{g \in \mathbb{L}^2([-\pi,\pi[) \,/\, \|g\|_{\mathcal{H}_s} < \infty\} \;\text{ with }\; \|g\|^2_{\mathcal{H}_s} = \sum_{\ell\in\mathbb{Z}} (1+|\ell|)^s \cdot |g_\ell|^2, \quad \text{and } s > 1,$$

for g a 2π-periodic function such that $g \in \mathbb{L}^2([-\pi,\pi[)$ and $g(\lambda) = \sum_{\ell\in\mathbb{Z}} g_\ell \, e^{i\ell\lambda}$. This space $\mathcal{H}_s$ is included in the space $C^\star$ of continuous and $2\pi-$periodic functions and $\|g\|_\infty = \sup_{[-\pi,\pi[} |g| \le \sqrt{c_s} \cdot \|g\|_{\mathcal{H}_s}$ with

$$c_s = \sum_{\ell\in\mathbb{Z}} (1+|\ell|)^{-s}. \tag{2}$$

As usual $\mathcal{H}'_s$ denotes the dual of $\mathcal{H}_s$ defined from the identity $\|T\|_{\mathcal{H}'_s} = \sup_{\|g\|_{\mathcal{H}_s}\le 1} |T(g)|$. Hence if $T \in \mathcal{H}'_s$:

$$\|T\|^2_{\mathcal{H}'_s} = \sup_{\|g\|_{\mathcal{H}_s}\le 1} |T(g)|^2 = \sum_{\ell\in\mathbb{Z}} (1+|\ell|)^{-s} \cdot |T(e^{i\ell\lambda})|^2.$$

We study the behavior of $J_n - J$ in the function space $\mathcal{H}_s$ or equivalently in the Hilbert space $\mathcal{H}'_s$.

2.2 Uniform Law of Large Numbers for the integrated periodogram

We develop a uniform law of large numbers (ULLN) for the integrated periodogram $(J_n(g))_g$. An important feature is that the results are only stated here in terms of cumulant sums; thus we need no additional assumption on the dependence of the sequence X.

Theorem 1 (Uniform SLLN) *If X satisfies Assumption M, then:*

$$\|J_n - J\|_{\mathcal{H}'_s} \xrightarrow[n\to\infty]{a.s.} 0.$$

2.3 Uniform Central Limit Theorem for the integrated periodogram

Now, we would like to establish a uniform central limit theorem (UCLT) for the integrated periodogram $J_n(.)$ on the space of functions $\mathcal{H}_s$. Now the Assumption M specifying the asymptotic behavior of the sequences of covariogram and cumulant are not sufficient. The dependence between the terms of the time series X has to be specified, and we will consider 2 cases. Before this, under Assumption M, we define for any $\lambda, \mu, \nu \in \mathbb{R}$, the bispectral density

$$f_4(\lambda,\mu,\nu) = \frac{1}{(2\pi)^3} \sum_{h=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \kappa_4(h,k,\ell) e^{i(h\lambda + k\mu + \ell\nu)}$$

the matrix $\Sigma = (\sigma_{\ell_i,\ell_j})_{1\le i,j\le m}$, where ℓ_i are distinct integer numbers, such that:

$$\sigma_{k,\ell} = \sum_{h\in\mathbb{Z}} \Big(R(h)R(h+\ell-k) + R(h+\ell)R(h-k) + \kappa_4(h,k,h+\ell) \Big), \tag{3}$$

and for g_1 and g_2 in $\mathcal{H}_s$, the limiting covariance $\Gamma(g_1,g_2)$:

$$\Gamma(g_1,g_2) = \frac{1}{\pi} \int_{-\pi}^{\pi} g_1(\lambda) g_2(\lambda) f^2(\lambda)\, d\lambda + 2\pi \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g_1(\lambda) g_2(\mu) f_4(\lambda,-\mu,\mu)\, d\lambda\, d\mu. \tag{4}$$

2.3.1 UCLT for causal time series

This first case follows a classical methodology: the UCLT results from the finite dimensional convergence and the tightness of the process $Z = (Z_n(g))_{g\in\mathcal{H}_s}$ where $Z_n(g) = \sqrt{n}\,(J_n(g) - J(g))$ for $n \in \mathbb{N}^*$ and $g \in \mathcal{H}_s$. Since $g \mapsto Z_n(g)$ is a linear functional, the finite dimensional convergence follows from a multidimensional central limit theorem for empirical covariances.

In the sequel, for $\ell \in \mathbb{Z}$, we will denote by $\mathcal{M}_0^{(\ell)}$ a σ-algebra such that

$$\mathcal{M}_0^{(\ell)} \supset \sigma\,(Y_{k,\ell},\ k \le 0) = \sigma\,(X_k X_{k+\ell},\ k \le 0)\,,$$

where $\sigma(W_i,\ i \in I)$ represents the σ-algebra of Ω generated by $(W_i)_{i\in I}$. The most classical example of such σ-algebra $\mathcal{M}_0^{(\ell)}$ is

$$\mathcal{B}_m = \sigma\,(X_k,\ k \le m)\,, \quad \text{when } m \ge \ell.$$

Lemma 1 *Let $(\ell_1, \ldots, \ell_m) \in \mathbb{Z}^m$ be arbitrary distinct integers ($m \in \mathbb{N}^*$). Let X satisfy Assumption M and be such that:*

$$\sum_{k\ge 0} \mathbb{E}\left|Y_{0,\ell_i}\,\mathbb{E}\,(Y_{k,\ell_i}\,|\,\mathcal{M}_0^{(\ell_i)})\right| < \infty \quad \textit{for all } \ i \in \{1,\ldots,m\}. \tag{5}$$

Then, if $\Sigma = (\sigma_{\ell_i,\ell_j})_{1\le i,j\le m}$ defined in (3) is a nonsingular matrix, with $\xrightarrow[n\to\infty]{\mathcal{D}}$ the weak convergence,

$$\left(\sqrt{n}\,(\widehat{R}_n(\ell_i) - \mathbb{E}\widehat{R}_n(\ell_i))\right)_{1\le i\le m} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_m(0,\Sigma). \tag{6}$$

Remarks. 1. The proof of an analogue CLT is provided in Hall and Heyde (1980), Theorem 5.4, page 136. Unfortunately this condition does not seem to be adapted to work out the forthcoming examples.

2. If for each $i \in \{1,\ldots,m\}$, $\displaystyle\sum_{k=0}^{\infty}\left(\mathbb{E}\Big(\mathbb{E}\,(Y_{k,\ell_i}\,|\,\mathcal{B}_0)\Big)^2\right)^{1/2} < \infty$, then (5) is satisfied. Thus, Lemma 1 is a generalization of a result of Rosenblatt (1985, Theorem 3, p. 58).

Theorem 2 *Under assumptions of Lemma 1, the Uniform Central Limit Theorem (UCLT) is satisfied:*

$$Z_n = \sqrt{n}\,(J_n - J) \xrightarrow[n\to\infty]{\mathcal{D}} Z \ \ \textit{in the space } \mathcal{H}_s', \tag{7}$$

with $(Z(g))_{g\in\mathcal{H}_s}$ the zero mean Gaussian process with covariance $\Gamma(g_1,g_2)$ defined in (4).

Examples of time series satisfying Theorem 2. This theorem is first applied to 3 classical examples of time series, which extend the known multidimensional CLT for integrated periodogram. Then, the UCLT is established for a very rich class of causal time series, the θ-weakly dependent processes.

1. Causal linear processes: let X be a linear and causal time series such that $X_n = \sum_{k=0}^{\infty} a_k\,\xi_{n-k}$ for $n \in \mathbb{Z}$, with real weights a_k satisfying $\displaystyle\sum_k k\,a_k^2 < \infty$ and $(\xi_k)_{k\in\mathbb{Z}}$ a sequence of zero mean independent identically distributed random variables such that $\mathbb{E}\xi_0^4 < \infty$. Then X satisfies Assumption M and from Rosenblatt (1985, p. 59), we have for all $\ell \in \mathbb{N}$, $\displaystyle\sum_{k=0}^{\infty} \|\mathbb{E}\,(Y_{k,\ell}\,|\,\mathcal{B}_\ell)\|_2 < \infty$ and thus (5) is also satisfied. Then Theorem 2 holds.

2. Gaussian processes: let the sequence $(X_n)_{n\in\mathbb{Z}}$ be a zero mean stationary Gaussian process such that $\sum_k R(k)^2 < \infty$. Then X satisfies Assumption M and for all $\ell \in \mathbb{Z}$ and $k \in \mathbb{N}$, $\mathbb{E}\left(|Y_{0,\ell}\mathbb{E}\left(Y_{k,\ell}\,|\,\mathcal{M}_0^{(\ell)}\right)|\right) \le \left|\mathbb{E}\left(X_0X_\ell X_k X_{k+\ell}\right) - R(\ell)^2\right|$. But $\mathbb{E}\left(X_0X_\ell X_k X_{k+\ell}\right) - R(\ell)^2 = R(k)^2 + R(k+\ell)R(k-\ell)$ for a zero mean stationary Gaussian process. Thus,

$$\sum_{k\ge 0}\mathbb{E}\left(|Y_{0,\ell}\mathbb{E}\left(Y_{k,\ell}\,|\,\mathcal{M}_0^{(\ell)}\right)|\right) \le \sum_{k\in\mathbb{Z}} R(k)^2 + \left(\sum_{k\in\mathbb{Z}} R^2(k+\ell)\right)^{1/2}\cdot\left(\sum_{k\in\mathbb{Z}} R^2(k-\ell)\right)^{1/2},$$

from the Cauchy-Schwarz inequality for ℓ^2 sequences. Therefore Theorem 2 holds.

3. Strong mixing processes: here, we consider the probability space $(\Omega, \mathcal{T}, \mathbb{P})$.

Corollary 1 *Let $X = (X_n)_{n\in\mathbb{Z}}$ be a sequence of random variables on $(\Omega, \mathcal{T}, \mathbb{P})$ satisfying Assumption M. Assume that X is a α'-mixing process, i.e. $\alpha'_n = \sup_{\ell\ge 0}\left\{\alpha\left(\sigma(X_n, X_{n+\ell})\,,\,\mathcal{B}_0\right)\right\} \underset{n\to\infty}{\longrightarrow} 0$, where $\alpha\left(\mathcal{A}\,,\,\mathcal{B}\right) = \sup_{A\in\mathcal{A},\ B\in\mathcal{B}}\left|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)\right|$ for $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$. Moreover, with $Q_{|X|}$ the quantile function of $|X|$ and $(\alpha'(u))^{-1} = \sum_{k\ge 0} 1\!\!I_{u\le\alpha'_k}$, assume that $\int_0^1 (\alpha'(u))^{-1}Q^4_{|X_0|}(u)\,du < \infty$. Then Theorem 2 holds.*

Remark. We note that $\alpha'_n \le \alpha_n = \left\{\alpha\left(\sigma(X_k,\, k\ge n)\,,\,\mathcal{B}_0\right)\right\}$. Hence this condition is weaker that the standard mixing coefficient in Rosenblatt (1985). However, no simple counter example seems to be available. Consequently, if X is a α-mixing process in the usual sense, that is, $\alpha_n \underset{n\to\infty}{\longrightarrow} 0$, then X is α'-mixing (for all $n \in \mathbb{N}$, $\alpha'_n \le \alpha_n$). Therefore, if X is a strongly α-mixing process satisfying Assumption M such that $\int_0^1 \alpha^{-1}(u)Q^4_{|X_0|}(u)\,du < \infty$, Theorem 2 also holds.

