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Exit from a Neighborhood of Zero for Weakly Damped Stochastic Nonlinear Schrödinger Equations

E. GAUTIER [1]

[1] CREST-INSEE, URA D2200 3 avenue Pierre Larousse, 92240 Malakoff, France et IRMAR, Ecole Normale Supérieure de Cachan, antenne de Bretagne, Campus de Ker Lann, avenue R. Schuman, 35170 Bruz, France.

Exit from a neighborhood of zero for weakly damped stochastic nonlinear Schrödinger equations

Eric GAUTIER[1,2]

Abstract: Exit from a neighborhood of zero for weakly damped stochastic nonlinear Schrödinger equations is studied. The small noise is either complex and of additive type or real and of multiplicative type. It is white in time and colored in space. The neighborhood is either in L^2 or in H^1. The small noise asymptotic of both the first exit times and the exit points are characterized.

Résumé: Nous étudions la sortie d'un voisinage de zéro pour des équations de Schrödinger non linéaires stochastiques faiblement amorties. Le petit bruit est complexe et additif ou réel et multiplicatif. Il est blanc en temps et coloré en espace. Le voisinage est un voisinage dans L^2 ou dans H^1. Nous caractérisons l'asymptotique des temps et points de sortie.

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[1]CREST-INSEE, URA D2200, 3 avenue Pierre Larousse, 92240 Malakoff, France

[2]IRMAR, Ecole Normale Supérieure de Cachan, antenne de Bretagne, Campus de Ker Lann, avenue R. Schuman, 35170 Bruz, France

1 Introduction

The study of the first exit time from a neighborhood of an asymptotically stable equilibrium point, the exit place determination or the transition between two equilibrium points in randomly perturbed dynamical systems is important in several areas of physics among which statistical and quantum mechanics, the natural sciences, financial macro economics... The problem is relevant in nonlinear optics; see for example [20]. We shall consider the case of weakly damped nonlinear Schrödinger equations. It is a model in nonlinear optics, hydrodynamics, biology, field theory, Fermi-Pasta-Ulam chains of atoms...

For a fixed noise amplitude and for diffusions, the first exit time and the distribution of the exit points on the boundary of the domain can be characterized respectively by the Dirichlet and Poisson equations. However, when the dimension is larger than one, we may seldom solve explicitly these equations and large deviation techniques are precious tools when the noise is assumed to be small; see for example [12, 15]. The techniques used in the physics literature is often called optimal fluctuations or instanton formalism and are closely related to the large deviations.

An energy then characterizes the transition between two states and the exit from a neighborhood of an asymptotically stable equilibrium point. The energy is derived from the rate function of the sample path large deviation principle (LDP). The paths that minimize this energy are the most likely exiting paths or transitions and when the infimum is unique the system shows an almost deterministic behavior. Note that the first order of the probability are that of the Boltzman theory and the amplitude of the small noise acts as the temperature. The deterministic dynamics is sometimes interpreted as the evolution at temperature 0 and the small noise as the small temperature nonequilibrium case. In the pioneering article [13], a nonlinear heat equation perturbed by a small noise of additive type is considered. Transitions in that case are the instantons of quantum mechanics. Also in [22], predictions for a noisy exit problem are confirmed both numerically and experimentally.

We will consider weakly damped nonlinear Schrödinger equations in $\mathbb{R}^d$. Equations are perturbed by a small noise. The noise is white in time and of additive or multiplicative type. We define it as the time derivative in the sense of distributions of a Hilbert space-valued Wiener process W. The two types of noises are physically relevant; see for example [9]. When the noise is of additive type, the Hilbert space is L^2 or H^1, spaces of complex valued functions. The evolution equation is then

$$i\mathrm{d}u^{\epsilon,u_0} = (\Delta u^{\epsilon,u_0} + \lambda|u^{\epsilon,u_0}|^{2\sigma}u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0})\mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W, \qquad (1.1)$$

where α and ϵ are positive and u_0 is an initial datum in L^2 (respectively H^1). When the noise is of multiplicative type, the Hilbert space is the Sobolev space based on L^2 of real valued functions $\mathrm{H}^s_{\mathbb{R}}$ for $s > \frac{d}{2}+1$ and the product is a Stratonovich product. In that case the equation may be written

$$i\mathrm{d}u^{\epsilon,u_0} = (\Delta u^{\epsilon,u_0} + \lambda|u^{\epsilon,u_0}|^{2\sigma}u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0})\mathrm{d}t + \sqrt{\epsilon}u^{\epsilon,u_0} \circ \mathrm{d}W. \qquad (1.2)$$

The Wiener process W is always assumed to be colored in space since the linear group does not have global regularizing properties and is an isometry on the Sobolev spaces based on L^2. The power σ in the nonlinearity satisfies $\sigma < \frac{2}{d}$ and thus solutions do not exhibit blow-up.

In [17] and [18] we have proved sample paths LDPs for the two types of noises but without damping and deduced the asymptotic of the tails of the blow-up times. In [17] we also deduced the tails of the mass, defined later, of the pulse at the end of a fiber optical line. We have thus evaluated the error probabilities in optical soliton transmission when the receiver records the signal on an infinite time interval. In [9] we have applied the LDP to the problem of the diffusion in position of the soliton and studied the tails of the random position. Our results are in perfect agreement with results from physics obtained via heuristic arguments. The damping term in the drift here is often physically relevant but small and neglected in the models. For example in [9], in the case of an additive noise, we have considered that the gain of the amplifiers is adjusted to compensate exactly for loss and that there remains only a spontaneous emission noise.

The flow in the equations above has Hamiltonian, gradient and random components. The mass

$$\mathbf{N}(u) = \int_{\mathbb{R}^d} |u|^2 \, dx$$

characterizes the gradient component. The Hamiltonian denoted by $\mathbf{H}(u)$, defined for functions in H^1, has a kinetic and a potential term, it may be written

$$\mathbf{H}(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \frac{\lambda}{2\sigma+2}\int_{\mathbb{R}^d} |u|^{2\sigma+2} \, dx.$$

Note that the vector fields associated to the mass and Hamiltonian are orthogonal. Recall that the mass and Hamiltonian are invariant quantities of the equation without noise and damping. Other quantities like the linear or angular momentum are also invariant for nonlinear Schrödinger equations. We could rewrite, for example equation (1.1) as

$$\mathrm{d}u^{\epsilon,u_0} = \left(\frac{\delta\mathbf{H}\left(u^{\epsilon,u_0}\right)}{\delta\overline{u^{\epsilon,u_0}}} - \frac{\alpha}{2}\frac{\delta\mathbf{N}\left(u^{\epsilon,u_0}\right)}{\delta u^{\epsilon,u_0}}\right)\mathrm{d}t - i\sqrt{\epsilon}\mathrm{d}W.$$

Without noise solutions are uniformly attracted to zero in L^2 and in H^1. We will prove that because of noise, the behavior is completely different. Though for finite times the probabilities of large excursions off neighborhoods of zero go to zero exponentially fast with ϵ, if we wait long enough, the time scale is exponential, such large fluctuations occur and exit from a neighborhood of zero takes place. In the L^2 case, we only consider noises of additive type where, because of noise, mass is injected or pumped randomly in the system. It would also be possible to treat rather general multiplicative noises as long as noise allows injection of mass. In H^1 we consider the two types of noises.

We use large deviation techniques to prove the corresponding result in our infinite dimensional setting. In [14], the case of a space variable in a unidimensional torus is treated for a particular SPDE and the regularizing property of the Heat semi-group is a central tool. Let us stress that the Schrödinger linear group is an isometry on every Sobolev space based on L^2. In [2], the neighborhood is defined for a strong topology of β-Hölder functions and is relatively compact for a weaker topology, the space variable is again in a bounded subset of $\mathbb{R}^d$. Note also that one particular difficulty in infinite dimensions, along with compactness, is that the linear group is strongly and not uniformly continuous. In this article the neighborhood is not relatively compact, we work on the all space $\mathbb{R}^d$, the nonlinearity is locally Lipschitz only in H^1 for $d = 1$.

However, there remain difficult problems from the control of nonlinear PDEs to prove for example that the upper and lower bounds on the exit time are equal. Also, it seems formally that, in the case of a noise of additive type which is white in time and in space, the escape off levels of the Hamiltonian less than a constant is intimately related to the solitary waves. We will not address these last issues in the present article.

2 Preliminaries

Throughout the paper the following notations will be used.

The set of positive integers and positive real numbers are denoted by $\mathbb{N}^*$ and $\mathbb{R}^*_+$. For $p \in \mathbb{N}^*$, L^p is the Lebesgue space of complex valued functions.

For k in $\mathbb{N}^*$, $\mathrm{W}^{k,p}$ is the Sobolev space of L^p functions with partial derivatives up to level k, in the sense of distributions, in L^p. For $p=2$ and s in $\mathbb{R}_+^*$, H^s is the Sobolev space of tempered distributions v of Fourier transform $\hat{v}$ such that $(1+|\xi|^2)^{s/2}\hat{v}$ belongs to L^2. We denote the spaces by $\mathrm{L}^p_{\mathbb{R}}$, $\mathrm{W}^{k,p}_{\mathbb{R}}$ and $\mathrm{H}^s_{\mathbb{R}}$ when the functions are real-valued. The space L^2 is endowed with the inner product $(u,v)_{\mathrm{L}2} = \mathfrak{Re}\int_{\mathbb{R}^d} u(x)\overline{v}(x)dx$. If I is an interval of $\mathbb{R}$, $(E, \|\cdot\|_E)$ a Banach space and r belongs to $[1,\infty]$, then $\mathrm{L}^r(I;E)$ is the space of strongly Lebesgue measurable functions f from I into E such that $t \to \|f(t)\|_E$ is in $\mathrm{L}^r(I)$.

The space of linear continuous operators from B into $\tilde{B}$, where B and $\tilde{B}$ are Banach spaces is $\mathcal{L}_c\left(B,\tilde{B}\right)$. When $B=H$ and $\tilde{B}=\tilde{H}$ are Hilbert spaces, such an operator is Hilbert-Schmidt when $\sum_{j\in\mathbb{N}} \|\Phi e_j^H\|_{\tilde{H}}^2 < \infty$ for every $(e_j)_{j\in\mathbb{N}}$ complete orthonormal system of H. The set of such operators is denoted by $\mathcal{L}_2(H,\tilde{H})$, or $\mathcal{L}_2^{s,r}$ when $H=\mathrm{H}^s$ and $\tilde{H}=\mathrm{H}^r$. When $H=\mathrm{H}^s_{\mathbb{R}}$ and $\tilde{H}=\mathrm{H}^r_{\mathbb{R}}$, we denote it by $\mathcal{L}_{2,\mathbb{R}}^{s,r}$. When $s=0$ or $r=0$ the Hilbert space is L^2 or $\mathrm{L}^2_{\mathbb{R}}$.

We also denote by B_ρ^0 and S_ρ^0 respectively the open ball and the sphere centered at 0 of radius ρ in L^2. We denote these by B_ρ^1 and S_ρ^1 in H^1. We will denote by $\mathcal{N}^0(A,\rho)$ the $\rho-$neighborhood of a set A in L^2 and $\mathcal{N}^1(A,\rho)$ the neighborhood in H^1. In the following we impose that compact sets satisfy the Hausdorff property.

We will use in Lemma 3.5 the integrability of the Schrödinger linear group which is related to the dispersive property. Recall that $(r(p),p)$ is an admissible pair if p is such that $2\le p<\frac{2d}{d-2}$ when $d>2$ ($2\le p<\infty$ when $d=2$ and $2\le p\le\infty$ when $d=1$) and $r(p)$ satisfies $\frac{2}{r(p)} = d\left(\frac{1}{2}-\frac{1}{p}\right)$.
For every $(r(p),p)$ admissible pair and T positive, we define the Banach spaces

$$Y^{(T,p)} = \mathrm{C}\left([0,T];\mathrm{L}^2\right)\cap \mathrm{L}^{r(p)}\left(0,T;\mathrm{L}^p\right),$$

and

$$X^{(T,p)} = \mathrm{C}\left([0,T];\mathrm{H}^1\right)\cap \mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}\right),$$

where the norms are the maximum of the norms in the two intersected Banach spaces. The Schrödinger linear group is denoted by $(U(t))_{t\ge0}$; it is defined on L^2 or on H^1. Let us recall the Strichartz inequalities, see [1],

(i) There exists C positive such that for u_0 in L^2, T positive and $(r(p),p)$ admissible pair,

$$\|U(t)u_0\|_{Y^{(T,p)}} \le C\,\|u_0\|_{\mathrm{L}^2},$$

(ii) For every T positive, $(r(p),p)$ and $(r(q),q)$ admissible pairs, s and ρ such that $\frac{1}{s}+\frac{1}{r(q)}=1$ and $\frac{1}{\rho}+\frac{1}{q}=1$, there exists C positive such that for f in $\mathrm{L}^s(0,T;\mathrm{L}^\rho)$,

$$\left\|\int_0^\cdot U(\cdot-s)f(s)ds\right\|_{Y^{(T,p)}} \le C\|f\|_{\mathrm{L}^s(0,T;\mathrm{L}^\rho)}.$$

Similar inequalities hold when the group is acting on H^1, replacing L^2 by H^1, $Y^{(T,p)}$ by $X^{(T,p)}$ and $\mathrm{L}^s(0,T;\mathrm{L}^\rho)$ by $\mathrm{L}^s\left(0,T;\mathrm{W}^{1,\rho}\right)$.

It is known that, in the Hilbert space setting, only direct images of uncorrelated space wise Wiener processes by Hilbert-Schmidt operators are well defined. However, when the semi-group has regularizing properties, the semi-group may act as a Hilbert-Schmidt operator and a white in space noise may be considered. It is not possible here since the Schrödinger group is an isometry on the Sobolev spaces based on L^2. The Wiener process W is thus defined as ΦW_c, where W_c is a cylindrical Wiener process on L^2 and Φ is Hilbert-Schmidt. Then $\Phi\Phi^*$ is the correlation operator of $W(1)$, it has finite trace.

