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Fiscally Stable Income Distributions under Majority Voting and Bargaining Sets

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Abstract

We explore two variants of the Bargaining Set in a simple majority game on income distributions in order to understand the apparent stability of tax schedules in democratic societies, despite the fact that the core of such games is empty (no majority Condorcet winner). Those variants are sharper than in the literature (Mas-Colell (1989), Shitovitz (1989), Zhou (1994)), by requiring that counterobjections try to garantee their initial income levels to all members of the minority who stand to lose in an objection. A first variant defines as usual an income disbribution to be stable if there is no objection against it that is "justified", i.e. for which there is no counterobjection satisfying the above requirement. A second variant alllows objecting majorities to look one more step ahead. An objection is "weakly justified" if, whenever there is a counterobjection, the objecting majority can beat it while guaranteeing their income levels to all of its members. An income distribution is stongly stable if there is no weakly justified objection against it.

These two variants generate sharper solution sets than when applied to large market games as in Mas-Colell (1989), Shitovitz(1989). An income distribution is stable if and only if its Lorenz curve has no point in common with the graph C of $f : [1/2; 1] \rightarrow [0; 1]$, with $f(b) = 1 - 1/(2b)$; for $b > 1/2$: It is strongly stable if and only if it is the egalitarian one.

JEL Classification numbers : C71, D31, D72, H24

Keywords : Inequality, income distribution, stable tax schedules, majority voting, cooperative games, core, bargaining set.

Distributions des revenus fiscalement stables pour le vote majoritaire et ensembles de marchandage

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Résumé

Nous explorons deux variantes de l'ensemble de marchandage dans un jeu simple de vote à la majorité sur des distributions de revenus, afin de comprendre la stabilité apparente des schémas de taxation dans les sociétés démocratiques, en dépit du fait que le noyau des jeux de ce type soit vide (absence de gagnant de Condorcet à la majorité). Ces variantes sont plus fines que dans la littérature (Mas-Colell (1989), Shitovitz (1989), Zhou (1994)), en imposant que toute contreobjection tente de garantir leurs niveaux de revenu initiaux à tous les membres de la minorité qui sont lésés dans une objection.

Une première variante définit de manière habituelle une distribution des revenus comme stable s'il n'y a pas d'objection à son encontre qui soit "justifiée", i.e. pour laquelle il n'y a pas de contreobjection satisfaisant au critère imposé ci-dessus. Une seconde variante permet aux majorités qui objectent de prévoir une étape supplémentaire. Une objection est "faiblement justifiée" si, lorsqu'il existe une contreobjection, la majorité qui objecte peut la battre tout en garantissant à tous ses membres leurs niveaux de revenus. Une distribution de revenu est fortement stable s'il n'y a pas d'objection faiblement justifiée à son encontre.

Ces deux variantes engendrent des ensembles de solutions plus stricts que s'ils étaient appliqués à des jeux de marchés comme dans Mas-Colell (1989), Shitovitz (1989). Une distribution des revenus est stable si et seulement si sa courbe de Lorenz n'a aucun point en commun avec le graphe C de $f : [1/2, 1] \to [0, 1]$ avec $f(b) = 1 - 1/(2b)$, pour $b > 1/2$. Elle est fortement stable si et seulement si elle est égalitaire.

Classification JEL : C71, D31, D72, H24

Mots-clés : Inégalité, distribution des revenus, schémas stables de taxation, vote majoritaire, jeux coopératifs, noyau, ensemble de marchandage.

1 Introduction

Tax schemes in democratic societies are commonly viewed as the outcome of the political process and are therefore often modelled as emerging , at least implicitly, from majority voting. As noted by many authors, this view is problematic since a majority winner typically does not exist, and electoral cycles should be the rule, when the policy space of tax schedules is multidimensional (see Bucovetsky (1991), Piketty (1993), Hindriks (2001) among many others). Such a prediction appears to be at odds with the casual observation that tax schedules seem most of the time fairly stable in democratic societies. Possible ways out followed in the literature involve restricting tax schedules to be linear (Roberts (1977)), quadratic in income (Cukierman and Meltzer (1991), De Donder and Hindriks (2003)) and/or to be ideal for some voter (Snyder and Kramer (1988)), introducing uncertainty about tax liabilities implied by a new tax proposal (Marhuenda and Ortuno-Ortin (1998)), and/or considering less demanding solution concepts, e.g. the uncovered set or the bipartisan set in simultaneous two-party competition games (Epstein (1997), De Donder and Hindriks (2003)).

