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A Classification of Two Factor Affine Diffusion Term Structure Models

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Abstract

Dai, Singleton (2000) introduced a typology of affine diffusion models when the domain of admissible values of the factors is an intersection of half planes and under some additional constraints on the parameters. This condition on the domain and the additional sufficient constraints are restrictive and can considerably diminish the practical interest of affine models. In this paper we successfully address the research agenda sketched by Duffie, Filipovic, Schachermayer (2003), Section 12.2, p. 50. A systematic investigation is performed and our paper provides a complete typology in the two factor case, without prior restrictions on the domain and on the parameters.

Keywords: Affine Term Structure Model, Parabolic Dynamics, Wishart Process, Domain Restrictions.

JEL: G13, G17

Une classification des modèles affines de structure par terme à deux facteurs

Résumé

Dai et Singleton (2000) ont introduit une typologie des modèles de diffusion affines, lorsque l'ensemble des valeurs possibles des facteurs est une intersection de demi-plans, et sous quelques contraintes complémentaires sur les paramètres. La condition sur le domaine et les contraintes suffisantes sur les paramètres sont restrictives et diminuent considérablement l'intérêt des modèles affines. Il reste done à effectuer une analyse plus systematique [Duffie, Filipovic, Schachermayer (2003), p. 50]. Dans ce papier nous présentons une typologie complete des modèles affines à deux facteurs sans restriction a priori sur le domaine et les paramètres.

Mots clés: Modèle affine de structure par terme, dynamique parabolique, processus de Wishart, restrictions de support.

1 Introduction

In a diffusion setting an affine term structure model assumes [see Duffie, Kan (1996), Duffie, Filipovic, Schachermayer (2003), Dai, Singleton (2003)]:

A1. The discount function is exponential affine in K latent factors, which are the components of the $(K,1)$ vector X_t:

$$\begin{aligned} D(t,T) &= E^Q\left[\exp - \int_t^T r_s ds | \underline{X_t}\right] \\ &= \exp - \left[A(T-t)' X_t + B(T-t)\right], \end{aligned}$$

where r_s denotes the short term interest rate, Q the risk-neutral probability, $\underline{X_t}$ the filtration generated by the factors, function A is $(K,1)$ and B is a scalar function.

A2. The factor process satisfies a K-dimensional stochastic differential system with linear drift and volatility:

$$dX_t = (\alpha X_t + \beta)\ dt + \left(H_0 + \sum_{k=1}^K H_k X_{kt}\right)^{1/2} dW_t,$$

where β is a $(K,1)$ vector, α is a (K,K) matrix, H_k, $k=0,1,\ldots,K$, are (K,K) symmetric matrices, and (W_t) is a K-dimensional Brownian motion under the risk-neutral distribution Q.

Then, functions A and B satisfy the following ordinary Riccati differential equations:

$$\begin{cases} \frac{dA(t)}{dt} = \alpha_0 + \alpha' A(t) - \frac{1}{2}\left[A(t)' H_1 A(t), \ldots, A(t)' H_K A(t)\right]', \\ \frac{dB(t)}{dt} = \beta_0 + \beta' A(t) - \frac{1}{2} A(t)' H_0 A(t), \end{cases}$$

with initial conditions $A(0)=0$, $B(0)=0$, corresponding to $D(t,t)=1$. In particular, the short term rate is: $r_t = \alpha_0' X_t + \beta_0$, where $\alpha_0 = \frac{dA(0)}{dt}$, $\beta_0 = \frac{dB(0)}{dt}$.

The parameters of the factor dynamics cannot be chosen arbitrarily, since the factor volatility matrix has to satisfy: i) the positive semidefiniteness condition:

$$H_0 + \sum_{k=1}^K H_k X_{kt} \gg 0,$$

for any values X_t in the factor domain $\mathcal{D}$ ($H \gg 0$ means that H is positive semidefinite), and

ii) the rejection condition that ensures the reflection of the volatility matrix towards a positive definite matrix when it becomes singular.

A limited number of parametric specifications satisfying both the positivity and rejection conditions have been derived in the literature.

i) A well-known specification corresponds to the Duffie-Kan model [Duffie, Kan (1996)] in which the volatility matrix is:

$$H_0 + \sum_{k=1}^{K} H_{1k} X_{kt} = Q \begin{pmatrix} a_1' X_t + c_1 & & 0 \\ & \ddots & \\ 0 & & a_K' X_t + c_K \end{pmatrix} Q',$$

and the factor domain is the intersection of half-planes:

$$\mathcal{D} = \left\{ a_k' X_t + c_k \geq 0,\ k = 1, \ldots, K \right\},$$

where a_K, $k = 1, \ldots, K$, are $(K, 1)$ vectors, c_k, $k = 1, \ldots, K$, are scalars, and Q is a (K, K) invertible matrix. Thus, up to a deterministic invertible transformation, the volatility matrix is diagonal and an affine function of the factors. For this specification, Dai, Singleton (2000) derive sufficient, but not necessary, parameter restrictions for which the positivity and rejection conditions are satisfied.

ii) Another specification has been recently introduced by Gourieroux and Sufana (2003) to extend the standard Cox-Ingersoll-Ross process [Cox, Ingersoll, Ross (1985)] to a multivariate framework. They consider factors corresponding to the different elements of a stochastic symmetric positive semidefinite matrix Y_t with dimension (n, n): $X_t = vech\,(Y_t)$, where the matrix Y_t follows a Wishart autoregressive (WAR) process (and more generally affine transformations of such factor processes). The number of Wishart factors is $n(n+1)/2$, and they vary in a nonlinear domain, which is not an intersection of half-planes. For instance, the domain restrictions in a three-factor Wishart model corresponding to $n = 2$ are: $X_{1,t} = Y_{11,t} \geq 0$, $X_{3,t} = Y_{22,t} \geq 0$, $X_{1,t}X_{3,t} - X_{2,t}^2 = Y_{11,t}Y_{22,t} - Y_{12,t}^2 \geq 0$; they involve quadratic restrictions due to the determinant condition.

The Wishart process shows that there exist important affine diffusion processes that are not members of the set of "standard" affine processes considered by Duffie, Kan (1996), and classified by Dai, Singleton (2000).[1] These nonstandard affine processes have a nonlinear state space and lead to more general specifications of the factor volatility matrix.

The general aim of this paper is to extend the classification of standard affine diffusion processes proposed by Dai, Singleton (2000) to the set of all possible affine diffusions. Whereas Dai, Singleton (2000) assume a priori that the state space is an intersection of half planes and consider only sufficient parameter restrictions to ensure the positivity and rejection conditions (as noted in Dai, Singleton (2000), p. 1949), we characterize all admissible state spaces, and give necessary and sufficient parameter restrictions. To highlight the problems and facilitate the comparison with the literature, we focus on the bidimensional case $K = 2$.

The contribution of this paper is the proof that in the bivariate case the only admissible nonlinear state spaces are parabolic domains. In particular, we prove that in the bivariate case the state space cannot be a hyperbolic or elliptic domain. A special example of affine diffusion process with parabolic domain is provided by Duffie, Filipovic, Schachermayer (2003) (Section 12.2, p. 50), who conjecture that this type of bivariate affine process is the only one with a nonlinear state space, but leave the systematic investigation for future research. This paper proves that this is indeed the case if $K = 2$, but the existence of Wishart processes indicates that there are additional affine processes with nonlinear domain when the number of factors is larger than two.[2] Also, we completely characterize all bivariate affine diffusion processes with nonlinear state space, and show that some of the parameter restrictions obtained by Dai, Singleton (2000) for standard affine diffusions are not necessary.

In Section 2, we present the general principle for deriving the pattern of the state space, and

[1] The set of standard affine processes does not include all possible affine processes due to the restrictive additional assumption on the state space introduced in Appendix 1 of Duffie, Kan (1996), p. 398.