We consider the following Cauchy problems

$$\begin{cases} i\mathrm{d}u^{\epsilon,u_0} &= (\Delta u^{\epsilon,u_0}+\lambda|u^{\epsilon,u_0}|^{2\sigma}u^{\epsilon,u_0}-i\alpha u^{\epsilon,u_0})\mathrm{d}t+\sqrt{\epsilon}\mathrm{d}W, \\ u^{\epsilon,u_0}(0) &= u_0 \end{cases} \tag{2.1}$$

with u_0 in L^2 and Φ in $\mathcal{L}_2^{0,0}$ or u_0 in H^1 and Φ in $\mathcal{L}_2^{0,1}$, and

$$\begin{cases} i\mathrm{d}u^{\epsilon,u_0} &= (\Delta u^{\epsilon,u_0}+\lambda|u^{\epsilon,u_0}|^{2\sigma}u^{\epsilon,u_0}-i\alpha u^{\epsilon,u_0})\mathrm{d}t+\sqrt{\epsilon}u^{\epsilon,u_0}\circ\mathrm{d}W, \\ u^{\epsilon,u_0}(0) &= u_0 \end{cases} \tag{2.2}$$

with u_0 in H^1 and Φ in $\mathcal{L}_{2,\mathbb{R}}^{0,s}$ where $s>\frac{d}{2}+1$. When the noise is of multiplicative type, we may write the equation in terms of a Itô product,

$$i\mathrm{d}u^{\epsilon,u_0}=(\Delta u^{\epsilon,u_0}+\lambda|u^{\epsilon,u_0}|^{2\sigma}u^{\epsilon,u_0}-i\alpha u^{\epsilon,u_0}-\frac{i\epsilon}{2}u^{\epsilon,u_0}F_\Phi)\mathrm{d}t+\sqrt{\epsilon}u^{\epsilon,u_0}\mathrm{d}W,$$

where $F_\Phi(x)=\sum_{j\in\mathbb{N}}\left(\Phi e_j(x)\right)^2$ for x in $\mathbb{R}^d$ and $(e_j)_{j\in\mathbb{N}}$ a complete orthonormal system of L^2. We consider mild solutions; for example the mild solutions of (2.1) satisfies

$$\begin{aligned} u^{\epsilon,u_0}(t)= \;& U(t)u_0-i\lambda\int_0^t U(t-s)(|u^{\epsilon,u_0}(s)|^{2\sigma}u^{\epsilon,u_0}(s)-i\alpha u^{\epsilon,u_0}(s))ds \\ & -i\sqrt{\epsilon}\int_0^t U(t-s)dW(s), \quad t>0. \end{aligned}$$

The Cauchy problems are globally well posed in L^2 and H^1 with the same arguments as in [6].

The main tools in this article are the sample paths LDPs for the solutions of the three Cauchy problems. They are uniform in the initial data. Unlike in [9, 17, 18] we use a Freidlin-Wentzell type formulation of the upper and lower bounds of the LDPs. Indeed it seems that the restriction that initial data be in compact sets in [18] is a real limitation in particular for stochastic NLS equations. Indeed the linear Schrödinger group is not compact due to the lack of smoothing effect and to the fact that we work on the whole space $\mathbb{R}^d$. This limitation disappears when we work with the Freidlin-Wentzell type formulation; we may now obtain bounds for initial data in balls of L^2 (respectively H^1) for ϵ small enough. Note that it is well known that in metric spaces and for non uniform LDPs the two formulations are equivalent. A proof will be given and we will stress, in the multiplicative case, on the slight differences with the proof of the result in [18].

We denote by $\mathbf{S}(u_0, h)$ the skeleton of equation (2.1) or (2.2), *i.e.* the mild solution of the controlled equation

$$\left\{ \begin{array}{l} i\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \alpha u\right) = \Delta u + \lambda |u|^{2\sigma} u + \Phi h, \\ u(0) = u_0 \end{array} \right.$$

where u_0 belongs to L^2 or H^1 in the additive case and the mild solution of

$$\left\{ \begin{array}{l} i\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \alpha u\right) = \Delta u + \lambda |u|^{2\sigma} u + u\Phi h, \\ u(0) = u_0 \end{array} \right.$$

where u_0 belongs to H^1 in the multiplicative case.
The rate functions of the LDPs are always defined as

$$I_T^{u_0}(w) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{S}(u_0,h)=w} \int_0^T \|h(s)\|_{\mathrm{L}^2}^2 ds.$$

We denote for T and a positive by $K_T^{u_0}(a) = \left(I_T^{u_0}\right)^{-1}([0,a])$ the levels of the rate function less or equal to a

$$K_T^{u_0}(a) = \left\{ w \in \mathrm{C}\left([0,T];\mathrm{L}^2\right) :\ w = \mathbf{S}(u_0,h),\ \frac{1}{2}\int_0^T \|h(s)\|_{\mathrm{L}^2}^2 ds \leq a \right\}.$$

We also denote by $d_{\mathrm{C}([0,T];\mathrm{L}^2)}$ the usual distance between sets of $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ and by $d_{\mathrm{C}([0,T];\mathrm{H}^1)}$ the distance between sets of $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$.

We also denote by $\tilde{\mathbf{S}}(u_0, f)$ the skeleton of equation (2.2) where we replace Φh by $\frac{\partial f}{\partial t}$ where f belongs to $\mathrm{H}_0^1\left(0,T;\mathrm{H}_{\mathbb{R}}^s\right)$, the subspace of $\mathrm{C}\left([0,T];\mathrm{H}_{\mathbb{R}}^s\right)$

of square integrable in time and with square integrable in time time derivative functions, null at $t = 0$. Also C_a denotes the set

$$\left\{ f \in \mathrm{H}^1_0\left(0, T; \mathrm{H}^s_{\mathbb{R}}\right) : \frac{\partial f}{\partial t} \in \mathrm{im}\Phi,\ I^W_T(f) = \frac{1}{2}\left\| \Phi^{-1}_{|(\ker\Phi)^{\perp}} \frac{\partial f}{\partial t} \right\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \le a \right\}$$

and $\mathcal{A}(d)$ the set $[2, \infty)$ when $d = 1$ or $d = 2$ and $\left[2, \frac{2(3d-1)}{3(d-1)}\right)$ when $d \ge 3$. The above I^W_T is the good rate function of the LDP for the Wiener process. The uniform LDP with the Freidlin-Wentzell formulation that we will need in the remaining is then as follows. In the additive case we consider the L^2 and H^1 case while in the multiplicative case we only consider the H^1 case because we will not need a L^2 result. Indeed in the case of the multiplicative noise the L^2 norm remains invariant.

Theorem 2.1 *In the additive case and in* L^2 *we have:*
for every a, ρ, T, δ *and* γ *positive,*

(i) *there exists* ϵ_0 *positive such that for every* ϵ *in* $(0, \epsilon_0)$, u_0 *such that* $\|u_0\|_{\mathrm{L}^2} \le \rho$ *and* $\tilde{a}$ *in* $(0, a]$,

$$\mathbb{P}\left(d_{\mathrm{C}([0,T];\mathrm{L}^2)}\left(u^{\epsilon,u_0}, K^{u_0}_T(\tilde{a})\right) \ge \delta\right) < \exp\left(-\frac{\tilde{a} - \gamma}{\epsilon}\right),$$

(ii) *there exists* ϵ_0 *positive such that for every* ϵ *in* $(0, \epsilon_0)$, u_0 *such that* $\|u_0\|_{\mathrm{L}^2} \le \rho$ *and* w *in* $K^{u_0}_T(a)$,

$$\mathbb{P}\left(\|u^{\epsilon,u_0} - w\|_{\mathrm{C}([0,T];\mathrm{L}^2)} < \delta\right) > \exp\left(-\frac{I^{u_0}_T(w) + \gamma}{\epsilon}\right).$$

In H^1*, the result holds for additive and multiplicative noises replacing in the above* $\|u_0\|_{\mathrm{L}^2}$ *by* $\|u_0\|_{\mathrm{H}^1}$ *and* $\mathrm{C}\left([0, T]; \mathrm{L}^2\right)$ *by* $\mathrm{C}\left([0, T]; \mathrm{H}^1\right)$.

Note that the extra condition
For every a *positive and* K *compact in* L^2*, the set* $K^K_T(a) = \bigcup_{u_0 \in K} K^{u_0}_T(a)$ *is a compact subset of* $\mathrm{C}\left([0, T]; \mathrm{L}^2\right)$
often appears to be part of a uniform LDP. It will not be used in the following. The proof of this result is given in the annex.

3 Exit from a domain of attraction in L^2

In this section we only consider the case of an additive noise. Recall that for the real multiplicative noise the mass is decreasing and thus exit is impossible.

We may easily check that the mass $\mathbf{N}(\mathbf{S}(u_0,0))$ of the solution of the deterministic equation satisfies

$$\mathbf{N}(\mathbf{S}(u_0,0)(t)) = \mathbf{N}(u_0)\exp(-2\alpha t). \tag{3.1}$$

With noise though, the mass fluctuates around the deterministic decay. Recall how the Itô formula applies to the fluctuation of the mass, see [6] for a proof,

$$\begin{array}{rl} \mathbf{N}(u^{\epsilon,u_0}(t)) - \mathbf{N}(u_0) = & -2\sqrt{\epsilon}\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \overline{u}^{\epsilon,u_0} dW dx \\ & -2\alpha \|u^{\epsilon,u_0}\|^2_{\mathrm{L}^2(0,t;\mathrm{L}^2)} + \epsilon t \|\Phi\|^2_{\mathcal{L}_2^{0,0}}. \end{array} \tag{3.2}$$

Assume that D is a bounded measurable subset of L^2 containing 0 in its interior and invariant by the deterministic flow, *i.e.*

$$\forall u_0 \in D,\ \forall t \geq 0,\ \mathbf{S}(u_0,0)(t) \in D;$$

it may be an open ball. There exists R positive such that $D \subset B_R$.
We define by

$$\tau^{\epsilon,u_0} = \inf\{t \geq 0 :\ u^{\epsilon,u_0}(t) \in D^c\}$$

the first exit time of the process u^{ϵ,u_0} off the domain D.

An easy information on the exit time is obtained as follows. The expectation of an integration via the Duhamel formula of the Itô decomposition, the process u^{ϵ,u_0} being stopped at the first exit time, gives $\mathbb{E}[\exp(-2\alpha\tau^{\epsilon,u_0})] = 1 - \frac{2\alpha R}{\epsilon\|\Phi\|^2_{\mathcal{L}_2^{0,0}}}$. Without damping we obtain $\mathbb{E}[\tau^{\epsilon,u_0}] = \frac{R}{\epsilon\|\Phi\|^2_{\mathcal{L}_2^{0,0}}}$. To get more precise information for small noises we use LDP techniques.

Let us introduce

$$\overline{e} = \inf\left\{I_T^0(w) :\ w(T) \in \overline{D}^c,\ T > 0\right\}.$$

When ρ is positive and small enough, we set

$$e_\rho = \inf\left\{I_T^{u_0}(w) : \|u_0\|_{\mathrm{L}^2} \leq \rho,\ w(T) \in (D_{-\rho})^c,\ T > 0\right\},$$

where $D_{-\rho} = D \setminus \mathcal{N}^0(\partial D, \rho)$ and ∂D is the the boundary of ∂D in L^2. We define then

$$\underline{e} = \lim_{\rho \to 0} e_\rho.$$

We shall denote in this section by $\|\Phi\|_c$ the norm of Φ as a bounded operator on L^2. Let us start with the following lemma.

Lemma 3.1 *$0 < \underline{e} \leq \overline{e}$.*

Proof. It is clear that $\underline{e} \leq \overline{e}$. Let us check that $\underline{e} > 0$. Let d denote the positive distance between 0 and ∂D. Take ρ small such that the distance between B_ρ^0 and $(D_{-\rho})^c$ is larger than $\frac{d}{2}$. Multiplying the evolution equation by $-i\overline{\mathbf{S}(u_0,h)}$, taking the real part, integrating over space and using the Duhamel formula we obtain

$$\begin{array}{l} \mathbf{N}\left(\mathbf{S}(u_0,h)(T)\right) - \exp\left(-2\alpha T\right)\mathbf{N}\left(u_0\right) \\ = 2\int_0^T \exp\left(-2\alpha(T-s)\right)\mathfrak{Im}\left(\int_{\mathbb{R}^d}\overline{\mathbf{S}\left(u_0,h\right)}\Phi h dx ds\right). \end{array}$$

If $\mathbf{S}(u_0,h)(T) \in (D_{-\rho})^c$ and correspond to the first escape off D then

$$\begin{array}{ll} \frac{d}{2} & \leq 2\|\Phi\|_c \int_0^T \exp\left(-2\alpha(T-s)\right)\|\mathbf{S}(u_0,h)(s)\|_{\mathrm{L}^2}\|h(s)\|_{\mathrm{L}^2} ds \\ & \leq 2R\|\Phi\|_c\left(\int_0^T \exp\left(-4\alpha(T-s)\right)ds\right)^{\frac{1}{2}}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}, \end{array}$$

thus

$$\frac{\alpha d^2}{8R^2\|\Phi\|_c^2} \leq \frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)},$$

and the result follows. □

Note that we would expect $\underline{e}$ and $\overline{e}$ to be equal. We should prove that, for a fixed level of energy, we may find ρ arbitrarily small and a control of energy less than the fixed level such that the controlled solution goes from 0 to u_0 in B_ρ^0 in finite time. We should also find a second control of energy smaller than the fixed level such that the controlled solution goes from $\partial D_{-\rho}$ to $\overline{D}^c$ in finite time. Note that control arguments for nonlinear Schrödinger equations where the control enters the equation as an external force or potential are used in [7, 8] in the study of the blow-up time for stochastic nonlinear Schrödinger equations. Here it seems more intricate and the arguments of [7, 8] do not seem to apply. If these two bounds were indeed equal, they would also correspond to

$$\begin{array}{rl} \mathcal{E}(D) = & \frac{1}{2}\inf\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} :\ \exists T>0 :\ \mathbf{S}(0,h)(T)\in\partial D\right\} \\ = & \inf_{v\in\partial D} V(0,v) \end{array}$$

where the quasi-potential is defined as

$$V(u_0,u_f) = \inf\left\{I_T^0(w) :\ w\in \mathrm{C}\left(\mathbb{R}^+;\mathrm{L}^2\right),\ w(0)=u_0,\ w(T)=u_f,\ T>0\right\}.$$

We shall prove in this section the two following results. The first theorem characterizes the first exit time from the domain.

