INSTITUT NATIONAL DE LA STATISTIQUE ET DES ETUDES ECONOMIQUES
Série des Documents de Travail du CREST
(Centre de Recherche en Economie et Statistique)

n° 2005-44

Affine Model for Credit Risk Analysis

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(First version : January 2003)
(Revised version : November 2005)

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Abstract

Continuous time affine models have been recently introduced in the theoretical financial literature on credit risk. They provide a coherent modelling, rather easy to implement, but have not yet encountered the expected success among practitioners and regulators. This is likely due to a lack of flexibility of these models, which often implied poor fit, especially compared to more ad hoc approaches proposed by the industry. The aim of this paper is to explain that this lack of flexibility is mainly due to the continuous time assumption. We develop a discrete time affine analysis of credit risk, explain how different types of factors can be introduced to capture separately the term structure of default correlation, of default heterogeneity, of correlation between default and loss-given-default; we also explain, why the factor dynamics are less constrained in discrete time and are able for instance to reproduce complicated cycle effects... These models are finally used to derive a CreditVaR and various decompositions of the spreads for corporate bonds or first-to-default basket.

Keywords : Term Structure, Credit Risk, Loss-Given-Default, Affine Model, Stochastic Discount Factor, Affine Process, Car Process, WAR process, Through-the-Cycle.

JEL number C41, G21.

1 Introduction

In the general strategy followed by the Basle Committee for monitoring the risk included in their financial investments, the banks have to compute a CreditVaR both for regulation and internal control [Basel Committee on Banking Supervision (2001)]. This Value-at-Risk defines the amount of reserve required to hedge the risk of their credit portfolios, including retail credits (consumer credits, mortgages, revolving credits, over-the-counter corporate loans),corporate bonds and credit derivatives, including mortgage backed securities. This paper focuses on portfolios of corporate credit derivatives which are traded on bond markets, and on the introduction of appropriate models for describing the associated risks. These models have to incorporate borrowers' heterogeneity with both industry-wide and firm-specific effects. This heterogeneity concerns the default intensity (resp. loss-given-default) patterns at various maturities, but also the so-called default correlation (resp. loss-given-default correlation), which accounts for what regulators call *concentration risk*.

The first generation of credit risk simulators proposed by the industry is already in place[4]. Even though these models fulfill the basic need for simple credit risk measurement, they have obvious limitations with yet unknown consequences. The following is a list of important empirical facts that are not accounted for in the current practical approaches :

- Riskfree interest rates evolve stochastically, but in most industry models they are assumed independent of the default phenomenon.
- Default correlation is often taken into account in a very crude way, implicitly assuming a similar default correlation for the different categories of firms, or for the short and long term credits, even though defaults are likely driven by general, sector specific, and firm specific factors, whose influences vary in time.
- The models have to use the available information, including microeconomic data on firms (such as size, industrial sector, financial ratios, rating by the agencies), and market information concerning the equity, or bond prices. It is well-known that the KMV simulator is based on

[4]See Crouhy, Galai and Mark (2000) for a comparative analysis of the major industry models, proposed by KMV, J.P. Morgan (Credit Metrics), or Credit Swiss First Boston (Credit Risk +).

Merton's model and incorporates observed equity prices. But more information is included in the corporate bond prices and this information is often not taken into account.

- The returns on credit portfolios are known to be heavily skewed to the left. Indeed, the introduction of a firm in a watch-list for downgrade by Moody's, or Standard & Poor's has an immediate impact on bond and equity prices, whereas the impact of an expected upgrade is slower.

- The loss-given-default component is considered in very crude ways. The uncertainty of loss-given-default, or the adverse links between default and recovery rates are often not taken into account, whereas they have important impact on the Credit VaR, and are needed for an advanced approach of credit risk [Basel Committee on Banking Supervision (2005)].

- Finally, the methodology has to provide coherent specifications of the risk-neutral dynamics, used for derivative pricing, and of the historical dynamics, used for risk prediction and determination of the CreditVar. It is especially important to understand the differences between the probability and price of default, or between default correlation and price of default dependence.

Our objective in this paper is to develop a consistent valuation model for portfolio of corporate bonds, which addresses the issues mentioned above. This model can be used for pricing, but also to compute a CreditVaR. In this model, two types of factors generate *default (or loss-given-default) correlation*. First, defaults are instantaneously correlated because all firms are influenced by systematic economic factors. They can capture the so-called aggregate state of the economy or some sectorial effects, if we consider a restricted set of firms. For example, a source of default correlation can be information about the term structure (i.e. short term yields, slope, curvature, etc.), and/or equity markets. It can also be due to the natural common dependence of same industry companies on various norms related to that industry alone. For example, profitability, intensity of competition, entrance of new players and introduction of new products, just to name a few, are potential sources of default correlation for firms in the same industry. Second, it is also necessary to link the default occurrences of a given corporate at different ages. This can be done by introducing corporate specific factors. These idiosyncratic factors can represent the effect of the strategy of the firm. Typically, an investment decision has an impact on many future financial results.

Understanding and modelling such instantaneous and serial default (resp. loss-given-default) dependencies is important, since this significantly impacts the overall credit risk in a portfolio. Whereas some factors are easily observable, other constitute a much more elusive, difficult to measure, set of factors. Fortunately, when such latent factors generate common default patterns, their values can be recovered from the observable corporate credit derivatives and Treasury-bond prices, something possible due to the *affine* character of the model.

Besides capturing the above empirical facts, the proposed specifications assess credit risk in a tractable manner. They also allow for pricing defaultable credit derivatives. For practical purposes, it is not necessary to get closed-form expressions of the future default probabilities and risk derivative prices, but only to explain how to compute them by simple numerical methods, for instance by recursive equations. Moreover, the computational burden should not become excessive when the number of credits included in the portfolio increases, since such portfolios may include thousands of different loans.

The specification is a discrete time affine specification and has to be compared to the continuous time affine literature, recently elaborated in [Duffie, Filipovic, Schachermayer (2003)]. The main potential benefits of a discrete time approach are threefold. (i) The regulator and some models (for retail credit and analysis of workout loss-given-default on corporates) prefer to work in a daily discrete time framework. (ii) A discrete time approach can be numerically easier to implement, since it requires the solution of recursive equations with a daily time unit. By comparison the continuous time affine specifications require the solution of multidimensional differential Riccati equations. These equations are solved numerically, often by considering approximate recursive systems at a small time unit, 5 mn, say. This approximation implies more computations, 288 times more in the above example, due to the smaller time unit needed to approximate continuous time. Moreover, contrary to the initial continuous time model, the approximate recursions are generally not compatible with no-arbitrage opportunity. (iii) The set of continuous time affine dynamics is rather restricted, as a consequence of the time coherency condition. Loosely speaking, the continuous time dynamics are built from Ornstein-Uhlenbeck processes, Cox-Ingersoll-Ross processes (or their multivariate extensions called Wishart processes), and special bifurcation processes. On the contrary, the discrete time specifications do not require time coherency for periods within the day. This increases considerably the type of admissible dynamics, including recursive systems, long memory, or complicated nonlinear effects. This is especially important in the context of default, where default probability and expected loss-given-default are procyclical.

In Section 2, we introduce the default arrival model. The survivor intensity rate depends on both systematic and firm specific factors. The model is compared with alternative specifications introduced in the literature to capture default correlation and with models written in continuous time. By assuming independent individual defaults, conditional on state variables, and by employing affine dynamics for the factors, we recursively compute the conditional joint survivor function in Section 3. The formulas are simplified when the effect of the state factors on the default intensity rate is time independent. The aim of Section 4 is to specify the pricing model. For this purpose we assume an exponential affine pricing kernel (stochastic discount factor or sdf), depending on general factors affecting default. Then, the term structure of Treasury-bonds, corporate bonds and first-to-default baskets can be derived recursively. In particular, we discuss the decomposition and interpretation of the term structure of the spread. Numerical examples are presented in Section 5. In particular, we discuss the pattern and evolution of the term structures through the cycle. The extension to loss-given-default is presented in Section 6, and Section 7 concludes.

2 The default arrival model

Let us consider a cohort of n obligor firms with the same birth date fixed by convention at $t = 0$. The birth date can be defined in different ways according to the problem of interest. It can correspond to the date of creation of the firms, if we focus on new industrial sectors, or to the date of the first rating by agencies such as Moody's or Standard and Poor's, if we consider the introduction on bond markets, or even an initial date corresponding to the period of analysis. We denote by $\tau_i, i = 1, \ldots, n$ the failure date for corporation i, that is the lifetime of this corporation. The aim of this section is to specify a joint historical distribution of the lifetimes compatible with various patterns of the term structure of default correlation. The distribution of the set of lifetimes is defined in two steps. First, we assume that the lifetimes are independent conditional on the past, present and future values of a set of systematic and corporate specific factors. Then, the future realization of the factors is integrated out to derive the joint distribution of lifetimes conditional on information available at time t and to create default dependence.

2.1 Assumptions

Assumption A.1 : There exist general (systematic) and corporate specific factors[5], denoted by $(Z_t), (Z_t^i), i = 1, \ldots, n$, respectively. These factors are independent, Markovian and their transitions are such that :

$$
\begin{aligned}
E[\exp(u' Z_{t+1}) | Z_t] &= \exp[a_g(u)' Z_t + b_g(u)], \\
E[\exp(u' Z_{t+1}^i) | Z_t^i] &= \exp[a_c(u)' Z_t^i + b_c(u)], i = 1, \ldots, n,
\end{aligned}
$$

where the above relations hold for all arguments u for which the expectation is well-defined. Moreover, we assume that the marginal distributions of $Z_o^i, i = 1, \ldots, n$ are identical.

Thus, the factors satisfy a compound autoregressive (Car) process [see Darolles, Gourieroux, Jasiak (2005)]. The conditional distributions are defined by means of the conditional Laplace transform, or moment generating function, restricted to real arguments u [6]. Since the functions a_g, b_g, a_c, b_c are not highly constrained a priori, the Car dynamics can be used to represent a large pattern of nonlinear serial dependence, including long memory, cycles, or default clustering under economic recession [see Jarrow, Yu (2001)] .

By Assumption A.1. the population is assumed homogenous, that is, the distributions of the corporate specific factor processes are independent of the firm. Thus, the cohort is both homogenous with respect to the birth date and to individual characteristics such as the industrial sector, or the initial rating. Finally, note that the general factor Z is defined for any date t, whereas the

[5]The model can be extended to also include sector specific factors. These additional factors are not introduced for expository purpose.

[6]We assume in the sequel that the real Laplace transform characterizes the factor distribution. This condition is satisfied for nonnegative or bounded variables [see Feller (1971)]. In a general case, it may require power conditional moments at any order and the possibility to get a series expansion of the Laplace transform in a neighborhood of zero.

firm specific factor Z^i can only exist until the default date τ_i of the i^{th} firm. This explains why the independence between idiosyncratic and systematic factors is assumed. Otherwise, complicated effects have to be taken into account at any firm's failure time [see e.g. Jarrow, Yu (2001), Gagliardini, Gourieroux (2003) for this extension in the case of two borrowers].

In this paper, instantaneous default correlation arises only because of the common risk factors that drive individual firms' default intensities. Equivalently, given those common factors, default arrivals of different firms become independent:

Assumption A.2 : Conditional on the realization path of the factors [7], $\underline{Z}, \underline{Z^i}, i = 1, \ldots, n$, default arrival times $\tau_i, i = 1, \ldots, n$ are independent. Moreover, the conditional survivor intensities are such that :

$$\begin{aligned} & P[\tau_i > t+1 | \tau_i > t, \underline{Z}, \underline{Z^j}, j = 1, \ldots, n] \\ = \; & P[\tau_i > t+1 | \tau_i > t, Z_{t+1}, Z^i_{t+1}] \\ = \; & \exp[-(\alpha_{t+1} + \beta'_{t+1} Z_{t+1} + \gamma'_{t+1} Z^i_{t+1})] \\ = \; & \exp(-\lambda^i_{t+1}), \text{ say}, \forall t, \end{aligned}$$

where $\alpha_{t+1}, \beta_{t+1}, \gamma_{t+1}$ are functions of the information included in the current and lagged factor values $\underline{Z_t}, \underline{Z^i_t}$.

Since the conditional survivor probability is smaller than unity, we get : $\lambda^i_t = \alpha_t + \beta'_t Z_t + \gamma'_t Z^i_t \geq 0, \forall t$. These restrictions imply conditions on both the sensitivity parameters and the factor distributions. For instance, they are satisfied if both factors and sensitivity coefficients are nonnegative. They can also be satisfied in a more general framework [see Gourieroux, Sufana (2003), Gourieroux, Jasiak, Sufana (2004)]. Indeed, it has been argued recently that some factors can be represented by the elements of a positive definite symmetric matrix Σ_t, say : $Z_t = vech\ (\Sigma_t)$, where vech denotes the operator stacking the different elements of Σ_t. In this case, a linear combination of components of Z can be written as : $\beta'_t Z_t = Tr(B_t \Sigma_t)$, where B_t is a symmetric matrix and the trace operator Tr computes the sum of diagonal elements. $Tr(B_t \Sigma_t)$ is nonnegative, if matrix B_t is positive definite. Moreover, it is known that the Wishart Autoregressive (WAR) process is a

[7] $\underline{Z} = (Z_t, \forall t)$, $\underline{Z^i} = (Z^i_t, \forall t)$, and, $\underline{Z_h} = (Z_t, t \leq h)$, $\underline{Z^i_h} = (Z^i_t, t \leq h)$.

special case of Car process, valued in the set of positive definite symmetric matrices (see e.g. Gourieroux (2005) for a survey on WAR process).