[2] Thus, the conjecture of Duffie, Filipovic, Schachermayer (2003) that the diffusion processes with parabolic domain are also the only nonstandard multifactor affine processes is not valid.

the *necessary and sufficient* parametric restrictions which ensure that the process will stay in its state space. In Section 3 we present a classification of bidimensional affine processes. The affine processes with parabolic state space are studied in more detail in Section 4. Section 5 concludes.

2 Principle for deriving the state space and the parametric restrictions

A two-factor affine process with components (x_t, y_t) satisfies a diffusion system:

$$\begin{aligned} \begin{pmatrix} dx_t \\ dy_t \end{pmatrix} &= \left[\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right] dt \\ &+ \begin{pmatrix} a_{11}x_t + b_{11}y_t + c_{11} & a_{12}x_t + b_{12}y_t + c_{12} \\ a_{12}x_t + b_{12}y_t + c_{12} & a_{22}x_t + b_{22}y_t + c_{22} \end{pmatrix}^{1/2} dW_t, \end{aligned} \tag{2.1}$$

with affine drift $\mu(x_t, y_t) = \alpha(x_t, y_t)' + \beta$, and volatility matrix $\Sigma(x_t, y_t) = Ax_t + By_t + C$.

Let us assume an initial date $t = 0$ and an initial value $(x_0, y_0)'$, such that the volatility matrix at $(x_0, y_0)'$ is positive semidefinite. Whenever the volatility matrix stays positive semidefinite, the standard sufficient conditions for the existence and uniqueness of a solution are satisfied, since the drift and the volatility matrix are affine [see Ikeda, Watanabe (1989), theorems 2.4, 3.1, p. 177-178]. The positive semidefiniteness of the volatility matrix is ensured if the process is rejected towards the interior of the domain when it reaches its boundary, following the standard argument used for the Cox-Ingersoll-Ross process [see Ikeda, Watanabe (1989), example 8.2, p. 236]. Thus, these rejection conditions are sufficient for the existence and uniqueness of the process. They are also necessary and sufficient conditions if any point on the boundary of the set of symmetric positive semidefinite matrices is reached with a strictly positive probability. It is beyond the scope of this paper to derive the conditions on the drift and volatility parameters which ensure that the assumption above is satisfied [see e.g. Kunita (1978), (1980), or Ikeda, Watanabe (1989), Chapter VI].[3]

[3] By analogy with the standard Cox-Ingersoll-Ross process [see Ikeda, Watanabe (1989), p. 237, and theorems VI.3.1, VI.3.2], the rejection from the boundary has to be sufficiently small.

The analysis involves several steps.

Step 1. Write the positivity conditions on $\Sigma(x_t, y_t)$, that are:

$$a_{11}x_t + b_{11}y_t + c_{11} \geq 0, \tag{2.2}$$

$$a_{22}x_t + b_{22}y_t + c_{22} \geq 0, \tag{2.3}$$

$$F(x_t, y_t) = \det \Sigma(x_t, y_t) \geq 0. \tag{2.4}$$

Let us introduce the bilinear form on symmetric matrices defined by $(A, B) = \frac{1}{2}(a_{11}b_{22} + a_{22}b_{11}) - a_{12}b_{12}$. We get $(A, A) = \det A$, and the third inequality constraint (2.4) involves the quadratic function:

$$F(x_t, y_t) = (x_t,\ y_t)\,\Lambda \begin{pmatrix} x_t \\ y_t \end{pmatrix} + 2(A, C)\,x_t + 2(B, C)\,y_t + (C, C), \tag{2.5}$$

where:

$$\Lambda = \begin{bmatrix} (A, A) & (A, B) \\ (A, B) & (B, B) \end{bmatrix}.$$

Step 2. Derive necessary and sufficient conditions on the volatility parameters A, B, C to ensure that the inequality restrictions (2.2), (2.3) and (2.4) define a nonempty domain for the pair (x, y).

Step 3. Discuss the form of the domain $\mathcal{D}$ and of its boundary $\widetilde{\mathcal{D}}$. The boundary is reached when $a_{11}x_t + b_{11}y_t + c_{11} = 0$ or $a_{22}x_t + b_{22}y_t + c_{22} = 0$, or $F(x_t, y_t) = 0$.

Since the positivity conditions involve two affine functions and a quadratic function of the factors, the admissible domains will be deduced from elliptic, parabolic, hyperbolic domains (according to the eigenvalues of the matrix Λ), or intersections of the previous domains with half planes, or even intersections of half planes in degenerate cases.

Step 4. Write the conditions for rejection towards the interior of the domain $\mathcal{D}$ for any (x, y) on the boundary $\widetilde{\mathcal{D}}$.

These rejection conditions can be written in a general framework. Let us assume that the boundary is reached for a pair (x, y) such that $F(x_t, y_t) = 0$. Then we have to consider the drift on $F(x_t, y_t)$, which is:

$$\text{Drift } F(x_t, y_t) = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix} \mu(x_t, y_t) + \frac{1}{2} Tr\left[\begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix} \Sigma(x_t, y_t)\right]. \tag{2.6}$$

From the expression (2.5), it is immediately seen that:

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 2(A, A)\, x_t + 2(A, B)\, y_t + 2(A, C), \\
\frac{\partial F}{\partial y} &= 2(A, B)\, x_t + 2(B, B)\, y_t + 2(B, C), \\
\frac{\partial^2 F}{\partial x^2} &= 2(A, A), \ \frac{\partial^2 F}{\partial y^2} = 2(B, B), \ \frac{\partial^2 F}{\partial x \partial y} = 2(A, B).
\end{aligned}$$

Thus, the drift is a quadratic function of x_t and y_t:

$$\text{Drift } F(x_t, y_t) = (x_t,\ y_t)\, \Lambda^* \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \lambda' \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \lambda_0, \text{ say,} \tag{2.7}$$

where Λ^*, λ and λ_0 are given in Appendix 1.

The rejection conditions require a zero volatility and a strictly positive drift of $F(x_t, y_t)$ on the boundary:

$$\Delta(x_t, y_t) = \left(\frac{\partial F}{\partial x}(x_t, y_t),\ \frac{\partial F}{\partial y}(x_t, y_t)\right) \Sigma(x_t, y_t) \left(\frac{\partial F}{\partial x}(x_t, y_t),\ \frac{\partial F}{\partial y}(x_t, y_t)\right)' = 0,$$

and

$$\text{Drift } F(x_t, y_t) > 0,$$

for any (x_t, y_t) such that $F(x_t, y_t) = 0$.

Similar conditions can be written when the boundary is reached for (x_t, y_t) such that $a_{11}x_t + b_{11}y_t + c_{11} = 0$:

$$(a_{11},\ b_{11})\, \Sigma(x_t, y_t)\, (a_{11},\ b_{11})' = 0, \text{ and Drift } (a_{11}x_t + b_{11}y_t + c_{11}) > 0,$$

for any (x_t, y_t) on the line $a_{11}x_t + b_{11}y_t + c_{11} = 0$. If the boundary is reached for (x_t, y_t) such that

$a_{22}x_t + b_{22}y_t + c_{22} = 0$, the conditions are:

$$(a_{22},\ b_{22})\,\Sigma\,(x_t, y_t)\,(a_{22},\ b_{22})' = 0, \ \text{ and } \ \text{Drift } \ (a_{22}x_t + b_{22}y_t + c_{22}) > 0,$$

for any (x_t, y_t) on the line $a_{22}x_t + b_{22}y_t + c_{22} = 0$.