Theorem 3.2 *For every u_0 in D and δ positive, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\tau^{\epsilon,u_0}\notin\left(\exp\left(\frac{\underline{e}-\delta}{\epsilon}\right),\exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right)\right)\le -L, \qquad (3.3)$$

and for every u_0 in D,

$$\underline{e}\le\underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\le\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\le\overline{e}. \qquad (3.4)$$

Moreover, for every δ positive, there exists L positive such that

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\sup_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,u_0}\ge\exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right)\le -L, \qquad (3.5)$$

and

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\sup_{u_0\in D}\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\le\overline{e}. \qquad (3.6)$$

The second theorem characterizes formally the exit points. We shall define for ρ positive small enough, N a closed subset of ∂D

$$e_{N,\rho}=\inf\left\{I_T^{u_0}(w):\|u_0\|_{\mathrm{L}^2}\le\rho,\ w(T)\in\left(D\setminus\mathcal{N}^0\left(N,\rho\right)\right)^c,\ T>0\right\}.$$

We then define

$$\underline{e}_N=\lim_{\rho\to 0}e_{N,\rho}.$$

Note that $e_\rho\le e_{N,\rho}$ and thus $\underline{e}\le\underline{e}_N$.

Theorem 3.3 *If $\underline{e}_N>\overline{e}$, then for every u_0 in D, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)\le -L.$$

Thus the probability of an escape off D via points of N such that $e_\rho\le e_{N,\rho}$ goes to zero exponentially fast with ϵ. Suppose that we were able to solve the previous control problem, then as noise goes to zero, the probability of an exit via closed subsets of ∂D where the quasi-potential is not minimal goes to zero. As the expected exit time is finite, an exit occurs almost surely. It is exponentially more likely that it occurs via infima of the quasi-potential. When there are several infima the exit measure is a probability measure on ∂D. When there exists only one infimum we may state the following corollary.

Corollary 3.4 *Assume that v^* in ∂D is such that for every δ positive and $N = \{v \in \partial D : \ \|v - v^*\|_{L^2} \geq \delta\}$ we have $\underline{e}_N > \overline{e}$ then*

$$\forall \delta > 0, \ \forall u_0 \in D, \ \exists L > 0 : \ \overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\|u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right) - v^*\|_{L^2} \geq \delta\right) \leq -L.$$

We need to prove a few lemmas before proving the two theorems.
Let us define

$$\sigma_\rho^{\epsilon,u_0} = \inf\left\{t \geq 0 : u^{\epsilon,u_0}(t) \in B_\rho^0 \cup D^c\right\},$$

where $B_\rho^0 \subset D$.

Lemma 3.5 *For every ρ and L positive with $B_\rho^0 \subset D$, there exists T and ϵ_0 positive such that for every u_0 in D and ϵ in $(0, \epsilon_0)$,*

$$\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T\right) \leq \exp\left(-\frac{L}{\epsilon}\right).$$

Proof. The result is straightforward if u_0 belongs to B_ρ^0. Suppose now that u_0 belongs to $D \setminus B_\rho^0$. From equation (3.1), the bounded subsets of L^2 are uniformly attracted to zero by the flow of the deterministic equation. Thus there exists a positive time T_1 such that for every u_1 in the $\frac{\rho}{8}-$neighborhood of $D \setminus B_\rho^0$ and $t \geq T_1$, $\mathbf{S}\left(u_1, 0\right)(t) \in B_{\frac{\rho}{8}}^0$. We shall choose $\rho < 8$ and follow three steps.

Step 1: Let us first recall why there exists $M' = M'(T_1, R, \sigma, \alpha)$ such that

$$\sup_{u_1 \in \mathcal{N}^0\left(D \setminus B_\rho^0, \frac{\rho}{8}\right)} \|\mathbf{S}(u_1, 0)\|_{Y(T_1, 2\sigma+2)} \leq M'. \tag{3.7}$$

From the Strichartz inequalities, there exists C positive such that

$$\begin{aligned}\|\mathbf{S}(u_1, 0)\|_{Y(t,2\sigma+2)} \leq \ & C\|u_1\|_{L^2} + C\left\||\mathbf{S}(u_1, 0)|^{2\sigma+1}\right\|_{L^{\gamma'}(0,t;L^{s'})} \\ & + C\alpha\|\mathbf{S}\left(u_1, 0\right)\|_{L^1(0,t;L^2)}\end{aligned}$$

where γ' and s' are such that $\frac{1}{\gamma'} + \frac{1}{r(\tilde{p})} = 1$ and $\frac{1}{s'} + \frac{1}{\tilde{p}} = 1$ and $(r(\tilde{p}), \tilde{p})$ is an admissible pair. Note that the first term is smaller than $C(R+1)$. From the Hölder inequality, setting

$$\frac{2\sigma}{2\sigma+2} + \frac{1}{2\sigma+2} = \frac{1}{s'}, \quad \frac{2\sigma}{\omega} + \frac{1}{r(2\sigma+2)} = \frac{1}{\gamma'},$$

we can write

$$\left\||\mathbf{S}(u_1, 0)|^{2\sigma+1}\right\|_{L^{\gamma'}(0,t;L^{s'})} \leq C\|\mathbf{S}(u_1, 0)\|_{L^{r(2\sigma+2)}(0,t;L^{2\sigma+2})}\|\mathbf{S}(u_1, 0)\|_{L^\omega(0,t;L^{2\sigma+2})}^{2\sigma}.$$

It is easy to check that since $\sigma < \frac{2}{d}$, we have $\omega < r(2\sigma+2)$. Thus it follows that

$$\|\mathbf{S}(u_1,0)\|_{Y(t,2\sigma+2)} \le C(R+1)+Ct^{\frac{\omega r(2\sigma+2)}{r(2\sigma+2)-\omega}}\|\mathbf{S}(u_1,0)\|^{2\sigma+1}_{Y(t,2\sigma+2)}+C\alpha\sqrt{t}\,\|\mathbf{S}(u_1,0)\|_{Y(t,2\sigma+2)}\,.$$

The function $x \mapsto C(R+1)+Ct^{\frac{\omega r(2\sigma+2)}{r(2\sigma+2)-\omega}}x^{2\sigma+1}+C\alpha\sqrt{t}x-x$ is positive on a neighborhood of zero. For $t_0=t_0(R,\sigma,\alpha)$ small enough, the function has at least one zero. Also, the function goes to ∞ as x goes to ∞. Thus, denoting by $M(R,\sigma)$ the first zero of the above function, we obtain by a classical argument that $\|\mathbf{S}(u_1,0)\|_{Y(t_0,2\sigma+2)} \le M(R,\sigma)$ for every u_1 in $\mathcal{N}^0\left(D\setminus B^0_\rho,\frac{\rho}{8}\right)$. Also, as for every t in $[0,T]$, $\mathbf{S}(u_1,0)(t)$ belongs to $\mathcal{N}^0\left(D\setminus B^0_\rho,\frac{\rho}{8}\right)$, repeating the previous argument, u_1 is replaced by $\mathbf{S}(u_1,0)(t_0)$ and so on, we obtain

$$\sup_{u_1\in\mathcal{N}^0\left(D\setminus B^0_\rho,\frac{\rho}{8}\right)}\|\mathbf{S}(u_1,0)\|_{Y(T_1,p)} \le M',$$

where $M'=\left\lceil\frac{T_1}{t_0}\right\rceil M$ proving (3.7).

Step2: Let us now prove that for T large enough, to be defined later, and larger than T_1, we have

$$\mathcal{T}_\rho=\left\{w\in \mathrm{C}\left([0,T];\mathrm{L}^2\right):\ \forall t\in[0,T],\ w(t)\in\mathcal{N}^0\left(D\setminus B^0_\rho,\frac{\rho}{8}\right)\right\}\subset K_T^{u_0}(2L)^c. \tag{3.8}$$

Since $K_T^{u_0}(2L)$ is included in the image of $\mathbf{S}(u_0,\cdot)$ it suffices to consider w in $\mathcal{T}_\rho$ such that $w=\mathbf{S}(u_0,h)$ for some h in $\mathrm{L}^2(0,T;\mathrm{L}^2)$. Take h such that $\mathbf{S}(u_0,h)$ belongs to $\mathcal{T}_\rho$ we have

$$\|\mathbf{S}(u_0,h)-\mathbf{S}(u_0,0)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)} \ge \|\mathbf{S}(u_0,h)(T_1)-\mathbf{S}(u_0,0)(T_1)\|_{\mathrm{L}^2} \ge \frac{3\rho}{4},$$

but also, necessarily, for the admissible pair $(r(2\sigma+2),2\sigma+2)$,

$$\|\mathbf{S}(u_0,h)-\mathbf{S}(u_0,0)\|_{Y(T_1,2\sigma+2)} \ge \frac{3\rho}{4}. \tag{3.9}$$

Denote by $\mathbf{S}^{M'+1}$ the skeleton corresponding to the following control problem

$$\left\{\begin{array}{l} i\left(\frac{\mathrm{d}u}{\mathrm{d}t}+\alpha u\right)=\Delta u+\lambda\theta\left(\frac{\|u\|_{Y(t,2\sigma+2)}}{M'+1}\right)|u|^{2\sigma}u+\Phi h,\\ u(0)=u_1\end{array}\right.$$

where θ is a C^∞ function with compact support, such that $\theta(x)=0$ if $x\ge 2$ and $\theta(x)=1$ if $0\le x\le 1$. Then (3.9) implies that

$$\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y(T_1,2\sigma+2)} \ge \frac{3\rho}{4}.$$

We shall now split the interval $[0,T_1]$ in many parts. We shall denote here by $Y^{s,t,2\sigma+2}$ for $s<t$ the space $Y^{t,2\sigma+2}$ on the interval $[s,t]$. Applying the Strichartz inequalities on a small interval $[0,t]$ with the computations in the proof of Lemma 3.3 in [5], we obtain

$$\begin{aligned}\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y^{(t,2\sigma+2)}} &\le C\alpha\sqrt{t}\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y^{(t,2\sigma+2)}}\\ &+C_{M'+1}t^{1-\frac{d\sigma}{2}}\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y^{(t,2\sigma+2)}}+C\sqrt{t}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}\end{aligned}$$

where $C_{M'+1}$ is a constant which depends on $M'+1$. Take t_1 small enough such that $C_{M'+1}t_1^{1-\frac{\sigma d}{2}}+C\alpha\sqrt{t_1}\le\frac{1}{2}$. We obtain then

$$\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y^{(t_1,2\sigma+2)}}\le 2C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t_1;\mathrm{L}^2)}.$$

In the case where $2t_1<T_1$, let us see how such inequality propagates on $[t_1,2t_1]$. We now have two different initial data $\mathbf{S}^{M'+1}(u_0,h)(t_1)$ and $\mathbf{S}^{M'+1}(u_0,0)(t_1)$. We obtain similarly

$$\begin{aligned}&\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y^{(t_1,2t_1,2\sigma+2)}}\\ &\le 2C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t_1;\mathrm{L}^2)}+2\left\|\mathbf{S}^{M'+1}(u_0,h)(t_1)-\mathbf{S}^{M'+1}(u_0,0)(t_1)\right\|_{\mathrm{H}^1}\\ &\le 2C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,T_1;\mathrm{L}^2)}+2\left\|\mathbf{S}^{M'+1}(u_0,h)(t_1)-\mathbf{S}^{M'+1}(u_0,0)(t_1)\right\|_{Y^{(0,t_1,2\sigma+2)}}.\end{aligned}$$

Then iterating on each interval of the form $[kt_1,(k+1)t_1]$ for k in $\left\{1,...,\left\lfloor\frac{T_1}{t_1}-1\right\rfloor\right\}$, the remaining term can be treated similarly, and using the triangle inequality we obtain that

$$\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y^{(T_1,2\sigma+2)}}\le 2^{\left\lceil\frac{T_1}{t_1}\right\rceil+1}C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t_1;\mathrm{L}^2)}.$$

We may then conclude that

$$\frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T_1;\mathrm{L}^2)}\ge M''$$

where $M''=\frac{\rho^2}{8C(t_1,T_1)\|\Phi\|_c^2}$ and $C(t_1,T_1)$ is a constant which depends only on t_1 and T_1. Note that we have used for later purposes that $\frac{3\rho}{2}>\frac{\rho}{2}$.

Similarly replacing $[0,T_1]$ by $[T_1,2T_1]$ and u_0 respectively by $\mathbf{S}(u_0,h)(T_1)$ and $\mathbf{S}(u_0,0)(T_1)$ in (3.9), the inequality still holds true. Thus thanks to the inverse triangle inequality we obtain on $[T_1, 2T_1]$

$$\begin{array}{l}\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y^{(T_1,2T_1,2\sigma+2)}}\\ =\left\|\mathbf{S}^{M'+1}\left(\mathbf{S}^{M'+1}(u_0,h)(T_1),h\right)-\mathbf{S}^{M'+1}\left(\mathbf{S}^{M'+1}(u_0,0)(T_1),0\right)\right\|_{Y^{(0,T_1,2\sigma+2)}}\\ \geq\frac{3\rho}{4}\end{array}$$

Thus from the inverse triangle inequality along with the fact that for both $\mathbf{S}^{M'+1}(u_0,h)(T_1)$ and $\mathbf{S}^{M'+1}(u_0,0)(T_1)$ as initial data the deterministic solutions belong to the ball $B^0_{\frac{\rho}{8}}$, we obtain

$$\left\|\mathbf{S}^{M'+1}\left(\mathbf{S}^{M'+1}(u_0,h)(T_1),h\right)-\mathbf{S}^{M'+1}\left(\mathbf{S}^{M'+1}(u_0,h)(T_1),0\right)\right\|_{Y^{(0,T_1,2\sigma+2)}}\geq\frac{\rho}{2}.$$

We finally obtain the same lower bound

$$\frac{1}{2}\|h\|^2_{\mathrm{L}^2(T_1,2T_1;\mathrm{L}^2)}\geq M''$$

as before.
Iterating the argument we obtain if $T > 2T_1$,

$$\frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,2T_1;\mathrm{L}^2)}=\frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T_1;\mathrm{L}^2)}+\frac{1}{2}\|h\|^2_{\mathrm{L}^2(T_1,2T_1;\mathrm{L}^2)}\geq 2M''.$$

Thus for j positive and $T > jT_1$, we obtain, iterating the above argument, that

$$\frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,jT_1;\mathrm{L}^2)}\geq jM''.$$

The result (3.8) is obtained for $T = jT_1$ where j is such that $jM'' > 2L$.