4. Causal θ-weakly dependent processes: the class of θ-weakly dependent processes was introduced by Doukhan and Louichi (1999) and developed in Dedecker and Doukhan (2003). It includes numerous kinds of causal times series, for instance the strong mixing processes (see other examples in Section 3). First, for $h : \mathbb{R}^u \to \mathbb{R}$ an arbitrary function, with $u \in \mathbb{N}^*$, denote:

$$\mathrm{Lip}\,h = \sup\left\{\frac{|\,h(y_1,\ldots,y_u) - h(x_1,\ldots,x_u)\,|}{|y_1 - x_1| + \cdots + |y_u - x_u|}\ \text{ for } (y_1,\ldots,y_u) \neq (x_1,\ldots,x_u)\right\}.$$

The time series $X = (X_n)_{n\in\mathbb{Z}}$ is so-called $\theta-$weakly dependent when there exists a sequence $(\theta_r)_{r\in\mathbb{N}}$ converging to 0 such that for all $r \in \mathbb{N}$, all function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\|f\|_\infty \le 1$, and all random variable $Z \in \mathcal{B}_0$ such that $\|Z\|_\infty < 1$,

$$|\mathrm{Cov}\left(f(X_{j_1}, X_{j_2}), Z\right)| \le 2\cdot\mathrm{Lip}\,f\cdot\theta_r \quad \text{for all } \ j_1, j_2 \ge r. \tag{8}$$

Corollary 2 *Let $X = (X_n)_{n\in\mathbb{Z}}$ a $\theta-$weakly dependent satisfying Assumption M. We also suppose that $\exists m > 4$, such that $\|X_0\|_m < \infty$ and $\sum_{k=0}^{\infty}\theta_k^{\frac{m-4}{m-1}} < \infty$. Then Theorem 2 holds.*

2.3.2 UCLT for non-causal weakly dependent time series

From the seminal paper of Doukhan and Louhichi (1999), a second class of weakly dependent processes can be considered. This class includes also non causal time series. Hence, a process

$X = (X_n)_{n\in\mathbb{Z}}$ with values in $\mathbb{R}^d$ is a so-called $\eta-$weakly dependent process when it exists a sequence $(\eta_r)_{r\in\mathbb{N}}$ converging to 0, satisfying:

$$\Big|\mathrm{Cov}\Big(g_1(X_{i_1},\ldots,X_{i_u}),g_2(X_{j_1},\ldots,X_{j_v})\Big)\Big| \le \Big(u\cdot(\mathrm{Lip}\,g_1)\cdot\|g_2\|_\infty + v\cdot(\mathrm{Lip}\,g_2)\cdot\|g_1\|_\infty\Big)\cdot\eta_r \quad (9)$$

$$\text{for all}\ \begin{cases} \bullet\ (u,v)\in\mathbb{N}^*\times\mathbb{N}^*;\\ \bullet\ (i_1,\ldots,i_u)\in\mathbb{Z}^u \text{ and } (j_1,\ldots,j_v)\in\mathbb{Z}^v \text{ with } i_1\le\cdots\le i_u<i_u+r\le j_1\le\cdots\le j_v\\ \bullet\ \text{functions } g_1:\mathbb{R}^{ud}\to\mathbb{R} \text{ and } g_2:\mathbb{R}^{vd}\to\mathbb{R} \text{ satisfying}\\ \qquad \|g_1\|_\infty\le\infty,\ \|g_2\|_\infty\le\infty,\ \mathrm{Lip}\,g_1<\infty \text{ and } \mathrm{Lip}\,g_2<\infty;\end{cases}$$

As a particular of the functional limit theorem presented in Bardet *et al.* (2005) a UCLT for integrated periodogram can also be established, and more precisely a convergence rate to the Gaussian law:

Theorem 3 *Let $X = (X_n)_{n\in\mathbb{Z}}$ satisfy Assumption M and be η-weakly dependent process. Suppose also that*

$$\exists m>4,\ \text{such that}\ \ \|X_0\|_m<\infty\ \ \text{and}\ \ \eta_n=\mathcal{O}(n^{-\alpha})\ \ \text{with}\ \ \alpha>\max\Big(3\,;\,\frac{2m-1}{m-4}\Big).$$

Then the UCLT (7) holds. Moreover, for $\phi:\mathbb{R}\to\mathbb{R}$ a $\mathcal{C}^3(\mathbb{R})$ function having bounded derivatives up to order 3, and for $g\in\mathcal{H}_s$:

$$\Big|\mathbb{E}\Big[\phi\Big(\sqrt{n}(J_n(g)-J(g))\Big)-\phi\big(\gamma(g)\cdot N\big)\Big]\Big| \le C\cdot n^{-\frac{t}{t+3}\left(\frac{\alpha(m-4)-2m+1}{2(m+1+\alpha\cdot m)}\right)}$$

where $C>0$, $t=\Big(\big(2\alpha\dfrac{m-2}{m-1}-1\big)\wedge\big(\dfrac{s-1}{2}\big)\Big)$, $N\sim\mathcal{N}(0,1)$ and $\gamma^2(g)=\Gamma(g,g)$ defined in (4).

Corollary 3 *Under the same assumptions as in Theorem 3, for $\ell\in\mathbb{Z}$ and $\phi:\mathbb{R}\to\mathbb{R}$ a $\mathcal{C}^3(\mathbb{R})$ function having bounded derivatives up to order 3,*

$$\Big|\mathbb{E}\Big[\phi\Big(\sqrt{n}(\widehat{R}_n(\ell)-R(\ell))\Big)-\phi\big(\sigma_\ell\cdot N\big)\Big]\Big| \le C\cdot n^{-\frac{\alpha(m-4-2m+1}{2(m+1+\alpha\cdot m)}},$$

with $C>0$, $N\sim\mathcal{N}(0,1)$ and

$$\sigma_\ell^2=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(\lambda\ell))f^2(\lambda)d\lambda+2\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos(\lambda\ell)\cos(\mu\ell)f_4(\lambda,-\mu,\mu)d\lambda d\mu.$$

Remark. The convergence rates in both of the functional limit theorems in Theorem 3 and Corollary 3 are obtained from the Bernstein's blocks method, and could be not optimal. However, under not too restrictive conditions ($\alpha\to\infty$ and $s\to\infty$), the convergence rates of those theorems can be $n^{-\lambda}$ with $\lambda<1/2$ as close to $1/2$ as one wants.

3 Applications to parametric estimation

Now we will apply the previous results to finite parameters estimates. Let $X=(X_n)_{n\in\mathbb{Z}}$ be a time series satisfying Assumption M. We also assume that the spectral density f of X can be written under the form:

$$f(\lambda)=f_{(\beta,\sigma^2)}(\lambda)=\sigma^2\cdot g_\beta(\lambda)\quad\text{for all}\ \ \lambda\in[-\pi,\pi[, \quad (10)$$

that is, f depends on a finite number of unknown parameters, a variance term σ^2 and a $\mathbb{R}^p$-vector β, where $\beta=(\beta^{(1)},\ldots,\beta^{(p)})$. Denote also σ^* and $\beta^*=(\beta^{(1)*},\ldots,\beta^{(p)*})$ the true value of σ and β. As a consequence, for all $\lambda\in[-\pi,\pi[$, we will now denote $\sigma^{*2}g_{\beta^*}(\lambda)$ the spectral density of X. We will also assume that β and g_β satisfy some of the following conditions:

- Condition C1: the true values σ^* and β^* are such that $\sigma^* > 0$ and β^* lies in a region $\mathcal{K} \subset \mathbb{R}^p$ where $\mathcal{K}$ is an open and relatively compact set.
- Condition C2: if β_1, β_2 are distinct elements of $\mathcal{K}$, the set $\{\lambda \in [-\pi, \pi[,\ g_{\beta_1}(\lambda) \neq g_{\beta_2}(\lambda)\}$ has positive Lebesgue measure.
- Condition C3: there is a normalization condition:

$$\int_{-\pi}^{\pi} \log(g_\beta(\lambda))\, d\lambda = 0 \quad \text{for all} \quad \beta \in \mathcal{K}.$$

- Condition C4: for all $\beta \in \mathcal{K}$, the function $\lambda \mapsto g_\beta^{-1}(\lambda) = \dfrac{1}{g_\beta(\lambda)} \in \mathcal{H}_s$.
- Condition C5: for all $\lambda \in [-\pi, \pi[$, the function $\beta \mapsto g_\beta^{-1}(\lambda)$ is continuous on $\mathcal{K}$.
- Condition C6: for all $\lambda \in [-\pi, \pi[$, the function $\beta \mapsto g_\beta^{-1}(\lambda)$ is twice continuously differentiable on $\mathcal{K}$.
- Condition C7: for all $\beta \in \mathcal{K}$ and $(i,j) \in \{1, \ldots, p\}$, the function $\lambda \mapsto \left(\dfrac{\partial^2 g_\beta^{-1}}{\partial \beta^{(i)} \partial \beta^{(j)}}\right)_\beta (\lambda) \in \mathcal{H}_s$.
- Condition C8: for all $\beta \in \mathcal{K}$, the function $\lambda \mapsto g_\beta(\lambda)$ is continuously differentiable on $[-\pi, \pi[$.

Let $(X_1, \ldots, X_n)$ be a sample from X. Define the Whittle maximum likelihood estimators of β^* and σ^{*2}, that are:

$$\widehat{\beta}_n = \text{Argmin}_{\beta \in \mathcal{K}} \left\{ J_n(g_\beta^{-1}) \right\} = \text{Argmin}_{\beta \in \mathcal{K}} \left\{ \int_{-\pi}^{\pi} \frac{I_n(\lambda)}{g_\beta(\lambda)}\, d\lambda \right\} \quad \text{and} \quad \widehat{\sigma}_n^2 = \frac{1}{2\pi} J_n(g_{\widehat{\beta}_n}^{-1}).$$

In the following paragraphs, we will show the strong consistency of the estimators $\widehat{\beta}_n$ and $\widehat{\sigma}_n^2$.

3.1 Asymptotic properties of the Whittle parametric estimators

Theorem 4 *Let X satisfy the assumptions of Theorem 1. Under Conditions C1-5, then*

$$\widehat{\beta}_n \xrightarrow[n\to\infty]{a.s.} \beta^* \quad \textit{and} \quad \widehat{\sigma}_n^2 \xrightarrow[n\to\infty]{a.s.} \sigma^{*2}.$$

Proof. From Theorem 1 and Conditions C4 and C5 (the function $\beta \mapsto g_\beta^{-1}$ is also uniformly continuous on $\mathcal{K}$ because $\mathcal{K}$ is a relatively compact set), with probability 1,

$$\lim_{n\to\infty} J_n(g_\beta^{-1}) = J(g_\beta^{-1}),$$

uniformly in β on $\mathcal{K}$. From Condition C2, we know that

$$J(g_\beta^{-1}) > \sigma^{*2} = J(g_{\beta^*}^{-1}) \quad \text{for all} \quad \beta \neq \beta^*$$

(see Lemma 2, in Hannan, 1973). Therefore (see the details of the proof of Theorem 1 in Hannan, 1973), $\widehat{\beta}_n = \text{Argmin}_{\beta \in \mathcal{K}} \left\{ J_n(g_\beta^{-1}) \right\}$ converges a.s. to β^* and $\widehat{\sigma}_n^2 = \dfrac{1}{2\pi} J_n(g_{\widehat{\beta}_n}^{-1})$ converges to $\sigma^{*2} = \dfrac{1}{2\pi} J(g_{\beta^*}^{-1})$. ∎

Remarks on the conditions C1-5 The C1-3 conditions are usual and can be found for example in Rosenblatt (1985) for mixing time series or in Fox and Taqqu (1986) for strong dependence times series. The condition C5 is weaker than the condition of differentiability generally required. The condition C4 is not usual and is linked with the uniform limit theorems 1 and 2.

Theorem 5 *Let X satisfy satisfy either the assumptions of Theorem 2 or those of Theorem 3. Under Conditions C1-7 and if the matrix $W^* = (w^*_{ij})_{1\leq i,j\leq p}$, with*

$$w^*_{ij} = \int_{-\pi}^{\pi} g^2_{\beta^*}(\lambda) \cdot \Big(\frac{\partial g_\beta^{-1}}{\partial \beta^{(i)}}\Big)_{\beta^*}(\lambda) \cdot \Big(\frac{\partial g_\beta^{-1}}{\partial \beta^{(j)}}\Big)_{\beta^*}(\lambda)\, d\lambda$$

is nonsingular, then

$$\sqrt{n}(\widehat{\beta}_n - \beta^*) \quad \xrightarrow[n\to\infty]{\mathcal{D}} \quad \mathcal{N}_p\Big(0\,,\, (\sigma^*)^{-4} \cdot (W^*)^{-1} \cdot Q^* \cdot (W^*)^{-1}\Big), \tag{11}$$

with the matrix $Q^* = (q^*_{ij})_{1\leq i,j\leq p}$ such that:

$$q^*_{ij} = 2\pi \left(\sigma^{*4} w^*_{ij} + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_4(\lambda,\mu,-\mu)\Big(\frac{\partial g_\beta^{-1}}{\partial \beta^{(i)}}\Big)_{\beta^*}(\lambda)\Big(\frac{\partial g_\beta^{-1}}{\partial \beta^{(j)}}\Big)_{\beta^*}(\mu)\, d\lambda\, d\mu \right).$$

Proof. Let $U_n(\beta) = J_n(g_\beta^{-1})$. From Conditions 2 and 6, $\beta \mapsto U_n(\beta)$ exists and is twice differentiable on $\mathcal{K}$. Denote $\frac{\partial}{\partial \beta}U_n(\beta)$ the vector $\Big(\frac{\partial}{\partial \beta^{(i)}}U_n(\beta)\Big)_{1\leq i\leq p}$ and $\frac{\partial^2}{\partial \beta^2}U_n(\beta)$ the $(p\times p)$ matrix $\Big(\frac{\partial^2}{\partial \beta^{(i)}\partial\beta^{(j)}}U_n(\beta)\Big)_{1\leq i,j\leq p}$. According to the mean value theorem,

$$\frac{\partial}{\partial \beta}U_n(\widehat{\beta}_n) = \frac{\partial}{\partial \beta}U_n(\beta^*) + \frac{\partial^2}{\partial \beta^2}U_n(\overline{\beta}_n)(\widehat{\beta}_n - \beta^*),$$

where $\|\overline{\beta}_n - \beta^*\|_p \leq \|\widehat{\beta}_n - \beta^*\|_p$ (with $\|.\|_p$ the euclidian norm in $\mathbb{R}^p$). Since $\widehat{\beta}_n$ minimizes $\beta \mapsto U_n(\beta)$, it follows that

$$\frac{\partial}{\partial \beta}U_n(\beta^*) = \left[-\frac{\partial^2}{\partial \beta^2}U_n(\overline{\beta}_n)\right](\widehat{\beta}_n - \beta^*). \tag{12}$$

But, from Theorem 4, $\widehat{\beta}_n \xrightarrow[n\to\infty]{a.s.} \beta^*$ and then $\overline{\beta}_n \xrightarrow[n\to\infty]{a.s.} \beta^*$. Consequently, from Condition C7 and Theorem 1 (Uniform Law of Large Number),

$$\frac{\partial^2}{\partial \beta^2}U_n(\overline{\beta}_n) \xrightarrow[n\to\infty]{a.s.} \Big(\int_{-\pi}^{\pi} \frac{\partial^2}{\partial \beta^{(i)}\partial\beta^{(j)}} g^{-1}_{\beta^*}(\lambda)\cdot \sigma^{*2} g_{\beta^*}(\lambda)\, d\lambda\Big)_{1\leq i,j\leq p} = \sigma^{*2}W^*,$$