3 Classification of bidimensional affine processes

The volatility matrix is singular when the determinant[4] $F\,(x, y) = \det\Sigma\,(x, y) = 0$. Since $F\,(x, y)$ is a polynomial of degree less than or equal to 2, different cases can be distinguished. In the nondegenerate cases, the condition $F\,(x, y) = 0$ corresponds to a hyperbola, parabola, or ellipse. The degenerate cases are obtained when either the polynomial is of degree 0 or 1, or the second degree polynomial corresponds to a product of affine functions. In all these degenerate situations, the domain is an intersection of hyperplanes, as studied in Duffie, Kan (1996), and in the main sections of Duffie, Filipovic, Schachermayer (2003).

The nondegenerate and degenerate cases are presented below.

3.1 Nondegenerate cases

Up to an affine invertible transformation, the condition $F\,(x, y) = 0$ corresponds to a hyperbola ($xy = 1$, say), a parabola ($y = x^2$, say), or an ellipse ($x^2 + y^2 = 1$, say). The property below is proved in Appendix 2.

Proposition 1 *To get a diffusion which is locally deterministic on $F\,(x, y) = 0$, the boundary of the domain $\mathcal{D}$ cannot be a hyperbola or an ellipse. It can be the parabola $y = x^2$, when*

$$\Sigma\,(x, y) = \begin{pmatrix} 1 & 2x \\ 2x & 4y \end{pmatrix}.$$

The counterexample provided by Duffie, Filipovic, Schachermayer (2003) (Section 12.2, p. 50) corresponds to this parabolic case. But their constraint on the drift implies an absorbing boundary.

[4] To simplify notation, the time subscripts of x and y are omitted.

Finally, the drift can be fixed to ensure the rejection property on the boundary of the domain (see Appendix 3).

Proposition 2 *The only nondegenerate bidimensional affine processes (up to invertible affine transformations) are such that:*

$$\Sigma(x,y) = \begin{pmatrix} 1 & 2x \\ 2x & 4y \end{pmatrix}, \quad \mu(x,y) = \begin{pmatrix} \alpha_{11}x + \beta_1 \\ \alpha_{21}x + \alpha_{22}y + \beta_2 \end{pmatrix},$$

where

$$\alpha_{22} > 2\alpha_{11}, \ \left(\alpha_{21} - 2\beta_1\right)^2 < 4\left(\alpha_{22} - 2\alpha_{11}\right)\left(\beta_2 - 1\right),$$

or

$$\alpha_{22} - 2\alpha_{11} = 0, \ \alpha_{21} = 2\beta_1, \ \beta_2 > 1.$$

3.2 Degenerate cases

In the degenerate case, it is known from Duffie, Kan (1996) that the volatility matrix can be written as:

$$\Sigma(x,y) = \begin{pmatrix} a_{11}x + b_{11}y + c_{11} & 0 \\ 0 & a_{22}x + b_{22}y + c_{22} \end{pmatrix},$$

up to an invertible affine transformation. Its determinant is:

$$F(x,y) = \left(a_{11}x + b_{11}y + c_{11}\right)\left(a_{22}x + b_{22}y + c_{22}\right).$$

Let us now study what arises when the boundary of the domain is reached. We must have:

$$\begin{aligned} V\left(a_{11}dx + b_{11}dy + c_{11}\right) &= \left[a_{11}^2\left(a_{11}x + b_{11}y + c_{11}\right) + b_{11}^2\left(a_{22}x + b_{22}y + c_{22}\right)\right] dt \\ &= b_{11}^2\left(a_{22}x + b_{22}y + c_{22}\right) dt = 0, \end{aligned}$$

if $a_{11}x + b_{11}y + c_{11} = 0$, and

$$\begin{aligned} V\left(a_{22}dx + b_{22}dy + c_{22}\right) &= \left[a_{22}^2\left(a_{11}x + b_{11}y + c_{11}\right) + b_{22}^2\left(a_{22}x + b_{22}y + c_{22}\right)\right] dt \\ &= a_{22}^2\left(a_{11}x + b_{11}y + c_{11}\right) dt = 0, \end{aligned}$$

if $a_{22}x + b_{22}y + c_{22} = 0$. Then two cases have to be distinguished:

Type 2: Both conditions $a_{11}x + b_{11}y + c_{11} = 0$ and $a_{22}x + b_{22}y + c_{22} = 0$ are jointly needed to define the boundary of the domain. When the boundary of the domain is reached, we can have either $a_{11}x + b_{11}y + c_{11} = 0$, or $a_{22}x + b_{22}y + c_{22} = 0$. In this situation, the volatility conditions above have to be jointly satisfied. This implies that either $a_{22}x + b_{22}y + c_{22}$ is proportional to $a_{11}x + b_{11}y + c_{11}$, or $b_{11} = a_{22} = 0$.

Type 3: The boundary of the domain is given by only one of the sets $a_{11}x + b_{11}y + c_{11} = 0$, or $a_{22}x + b_{22}y + c_{22} = 0$, but not by both of them. This situation arises when the two diagonal elements of $\Sigma(x, y)$ are proportional up to an additive constant.

Let us first assume $b_{11} = a_{22} = 0$. The volatility matrix reduces to

$$\Sigma(x,y) = \begin{pmatrix} a_{11}x + c_{11} & 0 \\ 0 & b_{22}y + c_{22} \end{pmatrix}.$$

Different types of dynamics can be distinguished according to the presence of the affine terms in the diagonal elements.

Type 2a: If $b_{11} = a_{22} = a_{11} = b_{22} = 0$, we get a constant diagonal volatility matrix, which can always be chosen as the identity matrix by an invertible affine transformation. Moreover, the constant term β of the drift can be set to zero by an appropriate translation of the variables x, y.[5] The process satisfies:

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \alpha_{11}x + \alpha_{12}y \\ \alpha_{21}x + \alpha_{22}y \end{pmatrix} dt + dW_t,$$

and is a bivariate Ornstein-Uhlenbeck process. This is the model $A_0(2)$ in the classification of Dai, Singleton (2000)[6].

Type 2b: One diagonal term is constant, whereas the other one is affine. Up to an affine invertible transformation, the volatility matrix is $\Sigma(x,y) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$, and the expected domain is $\mathcal{D} = \{x : x \geq 0\}$. The condition on the drift is: Drift $x|_{x=0} = \alpha_{12}y + \beta_1 > 0$, for any y. This

5 Whenever the matrix α is invertible.

6 The condition $\alpha_{12} = 0$ or $\alpha_{21} = 0$ in Dai, Singleton (2000) can be obtained by applying an appropriate orthogonal transformation to (x, y).

implies $\beta_1 > 0$, $\alpha_{12} = 0$. The process satisfies:

$$\left(\begin{array}{c} dx \\ dy \end{array}\right) = \left(\begin{array}{c} \alpha_{11}x + \beta_1 \\ \alpha_{21}x + \alpha_{22}y + \beta_2 \end{array}\right) dt + \left(\begin{array}{cc} \sqrt{x} & 0 \\ 0 & 1 \end{array}\right) dW_t,$$

where $\beta_1 > 0$. The first component is a Cox-Ingersoll-Ross process. This situation is a special case of the model $A_1(2)$ (with $a_{22} = 0$) in the classification of Dai, Singleton (2000) without sign restrictions on α_{11} and α_{21} (see Appendix 4).