Step 3: We may now conclude from the (i) of Theorem 2.1 since,

$$\begin{array}{rl}\mathbb{P}\left(\sigma^{\epsilon,u_0}_\rho>T\right) & =\mathbb{P}\left(\forall t\in[0,T],\ u^{\epsilon,u_0}(t)\in D\setminus B^0_\rho\right)\\ & =\mathbb{P}\left(d_{\mathrm{C}([0,T];\mathrm{L}^2)}\left(u^{\epsilon,u_0},\mathcal{T}^c_\rho\right)>\frac{\rho}{8}\right),\\ & \leq\mathbb{P}\left(d_{\mathrm{C}([0,T];\mathrm{L}^2)}\left(u^{\epsilon,u_0},K^{u_0}_T(2L)\right)\geq\frac{\rho}{8}\right),\end{array}$$

taking $a = 2L$, $\rho = R$ where $D \subset B_R$, $\delta = \frac{\rho}{8}$ and $\gamma = L$.

Note that if $\rho \geq 8$, we should replace $R + 1$ by $R + \frac{\rho}{8}$ and $M' + 1$ by $M' + \frac{\rho}{8}$. Anyway, we will use the lemma for small ρ. □

Lemma 3.6 *For every ρ positive such that $B^0_\rho \subset D$ and u_0 in D, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma^{\epsilon,u_0}_\rho\right)\in\partial D\right)\leq -L$$

Proof. Take ρ positive satisfying the assumptions of the lemma and take u_0 in D. When u_0 belongs to B_ρ^0 the result is straightforward. Suppose now that u_0 belongs to $D \setminus B_\rho^0$. Let T be defined as

$$T = \inf \left\{ t \geq 0 : \ \mathbf{S}\left(u_0, 0\right)(t) \in B_{\frac{\rho}{2}}^0 \right\},$$

then since $\mathbf{S}\left(u_0, 0\right)([0, T])$ is a compact subset of D, the distance d between $\mathbf{S}\left(u_0, 0\right)([0, T])$ and D^c is well defined and positive. The conclusion follows then from the fact that

$$\mathbb{P}\left(u^{\epsilon, u_0}\left(\sigma_\rho^{\epsilon, u_0}\right) \in \partial D\right) \leq \mathbb{P}\left(\left\|u^{\epsilon, u_0} - \mathbf{S}\left(u_0, 0\right)\right\|_{\mathrm{C}([0,T];\mathrm{L}^2)} \geq \frac{\rho \wedge d}{2}\right),$$

the LDP and the fact that, from the compactness of the sets $K_T^{u_0}(a)$ for a positive, we have

$$\inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2): \ \|\mathbf{S}(u_0,h) - \mathbf{S}(u_0,0)\|_{\mathrm{C}([0,T];\mathrm{L}^2)} \geq \frac{\rho \wedge d}{2}} \|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}^2 > 0.$$

We have used the fact that the upper bound of the LDP in the Freidlin-Wentzell formulation implies the classical upper bound. Note that this is a well known result for non uniform LDPs. Indeed we do not need a uniform LDP in this proof. □

The following lemma replaces Lemma 5.7.23 in [12]. Indeed, the case of a stochastic PDE is more intricate than that of a SDE since the linear group is only strongly and not uniformly continuous. However, it is possible to prove that the group on L^2 when acting on bounded sets of H^1 is uniformly continuous. We shall proceed in a different manner and thus we will not loose in regularity. Indeed, the Schrödinger group does not have regularizing properties and we would obtain a weaker result with extra assumptions on Φ and the initial data.

Lemma 3.7 *For every ρ and L positive such that $B_{2\rho}^0 \subset D$, there exists $T(L, \rho) < \infty$ such that*

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \sup_{u_0 \in S_\rho^0} \mathbb{P}\left(\sup_{t \in [0, T(L,\rho)]} \left(\mathbf{N}\left(u^{\epsilon, u_0}(t)\right) - \mathbf{N}\left(u_0\right)\right) \geq 3\rho^2\right) \leq -L$$

Proof. Take L and ρ positive. Note that for every ϵ in $(0, \epsilon_0)$ where $\epsilon_0 = \frac{\rho^2}{\|\Phi\|_{\mathcal{L}_2^{0,0}}^2}$, for $T(L, \rho) \leq 1$ we have $\epsilon T(L, \rho) \|\Phi\|_{\mathcal{L}_2^{0,0}}^2 < \rho^2$. Thus from equation

(3.2), we know that it is enough to prove that there exists $T(L,\rho)\leq 1$ such that for ϵ_1 small enough, $\epsilon_1<\epsilon_0$, and all $\epsilon<\epsilon_0$,

$$\epsilon\log\sup_{u_0\in S^0_\rho}\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}\left(-2\sqrt{\epsilon}\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\overline{u}^{\epsilon,u_0,\tau}dWdx\right)\geq 2\rho^2\right)\leq -L,$$

where $u^{\epsilon,u_0,\tau}$ is the process u^{ϵ,u_0} stopped at $\tau^{\epsilon,u_0}_{S^0_{2\rho}}$, the first time when u^{ϵ,u_0} hits $S^0_{2\rho}$. Setting $Z(t)=\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\overline{u}^{\epsilon,u_0,\tau}dWdx$, it is enough to show that

$$\epsilon\log\sup_{u_0\in S^0_\rho}\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}|Z(t)|\geq\frac{\rho^2}{\sqrt{\epsilon}}\right)\leq -L,$$

and thus to show exponential tail estimates for the process $Z(t)$. Our proof now follows closely that of [23][Theorem 2.1]. We introduce the function $f_l(x)=\sqrt{1+lx^2}$, where l is a positive parameter. We now apply the Itô formula to $f_l(Z(t))$ and the process decomposes into $1+E_l(t)+R_l(t)$ where

$$E_l(t)=\int_0^t\frac{2lZ(t)}{\sqrt{1+lZ(t)^2}}dZ(t)-\frac{1}{2}\int_0^t\left(\frac{2lZ(t)}{\sqrt{1+lZ(t)^2}}\right)^2d<Z>_t,$$

and

$$R_l(t)=\frac{1}{2}\int_0^t\left(\frac{2lZ(t)}{\sqrt{1+lZ(t)^2}}\right)^2d<Z>_t+\int_0^t\frac{l}{(1+lZ(t)^2)^{\frac{3}{2}}}d<Z>_t\,.$$

Moreover, given $(e_j)_{j\in\mathbb{N}}$ a complete orthonormal system of L^2,

$$<Z(t)>=\int_0^t\sum_{j\in\mathbb{N}}\left(u^{\epsilon,u_0,\tau},-i\Phi e_j\right)^2_{\mathrm{L}^2}(s)ds,$$

we prove with the Hölder inequality that $|R_l(t)|\leq 12l\rho^2\|\Phi\|^2_{\mathcal{L}_2^{0,0}}t$, for every u_0 in D. We may thus write

$$\begin{aligned}&\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}|Z(t)|\geq\tfrac{\rho^2}{\sqrt{\epsilon}}\right)\\&=\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}\exp\left(f_l(Z(t))\right)\geq\exp\left(f_l\left(\tfrac{\rho^2}{\sqrt{\epsilon}}\right)\right)\right)\\&\leq\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}\exp\left(E_l(t)\right)\geq\exp\left(f_l\left(\tfrac{\rho^2}{\sqrt{\epsilon}}\right)-1-12l\rho^2\|\Phi\|^2_{\mathcal{L}_2^{0,0}}T(L,\rho)\right)\right).\end{aligned}$$

The Novikov condition is also satisfied and $E_l(t)$ is such that $\left(\exp\left(E_l(t)\right)\right)_{t\in\mathbb{R}^+}$ is a uniformly integrable martingale. The exponential tail estimates follow from the Doob inequality optimizing on the parameter l. We may then write

$$\sup_{u_0\in S_\rho^0}\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}|Z(t)|\geq\frac{\rho^2}{\sqrt{\epsilon}}\right)\leq 3\exp\left(-\frac{\rho^2}{48\epsilon\|\Phi\|^2_{\mathcal{L}_2^{0,0}}T(L,\rho)}\right).$$

We now conclude setting $T(L,\rho)=\frac{\rho^2}{50\|\Phi\|^2_{\mathcal{L}_2^{0,0}}L}$ and choosing $\epsilon_1<\epsilon_0$ small enough. $\square$

Proof of Theorem 3.2. Let us first prove (3.6) and deduce (3.5). Fix δ positive and choose h and T_1 such that $\mathbf{S}(0,h)(T_1)\in\overline{D}^c$ and

$$I^0_{T_1}\left(\mathbf{S}(0,h)\right)=\frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\leq\overline{e}+\frac{\delta}{5}.$$

Let d_0 denote the positive distance between $\mathbf{S}(0,h)\left(T_1\right)$ and $\overline{D}$. With similar arguments as in [6] or with a truncation argument we may prove that the skeleton is continuous with respect to the initial datum for the L^2 topology. Thus there exists ρ positive, a function of h which has been fixed, such that if u_0 belongs to B^0_ρ then

$$\|\mathbf{S}\left(u_0,h\right)-\mathbf{S}(0,h)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)}<\frac{d_0}{2}.$$

We may assume that ρ is such that $B^0_\rho\subset D$. From the triangle inequality and the (ii) of Theorem 2.1, there exists ϵ_1 positive such that for all ϵ in $(0,\epsilon_1)$ and u_0 in B^0_ρ,

$$\begin{aligned}\mathbb{P}\left(\tau^{\epsilon,u_0}<T_1\right)&\geq\mathbb{P}\left(\|u^{\epsilon,u_0}-\mathbf{S}(0,h)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)}<d_0\right)\\&\geq\mathbb{P}\left(\|u^{\epsilon,u_0}-\mathbf{S}\left(u_0,h\right)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)}<\frac{d_0}{2}\right)\\&\geq\exp\left(-\frac{I^{u_0}_{T_1}(\mathbf{S}(u_0,h))+\frac{\delta}{5}}{\epsilon}\right).\end{aligned}$$

From Lemma 3.5, there exists T_2 and ϵ_2 positive such that for all ϵ in $(0,\epsilon_2)$,

$$\inf_{u_0\in D}\mathbb{P}\left(\sigma^{\epsilon,u_0}_\rho\leq T_2\right)\geq\frac{1}{2}.$$

Thus, for $T = T_1 + T_2$, from the strong Markov property we obtain that for all $\epsilon < \epsilon_3 < \epsilon_1 \wedge \epsilon_2$.

$$\begin{aligned} q = \inf_{u_0 \in D} \mathbb{P}\left(\tau^{\epsilon,u_0} \le T\right) &\ge \inf_{u_0 \in D} \mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} \le T_2\right) \inf_{u_0 \in B_\rho^0} \mathbb{P}\left(\tau^{\epsilon,u_0} \le T_1\right) \\ &\ge \frac{1}{2} \exp\left(-\frac{I_{T_1}^{u_0}(\mathbf{S}(u_0,h)) + \frac{\delta}{5}}{\epsilon}\right) \\ &\ge \exp\left(-\frac{I_{T_1}^{u_0}(\mathbf{S}(u_0,h)) + \frac{2\delta}{5}}{\epsilon}\right). \end{aligned}$$

Thus, for any $k \ge 1$, we have

$$\begin{aligned} \mathbb{P}\left(\tau^{\epsilon,u_0} > (k+1)T\right) &= \left[1 - \mathbb{P}\left(\tau^{\epsilon,u_0} \le (k+1)T \middle| \tau^{\epsilon,u_0} > kT\right)\right] \mathbb{P}\left(\tau^{\epsilon,u_0} > kT\right) \\ &\le (1-q)\mathbb{P}\left(\tau^{\epsilon,u_0} > kT\right) \\ &\le (1-q)^k. \end{aligned}$$

We may now compute, since $I_{T_1}^{u_0}\left(\mathbf{S}\left(u_0,h\right)\right) = I_{T_1}^{0}\left(\mathbf{S}\left(0,h\right)\right) = \frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$

$$\begin{aligned} \sup_{u_0 \in D} \mathbb{E}\left(\tau^{\epsilon,u_0}\right) &= \sup_{u_0 \in D} \int_0^\infty \mathbb{P}\left(\tau^{\epsilon,u_0} > t\right) dt \\ &\le T\left[1 + \sum_{k=1}^\infty \sup_{x \in D} \mathbb{P}\left(\tau^{\epsilon,u_0} > kT\right)\right] \\ &\le \frac{T}{q} \\ &\le T \exp\left(\frac{\overline{e} + \frac{3\delta}{5}}{\epsilon}\right). \end{aligned}$$

It implies that there exists ϵ_4 small enough such that for ϵ in $(0, \epsilon_4)$,

$$\sup_{u_0 \in D} \mathbb{E}\left(\tau^{\epsilon,u_0}\right) \le \exp\left(\frac{\overline{e} + \frac{4\delta}{5}}{\epsilon}\right). \tag{3.10}$$

Thus the Chebychev inequality gives that

$$\sup_{u_0 \in D} \mathbb{P}\left(\tau^{\epsilon,u_0} \ge \exp\left(\frac{\overline{e} + \delta}{\epsilon}\right)\right) \le \exp\left(-\frac{\overline{e} + \delta}{\epsilon}\right) \sup_{u_0 \in D} \mathbb{E}\left(\tau^{\epsilon,u_0}\right),$$

in other words

$$\sup_{u_0 \in D} \mathbb{P}\left(\tau^{\epsilon,u_0} \ge \exp\left(\frac{\overline{e} + \delta}{\epsilon}\right)\right) \le \exp\left(-\frac{\delta}{5\epsilon}\right). \tag{3.11}$$

Relations (3.10) and (3.11) imply (3.6) and (3.5).

