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Derivative Pricing with Multivariate Stochastic Volatility : Application to Credit Risk

C. GOURIEROUX [1]
R. SUFANA [2]

[1] CREST, CEPREMAP and University of Toronto. *(CREST, 15 Boulevard Gabriel Péri, 92245 Malakoff Cedex, France)*. Email : gouriero@ensae.fr
[2] University of Toronto, 150 St. George St., Toronto M5S 3G7, Canada. Email : razvan.sufana@utoronto.ca

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Christian Gouriéroux

CREST, CEPREMAP and University of Toronto

15 Bd Gabriel Péri, 92245 Malakoff, France

E-mail: gouriero@ensae.fr

Razvan Sufana

University of Toronto

150 St. George St., Toronto M5S 3G7, Canada

E-mail: razvan.sufana@utoronto.ca

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Abstract

This paper extends to the multiasset framework the closed-form solution for options with stochastic volatility derived in Heston (1993) and Ball and Roma (1994). This extension introduces a risk premium in the return equation and considers Wishart dynamics for the process of the stochastic volatility matrix, which is the multiasset analogue of the model of Cox, Ingersoll, and Ross (1985). This approach is used to extend Merton's model (Merton (1974)) for corporate default to a framework with stochastic liability, stochastic volatility and several firms.

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JEL: G12, G13

1 Introduction

The standard Black-Scholes model (Black and Scholes (1973)) is not flexible enough to reproduce some stylized facts observed on derivative prices such as the smile effect, that is a U-shaped relationship between the implied Black-Scholes volatility and the strike price (for any given residual maturity). It is well-known that a smile can be created by introducing stochastic volatility in the Black-Scholes model. This approach was introduced by Hull and White (1987) (see also Johnson and Shanno (1987), Scott (1987), Wiggins (1987), Melino and Turnbull (1990), Stein and Stein (1991)) and improved by Heston (1993), Ball and Roma (1994) who changed the volatility dynamics to ensure a positive volatility.

Another related empirical regularity widely documented in the empirical literature and not satisfied in the Black-Scholes framework concerns the leverage effect, that is the skewness of the univariate implied volatilities as function of the stock price. To account for this feature, stochastic volatility models often assume a correlation between the stock return and volatility shock. For example, Wiggins (1987) introduces such a correlation in a special case and Naik (1994) develops the general framework. Alternatively, a correlation can be implicitely introduced as in Hobson and Rogers (1998), who define instantaneous volatility in terms of past moments of the stock price. Recently, Chan, Kohn, and Kirby (2003) study the leverage effect in a multivariate framework.

The aim of this paper is to introduce and study a multiasset extension of Heston's model. In Section 2, we introduce the joint dynamics of asset prices and volatilities. The price equation includes a volatility-in-mean effect, whereas the volatility matrix is assumed to follow a Wishart autoregressive (WAR) process. The reason for introducing a volatility in mean effect is twofold. First, it is necessary to account for a risk premium if we want to get good historical fit. Second, by introducing interactions between covolatilities and expected returns, we expect to capture the tendency for volatility and stock price to move together even without assuming an instantaneous correlation between the stock return and volatility innovations.[1] A WAR specification of the volatility matrix is the direct multivariate extension of the CIR dynamics (Cox, Ingersoll, and Ross (1985)), and ensures that the volatility matrix is symmetric positive definite.

The closed-form expression of the conditional Laplace transform derived in Section 2 is used in Section 3 to obtain closed-form solutions for the prices of derivatives written on several assets. In Section 3 we also discuss the dynamics under the risk-neutral distributions and some stylized facts on European call options written on the stocks. The application to credit risk is discussed in Section 4 where Merton's model (Merton (1974)) is extended to a framework with stochastic firm liability, stochastic volatility and more than one firm. Section 5 concludes.

2 The joint dynamics of price and volatility

Let us consider a market with one riskfree asset and n risky assets. The riskfree rate is assumed to be constant and is denoted by r, whereas the infinitesimal geometric returns of the risky assets are represented in a n-dimensional vector $d\log S_t$ with S_t being the vector of asset prices at time t. The (infinitesimal) volatility matrix of the risky returns is denoted by Σ_t. It corresponds to a continuous-time process of stochastic symmetric positive definite matrices.

2.1 The model

The joint dynamics of $\log S_t$ and Σ_t is given by the stochastic differential system:

$$\begin{aligned} d\log S_t &= \left[\mu + \begin{pmatrix} Tr\left(D_1\Sigma_t\right) \\ \vdots \\ Tr\left(D_n\Sigma_t\right) \end{pmatrix}\right] dt + \Sigma_t^{1/2} dW_t^S, && (2.1) \\ d\Sigma_t &= \left(\Omega\Omega' + M\Sigma_t + \Sigma_t M'\right) dt + \Sigma_t^{1/2} dW_t^{\sigma} Q + Q' \left(dW_t^{\sigma}\right)' \Sigma_t^{1/2}, && (2.2) \end{aligned}$$

where W_t^S and W_t^σ are a n-dimensional vector and a (n,n) matrix, respectively, whose

[1] This tendency to move together is also captured by the increase in the number of assets which is considered.

elements are independent unidimensional standard Brownian motions, μ is a deterministic n-dimensional vector, whereas D_i, $i = 1, \ldots, n$, Ω, M, Q are (n, n) matrices with Ω invertible. Tr denotes the trace operator and $\Sigma_t^{1/2}$ is the positive square root of the volatility matrix Σ_t.

The volatility matrix is introduced in the drift to account for a risk premium. More explicitly, we get:

$$E_t \left(d \log S_{i,t} \right) = \left[\mu_i + Tr \left(D_i \Sigma_t \right) \right] dt, \;\; i = 1, \ldots, n, \tag{2.3}$$

where E_t denotes the expectation conditional on the information available at time t. The drift is an affine function of volatilities and covolatilities. To get the interpretation of the risk premium, we expect that $Tr \left(D_i \Sigma_t \right) \geq 0$, for any asset and realization of the volatility matrix. This condition is satisfied if[2]:

Assumption A.1: D_i is a symmetric positive definite matrix for any i.

The dynamics of the volatility matrix corresponds to the continuous-time Wishart autoregressive process (WAR) introduced in Gourieroux and Sufana (2003), Gourieroux, Jasiak,

[2] A symmetric positive definite matrix D can be written as: $D = \sum_{j=1}^{n} \lambda_j m_j m_j'$, where λ_j and m_j are the eigenvalues and eigenvectors of D, respectively. Thus we get:

$$\begin{aligned} Tr \left(D \Sigma_t \right) &= Tr \left(\sum_{j=1}^{n} \lambda_j m_j m_j' \Sigma_t \right) = \sum_{j=1}^{n} \lambda_j Tr \left(m_j m_j' \Sigma_t \right) \\ &= \sum_{j=1}^{n} \lambda_j Tr \left(m_j' \Sigma_t m_j \right) = \sum_{j=1}^{n} \lambda_j m_j' \Sigma_t m_j \geq 0, \end{aligned}$$

where the last equality and inequality follow since we can commute within the trace operator and since $\lambda_j > 0$, $\forall j$, and Σ_t is positive definite.

and Sufana (2004)[3,4,5]. The matricial stochastic system (2.2) ensures that the admissible values of Σ_t are symmetric positive definite matrices. The symmetry of $d\Sigma_t$ is immediately derived. Let us discuss more carefully its positivity. For this purpose, let us consider a quadratic form $a'\Sigma_t a$, say, where a is a n-dimensional vector. This quadratic form defines a one-dimensional process with drift:

$$E_t\left[d\left(a'\Sigma_t a\right)\right] = \left(a'\Omega\Omega' a + a'M\Sigma_t a + a'\Sigma_t M' a\right) dt, \tag{2.4}$$

and volatility (see Appendix 1):

$$V_t\left[d\left(a'\Sigma_t a\right)\right] = 4\left(a'\Sigma_t a\right)\left(a'Q'Qa\right) dt, \tag{2.5}$$

where V_t denotes the variance conditional on the information available at time t. Let us now consider what arises when Σ_t reaches the boundary of the set of symmetric positive definite matrices. There exists a nonzero vector a in the kernel of Σ_t which satisfies $a'\Sigma_t a = 0$, and also $\Sigma_t a = 0$. In this case, we have:

$$V_t\left[d\left(a'\Sigma_t a\right)\right] = 0,$$

and

$$E_t\left[d\left(a'\Sigma_t a\right)\right] = \left(a'\Omega\Omega' a\right) dt > 0.$$

Thus we get a reflection towards positivity when the boundary is reached (whenever Ω is invertible).

Finally note that the system defining Σ_t involves n^2 independent Brownian motions,

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3 See also Bru (1991), O'Connell (2003), Donati-Martin et alii (2003) for the special case of the unit root.

4 An alternative specification of the volatility matrix assumes that the inverse of Σ_t follows a Wishart process. This extends the inverted gamma distribution assumed in one-dimensional stochastic volatility models to get a closed form expression for the return density and to study its tail magnitude [see e.g. Praetz (1972), Clark (1973), Blattberg and Gonedes (1974)]. The direct Wishart specification used in our framework is more appropriate to get closed form expressions for the moment generating functions and to price derivatives.

5 The Wishart distribution, but not the Wishart process, is also used in Bayesian approaches of stochastic volatility models (Jacquier, Polson, and Rossi (1995)).

whereas the volatility matrix has dimension $n\,(n+1)\,/2$ due to the symmetry restrictions. Thus the Brownian matrix W_t^σ does not correspond to the process of standardized "innovations" of Σ_t and brings more information than Σ_t itself.

Example 1 In the one-dimensional framework ($n=1$) the system (2.1) and (2.2) becomes:

$$\begin{aligned} d\log S_t &= \left(\mu+\delta\sigma_t^2\right)dt+\sigma_t dW_t^S, \\ d\left(\sigma_t^2\right) &= \left(\omega^2+2m\sigma_t^2\right)dt+2q\sigma_t dW_t^\sigma. \end{aligned}$$

Thus the volatility process is a CIR process and the model reduces to Heston's specification (see Heston (1993), Ball and Roma (1994)).

2.2 Affine property

The joint process $(\log S_t, \Sigma_t)$ is an affine process, that is admits drift and volatility functions which are affine functions of $\log S_t$ and Σ_t (see Duffie and Kan (1996), Duffie, Filipovic, and Schachermayer (2003) for the definitions and analysis of affine processes). The affine property is clearly satisfied for the drifts of $\log S_t$, Σ_t and for the volatility of $\log S_t$. Let us now examine the volatility of the volatility. For any pair of vectors a and b we have (see Appendix 1):

$$V_t\left(a'd\Sigma_t b\right)=\left(a'\Sigma_t a\ b'Q'Qb+2a'\Sigma_t b\ b'Q'Qa+b'\Sigma_t b\ a'Q'Qa\right)dt. \tag{2.6}$$

This quantity is also affine with respect to Σ_t.

Thus it is possible to use the general theory of affine processes to derive:

i) the conditional Laplace transform of the process at any horizon,

ii) the set of risk-neutral distributions.

Let us first derive the conditional Laplace transform (moment generating function). In the property below the exponential affine expression of the conditional Laplace transform is a direct consequence of general results on affine processes (see Duffie, Filipovic, and Schachermayer (2003)). The form of the Riccati equations satisfied by the sensitivity coefficients is derived in Appendix 2.