Type 2c: When both diagonal terms admit affine components, the volatility matrix can be chosen as: $\Sigma(x,y) = \left(\begin{array}{cc} x & 0 \\ 0 & y \end{array}\right)$, and the expected domain is $\mathcal{D} = \{(x,y) : x \geq 0,\ y \geq 0\}$. The conditions on the drifts are:

$$\begin{aligned} \text{Drift } x|_{x=0} &= \alpha_{12}y + \beta_1 > 0, \text{ for any } y \geq 0, \\ \text{Drift } y|_{y=0} &= \alpha_{21}x + \beta_2 > 0, \text{ for any } x \geq 0. \end{aligned}$$

They imply $\alpha_{12} \geq 0$, $\alpha_{21} \geq 0$, $\beta_1 > 0$, $\beta_2 > 0$. We get:

$$\left(\begin{array}{c} dx \\ dy \end{array}\right) = \left(\begin{array}{c} \alpha_{11}x + \alpha_{12}y + \beta_1 \\ \alpha_{21}x + \alpha_{22}y + \beta_2 \end{array}\right) dt + \left(\begin{array}{cc} \sqrt{x} & 0 \\ 0 & \sqrt{y} \end{array}\right) dW_t,$$

where $\alpha_{12} \geq 0$, $\alpha_{21} \geq 0$, $\beta_1 > 0$, $\beta_2 > 0$. In the special case $\alpha_{12} = \alpha_{21} = 0$, we get two independent Cox-Ingersoll-Ross processes. This case corresponds to the model $A_2(2)$ in the classification of Dai, Singleton (2000), but without the sign restrictions on the components of $\alpha^{-1}\beta$ (see Appendix 4).

Type 2d: Let us now consider the situation where the diagonal elements are proportional:

$$\Sigma(x,y) = \left(\begin{array}{cc} a_{11}x + b_{11}y + c_{11} & 0 \\ 0 & \lambda(a_{11}x + b_{11}y + c_{11}) \end{array}\right),$$

where $\lambda > 0$, and at least one of the coefficients a_{11}, b_{11}, is different from zero. For instance, let us assume $a_{11} \neq 0$. Up to an affine invertible transformation on x, it is possible to choose:

$$\Sigma(x,y) = \left(\begin{array}{cc} x + b_{11}y & 0 \\ 0 & \lambda(x + b_{11}y) \end{array}\right), \quad \lambda > 0.$$

By applying the affine transformation:

$$X = \frac{1}{1 + \lambda b_{11}^2}(x + b_{11}y), \qquad Y = \frac{1}{\lambda^{1/2}(1 + \lambda b_{11}^2)}(-b_{11}\lambda x + y),$$

we get the equivalent form:

$$\Sigma(x,y) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix},$$

and the domain $\mathcal{D} = \{x : x \geq 0\}$.

Let us now consider the drift on x: Drift $x|_{x=0} = \alpha_{12}y + \beta_1$. The positivity condition for any y implies $\alpha_{12} = 0$, $\beta_1 > 0$. Thus we obtain the following bidimensional affine process:

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \alpha_{12}y + \beta_1 \\ \alpha_{21}x + \alpha_{22}y + \beta_2 \end{pmatrix} dt + \begin{pmatrix} \sqrt{x} & 0 \\ 0 & \sqrt{x} \end{pmatrix} dW_t,$$

where $\beta_1 > 0$. This case does not appear explicitly in the classification of Dai, Singleton (2000), but can be considered as a limiting case.

Type 3: This case occurs when the two affine forms are proportional up to an additive constant that is, when:

$$a_{22}x + b_{22}y + c_{22} = \lambda(a_{11}x + b_{11}y + c_{11}) + \mu, \text{ say.}$$

Without loss of generality, we assume that the domain is $\mathcal{D} = \{(x,y) : a_{11}x + b_{11}y + c_{11} \geq 0\}$, and that the second affine form is strictly positive on the domain. This implies $\lambda \geq 0$, $\mu > 0$.

The condition of zero local volatility on the boundary of the domain is:

$$\begin{aligned} V[d(a_{11}x + b_{11}y + c_{11})] &= \left[a_{11}^2(a_{11}x + b_{11}y + c_{11}) + b_{11}^2(a_{22}x + b_{22}y + c_{22})\right] dt \\ &= b_{11}^2\mu\, dt \text{ (on the boundary)} \\ &= 0. \end{aligned}$$

This implies $b_{11} = 0$. Therefore, the volatility matrix is necessarily of the form:

$$\Sigma(x,y) = \begin{pmatrix} a_{11}x + c_{11} & 0 \\ 0 & \lambda(a_{11}x + c_{11}) + \mu \end{pmatrix},$$

where $\lambda \geq 0$, $\mu > 0$.

The affine transformation:

$$X = \frac{1}{a_{11}^2}(a_{11}x + c_{11}), \quad Y = \frac{1}{\sqrt{\mu}}y,$$

provides the equivalent form:

$$\Sigma\left(x,y\right)=\left(\begin{array}{cc} x & 0 \\ 0 & 1+\rho x \end{array}\right),$$

with $\rho=\lambda a_{11}^{2}/\mu\geq 0$, and the domain $\mathcal{D}=\{x:x\geq 0\}$.

The condition on the drift is: Drift $x|_{x=0}=\alpha_{12}y+\beta_1>0$, for any y, which implies $\beta_1>0$, $\alpha_{12}=0$. Thus, the process satisfies:

$$\left(\begin{array}{c} dx \\ dy \end{array}\right)=\left(\begin{array}{c} \alpha_{11}x+\beta_1 \\ \alpha_{21}x+\alpha_{22}y+\beta_2 \end{array}\right)dt+\left(\begin{array}{cc} \sqrt{x} & 0 \\ 0 & \sqrt{1+\rho x} \end{array}\right)dW_t,$$

where $\beta_1>0$ and $\rho\geq 0$. This situation represents the model $A_1(2)$ in the classification of Dai, Singleton (2000). However, their set of sufficient conditions include the inequalities $\alpha_{21}\geq 0$, $\alpha_{11}<0$, which are not necessary.

The classification is summarized in Table 1 below, where each class is defined up to an invertible affine transformation.

Table 1. Classification of bidimensional affine processes (state space with nonempty interior).

D-S denotes the classification of Dai, Singleton (2000).

Type	Domain	Equation	Restrictions	D-S
1	$\mathcal{D} = \{y \geq x^2\}$, interior of a parabola	$\Sigma(x,y) = \begin{pmatrix} 1 & 2x \\ 2x & 4y \end{pmatrix}$, $\mu(x,y) = \begin{pmatrix} \alpha_{11}x + \beta_1 \\ \alpha_{21}x + \alpha_{22}y + \beta_2 \end{pmatrix}$	See Proposition 2	Missing
2a	$\mathcal{D} = \mathbb{R}^2$	$\Sigma(x,y) = Id$, $\mu(x,y) = \begin{pmatrix} \alpha_{11}x + \alpha_{12}y \\ \alpha_{21}x + \alpha_{22}y \end{pmatrix}$	$\alpha_{12} = 0$, or $\alpha_{21} = 0$	$A_0(2)$
2b	$\mathcal{D} = \{x \geq 0\}$	$\Sigma(x,y) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$, $\mu(x,y) = \begin{pmatrix} \alpha_{11}x + \beta_1 \\ \alpha_{21}x + \alpha_{22}y + \beta_2 \end{pmatrix}$	$\beta_1 > 0$	$A_1(2)$
2c	$\mathcal{D} = \{x \geq 0,\ y \geq 0\}$	$\Sigma(x,y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, $\mu(x,y) = \begin{pmatrix} \alpha_{11}x + \alpha_{12}y + \beta_1 \\ \alpha_{21}x + \alpha_{22}y + \beta_2 \end{pmatrix}$	$\alpha_{12} \geq 0$, $\alpha_{21} \geq 0$, $\beta_1 > 0$, $\beta_2 > 0$	$A_2(2)$
2d	$\mathcal{D} = \{x \geq 0\}$	$\Sigma(x,y) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, $\mu(x,y) = \begin{pmatrix} \alpha_{12}y + \beta_1 \\ \alpha_{21}x + \alpha_{22}y + \beta_2 \end{pmatrix}$	$\beta_1 > 0$	Limiting case
3	$\mathcal{D} = \{x \geq 0\}$	$\Sigma(x,y) = \begin{pmatrix} x & 0 \\ 0 & 1 + \rho x \end{pmatrix}$ $\mu(x,y) = \begin{pmatrix} \alpha_{11}x + \beta_1 \\ \alpha_{21}x + \alpha_{22}y + \beta_2 \end{pmatrix}$	$\beta_1 > 0$, $\rho \geq 0$	$A_1(2)$

4 The parabolic dynamics

In this section, we focus on affine models with parabolic domain (type 1). First, we discuss the positivity of the rates and the form of the term structure. Second, we apply a change of variable to provide another interpretation of the parabolic dynamics. Then, we study the limiting degenerate case in which the state space is the parabolic curve.

