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# Small Noise Asymptotic of the Timing Jitter in Soliton Transmission

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# Small noise asymptotic of the timing jitter in soliton transmission

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**Abstract:** We consider random perturbations of the focusing cubic one dimensional nonlinear Schrödinger equation. The noises, either additive or multiplicative, are white in time and colored in space. We study the small noise asymptotic of the tails of the center and mass of a pulse at a fixed coordinate when the initial datum is null or a soliton profile. Our main tools are large deviation principles (LDPs) at the level of paths. Upper and lower bounds are obtained from bounds for the optimal control problems derived from the rate function of the LDPs. Our results agree with results from physics which had been obtained with arguments which seem difficult to fully justify mathematically. Some results are new.

**Résumé:** Nous étudions des perturbations aléatoires d'équations de Schrödinger avec nonlinéarité cubique et focalisante en dimension 1. Les bruits, additifs ou multiplicatifs sont blancs en temps et colorés en espace. Nous étudions l'asymptotique de petits bruits des queues du centre et de la masse d'un signal en un point pour des données initiales nulles ou profils de soliton. Nos outils principaux sont des principes de grandes déviations (PGDs) trajectoriels. Nous obtenons des bornes supérieures et inférieures à partir de bornes pour les problèmes de contrôle optimal issus de la fonction de taux des PGDs. Nos résultats concordent avec des résultats de physique obtenus par des arguments difficiles à justifier mathématiquement. Plusieurs résultats sont nouveaux.

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*Key Words:* Large deviations, stochastic partial differential equations, nonlinear Schrödinger equation, solitons.

*Mots-clés:* Grandes déviations, équations aux dérivées partielles stochastiques, équation de Schrödinger non-linéaire, solitons.

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# 1 Introduction

The nonlinear Schrödinger (NLS) equation occurs as a generic model in many areas of physics and describes the propagation of slowly varying envelopes of a wave packet in media with both nonlinear and dispersive responses. The one-dimensional equation with a cubic focusing nonlinearity is for example a model in the context of long-haul transmission lines in fiber optics; see for example [25] for a derivation of the equation in that context. The variable $t$ stands for the space coordinate and $x$ for some retarded time. Resulting from a balance between the focusing nonlinearity and the dispersive linear part, localized (here in time) waves propagate, they are called solitons or solitary waves. The functions

$$\sqrt{2}A\mathrm{sech}(A(x-x_0)+2A\Omega t)\exp\left(-i(A^2-\Omega^2)t+i\Omega(x-x_0)+i\theta_0\right) \quad (1.1)$$

where $A>0$ is the amplitude, $\Omega$ is the group velocity or angular carrier frequency, $x_0$ and $\theta_0$ are respectively the initial position and phase, are solitons. In soliton based amplitude-shifted-keyed systems (ASK) communication systems, solitons are used as information carriers to transmit the datum 0 or 1. A 1 corresponds to the emission of a soliton at time 0 with null velocity $\Psi_A^0(x)=\sqrt{2}A\mathrm{sech}(Ax)$. It is produced by a laser beam. At the far end $T$ of the fiber a receiver records

$$\frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}}|u^{u_0}(T,x)|^2\,dx,\ u_0=0 \text{ or } u_0=\Psi_A^0,$$

$\left[-\frac{l}{2},\frac{l}{2}\right]$ is a window in time; $l$ may be chosen small since the wave $u^{u_0}$, solution of the NLS equation, is localized and remains centered. When the above quantity is above a threshold $I_d$ it is decided that a 1 has been emitted, otherwise it is decided that a 0 has been emitted.

However, it is physically more relevant to consider random perturbations and then error in transmission may occur. Phenomena such as a fluctuating dielectric permittivity, a deviating fiber radius or a random initial shape maybe taken into account in a perturbation term. Moreover noise is somehow intrinsic to such systems.

To counterbalance for loss in the fiber, regularly spaced amplifiers are placed along the line and the distance between amplifiers is small compared to the length of the line. If we suppose that the gain is adjusted to counterbalance exactly for loss, there remains a spontaneous emission noise. This could be justified theoretically thanks to Heisenberg's uncertainty principle.

This noise could be modeled as a random external force; see for example [15, 18, 32]. We could formally write the equation as

$$i\frac{\partial u^{\epsilon,u_0}}{\partial t} = \Delta u^{\epsilon,u_0} + |u^{\epsilon,u_0}|^2 u^{\epsilon,u_0} + \sqrt{\epsilon}\xi, \tag{1.2}$$

where $\epsilon$ stands for the small noise amplitude, $\xi$ is a complex Gaussian space-time noise and $u_0$ is the initial datum. The functions are complex valued. Note that this equation also appears in the context of anharmonic atomic chains in the presence of thermal fluctuation; see for example [7].

Other types of amplification among which Raman coupling to thermal phonon, see [16, 17, 30], and four-wave-mixing, see [16, 31], also lead to spontaneaous emission of noise. However in this case the noise enters as a real multiplicative noise. Note that in the case of the Raman amplification a Raman nonlinear response also appears in the equation and the Raman effect also contributes to the Kerr effect, *i.e.* the power law nonlinearity. It is assumed that the extra Raman nonlinear response may be neglected to a first approximation in a treatment of the noise effect on the frequency and thus, by dynamical coupling, on the position of the pulse since it produces essentially a deterministic shift in frequency. The evolution equation may be written formally as

$$i\frac{\partial u^{\epsilon,u_0}}{\partial t} = \Delta u^{\epsilon,u_0} + |u^{\epsilon,u_0}|^2 u^{\epsilon,u_0} + \sqrt{\epsilon}u^{\epsilon,u_0}\xi, \tag{1.3}$$

in that case the noise $\xi$ is a real Gaussian noise. Note that this model is also introduced in the context of crystals; see for example [3, 4, 5].

In the presence of noise, the soliton is progressively distorted by the noise, even though it is small, and with small probability an error in transmission may occur in the sense that 1 is discarded. Also, when the noise is additive, it may create from nothing a structure that might be mistaken as a 1.

When a 1 is emitted, it is assumed that two processes are mainly responsible for the loss of the signal: a decrease of the mass

$$\mathbf{N}\left(u^{\epsilon,\Psi_A^0}(T)\right) = \left\|u^{\epsilon,\Psi_A^0}(T)\right\|_{\mathrm{L}^2}^2$$

and a diffusion in position, characterized by the center of the pulse

$$\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(T)\right) = \int_{\mathbb{R}} x\left|u^{\epsilon,\Psi_A^0}(T,x)\right|^2 dx.$$

The fluctuation of the center results in a shift in the arrival time. It is called timing jitter. The event that for null initial datum a 1 is detected only results from a large fluctuation of the mass.

When the noise is of multiplicative type the mass is invariant and we shall only focus on the timing jitter.
Considering that the probability of sending a 1 is $\frac{1}{2}$, the bit error rate is defined as

$$\frac{1}{2}\mathbb{P}\left(\frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}}\left|u^{\epsilon,\Psi_A^0}(T,x)\right|^2 dx \leq I_d\right)+\frac{1}{2}\mathbb{P}\left(\frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}}\left|u^{\epsilon,0}(T,x)\right|^2 dx > I_d\right),$$

the probabilities that the measured quantities are below or above the threshold are conditional probabilities. Again, in the case of a multiplicative noise the second conditionnal probability is null. In practical applications, this bit error rate might be less than $10^{-9}$. Moreover it is widely admitted that the statistics are not Gaussian. Thus a statistical treatment for inference of the bit error rate requires a theoretical evaluation.

In the physics literature the amplitude of the noise is assumed to be small. Physical techniques often rely on an adiabatic perturbation theory where the pulse is approximated by a soliton ansatz with finite fluctuating collective variables; it requires that the noise is small.

Some articles from physics study the variance of the center; see for example of [7, 17, 25]. In the seminal paper [25] of Gordon and Haus it is obtained that the variance of the center is of the order of $T^3$ (superdiffusion, *i.e.* stronger than that of the Brownian motion which is linear) and that the fluctuation of the center is connected with a shift in the soliton carrier frequency. It is assumed that the timing jitter is the most troublesome and upper limit of the information rate is derived based on a Gaussian assumption. In [17], the only paper from physics we found on noise induced timing jitter when the noise is multiplicative, a Raman-modified NLS equation is considered; independent complex additive and real multiplicative noises appear both in the equation. The contribution of each noise to the variance of the center is of the order $T^3$. They however exhibit a different behavior in the initial amplitude $A$.

Other articles study the deviation from the Gaussian assumption. Again using the perturbation theory of solitons, see for example [26, 27], physicists have obtained that the statistics of the center may be non Gaussian when there is soliton interaction or filtering, see for example [15, 19, 20, 38, 33]. Otherwise it could be considered as Gaussian in the first order only; see for example [1, 15, 29]. In [34] as in [25] the model is a juxtaposition of deterministic evolutions with randomly perturbed initial initial data in between amplifiers. The log of the tails of the amplitude and center are evaluated numerically via an importance sampled Monte Carlo estimator. Simulations are obtained from a distribution where the small probability event is a central event; they are weighted by a likelihood ratio weight. It is obtained that the log of tails of the amplitude only differs significantly from that of Gaussian tails. Note that we may expect to use the numerical methodology based on a genealogical particle analysis developed in [8]. In this reference the importance sampling and Monte Carlo methodologies are compared to a particle system approach and it is applied to the estimation of probability of rare events due to polarization-mode dispersion in optical fibers.

In [15, 18, 32], probability density functions (PDF) are examinated. In [18] the PDF of the joint law of the mass and center at coordinate $T$, when the initial datum is a soliton profile, are approximated from a PDF of the random parameters of a solution described as a soliton with a finite set of fluctuating parameters. The parameters are assumed to evolve according to dynamically coupled SDEs. This latter PDF is obtained via a saddle point approximation of a corresponding finite dimensional Martin-Siggia-Rose effective action. The complete infinite dimensional effective action, see for example [28] is not treated. The PDF of the amplitude (a multiple of the mass with the parametrization) is obtained when the initial datum is null. The probability of loosing a 1 is numerically evaluated under the assumption of a very large window. In [15] the Fokker-Planck equation is used to obtain the PDF of the mass at $T$. In [32] a similar result is obtained. However the PDF of the marginal law of the center has not been evaluated.

Note that infinite dimensional effective actions in physics are intimately related to the rate function of a sample path large deviation principle (LDP). Paths minimizing the action for certain configurations of the system are called optimal fluctuations or instantons, see also for example [2, 37]. Note that in [21], where the large deviations approach is adopted, the problem of transitions between stable equilibrium configurations (tunnelling) of unforced nonlinear heat equations in the limit of small noise is studied. The most likely transitions are the instantons from quantum mechanics; they are saddle points of the equilibrium action functional related to the rate function of the sample path LDP. Exit from neighborhoods of zero for weakly damped stochastic NLS equations is studied in the article [24].