(see Lemma 3 of Fox and Taqqu, 1986). Moreover, from Theorem 2 and Condition C6,

$$\sqrt{n}\Big(\frac{\partial}{\partial \beta}U_n(\beta^*) - \frac{\partial}{\partial \beta}J(g^{-1}_{\beta^*})\Big) \quad \xrightarrow[n\to\infty]{\mathcal{D}} \quad \mathcal{N}_p(0,Q^*),$$

$$\text{and thus} \quad \sqrt{n}\frac{\partial}{\partial \beta}U_n(\beta^*) \quad \xrightarrow[n\to\infty]{\mathcal{D}} \quad \mathcal{N}_p(0,Q^*),$$

because $\frac{\partial}{\partial \beta}J(g^{-1}_{\beta^*}) = \int_{-\pi}^{\pi}\Big(\frac{\partial g_\beta^{-1}(\lambda)}{\partial \beta}\Big)_{\beta^*}\cdot \sigma^{*2} g_{\beta^*}(\lambda)\, d\lambda = \sigma^{*2}\frac{\partial}{\partial \beta}\Big(\int_{-\pi}^{\pi}\log(g_\beta^{-1}(\lambda))\, d\lambda\Big)_{\beta^*} = 0$ from Condition C2. Therefore, if the matrix W^* is nonsingular, from (12),

$$\sqrt{n}(\widehat{\beta}_n - \beta^*) \xrightarrow[n\to\infty]{\mathcal{D}} -(\sigma^*)^{-2}(W^*)^{-1}\cdot \mathcal{N}_p(0,Q^*),$$

and this completes the proof of Theorem 5. ∎

Theorem 6 *Let X satisfy either the assumptions of Theorem 2 or those of Theorem 3. Under Conditions C1-8, then*

$$\sqrt{n}(\widehat{\sigma}_n^2 - \sigma^*) \quad \xrightarrow[n\to\infty]{\mathcal{D}} \quad \mathcal{N}\Big(0\,,\,2\sigma^{*4} + 2\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_4(\lambda,\mu,-\mu) g_{\beta^*}^{-1}(\lambda) g_{\beta^*}^{-1}(\mu)\, d\lambda\, d\mu\Big). \tag{13}$$

Moreover, $\sqrt{n}(\widehat{\sigma}_n^2 - \sigma^*)$ and $\sqrt{n}(\widehat{\beta}_n - \beta^*)$ are jointly asymptotically normal with covariance:

$$\lim_{n\to\infty} \sqrt{n}\Big(Cov\,(\widehat{\sigma}_n^2\,,\,\widehat{\beta}_n^{(i)})\Big)_{1\le i\le p}$$

$$= \big(\sigma^{*2}\cdot W^*\big)^{-1} \cdot \Big(2\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_4(\lambda,\mu,-\mu) g_{\beta^*}^{-1}(\lambda) \Big(\frac{\partial}{\partial \beta^{(i)}} g_{\beta^*}^{-1}(\mu)\Big)\, d\lambda\, d\mu\Big)_{1\le i\le p}.$$

Proof. The Taylor's formula implies that:

$$U_n(\beta^*) = U_n(\widehat{\beta}_n) + (\beta^* - \widehat{\beta}_n)' \cdot \Big(\frac{\partial^2}{\partial \beta^2} U_n(\underline{\beta}_n)\Big) \cdot (\beta^* - \widehat{\beta}_n),$$

with probability 1, and with $\|\underline{\beta}_n - \beta^*\|_p < \|\widehat{\beta}_n - \beta^*\|_p$. From previous Theorem 5, it follows

$$\sqrt{n}(U_n(\beta^*) - \sigma^{*2}) = \sqrt{n}(U_n(\widehat{\beta}_n) - \sigma^{*2}) + \mathcal{O}_p(n^{-1/2}).$$

Under condition C8 (that implies $\sum |s| \cdot R(s) < \infty$), $\mathbb{E}\Big(U_n(\beta^*)\Big) = \sigma^{*2} + \mathcal{O}(\log n/n)$ (see for instance Rosenblatt, 1985) and thus $\sqrt{n}\Big(U_n(\beta^*) - \mathbb{E}\Big(U_n(\beta^*)\Big)\Big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\Big(0, \Gamma(g_{\beta^*}^{-1}, g_{\beta^*}^{-1})\Big)$ with $g_{\beta^*}^{-1} \in \mathcal{H}_s$. Therefore,

$$\sqrt{n}\Big(U_n(\widehat{\beta}_n) - \sigma^{*2}\Big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\Big(0, \Gamma(g_{\beta^*}^{-1}, g_{\beta^*}^{-1})\Big),$$

which implies relation (13). The end of the proof follows the same arguments as in Rosenblatt (1985). ∎

3.2 Examples of Whittle parametric estimates for different time series

Causal GARCH and ARCH(∞) processes

The famous and from now on classical GARCH(q', q) model was introduced by Engle (1982) and Bollerslev (1986) and is given by relations

$$X_k = \rho_k \cdot \xi_k \quad \text{with} \quad \rho_k^2 = a_0 + \sum_{j=1}^{q} a_j X_{k-j}^2 + \sum_{j=1}^{q'} c_j \rho_{k-j}^2, \tag{14}$$

where $(q', q) \in \mathbb{N}^2$, $a_0 > 0$, $a_j \ge 0$ and $c_j \ge 0$ for $j \in \mathbb{N}$ and $(\xi_k)_{k\in\mathbb{Z}}$ are i.i.d. random variables with zero mean (for an excellent survey about ARCH modelling, see Giraitis *et al.*, 2005). Under some additional conditions, the GARCH model can be written as a particular case of ARCH(∞) model (introduced in Robinson, 1991) that satisfied:

$$X_k = \rho_k \cdot \xi_k \quad \text{with} \quad \rho_k^2 = b_0 + \sum_{j=1}^{\infty} b_j X_{k-j}^2, \tag{15}$$

with a sequence $(b_j)_j$ depending on the family (a_j) and (c_j). Different sufficient conditions can be provided for obtaining a m-order stationary solution to (14) or (15). Notice that for both models

(14) or (15), the spectral density is a constant. As a consequence, the idea of Whittle estimation in the GARCH case (see Bollerslev, 1986) is based on the ARMA representation satisfied by $(X_k^2)_{k\in\mathbb{Z}}$. Indeed, if (X_k) is a solution of (14) or (15), then (X_k^2) can be written as a solution of a particular case of equation (16) satisfied by bilinear models (see Giraitis *et al.*, 2005). More precisely,

$$X_k^2 = \varepsilon_k\Big(\gamma\cdot b_0+\gamma\sum_{j=1}^{\infty} b_j X_{k-j}^2\Big)+\lambda_1\cdot b_0+\lambda_1\sum_{j=1}^{\infty} b_j X_{k-j}^2 \quad \text{for} \quad k\in\mathbb{Z},$$

with $\varepsilon_k = (\xi_k^2-\lambda_1)/\gamma$ for $k\in\mathbb{Z}$, $\lambda_1=\mathbb{E}\xi_0^2$ and $\gamma^2=\mathrm{Var}\,(\xi_0^2)$. Moreover, the time series $(Y_k)_{k\in\mathbb{Z}}$ defined by $Y_k = X_k^2-\lambda_1\cdot b_0\cdot\big(1-\lambda_1\sum_{j=1}^{\infty} b_j\big)^{-1}$ for $k\in\mathbb{Z}$, satisfies the equation (16) with parameter $c_0=0$ (as in Proposition 2). As a consequence, a sufficient condition for the stationarity of $(X_k^2)_{k\in\mathbb{Z}}$ with $\|X_0^2\|_m<\infty$ is

$$(\|\varepsilon_0\|_m+1)\cdot\sum_{j=1}^{\infty}|b_j|<1 \quad\Longleftrightarrow\quad \Big(\frac{\|\xi_0^2-\lambda_1\|_m}{\gamma}+1\Big)\cdot\sum_{j=1}^{\infty}|b_j|<1.$$

In a parametric frame, if $(X_k)_{k\in\mathbb{Z}}$ is a stationary solution of (15) and with $\beta=(\beta^{(1)},\dots,\beta^{(p)})\in\mathbb{R}^p$ such that $b_j=b_j(\beta)$ for $j\in\mathbb{N}$, and $\sigma^2=\mathbb{E}(X_0^2-\rho_0^2)=b_0^2(\beta)\cdot h(\lambda_1,\gamma,\sum_{j=1}^{\infty}b_j(\beta))$, where h is a positive real function, the spectral density of $(X_k^2)_{k\in\mathbb{Z}}$ is:

$$f_{(\beta,\sigma^2)}(\lambda)=\frac{\sigma^2}{2\pi}\cdot\Big|1-\sum_{j=1}^{\infty}b_j(\beta)\cdot e^{ij\lambda}\Big|^{-2},$$

Then a Whittle estimate of parameters β and σ^2 can be used for a ARCH(∞) process:

Proposition 1 *Let X be a stationary ARCH(∞) time series following equation (15), such that it exists $m>8$ satisfying $\mathbb{E}(|\xi_0|^m)<\infty$, with the condition of stationarity,*

$$\Big(\big(\frac{\|\xi_0^2-\lambda_1\|_{m/2}}{\|\xi_0^2-\lambda_1\|_2}+1\big)\ \wedge\ \|\xi_0\|_m^2\Big)\cdot\sum_{j=1}^{\infty}|b_j(\beta)|<1, \quad \textit{and one of the two following conditions:}$$

- *Geometric decay: $\forall j\in\mathbb{N}$, $0\le b_j(\beta)$ and $\exists\mu\in]0,1[$ such that $b_j(\beta)=\mathcal{O}(\mu^{-j})$;*
- *Riemannian decay: $\forall j\in\mathbb{N}$, $b_j(\beta)\ge 0$, $\exists\nu>\dfrac{2m-9}{m-8}$ such that $b_j(\beta)=\mathcal{O}(j^{-\nu})$.*

Then, under Conditions C1-7, the central limit theorems (11) and (13) are satisfied.

Corollary 4 *If there exists $m>8$ such that X is a m-order stationary GARCH(q', q) time series satisfying equation (14), then with $\beta=(a_1,\dots,a_q,c_1,\dots,c_{q'})$, the central limit theorems (11) and (13) are satisfied.*

Proof. First, Doukhan *et al* (2005) have shown that a ARCH(∞) process satisfies the θ-weak dependence property: in the "Geometric decay" case, $\theta_r=\mathcal{O}(e^{-c\sqrt{r}})$ with $c>0$ and in the "Riemannian decay" case, with $\nu>2$, $\theta_r=\mathcal{O}\big(r^{-\nu+1}\big)$. Now, after applying the following Lemma 6 (see section 5) for $h(x)=x^2$ and thus $a=2$, we deduce that $(X_k^2)_{k\in\mathbb{Z}}$ is a $\theta^{\frac{m-2}{m-1}}$-weak dependent time series. As a consequence, by denoting $\theta'=(\theta'_k)_{k\in\mathbb{Z}}$ the weak dependent sequence of X^2, in "Geometric decay" case, $\theta'_r=\mathcal{O}(e^{-c\sqrt{r}})$ with $c>0$ and in the "Riemannian decay" case, $\theta'_r=\mathcal{O}\big(r^{-\frac{(\nu-1)(m-2)}{m-1}}\big)$. The

result of Corollary 2 implies that 1/ in the "Geometric decay" case, for all μ, $(X_k^2)_{k\in\mathbb{Z}}$ satisfies the Uniform CLT (7), 2/ in the "Riemannian decay" case, $(X_k^2)_{k\in\mathbb{Z}}$ satisfies the Uniform CLT (7) if $\dfrac{(\nu-1)(m-2)}{m-1}\cdot\dfrac{m/2-4}{m/2-1}>1$, *i.e.* $\nu>\dfrac{2m-9}{m-8}$.
Finally, the proof of the corollary is a particular case of the "Geometric decay" case. ∎

Remarks. The question of the Whittle estimation of the parameter of a stationary solution of (14) was studied by Zaffaroni (2003) and improved by Giraitis and Robinson (2001). In this paper, the obtained results in term of the asymptotic normality of the Whittle estimate are better than Theorem (1), in the sense that: 1/ only the $m=8$ is required; 2/ the required conditions on the sequence $(b_j(\beta))$ in the general case of ARCH(∞) model are only $b_0(\beta)>0$ and $b_j(\beta)\geq 0$ for $j\in\mathbb{N}^*$ and the stationarity condition $\|\xi_0\|_m^2\cdot\sum_{j=1}^\infty|b_j(\beta)|<1$. However, the method developed in Giraitis and Robinson (2001) for establishing the central limit theorem satisfied by the periodogram is essentially *ad hoc* and can not be used for non causal or non linear time series. The recent book of Straumann (2005) also provides an up-to-date and complete overview to this question. Chapter 8 of this book is devoted to the results in Mikosch and Straumann (2002) that studied the case of intermediate moment conditions of order >4 and <8 for the special case of GARCH(1,1) processes; the convergence rates are proved to be slower than the present ones.