4.1 Positivity of the rates

In affine term structure models the rates are affine functions of the factors. For instance, the short term rate is:

$$r_t = \frac{dA_1(0)}{dt} x_t + \frac{dA_2(0)}{dt} y_t + \frac{dB(0)}{dt}$$

(see the introduction). The conditions for positive rates are easily derived for the different classes described in Table 1.

Table 2. Conditions for positivity of the rates.

Type	Conditions
1	$\frac{dA_2(0)}{dt} > 0$, $\left(\frac{dA_1(0)}{dt}\right)^2 - 4\frac{dA_2(0)}{dt}\frac{dB(0)}{dt} < 0$
2a	$\frac{dA_1(0)}{dt} = 0$, $\frac{dA_2(0)}{dt} = 0$, $\frac{dB(0)}{dt} > 0$
2b, 2d, 3	$\frac{dA_1(0)}{dt} > 0$, $\frac{dA_2(0)}{dt} = 0$, $\frac{dB(0)}{dt} > 0$
2c	$\frac{dA_1(0)}{dt} > 0$, $\frac{dA_2(0)}{dt} > 0$, $\frac{dB(0)}{dt} > 0$

For types 2b, 2c, 2d, 3, the rate is an affine function of the nonnegative factors x and / or y, with nonnegative coefficients (see Dai, Singleton (2000), Levendorskii (2004) for such a condition). For type 2a we get a degenerate flat term structure. Type 1 is much more interesting since the factor x

is not necessarily positive and its sensitivity coefficient can be strictly negative.

In the parabolic case, the Riccati equation followed by $A(t) = [A_1(t), A_2(t)]'$, useful to determine the term structure, becomes:

$$\begin{aligned}\frac{dA_1(t)}{dt} &= \frac{dA_1(0)}{dt} + \alpha_{11}A_1(t) + \alpha_{21}A_2(t) - 2A_1(t)A_2(t),\\ \frac{dA_2(t)}{dt} &= \frac{dA_2(0)}{dt} + \alpha_{22}A_2(t) - 2A_2(t)^2,\end{aligned}$$

with initial conditions: $A_1(0) = 0$, $A_2(0) = 0$. The second equation is a one-dimensional Riccati equation similar to the Riccati equation corresponding to the Cox-Ingersoll-Ross process. Therefore, this equation admits a closed form solution. Substitution of this solution into the first equation leads to a linear differential equation in A_1, which provides a closed form solution for A_1.

The solution A_2 of the Riccati equation is:

$$A_2(t) = \frac{\alpha_{22}+\gamma}{4} - \frac{\gamma}{2}\frac{1}{1 - \frac{\alpha_{22}-\gamma}{\alpha_{22}+\gamma}\exp(\gamma t)},$$

where $\gamma = \sqrt{8\alpha_{02} + \alpha_{22}^2}$ and $\alpha_{02} = \frac{dA_2(0)}{dt} > 0$. The solution A_1 is derived in Appendix 5 and is given by:

$$A_1(t) = \frac{d_2}{d_0\left(\delta + \frac{\gamma}{2}\right)} + \frac{1}{1 + d_0\exp(\gamma t)}\left[\frac{d_1}{\delta - \frac{\gamma}{2}} - \frac{d_2}{d_0\left(\delta + \frac{\gamma}{2}\right)} - \left(\frac{d_1}{\delta - \frac{\gamma}{2}} + \frac{d_2}{\delta + \frac{\gamma}{2}}\right)\exp\left(\left(\frac{\gamma}{2} - \delta\right)t\right)\right],$$

where

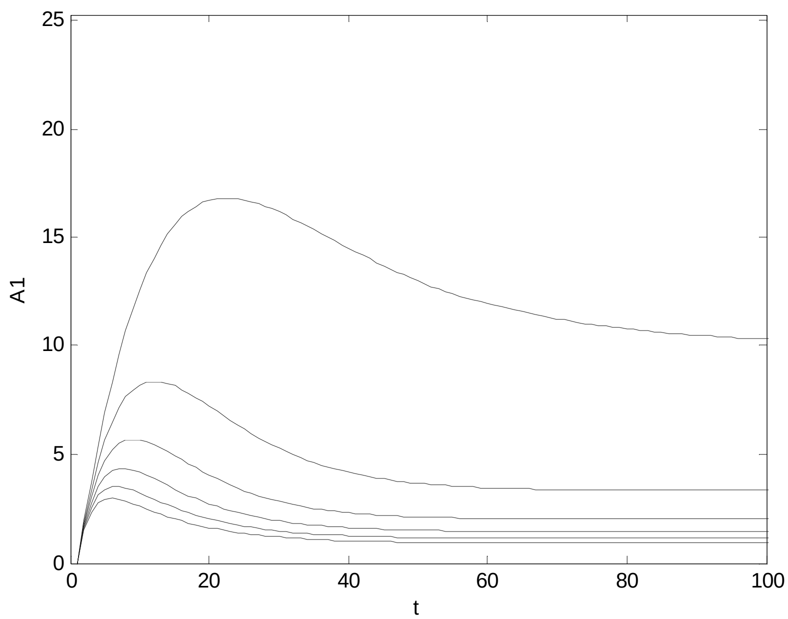
$$\begin{aligned}\delta &= \frac{\alpha_{22}}{2} - \alpha_{11},\\ d_0 &= -\frac{\alpha_{22}-\gamma}{\alpha_{22}+\gamma},\\ d_1 &= \alpha_{01} + \alpha_{21}\frac{\alpha_{22}-\gamma}{4},\\ d_2 &= d_0\left(\alpha_{01} + \alpha_{21}\frac{\alpha_{22}+\gamma}{4}\right).\end{aligned}$$

Note that $\gamma > 0$, $\delta \geq 0$, $d_0 > 0$, and:

$$\lim_{t\to\infty} A_1(t) = \frac{d_2}{d_0\left(\delta + \frac{\gamma}{2}\right)}.$$

In the standard two-factor affine framework, the term structures are derived from two independent Cox-Ingersoll-Ross processes (a special case of type 2c). Then, the component of the term structure sensitive to the factors corresponds to functions similar to A_2. The parabolic framework involves another type of term structure pattern associated with function A_1. Whereas function A_2 corresponding to the Cox-Ingersoll-Ross term structure has a monotonic increasing pattern, function A_1 can feature humps, as shown in Figure 1.

Figure 1. Plot of function A_1 for $\gamma = 0.1$, $d_0 = 2$, $d_1 = 5$, $d_2 = 1$, and $\delta = 0, 0.1, 0.2, 0.3, 0.4, 0.5$.