The survivor intensity depends on time by means of factors Z_{t+1}, Z_{t+1}^i and sensitivities $\alpha_{t+1}, \beta_{t+1}, \gamma_{t+1}$. This double time dependence can be interpreted in the following way. Let us assume for a while time-independent factors $Z_{t+1} = z, Z_{t+1}^i = z^i$, say, [8] meaning that the general environment is stable and the characteristics of the corporation such as its size, its financial ratios... stay the same. Even with constant factors, the survivor rate is not the same for a young corporation and an old one. This age effect is captured by the age dependent sensitivities. As far, the factor Z_t will capture for instance the calendar time effect and it is able to create various term structures of default correlation.

Assumption A.2 can easily be weakened to allow for bond dependent sensitivities. In the extended framework, if Z_t includes current and lagged values of a factor $Z_t = (\tilde{z}_t, \tilde{z}_{t-1})$, say, and, if the set of bonds is partitioned into two subsets with sensitivities $(\beta_{1,t+1}, 0), (0, \beta_{2,t+1})$, respectively, the model allows to distinguish primary and secondary bonds [see Jarrow, Yu (2001) IV,B for another constrained specification for this problem].

2.2 The link to continuous time.

The survival model is defined above in discrete time. However, the main stream of the theoretical literature on credit risk considers continuous time specifications[9], and it is useful to see which approach is the more flexible in practice.

Let us first recall the usual continuous time modelling for default arrivals [see e.g. Lando (1994), (1998), Duffie, Singleton (1999)]. In continuous time, the factors Z, Z^i are continuous time affine processes, whereas time-to-default can take a priori any positive real value.

[8] When both factors and sensitivities are age independent, we get the so-called multivariate mixed proportional hazard (MMPH) model described in Van den Berg (1997), (2001). An exponential affine factor representation of the individual heterogeneity in the MMPH framework is usually assumed in the applications to labour [see e.g. Flinn, Heckman (1982), Heckman, Walker (1990), Bonnal, Fougere, Serandon (1997)]. In the labour framework, the model of Section 2.1 allows for both a term structure of individual heterogeneity, and a description of the latent dynamic effort variable underlying moral hazard.

[9] But the simulators for credit risk proposed by the industry follow discrete time approaches for the same practical reasons as in our paper.

i) The continuous time factor is affine, iff the conditional Laplace transform is an exponential-affine function of the current value for any real horizon

$$E[\exp(u'Z_{t+h})|Z_t] = \exp[a_g(u,h)'Z_t + b_g(u,h)], \forall h \in (0,\infty). \tag{2.1}$$

These restrictions imply the condition of Assumption A.1. Thus, any time discretized continuous time affine process is Car, but there exist a lot of Car processes without continuous time counterpart. This point is developed below.

ii) The distribution of the default time conditional on the factor path is generally defined through the (stochastic) infinitesimal default intensity :

$$\tilde{\lambda}_t = \lim_{dt \to 0} \frac{1}{dt} P[t < \tau_i < t + dt | \tau_i > t, \underline{Z}, \underline{Z^i}].$$

The associated survivor probability at horizon 1 is :

$$P[\tau_i > t+1 | \tau_i > t, \underline{Z}, \underline{Z^i}] = \exp\left[-\int_t^{t+1} \tilde{\lambda}_u du\right].$$

If the time unit is small and the infinitesimal default intensity admits continuous path, this survivor probability can be approximated by $\exp(-\tilde{\lambda}_{t+1})$. Thus, the discrete time specification introduced in the section above is the counterpart of a continuous time model with affine stochastic default intensity [see Lando (1994), (1998), Duffie, Singleton (1999)] :

$$\tilde{\lambda}_t = \tilde{\alpha}_t + \tilde{\beta}_t' Z_t + \tilde{\gamma}_t' Z_t^i.$$

Let us now discuss the flexibility of continous time affine models and the modelling of default distribution by means of infinitesimal default intensity.

a) Lack of flexibility of continuous time affine processes

As already mentioned, the admissible affine dynamics are very restrictive in a continuous time framework.

i) Let us first consider the favorable case of Gaussian processes. In continuous time, the affine Gaussian processes are the multidimensional Ornstein-Uhlenbeck processes. Their time discretized versions are the Gaussian VAR(1) processes :

$$Z_t = \Phi Z_{t-1} + \mu + \varepsilon_t, \varepsilon_t \; IIN(0, \Omega),$$

where the autoregressive matrix has an exponential representation $\Phi = \exp A$. On the other hand, all Gaussian VAR(1) processes are Car processes.

The condition $\Phi = \exp(A)$ is restrictive. It implies that the eigenvalues $\lambda_j, j = 1, \ldots, J$ of the autoregressive matrix are real, strictly positive, and that the autocorrelation function of any linear combination of factors is $\rho(h) = \sum\limits_{j=i}^{J} \lambda_j^h P_j(h)$, where $P_j(h)$ are polynomials in h.

For instance, the following Gaussian processes are Car processes without continuous time counterpart : Gaussian white noise, Gaussian recursive system such as $Z_{1,t} = Z_{2,t-1} + \varepsilon_{1,t}, Z_{2,t} = \varepsilon_{2t}$, with $\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, Gaussian process with complex eigenvalues of the autoregressive matrix, that is, with autocorrelograms featuring amortized waves.

ii) The Car processes are easily extended to account for any autoregressive lag. For instance a Car (p) process is such that :

$$E[\exp(u'Z_{t+1})|Z_t] = \exp[a'_{og}(u)Z_t + \ldots + a'_{p-1,g}(u)Z_{t-p-1} + b_g(u)],$$

whereas such an extension is not possible in the continuous time framework. A Gaussian VAR(p) process, or an autoregressive gamma process ARG(p), which extends the CIR process to several lags, are also Car processes.

It is often advocated that higher order dynamics can also be deduced from continuous time specification, by considering a linear combination $\alpha' Z_t$, say of a multivariate affine process. This is true, but :

- In the Gaussian case, the pattern of the autocorrelation function of $\alpha' Z_t$, where Z_t corresponds to an Ornstein-Uhlenbeck process, is highly restricted.
- Moreover, an autoregressive gamma [ARG(p)] process has no continuous time counterpart, and, in particular, its transition density is very different from the transition density of a combination of independent CIR processes for instance.

iii) But, it is for capturing nonlinear dynamics that the class of Car processes offers much more possibilities than the class of time discretized continuous affine models. This is a consequence of the embeddability restriction (called infinitely decomposability condition in Duffie, Filipovic, Schachermayer (2003), p10) introduced to ensure the time coherency even for small time units. The conditions are similar to the restrictions exhibited in the linear Gaussian framework, and are easy to understand from the example of a Markov chain with transition matrix P. Let us denote by Z_t the random vector, whose components are the indicators of the different states of the chain at date t. We get :

$$\begin{aligned} & E[\exp(u'Z_{t+1})|Z_t] \\ = \quad & \exp[\sum_{k=1}^{K} \log(E\exp(u'Z_{t+1})|Z_{kt}=1)Z_{kt}]. \end{aligned}$$

Thus, a Markov chain is a Car process. But, this chain can be considered as a time discretized continuous time process if, and only if, the transition matrix has an exponential form $P = \exp(A)$.

b) Lack of flexibility of the specification by default intensity

The usual continuous time approach implicitly assumes the existence of an infinitesimal default intensity, and, therefore, time-to-default with continuous distribution. This assumption is not compatible with available individual data. Indeed, the default times often correspond to predetermined dates of repayment of the debt (as in Merton's model), implying duration distributions with a discrete component.

c) Artificial introduction of jump processes

Under domain assumptions, the continuous time affine models are essentially built from Ornstein-Uhlenbeck and CIR diffusion processes plus jump processes [see Duffie, Filipovic, Schachermayer (2003)]. In practice, the estimation of diffusion affine processes often reveals poor fit, and additional jumps are introduced without clear interpretation. Another advantage of discrete time processes is to avoid this distinction. In discrete time "every movement is a jump".

2.3 Alternative approaches of default correlation

Default dependence has been introduced in the specification above by means of time varying stochastic factors with joint influence on the underlying corporate default intensities. Other modelling approaches have been considered in the literature.

i) A first approach specifies directly a joint distribution for the set of times-to-default $\tau_1, \ldots, \tau_n$. This is usually done by introducing a copula in order to separate the treatment of marginal distribution from the dependence features [see e.g. Van den Berg (1997), Li (1999)]. Then, the joint cdf is given by :

$$F(t_1, \ldots, t_n) = C[G(t_1), G(t_2), \ldots, G(t_n)],$$

where C denotes the copula and G the common marginal cdf. In practice, the copula is selected in a parametric family, or depends on a functional parameter [see e.g. Gagliardini, Gourieroux (2005)]. For instance, it is possible to use an Archimedean copula, which admits a factor interpretation in a proportional hazard model framework [see e.g. Joe (1997)]. This modelling strategy has several drawbacks. First, when a parametric family is selected, the number of parameters is generally small and the family is not compatible with various patterns of the term structure of default. Second, the copula C provides the dependence between the durations at age 0. But, we are mainly interested in the dependence between durations for still alive obligors at any age h. It is difficult to study how the copulas evolve with age (i.e. the so-called term structure of default dependence), and even if they belong to the same family as the initial one.

ii) Another approach is based on a special interpretation of the survivor function introduced in [Bremaud (1981)]. Indeed, we have (in continuous time) :

$$\begin{aligned} P[\tau_i > t] &= \exp\left[-\int_o^t \tilde{\lambda}_u^i du\right] \\ &= P[\int_o^t \tilde{\lambda}_u^i du < E_i], \end{aligned}$$

where E_i is an exponential variate independent of the stochastic intensity. Thus, we get the same model by defining the duration in terms of stopping time as : $\tau_i = \inf\{\tau : \int_o^\tau \tilde{\lambda}_u^i du > E_i\}$.

It has been proposed in the literature to consider intensities, which are independent between individuals, and to extend the joint distribution of the thresholds $E_1, \dots, E_n$ by allowing for dependence between thresholds. Then, a copula is introduced for defining the joint distribution of $E_1, \dots, E_n$ [Schonbucher and Schubert (2001)].

The main drawback of the threshold approach is the introduction of default correlation by means of age independent factors $E_i, i = 1, \dots, n$, which does not allow to manage the term structure of default dependence. Indeed, let us consider stochastic baseline intensities independent of the individuals and independent of the thresholds. We get :

$$
\begin{aligned}
& P[\tau_i > t+1 | \tau_i > t, E_1, \dots, E_n] \\
= \; & P[\int_o^{t+1} \tilde{\lambda}_u^i du < E_i] / P[\int_o^t \tilde{\lambda}_u^i du < E_i] \\
= \; & \tilde{F}_i(t+1; E_i) / \tilde{F}_i(t; E_i) = A^i(t, E_i), \text{ say,}
\end{aligned}
$$

where $\tilde{F}_i(t; .)$ denotes the cumulative distribution function of the cumulated baseline hazard. We get a factor representation of the survivor intensity with time invariant factors. In some sense, the approach above is only a slight modification of the basic mixed multivariate proportional hazard (MMPH) model. Typically, with this strategy it is difficult to select a copula providing high default correlation in the short run and low default correlation in the long run. This situation arises in practice. For instance, at the beginning of the development of the dot com sector, the default probabilities and default correlations were both high. After some years, the remaining firms are more specialized and the default correlation has diminished.

3 The joint duration distribution

In this section, we study the joint distribution of times-to-default and its variation in time. We first consider the case of path dependent factor sensitivities, and then we particularize the results to the case of constant sensitivities.

3.1 General case

Under Assumptions A.1-A.2, it is possible to compute explicitly the joint conditional survivor function. This function depends on the information available at date t. When this information is complete, that is, includes the current and lagged values of the factors together with the information on corporate default, the conditional survivor function can be defined for any subset S of the Population at Risk, that is, the set PaR_t of firms, which are still operating at time t :

$$\begin{aligned} & S_t^c(h_i, i \in S) \\ = \;& P[\tau_i > t + h_i, i \in S | PaR_t, S \subseteq PaR_t, (\tau_j | j \in \overline{PaR_t}), \underline{Z_t}, \underline{Z_t^j}, j = 1, \ldots, n], \end{aligned}$$

where $PaR_t = \{i : \tau_i > t\}$ is the Population at Risk and $\overline{PaR_t}$ denotes its complement.

The property below is proved in Appendix 2.