Causal Bilinear processes

Now, assume that $X=(X_k)_{k\in\mathbb{Z}}$ is a bilinear process (see the seminal paper of Giraitis and Surgailis, 2002) satisfying the equation:

$$X_k=\xi_k\Big(a_0+\sum_{j=1}^\infty a_jX_{k-j}\Big)+c_0+\sum_{j=1}^\infty c_jX_{k-j}\quad\text{for } k\in\mathbb{Z},\tag{16}$$

where $(\xi_k)_{k\in\mathbb{Z}}$ are i.i.d. random variables with zero mean and such that $\|\xi_0\|_p<+\infty$ with $p\geq 1$, and a_j, c_j, $j\in\mathbb{N}$ are real coefficients. Assume $c_0=0$ and define the generating functions:

$$\begin{array}{ll} A(z)=\sum_{j=1}^\infty a_jz^j & C(z)=\sum_{j=1}^\infty c_jz^j\\ G(z)=(1-C(z))^{-1}=\sum_{j=0}^\infty g_jz^j & H(z)=A(z)G(z)=\sum_{j=1}^\infty h_jz^j.\end{array}$$

If $\|\xi_0\|_p\cdot\sum_{j=1}^\infty|h_j|<\infty$, for instance when $\|\xi_0\|_p\cdot\Big(\sum_{j=1}^\infty|a_j|+\sum_{j=1}^\infty|c_j|\Big)<1$ (see Giraitis and Surgailis, 2002), there exists a unique zero mean stationary and ergodic solution X in $\mathbb{L}^p(\Omega,\mathcal{A},\mathbb{P})$ of equation (16) (see Doukhan *et al.*, 2004). For $p\geq 2$, the covariogram of X is $R(k)=a_0^2\cdot\|\xi_0\|_2\cdot\Big(1-\sum_{j=1}^\infty h_j^2\Big)^{-1}\sum_{j=0}^\infty g_j\,g_{j+k}$ and satisfied $\sum_k|R(k)|<\infty$. If we assume that there exists $\beta=(\beta^{(1)},\ldots,\beta^{(p)})$ such that for all $k\in\mathbb{Z}$, $a_k=a_k(\beta)$ and $c_k=c_k(\beta)$, the spectral density of X exists and satisfies:

$$f_{(\beta,\sigma^2)}(\lambda)=\frac{a_0^2(\beta)\cdot\sigma^2}{2\pi\big(1-\sum_{j=1}^\infty h_j^2(\beta)\big)}\sum_{k=-\infty}^\infty\sum_{j=0}^\infty g_j(\beta)\,g_{j+k}(\beta)\,e^{-ik\lambda},$$

with $\sigma^2=\|\xi_0\|_2^2$. Like in Doukhan *et al.* (2004), we consider three different cases of the convergence rate to zero of the sequences (a_k) and (c_k). Then, using the previous results for θ-weak dependent time series, the Whittle estimate of parameters β and σ^2 satisfies the following proposition:
Proposition 2 *Let X be a stationary bilinear time series satisfying equation (16) with $c_0=0$, $\mathbb{E}(|\xi_0|^m)<\infty$ with $m>4$ and such that $\|\xi_0\|_p\cdot\Big(\sum_{j=1}^\infty|a_j|+\sum_{j=1}^\infty|c_j|\Big)<1$. Moreover, assume that X satisfies one of the 3 following conditions:*

– *Finite case :* $\exists J \in \mathbb{N}$ *such that* $\forall j > J$, $a_j(\beta) = c_j(\beta) = 0$;
– *Geometric decay :* $\exists \mu \in]0,1[$ *such that* $\sum_j |c_j(\beta)|\mu^{-j} \leq 1$ *and* $\forall j \in \mathbb{N}$, $0 \leq a_j(\beta) \leq \mu^j$;
– *Riemannian decay :* $\forall j \in \mathbb{N}$, $c_j(\beta) \geq 0$, *and* $\exists \nu_1 > \dfrac{2m-5}{m-4}$ *such that* $a_j(\beta) = \mathcal{O}(j^{-\nu_1})$ *and* $\exists \nu_2 > 0$ *such that* $\sum_j c_j(\beta) j^{1+\nu_2} < \infty$, *with* $\begin{cases} \nu_2 > \dfrac{(m-4)\delta}{(m-1)\delta - (m-4)\log 2} \\ \delta = \log\Big(1 + \dfrac{1 - \sum_j |c_j(\beta)|}{\sum_j c_j(\beta) j^{1+\nu_2}}\Big) > \log 2 \, \dfrac{(m-4)}{(m-1)} \end{cases}$.

Then, under Conditions C1-7, the central limit theorems (11) and (13) are satisfied.

Proof. The three different cases of the Proposition are studied in Doukhan *et al.* (2004) and the θ-weak dependence behavior of X is deduced in each case. In the "Finite" and the "Geometric decay" cases, $\theta_r = \mathcal{O}(e^{-c\sqrt{r}})$ with $c > 0$, which implies the conditions required in Corollary 2 and therefore, under Conditions C1-7, the central limit theorems (11) and (13) are satisfied.
In the "Riemannian decay" case, $\theta_r = \mathcal{O}\Big(\big(\frac{r}{\log r}\big)^d\Big)$ with $d = \max\Big(-(\nu_1 - 1)\,;\, -\dfrac{\nu_2 \cdot \delta}{\delta + \nu_2 \cdot \log 2}\Big)$. From Corollary 2, the Uniform CLT (7) is satisfied when $d \cdot \dfrac{m-1}{m-4} < -1$. Thus, if $1 - \nu_1 < -\dfrac{m-4}{m-1}$ and $-\dfrac{\nu_2 \cdot \delta}{\delta + \nu_2 \cdot \log 2} < -\dfrac{m-4}{m-1}$, *i.e.* $\nu_1 > \dfrac{2m-5}{m-4}$ and $\nu_2 > \dfrac{(m-4)\delta}{(m-1)\delta - (m-4)\log 2}$, Corollary 2 is satisfied, and under Conditions C1-7, the central limit theorems (11) and (13) hold. ∎

Non-causal (two-sided) linear processes

Let X be a zero mean stationary non causal (two-sided) linear time series satisfying:

$$X_k = \sum_{j=-\infty}^{\infty} a_j \xi_{k-j} \quad \text{for} \quad k \in \mathbb{Z}, \tag{17}$$

with $(a_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $(\xi_k)_{k\in\mathbb{Z}}$ a sequence of zero mean i.i.d. random variables such that $\mathbb{E}(\xi_0^2) = \sigma^2 < \infty$ and $\mathbb{E}(|\xi_0|^m) < \infty$ with $m \geq 4$. We assume that there exists $\beta = (\beta^{(1)}, \ldots, \beta^{(p)})$ such that for all $k \in \mathbb{Z}$, $a_k = a_k(\beta)$. Moreover, we assume that $(a_k(\beta))_{k\in\mathbb{Z}}$ is such that $a_k(\beta) = \mathcal{O}(|k|^{-a})$ with $a > 1$. Therefore the spectral density of X exists and satisfies:

$$f_{(\beta,\sigma^2)}(\lambda) = \frac{\sigma^2}{2\pi}\left|\sum_{k=-\infty}^{\infty} a_k(\beta) e^{-ik\lambda}\right|^2.$$

Then the results of the previous paragraph concerning non causal weak dependent processes can be applied.
Proposition 3 *Let X be a linear time series satisfying (17) with $(a_k(\beta))_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $(\xi_k)_{k\in\mathbb{Z}}$ a sequence of i.i.d. random variables with zero mean such that $\mathbb{E}(\xi_0^2) = \sigma^2$ and $\mathbb{E}(|\xi_0|^m) < \infty$ with $m > 4$. We assume that $(a_k(\beta))$ is such that:*

$$a_k(\beta) = \mathcal{O}(|k|^{-a}) \quad \textit{with} \quad a > \max\Big\{\frac{7}{2}\,;\, \frac{5m-6}{2(m-4)}\Big\}.$$

Then, under Conditions C1-7, the central limit theorems (11) and (13) are satisfied.

Proof. A η-weak dependence condition for non causal linear random fields could be found in Doukhan and Lang (2002, p. 3); under the previous assumptions, X is a η-weak dependent time series with the relation: $\eta_{2r}^2 = \mathcal{O}\big(\sum_{|k|>r} a_k^2(\beta)\big) \quad \Longrightarrow \quad \eta_r = \mathcal{O}\big(\frac{1}{r^{a-1/2}}\big)$. Proposition 3 is then a consequence of Theorem 3. ∎

Remarks. 1/ The Condition C8 of central limit theorem (13) is automatically satisfied by the convergence rate of (a_k) and therefore is not required in Proposition 3;

2/ To our knowledge, the known results about asymptotic behavior of Whittle parametric estimation for non-gaussian linear processes are essentially devoted to one-sided (causal) linear processes (see for instance, Hannan, 1973, Hall and Heyde, 1980, Rosenblatt, 1985, Brockwell and Davis, 1988). In such a case, the conditions on (a_k) are Conditions C1-7, with: $\sum_k k\, a_k^2 < \infty$ for the UCLT and the existence of $\sum_k k\, a_k e^{-ik\lambda}$ for Condition 8. It is such a case if $m = 4$ and $a_k = \mathcal{O}(|k|^{-a})$ with $a > 2$.

3/ There exist very few results in the case of two-sided linear processes. In Rosenblatt (2000, p. 52) a condition for strong mixing property for two-sided linear processes was given, but some restrictive conditions on the process were also required for obtaining a central limit theorem for Whittle estimators: the distribution of random variables ξ_k has to be absolutely continuous with respect to the Lebesgue measure with a bounded variation density, $m > 4 + 2\delta$ with $\delta > 0$ and a central limit theorem obtained with a tapered periodogram (under assumption also $\sum_{m=1}^{\infty} \alpha_{4,\infty}(m)^{\delta/(2+\delta)} < \infty$ where $\alpha_{4,\infty}(m) \geq \alpha_m$ denote a strong mixing coefficient define now with four points in the future instead of 2 for α'_m, the same remark following Corollary 1 still holds). The case of strongly dependent two-sided linear processes was also treated by Giraitis and Surgailis (1990) or Horvath and Shao (1999), however with more restrictive conditions than Conditions C1-7 and with $a_k = \mathcal{O}(|k|^{-a})$ for a fixed $-1 < a < 0$.

4/ In the case of causal linear processes, it is well known that:

$$\sqrt{n}(\widehat{\beta}_n - \beta^*) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}_p\big(0\,,\, 2\pi \cdot (W^*)^{-1}\big),$$

$\widehat{\sigma}_n^2$ is a consistent estimate of σ^4 and therefore $\sqrt{n}(\widehat{\sigma}_n^2 - \sigma^*) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\big(0\,, \sigma^{*4}\cdot\gamma_4\big)$, with γ_4 the fourth cumulant of the $(\xi_k)_{k\in\mathbb{Z}}$, and $\sqrt{n}(\widehat{\beta}_n - \beta^*)$ and $\sqrt{n}(\widehat{\sigma}_n^2 - \sigma^*)$ are asymptotically normal and independent.

Non-causal Volterra processes

Let $X = (X_t)_{t\in\mathbb{Z}}$ be the zero mean non causal (two-sided) and nonlinear time series, so called a non-causal Volterra process, such that for $t \in \mathbb{Z}$:

$$X_k = \sum_{p=1}^{\infty} Y_k^{(p)}, \quad \text{with} \quad Y_k^{(p)} = \sum_{\substack{j_1 < j_2 < \cdots < j_p \\ j_1,\ldots,j_p \in \mathbb{Z}}} a_{j_1,\ldots,j_p} \xi_{k-j_1} \cdots \xi_{k-j_p}, \tag{18}$$

where $(a_{j_1,\ldots,j_p}) \in \mathbb{R}$ for $p \in \mathbb{N}^*$ and $(j_1,\ldots,j_p) \in \mathbb{Z}^p$, and $(\xi_k)_{k\in\mathbb{Z}}$ a sequence of zero mean i.i.d. random variables such that $\mathbb{E}(\xi_0^2) = \sigma^2 < \infty$ and $\mathbb{E}(|\xi_0|^m) < \infty$ with $m > 4$. Such a Volterra process is a natural extension of the previous case of non-causal linear process. From Doukhan (2003), the existence of X and thus the stationarity in $\mathbb{L}^m$ relies on the assumption:

$$\sum_{p=0}^{\infty} \sum_{\substack{j_1 < j_2 < \cdots < j_p \\ j_1,\ldots,j_p \in \mathbb{Z}}} \big|a_{j_1,\ldots,j_p}\big|^m \, \|\xi_0\|_m^p < \infty.$$

Assume that there exists $\beta = (\beta^{(1)},\ldots,\beta^{(p)})$ such that for all $p \in \mathbb{N}^*$ and $(j_1,\ldots,j_p) \in \mathbb{Z}^p$, $a_{j_1,\ldots,j_p} = a_{j_1,\ldots,j_p}(\beta)$. Then the spectral density of X exists and satisfies:

$$f_{(\beta,\sigma^2)}(\lambda) = \sum_{p=1}^{\infty} \frac{(\sigma^2)^p}{2\pi} \sum_{k=-\infty}^{\infty} \sum_{\substack{j_1 < j_2 < \cdots < j_p \\ j_1,\ldots,j_p \in \mathbb{Z}}} a_{j_1,\ldots,j_p}(\beta) \cdot a_{j_1+k,\ldots,j_p+k}(\beta) \cdot e^{-ik\lambda}$$

(this formula is provided by the computation of the covariances of X; remark that the representation with strictly ordered indices $j_1 < j_2 < \cdots < j_p$ is fundamental). Certain conditions on the asymptotic behavior of the coefficients $a_{j_1,\dots,j_p}(\beta)$ provide the η-weak dependence property of X and then the asymptotic normality of estimators $(\widehat{\beta}_n, \widehat{\sigma}_n^2)$:

Proposition 4 *Let X be a non-causal zero mean stationary Volterra process verifying relation (18) where $(a^{(p)}_{j_1,\dots,j_p}) \in \mathbb{R}$ for $p \in \mathbb{N}^*$ and $(j_1, \dots, j_p) \in \mathbb{Z}^p$, and $(\xi_k)_{k\in\mathbb{Z}}$ a sequence of zero mean i.i.d. random variables such that $\mathbb{E}(\xi_0^2) = \sigma^2 < \infty$ and $\mathbb{E}(|\xi_0|^m) < \infty$ with $m > 4$.*

Moreover, assume that the process in in some finite order chaos (i.e. $a_{j_1,\dots,j_p}(\beta) = 0$ for $p > p_0$) and $a_{j_1,\dots,j_p}(\beta)$ is such that:

$$a_{j_1,\dots,j_p}(\beta) = \mathcal{O}\Big(\max_{1\le i \le p} \{|j_i|^{-a}\}\Big) \quad \text{with} \quad a > 4 + \max\Big\{0\,;\, \frac{11-m}{m-4}\Big\}.$$

Then, under Conditions C1-7, the central limit theorems (11) and (13) are satisfied.