Proposition 1 *The conditional Laplace transform of the joint process* $(\log S_t, \Sigma_t)$ *and of its cumulated values is defined by:*

$$\begin{aligned}\Psi_{t,h}\left(\gamma,\gamma_0,\widetilde{\gamma},C,c_0,\widetilde{C}\right) &= E_t \exp\left[\int_t^{t+h}\left(\gamma'\log S_u+\gamma_0\right)du+\widetilde{\gamma}'\log S_{t+h}\right.\\ &\left.+\int_t^{t+h} Tr\left(C\Sigma_u+c_0\right)du+Tr\left(\widetilde{C}\Sigma_{t+h}\right)\right], \qquad (2.7)\end{aligned}$$

where the coefficients $\gamma,\gamma_0,\widetilde{\gamma},C,c_0,\widetilde{C}$ *can be real or complex whenever the expectation exists. For the affine process given in (2.1) and (2.2), the conditional Laplace transform is:*

$$\Psi_{t,h}\left(\gamma,\gamma_0,\widetilde{\gamma},C,c_0,\widetilde{C}\right)=\exp\left[a\left(h\right)'\log S_t+Tr\left(B\left(h\right)\Sigma_t\right)+b\left(h\right)\right], \qquad (2.8)$$

where the functions a*,* B*, and* b *satisfy the system of Riccati equations:*

$$\begin{aligned}\frac{da\left(h\right)}{dh} &= \gamma, \qquad (2.9)\\ \frac{dB\left(h\right)}{dh} &= B\left(h\right)M+M'B\left(h\right)+2B\left(h\right)Q'QB\left(h\right)\\ &+\frac{1}{2}a\left(h\right)a\left(h\right)'+\sum\nolimits_{i=1}^{n}a_i\left(h\right)D_i+C, \qquad (2.10)\\ \frac{db\left(h\right)}{dh} &= a\left(h\right)'\mu+Tr\left[B\left(h\right)\Omega\Omega'\right]+\gamma_0+c_0, \qquad (2.11)\end{aligned}$$

with initial conditions: $a\left(0\right)=\widetilde{\gamma}$*,* $B\left(0\right)=\widetilde{C}$*,* $b\left(0\right)=0$*.*

Thus the differential system involves the parameters γ,γ_0,C,c_0, whereas $\widetilde{\gamma},\widetilde{C}$ define the initial conditions. Note that the differential equation for a admits the explicit solution:

$$a\left(h\right)=\gamma h+\widetilde{\gamma}.$$

Then the system in (2.9), (2.10) and (2.11) can be recursively solved. The second equation provides the matrix $B\left(h\right)$ and finally the form of $b\left(h\right)$ is obtained by substituting $a\left(h\right)$ and $B\left(h\right)$ by their expressions in the third equation.

The above Riccati equations admit a closed-form solution whenever $\gamma=0$ (see Appendix 4). Thus this is an example of multidimensional Riccati equations with a fundamental system of solutions (see Walcher (1986) for a general definition and Grasselli, Tebaldi (2004), Section

3.4.2, for a discussion of quadratic models, which does not include the present WAR process).

Proposition 2 *For $\gamma = 0$, we get:*

$$B(h) = B^* + \exp\left[(M + 2Q'QB^*)h\right]'$$
$$\left\{\left(\widetilde{C} - B^*\right)^{-1} + 2\int_0^h \exp\left[(M + 2Q'QB^*)u\right] Q'Q \exp\left[(M + 2Q'QB^*)u\right]' du\right\}^{-1}$$
$$\exp\left[(M + 2Q'QB^*)h\right],$$

where B^* satisfies:

$$M'B^* + B^*M + 2B^*Q'QB^* + \frac{1}{2}\widetilde{\gamma}\widetilde{\gamma}' + \sum\nolimits_{i=1}^n \widetilde{\gamma}_i D_i + C = 0.$$

The closed-form solution for $b(h)$ can be immediately deduced from the third differential equation:

$$b(h) = \left(\widetilde{\gamma}'\mu + \gamma_0 + c_0\right)h + Tr\left[\int_0^h B(u)\,du\ \Omega\Omega'\right].$$

The following corollary is a consequence of Propositions 1 and 2.

Corollary 1 *The conditional distribution of the asset return is characterized by:*

$$E_t\left[\exp\left(\widetilde{\gamma}'\log S_{t+h}\right)\right] = \exp\left[\widetilde{\gamma}'\log S_t + Tr\left(B(h)\Sigma_t\right) + b(h)\right],$$

where:

$$\begin{aligned} \frac{dB(h)}{dh} &= B(h)M + M'B(h) + 2B(h)Q'QB(h) + \frac{1}{2}\widetilde{\gamma}\widetilde{\gamma}' + \sum\nolimits_{i=1}^n \widetilde{\gamma}_i D_i, \\ \frac{db(h)}{dh} &= \widetilde{\gamma}'\mu + Tr\left[B(h)\Omega\Omega'\right], \end{aligned}$$

with initial conditions: $B(0) = 0$, $b(0) = 0$.

The closed-form solution for $B(h)$ is:

$$\begin{aligned} B(h) = {} & B^* + \exp\left[(M + 2Q'QB^*)h\right]' \\ & \left\{-(B^*)^{-1} + 2\int_0^h \exp\left[(M + 2Q'QB^*)u\right] Q'Q \exp\left[(M + 2Q'QB^*)u\right]' du\right\}^{-1} \\ & \exp\left[(M + 2Q'QB^*)h\right], \end{aligned}$$

where B^* satisfies:

$$M'B^* + B^*M + 2B^*Q'QB^* + \frac{1}{2}\widetilde{\gamma}\widetilde{\gamma}' + \sum\nolimits_{i=1}^{n} \widetilde{\gamma}_i D_i = 0.$$

The closed-form solution for $b\,(h)$ is:

$$b\,(h) = \widetilde{\gamma}'\mu h + Tr\left[\int_0^h B\,(u)\,du\ \Omega\Omega'\right].$$

Example 2 In the one-dimensional framework, which includes Heston (1993) and Ball and Roma (1994), we get:

$$E_t\left[\exp\left(\widetilde{\gamma}\log S_{t+h}\right)\right] = \exp\left[\widetilde{\gamma}\log S_t + B\,(h)\,\Sigma_t + b\,(h)\right],$$

where:

$$\begin{aligned} B\,(h) &= B^* + \frac{\exp\left[2\left(m+2q^2B^*\right)h\right]}{-\left(B^*\right)^{-1} + q^2\frac{\exp[2(m+2q^2B^*)h]-1}{m+2q^2B^*}}, \\ b\,(h) &= \left(\widetilde{\gamma}\mu + \omega^2 B^*\right)h + \frac{\omega^2}{2q^2}\log\left|1 + \frac{1-\exp\left[2\left(m+2q^2B^*\right)h\right]}{m/\left(q^2B^*\right)+2}\right|, \end{aligned}$$

and B^* is a solution of:

$$2mB^* + 2q^2B^{*2} + \frac{1}{2}\widetilde{\gamma}^2 + \widetilde{\gamma}\delta = 0.$$

The recursive equation can be used to find an expansion of the log-Laplace transform when $\widetilde{\gamma} = u\widetilde{\gamma}_0$, say, for u in a neighborhood of zero, and thus to deduce the first and second order conditional moments of asset returns. Indeed let us consider the expansion of $B\,(h)$ and $b\,(h)$:

$$\begin{aligned} B\,(h) &= B_1\,(h)\,u + B_2\,(h)\,u^2 + o\left(u^2\right), \\ b\,(h) &= b_1\,(h)\,u + b_2\,(h)\,u^2 + o\left(u^2\right). \end{aligned}$$

The system in $B\,(h)$ becomes:

$$\begin{aligned} \frac{dB_1\,(h)}{dh} &= B_1\,(h)\,M + M'B_1\,(h) + \sum\nolimits_{i=1}^{n}\widetilde{\gamma}_{0i}D_i, \\ \frac{dB_2\,(h)}{dh} &= B_2\,(h)\,M + M'B_2\,(h) + 2B_1\,(h)\,Q'QB_1\,(h) + \frac{1}{2}\widetilde{\gamma}_0\widetilde{\gamma}_0'. \end{aligned}$$

Thus we have just to solve recursively linear differential equations first with respect to B_1, then with respect to B_2 to deduce the affine expressions of the conditional mean and volatilities[6].

As usual the introduction of stochastic volatility increases the tail magnitude of the stock returns. In the present framework, it is easily checked that the Laplace transform admits a series expansion in a neighborhood of zero and the power moments of stock returns exist at any nonnegative order. Thus the tail increase due to WAR stochastic volatility does not imply the nonexistence of some moments.

The associated transition of the stock returns can be deduced by inverting the Fourier transform (the Laplace transform evaluated at pure imaginary arguments) or in a more direct way. Indeed, for a given volatility path, the return process is multivariate Gaussian. The conditional distribution of $\log S_{t+h}$ given (Σ_t) and $\log S_t$ is normal with mean:

$$\log S_t + \mu h + \begin{pmatrix} Tr\left(D_1 \int_t^{t+h} \Sigma_u du\right) \\ \vdots \\ Tr\left(D_n \int_t^{t+h} \Sigma_u du\right) \end{pmatrix},$$

and variance-covariance matrix $\int_t^{t+h} \Sigma_u du$. Thus the transition of $\log S_{t+h}$ given Σ_t and $\log S_t$ is deduced by integrating out the cumulated volatility $\int_t^{t+h} \Sigma_u du$ given Σ_t.

The general expression of the conditional Laplace transform can now be used to characterize the distribution of the integrated volatility $\int_t^{t+h} \Sigma_u du$, or of the average volatility $\frac{1}{h} \int_t^{t+h} \Sigma_u du$:

Corollary 2 *The conditional distribution of the integrated volatility is characterized by:*

$$E_t \left[\exp \int_t^{t+h} Tr\left(C\Sigma_u\right) du\right] = \exp\left[Tr\left(B\left(h\right)\Sigma_t\right) + b\left(h\right)\right],$$

[6] More generally, there exists a local analytic solution whose coefficients can be recursively computed as solutions of linear differential equations [Walcher (1991), p. 27 and Grasselli, Tebaldi (2004), Section 3.1].

where:

$$\begin{aligned}\frac{dB(h)}{dh} &= B(h)M + M'B(h) + 2B(h)Q'QB(h) + C,\\ \frac{db(h)}{dh} &= Tr\left[B(h)\Omega\Omega'\right],\end{aligned}$$

with initial conditions: $B(0) = 0,\ b(0) = 0.$

The closed-form solution for $B(h)$ *is:*

$$\begin{aligned}B(h) = {} & B^* + \exp\left[(M + 2Q'QB^*)h\right]' \\ & \left\{-(B^*)^{-1} + 2\int_0^h \exp\left[(M + 2Q'QB^*)u\right]Q'Q\exp\left[(M + 2Q'QB^*)u\right]' du\right\}^{-1} \\ & \exp\left[(M + 2Q'QB^*)h\right],\end{aligned}$$

where B^* *satisfies:*

$$M'B^* + B^*M + 2B^*Q'QB^* + C = 0.$$

The closed-form solution for $b(h)$ *is immediately deduced from the second differential equation:*

$$b(h) = Tr\left[\int_0^h B(u)\,du\ \Omega\Omega'\right].$$

3 Derivative pricing

The explicit expression of the conditional Laplace transform given in Proposition 2 can be used to price derivatives written on several assets by using the transform analysis (Duffie, Pan, and Singleton (2000)). It allows us to avoid the numerical approximations such as multibranches trees introduced in the multiasset framework (see e.g. Boyle (1988), Boyle, Evnine, and Gibbs (1989), Ho, Stapleton, and Subrahmanyam (1995), Chen, Chung, and Yang (2002)), or expansions around the constant volatility hypothesis (see e.g. Hull and White (1987)). Without loss of generality, the derivations can be performed assuming a zero riskfree rate.

3.1 Risk-neutral distribution

It is known from Girsanov theorem that the change of density for period $(t, t+h)$ between the historical and risk-neutral distributions is of the type:

$$m_{t,t+h} = \exp\left\{\int_t^{t+h} \left[\gamma_u' d\log S_u + Tr\left(C_u d\Sigma_u\right)\right] + \int_t^{t+h} \left(\gamma_{0u} du + c_{0u} du\right)\right\},$$

where γ_u, C_u, γ_{0u}, c_{0u} denote predetermined coefficients. The change of probabilities and thus the coefficients are constrained by both the unit mass restriction and the martingale condition on stock prices.