It may be claimed that the political instability predicted by the possible inexistence of a Condorcet majority winner relies upon a very myopic behavior of voters, who are assumed to vote against the current tax schedule and for a new tax proposal if and only if they gain in the short run from the corresponding change. One may argue that in a dynamic setting, voters are likely to be more forward looking and that "political conservatism" may arise in the sense that a majority of voters may not wish to vote against the status quo even though they would gain immediately from the change, because they fear that doing so would start a political escalation that would be harmful to them (Piketty, 1993).

This argument is actually closely related to the critique adressed to the core, and to the suggestion to look instead at the bargaining set, as a solution concept in cooperatives games (Davis and Maschler (1963, 1967), Aumann and Maschler (1964)). Coalitions that "object" to a tax schedule and thus to the implied expost income distribution are those majorities that vote for moving away from the status quo. When agents are shortsighted and consider only immediate gains and losses implied by such a move, an income distribution is in the core if no majority can object to it, i.e. if and only if it is a Condorcet majority winner. The inexistence of a Condorcet majority winner is accordingly equivalent to the fact that the core is empty in the corresponding majority game in coalitional form.

A critique addressed to the core as a solution concept is that it relies on a very myopic behavior of "majorities", that are assumed, when considering to make an objection to the status quo, not to take into account the possible reactions triggered by the move, of members of the minoriy who stand to lose in the objection. The idea underlying the Bargaining Set is that for an objection by a majority to the status quo to be actually carried out, there should be no "counterobjection" in which members of the minority try to maintain their initial income levels. The formalization of that idea by Mas-Colell (1989), Shitovitz (1989) for large market games in coalitional form, without or with atoms, generated the sharp result that their Bargaining Set was equal to the core in their contexts. It turns out, however, that the transposition of their definitions of the Bargaining Set to the simple majority game on income distributions considered here is too undiscriminating as every income distribution would become stable : formulating a counterobjection would then be too easy as it could be made by only a very small part of the minority (the same would be true with the variant introduced by Zhou (1994)).

We present here two stricter variants of the Bargaining Set in which we impose that any counterobjection must try to maintain the initial income of every member of the minority who stands to lose in the objection. An objection against the status quo is justified if there is no majority winning counterobjection in this sense. A first version of the Bargaining Set is obtained by defining as usual an income distribution as stable if there is no justified objection against it. A second, sharper version assumes that objecting majorities may look one more step ahead. An objection is weakly justified if, whenever there is a counterobjection, the objecting majority can reply with an income distribution that beats the counterobjection and that guarantees their income levels to all its members. An income distribution will be strongly stable if there is no weakly justified objection against it. These two variants of the Bargaining Set are smaller than under the definitions of Mas-Colell (1989), Shitovitz (1989), but still contain the core, so they would still lead to the core as the solution set in their contexts of large market games. These two variants are more sharply discriminating in the framework of the simple majority game on income distributions considered here. An income distribution is stable if and only if its Lorenz curve has no point in common with the graph C of $f : [1/2; 1] \to [0; 1]$; with $f(b) = 1 - 1/(2b)$; for $b > 1/2$ (Proposition 1). It is strongly stable if and only if it is the egalitarian one (Proposition 2).

Casual observation shows that this outcome does not seem to contradict

Lorenz curves for industrialized contries (Gottschalk and Smeeding (1999)), suggesting that the approach developped here may not be completely irrelevant. As the stricter variants of the Bargaining Set studied here generate sharper results in simple majority games on income distribution than in large market games, pursuing the analysis of these variants in general cooperative games may be of independent interest.

The focus of Section 2 is on the presentation of the results. Proofs are gathered in the Appendix. Concluding remarks are summarized in Section 3.