Proposition 1 : The conditional survivor function is given by :

$$\begin{aligned} & S_t^c(h_i, i \in S) \\ = \;& \exp[-\sum_{k=1}^{\bar{h}} n_{t+k}\alpha_{t+k} + B_g^{-[n][\beta]}(t, t+\bar{h}) + A_g^{-[n][\beta]}(t, t+\bar{h})' Z_t \\ + \;& \sum_{i \in S} B_c^{-[\gamma]}(t, t+h_i) + \sum_{i \in S} A_c^{-[\gamma]}(t, t+h_i)' Z_t^i], \end{aligned}$$

where, for any deterministic sequence[10] $[u] = (u_t)$, operators $A^{[u]}, B^{[u]}$ are recursively defined by :

$$\begin{aligned} A^{[u]}(t, t+h) &= a[u_{t+1} + A^{[u]}(t+1, t+h)], \\ B^{[u]}(t, t+h) &= b[u_{t+1} + A^{[u]}(t+1, t+h)] + B^{[u]}(t+1, t+h), \end{aligned}$$

[10]The [u] sequence may be infinite or not, provided that it is defined for the interval $t+1$ to $t+h$, corresponding to the values included in the recursive computation.

for $h > 0$, with terminal conditions :

$$A^{[u]}(t,t) = 0, B^{[u]}(t,t) = 0 \ , \ \forall t,$$

and where $\bar{h} = \max_{i \in S} h_i$, the deterministic sequence $[n]$ is defined by $n_{t+k} = \text{Card}\ \{h_i \geq k\ ,\ i \in S\}$, and the product sequence $[n][\lambda]$ by $n_t \lambda_t$.

This result shows, in particular, that, given the current and past values of the factors, the default history is not informative as a consequence of conditional independence :

$$S_t^c(h_i, i \in S) = P[\tau_i > t + h_i, i \in S | PaR_t, S \subset PaR_t, Z_t, Z_t^i, i \in PaR_t]$$

Proposition 1 shows that the conditional survivor function is easily computed numerically by means of discrete time recursive equations. This numerical implementation has to be compared with the practice of continuous time affine models. In continuous time, it is necessary to solve numerically differential Riccati equations, also by means of discrete time recursion. But, as mentioned in the introduction, these approximate iterations are done with a much smaller time unit and a much larger number of iterations. Moreover, these recursive systems are approximations, and are not compatible in general with no-arbitrage opportunity conditions. This is not the case of a coherent discrete-time approach.

Proposition 1 can be used to derive the distribution of a given time-to-default. It can also be used to derive the distribution of the time-to-first-failure in a basket of securities. These computations are needed for credit derivatives currently traded on the market [see Sections 4.4 and 4.5]. The credit default swaps (CDS) are options on a corporate default occurrence during a given period, whereas other options concern the occurrence of a default in a basket of securities (first-to- default basket security).

Let us consider a given obligor i. The conditional survivor function specific to this firm is $P[\tau_i > t + h | \tau_i > t, Z_t, Z_t^i]$, and corresponds to the joint survivor function with $S = \{i\}, h_i = h$, and $n_{t+k} = 1$ for $1 \leq k \leq h$.

Corollary 1 : We get :

$$
\begin{aligned}
& P(\tau_i > t+h | \tau_i > t, Z_t, Z_t^i) \\
= \ & \exp[-\sum_{k=1}^{h} \alpha_{t+k} + B_g^{-[\beta]}(t,t+h) + A_g^{-[\beta]}(t,t+h)' Z_t \\
+ \ & B_c^{-[\gamma]}(t,t+h) + A_c^{-[\gamma]}(t,t+h)' Z_t^i],
\end{aligned}
$$

where functions $A_g^{[u]}, A_c^{[u]}, B_g^{[u]}, B_c^{[u]}$ are recursively defined in Proposition 1.

Let us now consider the first-to-default time for a set S of still operating obligors, $\tau_S^* = \min_{\{i \in S \subseteq PaR_t\}} \tau_i$. The survivor function of τ_S^* is defined by :

$P[\tau_S^* > t+h | PaR_t, S \subseteq PaR_t, Z_t, Z_t^i, i \in S]$, and is equal to the joint survivor function evaluated at $h_i = h, \forall i \in S$. Thus, for all $1 \leq k \leq h$, we have $n_{t+k} = n =$ Card (S).

Corollary 2 : We get :

$$
\begin{aligned}
& P[\tau_S^* > t+h | PaR_t, S \subseteq PaR_t, Z_t, Z_t^i, i \in S] \\
= \ & \exp[-n \sum_{k=1}^{h} \alpha_{t+k} + B_g^{-n[\beta]}(t,t+h) + A_g^{-n[\beta]}(t,t+h)' Z_t \\
+ \ & n B_c^{-[\gamma]}(t,t+h) + A_c^{-[\gamma]}(t,t+h)' \textstyle\sum_{i \in S} Z_t^i],
\end{aligned}
$$

where $n[\beta]$ denotes the sequence $n\beta_{t+k}$.

Even if the expressions of the survivor functions given in Proposition 1 and Corollaries 1 and 2 seem rather cumbersome, they are easily implemented from recursive equations. The distribution of the first-to-default time does not depend on the state of individual obligors since the average state $\frac{1}{n} \sum_{i \in S} Z_t^i$ is a sufficient statistic. That is, the first-to-default probability of a security basket that contains some severely distressed corporations is identical to that of another basket where none of the corporations is distressed, but the average state is the same. This is a direct consequence of the homogeneity assumption.

3.2 Constant factor sensitivities

In the general framework of Assumption A.2, the conditional survivor probability for a given obligor i (for instance) varies with t because of the stochastic evolution of the factors Z_t, Z_t^i, but also because of the deterministic aging of sensitivity coefficients $\alpha_t, \beta_t, \gamma_t$. A special case arises when sensitivity functions α, β, γ are constant. In this framework, Corollaries 1 and 2 provide closed-form expressions for the conditional survivor probability.

Let us first consider the slope operator $A^{[u]}(t, t+h)$, when all the terms of the deterministic sequence $[u]$ are equal to a vector u. The recursive equation of Proposition 1 becomes :

$$\begin{aligned} A^u(t,t+h) &= a[u + A^u(t+1,t+h)] \\ &= a_u[A^u(t+1,t+h)], h > 0, \end{aligned}$$

where $a_u(s) = a(u+s)$ denotes a shifted version of function a and $A^u(t+h, t+h) = 0$. We deduce that :

$$A^u(t,t+h) = a_u^{oh}(0),$$

where a^{oh} denotes function $a(.)$ compounded h times. Similarly, for a constant sequence $u_t = u$, the intercept operator B^u is given by :

$$B^u(t,t+h) = \sum_{j=0}^{h-1} b_u(a_u^{oj}(0)).$$

Therefore, we derive the following corollary :

Corollary 3 : When the factor sensitivities are constant :

$$\begin{aligned} & P[\tau_i > t+h | \tau_i > t, Z_t, Z_t^i] \\ = \quad & \exp\{-h\alpha + \sum_{j=0}^{h-1} b_{g,-\beta}[(a_{g,-\beta}^{oj}(0)] + a_{g,-\beta}^{oh}(0)' Z_t + \sum_{j=0}^{h-1} b_{c,-\gamma}[a_{c,-\gamma}^{oj}(0)] + a_{c,-\gamma}^{oh}(0)' Z_t^i\}, \\ & P[\tau_S^* > t+h | PaR_t, S \subseteq PaR_t, Z_t, Z_t^i, i \in S] \\ = \quad & \exp\{-nh\alpha + \sum_{j=0}^{h-1} b_{g,-n\beta}[a_{g,-n\beta}^{oj}(0)] + a_{g,-n\beta}^{oh}(0)' Z_t + n \sum_{j=0}^{h-1} b_{c,-\gamma}[a_{c,-\gamma}^{oj}(0)] + a_{c,-\gamma}^{oh}(0)' \sum_{i \in S} Z_t^i\}, \end{aligned}$$

where the shifted functions are :

$$a_{g,-n\beta}(u) = a_g(u - n\beta), a_{c,-\gamma}(u) = a_c(u - \gamma), \text{ and so on.}$$

The expression involving τ_S shows the effects of the systematic and corporate specific factors. The survivor probability depends on the number n of firms in the basket, and on their average state $\sum_{i \in S} Z_t^i$. An interesting point is the way the number of corporations n appears in the shift of the a_g and b_g functions.

4 Affine term structure and credit risk

The joint historical distribution of factors and default is important for prediction purpose. However, for CreditVaR analysis, it has to be completed by specifying the risk-neutral distribution, or equivalently a stochastic discount factor (sdf). The sdf is used for pricing both future money value, that is, future dollar, and individual defaults. Indeed, it is not realistic to study independently the term structure of riskfree interest rates and default risk, which are both related to business cycles. For instance, in a period of high activity, we expect both an increase of the difference between the long and short term riskfree rates, and an improvement of credit quality [see e.g. the study by Duffee (1998)]. For this reason, we assume that systematic factors Z appearing in the conditional survivor probability can also influence the sdf. Moreover, we select a stochastic discount factor, which is an exponential-affine function of the general factors. As a consequence, the benchmark term structure of riskfree interest rates is affine.

4.1 Specification of the stochastic discount factor

The pricing model is completed by specifying the stochastic discount factor $M_{t,t+1}$ for period $(t, t+1)$. The sdf is the basis for pricing any derivative written on underlying factors and on default times. Typically, the price at t of a European derivative paying g_{t+h} at date $t+h$ is :

$$C_t(g, h) = E_t[M_{t,t+1} \dots M_{t+h-1,t+h} g_{t+h}]$$

$$= \quad E_t[M_{t,t+h} g_{t+h}], \text{ say,} \tag{4.1}$$

where E_t denotes the historical expectation, conditional on the information including the current and lagged values of the state variables, the knowledge of the Population-at-Risk at date t, and the default dates of the other firms. Pricing formula (4.1) will be first used to determine the prices of zero-coupon bonds and corporate zero-coupon corporate bonds with zero-recovery rate. For Treasury-bonds, we get :

$$B(t, t+h) = E_t(M_{t,t+h}), \forall t, h. \tag{4.2}$$

For the h-year zero-coupon corporate bond with zero-recovery rate corresponding to corporate $i \in PaR_t$, we get :

$$C_i(t, t+h) = E_t[M_{t,t+h} \mathbb{1}_{\tau_i > t+h}], \forall t, h. \tag{4.3}$$

For a first-to-default basket, which pays zero in case of at least one default, and 1$ otherwise, we get :

$$C_S(t, t+h) = E_t[M_{t,t+h} \mathbb{1}_{\tau_S^* > t+h}], \forall t, h. \tag{4.4}$$

The case where the recovery rate is nonzero will be treated in Section 6. To restrict the set of admissible risk corrected distributions and get closed form pricing formulas we select a sdf, which is exponential-affine in the systematic factors.

Assumption A.3 :

$$M_{t,t+1} = \exp[\nu_o + \nu' Z_{t+1}]. \tag{4.5}$$

It would have been possible to also introduce corporate specific factors in the expression of the sdf. However, the interpretation would become more complicated, since the set of alive corporate specific factors depends on the date. When a corporation fails, its specific factor ceases to exist. By introducing the effect of individual factors Z_i, we also introduce risk corrections for the number and structure of corporations, which is beyond the scope of the present paper [11].

[11] The number of corporations can have an effect on default correlation. If we consider an industrial sector with two firms only, the default of a firm will increase the monopolistic power of the remaining one, and likely diminish its default probability.

Also note that some components of ν [resp. β_t] can be zero. Therefore, general factors can influence the sdf, the default intensity, or both.

4.2 Zero-coupon Treasury-bonds.

The prices of the zero-coupon Treasury-bonds are given by :

$$\begin{aligned} B(t,t+h) &= E_t[M_{t,t+1} \dots M_{t+h-1,t+h}] \\ &= \exp(\nu_o h) E_t \exp[\nu' Z_{t+1} + \dots + \nu' Z_{t+h}]. \end{aligned}$$

The closed form expression of the price follows from Appendix 1.

Property 2 : The price of a zero-coupon Treasury-bond is :

$$B(t,t+h) = \exp[\nu_o h + \sum_{j=0}^{h-1} b_{g,\nu}(a_{g,\nu}^{oj}(0)) + a_{g,\nu}^{oh}(0)' Z_t].$$

In particular, the geometric yields defined by :

$$\begin{aligned} r(t,t+h) &= -\frac{1}{h} \ln B(t,t+h) \\ &= -\nu_o - \frac{1}{h} \sum_{j=0}^{h-1} b_{g,\nu}(a_{g,\nu}^{oj}(0)) - \frac{1}{h} a_{g,\nu}^{oh}(0)' Z_t, \end{aligned} \quad (4.6)$$

generate an affine space driven by the general factors. Thus, we get an affine term structure of riskfree interest rates [Duffie, Kan (1996)].

4.3 Zero-Coupon Corporate Bonds with Zero-Recovery Rate

The price is given by :

$$\begin{aligned} C_i(t,t+h) &= E_t[M_{t,t+1} \dots M_{t+h-1,t+h} \mathbf{1}_{\tau_i > t+h}] \\ &= E_t \left[\exp\left(\nu_o h + \nu' \textstyle\sum_{j=1}^h Z_{t+j} \right) \exp\left(-\textstyle\sum_{j=1}^h [\alpha_{t+j} + \beta'_{t+j} Z_{t+j} + \gamma'_{t+j} Z^i_{t+j}] \right) \right] \\ &= E_t \left[\exp\left(\nu_o h - \textstyle\sum_{j=1}^h \alpha_{t+j} + \sum_{j=1}^h [\nu - \beta_{t+j}]' Z_{t+j} - \sum_{j=1}^h \gamma'_{t+j} Z^i_{t+j} \right) \right]. \end{aligned}$$

By applying Lemma 1 in Appendix 1, we derive the term structure of corporate bonds with zero-recovery rate.