Let us now prove the lower bound on τ^{ϵ,u_0}. Take δ positive. Remind that we have proved that $\underline{e} > 0$. Take ρ positive small enough such that

$\underline{e} - \frac{\delta}{4} \leq e_\rho$ and $B_{2\rho}^0 \subset D$. We define the following sequences of stopping times, $\theta_0 = 0$ and for k in $\mathbb{N}$,

$$\begin{array}{ll} \tau_k & = \inf\left\{t \geq \theta_k : \ u^{\epsilon,u_0}(t) \in B_\rho^0 \cup D^c\right\}, \\ \theta_{k+1} & = \inf\left\{t > \tau_k : \ u^{\epsilon,u_0}(t) \in S_{2\rho}^0\right\}, \end{array}$$

where $\theta_{k+1} = \infty$ if $u^{\epsilon,u_0}(\tau_k) \in \partial D$. Fix $T_1 = T\left(\underline{e} - \frac{3\delta}{4}, \rho\right)$ given in Lemma 3.7. We know that there exists ϵ_1 positive such that for all ϵ in $(0, \epsilon_1)$, for all $k \geq 1$ and u_0 in D,

$$\mathbb{P}\left(\theta_k - \tau_{k-1} \leq T_1\right) \leq \exp\left(-\frac{\underline{e} - \frac{3\delta}{4}}{\epsilon}\right).$$

For u_0 in D and an m in $\mathbb{N}^*$, we have

$$\begin{array}{rl} \mathbb{P}\left(\tau^{\epsilon,u_0} \leq mT_1\right) \leq & \mathbb{P}\left(\tau^{\epsilon,u_0} = \tau_0\right) + \sum_{k=1}^m \mathbb{P}\left(\tau^{\epsilon,u_0} = \tau_k\right) \\ & + \mathbb{P}\left(\exists k \in \{1, ..., m\} : \ \theta_k - \tau_{k-1} \leq T_1\right) \\ = & \mathbb{P}\left(\tau^{\epsilon,u_0} = \tau_0\right) + \sum_{k=1}^m \mathbb{P}\left(\tau^{\epsilon,u_0} = \tau_k\right) \\ & + \sum_{k=1}^m \mathbb{P}\left(\theta_k - \tau_{k-1} \leq T_1\right). \end{array} \tag{3.12}$$

In other words the escape before mT_1 can occur either as an escape without passing in the small ball B_ρ^0 (if u_0 belongs to $D \setminus B_\rho^0$) or as an escape with k in $\{1, ...m\}$ significant fluctuations off B_ρ^0, *i.e.* crossing $S_{2\rho}^0$, or at least one of the m first transitions between S_ρ^0 and $S_{2\rho}^0$ happens in less than T_1. The latter is known to be arbitrarily small. Let us prove that the remaining probabilities are small enough for small ϵ.

For every $k \geq 1$ and T_2 positive, we may write

$$\mathbb{P}\left(\tau^{\epsilon,u_0} = \tau_k\right) \leq \mathbb{P}\left(\tau^{\epsilon,u_0} \leq T_2; \tau^{\epsilon,u_0} = \tau_k\right) + \mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T_2\right).$$

Fix T_2 as in Lemma 3.5 with $L = \underline{e} - \frac{3\delta}{4}$. Thus there exists ϵ_2 small enough such that for ϵ in $(0, \epsilon_2)$,

$$\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T_2\right) \leq \exp\left(-\frac{\underline{e} - \frac{3\delta}{4}}{\epsilon}\right).$$

Also, from the (i) of Theorem 2.1, we obtain that there exists ϵ_3 positive such that for every u_1 in B_ρ^0 and ϵ in $(0, \epsilon_3)$,

$$\begin{array}{rl} \mathbb{P}\left(\tau^{\epsilon,u_1} \leq T_2\right) & \leq \mathbb{P}\left(d_{C([0,T_2];L^2)}\left(u^{\epsilon,u_1}, K_{T_2}^{u_1}\left(e_\rho - \frac{\delta}{4}\right)\right) \geq \rho\right) \\ & \leq \exp\left(-\frac{e_\rho - \frac{\delta}{2}}{\epsilon}\right) \\ & \leq \exp\left(-\frac{\underline{e} - \frac{3\delta}{4}}{\epsilon}\right). \end{array}$$

Thus the above bound holds for $\mathbb{P}\left(\tau^{\epsilon,u_0} \leq T_2; \tau^{\epsilon,u_0} = \tau_k\right)$ replacing u_1 by $u^{\epsilon,u_0}\left(\tau_{k-1}\right)$ since as $k \geq 1$, $u^{\epsilon,u_0}\left(\tau_{k-1}\right)$ belongs to B_ρ^0 and $\tau_k - \tau_{k-1} \leq T_2$ and using the Markov property. The inequality (3.12) gives that for all ϵ in $(0,\epsilon_0)$ where $\epsilon_0 = \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3$,

$$\mathbb{P}\left(\tau^{\epsilon,u_0} \leq mT_1\right) \leq \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) + 3m \exp\left(-\frac{\underline{e} - \frac{3\delta}{4}}{\epsilon}\right).$$

Fix $m = \left\lceil \frac{1}{T_1} \exp\left(\frac{\underline{e}-\delta}{\epsilon}\right)\right\rceil$, then for all ϵ in $(0,\epsilon_0)$,

$$\begin{aligned}\mathbb{P}\left(\tau^{\epsilon,u_0} \leq \exp\left(\frac{\underline{e}-\delta}{\epsilon}\right)\right) &\leq \mathbb{P}\left(\tau^{\epsilon,u_0} \leq mT_1\right)\\ &\leq \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) + \frac{3}{T_1} \exp\left(-\frac{\delta}{4\epsilon}\right).\end{aligned}$$

We may now conclude with Lemma 3.6 and obtain the expected lower bound on $\mathbb{E}\left(\tau^{\epsilon,u_0}\right)$ from the Chebychev inequality. □

Proof of Theorem 3.3. Let N be closed subset of ∂D. When $\underline{e}_N = \infty$ we shall replace in the proof that follows $\underline{e}_N$ by an increasing sequence of positive numbers. Take δ such that $0 < \delta < \frac{\underline{e}_N - \overline{e}}{3}$, ρ positive such that $\underline{e}_N - \frac{\delta}{3} \leq e_{N,\rho}$ and $B_{2\rho}^0 \subset D$. Define the same sequences of stopping times $(\tau_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ as in the proof of Theorem 3.2.
Take $L = \underline{e}_N - \delta$ and T_1 and $T_2 = T(L,\rho)$ as in Lemma 3.5 and 3.7. Thanks to Lemma 3.5 and the uniform LDP, with a computation similar to the one following inequality (3.12), we obtain that for ϵ_0 small enough and $\epsilon \leq \epsilon_0$,

$$\begin{aligned}&\sup_{u_0\in S_{2\rho}^0} \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in N\right)\\ &\leq \sup_{u_0\in S_{2\rho}^0} \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in N, \sigma_\rho^{\epsilon,u_0} \leq T_1\right) + \sup_{u_0\in S_{2\rho}^0} \mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T_1\right)\\ &\leq \sup_{u_0\in B_{2\rho}^0} \mathbb{P}\left(d_{\mathrm{C}([0,T_1];\mathrm{L}^2)}\left(u^{\epsilon,u_0}, K_{T_1}^{u_0}\left(e_{N,\rho} - \frac{\delta}{3}\right)\right) \geq \rho\right)\\ &\quad + \sup_{u_0\in D} \mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T_1\right)\\ &\leq 2\exp\left(-\frac{\underline{e}_N - \delta}{\epsilon}\right).\end{aligned}$$

Possibly choosing ϵ_0 smaller, we may assume that for every positive integer l and every $\epsilon \leq \epsilon_0$,

$$\begin{aligned}\sup_{u_0\in D} \mathbb{P}\left(\tau_l \leq lT_2\right) \leq\ & l \sup_{u_0\in S_\rho^0} \mathbb{P}\left(\sup_{t\in[0,T_2]}\left(\mathbf{N}\left(u^{\epsilon,u_0}(t)\right) - \mathbf{N}\left(u_0\right)\right) \geq \rho\right)\\ \leq\ & l \exp\left(-\frac{\underline{e}_N - \delta}{\epsilon}\right).\end{aligned}$$

Thus if u_0 belongs to B_ρ^0

$$\begin{aligned}\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)\leq&\ \mathbb{P}\left(\tau^{\epsilon,u_0}>\tau_l\right)+\textstyle\sum_{k=1}^{l}\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N,\tau^{\epsilon,u_0}=\tau_k\right)\\ \leq&\ \mathbb{P}\left(\tau^{\epsilon,u_0}>lT_2\right)+\mathbb{P}\left(\tau_l\leq lT_2\right)\\ &+l\sup_{u_0\in S_{2\rho}^0}\mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right)\in N\right)\\ \leq&\ \mathbb{P}\left(\tau^{\epsilon,u_0}>lT_2\right)+3l\exp\left(-\frac{\underline{e}_N-\delta}{\epsilon}\right).\end{aligned}$$

Take now $l=\left\lceil\frac{1}{T_2}\exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right\rceil$ and use the upper bound (3.11), possibly choosing ϵ_0 smaller, we obtain that for $\epsilon\leq\epsilon_0$

$$\begin{aligned}\sup_{u_0\in B_\rho^0}\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)\leq&\ \exp\left(-\frac{\delta}{5\epsilon}\right)+\frac{4}{T_2}\exp\left(-\frac{\underline{e}_N-\overline{e}+2\delta}{\epsilon}\right)\\ \leq&\ \exp\left(-\frac{\delta}{5\epsilon}\right)+\frac{4}{T_2}\exp\left(-\frac{\delta}{\epsilon}\right).\end{aligned}$$

Finally, when u_0 is any function in D, we conclude thanks to

$$\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)\leq\mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right)\in\partial D\right)+\sup_{u_0\in B_\rho^0}\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)$$

and to Lemma 3.6. □

Remark 3.8 *Note that it has been proposed in [24] to introduce control elements in order to reduce or enhance exponentially the expected exit time or to act on the exiting points, for a limited cost. We may then think of optimizing on such external fields. However the problem is computationally involved since the optimal control problem requires double optimisation.*

4 Exit from a domain of attraction in H^1

We now consider a measurable bounded subset D of H^1 invariant by the flow of the deterministic equation; D and R are such that $D\subset B_R^1$. We consider both (2.1) and (2.2) where the noise is either of additive or of multiplicative type. In this section we are interested in both the fluctuation of the L^2 norm and that of the L^2 norm of the gradient. The Hamiltonian and a modified Hamiltonian will thus be of particular interest. We shall first distinguish the case where the nonlinearity is defocusing ($\lambda=-1$) where the Hamiltonian takes non negative values from the case where the nonlinearity is focusing ($\lambda=1$) where the Hamiltonian may take negative values.

We may prove, see for example [19], that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}\left(\mathbf{S}(u_0,0)(t)\right)+2\alpha\mathbf{\Psi}\left(\mathbf{S}(u_0,0)\right)=0,$$

where $\mathbf{S}(u_0, 0)$ is the solution of the deterministic weakly damped nonlinear Schrödinger equation with initial datum u_0 in H^1 and

$$\mathbf{\Psi}\left(\mathbf{S}(u_0, 0)\right) = \frac{1}{2}\left\|\nabla \mathbf{S}(u_0, 0)\right\|_{\mathrm{L}^2}^2 - \frac{\lambda}{2}\int_{\mathbb{R}^d} \left|\mathbf{S}(u_0, 0)(x)\right|^{2\sigma+2} dx.$$

Thus, when the nonlinearity is defocusing we have

$$0 \leq \mathbf{H}\left(\mathbf{S}(u_0, 0)(t)\right) \leq \mathbf{H}\left(u_0\right)\exp\left(-2\alpha t\right). \tag{4.1}$$

As it is done in [11], we shall consider in the focusing case a modified Hamiltonian denoted by $\tilde{\mathbf{H}}(u)$ defined for u in H^1 by

$$\tilde{\mathbf{H}}(u) = \mathbf{H}(u) + \beta(\sigma, d)C\left\|u\right\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}}$$

where the constant C is that of the third inequality in the following sequence of inequalities where we use the Gagliardo-Nirenberg inequality

$$\frac{1}{2\sigma+2}\|u\|_{\mathrm{L}^{2\sigma+2}}^{2\sigma+2} \leq C\|u\|_{\mathrm{L}^2}^{2\sigma+2-\sigma d}\|\nabla u\|_{\mathrm{L}^2}^{\sigma d} \leq \frac{1}{4}\|\nabla u\|_{\mathrm{L}^2}^2 + C\|u\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}},$$

and $\beta(\sigma, d) = \frac{2\sigma(2-\sigma d)}{(\sigma+2)(2-\sigma d)+2\sigma(4\sigma+3)} \vee 2$. When evaluated at the deterministic solution, the modified Hamiltonian satisfies

$$0 \leq \tilde{\mathbf{H}}\left(\mathbf{S}(u_0, 0)(t)\right) \leq \tilde{\mathbf{H}}\left(u_0\right)\exp\left(-2\alpha\frac{3(\sigma+1)}{4\sigma+3}t\right). \tag{4.2}$$

Also, when the nonlinearity is defocusing we now have, for every β positive,

$$0 \leq \tilde{\mathbf{H}}\left(\mathbf{S}(u_0, 0)(t)\right) \leq \tilde{\mathbf{H}}\left(u_0\right)\exp\left(-2\alpha t\right). \tag{4.3}$$

From the Sobolev inequalities, for ρ positive, the sets

$$\tilde{\mathbf{H}}_\rho = \left\{u \in \mathrm{H}^1 : \ \tilde{\mathbf{H}}(u) = \rho\right\} = \tilde{\mathbf{H}}^{-1}\left(\{\rho\}\right), \quad \rho > 0$$

are closed subsets of H^1 and

$$\tilde{\mathbf{H}}_{<\rho} = \left\{u \in \mathrm{H}^1 : \ \tilde{\mathbf{H}}(u) < \rho\right\} = \tilde{\mathbf{H}}^{-1}\left([0, \rho)\right) \quad \rho > 0$$

are open subsets of H^1.