In the present article we apply sample path LDPs to the study of the tails of the law of the mass and center of the pulse at the end of the fiber. We thus study cumulative distribution functions (CDFs) instead of PDFs but do not study the bulk of the distribution. As we will see, we are not able to treat mathematically the case of the space-time white noise which is mainly used in the physical models. We thus restrict ourselves to noises that are colored in space. In the case of a noise of additive type we will consider sequences of noises that mimic the white noise in the limit. The log of the tails in the limit of small noise are of the order of the opposite of the infima of a functional derived from the rate functions of the LDPs divided by the noise amplitude. The infima are optimal control problems. We give upper and lower bounds using energy inequalities and modulated solitons. The two bounds mostly differ up to multiplicative constants and the orders in $T$ and $A$ are compared to that of the physicists.

## 2 Notations and preliminaries

For $p \geq 1$, $\mathrm{L}^p$ is the classical Lebesgue space of complex valued functions on $\mathbb{R}$ and $\mathrm{W}^{1,p}$ is the associated Sobolev space of $\mathrm{L}^p$ functions with first order derivatives, in the sense of distributions, in $\mathrm{L}^p$. If $I$ is an interval of $\mathbb{R}$, $(E, \|\cdot\|_E)$ a Banach space and $r$ belongs to $[1,\infty]$, then $\mathrm{L}^r(I;E)$ is the space of strongly Lebesgue measurable functions $f$ from $I$ into $E$ such that $t \to \|f(t)\|_E$ is in $\mathrm{L}^r(I)$. The space $\mathrm{L}^2$ with the inner product defined by $(u,v)_{\mathrm{L}^2} = \mathfrak{Re} \int_{\mathbb{R}} u(x)\overline{v}(x)dx$ is a Hilbert space. The Sobolev spaces $\mathrm{H}^s$ are the Hilbert spaces of functions of $\mathrm{L}^2$ with partial derivatives up to order $s$ in $\mathrm{L}^2$. When $s$ is fractional it is defined classically via the Fourier transform. When the functions are real valued we specify it, for example we write $\mathrm{H}^s(\mathbb{R},\mathbb{R})$. The following Hilbert spaces of spatially localized functions

$$\Sigma = \left\{ f \in \mathrm{H}^1 : \ x \mapsto xf(x) \in \mathrm{L}^2 \right\},$$
$$\Sigma^{\frac{1}{2}} = \left\{ f \in \mathrm{H}^1 : \ x \mapsto \sqrt{|x|}f(x) \in \mathrm{L}^2 \right\}$$

are also introduced and endowed with the norms

$$\|f\|_{\Sigma}^2 = \|f\|_{\mathrm{H}^1}^2 + \|x \mapsto xf(x)\|_{\mathrm{L}^2}^2,$$
$$\|f\|_{\Sigma^{\frac{1}{2}}}^2 = \|f\|_{\mathrm{H}^1}^2 + \left\|x \mapsto \sqrt{|x|}f(x)\right\|_{\mathrm{L}^2}^2.$$

We denote by $\|\Phi\|_{\mathcal{L}_c(A,B)}$ the norm of $\Phi$ as a linear continuous operator from $A$ to $B$, where $A$ and $B$ are normed vector spaces. We recall that $\Phi$ is a Hilbert-Schmidt operator from $H$ to $\tilde{H}$, where $H$ and $\tilde{H}$ are Hilbert spaces, if it is a linear continuous operator such that, given a complete orthonormal system $(e_j^H)_{j=1}^{\infty}$ of $H$, $\sum_{j=1}^{\infty} \|\Phi e_j^H\|_{\tilde{H}}^2 < \infty$. We will denote by $\mathcal{L}_2(H,\tilde{H})$ the space of Hilbert-Schmidt operators from $H$ to $\tilde{H}$ endowed with the norm

$$\|\Phi\|_{\mathcal{L}_2(H,\tilde{H})} = \mathrm{tr}\left(\Phi\Phi^*\right) = \sum_{j=1}^{\infty} \|\Phi e_j^H\|_{\tilde{H}}^2.$$

We also recall that a cylindrical Wiener process $W_c$ in a Hilbert space $H$ is such that for any complete orthonormal system $(e_j)_{j=1}^{\infty}$ of $H$, there exists a sequence of independent Brownian motions $(\beta_j)_{j=1}^{\infty}$ such that $W_c = \sum_{j=1}^{\infty} \beta_j e_j$. This sum does not converge in $H$ but in any Hilbert space $U$ such that the embedding $H \subset U$ is Hilbert-Schmidt. The image of the process $W_c$ by a linear mapping $\Phi$ on $H$ is a well defined process in $H$ when the mapping is Hilbert-Schmidt on $H$, *i.e.* $\Phi \in \mathcal{L}_2(H) = \mathcal{L}_2(H, H)$. Then, $W = \Phi W_c$ is such that $W(1)$ is well defined with a covariance operator $\Phi\Phi^*$.

We recall that a rate function $I$ is a lower semicontinuous function and that a good rate function $I$ is a rate function such that for every positive $c$, $\{x : \ I(x) \le c\}$ is a compact set.

Let us now recall some mathematical aspects of the stochastic NLS equations. The equations, written as SPDEs in the Itô form, are in the additive case

$$i\mathrm{d}u^{\epsilon,u_0} - (\Delta u^{\epsilon,u_0} + |u^{\epsilon,u_0}|^2 u^{\epsilon,u_0})\mathrm{d}t = \sqrt{\epsilon}\mathrm{d}W, \tag{2.1}$$

and in the multiplicative case

$$i\mathrm{d}u^{\epsilon,u_0} - (\Delta u^{\epsilon,u_0} + |u^{\epsilon,u_0}|^2 u^{\epsilon,u_0})\mathrm{d}t = \sqrt{\epsilon}u^{\epsilon,u_0} \circ \mathrm{d}W. \tag{2.2}$$

The symbol $\circ$ stands for the Stratonovich product. In the case of equation (2.2), see [10], the mass

$$\mathbf{N}\left(u^{\epsilon,u_0}(t)\right) = \|u^{\epsilon,u_0}(t)\|_{\mathrm{L}^2}^2, \quad t > 0$$

is a conserved quantity. Precise assumptions on $\Phi$ such that $W = \Phi W_c$ are made below. These equations are supplemented with an initial datum

$$u^{\epsilon,u_0}(0) = u_0.$$

In this paper, we consider initial data in $\Sigma \subset \mathrm{H}^1$ and work with the solution constructed in [10]. Since we work with a subcritical non linearity, we could also consider solutions in $\mathrm{L}^2$ with initial data in $\mathrm{L}^2$. However, the $\mathrm{H}^1-$setting is preferred in order to be able to consider the spaces $\Sigma$ and $\Sigma^{\frac{1}{2}}$ and study the center of the pulse

$$\mathbf{Y}(u^{\epsilon,u_0}(t)) = \int_{\mathbb{R}} x|u^{\epsilon,u_0}(t,x)|^2 dx, \quad t > 0,$$

defined when $u^{\epsilon,u_0}(t)$ belongs to $\Sigma^{\frac{1}{2}}$.

We are concerned by weak solutions or equivalently by mild solutions which, in the additive case, satisfy

$$\begin{aligned} u^{\epsilon,u_0}(t) = \; & U(t)u_0 - i\int_0^t U(t-s)(|u^{\epsilon,u_0}(s)|^2 u^{\epsilon,u_0}(s))ds \\ & -i\sqrt{\epsilon}\int_0^t U(t-s)dW(s) \end{aligned} \tag{2.3}$$

where $(U(t))_{t\in\mathbb{R}}$ stands for the Schrödinger group, $U(t) = e^{-it\Delta},\ t \in \mathbb{R}$. The last term is called the stochastic convolution. In the multiplicative case, the mild equation is

$$\begin{aligned} u^{\epsilon,u_0}(t) = \; & U(t)u_0 - i\int_0^t U(t-s)(|u^{\epsilon,u_0}(s)|^2 u^{\epsilon,u_0}(s))ds \\ & -i\sqrt{\epsilon}\int_0^t U(t-s)u^{\epsilon,u_0}(s)dW(s) - \frac{i\epsilon}{2}\int_0^t U(t-s)F_\Phi u^{\epsilon,u_0}(s)ds \end{aligned} \tag{2.4}$$

where the stochastic integral is a Itô integral and, given $(e_j)_{j=1}^\infty$ an orthonormal basis of $\mathrm{L}^2$, $F_\Phi(x) = \sum_{j=1}^\infty (\Phi e_j)^2(x)$. The term $\frac{\epsilon}{2}F_\Phi(x)$ is the Itô correction.

The noise is the time derivative in the sense of distributions of the Wiener process $W$. It corresponds to a white noise in time. A space-time white noise would correspond to $\Phi$ equal to the identity. We cannot handle such rough noises and make the assumption that the two noises are colored in space. The basic limitation is that, unlike semi-groups like the Heat semi-group, the Schrödinger group is an isometry and does not allow smoothing in the Sobolev spaces based on $\mathrm{L}^2$. For instance, in the additive case, it can be seen that the stochastic convolution is a well defined process with paths in $\mathrm{L}^2$ if and only if $\Phi$ is a Hilbert-Schmidt operator on $\mathrm{L}^2$.

In fact, we make even stronger assumptions. In the additive case we assume that $W$ is a Wiener process on $\Sigma$. In the multiplicative case, it is imposed that $W$ is a Wiener process on $\mathrm{H}^s(\mathbb{R},\mathbb{R})$ where $s$ satisfies $s > \frac{3}{2}$.

We know that the Cauchy problem is globally well posed in $\mathrm{H}^1$; see [10] for a general discussion on the local well posedness and the global existence for more general nonlinearities and dimensions. Note that the present deterministic NLS equation is integrable thanks to the inverse scattering method. We will not use these techniques in the article. Results on the influence of the noise on the blow-up time, for more general nonlinearities and dimensions are given in [11, 12]. In [6, 14] the ideal white noise and results on the influence of a noise on the blow-up are studied numerically.

Sample path LDPs for stochastic NLS equations are proved in [22, 23]. These LDPs do not allow to treat the center of the solution and we shall consider LDPs in $C\left([0,T];\Sigma^{\frac{1}{2}}\right)$ where $T$ is positive (the length of the fiber line). The rate function of the LDP in the additive case is defined in terms of the mild solution of the control problem

$$\left\{\begin{array}{l} i\frac{\mathrm{d}u}{\mathrm{d}t}=\Delta u+|u|^2u+\Phi h, \\ u(0)=u_0 \;\in \Sigma \text{ and } h\in \mathrm{L}^2\left(0,T;\mathrm{L}^2\right). \end{array}\right. \tag{2.5}$$

We denote the solution by $u=\mathbf{S}^{a,u_0}(h)$. The mapping $h\to\mathbf{S}^{a,u_0}(h)$ is called the skeleton and (2.5) the skeleton equation.