Property 3 : The price of the corporate bond with zero-recovery rate is :

$$C_i(t,t+h) = \exp\left[\nu_o h - \sum_{j=1}^{h} \alpha_{t+j} + B_g^{\nu-[\beta]}(t,t+h) + A_g^{\nu-[\beta]}(t,t+h)'Z_t + B_c^{-[\gamma]}(t,t+h) + A_c^{-[\gamma]}(t,t+h)'Z_t^i\right].$$

The geometric yields $y_i(t,t+h) = -\frac{1}{h}\log C_i(t,t+h)$ $h = 1,2,\ldots$ generate an affine term structure, now driven by both the general and specific factors. The spread of the corporate bond with zero-recovery rate is given by :

$$\begin{aligned}
& s_i(t,t+h) = y_i(t,t+h) - r(t,t+h) \\
= \;& \frac{1}{h}\sum_{j=1}^{h}\alpha_{t+j} - \frac{1}{h}B_g^{\nu-[\beta]}(t,t+h) - \frac{1}{h}[A_g^{\nu-[\beta]}(t,t+h) - a_{g,\nu}^{oh}(0)]'Z_t - \frac{1}{h}B_c^{-[\gamma]}(t,t+h) \\
& + \frac{1}{h}\sum_{j=0}^{h-1} b_{g,\nu}(a_{g,\nu}^{oj}(0)) - \frac{1}{h}A_c^{-[\gamma]}(t,t+h)'Z_t^i].
\end{aligned}$$

From (4.2), (4.3), we know that :

$$\begin{aligned}
s_i(t,t+h) &= -\frac{1}{h}\log\frac{C_i(t,t+h)}{B(t,t+h)} \\
&= -\frac{1}{h}\log E_t^f[\mathbb{1}_{\tau_i > t+h}],
\end{aligned}$$

where E_t^f denotes the forward risk-adjusted measure for term h [see Merton (1973), Pedersen, Shiu (1994), Geman, El Karoui, Rochet (1995), for definition and use of the forward risk-adjusted measure]. Property 3 shows that the forward risk-adjusted measure is easy to use in the affine framework.

When the sensitivities $\alpha_t, \beta_t, \gamma_t$ are constant, we get :

$$
\begin{aligned}
C_i(t,t+h) &= \exp\left\{\nu_o h - \alpha h + \sum_{j=1}^{h-1} b_{g,\nu-\beta}[a^{oj}_{g,\nu-\beta}(0)] + a^{oh}_{g,\nu-\beta}(0)'Z_t\right. \\
&+ \left.\sum_{j=1}^{h-1} b_{c,\gamma}[a^{oj}_{c,-\gamma}(0)] + a^{oh}_{c,-\gamma}(0)'Z^i_t\right\},
\end{aligned}
$$

$$
\begin{aligned}
y_i(t,t+h) &= -\nu_o + \alpha - \frac{1}{h}\sum_{j=o}^{h-1} b_{g,\nu-\beta}[a^{oj}_{g,\nu-\beta}(0)] - \frac{1}{h}a^{oh}_{g,\nu-\beta}(0)'Z_t \\
&- \frac{1}{h}\sum_{j=0}^{h-1} b_{c,-\gamma}\left[a^{oj}_{c,-\gamma}(0)\right] - \frac{1}{h}a^{oh}_{c,-\gamma}(0)'Z^i_t,
\end{aligned}
$$

$$
\begin{aligned}
s_i(t,t+h) &= \alpha + \frac{1}{h}\sum_{j=0}^{h-1}\left[b_{g,\nu}\left[a^{oj}_{g,\nu}(0)\right] - b_{g,\nu-\beta}[a^{oj}_{g,\nu-\beta}(0)] - b_{c,-\gamma}[a^{oj}_{c,-\gamma}(0)]\right] \\
&+ \frac{1}{h}[a^{oh}_{g,\nu}(0) - a^{oh}_{g,\nu-\beta}(0)]'Z_t - \frac{1}{h}a^{oh}_{c,-\gamma}(0)'Z^i_t.
\end{aligned}
$$

4.4 Decomposition of the spread for a zero-coupon corporate bond with zero-recovery rate.

In the standard actuarial approach, default is assumed independent of riskfree rates, and is priced according to the historical probability [Fons (1994)]. Thus, the actuarial value of a corporate bond with zero-recovery rate is :

$$
C^a_i(t,t+h) = B(t,t+h)P[\tau_i > t+h|\tau_i > t, Z_t, Z^i_t]. \tag{4.7}
$$

By considering the associated actuarial yields, we get :

$$
\begin{aligned}
y^a_i(t,t+h) &= r(t,t+h) + \pi_i(t,t+h), \\
\text{where : } \pi_i(t,t+h) &= -\frac{1}{h}\log P[\tau_i > t+h|\tau_i > t, Z_t, Z^i_t] \\
&= -\frac{1}{h}\sum_{k=1}^{h}\log P[\tau_i > t+k|\tau_i > t+k-1, Z_t, Z^i_t] \\
&= \frac{1}{h}\sum_{h=1}^{h}\lambda^{i,f}_{t+k}, \text{ say,}
\end{aligned} \tag{4.8}
$$

can be interpreted as an average forward default intensity. The forward intensity $\lambda_{t+k}^{i,f} = -\log P(\tau_i > t+k | \tau_i > t+k-1, Z_t, Z_t^i)$ differs from the spot intensity $\lambda_{t+k}^{i} = -\log P[\tau_i > t+k | \tau_i > t+k-1, Z_{t+k-1}, Z_{t+k-1}^i]$, due to the time index of factor values. However, actuarial formula (4.7), which is frequently used by the markets to estimate the default probabilities from the spreads $s_i(t, t+h)$, is not valid in a general framework. The aim of this subsection is to derive a more accurate decomposition of the spread.

From Corollary 1, the average default intensity is given by :

$$\begin{aligned}
\pi_i(t, t+h) &= \frac{1}{h}\sum_{j=1}^{h} \alpha_{t+j} - \frac{1}{h} B_g^{-[\beta]}(t, t+h) - \frac{1}{h} A_g^{-[\beta]}(t, t+h)' Z_t \\
&- \frac{1}{h} B_c^{-[\gamma]}(t, t+h) - \frac{1}{h} A_c^{-[\gamma]}(t, t+h)' Z_t^i.
\end{aligned}$$

We deduce the following proposition :

Proposition 4 :

$$\begin{aligned}
& s_i(t, t+h) - \pi_i(t, t+h) \\
= & -\frac{1}{h}[A_g^{\nu-[\beta]}(t, t+h) - A_g^{-[\beta]}(t, t+h) - a_{g,\nu}^{oh}(0)]' Z_t \\
& -\frac{1}{h}[B_g^{\nu-[\beta]}(t, t+h) - B_g^{-[\beta]}(t, t+h) - \sum_{j=0}^{h-1} b_{g,\nu}[a_{g,\nu}^{oj}(0)]].
\end{aligned}$$

The average default intensity absorbs all idiosyncratic variability in spreads. Even if both the spread term structure $s_i(t, t+h)$, as well as the term structure of average default intensity $\pi_i(t, t+h)$ depend on the stochastic state of the i^{th} corporation, their difference does not, and is the same for all corporations of this industry.

The correcting term in the decomposition of the spread given in Proposition 4 measures the effect of the dependence between default and sdf, due to common factors. For instance, at short term horizon, we have :

$$
\begin{aligned}
s_i(t,t+1) - \pi_i(t,t+1) &= -\log \left\{ \frac{E_t[M_{t,t+1} \mathbb{1}_{\tau_i > t+1}]}{E_t(M_{t,t+1} E_t(\mathbb{1}_{\tau_i > t+1})} \right\} \\
&= -\log \left\{ 1 + \frac{Cov_t(M_{t,t+1}, \mathbb{1}_{\tau_i > t+1})}{E_t(M_{t,t+1}) E_t(\mathbb{1}_{\tau_i > t+1})} \right\}
\end{aligned}
.
$$

The correcting term can be of any sign. This term is positive [resp. negative], if the sdf and the default indicator are negatively correlated [resp. positively correlated]. However it is difficult to guess the sign of this dependence. Indeed, this sign depends on the conditioning set and the sign of the correlation can change (from negative to positive, or conversely), when this information set increases.

To summarize, Proposition 4 provides the following decomposition of the term structure of corporate bonds with zero-recovery rate :

term structure of corporate bonds with zero-recovery rate
= term structure of Treasury-bonds
+ term structure of average default intensity
+ term structure of dependence between sdf and default.

The correcting term takes a simplified form when the sensitivities are time independent.

Corollary 4 : When the sensitivities $\alpha_t, \beta_t, \gamma_t$ are time independent, we get :

$$
\begin{aligned}
& s_i(t,t+h) - \pi_i(t,t+h) \\
= \; & -\frac{1}{h} [a^{oh}_{g,\nu-\beta}(0) - a^{oh}_{g,-\beta}(0) - a^{oh}_{g,\nu}(0)]' Z_t \\
- \; & \frac{1}{h} \sum_{j=0}^{h-1} \left[b_{g,\nu-\beta}[a^{oj}_{g,\nu-\beta}(0)] - b_{g,-\beta}[a^{oj}_{g,-\beta}(0)] - b_{g,\nu}[a^{oj}_{g,\nu}(0)] \right].
\end{aligned}
$$

The correcting term is zero, whenever Z_t is partitioned into two independent subvectors Z_{1t}, Z_{2t}, and the conformable partitionings of ν and β are $(\nu_1', 0)'$ and $(0, \beta_2')'$, respectively. Thus, the correcting term vanishes when default and sdf are influenced by independent factors.

4.5 Term structures of yield and decomposition of the spread for a first-to-default basket

A first-to-default basket with time-to-maturity h, written on n firms provides at $t+h$ a payoff of 1\$, if no firms default before $t+h$, and 0\$, otherwise. The price at t of this basket is :

$$C(t,t+h) = E_t[M_{t,t+h} \mathbb{1}_{\tau^* > t+h}], \forall t, h,$$

where $\tau^* = \min_{i=1,\dots,n} \tau_i$.

Computations similar to those presented in the previous sections give the following results.

Proposition 5 :

$$\begin{aligned}
C(t,t+h) &= \exp[\nu_o h - n\sum_{j=1}^{h} \alpha_{t+j} + B_g^{\nu-n[\beta]}(t,t+h) + A_g^{\nu-n[\beta]}(t,t+h)' Z_t \\
&+ nB_c^{-[\gamma]}(t,t+h) + A_c^{[\gamma]}(t,t+h)'[Z_t^1 + \ldots + Z_t^n]], \\
s(t,t+h) &= \frac{n}{h}\sum_{j=1}^{h} \alpha_{t+j} - \frac{1}{h} B_g^{\nu-n[\beta]}(t,t+h) - \frac{1}{h}\left[A_g^{\nu-n[\beta]}(t,t+h) - a_{g,\nu}^{oh}(0)\right]' Z_t \\
&- \frac{n}{h} B_c^{-[\gamma]}(t,t+h) + \frac{1}{h}\sum_{j=0}^{h-1} b_{g,\nu}(a_{g,\nu}^{oh}(0)) \\
&- \frac{1}{h} A_c^{-[\gamma]}(t,t+h)'(Z_t^1 + \ldots + Z_t^n), \\
\pi(t,t+h) &= \frac{n}{h}\sum_{j=1}^{h} \alpha_{t+j} - \frac{1}{h} B_g^{-n[\beta]}(t,t+h) - \frac{1}{h} A_g^{-n[\beta]}(t,t+h)' Z_t
\end{aligned}$$

$$
\begin{aligned}
& - \frac{n}{h} B_c^{-[\gamma]}(t,t+h) - \frac{1}{h} A_c^{-[\gamma]}(t,t+h)(Z_t^1 + \ldots + Z_t^n), \\
s(t,t+h) & - \pi(t,t+h) = -\frac{1}{h}[A_g^{\nu-n[\beta]}(t,t+h) - A_g^{-n[\beta]}(t,t+h) - a_{g,\nu}^{oh}(0)]' Z_t \\
& - \frac{1}{h}[B_g^{\nu-n[\beta]}(t,t+h) - B_g^{-\nu[\beta]}(t,t+h) - \sum_{j=1}^{h-1} b_{g,\nu}[a_{g,\nu}^{oj}(0)].
\end{aligned}
$$

We still get affine term structures for first-to-default basket. The results are simplified if the sensitivities are constant.