Remark. A more tight dependence assumption is $\eta_r = \mathcal{O}\Big(r^{1-a}\Big)$ where a is submitted to the same restriction as before; recall that:

$$\eta_r \le 2 \sum_{k=1}^{\infty} \sum_{\substack{j_1 < j_2 < \cdots < j_k \\ j_1 < -r/2, \text{ or } j_k \ge r/2}} \Big| a^{(k)}_{j_1,\dots,j_k} \Big| \, \|\xi_0\|_1^k < \infty. \tag{19}$$

Proof. From Doukhan (2003), under the previous assumptions, X is a weakly dependent process with:

$$\eta_r \le \sum_{p=1}^{\infty} \sum_{\substack{j_1 < j_2 < \cdots < j_p \\ j_1 < -r/2, \text{ or } j_p \ge r/2}} \Big| a^{(p)}_{j_1,\dots,j_p} \Big| \, \|\xi_0\|_1^p < \infty \quad \Longrightarrow \quad \eta_r = \mathcal{O}\big(\frac{1}{r^{a+1}}\big). \tag{20}$$

Proposition 4 is then a consequence of Theorem 3. ∎

Non-causal (two-sided) bilinear and ARCH(∞) processes

The asymptotic normality of Whittle estimate can be obtained for non-causal bilinear and ARCH(∞) processes. Indeed, Doukhan *et al.* (2005), Lemma 2.1, introduced and proved the stationarity in $\mathbb{L}^k$ (for any $k \in]0, \infty]$) of the bilinear process $X = (X_k)_{k\in\mathbb{Z}}$ satisfying the equation:

$$X_k = \xi_k \cdot \Big(a_0 + \sum_{j\in\mathbb{Z}^*} a_j X_{k-j} \Big), \quad \text{for } k \in \mathbb{Z}, \tag{21}$$

where $(\xi_k)_{k\in\mathbb{Z}}$ are i.i.d. random bounded variables and $(a_k)_{k\in\mathbb{Z}}$ is a sequence of real numbers such that $\lambda = \|\xi_0\|_\infty \cdot \sum_{j\neq 0} |a_j| < 1$. We assume that there exists $\beta = (\beta^{(1)}, \dots, \beta^{(p)})$ such that for all $k \in \mathbb{Z}$, $a_k = a_k(\beta)$. Then the spectral density of X exists and is a function $f_{(\beta,\sigma^2)}$ depending only on parameters (β, σ^2). Unfortunately, the explicit formula of $f_{(\beta,\sigma^2)}$ is not known from now on.

By the same way as in the causal case, assume now that $Y = (Y_k)_{k\in\mathbb{Z}}$ satisfied the relation:

$$Y_k = \xi_k \cdot \sqrt{a_0 + \sum_{j\neq 0} a_j Y_{k-j}^2}, \quad \text{for } k \in \mathbb{Z}, \tag{22}$$

with the same assumptions on $(\xi_k)_{k\in\mathbb{Z}}$ and $(a_k)_{k\in\mathbb{Z}}$. Then, the time series $(Y_k^2)_{k\in\mathbb{Z}}$ satisfies the relation (21) and is a stationary process. Then, Y is a stationary process, so-called a two-sided ARCH(∞) process. The asymptotic normality of the Whittle estimate of parameters (β, σ^2) is satisfied using the η-weak dependence property of of X (or Y) depending on the behavior of the sequence $(a_k(\beta))_{k\in\mathbb{Z}}$:

Proposition 5 *If X is a stationary non causal bilinear (respectively non causal ARCH(∞)) process, i.e. a solution of (21) (respectively (22)), such that $\|\xi_0\|_\infty \cdot \sum_{j\neq 0}|a_j(\beta)| < 1$. We assume that the sequence $(a_k(\beta))_{k\in\mathbb{Z}}$ is such that:*

$$a_k = \mathcal{O}(|k|^{-a}) \quad \textit{with} \quad a > 4.$$

Then, under Conditions C1-7, the central limit theorems (11) and (13) hold.

Proof. The η-weak dependence property of a non causal bilinear process could be found in Doukhan *et al.* (2005): under the previous assumptions, X is a η-weak dependent time series with: $\eta_r = \mathcal{O}\Big(\sum_{2k<r} k\,\lambda^{k-1}\big(\sum_{|j|\geq r/k}|a_j|\big)\Big) \implies \eta_r = \mathcal{O}\big(\frac{1}{r^{a-1}}\big)$. Proposition 5 is then a consequence of Theorem 3. ∎

Remarks. The condition on the sequence $(\xi_k)_{k\in\mathbb{Z}}$, *i.e.* i.i.d. random bounded variables, is restricting. However, if it is only a sufficient condition for the existence of a non causal ARCH(∞) process, it seems to be very close to be also a necessary condition (see Doukhan *et al.*, 2005).

Non-causal linear processes with dependent innovations

Let $X = (X_n)_{n\in\mathbb{N}}$ be a zero mean stationary non causal (two-sided) linear time series satisfying equation (17) with a dependent innovation process. More precisely, let $(\xi_n)_{n\in\mathbb{Z}}$ be a weakly dependent fourth order centered stationary process verifying Assumption M and such that $\mathbb{E}(\xi_0^2) = \sigma^2 < \infty$. Assume that there exists $\beta = (\beta^{(1)}, \ldots, \beta^{(p)})$ such that for all $k \in \mathbb{Z}$, $a_k = a_k(\beta)$ with $a_k(\beta) = \mathcal{O}(|k|^{-a})$ and $a > 1$. Denoting $g_{(\beta,\sigma^2)}$ the spectral density of the process $(\xi_n)_{n\in\mathbb{Z}}$, the spectral density of X exists and satisfies:

$$f_{(\beta,\sigma^2)}(\lambda) \quad = \quad g_{(\beta,\sigma^2)}(\lambda) \cdot \left| \sum_{k=-\infty}^{\infty} a_k(\beta) e^{-ik\lambda} \right|^2.$$

For instance, the process $(\xi_n)_{n\in\mathbb{Z}}$ may be a causal or a non-causal ARCH(∞) or bilinear process. Following the results of Doukhan and Wintenberger (2005), if $(\xi_n)_{n\in\mathbb{Z}}$ is a η-weakly dependent process, then X is an η-weakly dependent process (with a sequence $(\eta_r)_r$ that can be deduced). As a consequence, the asymptotic normality of the Whittle estimate of parameters (β, σ^2) could be established:

Proposition 6 *Let X be a linear time series satisfying (17) with $(a_k(\beta))_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $(\xi_k)_{k\in\mathbb{Z}}$ a $\eta^{(\xi)}$-weakly dependent process with zero mean, a spectral density $g_{(\beta,\sigma^2)}$ depending only on parameters (β, σ^2), and such that $\mathbb{E}(\xi_0^2) = \sigma^2$ and $\mathbb{E}(|\xi_0|^m) < \infty$. Moreover, we assume that:*

$$a_k(\beta) = \mathcal{O}(|k|^{-a}), \qquad \eta_r^{(\xi)} = \mathcal{O}(r^{-b}), \qquad \textit{with} \quad b \cdot \frac{(a-2)(m-2)}{(a-1)(m-1)} > \max\Big\{3\,;\,\frac{2m-1}{m-4}\Big\}.$$

Then, under Conditions C1-7, the central limit theorems (11) and (13) are satisfied.

Proof. In Doukhan and Wintenberger (2005), it was proved that under assumptions $a_k(\beta) = \mathcal{O}(|k|^{-a})$ and $\eta_r^{(\xi)} = \mathcal{O}(r^{-b})$, then X is a η-weakly dependent process with $\eta_r = \mathcal{O}\Big(r^{-b\cdot\frac{(a-2)(m-2)}{(a-1)(m-1)}}\Big)$. Proposition 6 is then a consequence of Theorem 3. ∎

4 Appendix : proofs

4.1 Proof of Theorem 1

Lemma 2 *If X satisfies Assumption M, then:*

$$n \cdot \max_{\ell \geq 0} \Big(Var\Big(\widehat{R}_n(\ell)\Big)\Big) \leq \kappa_4 + 2\gamma.$$

Proof of Lemma 2. To prove this result, we use the identity

$$\mathrm{Cov}\,(Y_{0,\ell}, Y_{j,\ell}) = \kappa_4(\ell, j, j+\ell) + R(j)^2 + R(j+\ell)R(j-\ell)$$

and deduce from the stationarity of $(Y_{j,\ell})_{j\in\mathbb{Z}}$ when ℓ is a fixed integer:

$$\begin{aligned}
n \cdot \mathrm{Var}\Big(\widehat{R}_n(\ell)\Big) &\leq \frac{1}{n} \sum_{j=1\vee(1-\ell)}^{(n-\ell)\wedge n} \sum_{j'=1\vee(1-\ell)}^{(n-\ell)\wedge n} |\mathrm{Cov}\,(Y_{j,\ell}, Y_{j',\ell})| \\
&\leq \sum_{j\in\mathbb{Z}} |\mathrm{Cov}\,(Y_{0,\ell}, Y_{j,\ell})| \\
&\leq \sum_{j\in\mathbb{Z}} \big(|\kappa_4(\ell, j, j+\ell)| + 2R(j)^2\big) \\
&\leq \kappa_4 + 2\gamma,
\end{aligned}$$

with Cauchy-Schwarz inequality for ℓ^2-sequences. ∎

Lemma 3 *If X satisfies Assumption M, then:*

$$\mathbb{E}\|J_n - J\|^2_{\mathcal{H}'_s} \leq \frac{3}{n}\Big(\gamma + c_s \cdot (\kappa_4 + 2\gamma)\Big), \quad \textit{where } c_s \textit{ is defined in } (2).$$

Proof of Lemma 3. Let $g(\lambda) = \sum_{\ell\in\mathbb{Z}} g_\ell \, e^{i\ell\lambda} \in \mathcal{H}_s$. As in Doukhan and León (1989), we use the decomposition:

$$J_n(g) - J(g) = -T_1(g) - T_2(g) + T_3(g) \ \text{ with } \begin{cases} T_1(g) = \displaystyle\sum_{|\ell|\geq n} R(\ell)\, g_\ell, \\ T_2(g) = \displaystyle\frac{1}{n} \sum_{|\ell|<n} |\ell|\, R(\ell)\, g_\ell, \\ T_3(g) = \displaystyle\sum_{|\ell|<n} (\widehat{R}_n(\ell) - \mathbb{E}\widehat{R}_n(\ell))\, g_\ell \end{cases} \tag{23}$$

Remark that $T_3(g) = J_n(g) - \mathbb{E}J_n(g)$. Thus, we obtain the inequality:

$$\mathbb{E}\|J_n - J\|^2_{\mathcal{H}'_s} \leq 3(\|T_1\|^2_{\mathcal{H}'_s} + \|T_2\|^2_{\mathcal{H}'_s} + \mathbb{E}\|T_3\|^2_{\mathcal{H}'_s}).$$

Cauchy-Schwarz inequality yields:

$$\begin{aligned}
\|T_1\|^2_{\mathcal{H}'_s} &\leq \sum_{|\ell|\geq n} (1+|\ell|)^{-s}\, R(\ell)^2 \leq \frac{1}{n} \sum_{|\ell|\geq n} R(\ell)^2, \\
\|T_2\|^2_{\mathcal{H}'_s} &\leq \frac{1}{n^2} \sum_{|\ell|<n} |\ell|^2\, (1+|\ell|)^{-s}\, R(\ell)^2 \leq \frac{1}{n} \sum_{|\ell|<n} R(\ell)^2.
\end{aligned}$$