In the multiplicative case, the controlled equation is

$$i\frac{\mathrm{d}u}{\mathrm{d}t}=\Delta u+|u|^2u+u\Phi h, \tag{2.6}$$

whose mild solution is denoted by $u=\mathbf{S}^{m,u_0}(h)$. The mapping $\mathbf{S}^{m,u_0}$ is again called the skeleton and (2.6) the skeleton equation.

In this article, when describing properties which hold both in the additive and multiplicative cases, we use the symbol $\mathbf{S}(u_0,h)$ to denote either $\mathbf{S}^{a,u_0}(h)$ or $\mathbf{S}^{m,u_0}(h)$.

Let us now state the sample path LDPs. The proof is given in the annex.

**Theorem 2.1** *Assume that $\Phi$ belongs to $\mathcal{L}_2(\mathrm{L}^2,\Sigma)$ in the additive case and $\Phi\in\mathcal{L}_2(\mathrm{L}^2,\mathrm{H}^s(\mathbb{R},\mathbb{R}))$ with $s>3/2$ in the multiplicative case. Assume also that the initial datum $u_0$ is in $\Sigma$. Then the solutions of the stochastic nonlinear Schrödinger equations* (2.3) *and* (2.4) *are almost surely in $C([0,T];\Sigma^{\frac{1}{2}})$. Moreover, they define $C([0,T];\Sigma^{\frac{1}{2}})$ random variables and their laws $\left(\mu^{u^{\epsilon,u_0}}\right)_{\epsilon>0}$ satisfy a LDP of speed $\epsilon$ and good rate function*

$$I^{u_0}(w)=\frac{1}{2}\inf_{h\in\mathrm{L}^2(0,T;\mathrm{L}^2):\; w=\mathbf{S}(u_0,h)}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)},$$

*where $\mathbf{S}(u_0,\cdot)=\mathbf{S}^{a,u_0}(\cdot)$ in the additive case and $\mathbf{S}(u_0,\cdot)=\mathbf{S}^{m,u_0}(\cdot)$ in the multiplicative case, and with the convention that $\inf\emptyset=\infty$. It means that for every Borel set $B$ of $C\left([0,T];\Sigma^{\frac{1}{2}}\right)$, we have the lower bound*

$$-\inf_{w\in Int(B)}I^{u_0}(w)\le\underline{\lim}_{\epsilon\to0}\epsilon\log\mathbb{P}\left(u^{\epsilon,u_0}\in B\right)$$

*and the upper bound*

$$\overline{\lim}_{\epsilon\to0}\epsilon\log\mathbb{P}\left(u^{\epsilon,u_0}\in B\right)\le-\inf_{w\in\overline{B}}I^{u_0}(w).$$

These sample path LDPs allow for example to evaluate the probability that, originated from a soliton profile

$$\Psi_A^0(x) = \sqrt{2}A\mathrm{sech}\,(Ax)\,,$$

the random solution be significantly different from the deterministic soliton solution

$$\Psi_A(t,x) = \Psi_A^0(x)\exp\left(-iA^2t\right).$$

Indeed, for $T$, $\delta$ and $\eta$ positive and $\epsilon$ small enough, the LDP implies that

$$\exp\left(-\frac{C_1}{\epsilon}\right) \le \mathbb{P}\left(\left\|u^{\epsilon,\Psi_A^0} - \Psi_A\right\|_{\mathrm{C}([0,T];\Sigma^{\frac{1}{2}})} > \delta\right) \le \exp\left(-\frac{C_2}{\epsilon}\right),$$

where

$$C_1 = \inf_{w:\ \|w-\Psi_A\|_{\mathrm{C}([0,T];\Sigma^{\frac{1}{2}})} > \delta} I^{\Psi_A^0}(w) + \eta$$

and

$$C_2 = \inf_{w:\ \|w-\Psi_A\|_{\mathrm{C}([0,T];\Sigma^{\frac{1}{2}})} \ge \delta} I^{\Psi_A^0}(w) - \eta.$$

Recall that, since the rate function is a good rate function, if $B$ is a closed set and $\inf_{w\in B} I^{\Psi_A^0}(w) < \infty$, then there is an $f$ in $B$, optimal fluctuation, such that $I^{\Psi_A^0}(f) = \inf_{w\in B} I^{\Psi_A^0}(w)$. Thus if $B$ does not contain the deterministic solution then necessarily $\inf_{w\in B} I^{\Psi_A^0}(w) > 0$. Consequently $\eta$ may be chosen such that $C_2$ is positive and the above probability of a deviation from the deterministic path is exponentially small in the small $\epsilon$ limit.

In this article we are interested in estimating the probability of particular deviations from the deterministic paths. Namely, we wish to study how the mass and the center of a solution at coordinate $T$ deviate from their value in the "frozen" deterministic system (*i.e.* when $\epsilon = 0$). In the absence of noise, the mass is a conserved quantity and for initial data being either 0 or $\Psi_A^0$ the center remains equal to zero.

We know from [22] that we may push forward by continuity the LDP for the paths to a LDP for the mass at $T$ and obtain a LDP with speed $\epsilon$ and good rate function for an initial datum $u_0$

$$I_N^{u_0}(m) = \frac{1}{2} \inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{N}(\mathbf{S}^{a,u_0}(h)(T))=m} \left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\right\}.$$

In the case of a multiplicative noise, the mass is a conserved quantity. Thus, in this case, the mass cannot deviate from the deterministic behavior.

Similarly, the mapping $\mathbf{Y}$ is continuous from $\Sigma^{\frac{1}{2}}$ into $\mathbb{R}$. We may thus define by direct image the measures $\left(\mu^{\mathbf{Y}(u^{\epsilon,u_0}(T))}\right)_{\epsilon>0}$ for an initial datum $u_0$ in $\Sigma$. We obtain by contraction that they satisfy a LDP of speed $\epsilon$ and good rate function

$$I_Y^{u_0}(y) = \frac{1}{2} \inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{Y}(\mathbf{S}(u_0,h)(T))=y} \left\{ \|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \right\},$$

the skeleton $\mathbf{S}$ is either that of the additive or multiplicative case.

Let us briefly explain our strategy to estimate the probability of some event. Let us consider for instance the event $D_\epsilon = \left\{\mathbf{Y}\left(u^{\epsilon,0}(T)\right) \in [a,b]\right\}$ where $[a,b]$ is an interval which does not contain 0. We use the LDP to obtain

$$-\inf_{y\in(a,b)} I_Y^0(y) \le \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(D_\epsilon\right) \le \overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(D_\epsilon\right) \le -\inf_{y\in[a,b]} I_Y^0(y). \tag{2.7}$$

To estimate the upper bound, we use energy type inequalities. These give estimates of the minimum $\mathrm{L}^2$ norm of the control $h$ required to change the deterministic behavior and have the center in $[a,b]$ at time $T$. Namely, we obtain a constant $c$ such that

$$\text{if } \mathbf{Y}\left(S(u_0,h)(T)\right) \in [a,b] \text{ then } \frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \ge c.$$

This clearly implies

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(D_\epsilon\right) \le -c.$$

The second step is to find a particular function $h$ such that $\mathbf{Y}(S(u_0,h)(T)) \in (a,b)$ and $\tilde{c} = \frac{1}{2}\|h_J\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$ is as small as possible. Then

$$-\tilde{c} \le \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(D_\epsilon\right).$$

In this second step, we are led to solve a control problem.

The difficulty is to have sufficiently sharp energy estimates and to find a good solution to the control problem so that $c$ and $\tilde{c}$ are as close as possible. We see below that we are able to do so in some interesting situations and derive good estimates on such probabilities.

Note also that proceeding as in [22] for the mass, we may prove in the additive case that $\inf_{y\in J} I_Y^{u_0}(y) < \infty$ for every nonempty interval $J$ and any $u_0$ provided the range of $\Phi$ is dense. Indeed, for every real number $a$, a solution of the form $u(t,x) = (1+atx)u_0$ satisfies $\mathbf{Y}\left(u(T)\right) = \frac{aT\pi^2}{3}$. Plugging this solution into equation (2.5), we find a control such that the solution reaches any interval at time $T$. Using the continuity of $h \mapsto \mathbf{Y}\left(\mathbf{S}^{a,u_0}(h)(T)\right)$

from $L^2(0,T;L^2)$ into $\mathbb{R}$ and the density of the range of $\Phi$, we obtain $\inf_{y\in J} I_Y^{u_0}(y)<\infty$. This shows that in this case the two extreme bounds in (2.7) are finite implying that $\mathbb{P}(D_\epsilon)$ goes to zero exponentially fast when $\epsilon$ goes to 0.

**Remark 2.2** *Also, using similar arguments as in [22], we can prove that for every positive $R$ besides an at most countable set of points, we can replace* $\underline{\lim}$ *and* $\overline{\lim}$ *by* lim *in the LDP and obtain*

$$\begin{aligned}&\lim_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,u_0}(T)\right)\geq R\right)\\ &=-\tfrac{1}{2\epsilon}\inf_{h\in L^2(0,T;L^2):\ \mathbf{Y}(\mathbf{S}(u_0,h)(T))\geq R}\left\{\|h\|^2_{L^2(0,T;L^2)}\right\}\end{aligned}$$

$$\begin{aligned}&\lim_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,u_0}(T)\right)\leq -R\right)\\ &=-\tfrac{1}{2\epsilon}\inf_{h\in L^2(0,T;L^2):\ \mathbf{Y}(\mathbf{S}(u_0,h)(T))\leq -R}\left\{\|h\|^2_{L^2(0,T;L^2)}\right\}.\end{aligned}$$

*This uses the fact that a monotone and bounded function is continuous almost everywhere.*

We end this section with some remarks which will be useful in the development of our method when we consider the center of the solution. Let us consider an initial datum is $\Psi_A^0$. The probability of tail events of the center are related to the behavior of $\mathbf{Y}\left(S(\Psi_A^0,h)\right)$. If $h\neq 0$, $S(\Psi_A^0,h)(t)\neq\Psi_A$ and the center may move. An equation for the motion of the center is given in [39] in the case of an external potential. The first step consists in multiplying the controlled PDE by $-ix\overline{u}$, taking the real part, and integrating by part the term involving the Laplace operator. We then obtain for the controlled PDE associated to the multiplicative case

$$\left[\mathbf{Y}\left(\mathbf{S}^{m,\Psi_A^0}(h)(t)\right)\right]'=2\mathfrak{Re}\left(i\int_{\mathbb{R}}\overline{\mathbf{S}^{m,\Psi_A^0}(h)(t,x)}\partial_x\mathbf{S}^{m,\Psi_A^0}(h)(t,x)dx\right),\tag{2.8}$$

while in the additive case we obtain

$$\begin{aligned}\left[\mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0}(h)(t)\right)\right]' = \ & 2\mathfrak{Re}\left(i\int_{\mathbb{R}}\overline{\mathbf{S}^{a,\Psi_A^0}(h)(t,x)}\partial_x\mathbf{S}^{a,\Psi_A^0}(h)(t,x)dx\right)\\ &-2\mathfrak{Re}\left(i\int_{\mathbb{R}}x\overline{\mathbf{S}^{a,\Psi_A^0}(h)(t,x)}\left(\Phi h\right)(t,x)dx\right).\end{aligned}\tag{2.9}$$

The quantity

$$\mathbf{P}(u)=2\mathfrak{Re}\left(i\int_{\mathbb{R}}\overline{u}(x)\partial_x u(x)dx\right),\ u\in H^1.$$

on the right hand side of (2.8) and (2.9) is usually called the momentum.