Corollary 6 : If the sensitivities are constant, we get :

$$
\begin{aligned}
C(t,t+h) &= \exp[\nu_o h - n\alpha h + \sum_{j=0}^{h-1} b_{g,\nu-n\beta}(a_{g,\nu-n\beta}^{oj}(0)) + a_{g,\nu-n\beta}^{oh}(0)' Z_t \\
&+ n \sum_{j=1}^{h-1} b_{c,-\gamma}[a_{c,-\gamma}^{oj}(0)] + a_{c,-\gamma}^{oh}(0)'[Z_t^1 + \ldots + Z_t^n], \\
s(t,t+h) &= n\alpha + \frac{1}{h}[\sum_{j=1}^{h-1} b_{g,\nu}(a_{g,\nu}^{oj}(0))] - b_{g,\nu-n\beta}[a_{g,\nu-n\beta}^{oj}(0)] \\
&- n b_{c,-\gamma}[a_{c,-\gamma}^{oj}(0)] + \frac{1}{h}[a_{g,\nu}^{oh}(0) - a_{g,\nu-n\beta}^{oh}(0)' Z_t \\
&- \frac{1}{h} a_{c,-\gamma}^{oh}(0)'[Z_t^1 + \ldots + Z_t^n], \\
\pi(t,t+h) &= n\alpha - \frac{1}{h}\sum_{j=0}^{h-1}[b_{g,-n\beta}(a_{g,-n\beta}^{oj}(0)) + n b_{c,-\gamma}[a_{c,-\gamma}^{oj}(0)]] \\
&- \frac{1}{h} a_{g,-n\beta}^{oh}(0)' Z_t - \frac{1}{h} a_{c,-\gamma}^{oh}(0)'[Z_t^1 + \ldots + Z_t^n],
\end{aligned}
$$

$$
\begin{aligned}
s(t,t+h) \quad &- \quad \pi(t,t+h) = \frac{1}{h}\sum_{j=0}^{h-1}[b_{g,\nu}[a^{oj}_{g,\nu}(0)) + b_{g,-n\beta}(a^{oj}_{g,\nu-n\beta}(0)) \\
&- \quad b_{g,\nu-2\beta}[a^{oj}_{g,\nu-2\beta}(0)]] \\
&+ \quad \frac{1}{h}[a^{oh}_{g,\nu}(0) + a^{oh}_{g,-n\beta}(0) - a^{oh}_{g,\nu-n\beta}(0)]' Z_t.
\end{aligned}
$$

Moreover, the term structure of survivor default intensity $\pi(t,t+h)$ can be decomposed into two parts. The first component corresponds to the marginal effect of individual default risks, obtained under the independence assumption [12]:

$$
\pi^*(t,t+h) = \sum_{i=1}^{n} \pi_i(t,t+h).
$$

The second part corresponds to the residual $\pi(t,t+h) - \pi^*(t,t+h)$. The sign of the residual term depends on the type of dependence between corporate lifetimes. For illustration, let us consider two firms $n = 2$. We get :

$$
\begin{aligned}
& \pi(t,t+h) - \pi^*(t,t+h) \\
= \quad & -\frac{1}{h}\log P[\tau_1 > t+h, \tau_2 > t+h | \tau_1 > t, \tau_2 > t] \\
+ \quad & \frac{1}{h}\log P[\tau_1 > t+h | \tau_1 > t, \tau_2 > t] + \frac{1}{h}\log P[\tau_2 > t+h | \tau_1 > t, \tau_2 > t].
\end{aligned}
$$

This quantity is nonnegative if and only if :

$$
\begin{aligned}
& P[\tau_1 > t+h, \tau_2 > t+h | \tau_1 > t, \tau_2 > t] \leq P[\tau_1 > t+h | \tau_1 > t, \tau_2 > t] P[\tau_2 > t+h | \tau_1 > t, \tau_2 > t] \\
\Leftrightarrow \quad & \text{Cov}\,[1_{\tau_1 > t+h}, 1_{\tau_2 > t+h} | \tau_1 > t, \tau_2 > t] \leq 0.
\end{aligned}
$$

Thus, $\pi(t,t+h)$ is larger than $\pi^*(t,t+h)$, when lifetimes τ_1 and τ_2 feature negative dependence .

[12] keeping the same marginal risks.

To summarize, we have the following proposition.

Proposition 6 : The term structure of first-to-default basket yields can be decomposed as :

$$
\begin{aligned}
y(t,t+h) &= r(t,t+h)+\pi^*(t,t+h)+[\pi(t,t+h)-\pi^*(t,t+h)] \\
&+ [s(t,t+h)-\pi(t,t+h)] \qquad (4.9)
\end{aligned}
$$

corresponding, respectively, to :

(1) the term structure of Treasury-bond yields,
(2) the term structure of marginal defaults,
(3) the term structure of default correlation,
(4) the term structure of dependence between stochastic discount factor and default.

4.6 Factor observability

Up to now, we have not discussed the interpretation of the general and corporate specific factors, and especially their observability. It is well-known that the general factors included in the sdf Z^* (say) can be recovered from the observed Treasury-bond prices, due to the affine structure [see Duffie, Kan (1996)]. Equivalently, factors Z^* can be replaced by mimicking factors with yield interpretations [see Equation (4.6)].

A similar argument apply to general factors which influence default only, and to corporate specific factors. They can be recovered from the corporate yields, taking into account the observability of Z^*. Thus, the discussion of factor observability is equivalent to the discussion of observability of corporate term structure of yields, that is, the number and design of the liquid corporate bonds. More precisely, let us assume L_0 systematic factors and L_1 idiosyncratic factors per firm. Let us denote H [resp. H_i] the number of liquid Treasury bonds [resp.corporate bonds corresponding to firm i] at date t. The factor values are identifiable at date t under the order condition :

$$
\begin{aligned}
&H_i \geq L_1, i=1,\dots,n, \\
&H+\sum_{i=1}^{n}(H_i-L_1) \geq L_0.
\end{aligned}
$$

4.7 Determination of the CreditVaR.

The results above can be directly used to compute the CreditVaR of a portfolio of T-bonds and corporate bonds with zero-recovery rate. Let us consider at date t a portfolio involving a set S of firms. For corporation i, the portfolio includes a quantity $x_{i,t}(h)$ of corporate bonds with time-to-maturity $h, h = 1, \dots, H$. The current value of the credit portfolio is :

$$W_t = \sum_{i \in S} \sum_{h=1}^{H} x_{i,t}(h) C_i(t, t+h),$$

whereas its future value is :

$$W_{t+1} = \sum_{i \in S \cap PaR_{t+1}} \sum_{h=1}^{H} x_{i,t}(h) C_i(t+1, t+h),$$

where $S \cap PaR_{t+1}$ is the set of firms in S, which are still alive at $t+1$. The future portfolio value is doubly stochastic : first, Population-at-Risk at date $t+1$ is not known; second, the future term structure of corporate bonds has to be evaluated.

Due to the affine structure of the model, this future value can be written in terms of the factor values corresponding to date $t+1$:

$$W_{t+1} = \sum_{i \in S \cap PaR_{t+1}} \sum_{h=1}^{H} x_{it}(h) \exp[\bar{a}(h-1) + \bar{b}'(h-1) Z_{t+1} + \bar{c}'(h-1) Z_{t+1}^i],$$

say, where the coefficients $\bar{a}, \bar{b}, \bar{c}$ are deduced from the pricing formulas. The CreditVaR is a quantile of the conditional distribution of W_{t+1} given the information available at time t, that is, $Z_t, Z_t^i, i \in S \cap PaR_t$ (factor values, which are deduced from the observed term structures, see Section 4.6). This distribution can be approximated by Monte-Carlo, as follows :

i) First, draw the future value of the factor Z_{t+1}^s, [resp. $Z_{t+1}^{i,s}$] in the conditional distribution of Z_{t+1} given Z_t [resp. Z_{t+1}^i given Z_t^i].

ii) Second, simulate independently the default occurrence between t and $t+1$ for the different firms in $S \cap PaR_t$ and the drawn values of the factors. $S \cap PaR_{t+1}^s$ denotes the simulated set of surviving firms.

iii) Deduce the simulated future value of the portfolio by :

$$W^s_{t+1} = \sum_{i \in S \cap PaR^s_{t+1}} \sum_{h=1}^{H} x_{it}(h) \exp[\bar{a}(h-1) + \bar{b}'(h-1)Z^s_{t+1} + \bar{c}'(h-1)Z^{i,s}_{t+1}].$$

iv) Replicate the procedure for $s = 1, \ldots, \bar{s}$, where $\bar{s}$ is the total number of replications.

v) Approximate the CreditVaR by the associated empirical quantile of the sample distribution of $W^1_{t+1}, \ldots, W^{\bar{s}}_{t+1}$.

This procedure uses in a coherent way the historical distribution of factors and defaults, and the risk correction by sdf for the closed-form expression of future prices.

It requires conditional drawings of the future factor values in step 1. What about such simulations when the conditional probability distributions are only specified through the Laplace transform ? There is no general answer, but fortunately a large number of Car processes admit compound interpretations appropriate for simulation purpose [see Darolles, Gourieroux, Jasiak (2005)]. In such cases, it is not necessary to compute numerically the conditional cdf by inversion of Laplace transform in order to simulate. This compound interpretation is used to create artificial data from CIR process in the numerical experiment of Section 5.

5 Numerical Experiments

Two numerical experiments are performed in this section. First, we illustrate the decomposition of the spreads for a corporate bond with zero-recovery rate and a first-to-default basket. Second, we analyze the effect of a cyclical factor (which cannot be captured in continuous time model) on both the riskfree term structure and default probability.

5.1 Decomposition of the spread of a corporate bond with zero-recovery rate.

Let us consider a model with one systematic factor and one specific factor, and let us suppose that both factors follow autoregressive gamma processes, where functions a_g, b_g, a_c, b_c have the following expressions :

$$\begin{aligned}
a_g(u) &= \frac{\rho_g u}{1 - ud_g}, \rho_g > 0, d_g > 0, u > 1/d_g, \\
b_g(u) &= -\lambda_g \log(1 - ud_g), \lambda_g > 0, \\
a_c(u) &= \frac{\rho_c u}{1 - ud_c}, \rho_c > 0, d_c > 0, u < 1/d_c, \\
b_c(u) &= -\lambda_c \log(1 - ud_c), \lambda_c > 0.
\end{aligned}$$

We assume constant sensitivities with the following values :
sensitivities : $\alpha = 0.01, \beta = 2, \gamma = .1$; s.d.f.: $\nu_o = -.01, \nu = -.2$;
initial factor values : $Z_0 = .003, Z_0^i = .3$;
factor dynamics : $\rho_g = .9, d_g = .1, \lambda_g = .1, \rho_c = .9, d_c = .1, \lambda_c = .1$.

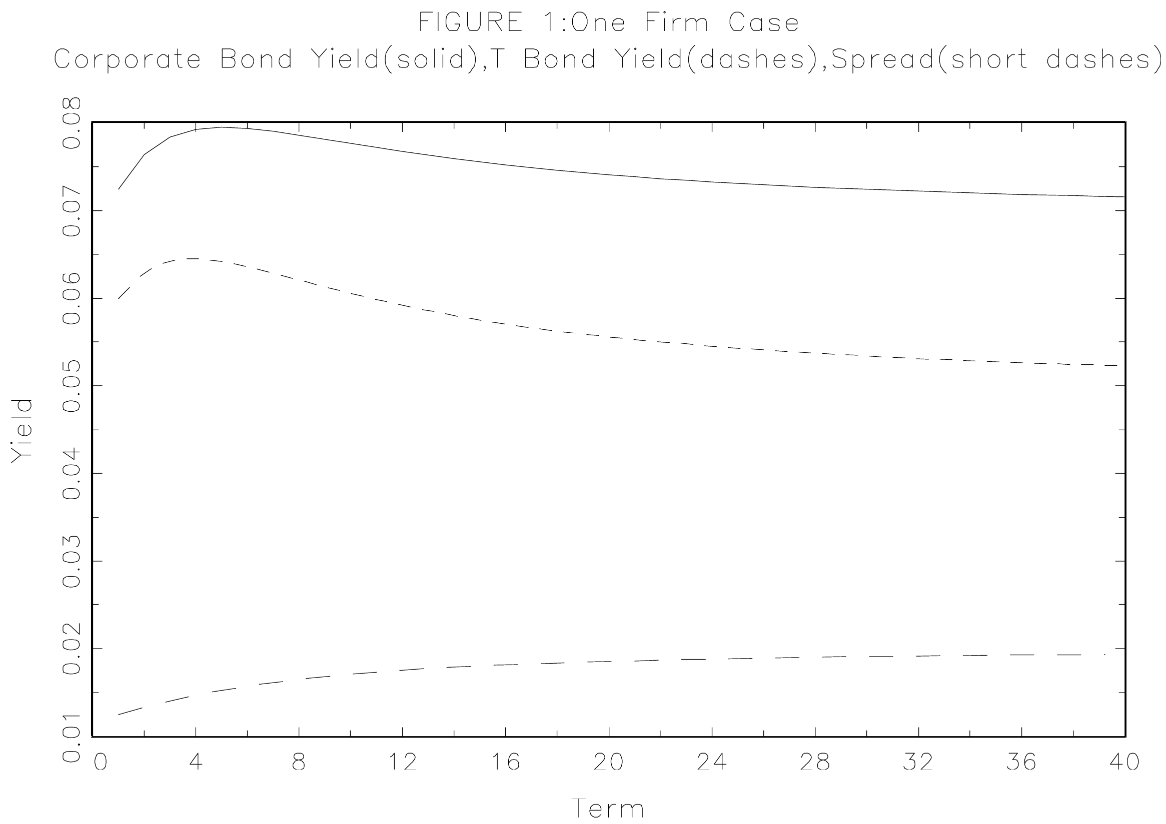


Figure 1 displays the term structures of the corporate yield, of the Treasury-bond yield and of the spread up to $h = 40$. Bump shapes of the corporate yield and of the spread, often observed on bond markets, can easily be reproduced even with fixed sensitivities.