Hence, $\|T_1\|^2_{\mathcal{H}'_s} + \|T_2\|^2_{\mathcal{H}'_s} \leq \frac{\gamma}{n}$. Lemma 2 entails

$$\begin{aligned}
\|T_3\|^2_{\mathcal{H}'_s} &\leq \sum_{|\ell|<n} (1+|\ell|)^{-s} \, (\widehat{R}_n(\ell) - \mathbb{E}\widehat{R}_n(\ell))^2, \\
\mathbb{E}\|T_3\|^2_{\mathcal{H}'_s} &\leq \sum_{|\ell|<n} (1+|\ell|)^{-s} \operatorname{Var}(\widehat{R}_n(\ell)) \leq \frac{1}{n} \sum_{|\ell|<n} (1+|\ell|)^{-s} \, (\kappa_4 + 2\gamma) \leq \frac{c_s(\kappa_4+2\gamma)}{n},
\end{aligned}$$

with c_s defined in (2). We combine those results to deduce Lemma 3. ∎

Proof of Theorem 1. We prove this strong law of large numbers from a weak $\mathbb{L}^2$-LLN and Lemma 3. The scheme of proof is analogue to the one in the standard strong LLN. Set $t > 0$. First, we know that for all random variables X and Y, we have $\mathbb{P}(X+Y \geq 2t) \leq \mathbb{P}(X \geq t) + \mathbb{P}(Y \geq t)$. Thus:

$$\begin{aligned}
\mathbb{P}\left(\max_{n\geq N} \|J_n - J\|_{\mathcal{H}'_s} \geq 2t\right) &\leq \sum_{k=[\sqrt{N}]}^{\infty} \mathbb{P}(\|J_{k^2} - J\|_{\mathcal{H}'_s} \geq t) \\
&+ \sum_{k=[\sqrt{N}]}^{\infty} \mathbb{P}\left(\max_{k^2 \leq n < (k+1)^2} \|J_n - J_{k^2}\|_{\mathcal{H}'} \geq t\right) \\
&\leq A_N + B_N. \qquad (24)
\end{aligned}$$

From Bienaymé-Tchebychev inequality, Lemma 3 implies that:

$$A_N \leq \frac{C_1}{t^2} \cdot \sum_{k \geq \sqrt{N}} \frac{1}{k^2}, \qquad (25)$$

with $C_1 \in \mathbb{R}_+$. Now set $\widetilde{R}_n(\ell) = \widehat{R}_n(\ell) - \mathbb{E}\widehat{R}_n(\ell)$. The fluctuation term B_N is more involved and its bound is based on the same type of decomposition as (23), because for $k^2 < n$:

$$J_n(g) - J_{k^2}(g) = -T'_1(g) + T'_2(g) - T'_3(g),$$

$$\begin{aligned}
\text{with now } \; T'_1(g) &= \sum_{k^2 \leq |\ell| < n} R(\ell) \, g_\ell, \\
T'_2(g) &= \frac{1}{n} \sum_{k^2 \leq |\ell| < n} |\ell| \, R(\ell) \, g_\ell, \\
\text{and } \; T'_3(g) &= \sum_{|\ell|<k^2} \widetilde{R}_{k^2}(\ell) \, g_\ell - \sum_{|\ell|<n} \widetilde{R}_n(\ell) \, g_\ell.
\end{aligned}$$

As previously: $\|T'_1\|^2_{\mathcal{H}'_s} + \|T'_2\|^2_{\mathcal{H}'_s} \leq \frac{\gamma}{k^2}$.

Set $L_k = \max_{k^2 \leq n < (k+1)^2} \|J_n - J_{k^2}\|_{\mathcal{H}'_s}$ and $T^*_k = \max_{k^2 \leq n < (k+1)^2} \|T'_3\|_{\mathcal{H}'_s}$. Then,

$$B_N \leq \sum_{k \geq \sqrt{N}} b_k, \quad \text{with} \quad b_k = \mathbb{P}(L_k \geq t) \leq \frac{\mathbb{E}(L_k^2)}{t^2}.$$

Now

$$\mathbb{E}(L_k^2) \leq 3(\|T'_1\|^2_{\mathcal{H}'_s} + \|T'_2\|^2_{\mathcal{H}'_s} + \mathbb{E}\|T^*_k\|^2_{\mathcal{H}'_s}) \leq \frac{3\gamma}{k^2} + 3 \cdot \mathbb{E}\|T^*_k\|^2_{\mathcal{H}'_s}.$$

Then, for $k^2 \le n < (k+1)^2$ and $\ell \in \mathbb{Z}$,

$$\begin{aligned}\widetilde{R}_n(\ell) &= \frac{k^2}{n}\widetilde{R}_{k^2}(\ell) + \Delta_{\ell,n,k} \\ \Delta_{\ell,n,k} &= \frac{1}{n}\sum_{(k^2\wedge(k^2-\ell))<h}^{n\wedge(n-\ell)} (X_hX_{h+\ell} - R(\ell)) = \frac{1}{n}\sum_{(k^2\wedge(k^2-\ell))<h}^{n\wedge(n-\ell)} Y_{h,\ell}.\end{aligned}$$

Remark that $\widetilde{R}_{k^2}(\ell) = 0$ if $k^2 \le |\ell| \le n$ and thus $\widetilde{R}_n(\ell) = \Delta_{\ell,n,k}$ in such a case. Also note that

$$\begin{aligned}\Delta^*_{\ell,k} &= \max_{k^2\le n<(k+1)^2} |\Delta_{\ell,n,k}| \le \frac{1}{k^2}\sum_{(k^2\wedge(k^2-\ell))<h}^{(k^2+2k)\wedge((k^2+2k)-\ell)} |Y_{h,\ell}| \\ \text{and thus } \ \mathbb{E}(\Delta^*_{\ell,k})^2 &\le \frac{1}{k^4}(2k)^2\cdot \max_{(h,\ell)\in\mathbb{Z}^2}\big(\mathbb{E}(|Y_{h,\ell}|^2)\big) \\ &\le \frac{4}{k^2}\,\mathbb{E}(|X_0|^4).\end{aligned}$$

Write

$$\begin{aligned}T'_3(g) &= \sum_{|\ell|<k^2} \widetilde{R}_{k^2}(\ell)\left(1-\frac{k^2}{n}\right)g_\ell - \sum_{|\ell|<n}\Delta_{\ell,n,k}\,g_\ell \\ |T^*_k(g)| &\le \frac{2}{k}\sum_{|\ell|<k^2} |\widetilde{R}_{k^2}(\ell)\,g_\ell| + \sum_{|\ell|<(k+1)^2}\Delta^*_{\ell,k}\,|g_\ell|,\end{aligned}$$

and we thus deduce

$$\mathbb{E}\|T^*_k\|^2_{\mathcal{H}'_s} \le 2c\cdot\left(\frac{4}{k^2}\max_{\ell\in\mathbb{Z}}\Big(\mathrm{Var}\,(\widehat{R}_{k^2}(\ell))\Big) + \max_{\ell\in\mathbb{Z}}\big(\mathbb{E}(\Delta^*_{\ell,k})^2\big)\right) \le \frac{c\cdot A}{k^2}$$

for a constant $A > 0$ depending on $\mathbb{E}|X_0|^4, \kappa_4$, and γ only. Hence $b_k \le 3(\gamma + A\cdot c)/(k^2t^2)$ is a summable series and, with $C_2 > 0$,

$$B_N \le \frac{C_2}{t^2}\cdot\sum_{k\ge\sqrt{N}}\frac{1}{k^2}. \tag{26}$$

Then, (24), (25) and (26) imply $\sup_{n\ge N}\|J_n - J\|_{\mathcal{H}'_s} \xrightarrow[N\to\infty]{\mathbb{P}} 0$ what is equivalent to $\|J_n - J\|_{\mathcal{H}'_s} \xrightarrow[n\to\infty]{a.s.} 0$. ∎

4.2 Proofs of the section 2.3.1

First let us recall the following classical lemma (see a proof in Rosenblatt (1985) [27], p. 58):
Lemma 4 *If X satisfies Assumption M and $(\ell,k)\in\mathbb{Z}^2$ be arbitrary integers, then*

$$n\cdot Cov(\widehat{R}_n(k),\widehat{R}_n(\ell)) \underset{n\to\infty}{\longrightarrow} \sigma_{k,\ell}.$$

Proof of Lemma 1. Under condition (5), the projective criterion, introduced in Dedecker and Rio (2000), *i.e.* $\mathbb{E}\Big|\sum_{k\ge0} Y_{0,\ell_i}\,\mathbb{E}\Big(Y_{k,\ell_i}\,|\,\sigma(Y_{j,\ell_i},\ j\le 0)\Big)\Big| < \infty$ for all $i\in\{1,\dots,m\}$, is satisfied. Therefore the central limit theorem is stated for each $\widehat{R}_n(\ell_i)$. Now, by considering a a linear combination of $(Y_{j,\ell_1},\dots,Y_{j,\ell_m})$, denoted Z_j, the projective criterion is also satisfied by $(Z_j)_{j\in\mathbb{Z}}$ yielding the multidimensional central limit theorem (6). ∎

Proof of Theorem 2. We first prove the following lemma:

Lemma 5 (Tightness) *If X satisfies Assumption M then the sequence of processes $(Z_n)_{n\in\mathbb{N}^*}$ is tight in $\mathcal{H}'_s$.*

Proof of Lemma 5. Following de Acosta (1970), for showing the tightness we only need to prove that the sequence is flatly concentrated, this means that

$$\mathbb{E}\Big(\|p_L Z_n\|^2_{\mathcal{H}'_s}\Big) \underset{L\to\infty}{\longrightarrow} 0,$$

where $p_L : \mathcal{H}'_s \to F'_L$ denotes the orthogonal projection on the closed linear subspace $F'_L \subset \mathcal{H}'_s$ generated by $(e_\ell)_{|\ell|\geq L}$ with $e_\ell(\lambda) = e^{i\ell\lambda}$ (also $F_L \subset \mathcal{H}_s$ denote the subspace generated by $(e_\ell)_{|\ell|\geq L}$). Then, for $L > 0$,

$$\|p_L Z_n\|_{\mathcal{H}'_s} = \sup_{\|g\|_{\mathcal{H}_s}<1,\ g\in F_L} |Z_n(g)|. \tag{27}$$

Thus, for $g = \sum_{|\ell|\geq L} g_\ell e_\ell \in F_L$ and $\|g\|_{\mathcal{H}_s} < 1$, using again the decomposition (23), we obtain

$$|Z_n(g)|^2 \leq 3n \cdot (|T_1(g)|^2 + |T_2(g)|^2 + |T_3(g)|^2).$$

First, from a Cauchy-Schwarz inequality, we have:

$$\begin{aligned}
n \cdot (|T_1(g)|^2 + |T_2(g)|^2) &\leq n \cdot \Bigg(\sum_{|\ell|\geq n\vee L} (1+|\ell|)^{-s} R(\ell)^2 \cdot \sum_{|\ell|\geq n\vee L} (1+|\ell|)^s g_\ell^2 \\
&\qquad + \mathbb{1}_{\{L<n\}} \cdot \frac{1}{n^2} \sum_{L\leq|\ell|<n} |\ell|^2 (1+|\ell|)^{-s} R(\ell)^2 \cdot \sum_{L\leq|\ell|<n} (1+|\ell|)^s g_\ell^2 \Bigg) \\
&\leq n \cdot (1+|n\vee L|)^{-s} \sum_{|\ell|\geq n\vee L} R(\ell)^2 + \mathbb{1}_{\{L<n\}} \cdot \sum_{L\leq|\ell|<n} |\ell|\,(1+|\ell|)^{-s}\, R(\ell)^2 \\
&\leq \gamma \cdot \Big(L\cdot(1+L)^{-s} \cdot \mathbb{1}_{\{L\geq n\}} + L\cdot(1+L)^{-s}\cdot \mathbb{1}_{\{L<n\}} \Big).
\end{aligned}$$

Thus, we obtain:

$$\sup_{\|g\|_{\mathcal{H}_s}<1,\ g\in F_L} n \cdot \big(|T_1(g)|^2 + |T_2(g)|^2\big) \underset{L\to\infty}{\longrightarrow} 0. \tag{28}$$

Also note that

$$\begin{aligned}
\sqrt{n}\, T_3(g) &= \sqrt{n} \sum_{|\ell|<n} (\widehat{R}_n(\ell) - \mathbb{E}\widehat{R}_n(\ell)) g_\ell \\
n\,|T_3(g)|^2 &\leq n \sum_{L\leq|\ell|<n} (1+|\ell|)^{-s} (\widehat{R}_n(\ell) - \mathbb{E}\widehat{R}_n(\ell))^2 \\
\mathbb{E}\Big(\sup_{\|g\|_{\mathcal{H}_s}<1,\ g\in F_L} n|T_3(g)|^2 \Big) &\leq \sum_{L\leq|\ell|<n} (1+|\ell|)^{-s} \cdot \sup_\ell \Big(n \operatorname{Var}(\widehat{R}_n(\ell)) \Big) \leq \sum_{|\ell|\geq L} (1+|\ell|)^{-s} (\kappa_4 + 3\gamma).
\end{aligned}$$

Since $\sum_{\ell\in\mathbb{Z}} (1+|\ell|)^{-s} < +\infty$, we deduce $\mathbb{E}\Big(\sup_{\|g\|_{\mathcal{H}_s}<1,\ g\in F_L} n|T_3(g)|^2 \Big) \underset{L\to\infty}{\longrightarrow} 0$. With (28) and (27), the proof is achieved. ∎

Now the proof of Theorem 2 can be achieved. Indeed, the tightness allows to establish the functional central limit theorem. Moreover, $J_n(g) - \mathbb{E}J_n(g) = \sum_{\ell\in\mathbb{Z}} g_\ell\,(\widehat{R}_n(\ell) - \mathbb{E}\,(\widehat{R}_n(\ell))$, and therefore, from (3), the limiting covariance of the process $(Z_n)_{n\geq 1}$ is given by (4). More details of the finite dimensional convergence of the process $(Z_n(g_1),\dots,Z_n(g_k))$ can be found in Rosenblatt (1985, Corollary 2, p. 61). ∎