As a consequence of (2.8) we see that in the multiplicative case, the center of the solution of the control problem cannot move unless its phase depends on the space variable. For instance, if the control is chosen so that the solution $\mathbf{S}^{a,\Psi_A^0}(h)(t)$ is a modulated soliton of type (1.1) with varying amplitude and group velocity,

$$\begin{aligned}\mathbf{S}^{a,\Psi_A^0}(h)(t) = \quad & \sqrt{2}A(t)\text{sech}(A(t)(x-x_0)+2A(t)\Omega(t)t) \\ & \exp\left(-i(A(t)^2-\Omega(t)^2)t+i\Omega(t)(x-x_0)+i\theta_0\right)\end{aligned}$$

we have the well known identity

$$\left[\mathbf{Y}\left(\mathbf{S}^{m,\Psi_A^0}(h)(t)\right)\right]' = -2\Omega(t)\mathbf{N}(\mathbf{S}^{m,\Psi_A^0}(h)(t)) = -8\Omega(t)A(t).$$

It will be convenient to choose controlled solutions of the form above. Since the initial datum is $\Psi_A^0$, we necessarily have $\Omega(0)=0$, hence $\Omega$ cannot be chosen constant. We will see that it is sufficient to have a constant amplitude $A$ in order to get sharp bounds. Thus we will use modulated solitons as solutions of the controlled problem with constant amplitude when studying the motion of the center.

The first idea to find a control giving a solution whose center or mass verify some desired property is to take the above modulated soliton and plug it into the skeleton equation. This gives an explicit form of the control in terms of the various parameters. Then, we compute the space-time $\mathrm{L}^2$ norm of this control. We obtain a function of the parameters which we can try to minimize thanks to the calculus of variations. This approach is not easy to perform, the function to minimize has a complicated form and is often singular. Thus, we also have chosen a simpler approach which consists in finding directly controls giving solutions with the desired properties. Note however that the calculus of variations approach has allowed us to guess the form of the modulated soliton we should choose.

Let us consider the following controlled nonlinear Schrödinger equation

$$i\frac{\mathrm{d}u}{\mathrm{d}t} = \Delta u + |u|^2u + \lambda(t)xu \tag{2.10}$$

with initial datum $\Psi_A^0$. The function $\lambda$ is taken in $\mathrm{L}^1(0,T;\mathbb{R})$. This corresponds to the multiplicative skeleton equation with $\Phi h = \lambda(t)x$ or to the additive one with $\Phi h = \lambda(t)xu$. We use well known transformation to compute explicitly the solution of (2.10) which we denote by $\Psi_{A,\lambda}$. We first may check that the functions $v_1$ and $v_2$ defined by $v_1(t,x)=\exp\left(i\left(\int_0^t\lambda(s)ds\right)x\right)u(t,x)$

and $v_2(t,x) = \exp\left(-i\int_0^t \left(\int_0^s \lambda(\tau)d\tau\right)^2 ds\right) v_1(t,x)$ (gauge transform) satisfy the PDEs

$$i\frac{\partial v_1}{\partial t} = \frac{\partial^2 v_1}{\partial x^2} + |v_1|^2 v_1 - \left(\int_0^t \lambda(s)ds\right)^2 v_1 - 2i\left(\int_0^t \lambda(s)ds\right)\frac{\partial v_1}{\partial x}$$

and

$$i\left(\frac{\partial v_2}{\partial t} + 2\left(\int_0^t \lambda(s)ds\right)\frac{\partial v_2}{\partial x}\right) = \frac{\partial^2 v_2}{\partial x^2} + |v_2|^2 v_2$$

with initial datum $\Psi_A^0$. We conclude using the methods of characteristics that $v_3$ defined by

$$v_3(t,x) = v_2\left(t, x + 2\int_0^t \int_0^s \lambda(u)duds\right)$$

is a solution of the usual NLS equation with initial datum $\Psi_A^0$. Thus we obtain that $v_3(t,x) = \Psi_A(t,x)$ and that the solution of the Cauchy problem associated to (2.10) is

$$\Psi_{A,\lambda}(t,x) = \sqrt{2}A\mathrm{sech}\left(A\left(x - 2\int_0^t\int_0^s \lambda(\tau)d\tau ds\right)\right)$$
$$\exp\left[-iA^2t + i\int_0^t\left(\int_0^s \lambda(\tau)d\tau\right)^2 ds - ix\int_0^t \lambda(s)ds + 2i\left(\int_0^t \lambda(s)ds\right)\left(\int_0^t\int_0^s \lambda(\tau)d\tau ds\right)\right].$$

We obtain a modulated soliton with group velocity given by $\Omega(t) = \int_0^t \lambda(s)ds$. In the additive case, it is possible to obtain a control such that the solution has same center and group velocity and such that the space-time $\mathrm{L}^2$ norm of the control is simpler to compute. It is obtained thanks to the observation that using the gauge transform the solution of the Cauchy problem

$$\begin{cases} i\frac{\mathrm{d}v}{\mathrm{d}t} = \Delta v + |v|^2 v + \lambda(t)\left(x - 2\int_0^t\int_0^s \lambda(\tau)d\tau ds\right)v \\ v(0) = \Psi_A^0, \end{cases} \quad (2.11)$$

is given by

$$\tilde{\Psi}_{A,\lambda}(t,x) = \exp\left(2i\int_0^t \lambda(s)\int_0^s\int_0^\tau \lambda(\sigma)d\sigma d\tau ds\right)\Psi_{A,\lambda}(t,x).$$

**Remark 2.3** *Note that, for the controls chosen above, relation* (2.8) *holds also in the additive case. Thus the second term in* (2.9) *which, at first glance, could be useful to act on the center is in fact useless.*

*Also, it could be thought that the choice of more complicated group velocities could be useful. We have tried to consider a space dependent group velocity but the calculus of variations approach shows that optimality is reached when it does not depend on space.*

# 3 Tails of the the mass and center with additive noise

In the case of an additive noise, both the mass and center may deviate from the deterministic behavior and result in error in transmission.

We shall study tails and thus the probability of a deviation from the mean. The constant $R$ will quantify this deviation. We are not really interested in large $R$. In practice $R$ may be assumed to be in $(0, 4)$. But, since $\epsilon$ goes to zero and the factor in the exponential should be multiplied by $\frac{1}{\epsilon}$ while $R$ is of order 1. It results in very unlikely events. These significant excursions of the mass and position are exactly large deviation events.

Moreover another parameter is particularly interesting. It is $T$ the length of the fiber optical line. It is assumed to be large. For example we could think of a fiber optical line between Europe and America.

We first recall the results obtained in [22] for the tails of mass of the pulse at the end of the line. The initial datum may be $u_0 = 0$ or $u_0 = \Psi$ where $\Psi(x) = \sqrt{2}\text{sech}(x)$. We could consider a soliton profile with any amplitude $A$ as well but for simplicity, we consider the case $A = 1$. However we consider the parameter $A$ for the timing jitter in order to compare with results from physics.

Let us begin with upper bounds of the tails. As already mentioned, they are obtained thanks to energy estimates. For the second bound we consider the case of the emission of a signal. In that case only a decrease of the mass is troublesome and causes in error in transmission. Thus the bound given only accounts for a significant decrease of the mass.

**Proposition 3.1** *For every positive $T$ and $R$ ($R$ in $(0, 4)$ for the second inequality) and every operator $\Phi$ in $\mathcal{L}_2(\mathrm{L}^2, \mathrm{H}^1)$, the following inequalities hold*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,0}(T)\right) \geq R\right) \leq -\frac{R}{8T\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}},$$

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,\Psi}(T)\right) - 4 < -R\right) \leq -\frac{R^2}{2T\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}(4+R)^2}.$$

**Proof.** We only give a sketch of the proof. Details can be found in [22]. We treat the first inequality. The proof for the second inequality is similar. Multiplying by $-i\overline{u}$ the equation

$$i\frac{\mathrm{d}u}{\mathrm{d}t} - \Delta u - \lambda|u|^2 u = \Phi h,$$

integrating over time and space and taking the real part gives, for $t \in [0,T]$,

$$\|\mathbf{S}^{a,0}(h)(t)\|_{\mathrm{L}^2}^2 - \|u_0\|_{\mathrm{L}^2}^2 = 2\mathfrak{Re}\left(-i\int_0^t\int_{\mathbb{R}}\left((\Phi h)(s,x)\overline{\mathbf{S}^{a,0}(h)(s,x)}\right)dxds\right). \tag{3.1}$$

We first integrate once more with respect to $t \in [0,T]$ and use the Cauchy-Schwarz inequality to obtain

$$\left(\int_0^T \|\mathbf{S}^{a,0}(h)(s)\|_{\mathrm{L}^2}^2 ds\right)^{1/2} \leq 2T\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}\left(\int_0^T \|h(s)\|_{\mathrm{L}^2}^2 ds\right)^{1/2}.$$

Then, taking $t=T$ in (3.1), using again the Cauchy-Schwarz inequality and the above bound, we deduce

$$\|\mathbf{S}^{a,0}(h)(T)\|_{\mathrm{L}^2}^2 \leq 4T\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}^2\int_0^T \|h(s)\|_{\mathrm{L}^2}^2 ds.$$

It follows

$$\begin{aligned} I_N^0(m) &= \frac{1}{2}\inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{N}(\mathbf{S}^{a,0}(h)(T))=m}\left\{\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}^2\right\} \\ &\geq \frac{x}{8T\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}^2}. \end{aligned}$$

Now, by the LDP on the mass, we have

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,0}(T)\right)\geq R\right) \leq -\inf_{x\in[R,\infty]} I_N^{u_0}(m)$$

and the result follows. □

Let us now consider lower bounds. We use modulated solitons as solutions of the controlled equation. We have found that it is sufficient that only the amplitude is varying. We take the solution of (2.5) of the form

$$\sqrt{2}A(t)\exp\left(-i\int_0^t A^2(s)ds\right)\operatorname{sech}(A(t)x). \tag{3.2}$$

The singular Euler-Lagrange equation given by the calculus of variations when minimizing the energy of the controls giving such solutions has allowed to guess a good parametrization when the initial datum is either 0 or $\Psi$. Define the following sets of time dependent functions, depending on a set of parameters $D$,

$$\mathcal{A}_D^1 = \left\{A\,:\,[0,T]\to\mathbb{R},\ \text{there exists } \tilde{R}\in D \text{ such that } A(t)=\tilde{R}\left(\frac{t}{2T}\right)^2\right\}$$

and

$$\mathcal{A}_D^2 = \Big\{ \quad A : [0,T] \to \mathbb{R}, \text{ there exists } \tilde{R} \in D \text{ such that}$$
$$A(t) = \left(8 - \tilde{R} - 4\sqrt{4-\tilde{R}}\right)\left(\frac{t}{2T}\right)^2 + \left(-4 + 2\sqrt{4-\tilde{R}}\right)\frac{t}{2T} + 1 \Big\}.$$

Modulated amplitude taken in $\mathcal{A}_D^1$ or $\mathcal{A}_D^2$ set are associated to controls in the set

$$\mathcal{C}_D^i = \Big\{ h \in \mathrm{L}^2(0,T;\mathrm{L}^2), \text{ there exists } A \in \mathcal{A}_D^i$$
$$h(t,x) = i\frac{A'(t)}{A(t)}\Psi_A(t,x) - i\sqrt{2}A'(t)\exp\left(-i\int_0^t A^2(s)ds\right)A(t)x\frac{\sinh}{\cosh^2}\left(A(t)x\right)\Big\}$$

where $i = 1$ or $i = 2$.