Also, $\tilde{\mathbf{H}}$ is such that

$$\frac{1}{2}\|\nabla u\|_{\mathrm{L}^2}^2 + \beta C\|u\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}} \leq \tilde{\mathbf{H}}(u) \leq \frac{3}{4}\|\nabla u\|_{\mathrm{L}^2}^2 + (\beta+1)C\|u\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}} \tag{4.4}$$

when the nonlinearity is defocusing and

$$\frac{1}{4}\|\nabla u\|_{\mathrm{L}^2}^2 + C\|u\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}} \leq \tilde{\mathbf{H}}(u) \leq \frac{1}{2}\|\nabla u\|_{\mathrm{L}^2}^2 + \beta(\sigma,d)C\|u\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}} \quad (4.5)$$

when it is focusing. Thus the sets $\tilde{\mathbf{H}}_{<\rho}$ for ρ positive are bounded in H^1 and a bounded set in H^1 is bounded for $\tilde{\mathbf{H}}$.

We will no longer distinguish the focusing and defocusing cases and will take the same value of β, *i.e.* $\beta(\sigma,d)$. Also to simplify the notations we will sometimes drop the dependence of the solution in ϵ and u_0.

The fluctuation of $\tilde{\mathbf{H}}\left(u^{\epsilon,u_0}(t)\right)$ is of particular interest. We have the following result when the noise is of additive type.

Proposition 4.1 *When u denotes the solution of equation* (2.1), $(e_j)_{j\in\mathbb{N}}$ *a complete orthonormal system of* L^2, *the following decomposition holds*

$$\begin{aligned}
\tilde{\mathbf{H}}(u(t)) \;&= \tilde{\mathbf{H}}(u_0)\\
&-2\alpha\int_0^t \mathbf{\Psi}(u(s))\,ds - 2\beta C\left(1+\tfrac{2\sigma}{2-\sigma d}\right)\alpha\int_0^t \|u(s)\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}}ds\\
&+\sqrt{\epsilon}\Big(\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \nabla\overline{u}(s)\nabla dW(s)dx - \lambda\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t |u(s)|^{2\sigma}\,\overline{u}(s)dW(s)dx\\
&\qquad +2\beta C\left(1+\tfrac{2\sigma}{2-\sigma d}\right)\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \|u(s)\|_{\mathrm{L}^2}^{\frac{4\sigma}{2-\sigma d}}\,\overline{u}(s)dW(s)dx\Big)\\
&-\tfrac{\lambda\epsilon}{2}\sum_{j\in\mathbb{N}}\int_0^t\int_{\mathbb{R}^d}\left[|u(s)|^{2\sigma}|\Phi e_j|^2 + 2\sigma|u(s)|^{2\sigma-2}\left(\mathfrak{Re}(\overline{u}(s)\Phi e_j)\right)^2\right]dxds\\
&+\tfrac{\epsilon}{2}\|\nabla\Phi\|_{\mathcal{L}_2^{0,0}}^2 t + \epsilon\beta C\left(1+\tfrac{2\sigma}{2-\sigma d}\right)\|\Phi\|_{\mathcal{L}_2^{0,0}}^2\int_0^t \|u(s)\|_{\mathrm{L}^2}^{\frac{4\sigma}{2-\sigma d}}ds\\
&+\epsilon\beta C\tfrac{4\sigma}{2-\sigma d}\left(1+\tfrac{2\sigma}{2-\sigma d}\right)\sum_{j\in\mathbb{N}}\int_0^t \|u(s)\|_{\mathrm{L}^2}^{2\left(\frac{2\sigma}{2-\sigma d}-1\right)}\left(\mathfrak{Re}\int_{\mathbb{R}^d}\overline{u}(s)\Phi e_j dx\right)^2 ds
\end{aligned}$$

Proof. The result follows from the Itô formula. The main difficulty is in justifying the computations. We may proceed as in [6]. □

Also, when the noise is of multiplicative type we obtain the following proposition.

Proposition 4.2 *When u denotes the solution of equation* (2.1), $(e_j)_{j\in\mathbb{N}}$ *a complete orthonormal system of* L^2, *the following decomposition holds*

$$\begin{aligned}
\tilde{\mathbf{H}}(u(t)) = \;&\tilde{\mathbf{H}}(u_0)\\
&-2\alpha\int_0^t \mathbf{\Psi}(u(s))\,ds - 2\beta C\left(1+\tfrac{2\sigma}{2-\sigma d}\right)\alpha\int_0^t \|u(s)\|_{\mathrm{L}^2}^{2+\frac{4\sigma}{2-\sigma d}}ds\\
&+\sqrt{\epsilon}\,\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t u(s)\nabla\overline{u}(s)\nabla dW(s)dx\\
&+\tfrac{\epsilon}{2}\sum_{j\in\mathbb{N}}\int_0^t\int_{\mathbb{R}^d}|u(s)|^2|\nabla\Phi e_j|^2dxds.
\end{aligned}$$

The first exit time τ^{ϵ,u_0} from the domain D in H^1 is defined as in Section 2. Note that the domain D may be a domain of attraction of the form $\tilde{\mathbf{H}}_{<a}$ where a is positive. We also define

$$\overline{e} = \inf\left\{I_T^0(w) :\ w(T) \in \overline{D}^c,\ T > 0\right\},$$

and for ρ positive small enough

$$e_\rho = \inf\left\{I_T^{u_0}(w) :\ \tilde{\mathbf{H}}\left(u_0\right) \leq \rho,\ w(T) \in \left(D_{-\rho}\right)^c,\ T > 0\right\},$$

where $D_{-\rho} = D \setminus \mathcal{N}^1\left(\partial D, \rho\right)$. Then we set

$$\underline{e} = \lim_{\rho \to 0} e_\rho.$$

Also, for ρ positive small enough, N a closed subset of the boundary of D, we define

$$e_{N,\rho} = \inf\left\{I_T^{u_0}(w) :\ \tilde{\mathbf{H}}\left(u_0\right) \leq \rho,\ w(T) \in \left(D \setminus \mathcal{N}^1\left(N, \rho\right)\right)^c,\ T > 0\right\}$$

and

$$\underline{e}_N = \lim_{\rho \to 0} e_{N,\rho}.$$

We finally also introduce

$$\sigma_\rho^{\epsilon,u_0} = \inf\left\{t \geq 0 : u^{\epsilon,u_0}(t) \in \tilde{\mathbf{H}}_{<\rho} \cup D^c\right\},$$

where $\tilde{\mathbf{H}}_{<\rho} \subset D$.

Again we have the following inequalities.

Lemma 4.3 $0 < \underline{e} \leq \overline{e}$.

Proof. We only have to prove the first inequality. Integrating the equation describing the evolution of $\tilde{\mathbf{H}}\left(\mathbf{S}\left(u_0, h\right)(t)\right)$ via the Duhamel formula where the skeleton is that of the equation with an additive noise we obtain

$$\begin{aligned}
&\tilde{\mathbf{H}}\left(\mathbf{S}(u_0, h)(T)\right) - \exp\left(-2\alpha\tfrac{3(\sigma+1)}{4\sigma+3}T\right)\tilde{\mathbf{H}}\left(u_0\right) \\
&\leq \int_0^T \exp\left(-2\alpha\tfrac{3(\sigma+1)}{4\sigma+3}(T-s)\right)\Big[\mathfrak{Im}\int_{\mathbb{R}^d}\left(\nabla\mathbf{S}(u_0, h)\nabla\overline{\Phi h}\right)(s,x)dx \\
&\quad -\lambda\mathfrak{Im}\int_{\mathbb{R}^d}\left(|\mathbf{S}(u_0, h)|^{2\sigma}\mathbf{S}(u_0, h)\overline{\Phi h}\right)(s,x)dx \\
&\quad -2C\beta\left(1 + \tfrac{2\sigma}{2-\sigma d}\right)\mathfrak{Im}\int_{\mathbb{R}^d}\left(\mathbf{S}\left(u_0, h\right)\overline{\Phi h}\right)(s,x)dx\Big]\,ds,
\end{aligned}$$

with a focusing or defocusing nonlinearity. Let d denote the positive distance between 0 and ∂D. Take ρ such that the distance between B_ρ^1 and $\left(D_{-\rho}\right)^c$ is larger than $\frac{d}{2}$. We then have, from the fact that the Sobolev injection from H^1 into $\mathrm{L}^{2\sigma+2}$,

$$\begin{aligned}\frac{d}{2} \leq \int_0^T \exp\left(-2\alpha\tfrac{3(\sigma+1)}{4\sigma+3}(T-s)\right) \Big[& R\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{H}^1)}\|h\|_{\mathrm{L}^2} \\ & +CR^{2\sigma+1}\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{H}^1)}\|h\|_{\mathrm{L}^2} \\ & +2C\beta\left(1+\tfrac{2\sigma}{2-\sigma d}\right)R\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}\|h\|_{\mathrm{L}^2}\Big] ds,\end{aligned}$$

We conclude as in Lemma 3.1 and use that from the choice of β the complementary of a ball is included in the complementary of a set $\tilde{\mathbf{H}}_{<a}$. In the case of the skeleton of the equation with a multiplicative noise, it is enough to replace the term in bracket in the right hand side of the above formula by $\mathfrak{Im}\int_{\mathbb{R}^d}\left(\nabla\mathbf{S}\left(u_0,h\right)\overline{\mathbf{S}\left(u_0,h\right)\nabla\Phi h}\right)(s,x)dx$. Recall that we can proceed as in the additive case since we have imposed that Φ belongs to $\mathcal{L}_{2,\mathbb{R}}^{0,s}$ where $s>\frac{d}{2}+1$, in particular Φ belongs to $\mathcal{L}_c\left(\mathrm{L}^2,\mathrm{W}^{1,\infty}\right)$. □

The theorems of Section 2 still hold for a domain of attraction in H^1 and a noise of additive and multiplicative type.

Theorem 4.4 *For every u_0 in D and δ positive, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to0}\epsilon\log\mathbb{P}\left(\tau^{\epsilon,u_0}\notin\left(\exp\left(\frac{\underline{e}-\delta}{\epsilon}\right),\exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right)\right)\leq -L, \tag{4.6}$$

and for every u_0 in D,

$$\underline{e}\leq\underline{\lim}_{\epsilon\to0}\epsilon\log\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\leq\overline{\lim}_{\epsilon\to0}\epsilon\log\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\leq\overline{e}. \tag{4.7}$$

Moreover, for every δ positive, there exists L positive such that

$$\overline{\lim}_{\epsilon\to0}\epsilon\log\sup_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,u_0}\geq\exp\left(\frac{\overline{e}+\delta}{\epsilon}\right)\right)\leq -L, \tag{4.8}$$

and

$$\overline{\lim}_{\epsilon\to0}\epsilon\log\sup_{u_0\in D}\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\leq\overline{e}. \tag{4.9}$$

Theorem 4.5 *If $\underline{e}_N>\overline{e}$, then for every u_0 in D, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to0}\epsilon\log\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)\leq -L.$$

Again we may deduce the corollary

Corollary 4.6 *Assume that v^* in ∂D is such that for every δ positive and $N = \{v \in \partial D : \ \|v - v^*\|_{L^2} \geq \delta\}$ we have $\underline{e}_N > \overline{e}$ then*

$$\forall \delta > 0,\ \forall u_0 \in D,\ \exists L > 0 : \ \overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\|u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right) - v^*\|_{L^2} \geq \delta\right) \leq -L.$$

The proof of these results still relies on three lemmas and the uniform LDP. Let us now state the lemmas for both a noise of additive and of multiplicative type.

Lemma 4.7 *For every ρ and L positive with $\tilde{\mathbf{H}}_{<\rho} \subset D$, there exists T and ϵ_0 positive such that for every u_0 in D and ϵ in $(0, \epsilon_0)$,*

$$\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T\right) \leq \exp\left(-\frac{L}{\epsilon}\right).$$

Proof. We proceed as in the proof of Lemma 3.5.
Let d denote the positive distance between 0 and $D \setminus \tilde{\mathbf{H}}_{<\rho}$. Take α positive such that $\alpha\rho < d$. The domain D is uniformly attracted to 0, thus there exists a time T_1 such that for every initial datum u_1 in $\mathcal{N}^1\left(D \setminus \tilde{\mathbf{H}}_{<\rho}, \frac{\alpha\rho}{8}\right)$, for $t \geq T_1$, $\mathbf{S}\left(u_1, 0\right)(t)$ belongs to $B^1_{\frac{\alpha\rho}{8}}$.

We could also prove, see [6], that there exists a constant M' which depends on T_1, R, σ and α such that

$$\sup_{u_1 \in \mathcal{N}^1\left(D \setminus \tilde{\mathbf{H}}_{<\rho}, \frac{\alpha\rho}{8}\right)} \|\mathbf{S}\left(u_1, 0\right)\|_{X^{(T_1, 2\sigma+2)}} \leq M'. \tag{4.10}$$

The Step 2, corresponding to that of Lemma 3.5, in the proof in the additive case uses the truncation argument, upper bounds similar to that in [6] derived from the Strichartz inequalities on smaller intervals; we shall also replace in the proof of Lemma 3.5 $\frac{\rho}{8}$ by $\frac{\alpha\rho}{8}$.

In Step 2 for the multiplicative case, we also introduce the truncation in front of the term $u\Phi h$ in the controlled PDE.