4.2 Change of variable

Let us consider the affine dynamics derived in Proposition 2 and introduce the change of variable $z = y - x^2$. The bivariate process (x, z) is a diffusion process with volatility matrix:

$$\Sigma^* (x, z) = \begin{pmatrix} 1 & 0 \\ -2x & 1 \end{pmatrix} \begin{pmatrix} 1 & 2x \\ 2x & 4y \end{pmatrix} \begin{pmatrix} 1 & -2x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4\left(y - x^2\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4z \end{pmatrix}.$$

The drift on z is:

$$\text{Drift } z \quad = \quad 4\left(-2\alpha_{11}x^2 - 2\beta_1 x + \alpha_{21}x + \alpha_{22}y + \beta_2 - 1\right)$$

$$= \quad 4\alpha_{22}z + 4\left[(\alpha_{22} - 2\alpha_{11})\,x^2 + (\alpha_{21} - 2\beta_1)\,x + \beta_2 - 1\right].$$

We observe that in general the process (x, z) admits an affine volatility matrix and a quadratic drift. However, the drift is still affine if $\alpha_{22} = 2\alpha_{11}$, which implies $\alpha_{21} = 2\beta_1$ from Proposition 2. Therefore, in this special case, another affine process can be recovered after a nonlinear quadratic transformation of the factors.

Proposition 3 *Let us consider the affine process:*

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \alpha_{11}x + \beta_1 \\ 2\beta_1 x + 2\alpha_{11}y + \beta_2 \end{pmatrix} dt + \begin{pmatrix} 1 & 2x \\ 2x & 4y \end{pmatrix}^{1/2} dW_t,$$

with state space $\mathcal{D} = \{(x, y) : y > x^2\}$. Then the process $(x, z = y - x^2)$, is still affine:

$$\begin{pmatrix} dx \\ dz \end{pmatrix} = \begin{pmatrix} \alpha_{11}x + \beta_1 \\ 8\alpha_{11}z + 4\,(\beta_2 - 1) \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & 4z \end{pmatrix}^{1/2} dW_t,$$

with state space $\mathcal{D}^* = \{(x, z) : z > 0\}$. Thus the components x and z are independent, x follows an Ornstein-Uhlenbeck process, whereas z follows a Cox-Ingersoll-Ross process.

The construction of this process is similar to the construction of the Wishart process for dimension $n\,(n+1)\,/2$, where n is an integer [see Gourieroux, Sufana (2003)][7].

4.3 Degenerate parabolic dynamics

The characterizations of the state spaces of affine processes has been performed under the assumption of a state space with nonempty interior. There exists also nonlinear state spaces with empty interior such as: $\mathcal{D} = \{(x, y) : y = x^2\}$. A simple example is derived as follows.

Let us consider an Ornstein-Uhlenbeck process:

$$dx = (\alpha x + \beta)\,dt + c dW_t,$$

[7] Note that a Wishart process cannot be defined in the bidimensional case, since 2 cannot be written as $n\,(n+1)\,/2$.

and the joint process $(x, y = x^2)$. This is a bivariate diffusion process such that:

$$\begin{cases} dx = (\alpha x + \beta)\, dt + c dW_t, \\ \\ dy = \left[2x\,(\alpha x + \beta) + c^2\right] dt + 2cx dW_t, \end{cases}$$

or, equivalently,

$$\begin{cases} dx = (\alpha x + \beta)\, dt + c dW_t, \\ \\ dy = (2\alpha y + 2\beta x + c^2)\, dt + 2cx dW_t. \end{cases}$$

This is a bidimensional affine process with state space $\mathcal{D} = \{(x, y) : y = x^2\}$. This type of transformation has been used to construct the so-called quadratic term structure models (see e.g. Leippold, Wu (2002)).

5 Concluding remarks

This paper extends the classification of standard affine diffusion processes of Dai, Singleton (2000) in the bidimensional case by considering all nonlinear state spaces, and by providing necessary and sufficient parameter restrictions that ensure the positivity and rejection conditions. We prove that in the bivariate case the only admissible nonlinear state spaces are parabolic domains. Thus the conjecture of Duffie, Filipovic, Schachermayer (2003) that this type of bivariate affine process is the only one with a nonlinear state space is valid in the two-factor case. However, their conjecture is not valid in a framework with more than two factors, since Wishart processes are additional examples of affine processes with nonlinear domain. Also, we completely characterize all bivariate affine diffusion processes with nonlinear state space, and show that some of the parameter restrictions obtained by Dai, Singleton (2000) for standard affine diffusions are not necessary.

Appendix 1. Drift of $F(x_t,\ y_t)$

The drift of $F(x_t,\ y_t)$ is:

$$\text{Drift } F(x_t, y_t) = (x_t,\ y_t)\, \Lambda^* \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \lambda' \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \lambda_0,$$

where

$$\Lambda^* = 2\Lambda\beta,$$
$$\lambda' = 2\left[(A,C),(B,C)\right]\beta + 2\alpha'\Lambda + \left[Tr\,(\Lambda A),Tr\,(\Lambda B)\right],$$
$$\lambda_0 = 2\left[(A,C),(B,C)\right]\alpha + Tr\,(\Lambda C)\,.$$

Appendix 2. Conditions for a locally deterministic system on the boundary

In this appendix, we focus on nondegenerate cases. Up to an affine invertible transformation, the condition $F(x,y) = \det\Sigma(x,y) = 0$ can correspond to:

i) a hyperbola : $xy = 1$ (say);

ii) a parabola : $y = x^2$ (say);

iii) an ellipse : $x^2 + y^2 = 1$ (say).

In each case, we consider the restrictions implied on the volatility parameters by the equality:

$$\Delta(x,y) = \left(\frac{\partial F}{\partial x}(x,y)\,,\ \frac{\partial F}{\partial y}(x,y)\right)\Sigma(x,y)\left(\frac{\partial F}{\partial x}(x,y)\,,\ \frac{\partial F}{\partial y}(x,y)\right)' = 0,$$

which has to be satisfied for the pairs (x,y) such that $F(x,y) = 0$.

1. Hyperbolic case

When $F(x,y) = k(xy-1)$, $k \neq 0$, we get: $\frac{\partial F}{\partial x}(x,y) = ky$, $\frac{\partial F}{\partial y}(x,y) = kx$. The condition $\Delta(x,y) = 0$ is equivalent to:

$$(a_{11}x + b_{11}y + c_{11})\,y^2 + (a_{22}x + b_{22}y + c_{22})\,x^2 + 2\,(a_{12}x + b_{12}y + c_{12})\,xy = 0,$$

or, since $xy = 1$:

$$b_{11}x^{-3} + c_{11}x^{-2} + (a_{11} + 2b_{12})\,x^{-1} + 2c_{12} + (b_{22} + 2a_{12})\,x + c_{22}x^2 + a_{22}x^3 = 0,$$

for any x in an open set. We deduce the necessary conditions:

$$b_{11} = c_{11} = c_{12} = c_{22} = a_{22} = 0,\ \ a_{11} + 2b_{12} = 0,\ \ b_{22} + 2a_{12} = 0.$$

In particular, $C = 0$ and this condition is not compatible with: $F(0,0) = \det C = -k \neq 0$. Thus, the boundary cannot correspond to a hyperbola.