We have the following proposition whose proof follows from the lower bound of the LDP for the mass. The proof is given in [22]. It uses that the infimum of the rate function is smaller than the infimum on the smaller sets of controls $\mathcal{C}_D^1$ and $\mathcal{C}_D^2$ corresponding to well-chosen modulated amplitudes. The assumptions can easily be fulfilled. They are made to be as close as possible to the space-time white noise considered in physics that we are not able to treat mathematically.

**Proposition 3.2** *Let $T$ and $R$ be positive numbers ($R$ in $(0,4)$ for the second inequality), take $D$ dense in $[R, R+1]$ and a sequence of operators $(\Phi_n)_{n\in\mathbb{N}}$ in $\mathcal{L}_2\left(\mathrm{L}^2,\mathrm{L}^2\right)$ such that for every $h \in \mathcal{C}_D^1$ we have $\Phi_n h$ converges to $h$ in $\mathrm{L}^1\left(0,T;\mathrm{L}^2\right)$. Then we obtain*

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,0,n}(T)\right) \geq R\right) \geq -\frac{R(12+\pi^2)}{18T}.$$

*Replacing in the above $\mathcal{C}_D^1$ by $\mathcal{C}_D^2$ we obtain*

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,\Psi,n}(T)\right) - 4 < -R\right) \geq -\frac{2(8-R-4\sqrt{4-R})(12+\pi^2)}{36T}.$$

*The exponent $n$ is there to recall that $\Phi$ is replaced by $\Phi_n$,*

Note that the result in Proposition 3.1 depends on $\Phi$ only through its norm as a bounded operator in $\mathrm{L}^2$. It is not difficult to see that there exists sequences of operators $(\Phi_n)_{n\in\mathbb{N}}$ satisfying the assumptions of Proposition 3.2, *i.e.* which are Hilbert-Schmidt from $\mathrm{L}^2$ to $\Sigma$ and $\Phi_n$ approximates the identity on the good set of controls, and are uniformly bounded as operators on $L^2$ by a constant independent on $T$. For such sequences of operators, the upper and lower bounds given above agree up to constants in their behavior in large $T$.

It is obtained in [18], for the ideal white noise and using the heuristic arguments recalled in the introduction, that the probability density function of the amplitude of the pulse at coordinate $T$ when the initial datum is null is asymptotically that of an exponential law of parameter $\frac{\epsilon T}{2}$. The amplitude is a constant times the mass for the modulated soliton solutions considered [18]. Integrating this density over $[\frac{R}{2}, \infty)$ and taking into account the different normalisation, we obtain $\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,0}(T)\right) \geq R\right) = -\frac{R}{T}$. It is in between our two bounds and very close to our lower bound. A surprising fact is that, we obtain our result by parameterizing only the amplitude whereas in [18] a much more general parametrization is used. Both bounds exhibit the right behavior in $R$ and $T$. Moreover, the order in $R$ confirms physical and numerical results that the law is not Gaussian. On a log scale the order in $R$ is that of tails of an exponential law. In such a case the Gaussian approximation leads to incorrect tails and error estimates.

Let us now comment on our results in the case of a soliton as initial datum. In [18], the error probability when the size of the measurement window is of the order of the coordinate $T$ is obtained. It is given by $\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\mathbf{N}\left(u^{\epsilon,\Psi}(T)\right) - 4 < -R\right) = -\frac{c(R)}{T}$, with a constant $c(R)$. It exhibits the same behavior in $T$ as in our calculations. The discussion on the behavior with respect to $R$ is less clear. Our bounds are not of the same order. In [15, 32] the PDF of the mass at coordinate $T$ for a soliton profile as initial datum is not Gaussian. The numerical simulations in [34] also exhibit a significant difference between the log of the tails of the amplitude and that of a Gaussian law. Our lower bound indicates that again the tails are larger than Gaussian tails. Thus we give a rigorous proof of the fact that a Gaussian approximation is incorrect.

Finally, it is natural to obtain that the tails of the mass are increasing functions of $T$ since the higher is $T$, the less energy is needed to form a signal whose mass gets above a fixed threshold at $T$. Replacing above by under, the same holds in the case of a soliton as initial datum.

**Remark 3.3** *The* $H^1$ *setting is not required here. We could as well work with* $L^2$ *solutions and a LDP in* $L^2$*. However, it is required to work in* $H^1$ *for the study of the center below.*

We now estimate the tails of the center. As for the mass, the rate is hard to handle since it involves an optimal control problem for controlled NLS equations. We again deduce the asymptotic of the tails from the LDP looking at upper and lower bounds. We consider that the initial datum is $\Psi_A^0$ since only in this case the timing jitter might be troublesome.

Let us begin with an upper bound. It is deduced from the equation of motion of the center in the controlled NLS equation (2.9).

**Proposition 3.4** *For every positive $T$, $A$ and $R$ and every operator $\Phi$ in $\mathcal{L}_2\left(\mathrm{L}^2,\Sigma\right)$, the following inequality holds*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(T)\right)\geq R\right)\leq -\frac{R^2}{8T(2T+1)^2\left(4A+\frac{R}{2T+1}\right)\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}}.$$

**Proof.** Differentiating the momentum of the solution with respect to time and replacing the time derivative of the solution with the corresponding terms of the equation we obtain

$$\left[\mathbf{P}\left(\mathbf{S}^{a,\Psi_A^0}(h)(t)\right)\right]'=4\mathfrak{Re}\int_{\mathbb{R}}\mathbf{S}^{a,\Psi_A^0}(h)(t,x)\left(\partial_x\overline{\Phi h}\right)(t,x)dx.$$

Indeed by successive integration by parts all terms cancel besides the one involving the forcing term. Since $\mathbf{Y}\left(\Psi_A^0\right)=0$ and $\mathbf{P}\left(\Psi_A^0\right)=0$, thanks to (2.9), we obtain the identity

$$\begin{aligned}\mathbf{Y}(\mathbf{S}^{a,\Psi_A^0}(h)(t))= \;& 4\mathfrak{Re}\left(\int_0^t\int_0^s\int_{\mathbb{R}}\overline{\mathbf{S}^{a,\Psi_A^0}(h)(\sigma,x)}\left(\partial_x\Phi h\right)(\sigma,x)dxd\sigma ds\right)\\ &-2\mathfrak{Re}\left(i\int_0^t\int_{\mathbb{R}}x\overline{\mathbf{S}^{a,\Psi_A^0}(h)(s,x)}\left(\Phi h\right)(s,x)dxds\right).\end{aligned}$$

From this identity it follows that the controls $h$ in the minimizing set of the LDP applied to the event we consider necessarily satisfy

$$\begin{aligned}R\leq\mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0}(h)(T)\right)\leq\;& 4T\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{H}^1)}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\|\mathbf{S}^{a,\Psi_A^0}(h)\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\ &+2\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\|\mathbf{S}^{a,\Psi_A^0}(h)\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}.\end{aligned}$$

Moreover, arguing as in the proof of Proposition 3.1, see also [22],

$$\begin{aligned}\|\mathbf{S}^{a,\Psi_A^0}(h)\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\leq\;& T\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\\ &\left(1+\sqrt{1+\frac{4A}{T\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}}}\right).\end{aligned}$$

A lower bound on $\frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$ follows easily since the function $x\mapsto x\left(1+\sqrt{1+\frac{4}{x}}\right)$ is increasing on $\mathbb{R}_+^*$. The result follows. □

A lower bound is obtained considering controls suggested at the end of Section 2 and minimizing on the smaller set of controls. We define the following set of control for $A$, $T$ positive and $D$ a subset of $(0,\infty)$

$$\mathcal{H}^D_{A,T} = \quad \{h \in \mathrm{L}^2(0,T;\mathrm{L}^2),\ h(t,x) = \lambda(t)\left(x - 2\int_0^t\int_0^s \lambda(\tau)d\tau ds\right)\tilde{\Psi}_{A,\lambda}(t,x), \text{ with } \lambda(t) = \tfrac{3\tilde{R}(T-t)}{8AT^3},\ \tilde{R}\in D\}$$

**Proposition 3.5** *Let $T$, $A$ and $R$ be positive. Assume that, for a dense set $D$ of $[R,R+1]$, $(\Phi_n)_{n\in\mathbb{N}}$ is a sequence of operators in $\mathcal{L}_2\left(\mathrm{L}^2,\Sigma\right)$ such that for any $h$ in $\mathcal{H}^D_{T,A}$, $\Phi_n h$ converges to $h$ in $\mathrm{L}^1(0,T;\Sigma)$. Then we have the following inequality where the $n$ in the exponent recalls that $\Phi$ is replaced by $\Phi_n$,*

$$\underline{\lim}_{n\to\infty,\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi^0_A,n}(T)\right)\geq R\right)\geq -\frac{\pi^2R^2}{128T^3A^3}.$$

**Proof.** By the LDP for the center $\mathbf{Y}$, we know that for a fixed $n$ a lower bound is given by

$$-\inf_{y>R} I^{\Psi^0_A}_{Y,n}(y)$$

where

$$I^{\Psi^0_A}_{Y,n}(y) = \frac{1}{2}\inf_{h\in\mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{Y}\ \mathbf{S}^{a,\Psi^0_A,n}(h)(T)\ =y}\left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\right\}.$$