Let E_t^* denote the conditional expectation under the risk-neutral probability. The property below is proved in Appendix 3.

Proposition 3 *Under the risk-neutral distribution, the joint process* $(\log S_t, \Sigma_t)$ *satisfies a stochastic differential system with volatility equal to the historical volatility and a modified drift:*

$$\begin{aligned}
E_t^*\left(d\log S_t\right) &= E_t\left(d\log S_t\right) + \Sigma_t\gamma_t \ dt \\
&= -\frac{1}{2}\left[Tr\left(e_i e_i'\Sigma_t\right)\right] dt \\
&= -\frac{1}{2}\left[\sigma_{ii,t}\right] dt, \qquad (3.1) \\
E_t^*\left(d\Sigma_t\right) &= E_t\left(d\Sigma_t\right) + Cov_t\left[Tr\left(C_t d\Sigma_t\right), d\Sigma_t\right] \\
&= E_t\left(d\Sigma_t\right) + 2\left(\Sigma_t C_t Q'Q + Q'QC_t'\Sigma_t\right) dt, \qquad (3.2)
\end{aligned}$$

where e_i *denotes the canonical vector with zero components except the* i^{th} *component equal to* 1 *and* $[\sigma_{ii,t}]$ *denotes the vector with the* i^{th} *element equal to* $\sigma_{ii,t}$.

The risk premium on the Brownian motion of the return equation is fixed by the martingale condition (see Appendix 3), whereas the risk premia corresponding to the volatilities-covolatilities (that are C_t) can be fixed arbitrarily as a consequence of market incompleteness (see e.g. Garman (1976)).

The stochastic system under the risk-neutral probability has the same form as the stochastic system under the historical probability if:

$$C_t = C \text{ is constant.} \tag{3.3}$$

Indeed the former differential system corresponds to:

$$E_t^* \left(d \log S_t \right) = \left(\mu^* + \left[Tr \left(D_i^* \Sigma_t \right) \right] \right) dt,$$

where $\mu^* = 0$, $D_i^* = -\frac{1}{2} e_i e_i'$, and

$$E_t^* \left(d\Sigma_t \right) = \left(\Omega^* \Omega^{*\prime} + M^* \Sigma_t + \Sigma_t M^{*\prime} \right) dt,$$

where $\Omega^* \Omega^{*\prime} = \Omega \Omega'$, $M^* = M + 2Q'QC'$. In this case, the intercept in the volatility drift stays the same, whereas the matrix of "mean-reverting parameters" can be fixed arbitrarily.

3.2 Conditional Laplace transform under the risk-neutral distribution

The risk-neutral conditional Laplace transform $\Psi_{t,h}^* \left(\gamma, \gamma_0, \widetilde{\gamma}, C, c_0, \widetilde{C} \right)$ is defined as in equation (2.7), with E_t replaced by E_t^*. Under condition (3.3) above, it can be directly deduced from Proposition 1, after replacing the historical parameters by the risk-neutral ones.

Proposition 4 *The conditional Laplace transform of the joint process* $(\log S_t, \Sigma_t)$ *and of its integrated values under the risk-neutral distribution is:*

$$\begin{aligned} \Psi_{t,h}^* \left(\gamma, \gamma_0, \widetilde{\gamma}, C, c_0, \widetilde{C} \right) &= E_t^* \exp \left[\int_t^{t+h} \left(\gamma' \log S_u + \gamma_0 \right) du + \widetilde{\gamma}' \log S_{t+h} \right. \\ &\quad \left. + \int_t^{t+h} Tr \left(C \Sigma_u + c_0 \right) du + Tr \left(\widetilde{C} \Sigma_{t+h} \right) \right] \qquad (3.4) \\ &= \exp \left[a^* (h)' \log S_t + Tr \left(B^* (h) \Sigma_t \right) + b^* (h) \right], \qquad (3.5) \end{aligned}$$

where the functions a^*, B^*, and b^* satisfy the system of Riccati equations:

$$\begin{aligned} \frac{da^* (h)}{dh} &= \gamma, \qquad (3.6) \\ \frac{dB^* (h)}{dh} &= B^* (h) M^* + M^{*\prime} B^* (h) + 2 B^* (h) Q'Q B^* (h) \\ &\quad + \frac{1}{2} a^* (h) a^* (h)' - \frac{1}{2} diag \left(a^* (h) \right) + C, \qquad (3.7) \end{aligned}$$

$$\frac{db^*(h)}{dh} = Tr\left[B^*(h)\,\Omega\Omega'\right] + \gamma_0 + c_0, \tag{3.8}$$

with initial conditions: $a^*(0) = \widetilde{\gamma}$, $B^*(0) = \widetilde{C}$, $b^*(0) = 0$.

The closed-form solutions for a^*, B^* and b^* are similar to the solutions for a, B and b derived in Section 2.

Proposition 5 *For $\gamma = 0$, we get:*

$$\begin{aligned}
a^*(h) &= \widetilde{\gamma}, \\
B^*(h) &= B^* + \exp\left[(M^* + 2Q'QB^*)\,h\right]' \\
&\quad \left\{\left(\widetilde{C} - B^*\right)^{-1} + 2\int_0^h \exp\left[(M^* + 2Q'QB^*)\,u\right] Q'Q \exp\left[(M^* + 2Q'QB^*)\,u\right]' du\right\}^{-1} \\
&\quad \exp\left[(M^* + 2Q'QB^*)\,h\right], \\
b^*(h) &= (\gamma_0 + c_0)\,h + Tr\left[\int_0^h B^*(u)\,du\ \Omega\Omega'\right],
\end{aligned}$$

where B^* satisfies:

$$M^{*\prime}B^* + B^*M^* + 2B^*Q'QB^* + \frac{1}{2}\widetilde{\gamma}\widetilde{\gamma}' - \frac{1}{2}diag\left(\widetilde{\gamma}\right) + C = 0.$$

Propositions 4 and 5 can be used to compute the price of a European derivative with exponential payoff jointly written on $\log S_t$ and Σ_t.

Corollary 3 *The price at time t of the derivative with residual maturity h and payoff* $\exp\left[\widetilde{\gamma}'\log S_{t+h} + Tr\left(\widetilde{C}\Sigma_{t+h}\right)\right]$ *is:*

$$\begin{aligned}
\Pi\left(t, h; \widetilde{\gamma}, \widetilde{C}\right) &= \Psi^*_{t,h}\left(0, 0, \widetilde{\gamma}, 0, 0, \widetilde{C}\right) \\
&= \exp\left[\widetilde{\gamma}'\log S_t + Tr\left(B^*(h)\,\Sigma_t\right) + b^*(h)\right],
\end{aligned}$$

where:

$$\begin{aligned}
B^*(h) &= B^* + \exp\left[(M^* + 2Q'QB^*)\,h\right]' \\
&\quad \left\{\left(\widetilde{C} - B^*\right)^{-1} + 2\int_0^h \exp\left[(M^* + 2Q'QB^*)\,u\right] Q'Q \exp\left[(M^* + 2Q'QB^*)\,u\right]' du\right\}^{-1} \\
&\quad \exp\left[(M^* + 2Q'QB^*)\,h\right],
\end{aligned}$$

$$b^*(h) = Tr\left[\int_0^h B^*(u)\,du\;\Omega\Omega'\right],$$

and B^* satisfies:

$$M^{*\prime}B^* + B^*M^* + 2B^*Q'QB^* + \frac{1}{2}\widetilde{\gamma}\widetilde{\gamma}' - \frac{1}{2}diag\left(\widetilde{\gamma}\right) = 0.$$

3.3 Stylized facts and financial puzzles

The multivariate stochastic volatility model of Sections 2 and 3 provides a convenient framework to understand some stylized facts on derivative prices.

As an illustration let us assume a zero riskfree rate, and consider two assets ($n = 2$) and a European call option written on the first asset. Its price at date t is:

$$\begin{aligned}
& g\left(t,h,k;\Sigma_t\right) \\
= \;& S_{1,t}E^*\left[\left(\frac{S_{1,t+h}}{S_{1,t}} - k\right)^+ |\Sigma_t\right] \\
= \;& S_{1,t}E^*\left\{\left[\exp\left(\log S_{1,t+h} - \log S_{1,t}\right) - k\right]^+ |\Sigma_t\right\} \\
= \;& S_{1,t}E^*\left\{E^*\left\{\left[\exp\left(\log S_{1,t+h} - \log S_{1,t}\right) - k\right]^+ |\underline{\Sigma_{t,t+h}}\right\} |\Sigma_t\right\} \\
= \;& S_{1,t}E^*\left\{E^*\left\{\left[\exp\left[-\frac{1}{2}\int_t^{t+h}\sigma_{11,u}du + \left(\int_t^{t+h}\sigma_{11,u}du\right)^{1/2}\xi\right] - k\right]^+ |\underline{\Sigma_{t,t+h}}\right\} |\Sigma_t\right\},
\end{aligned}$$

where $\underline{\Sigma_{t,t+h}}$ denotes the volatility path between dates t, $t+h$, ξ is a standard normal variable independent of the volatility process (Σ_t), k is the moneyness strike and h is the residual maturity. As usual, the call price is deduced from the one-dimensional Black-Scholes formula. If $X \sim N\left(m, s^2\right)$, it is well-known that:

$$\begin{aligned}
\Psi\left(k,m,s^2\right) &= E\left[\left(\exp X - k\right)^+\right] \\
&= \left(E\exp X\right)N\left(d_1\right) - kN\left(d_2\right),
\end{aligned}$$

where

$$d_1 = \frac{\log\left[\left(E\exp X\right)/k\right] + s^2/2}{s}, \qquad d_2 = d_1 - s,$$

and N denotes the cumulative distribution function of the standard normal distribution.

The option price becomes:

$$
\begin{aligned}
g(t,h,k;\Sigma_t) &= S_{1,t}E^*\left[\Psi\left[k,-\frac{1}{2}\int_t^{t+h}\sigma_{11,u}du,\int_t^{t+h}\sigma_{11,u}du\right]|\Sigma_t\right] \\
&= S_{1,t}E^*\left[N(d_1)-kN(d_2)\,|\Sigma_t\right],
\end{aligned}
\tag{3.9}
$$

where

$$
d_1=\frac{-\log(k)+\frac{1}{2}\int_t^{t+h}\sigma_{11,u}du}{\left(\int_t^{t+h}\sigma_{11,u}du\right)^{1/2}},\qquad d_2=d_1-\left(\int_t^{t+h}\sigma_{11,u}du\right)^{1/2}.
$$

Thus for any parameter values, we can easily simulate the joint path of the fundamental volatility factor Σ_t, the stock price $S_{1,t}$, and the option price $g(t,h,k;\Sigma_t)$.

A number of stylized facts are observed from approaches which consider separately the different stocks, and in particular introduce different measures of volatility for asset 1. Among these measures are:

i) the Black-Scholes implied volatility associated with the option price $g(t,h,k;\Sigma_t)$,

ii) the realized volatility at a higher frequency, computed as a sample historical variance of high frequency returns within the period $(t,t+1)$,

iii) the GARCH(1,1) volatility forecast to approximate $\eta_{1t}^2=V\left[\log S_{1,t+1}\mid S_{1,t},\ S_{1,t-1},\ldots\right]$ based on annual data.