The end of the proof is identical to that of Lemma 3.5, the LDP is the LDP in $C\left([0, T]; H^1\right)$, for additive or multiplicative noises. □

Lemma 4.8 *For every ρ positive such that $\tilde{\mathbf{H}}_\rho \subset D$ and u_0 in D, there exists L positive such that*

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) \leq -L$$

Proof. It is the same proof as for Lemma 3.6. We only have to replace $B^0_{\frac{\rho}{2}}$ by any ball in H^1 centered at 0 and included in $\tilde{\mathbf{H}}_{<\rho}$ and use the LDP in $C\left([0, T]; H^1\right)$. □

Lemma 4.9 *For every ρ and L positive such that $\tilde{\mathbf{H}}_{2\rho} \subset D$, there exists $T(L,\rho) < \infty$ such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \sup_{u_0\in\tilde{\mathbf{H}}_\rho} \mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}\left(\tilde{\mathbf{H}}\left(u^{\epsilon,u_0}(t)\right) - \tilde{\mathbf{H}}\left(u_0\right)\right) \geq \rho\right) \leq -L$$

Proof. Integrating the Itô differential relation using the Duhamel formula allows to get rid of the drift term that is not originated from the bracket. Indeed, the event

$$\left\{\sup_{t\in[0,T(L,\rho)]}\left(\tilde{\mathbf{H}}\left(u^{\epsilon,u_0}(t)\right) - \tilde{\mathbf{H}}\left(u_0\right)\right) \geq \rho\right\}$$

is included in

$$\left\{\sup_{t\in[0,T(L,\rho)]}\left(\tilde{\mathbf{H}}\left(u^{\epsilon,u_0}(t)\right) - \exp\left(-2\alpha\left(\frac{3(\sigma+1)}{4\sigma+3}\right)T(L,\rho)\right)\tilde{\mathbf{H}}\left(u_0\right)\right) \geq \rho\right\}.$$

Then, setting $c(\sigma) = \frac{3(\sigma+1)}{4\sigma+3}$ and $m(\sigma,d) = 1+\frac{2\sigma}{2-\sigma d}$, dropping the exponents ϵ and u_0 to have more concise formulas, we obtain in the additive case

$$\begin{aligned}
&\tilde{\mathbf{H}}\left(u(t)\right) - \exp\left(-2\alpha c(\sigma)t\right)\tilde{\mathbf{H}}\left(u_0\right)\\
&\leq \sqrt{\epsilon}\Big(\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \exp\left(-2\alpha c(\sigma)(t-s)\right)\nabla\overline{u}(s)\nabla dW(s)dx\\
&-\lambda\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \exp\left(-2\alpha c(\sigma)(t-s)\right)|u(s)|^{2\sigma}\,\overline{u}(s)dW(s)dx\\
&+2\beta C m(\sigma,d)\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \exp\left(-2\alpha c(\sigma)(t-s)\right)\|u(s)\|_{\mathrm{L}^2}^{\frac{4\sigma}{2-\sigma d}}\,\overline{u}(s)dW(s)dx\Big)\\
&-\frac{\lambda\epsilon}{2}\sum_{j\in\mathbb{N}}\int_0^t \exp\left(-2\alpha c(\sigma)(t-s)\right)\int_{\mathbb{R}^d}\Big[|u(s)|^{2\sigma}|\Phi e_j|^2\\
&\qquad\qquad + 2\sigma|u(s)|^{2\sigma-2}\left(\mathfrak{Re}(\overline{u}(s)\Phi e_j)\right)^2\Big]dxds\\
&+\frac{\epsilon}{4\alpha c(\sigma)}\left(1-\exp\left(-2\alpha c(\sigma)t\right)\right)\|\nabla\Phi\|^2_{\mathcal{L}_2^{0,0}}\\
&+\epsilon\beta C m(\sigma,d)\|\Phi\|^2_{\mathcal{L}_2^{0,0}}\int_0^t \exp\left(-2\alpha c(\sigma)(t-s)\right)\|u(s)\|_{\mathrm{L}^2}^{\frac{4\sigma}{2-\sigma d}}ds\\
&+\epsilon\beta C\frac{4\sigma}{2-\sigma d}m(\sigma,d)\sum_{j\in\mathbb{N}}\int_0^t \exp\left(-2\alpha c(\sigma)(t-s)\right)\|u(s)\|_{\mathrm{L}^2}^{2\left(\frac{2\sigma}{2-\sigma d}-1\right)}\left(\mathfrak{Re}\int_{\mathbb{R}^d}\overline{u}(s)\Phi e_j dx\right)^2 ds.
\end{aligned}$$

We again use a localization argument and replace the process u by the process u^τ stopped at the first exit time off $\tilde{\mathbf{H}}_{<2\rho}$. We use (4.4) and (4.5) and obtain

$$\|u^\tau\|^2_{\mathrm{H}^1} \leq 8\rho + \left(\frac{2\rho}{C\sigma}\right)^{\frac{1}{1+\frac{2\sigma}{2-\sigma d}}}.$$

We denote the right hand side of the above by $b(\rho,\sigma,d)$.
From the Hölder inequality along with the Sobolev injection of H^1 into $\mathrm{L}^{2\sigma+2}$ we obtain the following upper bound for the drift

$$\begin{array}{l}\frac{\epsilon}{4\alpha c(\sigma)}\left[(1+2\sigma)c(1,2\sigma+2)^{2\sigma+2}\|\Phi\|^2_{\mathcal{L}_2^{0,1}}b(\rho,\sigma,d)^{2\sigma}+\|\nabla\Phi\|^2_{\mathcal{L}_2^{0,0}}\right]\\ +\frac{\epsilon\beta C}{2\alpha c(\sigma)}m(\sigma,d)\left(1+\frac{4\sigma}{2-\sigma d}\right)\|\Phi\|^2_{\mathcal{L}_2^{0,0}}b(\rho,\sigma,d)^{\frac{4\sigma}{2-\sigma d}}\end{array}$$

where we denote by $c(1,2\sigma+2)$ the norm of the continuous injection of H^1 into $\mathrm{L}^{2\sigma+2}$.
Thus, choosing ϵ small enough, it is enough to show the result for the stochastic integral remplacing ρ by $\frac{\rho}{2}$. Also it is enough to show the result for each of the three stochastic integrals replacing $\frac{\rho}{2}$ by $\frac{\rho}{6}$. With the same one parameter families and similar computations as in the proof of Lemma 3.7, we know that it is enough to obtain upper bounds of the brackets of the stochastic integrals

$$\begin{array}{l}Z_1(t)=\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(2\alpha c(\sigma)s\right)\nabla\overline{u^\tau}(s)\nabla dW(s)dx\\ Z_2(t)=\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(2\alpha c(\sigma)s\right)|u^\tau(s)|^{2\sigma}\,\overline{u^\tau}(s)dW(s)dx\\ Z_3(t)=2\beta Cm(\sigma,d)\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(2\alpha c(\sigma)s\right)\|u^\tau(s)\|_{\mathrm{L}^2}^{\frac{4\sigma}{2-\sigma d}}\,\overline{u^\tau}(s)dW(s)dx.\end{array}$$

We then obtain

$$\begin{array}{l}d<Z_1>_t\leq\exp\left(4\alpha c(\sigma)t\right)\sum_{j\in\mathbb{N}}\left(\nabla u^\tau(t),-i\nabla\Phi e_j\right)^2_{\mathrm{L}^2}dt\\ d<Z_2>_t\leq\exp\left(4\alpha c(\sigma)t\right)\sum_{j\in\mathbb{N}}\left(|u^\tau(t)|^{2\sigma}u^\tau(t),-i\Phi e_j\right)^2_{\mathrm{L}^2}dt\\ d<Z_3>_t\leq 4\beta^2C^2m(\sigma,d)^2\exp\left(4\alpha c(\sigma)t\right)\|u^\tau(t)\|_{\mathrm{L}^2}^{\frac{8\sigma}{2-\sigma d}}\sum_{j\in\mathbb{N}}\left(u^\tau(t),-i\Phi e_j\right)^2_{\mathrm{L}^2}dt.\end{array}$$

Using the Hölder inequality and, for Z_2, the continuous Sobolev injection of H^1 into $\mathrm{L}^{2\sigma+2}$ we obtain

$$\begin{array}{l}d<Z_1>_t\leq\exp\left(4\alpha c(\sigma)t\right)\|\Phi\|^2_{\mathcal{L}_2^{0,1}}b(\rho,\sigma,d)dt\\ d<Z_2>_t\leq\exp\left(4\alpha c(\sigma)t\right)c(1,2\sigma+2)^{2(2\sigma+2)}\|\Phi\|^2_{\mathcal{L}_2^{0,1}}b(\rho,\sigma,d)^{2\sigma+1}dt\\ d<Z_3>_t\leq 4\beta^2C^2m(\sigma,d)^2\exp\left(4\alpha c(\sigma)t\right)b(\rho,\sigma,d)^{\left(1+\frac{4\sigma}{2-\sigma d}\right)}\|\Phi\|^2_{\mathcal{L}_2^{0,1}}dt.\end{array}$$

We can then bound each of the three remainders $\left(R_l^i(t)\right)_{i=1,2,3}$ similar to that of Lemma 3.7 using the inequality $R_l^i(t)\leq 3l\int_0^t d<Z_i>_t$.
We conclude that it is possible to choose $T(L,\rho)$ equal to

$$\frac{1}{4\alpha c(\sigma)}\log\left(\frac{\alpha c(\sigma)\rho^2}{90b(\rho,\sigma,d)\|\Phi\|^2_{\mathcal{L}_2^{0,1}}\max\ \ 1,c(1,2\sigma+2)^{2(2\sigma+1)}b(\rho,\sigma,d)^{2\sigma},4\beta^2C^2m(\sigma,d)^2b(\rho,\sigma,d)^{\frac{4\sigma}{2-\sigma d}}}\right).$$

When the noise is of multiplicative type we obtain

$$\begin{array}{l}\tilde{\mathbf{H}}\left(u(t)\right)-\exp\left(-2\alpha c(\sigma)t\right)\tilde{\mathbf{H}}\left(u_0\right)\\ \leq\sqrt{\epsilon}\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)u(s)\nabla\overline{u}(s)\nabla dW(s)dx\\ \quad+\frac{\epsilon}{2}\sum_{j\in\mathbb{N}}\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)\int_{\mathbb{R}^d}|u(s)|^2|\nabla\Phi e_j|^2dxds.\end{array}$$

Again we use a localization argument and consider the process u stopped at the exit off $\tilde{\mathbf{H}}_{2\rho}$. As Φ is Hilbert-Schmidt from L^2 into $\mathrm{H}^s_{\mathbb{R}}$, the second term of the right hand side is less than $\frac{\epsilon}{4\alpha c(\sigma)}\|\Phi\|^2_{\mathcal{L}_2^{0,s}}b(\rho,\sigma,d)$ and for ϵ small enough, it is enough to prove the result for the stochastic integral replacing ρ by $\frac{\rho}{2}$. We know that it is enough to obtain an upper bound of the bracket of

$$Z(t)=\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(2\alpha c(\sigma)s\right)u^{\tau}(s)\nabla\overline{u}^{\tau}(s)\nabla dW(s)dx.$$

We obtain

$$d<Z>_t\leq\exp\left(4\alpha c(\sigma)t\right)\sum_{j\in\mathbb{N}}\left(\nabla u^{\tau}(t),-iu^{\tau}(t)\nabla\Phi e_j\right)^2_{\mathrm{L}^2}dt.$$

Denoting by $c(s,\infty)$ the norm of the Sobolev injection of $\mathrm{H}^s_{\mathbb{R}}$ into $\mathrm{W}^{1,\infty}_{\mathbb{R}}$ we deduce that

$$d<Z>_t\leq\exp\left(4\alpha c(\sigma)t\right)c(s,\infty)^2\|\Phi\|^2_{\mathcal{L}_2^{0,s}}b(\rho,\sigma,d)^2dt.$$

Finally, we conclude that we may choose

$$T(L,\rho)=\frac{1}{4\alpha c(\sigma)}\log\left(\frac{\alpha c(\sigma)\rho^2}{10b(\rho,\sigma,d)^2c(s,\infty)^2\|\Phi\|^2_{\mathcal{L}_2^{0,s}}L}\right).$$

□

We may now prove Theorem 4.6 and 4.7.
Here are some of the specific aspects of the proof of Theorem 4.6.
Proof of Theorem 4.6. There is no difference in the proof of the upper bound on τ^{ϵ,u_0}. Let us thus focus on the lower bound. Take δ positive. Since $\underline{e}>0$, we now choose ρ positive such that $\underline{e}-\frac{\delta}{4}\leq e_\rho$, $\tilde{\mathbf{H}}_{2\rho}\subset D$ and $\tilde{\mathbf{H}}_{2\rho}\subset D^c_{-\rho}$. We define the sequences of stopping times $\theta_0=0$ and for k in $\mathbb{N}$,

$$\begin{aligned}\tau_k&=\inf\left\{t\geq\theta_k:\ u^{\epsilon,u_0}(t)\in\tilde{\mathbf{H}}_{<\rho}\cup D^c\right\},\\ \theta_{k+1}&=\inf\left\{t>\tau_k:\ u^{\epsilon,u_0}(t)\in\tilde{\mathbf{H}}_{2\rho}\right\},\end{aligned}$$

where $\theta_{k+1}=\infty$ if $u^{\epsilon,u_0}(\tau_k)\in\partial D$. Let us fix $T_1=T\left(\underline{e}-\frac{3\delta}{4},\rho\right)$ given by Lemma 4.9. We now use that for u_0 in D and m a positive integer,

$$\begin{aligned}\mathbb{P}\left(\tau^{\epsilon,u_0}\le mT_1\right)\le\ & \mathbb{P}\left(\tau^{\epsilon,u_0}=\tau_0\right)+\textstyle\sum_{k=1}^m\mathbb{P}\left(\tau^{\epsilon,u_0}=\tau_k\right)\\ &+\textstyle\sum_{k=1}^m\mathbb{P}\left(\theta_k-\tau_{k-1}\le T_1\right)\end{aligned}\tag{4.11}$$

and conclude as in the proof of Theorem 3.2. □

We may also check that the proof of Theorem 3.3 also applies to prove Theorem 4.5, the LDPs are those in H^1 and the sequences of stopping times are those defined above.

Again the control argument to prove that $\underline{e}=\overline{e}$ seems difficult. We may however apply in the H^1 case the Sobolev injection in order to treat the nonlinearity.