Again, the $n$ is there to recall that in the skeleton equation, $\Phi$ is replaced by $\Phi_n$. To minorize this quantity, we first treat the case $\Phi = I$. Note that the stochastic equation has no meaning in this case but the skeleton equation has a well defined solution provided $h\in\mathrm{L}^2(0,T;\mathrm{L}^2)$. We denote by $\mathbf{S}^{a,\Psi^0_A}_{WN}$ the skeleton when $\Phi = I$. It is not difficult to see that $\mathbf{S}^{a,\Psi^0_A}_{WN}(h)$ belongs to $\mathrm{L}^\infty([0,T];\Sigma)$ when $h$ belong to $\mathrm{L}^1(0,T;\Sigma)$. A standard argument to prove this is to compute the second derivative with respect to time of the variance $\mathbf{V}(u) = \int_{\mathbb{R}} x^2|u(t,x)|^2dx$ when $u = \mathbf{S}^{a,\Psi^0_A}_{WN}(h)$. It is also standard to prove that, for each $t$, the mapping $h\to\mathbf{S}^{a,\Psi^0_A}_{WN}(h)(t)$ is weakly continuous from $\mathrm{L}^1(0,T;\Sigma)$ to $\Sigma$ and strongly continuous from $\mathrm{L}^1(0,T;\Sigma)$ to $\mathrm{H}^1$ . Therefore, since $Y$ is weakly continuous on $\Sigma$, thanks to our assumptions, we know that for $h\in\mathcal{H}^D_{T,A}$

$$\mathbf{Y}\left(\mathbf{S}^{a,\Psi^0_A,n}(h)(T)\right) = \mathbf{Y}\left(\mathbf{S}^{a,\Psi^0_A}_{WN}(\Phi_n h)(T)\right)\to\mathbf{Y}\left(\mathbf{S}^{a,\Psi^0_A}_{WN}(h)(T)\right) \text{ when } n\to\infty. \tag{3.3}$$

Let $\tilde{\mathcal{H}}_{T,A}$ be the same set of controls as above but where $\lambda$ is only assumed to belong to $\mathrm{L}^2(0,T;\mathbb{R})$

$$\tilde{\mathcal{H}}_{T,A} = \{h \in \mathrm{L}^2(0,T;\mathrm{L}^2),\ h(t,x) = \lambda(t)\left(x - 2\int_0^t\int_0^s \lambda(\tau)d\tau ds\right)\tilde{\Psi}_{A,\lambda}(t,x),\quad \lambda \in \mathrm{L}^2(0,T;\mathbb{R})\}.$$

Clearly,

$$\begin{aligned}
&\inf_{h\in\mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{Y}\ \ \mathbf{S}_{WN}^{a,\Psi_A^0}(h)(T)\ \ \geq \tilde{R}} \|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\
\leq &\inf_{h\in\tilde{\mathcal{H}}_{T,A}:\ \mathbf{Y}\ \ \mathbf{S}_{WN}^{a,\Psi_A^0}(h)(T)\ \ \geq \tilde{R}} \|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \\
= &\inf_{\lambda\in\mathrm{L}^2(0,T;\mathbb{R}),\int_0^T\int_0^t \lambda(s)dsdt\geq \frac{\tilde{R}}{8A}} \frac{\pi^2}{3A}\int_0^T \lambda^2(t)dt
\end{aligned}$$

Note that the contraint $\int_0^T\int_0^t \lambda(s)dsdt \geq \frac{\tilde{R}}{8A}$, is not a boundary condition as in the usual calculus of variations. To solve this minimization problem, we use the quantity $\mathcal{L}_{T,A,\tilde{R}}(\lambda)$ defined by

$$\mathcal{L}_{T,A,\tilde{R}}(\lambda) = \frac{\pi^2}{3A}\int_0^T \lambda^2(t)dt - \gamma\int_0^T\int_0^t \lambda(s)dsdt,$$

where $\gamma$ belongs to $\mathbb{R}$. We then impose that our guess $\lambda^*_{T,A,\tilde{R}}$ is a critical point of $\mathcal{L}_{T,A,\tilde{R}}(\lambda)$ and that it satisfies the constraint $\int_0^T\int_0^t \lambda(s)dsdt = \frac{\tilde{R}}{8A}$. We obtain

$$\lambda^*_{T,A,\tilde{R}}(t) = \frac{3\tilde{R}(T-t)}{8AT^3}.$$

We do not claim that the minimization problem is solved, we simply write

$$\begin{aligned}
&\inf_{\lambda\in\mathrm{L}^1(0,T;\mathbb{R}),\int_0^T\int_0^t \lambda(s)dsdt\geq \frac{\tilde{R}}{8A}} \frac{\pi^2}{3A}\int_0^T \lambda^2(t)dt \\
\leq &\frac{\pi^2}{3A}\int_0^T \lambda^*_{T,A,\tilde{R}}(t)dt = \frac{\pi^2\tilde{R}^2}{64A^3T^3}
\end{aligned}$$

Let us set

$$h^*_{\tilde{R}}(t,x) = \lambda^*_{T,A,\tilde{R}}(t)\left(x - 2\int_0^t\int_0^s \lambda^*_{T,A,\tilde{R}}(\tau)d\tau ds\right)\tilde{\Psi}_{A,\lambda^*_{T,A,\tilde{R}}}(t,x).$$

By (3.3), we have for $\tilde{R} \in D$,

$$\mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0,n}(h^*_{\tilde{R}})(T)\right) \to \mathbf{Y}\left(\mathbf{S}_{WN}^{a,\Psi_A^0}(h^*_{\tilde{R}})(T)\right) \text{ when } n\to\infty.$$

Therefore, for $n$ large enough,

$$\mathbf{Y}\left(\mathbf{S}^{a,\Psi_A^0,n}(h^*_{\tilde{R}})(T)\right) > R.$$

We deduce

$$\inf_{x>R} I_{Y,n}^{\Psi_A^0}(x) \leq \frac{\pi^2\tilde{R}^2}{64A^3T^3}.$$

Since this is true for $\tilde{R}$ in a dense set of $[R,R+1]$ we deduce the result. □

The upper and lower bounds given in Proposition 3.4 and 3.5 are in perfect agreement in their behavior with respect to $R$ and to $T$ when $T$ is large. Indeed, for $T$ large, the upper bound in Proposition 3.4 is close to $\frac{R^2}{128T^3A\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}}$. However, we have to be careful before doing such a comparison. Indeed, the bounds can be compared only if we are able to consider a sequence of operators $(\Phi_n)_{n\in\mathbb{N}}$ satisfying the assumptions of Proposition 3.5 and such that $\|\Phi_n\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}$ is bounded uniformly in $n$.

It seems possible to construct such a sequence. For instance we may choose $\widetilde{\Phi}$ in $\mathcal{L}_2(\mathrm{L}^2,\Sigma)$ such that $\widetilde{\Phi}k = k$ for $k$ in $K_A$, the closure in $\mathrm{L}^2$ of the vector space spanned by $\{(x-a)\mathrm{sech}\,(A(x-b))\,,\ a\in[0,1],\ b\in[0,1]\}$. We believe that $K_A$ is embedded in $\Sigma$ in a Hilbert-Schmidt way. For $T$ and $A$ sufficiently large and $D\subset[R,R+1]$, each $h$ in the set $\mathcal{H}^D_{A,T}$ is such that $h(t)\in K_A$ for $t\in[0,T]$, thus $\widetilde{\Phi}h = h$ and we can take $\Phi_n = \widetilde{\Phi}$ in Proposition 3.5. In this case, the two bounds are comparable and are of the same order in $R$ and $T$. Note that $\|\Phi_n\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}$ is independent on $R$ and $T$.

In fact, many such sequences probably exist. Therefore, it seems that the bounds can be compared in many circumstances. Roughly speaking, the fact that this can be done means that we are treating noises which are sufficiently localized around the soliton $\Psi_A^0$.

If the sequence $(\Phi_n)_{n\in\mathbb{N}}$ converges pointwise to the identity, *i.e.* if we wish to understand what happens in the white noise limit, then this localization assumption does not hold. In this case, the lower bound is meaningful whereas the upper bound converges to zero and provides no information.

The comparison of the behavior of the bounds with respect to $A$ is less clear. The two bounds seem contradictory for large $A$. This is due to the fact that it is not possible to choose a sequence of operators $(\Phi_n)_{n\in\mathbb{N}}$ satisfying the assumptions of Proposition 3.5 and such that $\|\Phi_n\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}$ is uniformly bounded with respect to $A$. Indeed such a sequence necessarily satisfies

$$\|h\|_{\mathrm{L}^1(0,T;\Sigma)} \leq \underline{\lim}_{n\to\infty}\|\Phi_n\|_{\mathcal{L}_c(\mathrm{L}^2,\Sigma)}\|h\|_{\mathrm{L}^1(0,T;\mathrm{L}^2)}$$

for any $h \in \mathcal{H}^D_{T,A}$. It is easily seen that for $A$ and $T$ sufficiently large, the ratio of $\|h\|_{L^1(0,T;\Sigma)}$ and $\|h\|_{L^1(0,T;L^2)}$ is of the order $A$.

In fact this shows that the upper bound in Proposition 3.4 is always larger than a constant times $\frac{R^2}{T^3A^3}$ for a sequence satisfying the assumptions of Proposition 3.5. Thus there is no contradiction.

We can probably go further. Indeed, there may exist sequences of operators satisfying the assumptions of Proposition 3.5 and such that $\|\Phi_n\|_{\mathcal{L}_c(L^2,\Sigma)} \leq cA$ for some constant $c$. In this case the bounds are of the same order with respect to $A$, $R$ and $T$. An example could be constructed in the same way as above. It suffices to take $\Phi_n$ equal to the identity on $K_A$ and zero on a complementary space. Indeed, it can be shown that $\|h\|_\Sigma \leq cA\|h\|_{L^2}$ for some constant $c$.

Therefore, the two bounds are also comparable in their behavior with respect to $A$ under a localization assumption on the noise.

Let us now compare our result with the results obtained in the physics literature. First, we note that we obtain that on a log scale the tails are equivalent to Gaussian tails. This is indeed the kind of result obtained by arguments from the physical theory of perturbation of solitons.

**Remark 3.6** *We are missing the pre exponential factors to conclude whether or not the tails are Gaussian. We could think of using sharp Laplace asymptotics to obtain these factors.*

Now, suppose the law were indeed Gaussian, then the asymptotic of the tails may be written in terms of the variance. By doing so, we find that the variance of the timing jitter is of the order $T^3$. It agrees perfectly with the initial results of [25]. Also the order in both $A$ and $T$ seems to agree perfectly with the orders of the contribution of the additive noise to the variance of the timing jitter in equation (3.18) in [17]. Note however that in [25, 29], where the model is instead a juxtaposition of deterministic evolutions with random initial data in between amplifiers, the order in $A$ seems to be $-\frac{c}{A}$.

We end this section noticing that our result confirms the fact that, in the presence of additive noise, the timing jitter is more troublesome than the fluctuation of the mass when we consider the problem of losing a signal. Indeed we have found that the error probability due to timing jitter is of the order of $\exp\left(-\frac{c_1(R)}{\epsilon T^3}\right)$ and an error probability due to the fluctuation of the mass is of the order of $\exp\left(-\frac{c_2(R)}{\epsilon T}\right)$ which is clearly negligible compared to the first for large $T$. Recall that $T$ represents the length of a fiber optical line and is thus assumed to be very large.