We use simulations to investigate the implications of the multivariate stochastic volatility model proposed in this paper. We consider two risky assets driven by the differential system (2.1) - (2.2) and simulate log-prices over 50 years, with a time step of 0.01 (which provides 100 prices per year). The parameter values are:

$$
\mu=\begin{pmatrix}0.1\\0.035\end{pmatrix},\ D_1=D_2=0,
$$

for the stock price equation, and

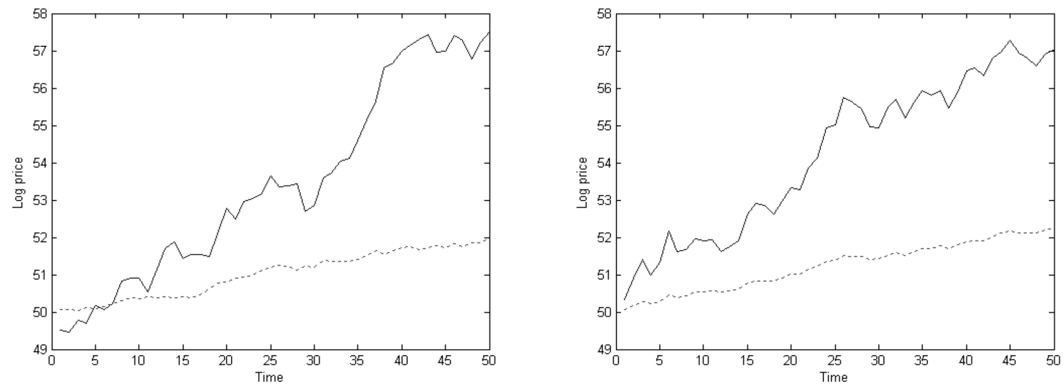
$$
K=4,\ M_d=0,\ \Sigma_d=\begin{bmatrix}0.035 & \rho\sqrt{(0.035)(0.001)}\\ \rho\sqrt{(0.035)(0.001)} & 0.001\end{bmatrix},
$$

for the time discretized Wishart process[7]. Each simulation is performed for two extreme

[7] As shown in Gourieroux, Jasiak, and Sufana (2004), when the degree of freedom K is integer, the time dis-

values of the latent correlation: $\rho = 0$ and $\rho = 0.95$.

Figure 1 shows the end-of-year log asset prices and Figure 2 plots the associated simple annual returns. The two assets have different expected return and volatility. Asset 1 has a higher expected return to compensate for its higher volatility.



a) Latent correlation $= 0$ b) Latent correlation $= 0.95$

Figure 1. End-of-year log price of asset 1 (solid line) and asset 2 (dotted line).

---

cretization of the continuous-time Wishart process:

$$d\Sigma_t \quad = \left(KQQ' + M\Sigma_t + \Sigma_t M'\right) dt + \Sigma_t^{1/2} dW_t^{\sigma} Q' + Q \left(dW_t^{\sigma}\right)' \Sigma_t^{1/2},$$

is the discrete-time Wishart process:

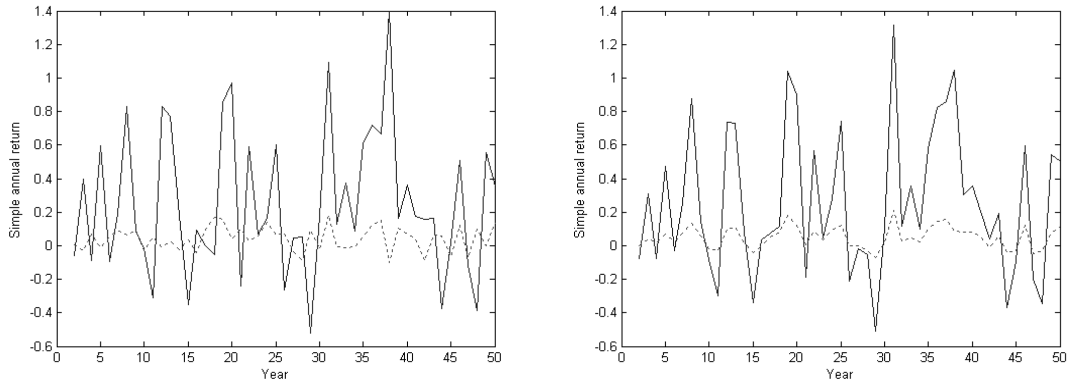
$$\Sigma_t = \sum_{k=1}^{K} x_{kt} x'_{kt},$$
$$x_{k,t+s} = M_d x_{kt} + \varepsilon_{k,t+s}, \; \varepsilon_{k,t+s} \sim N\left(0, \Sigma_d\right),$$

where

$$M_d = \exp\left(Ms\right), \; \Sigma_d = \int_0^s \exp\left(Mu\right) QQ' \left[\exp\left(Mu\right)\right]' du,$$

and s is the time step.

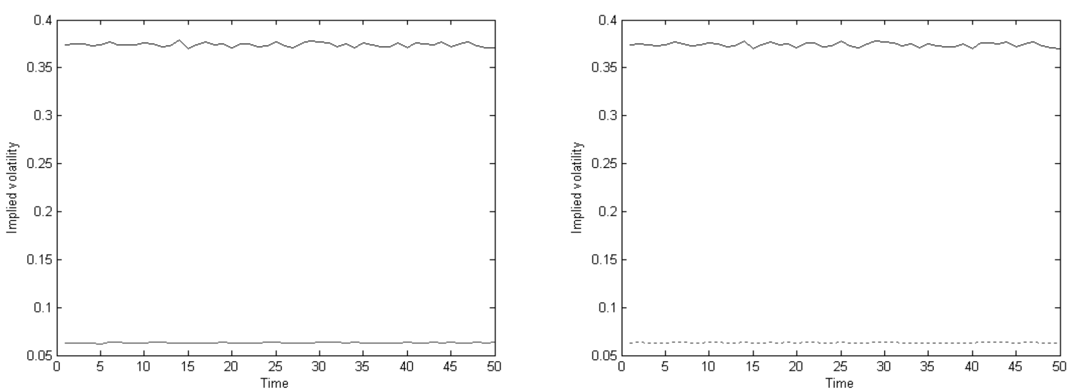
The latent correlation ρ represents the conditional correlation between the two latent components of the process (x_{kt}). In particular, if $\rho = 0$, the two latent components are conditionally independent.



a) Latent correlation = 0 b) Latent correlation = 0.95

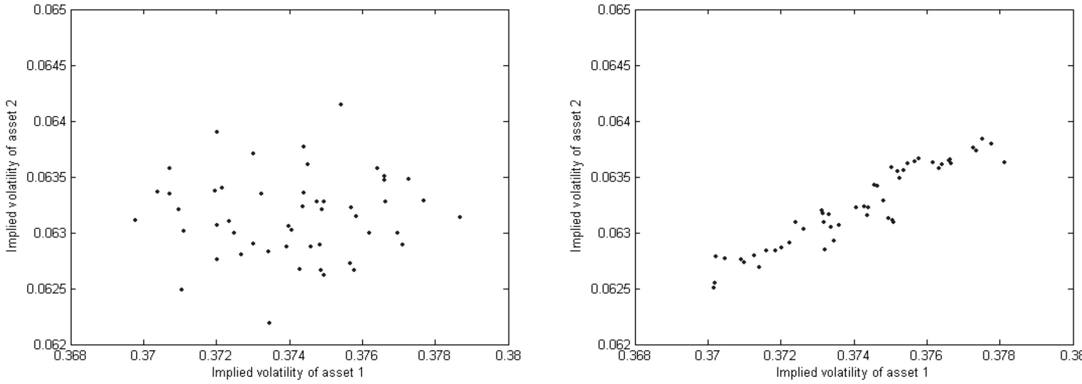
Figure 2. Simple annual return of asset 1 (solid line) and asset 2 (dotted line).

End-of-year at-the-money call prices are computed using equation (3.9) with moneyness strike $k = 1$ and horizon $h = 1$ and the corresponding implied volatilities, obtained by inverting the Black-Scholes formula, are presented in Figures 3 and 4.



a) Latent correlation = 0 b) Latent correlation = 0.95

Figure 3. Implied volatility of asset 1 (solid line) and asset 2 (dottedline).



a) Latent correlation = 0 b) Latent correlation = 0.95

Figure 4. Implied volatility of asset 1 versus implied volatility of asset 2.

Since the stochastic volatility is driven by three factors, the implied Black-Scholes volatilities of the two assets do not satisfy a deterministic relationship. They are stochastic with a nondegenerate joint distribution. The pattern of this distribution depends on the latent correlation. When $\rho = 0$, we observe in Figure 4a) that the implied volatilities are almost independent, whereas Figure 4b) shows a regression line when $\rho = 0.95$. Figure 5 plots the sample correlation between the 49 implied volatilities of asset 1 and the 49 implied volatilities of asset 2 as a function of the latent correlation ρ. For each value of ρ, the sample correlation is computed as the ratio of the sample covariance of the two assets to the square root of the product of the sample variances.

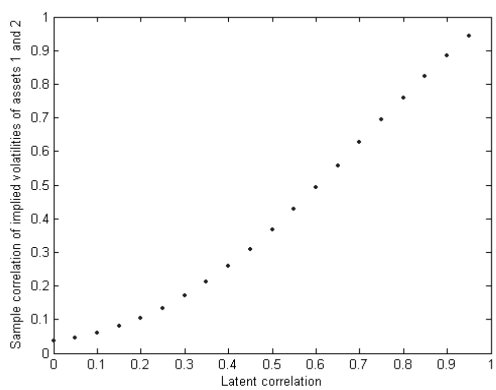
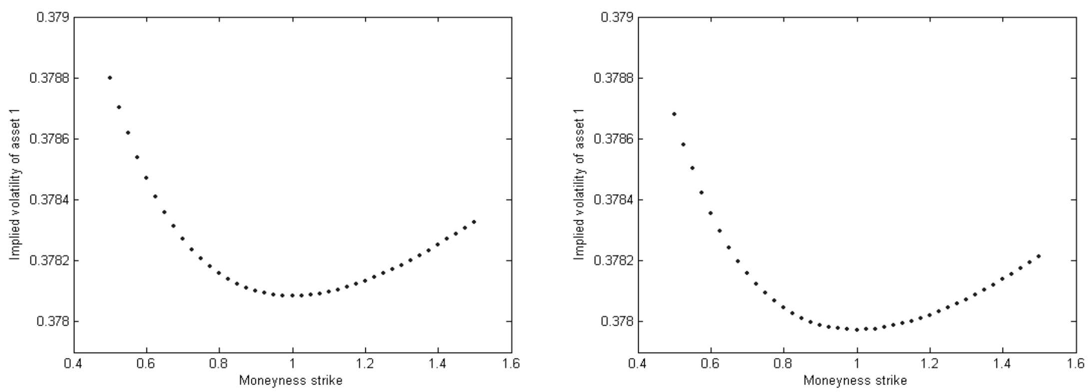


Figure 5. Sample correlation of implied volatilities of assets 1 and 2 versus latent correlation.

3.3.1 Skewed implied volatility

The Black-Scholes implied volatilities can be a skewed function of the (moneyness) strike, and this stylized fact is usually reproduced in the standard one-asset stochastic volatility model by introducing a correlation between the Brownian motions of the price and volatility equations.

In the present multiasset framework, the two (multivariate) Brownian motions have been assumed independent. However, the independence of innovations conditional on the information set $\underline{\Sigma_t}$, is compatible with a dependence of innovations conditional on the smaller information set $\underline{S_{1,t}}$, $\underline{\sigma_{11t}}$, say. Thus it is not surprising to reproduce an asymmetric volatility smile in the framework of a bivariate model with independent innovations. This property is illustrated in Figure 6, where the implied volatilities of asset 1 for the second year are reported as a function of moneyness strike k.