2 Stable Income Distributions

We assume that the initial (pre-tax) distribution of income is exogenous, so that the ex-post income distribution is entirely determined by the tax schedule (no incentive problem) : voting over tax schemes is equivalent to voting directly over income distributions. We consider a continuum of individuals indexed by a in the closed interval $A = [0, 1]$ endowed with the Lebesgue measure $\mu(da)$, and non-atomic income distributions described by (measurable) densities $x(a)$, total income being normalized to unity, $\int_A x(a) = 1$. "Coalitions" are measurable subsets of A. The income distribution $x_2(a)$ is said to be preferred or indifferent (resp. preferred) to the income distribution $x_1(a)$ through majority voting, noted $x_2 R_M x_1$ (resp. $x_2 P_M x_1$), if the size of the set of voters who gain when moving from $x_1(a)$ to $x_2(a)$, i.e. $\mu\{a \in A \mid x_2(a) > x_1(a)\}$, is larger than or equal to (resp. larger than) the size of the set of agents who lose, i.e. $\mu\{a \in A \mid x_2(a) < x_1(a)\}$.

By definition, an objection (S, x_2) by the majority S (a measurable subset, or "coalition", of A) to the income distribution (status quo) $x_1(a)$ is an income distribution $x_2(a)$ such that 1. $x_2 P_M x_1$ and 2. $x_2(a) \geq x_1(a)$ on S with $\int_S x_2 > \int_S x_1$ and $x_2(a) < x_1(a)$ on $T = A \setminus S$. An income distribution $x_1(a)$ is in the core if and only if there is no objection to it, i.e. if and only if it is a majority Condorcet winner ($x_1 R_M x_2$ for every other income distribution). Clearly the core is empty (there is no Condorcet winner) in this simple majority game.

This conclusion carries over even if one assumes that forming coalitions involves a fixed cost $\varepsilon \geq 0$, whenever $\varepsilon < 1/2$. If forming a coalition entails the cost ε, one can design an objection (S, x_2) to the status quo $x_1(a)$ if and only if $\mu(S) > 1/2$ and $1 - \varepsilon \geq \int_S x_2 > \int_S x_1$. An income distribution $x_1(a)$ will be in the $\varepsilon-$core whenever it is impossible to find such a "costly" objection to it. Now recall that the Lorenz curve of any income distribution $x(a)$ is defined as the graph of the function $L_x : [0;1] \to [0;1]$ where $L_x(b) = \inf_S \left\{ \int_S x \mid \mu(S) = b \right\}$: This Lorenz curve is continuous, convex, non-decreasing from 0 to 1. It satisfies $L_x(b) \leq b$ with strict inequality for $0 < b < 1$ if and only if $x(a)$ differs from the egalitarian income distribution $x(a) \equiv 1$ for all a: So there is "costly" objection $(S; x_2)$ to the income distribution $x_1(a)$ if and only if $L_{x_1}(1/2) \leq L_{x_1}(\mu(S)) < 1 - \varepsilon$: Equivalently, $x_1(a)$ is in the $\varepsilon -$ core if and only if $L_{x_1}(1/2) \geq 1 - \varepsilon$: So the necessary and sufficient condition for a non-empty $\varepsilon -$ core is that the cost be large enough, $\varepsilon \geq 1/2$: The only income distribution to be in all $\varepsilon -$ cores whenever they are non-empty (the "least core" (Einy, Holzman and Monderer (1999)) is the egalitarian one, $x_1(a) \equiv 1$ for all a:[1]

A weakness of the core as a solution concept is that it assumes that when a majority S considers an objection $x_2(a)$ to the status quo $x_1(a)$; it does not take into account the possible reactions of members of the minority triggered by the move. The idea underlying the Bargaining Set is that for an objection $(S; x_2)$ to be implemented, there should be no "counterojection". The formalization of that idea by Mas-Colell (1989) for large market games in coalitional form, when transposed in the present framework, leads to the notion that $(U; x_3)$ is a counterobjection to the objection $(S; x_2)$ if the income distribution $x_3(a)$ can attract the votes of a new majority U, a measurable subject of A; while guaranteeing their initial incomes $x_1(a)$ to members of the old minority T who were losing in the objection if they wish to join the new majority : 1. $x_3 P_M x_2$ with $x_3(a) \geq x_2(a)$ on U; $x_2(a) > x_3(a)$ on $V = S \setminus U$; and 2. $x_3(a) \geq x_1(a)$ on $U \cap T$: According to such a definition, an objection $(S; x_2)$ to $x_1(a)$ is said to be justified when there is no counterobjection to it, while an income distribution $x_1(a)$ belongs to the Bargaining Set whenever there is no justified counterobjection to it.