2. Parabolic case

Let us now assume: $F(x,y) = k(y - x^2)$, $k \neq 0$. We get: $\frac{\partial F}{\partial x}(x,y) = -2kx$, $\frac{\partial F}{\partial y}(x,y) = k$. The condition $\Delta(x,y) = 0$ becomes:

$$4(a_{11}x + b_{11}y + c_{11})x^2 + (a_{22}x + b_{22}y + c_{22}) - 4(a_{12}x + b_{12}y + c_{12})x = 0,$$

or, since $y = x^2$:

$$(4b_{11})x^4 + (4a_{11} - 4b_{12})x^3 + (4c_{11} + b_{22} - 4a_{12})x^2 + (a_{22} - 4c_{12})x + c_{22} = 0,$$

for any x in an open set. We deduce:

$$b_{11} = 0,\;\; a_{11} = b_{12},\;\; 4c_{11} + b_{22} - 4a_{12} = 0,\;\; a_{22} = 4c_{12},\;\; c_{22} = 0,$$

and the volatility matrix can be written as:

$$\Sigma(x,y) = \begin{pmatrix} a_{11}x + c_{11} & a_{12}x + a_{11}y + c_{12} \\ a_{12}x + a_{11}y + c_{12} & 4c_{12}x + (4a_{12} - 4c_{11})y \end{pmatrix}.$$

Let us now write the conditions to ensure that $\det \Sigma(x,y) = k(y - x^2)$. We get:

$$\begin{aligned} \det \Sigma(x,y) &= (a_{11}x + c_{11})\left[4c_{12}x + (4a_{12} - 4c_{11})y\right] - (a_{12}x + a_{11}y + c_{12})^2 \\ &= -a_{11}^2 y^2 + x^2\left(-a_{12}^2 + 4a_{11}c_{12}\right) + xy\left[-2a_{11}a_{12} + a_{11}(4a_{12} - 4c_{11})\right] \\ &\quad + x(-2a_{12}c_{12} + 4c_{11}c_{12}) + y\left[-2a_{11}c_{12} + c_{11}(4a_{12} - 4c_{11})\right] - c_{12}^2. \end{aligned}$$

We deduce:

$$a_{11}^2 = 0, \quad -a_{12}^2 + 4a_{11}c_{12} = -k, \quad -2a_{11}a_{12} + a_{11}(4a_{12} - 4c_{11}) = 0,$$

$$c_{12}(-2a_{12} + 4c_{11}) = 0, \quad -2a_{11}c_{12} + c_{11}(4a_{12} - 4c_{11}) = k,\;\; c_{12}^2 = 0.$$

The system is equivalent to:

$$a_{11} = 0,\; c_{12} = 0,\; a_{12}^2 = k,\; c_{11}(4a_{12} - 4c_{11}) = a_{12}^2.$$

The last equality is equivalent to $(a_{12} - 2c_{11})^2 = 0 \Leftrightarrow a_{12} = 2c_{11}$. Finally the form of the volatility matrix is:

$$\Sigma(x,y) = \begin{pmatrix} c_{11} & 2c_{11}x \\ 2c_{11}x & 4c_{11}y \end{pmatrix},$$

where $c_{11} > 0$. Note that c_{11} can be taken equal to 1 by considering the linear transformation: $x/c_{11}^{1/2} \to x$, $y/c_{11} \to y$.

3. Elliptic case

In this case, $F(x,y) = k(x^2+y^2-1)$, $k \neq 0$. We get: $\frac{\partial F}{\partial x}(x,y) = 2kx$, $\frac{\partial F}{\partial y}(x,y) = 2ky$. The condition $\Delta(x,y) = 0$ becomes:

$$(a_{11}x + b_{11}y + c_{11})\,x^2 + (a_{22}x + b_{22}y + c_{22})\,y^2 - 2\,(a_{12}x + b_{12}y + c_{12})\,xy = 0,$$

or, since $y^2 = 1 - x^2$:

$$(a_{11} - a_{22} + 2b_{12})\,x^3 + (b_{11} - b_{22} - 2a_{12})\,x^2y + (c_{11} - c_{22})\,x^2 - 2c_{12}xy + (a_{22} - 2b_{12})\,x + b_{22}y + c_{22} = 0.$$

Let us consider the change of variables:

$$\begin{aligned} x &= \sin\theta = \frac{2t}{1+t^2}, \\ y &= \cos\theta = \frac{1-t^2}{1+t^2}, \end{aligned}$$

where $t = \tan\frac{\theta}{2}$, $-\pi < \theta < \pi$. This leads to:

$$\begin{aligned} & (c_{22} - b_{22})\,t^6 + [4c_{12} + 2(a_{22} - 2b_{12})]\,t^5 + [-4(b_{11} - b_{22} - 2a_{12}) + 4(c_{11} - c_{22}) - b_{22} + 3c_{22}]\,t^4 \\ & + [8(a_{11} - a_{22} + 2b_{12}) + 4(a_{22} - 2b_{12})]\,t^3 + [4(b_{11} - b_{22} - 2a_{12}) + 4(c_{11} - c_{22}) + b_{22} + 3c_{22}]\,t^2 \\ & + [-4c_{12} + 2(a_{22} - 2b_{12})]\,t + b_{22} + c_{22} \\ = \;& 0, \end{aligned}$$

for any t in an open set. We deduce:

$$b_{22} = c_{22} = 0,\; c_{11} = 0,\; c_{12} = a_{22} - 2b_{12} = 0.$$

In particular, $C = 0$, and this condition is not compatible with: $F(0,0) = \det C = -k \neq 0$. Thus, the boundary cannot correspond to an ellipse.

Appendix 3. Conditions for rejection on the boundary for the parabolic case

In the parabolic case $\Sigma(x,y) = \begin{pmatrix} 1 & 2x \\ 2x & 4y \end{pmatrix}$. We get: $F(x,y) = 4(y - x^2)$, which is positive if, and only if, $y \geq x^2$. The first and second order derivatives of the determinant are:

$$\frac{\partial F}{\partial x} = -8x, \quad \frac{\partial F}{\partial y} = 4, \quad \frac{\partial^2 F}{\partial x^2} = -8, \quad \frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 F}{\partial x \partial y} = 0.$$

For $y = x^2$ the drift on the determinant is:

$$\begin{aligned} \text{Drift } F(x_t, y_t) &= -8x(\alpha_{11}x + \alpha_{12}y + \beta_1) + 4(\alpha_{21}x + \alpha_{22}y + \beta_2) - 4 \\ &= 4\left[-2\alpha_{12}x^3 + (\alpha_{22} - 2\alpha_{11})x^2 + (\alpha_{21} - 2\beta_1)x + \beta_2 - 1\right]. \end{aligned}$$

The drift is positive for any x if, and only if, one of the following sets of conditions is satisfied:

$$\alpha_{12} = 0, \;\; \alpha_{22} > 2\alpha_{11}, \;\; (\alpha_{21} - 2\beta_1)^2 < 4(\alpha_{22} - 2\alpha_{11})(\beta_2 - 1),$$

or

$$\alpha_{12} = 0, \;\; \alpha_{22} - 2\alpha_{11} = 0, \;\; \alpha_{21} = 2\beta_1, \;\; \beta_2 > 1.$$

Appendix 4. Dai-Singleton classification

In this appendix, we particularize the classification derived by Dai, Singleton (2000) [with restrictions (11) - (19), p.1948 - 1949][8] to the one and two factor frameworks. They consider a model satisfying the Duffie-Kan condition on the volatility matrix:

$$\begin{aligned} \begin{pmatrix} dx_t \\ dy_t \end{pmatrix} &= \left[\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}\right] dt \\ &\quad + \begin{pmatrix} a_{11}x_t + b_{11}y_t + c_{11} & 0 \\ 0 & a_{22}x_t + b_{22}y_t + c_{22} \end{pmatrix}^{1/2} dW_t. \end{aligned}$$

Different families of models are defined according to the number m of linearly independent combinations of state variables involved in the volatility matrix. Then sufficient restrictions are introduced on the parameters to ensure the positive definiteness of the volatility matrix.

[8] Note that their conditions (17), p. 1949, differ from the condition C4 given in Appendix B.

i) One factor framework

Family $A_0(1)$ corresponding to $m = 0$

Up to an invertible affine transformation, the model can be written as:

$$dx_t = \alpha x_t \, dt + dW_t,$$

with no restriction on α. This is an Ornstein-Uhlenbeck process with zero mean and unitary volatility.