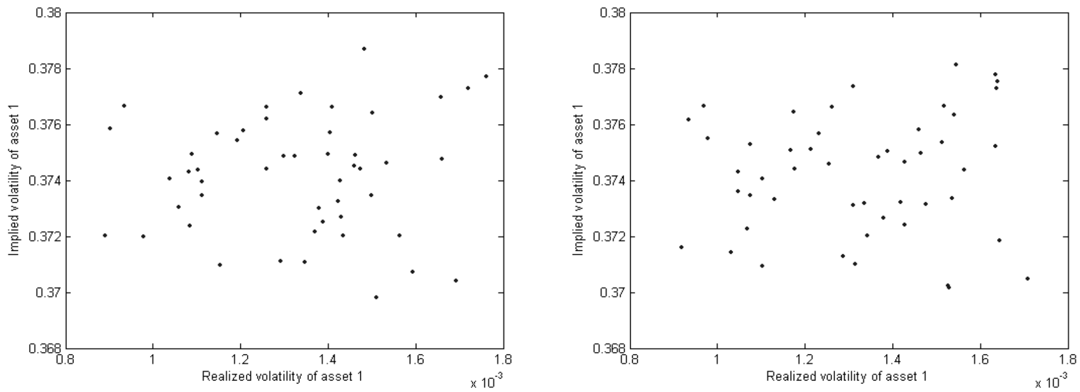


a) Latent correlation = 0 b) Latent correlation = 0.95

Figure 6. Implied volatility of asset 1 versus moneyness strike.

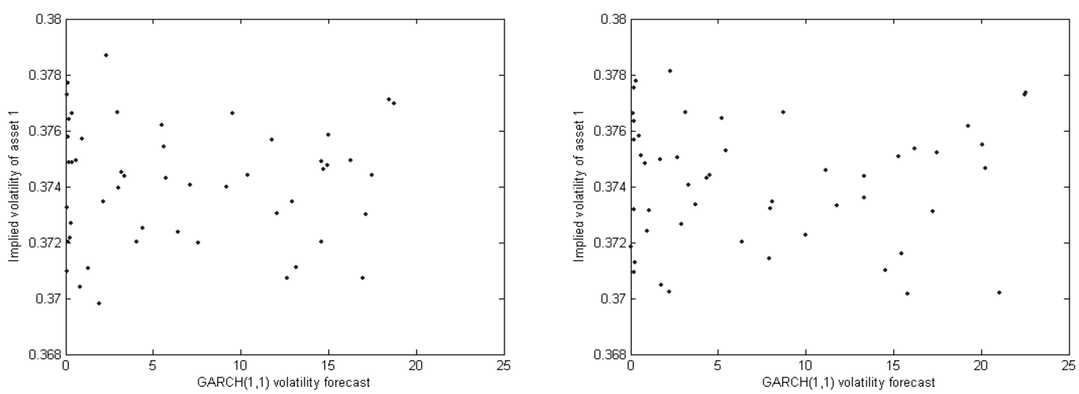
3.3.2 Relation between the option price and volatility

In the one-dimensional Black-Scholes model, the call price is an increasing function of the (marginal) volatility. However, this property is not always satisfied in a more complicated framework (see El Karoui, Jeanblanc, and Shreve (1998)). In the multivariate stochastic volatility model it is expected that the call price is an increasing function of the fundamental risk Σ_t, but this does not imply that it will be an increasing function of a "marginal" volatility, computed with a restricted information set. As an illustration, we provide below the dependence between the at-the-money implied volatility and the realized and GARCH(1,1) volatilities, respectively. Note that a realized volatility is an approximation of a marginal volatility, and is computed without taking into account the information of lagged returns. Similarly, the GARCH(1,1) volatility is computed by considering the information on lagged returns of stock 1 only.



a) Latent correlation = 0 b) Latent correlation = 0.95

Figure 7. Implied volatility of asset 1 versus realized volatility of asset 1. The sample correlation coefficient between these two variables is: a) 0.0692, b) 0.0814.



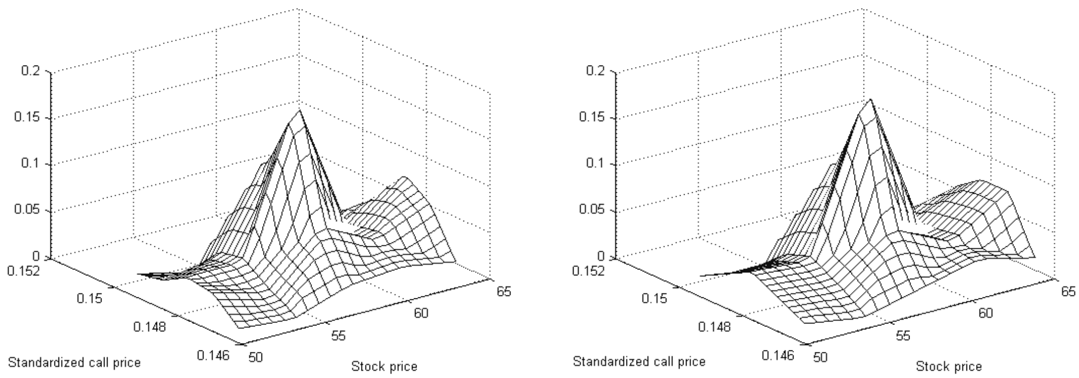
a) Latent correlation = 0 b) Latent correlation = 0.95

Figure 8. Implied volatility of asset 1 versus GARCH(1,1) volatility forecast. The sample correlation coefficient between these two variables is: a) 0.0016, b) -0.0026.

It is immediately noted that the implied volatility is weakly related with both the realized and GARCH(1,1) volatilities. Moreover, they do not vary around the same volatility level. Thus both realized and GARCH(1,1) volatilities are poor proxies of the implied Black-Scholes volatilities.

3.3.3 Relation between call and stock prices

Figure 9 reveals a more complex relationship between the standardized call price (call price divided by stock price) and the stock price than is implied by the Black-Scholes model.



a) Latent correlation = 0 b) Latent correlation = 0.95

Figure 9. Joint density of the standardized call price and stock price (in logarithm).

In the standard Black-Scholes model, the ratio of the call price to the stock price depends only on the design of the call and on the return volatility. But it is constant with respect to the stock price.

4 Application to Credit Risk

The multivariate stochastic volatility model can in particular be applied to credit risk analysis by considering the asset values and liabilities of the firms as the basic contingent claims. The model is described in Section 4.1, whereas simulation results are presented in Section 4.2.

4.1 The model

A new interest in multiasset derivatives has been shown recently in relation with credit risk.

Indeed in the standard framework of the firm value model introduced by Merton (1974), the potential time to default h, say, is predetermined, and the stock, bonds, credit default swaps corresponding to a given firm i are defined from its asset $A_{i,t+h}$ and liability $L_{i,t+h}$ at date $t+h$. More precisely, with a zero riskfree rate, the value at date t of a zero-coupon bond with residual maturity h issued by the firm i is[8]:

$$B_i\left(t,t+h\right) = E_t^*\left[\frac{A_{i,t+h}}{L_{i,t+h}}\mathbf{1}_{A_{i,t+h}<L_{i,t+h}} + \mathbf{1}_{A_{i,t+h}>L_{i,t+h}}\right],$$

where E_t^* denotes the conditional expectation with respect to the risk-neutral probability and the first component takes into account the recovery rate when default occurs.

The value at date t of the equity is:

$$S_{i,t} = E_t^*\left[\left(A_{i,t+h} - L_{i,t+h}\right)^+\right],$$

whereas the value of the credit default swap with residual maturity h is:

$$CDS_i\left(t,t+h\right) = E_t^*\left(\mathbf{1}_{A_{i,t+h}<L_{i,t+h}}\right).$$

Therefore all financial assets defined above are written on the underlying variables A, L, or, equivalently, on the variables $\log A$, $\log L$. In the basic Merton's model the debt amount L is assumed predetermined, which allows the use of the one-dimensional Black-Scholes model on variable A, and the same assumption is made in the practical approach developped by Moody's KMV for credit risk (see e.g. Crosbie and Bohn (2003)). As a consequence, all derivative prices are deterministic functions of the asset value at date t. However, the corporate liability is clearly as varying as the asset value and both underlying variables move together.

An extension to the framework of stochastic liability has been done by (Stapleton and Subrahmanyam (1984)) with a multivariate Black-Scholes model. The results of Sections 2 and 3 allow for the direct extension to the stochastic volatility and multifirm framework.

Let us first consider a given firm. We can represent the joint dynamics of the asset value

[8] The computations are performed with a zero riskfree rate.

and liability by:

$$\begin{pmatrix} d\log A_t \\ d\log L_t \end{pmatrix} = \begin{bmatrix} \mu_A + Tr\left(D_A\Sigma_t\right) \\ \mu_L + Tr\left(D_L\Sigma_t\right) \end{bmatrix} dt + \Sigma_t^{1/2} dW_t^S,$$
$$d\Sigma_t = \left(\Omega\Omega' + M\Sigma_t + \Sigma_t M'\right) dt + \Sigma_t^{1/2} dW_t^\sigma Q + Q'\left(dW_t^\sigma\right)' \Sigma_t^{1/2}.$$

This model can easily be extended to several firms, in order to distinguish the firm idiosyncratic effects from the general effects, creating default dependence. Let us consider for expository purposes a homogeneous portfolio where the n firms can be considered as equivalent. The model will be written as:

$$\begin{pmatrix} d\log A_{i,t} \\ d\log L_{i,t} \end{pmatrix} = \begin{bmatrix} \mu_A + Tr\left(D_A^G\Sigma_t\right) + Tr\left(D_A^C\Sigma_{i,t}\right) \\ \mu_L + Tr\left(D_L^G\Sigma_t\right) + Tr\left(D_L^C\Sigma_{i,t}\right) \end{bmatrix} dt + \Sigma_t^{1/2} dW_t^S + \Sigma_{i,t}^{1/2} dW_{i,t}^S, \tag{4.1}$$

where the general risk factor satisfies:

$$d\Sigma_t = \left(\Omega^G\Omega^{G\prime} + M^G\Sigma_t + \Sigma_t M^{G\prime}\right) dt + \Sigma_t^{1/2} dW_t^\sigma Q^G + Q^{G\prime}\left(dW_t^\sigma\right)' \Sigma_t^{1/2}, \tag{4.2}$$

and the idiosyncratic risk factors are such that:

$$d\Sigma_{i,t} = \left(\Omega^C\Omega^{C\prime} + M^C\Sigma_{i,t} + \Sigma_{i,t} M^{C\prime}\right) dt + \Sigma_{i,t}^{1/2} dW_{i,t}^\sigma Q^C + Q^{C\prime}\left(dW_{i,t}^\sigma\right)' \Sigma_{i,t}^{1/2}. \tag{4.3}$$

As usual the idiosyncratic and general innovations W_t^S, $W_{i,t}^S$, $i = 1, \ldots, n$, W_t^σ, $W_{i,t}^\sigma$, $i = 1, \ldots, n$, are assumed independent.