Let us now make an interesting comment. Assume that we are able to prove Theorem 4.4 with $\overline{e}=\underline{e}$ at least for an additive noise. The exit points are then characterized by the infimum of the quasi potential on the boundary of the domain of attraction. Under assumptions such that Φ commutes with the Laplacian and that Φ does not change the phase, we have an explicit expression of the quasi potential since the vector-field in the drift is the sum of a gradient vector-field and a vector-field which is orthogonal to the first one, see for example [14, 15]. These assumptions on Φ are such that we can mimick the computations for the ideal white noise. The quasipotential is proportional to $\mathbf{N}_H(u)=\left\|\left(\Phi_{|\ker\Phi^\perp}\right)^{-1}u\right\|_{\mathrm{L}^2}^2$. Indeed, the rate function of the LDP applied to u, for T positive, may be written for γ in $\mathbb{R}$,

$$\begin{aligned}&I_T^{u_0}(u)\\ &=\tfrac12\int_0^T\left\|\left(\left(\Phi_{|\ker\Phi^\perp}\right)^{-1}\left(i\tfrac{\partial u}{\partial t}+i\alpha(1-\gamma)u+i\alpha\gamma u-\Delta u-\lambda|u|^{2\sigma}u\right)\right)(s)\right\|_{\mathrm{L}^2}^2ds\\ &=\tfrac12\int_0^T\left\|\left(\left(\Phi_{|\ker\Phi^\perp}\right)^{-1}\left(i\tfrac{\partial u}{\partial t}+i\alpha(1-\gamma)u-\Delta u-\lambda|u|^{2\sigma}u\right)\right)(s)\right\|_{\mathrm{L}^2}^2ds\\ &\quad+\tfrac{\alpha\gamma}{2}\left[\mathbf{N}_H(u(T))-\mathbf{N}_H(u_0)\right]+\alpha^2\left(\tfrac{\gamma^2}{2}+(1-\gamma)\gamma\right)\int_0^T\mathbf{N}_H(u(s))ds.\end{aligned}$$

The last term is equal to zero if and only if $\gamma=2$ or $\gamma=0$. When $\gamma=2$ we obtain

$$\begin{aligned}V(0,u_f)=\ &\inf\Big\{\tfrac12\int_0^T\left\|\left(\left(\Phi_{|\ker\Phi^\perp}\right)^{-1}\left(\tfrac{\partial u}{\partial t}-\alpha u+i\Delta u+i\lambda|u|^{2\sigma}u\right)\right)(s)\right\|_{\mathrm{L}^2}^2ds\\ &\qquad+\alpha\mathbf{N}_H(u_f):\ u(0)=0,\ u(T)=u_f,\ T>0\Big\}\\ \ge\ &\alpha\mathbf{N}_H(u_f).\end{aligned}$$

In order to prove the converse inequality, we should prove that there exists a sequence of functions satisfying the boundary conditions such that the first term is arbitrarily small; it is another control problem. Assume that we are able to solve it, then the quasi potential is indeed proportional to the mass.

Suppose now that the domain of attraction is a set of the form $\tilde{\mathbf{H}}_{<\rho}$ for ρ positive. Exit points are points of the level set $\tilde{\mathbf{H}}_{\rho}$ that minimize $\mathbf{N}_H$. Since $\mathbf{N}_H$ is also the square of the norm of the reproducing kernel Hilbert space of the law of $W(1)$, or because Φ is Hilbert-Schmidt, we know that infima do exist. Also because they satisfy $\tilde{\mathbf{H}}_{<\rho}(u) = \rho$ they are different from 0. Note that in the ideal white noise case infima do not exist and the infimum is 0. By a standard minimization argument we deduce that the exit points satisfy for some ω in $\mathbb{R}$,

$$\left(\left(\left(\Phi_{|\ker\Phi^{\perp}}\right)^{-1}\right)^{*}\left(\Phi_{|\ker\Phi^{\perp}}\right)^{-1} + \omega 2\beta C\|u\|_{\mathrm{L}^2}^{\frac{4\sigma}{2-\sigma d}}\right) u = \omega\left(\Delta u + \lambda |u|^{2\sigma} u\right).$$

The case where $\omega = 0$ corresponds to $u = 0$; we may thus assume that $\omega \neq 0$. When $\Phi = I$ and $\lambda = 1$, this equation has solutions which are solitary waves profiles.

If we could approximate the white noise in a suitable sense and justify all of the above rigorously, it would give an important information on the dynamical behavior of the solutions of the nonlinear equation under the influence of a noise. Indeed, it would give an indication that the energy injected by the noise organizes and creates solitary waves. Note that such behavior has been observed numerically in [10] on the Korteweg-de Vries equation.

5 Annex - proof of Theorem 2.1

The following lemma proves to be at the core of the proof of the uniform LDPs. It is often called Azencott lemma or Freidlin-Wentzell inequality. The differences with the result of [18] are that here the initial data are the same for the random process and the skeleton and that the "for every ρ positive" stands before "there exists ϵ_0 and γ positive". We shall only stress on the differences in the proof.

Lemma 5.1 *For every a, L, T, δ and ρ positive, f in C_a, p in $\mathcal{A}(d)$, there exists ϵ_0 and γ positive such that for every ϵ in $(0, \epsilon_0)$, $\|u_0\|_{\mathrm{H}^1} \leq \rho$,*

$$\epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} \geq \delta;\ \|\sqrt{\epsilon}W - f\|_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)} < \gamma\right) \leq -L.$$

Proof. There are still three steps in the proof of this result. The first step is a change of measure to center the process around f. It uses the Girsanov theorem and is the same as in [18].
The second step is a reduction to estimates for the stochastic convolution. It strongly involves the Strichartz inequalities but it is slightly different than in [18]. The truncation argument has to hold for all $\|u_0\|_{\mathrm{H}^1} \leq \rho$. Thus we use the fact that there exists $M = M(T, \rho, \sigma)$ positive such that

$$\sup_{u_1 \in B_\rho^1} \left\| \tilde{\mathbf{S}}(u_1, f) \right\|_{X^{(T,p)}} \leq M.$$

The proof of this fact follows from the computations in [6], we will recall the arguments in L^2 in the proof of Lemma 3.5. The result in H^1 will again be used in the proof of Lemma 4.7. As the initial data are the same for the random process and the skeleton, the remaining of the argument does not require restrictions on ρ.
The third step corresponds to estimates for the stochastic convolution. It is the same as in [18].
Note that the extra damping term in the drift is treated easily thanks to the Strichartz inequalities. $\square$

We shall now prove Theorem 2.1.
Proof of Theorem 2.1. Let us start with the case of an additive noise. Recall that, in that case, the mild solution of the stochastic equation could be written as a function of the perturbation in the convolution form. Let $v^{u_0}(Z)$ denote the solution of

$$\begin{cases} i\frac{\partial v}{\partial t} - \left(\Delta v + |v - iZ|^{2\sigma}(v - iZ) - i\alpha(v - iZ)\right) = 0, \\ v(0) = u_0, \end{cases}$$

or equivalently a fixed point of the functional $\mathcal{F}_Z$ such that

$$\begin{aligned} \mathcal{F}_Z(v)(t) = \;& U(t)u_0 - i\lambda \int_0^t U(t-s)\left(|(v - iZ)(s)|^{2\sigma}(v - iZ)(s)\right) ds \\ & -\alpha \int_0^t U(t-s)(v - iZ)(s) ds, \end{aligned}$$

where Z belongs to $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ (respectively $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$). If u^{ϵ,u_0} is defined as $u^{\epsilon,u_0} = v^{u_0}\left(Z^\epsilon\right) - iZ^\epsilon$ where Z^ϵ is the stochastic convolution $Z^\epsilon(t) = \sqrt{\epsilon}\int_0^t U(t-s)dW(s)$ then u^{ϵ,u_0} is a solution of the stochastic equation. Consequently, if $\mathcal{G}\left(\cdot, u_0\right)$ denotes the mapping from $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ (respectively $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$) to $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ (respectively $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$) defined by $\mathcal{G}\left(Z, u_0\right) = v^{u_0}(Z) - iZ$, we obtain $u^{\epsilon,u_0} = \mathcal{G}\left(Z^\epsilon, u_0\right)$. We may also check with arguments similar to that of [6, 17], involving the Strichartz inequalities that the mapping $\mathcal{G}$ is equicontinuous in its first arguments for second arguments in bounded sets of L^2 (respectively H^1). The result now follows from Proposition 5 in [25].

Let us now consider the case of a multiplicative noise. Initial data belong to H^1 and we consider paths in H^1. The proof is very close to that in [18].

The main tool is again the Azencott lemma or almost continuity of the Itô map. We need the slightly different result from that in [18].
Let us see how the above lemma implies (i) and (ii).

We start with the upper bound (i). Take a, ρ, T and δ positive. Take $L > a$. For $\tilde{a}$ in $(0, a]$, we denote by

$$A_{\tilde{a}}^{u_0} = \left\{ v \in \mathrm{C}\left([0,T];\mathrm{H}^1\right) : \ d_{\mathrm{C}([0,T];\mathrm{H}^1)}\left(v, K_T^{u_0}(\tilde{a})\right) \geq \delta \right\}.$$

Note that we have $A_a^{u_0} \subset A_{\tilde{a}}^{u_0}$ and $C_{\tilde{a}} \subset C_a$. Take $\tilde{a} \in (0, a]$ and f such that $I_T^W(f) < \tilde{a}$.

We shall now apply the Azencott lemma and choose $p = 2$. We obtain $\epsilon_{\rho,f,\delta}$ and $\gamma_{\rho,f,\delta}$ positive such that for every $\epsilon \leq \epsilon_{\rho,f,\delta}$ and u_0 such that $\|u_0\|_{\mathrm{H}^1} \leq \rho$,

$$\epsilon \log \mathbb{P}\left(\left\| u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f) \right\|_{X^{(T,p)}} \geq \delta ; \left\| \sqrt{\epsilon} W - f \right\|_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)} < \gamma_{\rho,f,\delta} \right) \leq -L.$$

Let us denote by $O_{\rho,f,\delta}$ the set $O_{\rho,f,\delta} = B_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)}(f, \gamma_{\rho,f,\delta})$. The family $(O_{\rho,f,\delta})_{f \in C_a}$ is a covering by open sets of the compact set C_a, thus there exists a finite sub-covering of the form $\bigcup_{i=1}^N O_{\rho,f_i,\delta}$. We can now write

$$\begin{aligned}
\mathbb{P}\left(u^{\epsilon,u_0} \in A_{\tilde{a}}^{u_0}\right) \leq\ & \mathbb{P}\left(\left\{ u^{\epsilon,u_0} \in A_{\tilde{a}}^{u_0} \right\} \cap \left\{ \sqrt{\epsilon} W \in \textstyle\bigcup_{i=1}^N O_{\rho,f_i,\delta} \right\} \right) \\
& + \mathbb{P}\left(\sqrt{\epsilon} W \notin \textstyle\bigcup_{i=1}^N O_{\rho,f_i,\delta} \right) \\
\leq\ & \textstyle\sum_{i=1}^N \mathbb{P}\left(\left\{ u^{\epsilon,u_0} \in A_{\tilde{a}}^{u_0} \right\} \cap \left\{ \sqrt{\epsilon} W \in O_{\rho,f_i,\delta} \right\} \right) \\
& + \mathbb{P}\left(\sqrt{\epsilon} W \notin C_a \right) \\
\leq\ & \textstyle\sum_{i=1}^N \mathbb{P}\left(\left\{ \left\| u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f) \right\|_{X^{(T,p)}} \geq \delta \right\} \cap \left\{ \sqrt{\epsilon} W \in O_{\rho,f_i,\delta} \right\} \right) \\
& + \exp\left(-\tfrac{a}{\epsilon} \right),
\end{aligned}$$

for $\epsilon \leq \epsilon_0$ for some ϵ_0 positive. We used that

$$d_{\mathrm{C}([0,T];\mathrm{H}^1)}\left(\tilde{\mathbf{S}}(u_0, f), A_{\tilde{a}}^{u_0} \right) \geq \delta,$$

which is a consequence of the definition of the sets $A_{\tilde{a}}^{u_0}$.

As a consequence, for $\epsilon \leq \epsilon_0 \wedge \left(\min_{i=1,..,N} \epsilon_{u_0,f_i}\right)$ we obtain for u_0 in B_ρ^1,

$$\mathbb{P}\left(u^{\epsilon,u_0} \in A_{\tilde{a}}^{u_0}\right) \leq \ N \exp\left(-\frac{L}{\epsilon} \right) + \exp\left(-\frac{a}{\epsilon} \right),$$

and for ϵ_1 small enough, for every $\epsilon \in (0, \epsilon_1)$,

$$\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0} \in A_{\tilde{a}}^{u_0}\right) \leq \epsilon \log 2 + (\epsilon \log N - L) \vee (-a).$$

If ϵ_1 is also chosen such that $\epsilon_1 < \frac{\gamma}{\log(2)} \wedge \frac{L-a}{\log(N)}$ we obtain

$$\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0} \in A_{\tilde{a}}^{u_0}\right) \leq -\tilde{a} - \gamma,$$

which holds for every u_0 such that $\|u_0\|_{\mathrm{H}^1} \leq \rho$.

We consider now the lower bound (ii). Take a, ρ, T and δ positive. The continuity of $\tilde{\mathbf{S}}(u_0, \cdot)$, to be proved as in [18], along with the compactness of C_a give that for u_0 such that $\|u_0\|_{\mathrm{H}^1} \leq \rho$ and w in $K_T^{u_0}(a)$, there exists f such that $w = \tilde{\mathbf{S}}(u_0, f)$ and $I_T^{u_0}(w) = I_T^W(f)$. Take $L > I^{u_0}(w)$. Choose $\epsilon_{\rho,f,\delta}$ positive and $O_{\rho,f,\delta}$, the ball centered at f of radius $\gamma_{\rho,f,\delta}$ defined as previously, such that for every $\epsilon \leq \epsilon_{\rho,f,\delta}$ and u_0 such that $\|u_0\|_{\mathrm{H}^1} \leq \rho$,

$$\epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} \geq \delta; \left\|\sqrt{\epsilon}W - f\right\|_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)} < \gamma_{\rho,f,\delta}\right) \leq -L.$$

We obtain

$$\begin{aligned} \exp\left(-\frac{I_T^W(f)}{\epsilon}\right) & \leq \mathbb{P}\left(\sqrt{\epsilon}W \in O_{\rho,f,\delta}\right) \\ & \leq \mathbb{P}\left(\left\{\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} \geq \delta\right\} \cap \left\{\sqrt{\epsilon}W \in O_{\rho,f,\delta}\right\}\right) \\ & \quad + \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} < \delta\right). \end{aligned}$$

Thus, for $\epsilon \leq \epsilon_{\rho,f,\delta}$, for every u_0 such that $\|u_0\|_{\mathrm{H}^1} \leq \rho$,

$$-I^{u_0}(w) \leq \epsilon \log 2 + \left(\epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} < \delta\right)\right) \vee (-L)$$

and for ϵ_1 small enough and such that $\epsilon_1 \log(2) < \gamma$, for every ϵ positive such that $\epsilon < \epsilon_1$, for every u_0 such that $\|u_0\|_{\mathrm{H}^1} \leq \rho$,

$$-I^{u_0}(w) - \gamma \leq \epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} < \delta\right).$$

It ends the proof of (i) and (ii). □

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