**Remark 3.7** *From an engineering point of view it is possible to exponentially reduce the probability of undesired deviations of the center by introducing inline control elements; see for example [18]. We could also use ideas given in [36] and optimize on such external fields for a limited cost or penalty functional. The new optimal control problem requires then double optimization.*

**Remark 3.8** *Note that the methodology developed herein could also be applied to the determination of the small noise asymptotic of the tails of the position of an isolated vortex, defined by* $\oint \nabla \arg u(t,x) \cdot d\mathbf{l}$*, in a Bose condensates or superfluid Helium as in [35]. There the physical perturbation approach along with the Fokker-Planck equation are used. The small noise acts as the small temperature.*

# 4 Tails of the center in the multiplicative case

In the case of the multiplicative noise, the mass is a conserved quantity and we restrict our attention to the case of the law of the center of the pulse when the initial datum is the soliton profile $\Psi_A^0$.

Again, let us begin with upper bounds obtained from an equation for the motion of the center in the controlled NLS equation.

From relation (2.8) and integration by parts, we obtain the equation in [39],

$$\left[\mathbf{Y}(\mathbf{S}^{m,\Psi_A^0}(h)(t))\right]'' = 2\int_{\mathbb{R}} |\mathbf{S}^{m,\Psi_A^0}(h)(t,x)|^2 \left(\partial_x \Phi h\right)(t,x)dx. \qquad (4.1)$$

We may thus deduce the next proposition.

**Proposition 4.1** *For every positive $T$, $A$ and $R$ and every operator $\Phi$ in $\mathcal{L}_2\left(\mathrm{L}^2, \mathrm{H}^s(\mathbb{R},\mathbb{R})\right)$, where $s > \frac{3}{2}$ the following inequality holds*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(T)\right) \geq R\right) \leq -\left(\frac{3}{16}\right)^2 \frac{R^2}{2A^2T^3\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R}))}}.$$

**Proof.** From equation (4.1), the fact that $\mathbf{Y}\left(\mathbf{S}^{m,\Psi_A^0}(h)\right)'(0) = \mathbf{P}(\Psi_A^0) = 0$, that for such values of $s$ the injection of $\mathrm{H}^s(\mathbb{R},\mathbb{R})$ into $\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R})$ is continuous and that the mass is conserved and thus remains equal to 4 we obtain that

$$\begin{aligned}\mathbf{Y}\left(\mathbf{S}^{m,\Psi_A^0}(h)(t)\right)' &\leq 8A\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R}))}\|h\|_{\mathrm{L}^1(0,t;\mathrm{L}^2)}\\ &\leq 8A\sqrt{t}\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R}))}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}\end{aligned}$$

Then, since $\mathbf{Y}\left(\Psi_A^0\right)=0$, we obtain integrating the above inequality that

$$R \leq \mathbf{Y}\left(\mathbf{S}^{m,\Psi_A^0}(h)(T)\right) \leq \frac{16AT^{\frac{3}{2}}}{3}\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R}))}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$$

and the conclusion follows. □

Let us consider now lower bounds. We need to find controls which have the desired effect on the center. We have seen that in the additive case, good controls are given by functions in $\mathcal{H}_{A,T}^D$. Recalling the transformations on the equation made at the end of Section 2, we can equivalently take controls of the form $\lambda(t)x\Psi_{A,\lambda}$ which correspond to the solution $\Psi_{A,\lambda}$. Thus, in the multiplicative case, a good control is given by $h(t,x)=\lambda(t)x$. Unfortunately these controls do not belong to the range of $\Phi$ nor to $\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$ and are not admissible.

We have tried to approximate these controls by admissible ones. Since the control is multiplied by $\Psi_{A,\lambda}$ in the equation, it seems that it has no effect outside a set centered around the center of $\Psi_{A,\lambda}$ and that we could replace $\lambda(t)x$ by a truncation. We have not been able to get any information by such arguments. We have tried several other choices of control corresponding to various modulated solitons especially with a phase nonlinear in $x$. They never yielded the right order of the lower bound with respect to $A$ or $T$. We therefore impose a new assumption that $\Phi$ takes its values in $\mathrm{H}^s(\mathbb{R},\mathbb{R})\oplus x\mathrm{L}^1(0,T;\mathbb{R})$. In other words we consider the slightly different equation

$$i\mathrm{d}\tilde{u}^{\epsilon,u_0}=\left(\Delta\tilde{u}^{\epsilon,u_0}+|\tilde{u}^{\epsilon,u_0}|^2\tilde{u}^{\epsilon,u_0}\right)\mathrm{d}t+\tilde{u}^{\epsilon,u_0}\circ\sqrt{\epsilon}\mathrm{d}W(t)+\sqrt{\epsilon}x\tilde{u}^{\epsilon,u_0}\circ\mathrm{d}\beta(t) \tag{4.2}$$

where $\beta$ is a standard Brownian motion independant of $W$ and the corresponding controlled PDE

$$\begin{aligned} i\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{S}^{u_0}(h_1,h_2)= \; & \Delta\tilde{S}^{u_0}(h_1,h_2)+|\tilde{S}^{u_0}(h_1,h_2)|^2\tilde{S}^{u_0}(h_1,h_2)\\ & +\tilde{S}^{u_0}(h_1,h_2)\Phi h_1+x\tilde{S}^{u_0}(h_1,h_2)h_2 \end{aligned}$$

where $h_1$ belongs to $\mathrm{L}^2\left(0,T;\mathrm{L}^2\right)$ and $h_2$ belongs to $\mathrm{L}^2(0,T;\mathbb{R})$, the initial datum is $u_0$ and in the sequel $u_0=\Psi_A^0$. We may guess by successive applications of the Itô formula, multiplying $\tilde{u}^{\epsilon,u_0}$ by the random phase term $\exp\left(ix\sqrt{\epsilon}\beta(t)\right)$, and similar transformations as in Section 2 (stochastic gauge transform, stochastic methods of characteristics...) that we should consider the function

$$\exp\left(ix\sqrt{\epsilon}\beta(t)-i\epsilon\int_0^t\beta^2(s)ds\right)\tilde{u}^{\epsilon,u_0}\left(t,x+2\sqrt{\epsilon}\int_0^t\beta(s)ds\right).$$

It indeed satisfies equation (2.2) with same initial datum. We deduce that

$$\tilde{u}^{\epsilon,u_0}(t,x) = \exp\left(-ix\sqrt{\epsilon}\beta(t) + i\epsilon\int_0^t \beta^2(s)ds + 2i\epsilon\beta(t)\int_0^t \beta(s)ds\right) u^{\epsilon,u_0}\left(t, x - 2\sqrt{\epsilon}\int_0^t \beta(s)ds\right).$$

A similar computation shows that

$$\tilde{S}^{u_0}(h_1,h_2)(t,x) = \exp\left(-ix\sqrt{\epsilon}\int_0^t h_2(s)ds + i\int_0^t \left(\int_0^s h_2(u)du\right)^2 ds \right. \\ \left. +2i\int_0^t h_2(s)ds\int_0^t\int_0^s h_2(u)duds\right)\mathbf{S}^{m,u_0}(h_1)\left(t, x-2\int_0^t\int_0^s h_2(u)du\right).$$

The functions $\tilde{u}^{\epsilon,u_0}$ and $\tilde{S}^{u_0}(h_1,h_2)$ are well defined functions of $\mathrm{L}^2(0,T;\Sigma)$ and we may compute their centers. We obtain a lower bound of the asymptotic of the tails of the center of the new solutions.

**Proposition 4.2** *For every positive $T$, $A$ and $R$ and every operator $\Phi$ in $\mathcal{L}_2\left(\mathrm{L}^2, \mathrm{H}^s(\mathbb{R},\mathbb{R})\right)$ where $s > \frac{3}{2}$ the following inequality holds*

$$\underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(T)\right) \geq R\right) \geq -\frac{3R^2}{128A^2T^3}.$$

**Proof.** Consider the mapping $F$ from $\mathrm{C}\left([0,T];\Sigma^{\frac{1}{2}}\right)\times\mathrm{C}\left([0,T];\mathbb{R}\right)$ into $\mathbb{R}$ such that

$$F(u,b) = \int_{\mathbb{R}}|x|\left|u\left(T, x-2\int_0^T b(s)ds\right)\right|^2 dx.$$

Take $u$ and $u'$ in $\mathrm{C}\left([0,T];\Sigma^{\frac{1}{2}}\right)$ and $b$ and $b'$ in $\mathrm{C}\left([0,T];\mathbb{R}\right)$, then by the triangle and inverse triangle inequalities and the change of variables we obtain

$$\begin{aligned}
&|F(u,b)-F(u',b')| \\
&\leq \int_{\mathbb{R}}\left|\left|x+2\int_0^T b(s)ds\right| - \left|x+2\int_0^T b'(s)ds\right|\right| |u(T,x)|^2 dx \\
&\quad + \left|\int_{\mathbb{R}}\left|x+2\int_0^T b'(s)ds\right|\left(|u(T,x)|^2 - |u'(T,x)|^2\right)dx\right| \\
&\leq 2\left|\int_0^T b(s)ds - \int_0^T b'(s)ds\right|\int_{\mathbb{R}}|u(T,x)|^2 dx \\
&\quad + \int_{\mathbb{R}}|x|\,||u(T,x)|-|u'(T,x)||\left(|u(T,x)|+|u'(T,x)|\right)dx \\
&\quad +2\left|\int_0^T b'(s)ds\right|\int_{\mathbb{R}}||u(T,x)|-|u'(T,x)||\left(|u(T,x)|+|u'(T,x)|\right)dx
\end{aligned}$$

we conclude from the inverse triangle and Hölder inequalities that $F$ is continuous. We may then push forward the LDP for the paths of $u^{\epsilon,\Psi_A^0}$ and of the Brownian motion by the mapping $F$ using a slight modification of the result of exercise 4.2.7 of [9] and obtain a LDP for the laws of $\mathbf{Y}\left(\tilde{u}^{\epsilon,\Psi_A^0}(T)\right)$ which is that of $F\left(u^{\epsilon,\Psi_A^0},\sqrt{\epsilon}\beta\right)$ of speed $\epsilon$ and good rate function defined as a function of the rate function of the original solutions and of the rate function $I_\beta$ of the sample path LDP for the Brownian motion

$$\begin{aligned}
\tilde{I}_Y^{\Psi_A^0}(x) = & \inf_{(u,b):\ F(u,b)=x}\left(I^{u_0}(u)+I_\beta(b)\right)\\
\le & \tfrac{1}{2}\inf_{(h_1,h_2):\ F\left(\mathbf{S}^{m,\Psi_A^0}(h_1),\int_0^\cdot h_2(s)ds\right)=x}\left\{\|h_1\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}+\|h_2\|^2_{\mathrm{L}^2(0,T;\mathbb{R})}\right\}\\
\le & \tfrac{1}{2}\inf_{(h_1,h_2):\ \mathbf{Y}\left(\tilde{\mathbf{S}}^{\Psi_A^0}(h_1,h_2)(T)\right)=x}\left\{\|h_1\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}+\|h_2\|^2_{\mathrm{L}^2(0,T;\mathbb{R})}\right\}.
\end{aligned}$$

Thus considering solely controls of the from $(0,h_2)$, we minimize in $h_2$ for $\gamma$ in $\mathbb{R}$,

$$\int_0^T h_2^2(t)dt-\gamma\int_0^T\int_0^t h_2(s)ds,$$

where we impose that

$$\mathbf{Y}\left(\Psi_{A,h_2}(T)\right)=8A\int_0^T\int_0^t h_2(s)ds=\tilde{R}>R.$$

The conclusion follows. □

**Remark 4.3** *We may check that* $\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}\right)=\mathbf{Y}\left(\tilde{u}^{\epsilon,\Psi_A^0}\right)-8\sqrt{\epsilon}\int_0^T\beta(s)ds$ *and that* $\int_0^T\beta(s)ds$ *is a centered Gaussian random variable with variance* $\frac{T^3}{3}$.