4.2 Simulations

Let us consider two firms with no general risk factor. The parameter values are:

$$\mu = \begin{pmatrix} 0.05 \\ 0.05 \end{pmatrix}, \ D_A^G = 0, \ D_L^G = 0, \ D_A^C = 0, \ D_L^C = 0,$$

for the asset-liability equation,

$$K = 4, \ M_d = 0, \ \Sigma_{d,1} = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix},$$

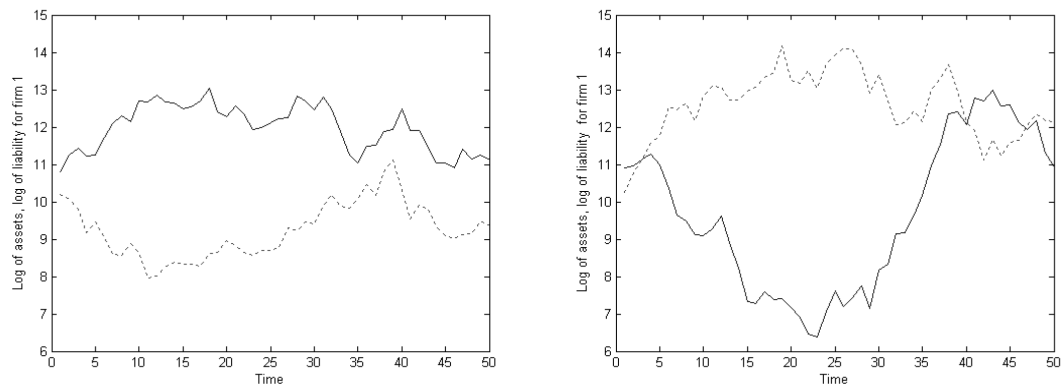
for the time discretized Wishart risk process of firm 1, and

$$K = 4, \ M_d = 0, \ \Sigma_{d,2} = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.02 \end{pmatrix},$$

for the time discretized Wishart risk process of firm 2.[9] We simulate the evolution of asset values and liabilities for 50 years with a time step of 0.01. The initial state of each firm at time 0.01 is assumed to be $\begin{pmatrix} \log A \\ \log L \end{pmatrix} = \begin{pmatrix} 11 \\ 10 \end{pmatrix}$. Each simulation is also performed for the case when the firms are driven by the same Wishart process with parameter values:

$$K = 4, \; M_d = 0, \; \Sigma_d = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix}.$$

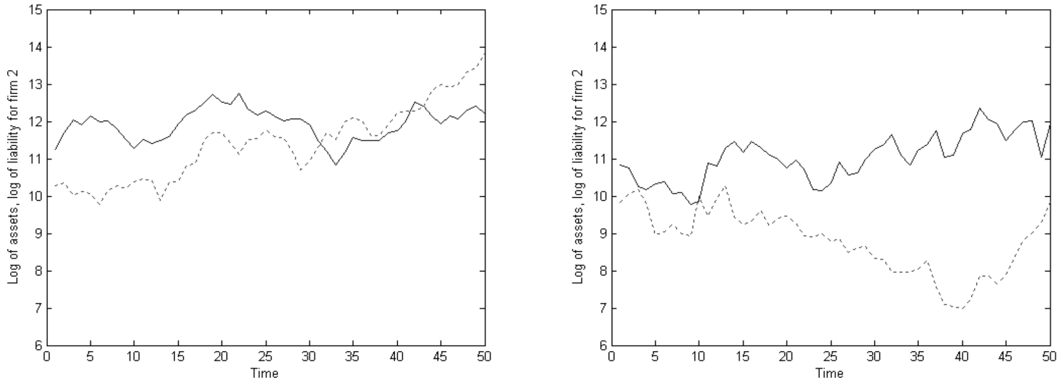
Figures 10 and 11 plot the end-of-year values of the joint process $(\log A \,, \log L)$ to illustrate a fundamental difference between this framework and Merton's model: this multivariate model allows a firm's liability to vary stochastically over time while in the Merton's model it is assumed predetermined (that is with deterministic evolution in Moody's KMV implementation). In our framework, the behavior of the liability can be similar or very different from that of the asset value, which is closer to the actual behavior of these variables. In Merton's model, default occurs at the first crossing of asset and liability curves.



a) Independent WAR processes b) Same WAR process

Figure 10. Log-asset (solid line) and log-liability (dotted line) for firm 1.

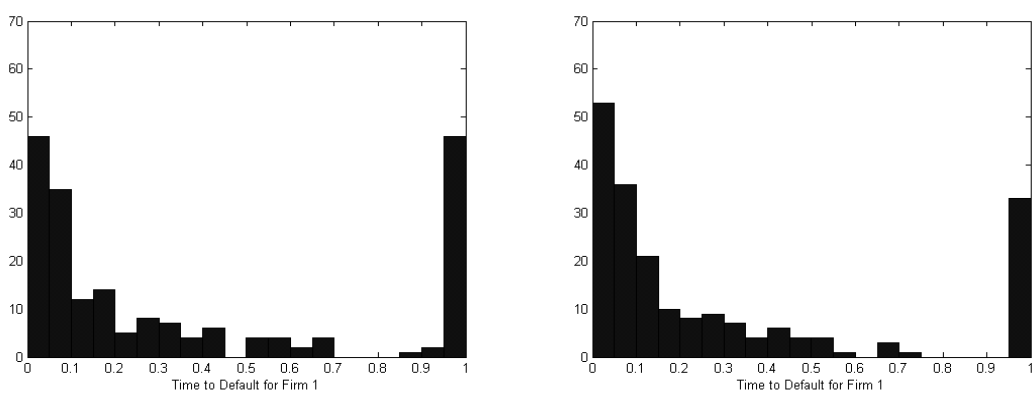
[9] See footnote 7 for the relationship between the parameters Ω^C, M^C, Q^C, of the continuous-time Wishart process in equation (4.3), and the parameters K, M_d, Σ_d, of the corresponding time discretized Wishart process.



a) Independent WAR processes b) Same WAR process

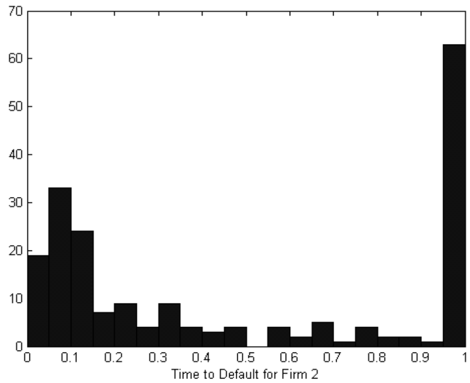
Figure 11. Log-asset (solid line) and log-liability (dotted line) for firm 2.

By replicating asset and liability paths, we can simulate the joint distribution of the times to default of the two firms. The number of replications is set to 200. The (right censored) marginal distributions of the time to default are provided in Figures 12 and 13.

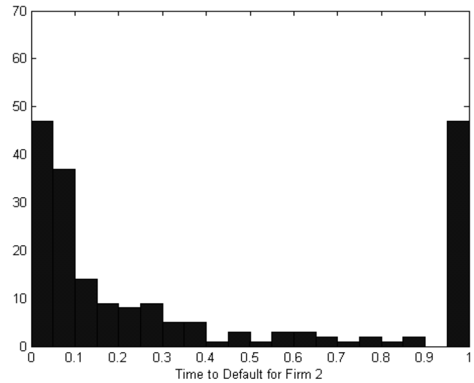


a) Independent WAR processes b) Same WAR process

Figure 12. Distribution of the time to default for firm 1. The sample mean of the time to default for firm 1 is: a) 0.3710, b) 0.2966.



a) Independent WAR processes

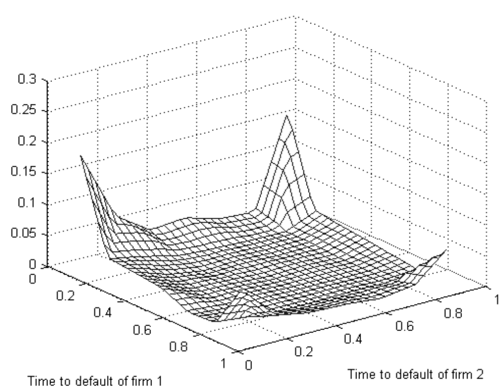


b) Same WAR process

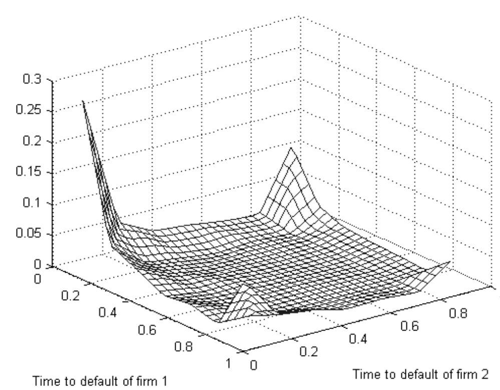
Figure 13. Distribution of the time to default for firm 2. The sample mean of the time to default for firm 2 is: a) 0.4785, b) 0.3711.

It is seen in Figure 13a) that the effect of stochastic volatility can provide marginal distributions of the time to default significantly different from the standard exponential distribution. In particular, the introduction of positive correlation between firms reduces the average time to default.

Of course it is more interesting to consider the joint distribution of the times to default. Indeed these duration variables are likely more dependent when there exists a common volatility factor between the two firms. The joint density plots are displayed in Figure 14 and are based on 800 replications.



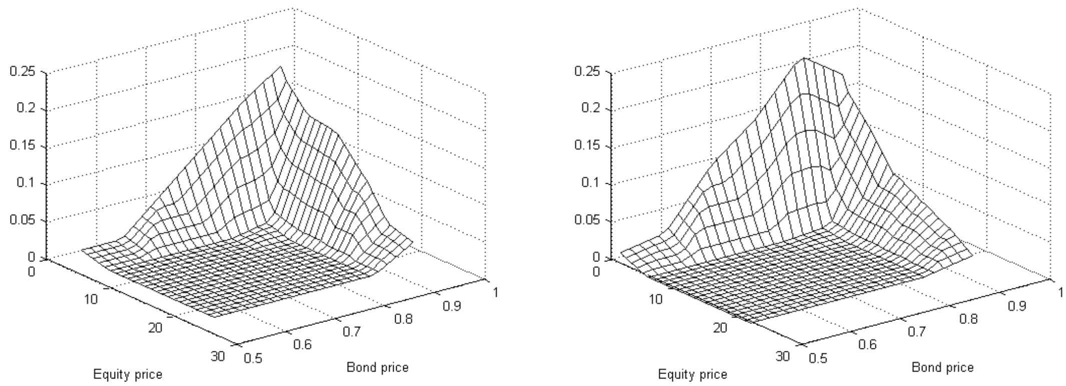
a) Independent WAR processes



b) Same WAR process

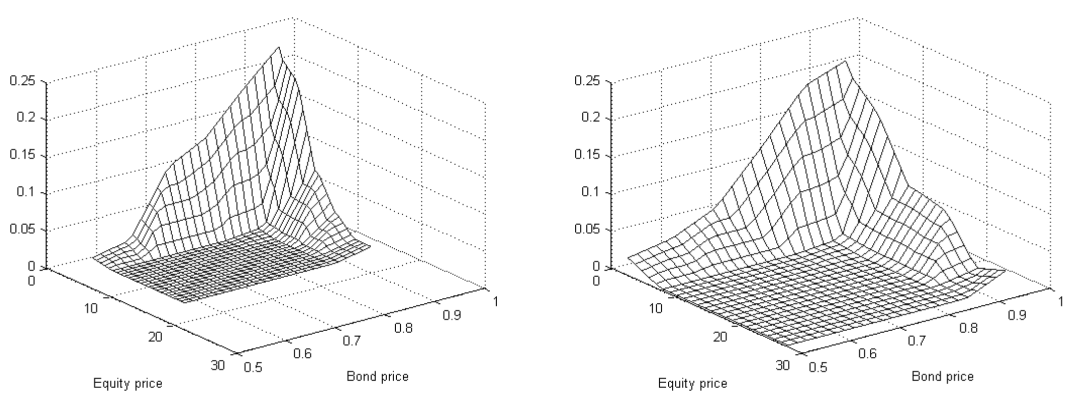
Figure 14. Joint density of the time to default of firm 1 and time to default of firm 2.

Finally we can consider the various credit derivative prices. The bond, equity and CDS end-of-year prices are computed for each firm for residual maturity $h = 10$. In the standard Merton's one-factor model, two credit derivative prices are in a nonlinear deterministic relationship. In our multifactor framework, the joint distribution of credit derivative prices is no longer degenerate. Bivariate density plots are provided in Figures 15 - 18.