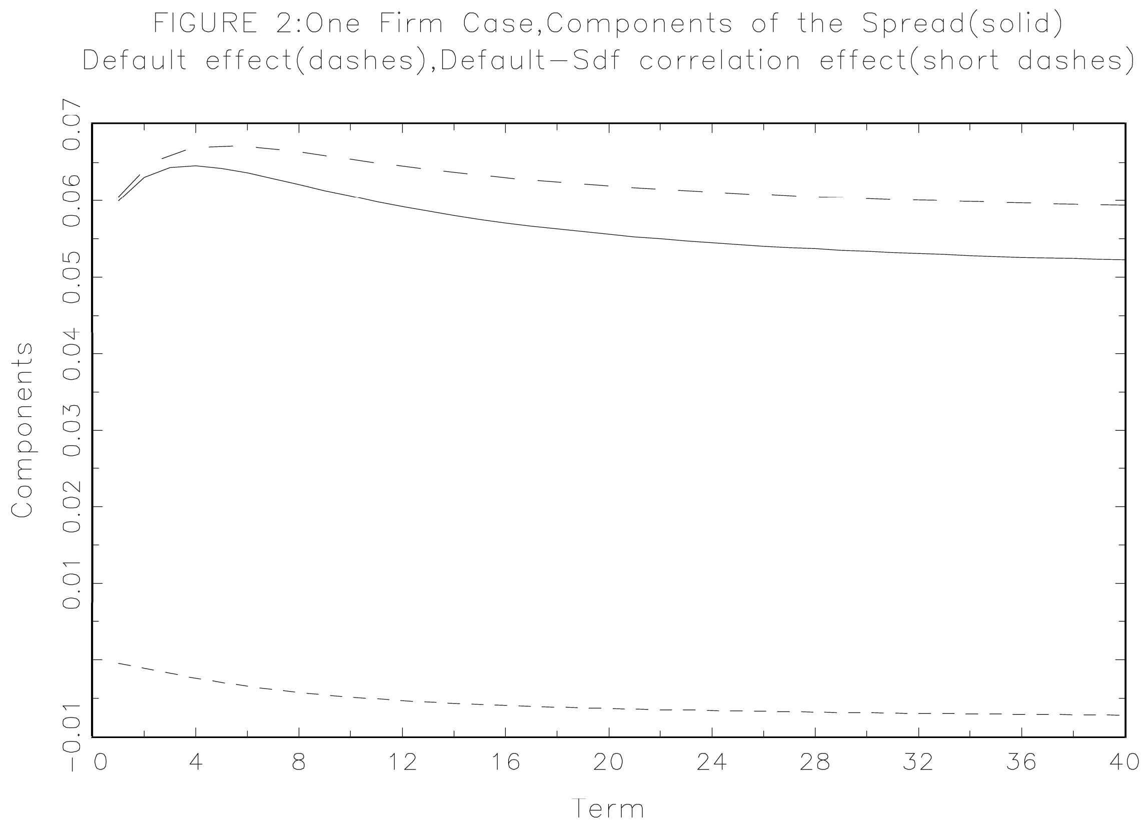


Figure 2 displays the decomposition of the spread into a default effect and a default-sdf correlation effect. In the experiment, the latter effect is negative and rather small in absolute value .

5.2 Decomposition of the first-to-default basket spread

Let us now consider a first-to-default basket on n firms. We still assume a general factor and univariate specific factors, following independent autoregressive gamma processes. We also assume fixed sensitivities. The values of the parameters are :

portfolio size : $n = 3$; sensitivities : $\alpha = .01, \beta = .05, \gamma = .01$;
s.d.f. : $\nu_o = -.15, \nu = .05$; initial values : $Z_0 = 1, Z_0^i = 1, i = 1, 2, 3$;
factor dynamics : $\rho_g = .9, d_g = .1, \lambda_g = 1; \rho_c = .9, d_c = .1, \lambda_c = 1$.

FIGURE 3:Portfolio case
Corporate Bond Yield(solid),T Bond Yield(dashes),Spread(short dashes)

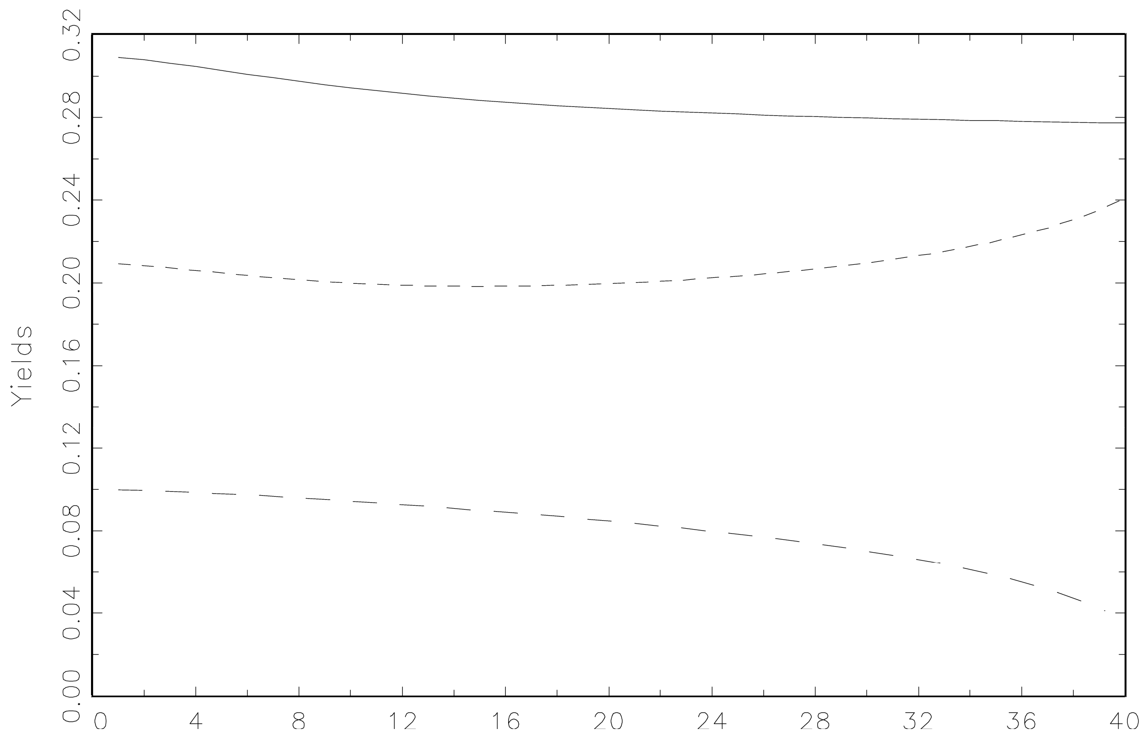
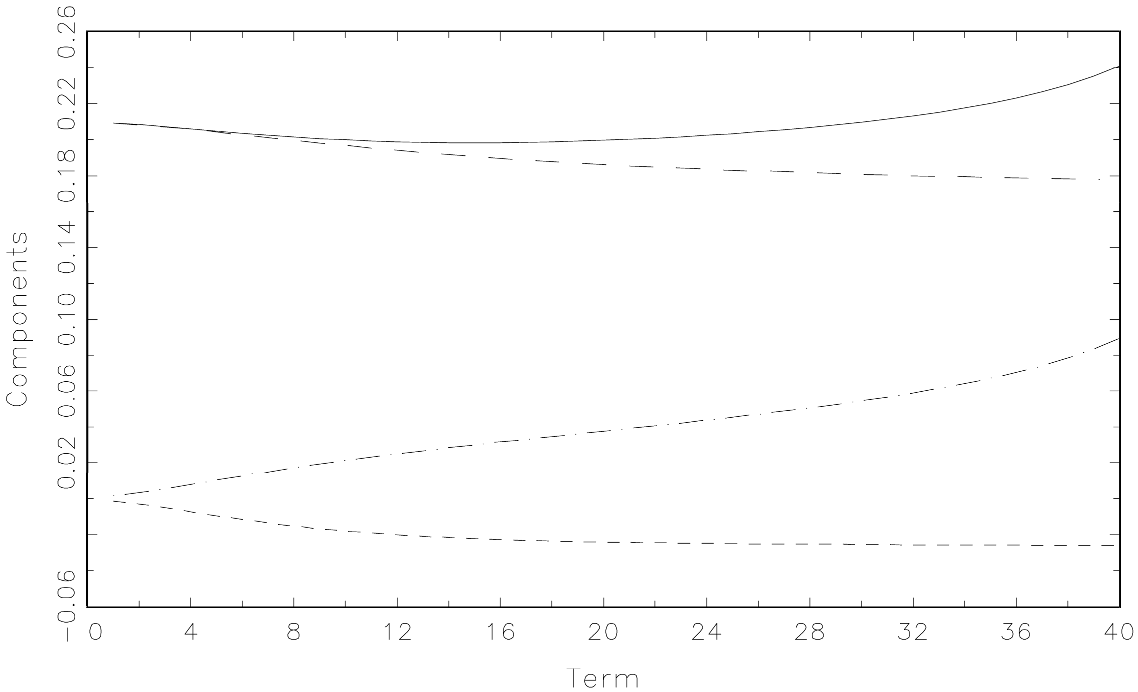


Figure 3 displays the term structures of the first-to-default yield, of the Treasury-bond yield and of the spread. In the example, both yields are decreasing, whereas the spread is first decreasing, and then increasing.

FIGURE4:Portfolio Case,Components of the Spread(solid)
Marginal Default Effect(dashes),Defaults correlation effect(short dashes)
Default−Sdf correlation effect(dots and dashes))



The decomposition of the spread is provided in Figure 4. The main component corresponds to the marginal default effect. The default correlation effect is negative (since the durations τ_1, τ_2, τ_3 feature positive quadrant dependence), whereas the default sdf correlation effect is positive.

5.3 The term structure through the cycle

In this section, we assume that the general factor follows an Autoregressive Gamma (ARG) process of order 4. In this case, using the vector $Z_t = (\tilde{Z}_t, \tilde{Z}_{t-1}, \tilde{Z}_{t-2}, \tilde{Z}_{t-3})'$ in order to be in the one lag general setting, we have :

$$\begin{aligned} a_g(u) &= \left[\frac{\varphi_1}{1-u_1 d_g} + u_2, \frac{\varphi_2}{1-u_1 d_g} + u_3, \frac{\varphi_3}{1-u_1 d_g} + u_4, \frac{\varphi_4}{1-u_1 d_g}\right] \\ b_g(u) &= -\lambda_g \log(1-u_1 d_g), \end{aligned}$$

where $\varphi_i > 0, d_g > 0, \lambda_g > 0, u_1 > 1/d_g$.

The stationarity condition is : $\sum_{i=1}^{4} \varphi_i < 1$. The process $\tilde{Z}_t$ is positive and admits a weak $AR(4)$ representation, which is :

$$\tilde{Z}_t = d_g \lambda_g + \sum_{i=1}^{4} \varphi_i \tilde{Z}_{t-i} + \varepsilon_t.$$

We adopt the following values for the parameters :

$$\begin{aligned} \varphi_1 &= 0.2, \varphi_2 = \varphi_3 = \varphi_4 = 0.1, \\ d_g &= 0.1, \lambda_g = 0.3. \end{aligned}$$

The numerical values of the parameters in the sdf are $\nu_0 = -0.05, \nu_i = -0.2, (i = 1, \ldots, 4)$. The unconditional mean of the factor process is $d_g \lambda_g / (1 - \sum_{i=1}^{4} \varphi_i) = 0.06$. The autoregressive dynamics has been selected to contain a cycle effect of period 4, as seen from the spectral density of the process reported on Figure 5. Such an AR process with complex roots has no equivalent in continuous time.

There can exist a double effect of a cyclical factor on the term structures. On the one hand, the cyclical component of the factor can influence the pattern of the term structure. Intuitively, this effect is larger in the short run than in the long run, since the long run rate is an average of forward short run rates, and the cycle effect is reduced by time averaging. On the other hand, the level and pattern of the term structure can also depend on the current situation, that is, if we are currently in a recession, or an expansion period. To capture this effect we consider four environments $(\tilde{Z}_t, \tilde{Z}_{t-1}, \tilde{Z}_{t-2}, \tilde{Z}_{t-3})$ selected along a periodic history of period 4 with mean 6 %. The environments are :

HMLM = (10%, 6%, 2%, 6%), MLMH = (6%, 2%, 6%, 10 %),

LMHM = (2%, 6%, 10%, 6%), MHML = (6%, 10%, 6%, 2 %), respectively with H = high, M = Medium, L = low.

Figure 6 provides the term structures for the successive situations, in the cycle. The short run impacts can be in opposite directions, whereas the long run interest rates stay identical.