The corresponding upper bound for this modified stochastic NLS equation is

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathbf{Y}\left(u^{\epsilon,\Psi_A^0}(T)\right)\ge R\right)\le-\left(\frac{3}{16}\right)^2\frac{R^2}{A^2T^3\left(\|\Phi\|^2_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{W}^{1,\infty}(\mathbb{R},\mathbb{R}))}\vee 1\right)}.$$

Note that the lower bound do not require to consider a sequence of operators $(\Phi_n)_{n\in\mathbb{N}}$ and we may indeed compare the upper and lower bounds. They are of the same order in $T$ and in $A$. Note also that, as in the additive case, we obtain that on a log scale the tails are equivalently that of Gaussian tails. Also, our tails are of the order in $T$ that we expect from the contribution of the multiplicative noise to the variance of the timing jitter in equation (3.18) in [17].

However, concerning the amplitude, it is not of the order of $-\frac{c}{A^4}$ as we would expect from [17]. This is probably due to the fact that we have considered a colored noise with a term $x\frac{\mathrm{d}}{\mathrm{d}t}\beta$ that grows linearly in time (the $x$ variable). We have explained that, otherwise, we fail to obtain a lower bound. We have obtained, that for large $A$, and thus for even more localized in time solitons, the tails of the center in the additive noise are larger than that in the multiplicative noise. Note that it is predicted in [17] that the quantum Raman noise is a dominant source of fluctuations in phase and arrival time for sub-picosecond solitons and that on the other hand for longer solitons, Raman effects are reduced compared to the usual Gordon-Haus jitter. It seems at first glance to be in contradiction with our results but their result is obtained for $A = 1$ and time corresponds to the typical pulse duration considered for scaling purposes in order to obtain the NLS equation; also our order in $A$ differs from theirs.

# 5 Annex - proof of Theorem 2.1

We denote herein by $\mathbf{V}(f) = \int_{\mathbb{R}} |x|^2|f(x)|^2dx$ the variance defined for $f$ in $\Sigma$.

Let us start with the additive case. We denote by $v^{u_0}(z)$ the solution of

$$\begin{cases} i\frac{\mathrm{d}v}{\mathrm{d}t} = \Delta v + \lambda|v - iz|^{2\sigma}(v - iz) \\ u(0) = u_0 \in \Sigma \end{cases},$$

where $z$ belongs to $X^{(T,2\sigma+2,\Sigma)} = \mathrm{C}([0,T];\Sigma) \cap \mathrm{L}^r\left(0,T;\mathrm{W}^{1,2\sigma+2}\right)$ and $r$ is such that $\frac{2}{r} = \frac{1}{2} - \frac{1}{2\sigma+2}$. We also denote by $\mathcal{G}^{u_0}$ the mapping

$$z \mapsto v^{u_0}(z) - iz,$$

it is such that $u^{\epsilon,u_0} = \mathcal{G}^{u_0}(\sqrt{\epsilon}Z)$ where $Z$ is the stochastic convolution defined by $Z(t) = \int_0^t U(t-s)dW(s)$.

We can check from similar arguments as those of the proof of Proposition 1 in [22] that the stochastic convolution is a $X^{(T,2\sigma+2,\Sigma)}$ random variable whose law $\mu^Z$ is a centered Gaussian measure. Let $z$ belong to $X^{(T,2\sigma+2,\Sigma)}$, take $s < t < T$, the triangle along with the Hölder inequalities then allow to compute

$$\begin{aligned}
&\left|\int_{\mathbb{R}} |x| \left(|\mathcal{G}^{u_0}(z)(t,x)|^2 - |\mathcal{G}^{u_0}(z)(s,x)|^2\right) dx\right| \\
&\leq \int_{\mathbb{R}} |x|(|\mathcal{G}^{u_0}(z)(t,x)| + |\mathcal{G}^{u_0}(z)(s,x)|)|(|\mathcal{G}^{u_0}(z)(t,x)| - |\mathcal{G}^{u_0}(z)(s,x)|)|dx \\
&\leq \|\mathcal{G}^{u_0}(z)(t) - \mathcal{G}^{u_0}(z)(s)\|_{\mathrm{L}^2}\sqrt{\mathbf{V}(|\mathcal{G}^{u_0}(z)(t)| + |\mathcal{G}^{u_0}(z)(s)|)} \\
&\leq 2\sqrt{2}\|\mathcal{G}^{u_0}(z)(t) - \mathcal{G}^{u_0}(z)(s)\|_{\mathrm{L}^2} \\
&\quad \times \left(\sqrt{\mathbf{V}(v^{u_0}(z)(t))} + \sqrt{\mathbf{V}(v^{u_0}(z)(s))} + \sqrt{\mathbf{V}(z(t))} + \sqrt{\mathbf{V}(z(s))}\right).
\end{aligned}$$

The application of the Gronwall inequality in the proof of Proposition 3.5 in [11], along with the Sobolev injection allow to prove that $\mathcal{G}^{u_0}(z)$ belongs to $\mathrm{C}([0,T];\Sigma^{\frac{1}{2}})$. The computation above also shows that the mapping $\mathcal{G}^{u_0}$ is continuous from $X^{(T,2\sigma+2,\Sigma)}$ to $\mathrm{C}([0,T];\Sigma^{\frac{1}{2}})$. The general result on LDP for Gaussian measures gives the LDP for the measures $\mu^{Z_\epsilon}$, the direct images of $\mu^Z$ under the transformation $x \mapsto \sqrt{\epsilon}x$ on $X^{(T,2\sigma+2,\Sigma)}$. We conclude with the contraction principle.

In the multiplicative case, it is also required to revisit the proof of the LDP in [23]. Note that in the following when $\Phi h$ is replaced by $\frac{\partial f}{\partial t}$ where $f$ belongs to $\mathrm{H}^1_0\left(0,T;\mathrm{H}^s(\mathbb{R},\mathbb{R})\right)$ which is the subspace of $\mathrm{C}\left([0,T];\mathrm{H}^s(\mathbb{R},\mathbb{R})\right)$ of functions null at time 0, square integrable in time and with square integrable in time time derivative. The skeleton is then denoted by $\tilde{\mathbf{S}}^{m,u_0}(f)$.
We may check using the above calculation and the fact that for every $t \in [0,T]$, $\tilde{\mathbf{S}}^{m,u_0}(f)(t)$ belongs to $\Sigma$ that

$$\mathbf{V}\left(\tilde{\mathbf{S}}^{m,u_0}(f)(t)\right) \le \left(4\|\tilde{\mathbf{S}}^{m,u_0}(f)(t)\|^2_{\mathrm{C}([0,T];\mathrm{H}^1)} + \mathbf{V}(u_0)\right)e^T,$$

see the arguments of the proof of Proposition 3.2 in [12] used for the skeleton, that the skeleton is continuous from the sets of levels of the rate function of the Wiener process less or equal to a positive constant, with the topology induced by that of $\mathrm{C}\left([0,T];\mathrm{H}^s(\mathbb{R},\mathbb{R})\right)$, to $\mathrm{C}\left([0,T];\Sigma^{\frac{1}{2}}\right)$. The only difference in the proof of Proposition 4.1 in [23], the Azencott lemma (also called Freidlin-Wentzell inequality or almost continuity of the Itô map) is in step 2. It is the reduction to estimates on the stochastic convolution. We use

$$\mathbf{V}\left(v^{\epsilon,\tilde{u}_0}(t)\right) \le \left(4\|v^{\epsilon,\tilde{u}_0}(t)\|^2_{\mathrm{C}([0,T];\mathrm{H}^1)} + \mathbf{V}(\tilde{u}_0)\right)e^T,$$

see the proof of Proposition 3.2 in [12], where $v^{\epsilon,\tilde{u}_0}$ satisfies $v^{\epsilon,\tilde{u}_0}(0) = \tilde{u}_0$ and

$$idv^{\epsilon,\tilde{u}_0} = \left(\Delta v^{\epsilon,\tilde{u}_0} + \lambda|v^{\epsilon,\tilde{u}_0}|^{2\sigma}v^{\epsilon,\tilde{u}_0} + \frac{\partial f}{\partial t}v^{\epsilon,\tilde{u}_0} - \frac{i\epsilon}{2}F_\Phi v^{\epsilon,\tilde{u}_0}\right)dt + \sqrt{\epsilon}v^{\epsilon,\tilde{u}_0}dW_\epsilon,$$

$f(\cdot) = \int_0^{\cdot}\Phi h(s)ds$, $W_\epsilon(t) = W(t) - \frac{1}{\sqrt{\epsilon}}\int_0^t \frac{\partial f}{\partial s}ds = W(t) - \frac{1}{\sqrt{\epsilon}}\int_0^t \Phi h(s)ds$, $F_\Phi(x) = \sum_{j=1}^\infty\left(\Phi e_j(x)\right)^2$ and $(e_j)_{j=1}^\infty$ is any complete orthonormal system of $\mathrm{L}^2$. The bound remains the same as in [12] because of the cancelation of the extra term in the application of the Itô formula and the cancelation of the Itô-Stratonovich correction with the second order Itô correction term when the Itô formula is applied to the truncated variance $V_r(v) = \int_{\mathbb{R}}\exp(-r|x|^2)|x|^2|v(x)|^2dx$. □

**Remark 5.1** *Uniform LDPs hold (uniform with respect to initial data in balls) in the Freidlin-Wentzell formulation or compact sets in the present formulation with* $\underline{\lim}$ *and* $\overline{\lim}$*. More general nonlinearities and dimensions and the case where blow-up may occur could be considered. It is still possible to state the result in spaces of exploding paths with a projective limit topology accounting for the various integrability. Uniformity could be useful since in optical experiments the initial pulse is a laser output and it is known up to a certain level of uncertainty.*

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