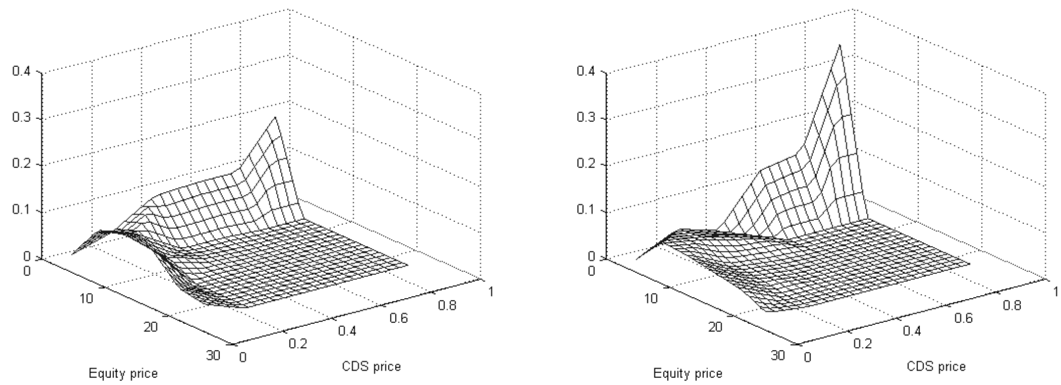
a) Independent WAR processes   b) Same WAR process

Figure 15. Joint density of the bond price and equity price for firm 1.



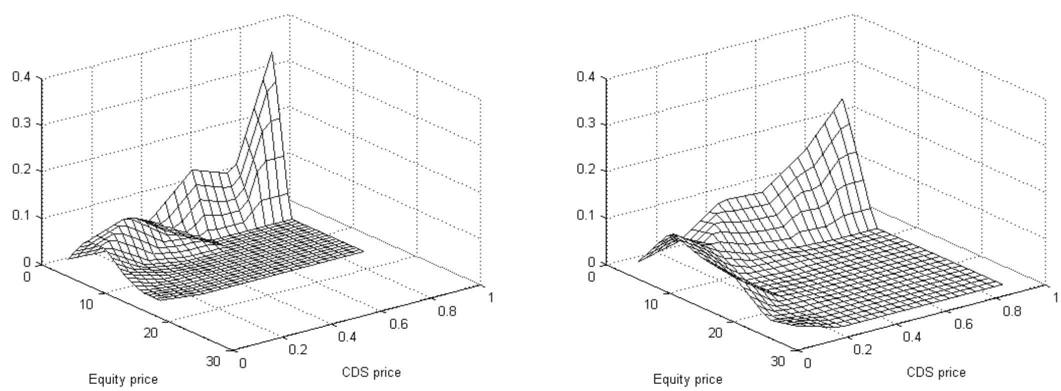
a) Independent WAR processes   b) Same WAR process

Figure 16. Joint density of the bond price and equity price for firm 2.



a) Independent WAR processes b) Same WAR process

Figure 17. Joint density of the CDS price and equity price for firm 1.



a) Independent WAR processes b) Same WAR process

Figure 18. Joint density of the CDS price and equity price for firm 2.

We clearly observe some stochastic decreasing relationship between bond and equity prices and between CDS and equity prices.

5 Conclusion

In the Black-Scholes model with CIR stochastic volatility, a closed-form solution for option prices can be derived (Heston (1993) and Ball and Roma (1994)). In the present paper we have considered a multiasset extension of this approach, where risk premia are introduced in the return equations and the CIR volatility process is replaced by a Wishart process for stochastic volatility matrices. Then the approach has been used to extend the standard Merton's model for credit risk (Merton (1974)) by allowing for stochastic corporate liability, stochastic volatility and more than one firm. These extensions show that the Wishart process is a convenient tool for modelling the dynamics of volatility matrices (Gourieroux, Jasiak, and Sufana (2004)).

As noted in Ball and Roma (1994), derivative pricing in models with stochastic volatility is similar to bond pricing. Thus it is not surprising that the Wishart process can be used to define new affine models for the term structure of interest rates, called Wishart quadratic term structure models (Gourieroux and Sufana (2003)), or for introducing a coherent pricing approach for bonds, stocks and currencies (Gourieroux, Monfort, and Sufana (2004)).

Appendix 1. Volatility of Σ_t

We have:

$$\begin{aligned}
V_t\left(a'd\Sigma_t b\right) &= V_t\left[a'\Sigma_t^{1/2}dW_t^\sigma Qb + a'Q'\left(dW_t^\sigma\right)'\Sigma_t^{1/2}b\right] \\
&= V_t\left(a'\Sigma_t^{1/2}dW_t^\sigma Qb + b'\Sigma_t^{1/2}dW_t^\sigma Qa\right) \\
&= a'\Sigma_t^{1/2}V_t\left(dW_t^\sigma Qb\right)\Sigma_t^{1/2}a + a'\Sigma_t^{1/2}Cov_t\left(dW_t^\sigma Qb, dW_t^\sigma Qa\right)\Sigma_t^{1/2}b \\
&\quad + b'\Sigma_t^{1/2}Cov_t\left(dW_t^\sigma Qa, dW_t^\sigma Qb\right)\Sigma_t^{1/2}a + b'\Sigma_t^{1/2}V_t\left(dW_t^\sigma Qa\right)\Sigma_t^{1/2}b.
\end{aligned}$$

Since $Cov_t\left(dW_t^\sigma a, dW_t^\sigma b\right) = a'b\ Id\ dt$, we obtain:

$$\begin{aligned}
V_t\left(a'd\Sigma_t b\right) &= \Big[a'\Sigma_t^{1/2}\left(b'Q'Qb\ Id\right)\Sigma_t^{1/2}a + a'\Sigma_t^{1/2}\left(b'Q'Qa\ Id\right)\Sigma_t^{1/2}b \\
&\quad + b'\Sigma_t^{1/2}\left(a'Q'Qb\ Id\right)\Sigma_t^{1/2}a + b'\Sigma_t^{1/2}\left(a'Q'Qa\ Id\right)\Sigma_t^{1/2}b\Big]\,dt \\
&= \left(a'\Sigma_t a\ b'Q'Qb + 2a'\Sigma_t b\ b'Q'Qa + b'\Sigma_t b\ a'Q'Qa\right)dt. \qquad \text{(A.1.1)}
\end{aligned}$$

We deduce from the result above that:

$$V_t\left(a'd\Sigma_t a\right) = 4a'\Sigma_t a\ a'Q'Qa\ dt. \qquad \text{(A.1.2)}$$

Similar computations provide the conditional covariance between two quadratic forms based on $d\Sigma_t$:

$$Cov_t\left(a'd\Sigma_t a, b'd\Sigma_t b\right) = 4\ a'\Sigma_t b\ a'Q'Qb\ dt, \qquad \text{(A.1.3)}$$

and

$$Cov_t\left(a'd\Sigma_t a, b'd\Sigma_t c\right) = 2\left[a'\Sigma_t b\ a'Q'Qc + a'\Sigma_t c\ a'Q'Qb\right]dt. \qquad \text{(A.1.4)}$$

From (A.1.3) we deduce the volatility of $Tr\left(Dd\Sigma_t\right)$, where D is a symmetric positive definite matrix. Since D can be decomposed as $D = \sum_{i=1}^n a_i a_i'$, we get:

$$Tr\left(Dd\Sigma_t\right) = \sum_{i=1}^n Tr\left(a_i a_i' d\Sigma_t\right) = \sum_{i=1}^n a_i' d\Sigma_t a_i.$$

We can write:

$$V_t\left[Tr\left(Dd\Sigma_t\right)\right] = V_t\left(\sum_{i=1}^n a_i' d\Sigma_t a_i\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{n} Cov_t\left(a_i' d\Sigma_t a_i, a_j' d\Sigma_t a_j\right) \\
&= 4\sum_{i=1}^{n}\sum_{j=1}^{n} a_i'\Sigma_t a_j \ a_i' Q' Q a_j \ dt \\
&= 4\sum_{i=1}^{n}\sum_{j=1}^{n} a_j'\Sigma_t a_i \ a_i' Q' Q a_j \ dt \\
&= 4\sum_{j=1}^{n} a_j'\Sigma_t D Q' Q a_j \ dt,
\end{aligned}
$$

which implies that:

$$
V_t\left[Tr\left(Dd\Sigma_t\right)\right] = 4Tr\left(D\Sigma_t D Q' Q\right) dt. \tag{A.1.5}
$$

The result above is modified if D is not positive definite. The decomposition becomes $D = \sum_{i=1}^{n} \varepsilon_i a_i a_i'$, where $\varepsilon_i = \pm 1$. A similar computation provides:

$$
V_t\left[Tr\left(Dd\Sigma_t\right)\right] = 4Tr\left(D^+\Sigma_t D^+ Q' Q\right) dt,
$$

where $D^+ = \sum_{i=1}^{n} a_i a_i'$ is derived from D by replacing all the eigenvalues by their absolute values.

Similarly, using (A.1.4) we have:

$$
\begin{aligned}
Cov_t\left(Tr\left(Dd\Sigma_t\right), d\Sigma_t\right) &= \sum_{i=1}^{n} Cov_t\left(a_i' d\Sigma_t a_i, d\Sigma_t\right) \\
&= \left[\sum_{i=1}^{n} Cov_t\left(a_i' d\Sigma_t a_i, e_k' d\Sigma_t e_l\right)\right] \\
&= 2\left[\sum_{i=1}^{n} a_i'\Sigma_t e_k \ a_i' Q' Q e_l + \sum_{i=1}^{n} a_i'\Sigma_t e_l \ a_i' Q' Q e_k\right] dt \\
&= 2\left[\sum_{i=1}^{n} e_k'\Sigma_t a_i \ a_i' Q' Q e_l + \sum_{i=1}^{n} e_l'\Sigma_t a_i \ a_i' Q' Q e_k\right] dt \\
&= 2\left[e_k'\Sigma_t D Q' Q e_l + e_l'\Sigma_t D Q' Q e_k\right] dt \\
&= 2\left(\Sigma_t D Q' Q + Q' Q D' \Sigma_t\right) dt.
\end{aligned}
$$

Appendix 2. Conditional Laplace transform

The exponential affine expression of the conditional Laplace transform is a well-known property of affine processes (see e.g. Duffie, Filipovic, and Schachermayer (2003)). Therefore we only look for the generalized Riccati equations satisfied by functions a, B, and b. We have:

$$\begin{aligned}
\Psi_{t,h+dt} &= E_t\left\{\exp\left[\int_t^{t+dt}(\gamma'\log S_u+\gamma_0)\,du+\int_t^{t+dt}Tr\,(C\Sigma_u+c_0)\,du\right]\Psi_{t+dt,h}\right\} \\
&\simeq E_t\{\exp\left[(\gamma'\log S_t+\gamma_0)\,dt+Tr\,(C\Sigma_t+c_0)\,dt\right] \\
&\quad \exp\left[a\,(h)'\log S_{t+dt}+Tr\,(B\,(h)\,\Sigma_{t+dt})+b\,(h)\right]\} \\
&= \exp\left[(\gamma'\log S_t+\gamma_0)\,dt+Tr\,(C\Sigma_t+c_0)\,dt\right] \\
&\quad \exp\left[a\,(h)'\log S_t+Tr\,(B\,(h)\,\Sigma_t)+b\,(h)\right] \\
&\quad E_t\exp\left[a\,(h)'\,d\log S_t+Tr\,(B\,(h)\,d\Sigma_t)\right] \\
&= \exp\left[(\gamma'\log S_t+\gamma_0)\,dt+Tr\,(C\Sigma_t+c_0)\,dt\right] \\
&\quad \exp\left[a\,(h)'\log S_t+Tr\,(B\,(h)\,\Sigma_t)+b\,(h)\right] \\
&\quad \exp\left\{E_t\left[a\,(h)'\,d\log S_t+Tr\,(B\,(h)\,d\Sigma_t)\right]\right. \\
&\quad \left.+\frac{1}{2}V_t\left[a\,(h)'\,d\log S_t+Tr\,(B\,(h)\,d\Sigma_t)\right]\right\}.
\end{aligned}$$