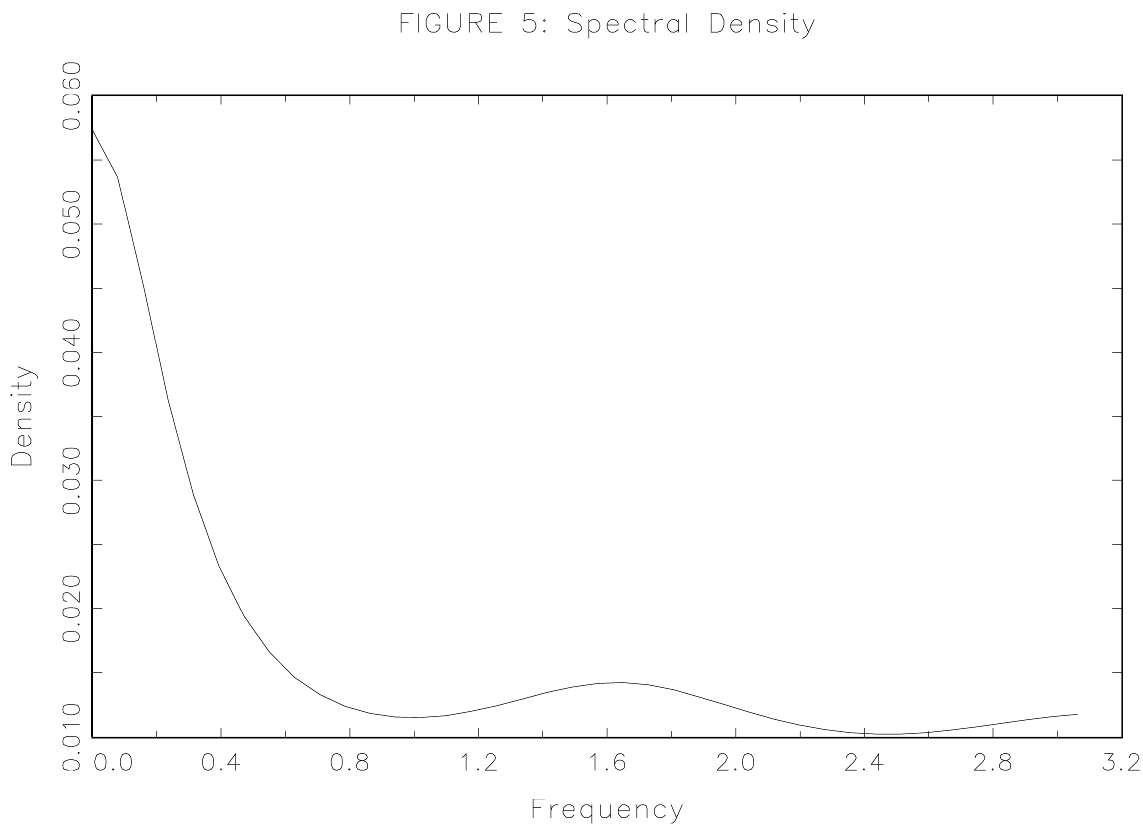
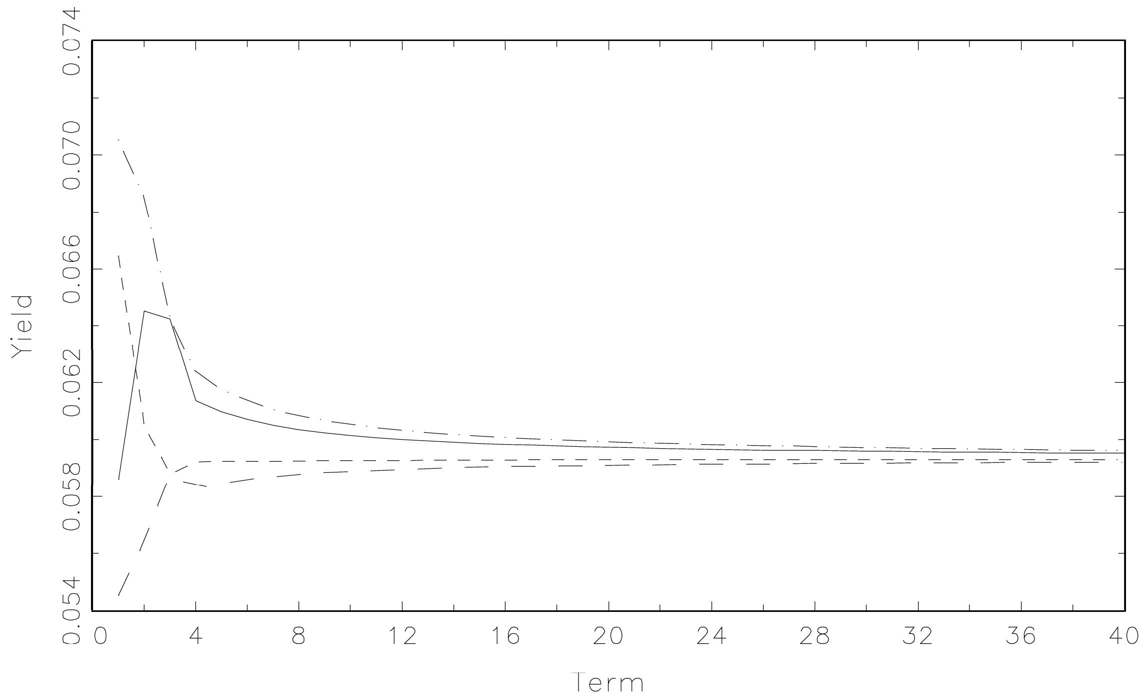


FIGURE 5: Spectral Density
Density
0.010
0.020
0.030
0.040
0.050
0.060
0.0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
Frequency

FIGURE 6 : T Bond Yields,Business Cycle Impact,
Positions at t,t−1,t−2,t−3 (H:High,M:Medium,L:Low)
Solid :HMLM,Dashes:MLMH,Short dashes:LMHM,Dots and dashes:MHML



6 Loss-Given-Default

6.1 Extension of affine model for credit risk.

The affine specification for credit risk can be extended to account for recovery rates. Let us consider a given period $(t, t+1)$. For a default occurring in this period, the loss-given-default is equal to one minus the recovery rate, and denoted by $LGD_{t,t+1}$. Before default, the LGD variable and the associated recovery rate are stochastic variables with values[13] between 0 and 1.

For derivative pricing, the introduction of LGD allows for distinguishing between corporate bonds with different seniorities : junior subordinated, subordinated, senior subordinated, for instance. More precisely, let us consider zero-coupon corporate bonds with time-to-maturity h. The zero-coupon corporate bonds with zero-recovery rate is the derivative with payoff 1\$ at $t+h$, if the corporate is still alive at this date, 0\$, otherwise. Similarly, the zero-coupon corporate bond with recovery corresponds to a payoff 1\$, if the corporate is still alive at $t+h$, and to a payoff $1 - LGD_{t+k,t+k+1}$ received at $t+k+1$, if default occurs between $t+k$ and $t+k+1$, where the definition of LGD depends on the seniority level. Thus, the prices of corporate bonds without recovery and with recovery are, respectively[14] :

$$C(t, t+h) = E_t \left\{ \Pi_{j=0}^{h-1} \left[M_{t+j,t+j+1}(1 - D_{t+j,t+j+1}) \right] \right\},$$

and

$$\begin{aligned} C^*(t, t+h) &= \sum_{k=1}^{h} E_t \left\{ \Pi_{j=0}^{k-2} \left[M_{t+j,t+j+1}(1 - D_{t+j,t+j+1}) \right] M_{t+k-1,t+k} D_{t+k-1,t+k} [1 - LDG_{t+k-1,t+k} \right\} \\ &+ E_t \left\{ \Pi_{j=0}^{k-1} \left[M_{t+j,t+j+1}(1 - D_{t+j,t+j+1}) \right] \right\}, \end{aligned}$$

where $M_{t,t+1}$ denotes the sdf and $D_{t,t+1}$ the default indicator for period $(t, t+1)$.

As above a factor representation can be introduced. This representation assumes that the sdf depends on factor values only, and that default

[13] In practice, observed LGD can be negative, or larger than one, as a consequence of penalties and recovery costs. Following a suggestion of the Basle Committee, they are truncated to (0,1) for regulatory purpose.

[14] We have chosen specifications in which the recovery is paid at default time, and not at the contractual maturity (see e.g. Duffie, Singleton (1994)-(2003), Baho, Bernasconi (2003), for other specifications).

and loss-given-default are independent given the factor value [but generally dependent, when the factor is integrated out]. Thus, this approach can be considered as an extension of the approach of Brennan, Schwartz (1980), Duffee (1998), in which the creditor receives an endogenous random fraction of face value immediately upon-default. This approach allows for a symmetric treatment of probability of default and loss-given-default. It avoids additional assumptions, such as the so-called "recovery of market value" [Duffie, Singleton (1999)], in which "the expected risk-neutral recovery rate is a predetermined fraction of the risk-neutral expected survival market value". In fact, such an hypothesis can easily be tested in our framework.

The derivative prices can be rewritten in terms of survivor intensity and expected loss-given-default, computed conditional on factor values. We get :

$$\begin{aligned} C(t,t+h) &= E_t\left\{\Pi_{j=0}^{h-1}\left[M_{t+j,t+j+1}(1-PD_{t+j,t+j+1})\right]\right\} \\ &= E_t\left[\Pi_{j=0}^{h-1}(M_{t+j,t+j+1}\pi_{t+j,t+j+1})\right], \end{aligned}$$

and

$$\begin{aligned} C^*(t,t+h) &= \sum_{k=1}^{h} E_t\left\{\Pi_{j=0}^{k-2}[M_{t+j,t+j+1}(1-PD_{t+j,t+j+1})M_{t+k-1,t+k}PD_{t+k-1,t+k}(1-ELGD_{t+k-1,t+k})\right] \\ &+ E_t\left\{\Pi_{j=0}^{k-1}\left[M_{t+j,t+j+1}(1-PD_{t+j,t+j+1})\right]\right\} \\ &= \sum_{h=1}^{h} E_t\left\{\Pi_{j=0}^{k-2}\left[M_{t+j,t+j+1}\pi_{t+j,t+j+1}\right]M_{t+k-1,t+k}(1-\pi_{t+k-1,t+k})(1-ELGD_{t+k-1,t+k})\right\} \\ &+ E_t\left[\Pi_{j=0}^{k-1}(M_{t+j,t+1}\pi_{t+j,t+j+1})\right], \end{aligned}$$

where PD and ELGD denote the default probability and the expected loss-given-default, respectively.

The model is completed by introducing exponential affine specifications, and factors with Car dynamics in order to use the closed form expressions of the conditional Laplace transform of the factors. At this step, the exponential-affine specification can be written either for the expected loss-given-default, or for the expected recovery rate. In this model the "risk-neutral" expected loss-given default and the risk-neutral expected survival market value are both exponential affine functions of the current values of the factors. The price of corporate bonds with recovery are available in a quasi-explicit form, using straightforward generalization of the recursive methods presented above [see Baho, Bernasconi (2003) for details]. Moreover, the recovery at market value assumption is easily tested in this framework by checking if the coefficients of the factors are the same and just the intercepts differ.

Finally, more detailed spread decomposition can be derived for corporate bonds to highlight the term structure of loss-given-default, the term structure of dependence between default probability and expected loss-given-default, and so on.

6.2 Correlations

Any joint modelling of default and loss-given-default must reproduce the adverse correlations observed on historical data [Altman et al. (2003), Basel Committee on Banking Supervision (2005)]. The recovery rates are in average lower and the default probabilities higher in recessions explaining the positive [resp. negative] correlations observed between default and loss-given-default [resp. default and recovery rates]. Therefore, any exponential affine intensity model has to allow for this dependence feature. Let us discuss this point on the example of a 3-factor model, where :

$$\begin{aligned} PD &= \exp(\alpha_0 + \alpha_1 Z_1 + \alpha_2 Z_2 + \alpha_3 Z_3), \\ ELGD &= \exp(\beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_3 Z_3), say. \end{aligned}$$

Standard affine dynamics are as follows :

i) a Gaussian VAR(1) (Ornstein-Uhlenbeck) process of (Z_1, Z_2, Z_3);
ii) independent autoregressive gamma (CIR) processes;
iii) a WAR process for the stochastic matrix $\begin{pmatrix} Z_1 & Z_2 \\ Z_2 & Z_3 \end{pmatrix}$.

The choice of a Gaussian factor process does not ensure PD and ELGD between 0 and 1. The choice of independent CIR process ensures PD and ELGD in (0,1) if, and, only if, $\alpha_0 = \beta_0$, $\alpha_1 < 0, \alpha_2 < 0, \alpha_3 < 0, \beta_1 < 0, \beta_2 < 0, \beta_3 < 0$. Thus, a shock on a factor, Z_1, say, the other types of factors being fixed has the same type of effect on PD and ELGD. Automatically, we get conditional and unconditional positive links between the two exponential affine functions of the CIR factors.

We will now discuss positive and negative dependence for models with Wishart factors. Since Wishart processes concern stochastic volatility matrices, we introduce matrix notations :

$$PD = \exp[-Tr(AZ)], ELGD = \exp[-Tr(BZ)],$$

$$\text{where} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix},$$

are symmetric positive definite matrices. The stochastic matrix factor Z follows marginally [15] a standard Wishart distribution with K degrees of freedom and Laplace transform :

$$E \exp[-Tr(UZ)] = det\ (Id + 2U)^{-K/2},$$

where U is a symmetric matrix.