Proof of Corollary 1. From Rio's inequality (1994) and the stationarity of X, for all $\ell,k\in\mathbb{N}$, we have:

$$\begin{aligned}\|Y_{0,\ell}\,\mathbb{E}\left(Y_{k+\ell,\ell}\mid\mathcal{B}_\ell\right)\|_1 &\leq 2\int_0^{\alpha(\sigma(Y_{k+\ell,\ell}),\mathcal{B}_\ell)} Q_{|Y_{0,\ell}|}(u)\,Q_{|Y_{k+\ell,\ell}|}(u)\,du\\ &\leq 2\int_0^{\alpha(\sigma(X_\ell,X_{k+\ell}),\mathcal{B}_0)} Q^2_{|Y_{0,\ell}|}(u)\,du, \qquad \text{from Schwartz inequality.}\end{aligned}$$

Therefore, for all $\ell,k\in\mathbb{N}$,

$$\|Y_{0,\ell}\,\mathbb{E}\left(Y_{k+\ell,\ell}\mid\mathcal{B}_\ell\right)\|_1 \leq 2\int_0^{\alpha'_k} Q^2_{|Y_{0,\ell}|}(u)\,du.$$

Consequently, for all $\ell\in\mathbb{N}$

$$\begin{aligned}\sum_{k\geq 0}\|Y_{0,\ell}\,\mathbb{E}\left(Y_{k+\ell,\ell}\mid\mathcal{B}_\ell\right)\|_1 &\leq 2\int_0^1\left(\sum_{k\geq 0}\mathbb{1}_{u\leq\alpha'_k}\right)Q^2_{|Y_{0,\ell}|}(u)\,du,\\ &\leq 2\int_0^1(\alpha'(u))^{-1}Q^2_{|Y_{0,\ell}|}(u)\,du.\end{aligned}$$

But Lemma 2.1 in Rio (2000) provides:

$$\begin{aligned}\int_0^1(\alpha'(u))^{-1}Q^2_{|Y_{0,\ell}|}(u)du &\leq \int_0^1(\alpha'(u))^{-1}\Big(Q_{|X_0|}(u)Q_{|X_\ell|}(u)+Q_{|R(\ell)|}(u)\Big)^2\,du\\ &\leq \int_0^1(\alpha'(u))^{-1}(Q^2_{|X_0|}(u)+|R(\ell)|)^2\,du,\end{aligned}$$

and therefore if $\displaystyle\int_0^1(\alpha'(u))^{-1}Q^4_{|X_0|}(u)\,du<\infty$, then $\displaystyle\sum_{k\geq 0}\|Y_{0,\ell}\,\mathbb{E}\left(Y_{k,\ell}\mid\mathcal{B}_\ell\right)\|_1<+\infty$ for all $\ell\in\mathbb{N}$. ∎

Proof of Corollary 2. We truncate the variables $X_j = f_M(X_j)+g_M(X_j)$ where, for $x\in\mathbb{R}$, we set $f_M(x)=(x\wedge M)\vee(-M)$ (then $f_M\in[-M,M]$) and $g_M(x)=x-f_M(x)=x\cdot\mathbb{1}_{|x|\geq M}$. Note that $\mathrm{Lip}\, f_M=1$ but $\|f_M\|_\infty=M$.

Then, $Y_{k,\ell}=\Big(f_M(X_k)\,f_M(X_{k+\ell})-R'(\ell)\Big)+U_{k,\ell,M}$ with:

- $R'(\ell)=\mathrm{Cov}\Big(f_M(X_k),f_M(X_{k+\ell})\Big)$;
- $U_{k,\ell,M}=g_M(X_k)f_M(X_{k+\ell})+f_M(X_k)g_M(X_{k+\ell})+g_M(X_k)g_M(X_{k+\ell})+R'(\ell)-R(\ell)$.

Therefore, $\mathbb{E}(U_{k,\ell,M}) = 0$ and, for m such that $\|X_0\|_m < \infty$, we derive

$$
\begin{aligned}
\|U_{k,\ell,M}\|_1 \le\ & M\mathbb{E}\left|X_k \cdot \mathbb{1}_{|X_k|>M}\right| + M\mathbb{E}\left|X_{k+\ell} \cdot \mathbb{1}_{|X_{k+\ell}|>M}\right| \\
&\qquad + \mathbb{E}\left((X_k \cdot \mathbb{1}_{|X_k|>M})^2\right) + \left|\mathbb{E}\left(X_0 X_\ell - f_M(X_0) f_M(X_\ell)\right)\right| \\
\le\ & 2M\|X_0\|_m \Big(\mathbb{P}(X_0 > M)\Big)^{1-1/m} + \|X_0\|_m^2 \Big(\mathbb{P}(X_0 > M)\Big)^{1-2/m} \\
&\qquad + \mathbb{E}\,|f_M(X_0) g_M(X_\ell)| + \mathbb{E}\,|g_M(X_0) f_M(X_\ell)| + \mathbb{E}\,|g_M(X_0) g_M(X_\ell)| \\
\le\ & 4M\|X_0\|_m \Big(\frac{\mathbb{E}\,|X_0|^m}{M^m}\Big)^{1-1/m} + 2\|X_0\|_m^2 \Big(\frac{\mathbb{E}\,|X_0|^m}{M^m}\Big)^{1-2/m} \\
\le\ & 6 \cdot M^{2-m} \cdot \|X_0\|_m^m, \qquad (29)
\end{aligned}
$$

from Hölder and Markov inequalities. By the same procedure, we also obtain:

$$
\begin{aligned}
\|U_{k,\ell,M}\|_2^2 \ &\le\ 6\Big(\mathbb{E}\, g_M^2(X_k) f_M^2(X_{k+\ell}) + \mathbb{E}\, f_M^2(X_k) g_M^2(X_{k+\ell}) + \mathbb{E}\, g_M^2(X_k) g_M^2(X_{k+\ell})\Big) \\
&\le\ 18 \cdot M^{4-m} \cdot \|X_0\|_m^m. \qquad (30)
\end{aligned}
$$

Let h_M be the function such that $h_M(x,y) = f_M(x) f_M(y) - R'(\ell)$ for all $(x,y) \in \mathbb{R}^2$. Note that $\|h_M\|_\infty \le 2M^2$ and $\operatorname{Lip} h_M = M$. Moreover, for all random variable W in $\mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$,

$$
\|\mathbb{E}\,(W \,|\, \mathcal{B}_\ell)\|_1 = \sup_{Z \in \mathbb{L}^\infty(\Omega, \mathcal{B}_\ell, \mathbb{P})\ ,\ \|Z\|_\infty \le 1} \Big|\mathbb{E}\,(W \cdot Z)\Big|.
$$

Therefore,

$$
\begin{aligned}
\Big\|h_M(X_0, X_\ell) \cdot \mathbb{E}\,\Big(h_M(X_k, X_{k+\ell}) \,|\, \mathcal{B}_\ell\Big)\Big\|_1 \ &=\ \Big\|\mathbb{E}\,(h_M(X_0, X_\ell) \cdot h_M(X_k, X_{k+\ell}) \,|\, \mathcal{B}_\ell)\Big\|_1 \\
&=\ \sup_{Z \in \mathbb{L}^\infty(\Omega, \mathcal{B}_\ell, \mathbb{P})\ ,\ \|Z\|_\infty \le 1} \Big|\mathbb{E}\,\Big(Z \cdot h_M(X_0, X_\ell) \cdot h_M(X_k, X_{k+\ell})\Big)\Big| \\
&\le\ \sup_{Z' \in \mathbb{L}^\infty(\Omega, \mathcal{B}_\ell, \mathbb{P})\ ,\ \|Z'\|_\infty \le 2M^2} \Big|\operatorname{Cov}\,\Big(h_M(X_k, X_{k+\ell})\,,\, Z'\Big)\Big|.
\end{aligned}
$$

Consequently, from the inequality (8) and the stationarity of X, for all $k \ge 0$,

$$
\Big\|h_M(X_0, X_\ell) \cdot \mathbb{E}\,\Big(h_M(X_k, X_{k+\ell}) \,|\, \mathcal{B}_\ell\Big)\Big\|_1 \le 2 \cdot M^3 \cdot \theta_{k-|\ell|}. \qquad (31)
$$

Thus,

$$
\begin{aligned}
\Big\|Y_{0,\ell} \cdot \mathbb{E}\,\Big(Y_{k,\ell} \,|\, \mathcal{B}_\ell\Big)\Big\|_1 \ &\le\ \Big\|(U_{0,\ell,M} \cdot \mathbb{E}\,\Big(U_{k,\ell,M} \,|\, \mathcal{B}_\ell\Big)\Big\|_1 + \Big\|h_M(X_0, X_\ell) \cdot \mathbb{E}\,\Big(U_{k,\ell,M} \,|\, \mathcal{B}_\ell\Big)\Big\|_1 \\
&\qquad + \Big\|U_{0,\ell,M} \cdot \mathbb{E}\,\Big(h_M(X_k, X_{\ell+k}) \,|\, \mathcal{B}_\ell\Big)\Big\|_1 + \Big\|h_M(X_0, X_\ell) \cdot \mathbb{E}\,\Big(h_M(X_k, X_{k+\ell}) \,|\, \mathcal{B}_\ell\Big)\Big\|_1 \\
&\le\ \Big\|U_{k,\ell,M}\Big\|_2^2 + 4M^2 \Big\|U_{k,\ell,M}\Big\|_1 + \Big\|h_M(X_0, X_\ell) \cdot \mathbb{E}\,\Big(h_M(X_k, X_{k+\ell}) \,|\, \mathcal{B}_\ell\Big)\Big\|_1 \\
&\le\ 34 \cdot M^{4-m} \cdot \|X_0\|_m^m + 2 \cdot M^3 \cdot \theta_{k-|\ell|},
\end{aligned}
$$

from (29), (30) and (31). With the choice $M = \theta_{k-|\ell|}^{-1/(m-1)}$ we prove that if

$$
\sum_{k=0}^{\infty} \theta_k^{1-3/(m-1)} < \infty, \qquad \|X_0\|_m < \infty
$$

for some $m > 4$, yielding the Uniform CLT (7). ∎

4.3 Proofs of Theorem 3 and Corollary 3

Proof of Theorem 3: Set $g \in \mathcal{H}_s$ with $g(\lambda) = \sum_{\ell\in\mathbb{Z}} g_\ell e^{i\lambda\ell}$, let $k \in \mathbb{N}^*$ and define $g^{(k)}(\lambda) = \sum_{|\ell|<k} g_\ell e^{i\lambda\ell}$. From the functional limit theorem established in Bardet *et al.* (2005) and applied to g, one obtains for all $\mathcal{C}^3(\mathbb{R})$ function ϕ with bounded derivatives up to order 3 :

$$\Big|\mathbb{E}\Big[\phi\Big(\sqrt{n}(J_n(g)-J(g))\Big)-\phi\Big(\sigma(g)\cdot N\Big)\Big]\Big| \le D_{1,n}^{(k)}+D_{2,n}^{(k)}+D_{3,n}^{(k)},$$

with

$$\begin{aligned} D_{1,n}^{(k)} &= \Big|\mathbb{E}\Big[\phi\Big(\sqrt{n}(J_n(g^{(k)})-J(g^{(k)}))\Big)-\phi\Big(\sigma(g^{(k)})\cdot N\Big)\Big]\Big| \\ D_{2,n}^{(k)} &= \Big|\mathbb{E}\Big[\phi\Big(\sigma(g^{(k)})\cdot N\Big)-\phi\Big(\sigma(g)\cdot N\Big)\Big]\Big| \\ D_{3,n}^{(k)} &= \Big|\mathbb{E}\Big[\phi\Big(\sqrt{n}(J_n(g^{(k)})-J(g^{(k)}))\Big)-\phi\Big(\sqrt{n}(J_n(g)-J(g))\Big)\Big]\Big|. \end{aligned}$$

Term $D_{1,n}^{(k)}$ **:** For $i=1,\dots,n$, set $x_i=(X_{i+\ell})_{|\ell|<k}$ a stationary random vector in $\mathbb{R}^{2k-1}$. The function:

$$h(x_i)=\sum_{|\ell|<k} g_\ell(X_iX_{i+\ell}-R(\ell)) \;\text{ for } i=1,\dots,n,$$

satisfies the assumption H (defined above) with $a=2$ and $A=A(2k-1)=2k-1$. Define also:

$$S_n^{(k)}=\frac{1}{\sqrt{n}}\sum_{i=1}^n h(x_i)=\sqrt{n}(J_n(g^{(k)})-J(g^{(k)})).$$

By applying Theorem 1 of Bardet *et al.* (2005) to this function h, one obtains with $C_1>0$ and $\lambda=\dfrac{\alpha(m-4)-2m+1}{2(m+1+\alpha\cdot m}$:

$$D_{1,n}^{(k)} \le C_1\cdot k^3\cdot n^{-\lambda}, \tag{32}$$

Term $D_{2,n}^{(k)}$ **:** With the same method used for obtaining the bound of $\Delta_{4,n}$ in the previous proof, we have:

$$D_{2,n}^{(k)} \le \|\phi''\|_\infty\cdot\Big|\sigma^2(g)-\sigma^2(g^{(k)})\Big|.$$