Using the definitions of the affine processes in equations (2.1), (2.2), and the result (A.1.5) derived in Appendix 1, we have:

$$\begin{aligned}
\Psi_{t,h+dt} &= \exp\left[(\gamma'\log S_t+\gamma_0)\,dt+Tr\,(C\Sigma_t+c_0)\,dt\right] \\
&\quad \exp\left[a\,(h)'\log S_t+Tr\,(B\,(h)\,\Sigma_t)+b\,(h)\right] \\
&\quad \exp\left\{a\,(h)'\left[\mu+\begin{pmatrix}Tr\,(D_1\Sigma_t)\\ \vdots \\ Tr\,(D_n\Sigma_t)\end{pmatrix}\right]dt\right. \\
&\quad +Tr\left[B\,(h)\,(\Omega\Omega'+M\Sigma_t+\Sigma_tM')\right]dt \\
&\quad \left.+\frac{1}{2}a\,(h)'\,\Sigma_t a\,(h)\,dt+2Tr\left[B\,(h)\,\Sigma_t B\,(h)\,Q'Q\right]dt\right\}.
\end{aligned}$$

Since another expression for $\Psi_{t,h+dt}$ is:

$$\Psi_{t,h+dt}=\exp\left[a\,(h+dt)'\log S_t+Tr\,(B\,(h+dt)\,\Sigma_t)+b\,(h+dt)\right],$$

the terms multiplying $\log S_t$, Σ_t and the intercept must be the same in the two expressions above. Identifying the corresponding terms and taking $dt \longrightarrow 0$, we deduce the differential equations:

$$\begin{aligned}
\frac{da(h)}{dh} &= \gamma, \\
\frac{dB(h)}{dh} &= B(h)M + M'B(h) + 2B(h)Q'QB(h) \\
&\quad + \frac{1}{2}a(h)a(h)' + \sum\nolimits_{i=1}^{n} a_i(h)D_i + C, \\
\frac{db(h)}{dh} &= a(h)'\mu + Tr\left[B(h)\Omega\Omega'\right] + \gamma_0 + c_0.
\end{aligned}$$

The initial conditions follow from:

$$\Psi_{t,0} = \exp\left[\widetilde{\gamma}' \log S_t + Tr\left(\widetilde{C}\Sigma_t\right)\right].$$

Appendix 3. Drift of the process $(\log S_t, \Sigma_t)$ under the risk-neutral distribution

It is well-known that the volatility of the process is the same under the risk-neutral and historical distributions. Moreover, since

$$m_{t,t+dt} = \exp\left[\gamma_t' d\log S_t + Tr\left(C_t d\Sigma_t\right) + \left(\gamma_{0t} + c_{0t}\right)dt\right],$$

the risk-neutral drift of $\log S_t$ is:

$$\begin{aligned}
E_t^*\left(d\log S_t\right) &= \frac{E_t\left(m_{t,t+dt} d\log S_t\right)}{E_t\left(m_{t,t+dt}\right)} \\
&\simeq \frac{E_t\left\{\left[1 + \gamma_t' d\log S_t + Tr\left(C_t d\Sigma_t\right)\right] d\log S_t\right\}}{E_t\left[1 + \gamma_t' d\log S_t + Tr\left(C_t d\Sigma_t\right)\right]} \\
&\simeq \left\{E_t\left(d\log S_t\right) + E_t\left(\gamma_t' d\log S_t d\log S_t\right) + E_t\left[Tr\left(C_t d\Sigma_t\right) d\log S_t\right]\right\} \\
&\quad \left\{1 - E_t\left(\gamma_t' d\log S_t\right) - E_t\left[Tr\left(C_t d\Sigma_t\right)\right]\right\} \\
&= E_t\left(d\log S_t\right) + E_t\left(\gamma_t' d\log S_t d\log S_t\right) + E_t\left[Tr\left(C_t d\Sigma_t\right) d\log S_t\right] \\
&\simeq E_t\left(d\log S_t\right) + Cov_t\left(d\log S_t, \gamma_t' d\log S_t\right)
\end{aligned}$$

$$= E_t\left(d\log S_t\right) + \Sigma_t \gamma_t \; dt,$$

since $d\log S_t$ and $d\Sigma_t$ are conditionally independent. Similarly, the risk-neutral drift of Σ_t is:

$$\begin{aligned}
E_t^*\left(d\Sigma_t\right) &= \frac{E_t\left(m_{t,t+dt} d\Sigma_t\right)}{E_t\left(m_{t,t+dt}\right)} \\
&\simeq \frac{E_t\left\{\left[1+\gamma_t' d\log S_t + Tr\left(C_t d\Sigma_t\right)\right] d\Sigma_t\right\}}{E_t\left[1+\gamma_t' d\log S_t + Tr\left(C_t d\Sigma_t\right)\right]} \\
&\simeq E_t\left(d\Sigma_t\right) + E_t\left(\gamma_t' d\log S_t d\Sigma_t\right) + E_t\left[Tr\left(C_t d\Sigma_t\right) d\Sigma_t\right] \\
&\simeq E_t\left(d\Sigma_t\right) + Cov_t\left[Tr\left(C_t d\Sigma_t\right), d\Sigma_t\right].
\end{aligned}$$

Moreover, Ito's formula and the martingale condition imply:

$$\begin{aligned}
E_t^*\left(d\log S_{it}\right) &\simeq E_t^* \frac{dS_{it}}{S_{it}} - \frac{1}{2} E_t^* \frac{\left(dS_{it}\right)^2}{\left(S_{it}\right)^2} \\
&\simeq -\frac{1}{2}\sigma_{ii,t} dt.
\end{aligned}$$

Thus the risk premium γ_t is fixed to:

$$\gamma_t = \Sigma_t^{-1}\left[-\frac{1}{2}\sigma_{ii,t} - \mu_i - Tr\left(D_i \Sigma_t\right)\right].$$

Finally, the sum of the coefficients γ_{0t}, c_{0t} is fixed by the unit mass restriction and given by:

$$\begin{aligned}
\gamma_{0t} + c_{0t} &= -\gamma_t'\left(\mu + \left[Tr\left(D_i \Sigma_t\right)\right]\right) - Tr\left[C_t\left(\Omega\Omega' + M\Sigma_t + \Sigma_t M'\right)\right] \\
&\quad -\frac{1}{2}\gamma_t' \Sigma_t \gamma_t - 2Tr\left[C_t \Sigma_t C_t Q' Q\right].
\end{aligned}$$

Appendix 4. An explicit solution to the Riccati equations

The aim of this appendix is to find the solutions of the differential system:

$$\frac{dX\left(h\right)}{dh} = A' X\left(h\right) + X\left(h\right) A + 2X\left(h\right) \Lambda X\left(h\right) + C_1, \tag{A.4.1}$$

with initial condition: $X\left(0\right) = C_0$, where Λ, C_1, C_0 are symmetric matrices, $\Lambda \gg 0$, and A is a square matrix.

In a first step, we explain how to eliminate the constant C_1. Then in a second step, we solve the system with $C_1 = 0$.

First step:

Lemma 1 *Let X^* be a solution of the system:*

$$A'X^* + X^*A + 2X^*\Lambda X^* + C_1 = 0.$$

Then the process: $Z(h) = X(h) - X^*$ satisfies:

$$\frac{dZ(h)}{dh} = A^{*\prime}Z(h) + Z(h)A^* + 2Z(h)\Lambda Z(h),$$

with $Z(0) = C_0^*$, where $A^* = A + 2\Lambda X^*$, $C_0^* = C_0 - X^*$.

Proof. The Lemma is obtained by replacing $X(h)$ by $Z(h) + X^*$ in equation (A.4.1). ■

Second step:

Lemma 2 *The solution of the system:*

$$\frac{dZ(h)}{dh} = A^{*\prime}Z(h) + Z(h)A^* + 2Z(h)\Lambda Z(h),$$

with $Z(0) = C_0^*$, is:

$$Z(h) = \exp(A^*h)'\left[C_0^{*-1} + 2\int_0^h \exp(A^*u)\Lambda\exp(A^*u)'\,du\right]^{-1}\exp(A^*h).$$

Proof. Let us consider the process $\Lambda(h)$ defined by:

$$Z(h) = \exp(A^*h)'\Lambda(h)\exp(A^*h).$$

The derivative of $Z(h)$ is:

$$\begin{aligned}\frac{dZ(h)}{dh} &= A^{*\prime}\exp(A^*h)'\Lambda(h)\exp(A^*h)\\ &\quad+\exp(A^*h)'\Lambda(h)\exp(A^*h)A^* + \exp(A^*h)'\frac{d\Lambda(h)}{dh}\exp(A^*h)\\ &= A^{*\prime}Z(h) + Z(h)A^* + \exp(A^*h)'\frac{d\Lambda(h)}{dh}\exp(A^*h).\end{aligned}$$

Comparing with the initial equation, we get:

$$\begin{aligned} & \exp\left(A^{*}h\right)' \frac{d\Lambda\left(h\right)}{dh} \exp\left(A^{*}h\right) \\ = \; & 2\exp\left(A^{*}h\right)' \Lambda\left(h\right) \exp\left(A^{*}h\right) \Lambda \exp\left(A^{*}h\right)' \Lambda\left(h\right) \exp\left(A^{*}h\right). \end{aligned}$$

The result follows by integrating the differential system:

$$\frac{d\Lambda\left(h\right)}{dh} = 2\Lambda\left(h\right) \exp\left(A^{*}h\right) \Lambda \exp\left(A^{*}h\right)' \Lambda\left(h\right),$$

with initial condition $\Lambda\left(0\right) = C_0^{*}$. ■

Third step:

The application of Lemmas 1 and 2 provide the general solution of equation (A.4.1):

$$\begin{aligned} X\left(h\right) \; = \; & X^{*} + \exp\left[\left(A + 2\Lambda X^{*}\right)h\right]' \\ & \left\{\left(C_0 - X^{*}\right)^{-1} + 2\int_0^h \exp\left[\left(A + 2\Lambda X^{*}\right)u\right] \Lambda \exp\left[\left(A + 2\Lambda X^{*}\right)u\right]' du\right\}^{-1} \\ & \exp\left[\left(A + 2\Lambda X^{*}\right)h\right], \end{aligned}$$

where X^{*} satisfies:

$$A'X^{*} + X^{*}A + 2X^{*}\Lambda X^{*} + C_1 = 0.$$

Finally note that the equation defining X^{*} can also be written as:

$$\left[\left(2\Lambda\right)^{1/2} X^{*} + \left(2\Lambda\right)^{-1/2} A\right]' \left[\left(2\Lambda\right)^{1/2} X^{*} + \left(2\Lambda\right)^{-1/2} A\right] + C_1 - A'\left(2\Lambda\right)^{-1} A = 0.$$

We deduce the following lemma.

Lemma 3 *If the matrix A is symmetric, two cases can be distinguished.*

i) If $C_1 \gg A'\left(2\Lambda\right)^{-1} A$, the equation defining X^{*} has no solution.

ii) If $C_1 \ll A'\left(2\Lambda\right)^{-1} A$, there is a multiplicity of solutions:

$$X^{*} = \left(2\Lambda\right)^{-1/2} \left[A'\left(2\Lambda\right)^{-1} A - C_1\right]^{1/2} - \left(2\Lambda\right)^{-1} A,$$

where $\left[A'\left(2\Lambda\right)^{-1} A - C_1\right]^{1/2}$ denotes any square root of the matrix $A'\left(2\Lambda\right)^{-1} A - C_1$, not necessarily the positive one.

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