Since A and B are symmetric positive definite matrices, PD and ELGD are between 0 and 1. Let us now consider their explicit expressions :

$$\begin{aligned} PD &= \exp[-(a_{11}Z_{11} + a_{22}Z_{22} + 2a_{12}Z_{12})], \\ \text{ELGD} &= \exp[-(b_{11}Z_{11} + b_{22}Z_{22} + 2b_{12}Z_{12}]. \end{aligned}$$

The sensitivity factor coefficients are constrained by :

$$a_{11} > 0, a_{11}a_{22} - a_{12}^2 > 0, b_{11} > 0, b_{11}b_{22} - b_{12}^2 > 0.$$

These constraints are compatible with opposite sign for a_{12} and b_{12}. Then, a shock on the stochastic volatility Z_{12} (or equivalently on the stochastic correlation), given the stochastic volatilities Z_{11}, Z_{22}, implies opposite effects on $\exp[-Tr(AZ)]$ and $\exp[-Tr(BZ)]$. Thus, we get a negative correlation for given volatilities. Let us now show that, historically, the effects of shocks on stochastic covolatility cannot dominate the effects due to shocks on volatilities Z_{11} and Z_{22}.

More precisely, the negative dependence given stochastic volatilities is compatible with a positive unconditional dependence. Let us consider :

[15] The standard Wishart distribution is the marginal (invariant) distribution of a Wishart autoregressive process.

$$
\begin{aligned}
&Cov[DP,\ LGD]\\
&= Cov(\exp[-Tr(AZ)], \exp(-Tr(BZ)])\\
&= E\exp\{-Tr(A+B)Z) - E\exp[-Tr(AZ)]E\exp[-Tr(BZ)]\\
&= \frac{1}{det(Id+2(A+B)]} - \frac{1}{det(Id+2A)}\frac{1}{det(Id+2B)}.
\end{aligned}
$$

The covariance is positive if, and only if,

$$\Delta = det(Id+2A)\ det(Id+2B) - det(Id+2(A+B)] \geq 0.$$

Without loss of generality, we can assume $a_{12} = 0$. We get :

$$
\begin{aligned}
\Delta\ &= 8(b_{11}b_{22} - b_{12}^2)(a_{11} + a_{12} + 2a_{11}a_{22})\\
&+4[a_{11}b_{11} + a_{22}b_{22} + 2a_{11}a_{22}(b_{11} + b_{22})] \geq 0,
\end{aligned}
$$

by the positiveness condition on matrices A and B.

6.3 Marginal distribution of LGD.

Finally, it is interesting to mention that in the exponential- affine specification with ARG or Wishart factors, the marginal and conditional distributions of LGD can admit a variety of patterns similar to the standard pattern of the beta distribution. More precisely, this distribution corresponds to the distribution of $Y = \exp(-bZ)$, where Z follows a gamma distribution with ν degree of freedom. The associated density function is :

$$f(y) = \frac{1}{\Gamma(\nu)}\frac{1}{b^\nu}(-\log y)^{\nu-1}y^{1/b-1}\mathbb{1}_{(0,1)}(y).$$

These densities are either bell-shaped (if $\nu < 1, b < 1$), or monotonic with a mode on one of the boundary (if $\nu < 1, b > 1$, or $\nu > 1, b < 1$), or admit U-pattern with infinite values on both 0 and 1 (if $\nu > 1$ and $b > 1$). Thus, this family is able to provide good fit to the observed LGD distributions.

7. Concluding remarks

We have described in this paper the affine specification for the analysis of credit risk in a discrete time framework. This framework assumes stochastic discount factor, survivor intensities and loss-given-default (or recovery rates), which are exponential-affine functions of affine factor processes. The affine framework offers a coherent description of the Treasury-bond, corporate bond prices with or without recovery, and first-to-default basket, and tractable methods for predicting the risk included in a portfolio of corporate bonds, while taking into account any possible dependence between default and loss-given-default.

The discrete time affine specification is more flexible than the continuous time affine specification.

i) By considering a much larger set of affine dynamics, it provides better data fit, can take into account the procyclical effect existing in default probabilities, loss-given-default, and riskfree rates, or allow for lagged causality and recursive effects.

ii) From a numerical point of view, the discrete time affine specification is easy to implement and avoids the numerical approximation of the standard Riccati equations. This allows a diminution of the number of iterations needed to derive the credit derivative prices, while being still compatible with no- arbitrage opportunity.

iii) It seems more appropriate for estimation purpose, since the data on failure are available in discrete time (monthly), and for simulation purpose, since the Credit VaR has to be evaluated in discrete time too.

The discrete time affine model can also be used to analyse other events, such as the up-or downgrades by rating agencies. Such an analysis has to be done before using the basic migration models.

After having surveyed practitioners and academic research, the LGD working group of the Basel Committee pointed out the three following findings (Basel Committee on Banking Supervision (2005)] :

First, "the potential for recovery rates to be lower than average during times of high default rates",

Second, "data limitation posing an important challenge",

Third, "the little consensus for incorporating downturn conditions in LGD".

As seen in the paper, the affine model in discrete time is sufficiently flexible to get the expected sign of conditional dependency between LGD and default (point 1), or to incorporate and predict downturns by the introduction of unobservable factors with cyclical dynamics (point 3).

Nethertheless, data limitation is still a challenge. If databases on rating migrations are sufficient to estimate nonlinear dynamic factor models, and get reliable results on default [see e.g. Gagliardini, Gourieroux (2005)], the lack of a sufficient number of detailed data on LGD is clearly a problem. Currently, available microeconomic data on LGD can mainly be used to get unconditional information on the link between LGD and default, or on the type of unconditional LGD distribution, but not to get reliable results after conditioning on observable, or unobservable factors. To circumvent this difficulty, it has been proposed to use market data on corporate bonds. But, a new technical difficulty arises. The credit derivative returns are submitted to parametric domain restrictions, which may have to be taken into account to avoid inconsistent, or inefficient calibration procedures [Gourieroux, Monfort (2005)]. This question is clearly beyond the scope of the present paper.

Appendix 1

Conditional multivariate Laplace transform for a Car process

Let us consider a multivariate Car process, which satisfies :

$E[\exp(u'Z_{t+1})|Z_t] = \exp[a(u)'Z_t + b(u)]$.

For any deterministic sequence [u] of vectors $\{u_s, s = 1, \ldots\}$, let us define the transformation:

$$E[\exp(u'_{t+1}Z_{t+1} + \ldots + u'_{t+h}Z_{t+h})|Z_t],$$

which provides the conditional joint Laplace transform of $Z_{t+1}, \ldots, Z_{t+h}$. The following lemma holds:

Lemma 1 : For any deterministic sequence [u] of vectors $\{u_s, s = 1, \ldots\}$, we have :

$$\begin{aligned} & E[\exp(u'_{t+1}Z_{t+1} + \ldots + u'_{t+h}Z_{t+h})|Z_t] \\ = & \exp[A^{[u]}(t, t+h)Z_t + B^{[u]}(t, t+h)], \end{aligned}$$

where the operators $A^{[u]}$ and $B^{[u]}$ depend on functions $a(.)$, $b(.)$ as well as on sequence [u], and satisfy the backward recursion :

$$\begin{aligned} A^{[u]}(t, t+h) &= a[u_{t+1} + A^{[u]}(t+1, t+h)], \\ B^{[u]}(t, t+h) &= b[u_{t+1} + A^{[u]}(t+1, t+h)] + B^{[u]}(t+1, t+h), \end{aligned}$$

for $h > 0$, with terminal conditions :

$$A^{[u]}(t, t) = 0, B^{[u]}(t, t) = 0 \ , \forall \ t.$$

Proof : We have :

$$
\begin{aligned}
& E[\exp(u'_{t+1}Z_{t+1}+\ldots+u'_{t+h}Z_{t+h})|Z_t] \\
= \quad & E\left(E[\exp(u'_{t+1}Z_{t+1}+\ldots+u'_{t+h}Z_{t+h})|Z_{t+1}]|Z_t\right) \\
= \quad & E(\exp[u'_{t+1}Z_{t+1}+A^{[u]}(t+1,t+h)'Z_{t+1}+B^{[u]}(t+1,t+h)]|Z_t) \\
= \quad & E(\exp([u_{t+1}+A^{[u]}(t+1,t+h)]'Z_{t+1})|Z_t)\exp(B^{[u]}(t+1,t+h)) \\
= \quad & \exp\{a[u_{t+1}+A^{[u]}(t+1,t+h)]'Z_t+B^{[u]}(t+1,t+h)+b[u_{t+1}+A^{[u]}(t+1,t+h)]\}.
\end{aligned}
$$

The recursion follows by identification.

Moreover, the terminal conditions are satisfied, since :

$$A^{[u]}(t,t+1)=a(u_{t+1}), B^{[u]}(t,t+1)=b(u_{t+1}),$$

are deduced from the recursive equations applied with $A^{[u]}(t+1,t+1)=0, B^{[u]}(t+1,t+1)=0$.

QED

Appendix 2

Conditional survivor functions

i) Let us first compute the survivor function of one firm i, given $\underline{Z}$ and $\underline{Z}^i$. We deduce from Assumption A.2 that :

$$P[\tau_i>h|\underline{Z},\underline{Z^i}]=\Pi_{t=1}^h\exp\{-(\alpha_t+\beta'_tZ_t+\gamma'_tZ_t^i)\},$$

where $\underline{Z}=(Z_t,\forall t\geq 0)$, $\underline{Z^i}=(Z_t^i,\forall t\geq 0)$.

In particular, $P(\tau_i>h|\underline{Z},\underline{Z^i})=P(\tau_i>h|\underline{Z}_h,\underline{Z^i}_h)$,

where $\underline{Z}_h=(Z_t,t\leq h),\underline{Z_h^i}=(Z_t^i,t\leq h)$. We also deduce that :

$$P[\tau_i\leq h|\underline{Z},\underline{Z^i}]=P[\tau_i\leq h|\underline{Z}_h,\underline{Z^i}_h],$$

$$P[\tau_i=h|\underline{Z},\underline{Z^i}]=P[\tau_i=h_i|\underline{Z}_h,\underline{Z^i}_h].$$

ii) Then, the joint survivor function given the entire realization of the factor path is :

$$
\begin{aligned}
& P[\tau_i > h_i, i \in S | \underline{Z}, \underline{Z^j}, j = 1, \ldots, n] \\
= \;& P[\tau_i > h_i, i \in S | \underline{Z}, \underline{Z^j}, j \in S] \\
= \;& \Pi_{i \in S} P[\tau_i > h_i | \underline{Z}, \underline{Z^i}] \\
= \;& \Pi_{i \in S} \Pi_{j=1}^{h_i} \exp\{-(\alpha_j + \beta_j' Z_j + \gamma_j' Z_j^i)\}.
\end{aligned}
$$

iii) We deduce that :

$$
\begin{aligned}
& P[\tau_i > t + h_i, i \in S | PaR_t, S \subseteq PaR_t, \tau_j, j \in \overline{PaR_t}, \underline{Z}_t, \underline{Z^j}_t, j = 1, \ldots, n] \\
= \;& \frac{P[\tau_i > t + h_i, i \in S, \tau_k > t, k \in PaR_t - S | \underline{Z}_t, \underline{Z^j}_t, j \in PaR_t]}{P[\tau_i > t, i \in PaR_t | \underline{Z}_t, \underline{Z^j}_t, j \in PaR_t]} \\
= \;& \frac{E\left[\Pi_{i \in S} \Pi_{j=1}^{t+h_i} \exp\left(-(\alpha_j + \beta_j' Z_j + \gamma_j' Z_j^i\right) | \underline{Z_t}, \underline{Z_t^j}, j \in S\right]}{\Pi_{i \in S} \Pi_{j=1}^{t} \exp[-(\alpha_j + \beta_j' Z_j + \gamma_j' Z_j^i]} \\
= \;& E\left[\Pi_{i \in S} \Pi_{j=1}^{h_i} \exp\left(-(\alpha_{t+j} + \beta_{t+j}' Z_{t+j} + \gamma_{t+j}' Z_{t+j}^i)\right) | \underline{Z_t}, \underline{Z_t^j}, j \in S\right] \\
= \;& E[\Pi_{i \in S} \Pi_{k=1}^{h_i} \exp[-\beta_{t+k}' Z_{t+k} - \alpha_{t+k}] | Z_t] \\
& \Pi_{i \in S} E[\Pi_{k=1}^{h_i} \exp(-\gamma_{t+k}' Z_{t+k}^i) | Z_t^i], \text{ from Assumption A.1,} \\
= \;& E[\Pi_{k=1}^{\bar{h}} \exp(-n_{t+k} \beta_{t+k}' Z_{t+k}) | Z_t] \\
& \Pi_{i \in S} E[\Pi_{k=1}^{h_i} \exp[-\gamma_{t+k}' Z_{t+k}^i] | Z_t^i] \\
& \Pi_{k=1}^{\bar{h}} \exp[-n_{t+k} \alpha_{t+k}], \qquad\qquad (A.1)
\end{aligned}
$$

where : $\bar{h} = \max_{i \in S} h_i$ and n_{t+k} denotes the number of firms in set S with $h_i > k$. Therefore, the conditional survivor function can be deduced from the conditional Laplace transform of Car process Z (see Appendix 1, Lemma 1).

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