But, from the expression (4), we deduce:

$$\begin{aligned}\Big|\sigma^2(g)-\sigma^2(g^{(k)})\Big| \le \Big|&\frac{1}{\pi}\int_{-\pi}^{\pi}(g^2(\lambda)-(g^{(k)}(\lambda))^2)f^2(\lambda)\,d\lambda \\ &+2\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(g(\lambda)g(\mu)-g^{(k)}(\lambda)g^{(k)}(\mu))f_4(\lambda,-\mu,\mu)d\lambda d\mu\Big|.\end{aligned}$$

With $g\in\mathcal{H}_s$, we have:

$$\|g-g^{(k)}\|_\infty \le \Big(\sum_{|\ell|\ge k}(1+|\ell|)^{-s}\Big)^{1/2}\Big(\sum_{|\ell|\ge k}(1+|\ell|)^s g_\ell^2\Big)^{1/2} \le \Big(\sum_{|\ell|\ge k}(1+|\ell|)^{-s}\Big)^{1/2}\|g\|_{\mathcal{H}_s}.$$

Consequently, with also $\|g+g^{(k)}\|_\infty \le 2\Big(\sum_{\ell\in\mathbb{Z}}(1+|\ell|)^{-s}\Big)^{1/2}\cdot\|g\|_{\mathcal{H}_s}$, there exists $C_2>0$ such that:

$$D_{2,n}^{(k)} \le C_2\cdot(k^{\frac{1-s}{2}}), \tag{33}$$

Term $D_{3,n}^{(k)}$ **:** First, from a Taylor development,

$$D_{3,n}^{(k)} \leq \frac{1}{2} \cdot \|\phi''\|_\infty \cdot n \cdot \mathbb{E}\Big(J_n(g-g^{(k)}) - J(g-g^{(k)})\Big)^2.$$

With the same decomposition as in the proof of Lemma 3, one obtains:

$$\mathbb{E}\Big(J_n(g-g^{(k)}) - J(g-g^{(k)})\Big)^2 \quad \leq \quad 3\Big(\Big(\sum_{|\ell|\geq n} R(\ell) g_\ell\Big)^2 + \Big(\frac{1}{n}\sum_{k\leq|\ell|<n} |\ell| R(\ell) g_\ell\Big)^2 + \\ + \Big\| \sum_{k\leq|\ell|<n} g_\ell\Big(\widehat{R}_n(\ell) - \mathbb{E}(\widehat{R}_n(\ell))\Big)\Big\|_2^2\Big).$$

First,

$$\Big(\sum_{|\ell|\geq n} R(\ell) g_\ell\Big)^2 \leq (1+n)^{-s} \cdot \sum_{|\ell|\geq n} R(\ell)^2 \cdot \sum_{|\ell|\geq n} (1+|\ell|)^s g_\ell^2 \leq \frac{1}{n} \cdot \|g\|^2_{\mathcal{H}_s} \cdot \sum_{|\ell|\geq n} R(\ell)^2.$$

Using the weak dependence of $(X_i)_i$ and with the same method as in the proof of Lemma 2 in Bardet *et al.* (2005) adapted to the function $h(x)=x$ (therefore with $a=1$),

$$|R(\ell)| \leq c \cdot \eta_{|\ell|}^{\frac{m-2}{m-1}} \leq c \cdot |\ell|^{-\alpha\,\frac{m-2}{m-1}},$$

from the rate $\eta_{|\ell|} = \mathcal{O}(|\ell|^{-\alpha})$ with $\alpha > 3$. As a consequence,

$$\sum_{|\ell|\geq n} R(\ell)^2 \leq c \cdot n^{1-2\alpha\,\frac{m-2}{m-1}} \quad \text{and} \quad \Big(\sum_{|\ell|\geq n} R(\ell) g_\ell\Big)^2 \leq c \cdot n^{-2\alpha\,\frac{m-2}{m-1}}.$$

In the same way,

$$\Big(\frac{1}{n}\sum_{k\leq|\ell|<n} |\ell| R(\ell) g_\ell\Big)^2 \leq \frac{1}{n} \cdot \|g\|^2_{\mathcal{H}_s} \cdot \sum_{k\leq|\ell|<n} R(\ell)^2 \leq \frac{c}{n} \cdot k^{1-2\alpha\,\frac{m-2}{m-1}}.$$

Finally,

$$\begin{aligned}
\Big\| \sum_{k\leq|\ell|<n} g_\ell\Big(\widehat{R}_n(\ell) - \mathbb{E}(\widehat{R}_n(\ell))\Big)\Big\|_2^2 \quad &\leq \quad \Big(\sum_{k\leq|\ell|<n} |g_\ell| \Big(\mathrm{Var}\,(\widehat{R}_n(\ell))\Big)^{1/2}\Big)^2 \\
&\leq \quad \max_{\ell\in\mathbb{Z}} \Big(\mathrm{Var}\,(\widehat{R}_n(\ell))\Big) \cdot \|g\|^2_{\mathcal{H}_s} \cdot \sum_{k\leq|\ell|<n} (1+|\ell|)^{-s} \\
&\leq \quad \frac{1}{n} \cdot (\kappa_4 + 2\gamma) \cdot \|g\|^2_{\mathcal{H}_s} \cdot \sum_{k\leq|\ell|<n} (1+|\ell|)^{-s} \quad \text{from Lemma 2.}
\end{aligned}$$

Finally, there exists $C_3 > 0$ such that:

$$D_{3,n}^{(k)} \leq C_3 \cdot (k^{1-2\alpha\,\frac{m-2}{m-1}} + k^{1-s}). \tag{34}$$

Now, with (32), (33) and (34), we deduce by considering $t = \Big(2\alpha\,\dfrac{m-2}{m-1} - 1\Big) \wedge \dfrac{s-1}{2}$ and selecting k such that $k^{t+3} = n^\lambda$, that there exists $C > 0$ such that:

$$\Big|\mathbb{E}\Big[\phi\Big(\sqrt{n}(J_n(g) - J(g))\Big) - \phi\Big(\sigma(g)\cdot N\Big)\Big]\Big| \leq C \cdot n^{-\frac{t}{t+3}\lambda}. \ \blacksquare$$

Proof of Corollary 3. Set $g(\lambda) = e^{i\ell\lambda}$ in Theorem 3. Since this function belongs to each space $\mathcal{H}_s$, it follows that the terms $D_{2,n}^{(k)}$ and $D_{3,n}^{(k)}$ both vanish and the result follows from the bound (32). $\blacksquare$

5 Appendix : a useful lemma

For a weakly dependent process, the following auxiliary lemma shows that a function of this process is also a weakly dependent process and provides a relation between the two weak dependence sequences. For an example of the use of such a result, see the paragraph devoted to causal ARCH(∞) time series.

Lemma 6 *Let $(X_i)_{i\in\mathbb{Z}}$ be a $\mathbb{L}^p$-stationary time series with $p>0$, $h:\mathbb{R}\to\mathbb{R}$ be a function such that $|h(x)|\le c\cdot|x|^a$ and $|h(x)-h(y)|\le c\cdot|x-y|\cdot(|x|^{a-1}+|y|^{a-1})$ for $(x,y)\in\mathbb{R}^2$ with $0<c$ and $0<a<p$. Let $(Y_i)_{i\in\mathbb{Z}}$ be the stationary times series defined by $Y_i=h(X_i)$ for $i\in\mathbb{Z}$. Then:*

- *If $(X_i)_{i\in\mathbb{Z}}$ is θ-weakly dependent time series, then $(Y_i)_{i\in\mathbb{Z}}$ is a stationary θ^Y-weakly dependent time series, such that $\forall r\in\mathbb{N}$, $\theta_r^Y=C\cdot\theta_r^{\frac{p-a}{p-1}}$ with a constant $C>0$;*
- *If $(X_i)_{i\in\mathbb{Z}}$ is η-weakly dependent time series, then $(Y_i)_{i\in\mathbb{Z}}$ is a η^Y-weakly dependent time series, such that $\forall r\in\mathbb{N}$, $\eta_r^Y=C\cdot\eta_r^{\frac{p-a}{p-1}}$ with a constant $C>0$.*

Proof. Let $f:\mathbb{R}^u\to\mathbb{R}$ and $g:\mathbb{R}^v\to\mathbb{R}$ two real functions such that $\mathrm{Lip}\,f<\infty$, $\|f\|_\infty\le 1$, $\mathrm{Lip}\,g<\infty$, $\|g\|_\infty\le 1$. Denote $x^{(M)}=(x\wedge M)\vee(-M)$ for $x\in\mathbb{R}$. For simplicity we first assume that $v=2$. Let $i_1,\ldots,i_u,j_1,\ldots,j_v\in\mathbb{Z}^{u+v}$ such that $i_1,\ldots,i_u\ge r$ and $j_1,\ldots,j_v\le 0$ and denote $x_{\mathbf{i}}=(X_{i_1},\ldots,X_{i_u})$ and $x_{\mathbf{j}}=(X_{j_1},\ldots,X_{j_v})$. We then define functions $F:\mathbb{R}^u\to\mathbb{R}$, $F^{(M)}:\mathbb{R}^u\to\mathbb{R}$ and $G:\mathbb{R}^v\to\mathbb{R}$, $G^{(M)}:\mathbb{R}^v\to\mathbb{R}$ through the relations: $F(x_{\mathbf{i}})=f(h(X_{i_1}),\ldots,h(X_{i_u}))$, $F^{(M)}(x_{\mathbf{i}})=f(h(X_{i_1}^{(M)}),\ldots,h(X_{i_u}^{(M)}))$ and $G(x_{\mathbf{j}})=g(h(X_{j_1}),\ldots,h(X_{j_v}))$, $G^{(M)}(x_{\mathbf{j}})=g(h(X_{j_1}^{(M)}),\ldots,h(X_{j_v}^{(M)}))$. Then:

$$
\begin{aligned}
|\mathrm{Cov}\,(F(x_{\mathbf{i}}),G(x_{\mathbf{j}}))| &\le |\mathrm{Cov}\,(F(x_{\mathbf{i}}),G(x_{\mathbf{j}})-G^{(M)}(x_{\mathbf{j}}))|+|\mathrm{Cov}\,(F(x_{\mathbf{i}}),G^{(M)}(x_{\mathbf{j}}))|\\
&\le 2\mathbb{E}|G(x_{\mathbf{j}})-G^{(M)}(x_{\mathbf{j}}))|+2\mathbb{E}|F(x_{\mathbf{i}})-F^{(M)}(x_{\mathbf{i}})|+|\mathrm{Cov}\,(F^{(M)}(x_{\mathbf{i}}),G^{(M)}(x_{\mathbf{j}}))|.
\end{aligned}
$$

The last relation comes from $\|f\|_\infty\le 1$. But we also have

$$
\begin{aligned}
\mathbb{E}|G(x_{\mathbf{j}})-G^{(M)}(x_{\mathbf{j}}))| &\le v\cdot\mathrm{Lip}\,g\cdot\mathbb{E}|h(X_0)-h(X_0^{(M)})|\\
&\le 2c\cdot v\cdot\mathrm{Lip}\,g\cdot\mathbb{E}\big(|X_0|^a\cdot\mathbb{1}_{|X_0|>M}\big)\qquad\text{(from the assumptions on } h\text{)},\\
&\le 2c\cdot v\cdot\mathrm{Lip}\,g\cdot\|X_0\|_p\cdot M^{a-p}\qquad\text{(from Markov inequality)}.
\end{aligned}
$$

The same thing holds for F. Moreover, the functions $F^{(M)}$ and $G^{(M)}$ satisfy $\mathrm{Lip}\,F^{(M)}=\mathrm{Lip}\,F^{(M)}=c\cdot M^{a-1}$, with $c>0$, and $\|F^{(M)}\|_\infty\le 1$, $\|G^{(M)}\|_\infty\le 1$. Thus, from the definition of the weak dependence of X:

$$
\begin{aligned}
\Big|\mathrm{Cov}\,\big(F^{(M)}(x_{\mathbf{i}}),G^{(M)}(x_{\mathbf{j}})\big)\Big| &\le C\cdot v\cdot\mathrm{Lip}\,f\cdot M^{a-1}\theta_r,\ \text{with } u=2,\text{ under condition }\theta;\\
&\le C\cdot(v\cdot\mathrm{Lip}\,f+u\cdot\mathrm{Lip}\,g)\cdot M^{a-1}\eta_r,\quad\text{under condition }\eta.
\end{aligned}
$$

Finally, we obtain respectively:

$$
\begin{aligned}
|\mathrm{Cov}\,(F(x_{\mathbf{i}}),G(x_{\mathbf{j}}))| &\le C\cdot v\cdot\mathrm{Lip}\,f\cdot\big(M^{a-1}\cdot\theta_r+M^{a-p}\big)\\
&\le C\cdot(v\cdot\mathrm{Lip}\,f+u\cdot\mathrm{Lip}\,g)\big(M^{a-1}\cdot\eta_r+M^{a-p}\big).
\end{aligned}
$$

By the optimal choice of $M=\theta_r^{1/(1-p)}$, we obtain respectively:

$$
\begin{aligned}
|\mathrm{Cov}\,(f(Y_{i_1},\ldots,Y_{i_u}),g(Y_{i_1},\ldots,Y_{i_v}))| &\le C\cdot v\cdot\mathrm{Lip}\,f\cdot\theta_r^{\frac{p-a}{p-1}};\\
&\le C\cdot(v\cdot\mathrm{Lip}\,f+u\cdot\mathrm{Lip}\,g)\cdot\eta_r^{\frac{p-a}{p-1}}.\quad\blacksquare
\end{aligned}
$$

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