Family $A_1(1)$ corresponding to $m = 1$

Up to an invertible affine transformation, the model can be written as:

$$dx_t = \alpha (\theta - x_t) \ dt + \sqrt{x_t} dW_t,$$

where the parameters are constrained by $\alpha\theta > 0$ and $\theta > 0$. This is a special case of the Cox-Ingersoll-Ross process, but the condition $\theta > 0$ is not necessary. In fact since the inequalities $\alpha\theta > 0$ and $\theta > 0$ are equivalent to $\alpha\theta > 0$ and $\alpha > 0$, Dai and Singleton implicitly imposed an additional stationarity condition.

ii) Two factor framework

Family $A_0(2)$ corresponding to $m = 0$

Up to an invertible affine transformation, the model can be written as:

$$\begin{pmatrix} dx_t \\ dy_t \end{pmatrix} = \begin{pmatrix} \alpha_{11} x_t + \alpha_{12} y_t \\ \alpha_{21} x_t + \alpha_{22} y_t \end{pmatrix} dt + dW_t,$$

where the parameters are constrained by either $\alpha_{12} = 0$, or $\alpha_{21} = 0$.

Family $A_1(2)$ corresponding to $m = 1$

Up to an invertible affine transformation, the model can be written as:

$$\begin{pmatrix} dx_t \\ dy_t \end{pmatrix} = \begin{pmatrix} \alpha_{11} (x_t - \theta) \\ \alpha_{21} (x_t - \theta) + \alpha_{22} y_t \end{pmatrix} dt + \begin{pmatrix} x_t & 0 \\ 0 & 1 + a_{22} x_t \end{pmatrix}^{1/2} dW_t,$$

where the parameters are constrained by:

$$\theta \geq 0,\ \ \alpha_{11}\theta < 0,\ \ \alpha_{21} \geq 0,\ \ a_{22} \geq 0.$$

Family $A_2(2)$ **corresponding to** $m = 2$

Up to an invertible affine transformation, the model can be written as:

$$\begin{pmatrix} dx_t \\ dy_t \end{pmatrix} = \begin{pmatrix} \alpha_{11}x_t + \alpha_{12}y_t + \beta_1 \\ \alpha_{21}x_t + \alpha_{22}y_t + \beta_2 \end{pmatrix} dt + \begin{pmatrix} x_t & 0 \\ 0 & y_t \end{pmatrix}^{1/2} dW_t,$$

where the parameters are constrained by:

$$\alpha_{21} \geq 0,\ \ \alpha_{12} \geq 0,\ \ \beta_1 > 0,\ \ \beta_2 > 0,$$

and the components of $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}^{-1} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ are negative.

Appendix 5. Closed-form solution for the parabolic case

In this appendix, we derive the closed-form solution for function A_1 in the parabolic case. The linear differential equation satisfied by function A_1 is of the type:

$$\frac{dA_1(t)}{dt} = f[\exp(\gamma t)] + A_1(t)\, g[\exp(\gamma t)], \tag{A.5.1}$$

with initial condition $A_1(0) = 0$, where $f[\exp(\gamma t)] = \alpha_{01} + \alpha_{21}A_2(t)$ and $g[\exp(\gamma t)] = \alpha_{11} - 2A_2(t)$.

Let us introduce the function A_1^* such that $A_1(t) = A_1^*(z)$, where $z = \exp(\gamma t)$. This function satisfies the linear differential equation:

$$\frac{dA_1^*(z)}{dz} = \frac{f(z)}{\gamma z} + A_1^*(z)\,\frac{g(z)}{\gamma z}, \tag{A.5.2}$$

with initial condition $A_1^*(1) = 0$. The solution of equation (A.5.2) is:

$$A_1^*(z) = \int_1^z \frac{f(u)}{\gamma u} \exp\left(-\int_{\cdot}^{u} \frac{g(v)}{\gamma v} dv\right) du \exp\left(\int_{\cdot}^{z} \frac{g(v)}{\gamma v} dv\right).$$

In our framework, we have:

$$g(z) = \alpha_{11} - \frac{\alpha_{22} + \gamma}{2} + \frac{\gamma}{1 - \frac{\alpha_{22}-\gamma}{\alpha_{22}+\gamma} z},$$

$$\frac{g(z)}{\gamma z} = \frac{1}{z}\left[\frac{1}{\gamma}\left(\alpha_{11} - \frac{\alpha_{22}+\gamma}{2}\right)+1\right] + \frac{\frac{\alpha_{22}-\gamma}{\alpha_{22}+\gamma}}{1-\frac{\alpha_{22}-\gamma}{\alpha_{22}+\gamma}z},$$

$$\exp\left(\int_{\cdot}^{z}\frac{g(v)}{\gamma v}dv\right) = \frac{z^{\frac{1}{\gamma}\left(\alpha_{11}-\frac{\alpha_{22}+\gamma}{2}\right)+1}}{1-\frac{\alpha_{22}-\gamma}{\alpha_{22}+\gamma}z}, \tag{A.5.3}$$

since $1-\frac{\alpha_{22}-\gamma}{\alpha_{22}+\gamma}z > 0$. Since

$$\frac{f(z)}{\gamma z} = \frac{\alpha_{01}+\alpha_{21}\frac{\alpha_{22}+\gamma}{4}}{\gamma z} - \frac{\alpha_{21}}{2}\frac{1}{\left(1-\frac{\alpha_{22}-\gamma}{\alpha_{22}+\gamma}z\right)z},$$

we get:

$$\begin{aligned} I(z) &= \int_{\cdot}^{z}\frac{f(u)}{\gamma u}\exp\left(-\int_{\cdot}^{u}\frac{g(v)}{\gamma v}dv\right)du \\ &= \left(\frac{\alpha_{01}+\alpha_{21}\frac{\alpha_{22}+\gamma}{4}}{\gamma} - \frac{\alpha_{21}}{2}\right)\frac{z^{-\frac{1}{\gamma}\left(\alpha_{11}-\frac{\alpha_{22}+\gamma}{2}\right)-1}}{-\frac{1}{\gamma}\left(\alpha_{11}-\frac{\alpha_{22}+\gamma}{2}\right)-1} \\ &\quad -\frac{\alpha_{22}-\gamma}{\alpha_{22}+\gamma}\frac{\alpha_{01}+\alpha_{21}\frac{\alpha_{22}+\gamma}{4}}{\gamma}\frac{z^{-\frac{1}{\gamma}\left(\alpha_{11}-\frac{\alpha_{22}+\gamma}{2}\right)}}{-\frac{1}{\gamma}\left(\alpha_{11}-\frac{\alpha_{22}+\gamma}{2}\right)}, \end{aligned} \tag{A.5.4}$$

and

$$A_1^*(z) = [I(z) - I(1)]\exp\left(\int_{\cdot}^{z}\frac{g(v)}{\gamma v}dv\right). \tag{A.5.5}$$

Substituting results (A.5.3) and (A.5.4) into equation (A.5.5), we obtain the solution for function A_1:

$$A_1(t) = \frac{d_2}{d_0\left(\delta+\frac{\gamma}{2}\right)} + \frac{1}{1+d_0\exp(\gamma t)}\left[\frac{d_1}{\delta-\frac{\gamma}{2}} - \frac{d_2}{d_0\left(\delta+\frac{\gamma}{2}\right)} - \left(\frac{d_1}{\delta-\frac{\gamma}{2}}+\frac{d_2}{\delta+\frac{\gamma}{2}}\right)\exp\left(\left(\frac{\gamma}{2}-\delta\right)t\right)\right],$$

where:

$$\begin{aligned} \delta &= \frac{\alpha_{22}}{2} - \alpha_{11}, \\ d_0 &= -\frac{\alpha_{22}-\gamma}{\alpha_{22}+\gamma}, \\ d_1 &= \alpha_{01} + \alpha_{21}\frac{\alpha_{22}-\gamma}{4}, \\ d_2 &= d_0\left(\alpha_{01}+\alpha_{21}\frac{\alpha_{22}+\gamma}{4}\right). \end{aligned}$$

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