It is clear that the Bargaining Set defined in such a way is larger than the core since it makes harder for a majority S to design an objection to the status quo $x_1(a)$: Mas-Colell (1989) showed that the Bargaining Set defined along this line is precisely equal to the core (hence to the set of competitive allocations) in large market economies with a continuum of agents without atoms. [2] He noted nevertheless that the "really serious problem" with such a definition was that it seemed to generate in other contexts solution sets that were much too large by comparison to the core. It is not difficult to verify that this is indeed the case in the present framework : every income distribution $x_1(a)$ is in the Bargaining Set as defined above. The basic reason is that it is much too easy to design a counterobjection to any objection $(S; x_2)$ to $x_1(a)$:

By construction, $\mu(S) > 1/2$ and $\int_S x_2 > \int_S x_1 \geq 0$. So there exists $W \subset S$ such that $\mu(S) > \mu(W) > 1/2$ and $x_2(a) > 0$ on $S \setminus W$. The new majority making the counterobjection may then be $U = W \cup Z$ where Z is a subset of the old minority T small enough to ensure that $\int_{S \setminus W} x_2 > \int_Z x_1$. The corresponding counterobjection is obtained by setting $x_3 = 0$ on $A \setminus U$, giving their initial incomes $x_3(a) = x_1(a)$ to all members of Z and spreading what is left $\int_{S \setminus W} x_2 + \int_{T \setminus Z} x_2 - \int_Z x_1 > 0$ to the members of W so as to guarantee $x_3(a) > x_2(a)$ to each of them. This construction makes clear that the result (every income distribution is in the Bargaining Set defined along this line) still holds even if one requires that the coalition U making a counterobjection has a non-empty intersection with T and/or with S, as in the variants put forward in Shitovitz (1989), Zhou (1994).

The origin of the phenomenon is that the new majority U making the counterobjection to the objection (S, x_2), has too easy a job because it is allowed when doing so to include only a possibly very small part Z of the minority T that is standing to lose in the objection. We propose a variant here, in the spirit of the original definitions of Davis and Maschler (1963, 1967), and of Aumann and Maschler (1964), in which the objection (S, x_2) to the status quo $x_1(a)$ is interpreted as an objection against the whole minority $T = A \setminus S$. We require accordingly that any counterobjection (U, x_3) guarantees its initial income $x_1(a)$ to all members of the whole minority T who is standing to lose in the objection.

Definition 1. An objection (S, x_2) by the majority S to the minority $T = A \setminus S$ at the income distribution (status quo) $x_1(a)$ is an income distribution $x_2(a)$ such that 1. $x_2 P_M x_1$ and 2. $x_2(a) \geq x_1(a)$ on S with $\int_S x_2 > \int_S x_1 \geq 0$ and $x_2(a) < x_1(a)$ on T.

A counterobjection (U, x_3) to the objection (S, x_2) is an income distribution $x_3(a)$ proposed by a new majority U that includes the whole losing minority T and guarantees their initial incomes to all its members : 1. $x_3 P_M x_2$ and 2. $x_3(a) \geq x_2(a)$ on U, with $T \subset U$, $x_2(a) > x_3(a)$ on $V = S \setminus U$ and $x_3(a) \geq x_1(a)$ on T.

The objection (S, x_2) against the minority T at the income distribution $x_1(a)$ is justified (noted $x_2 D^1 x_1$) if there is no counterobjection (U, x_3) to it. The income distribution $x_1(a)$ is said to be stable if there is no justified objection (S, x_2) against it.

The stricter variant of the Bargaining Set should lead in principle to a smaller solution set, since it makes formulating a counterobjection harder, while it should still contain the core. We note in passing that our requirement that a coalition U making a counterobjection to an objection $(S; x_2)$ must guarantee their initial incomes $x_1(a)$ to all agents left out in the first round in the minority $T = A \setminus S$; when transposed back to large market economies without or with atoms as in Mas-Colell (1989), Shitovitz (1989), should not make a difference there as our stricter requirement must lead again in their contexts to the equality of the (in principle, smaller) Bargaining Set with the core. [3] The following result shows that adding this stricter requirement makes a significant difference in the framework of the simple majority game considered here.

Proposition 1. The income distribution $x_1(a)$ is stable in the sense of Definition 1 if and only if its Lorenz curve (the graph of the function $L_{x_1} : [0; 1] \to [0; 1]$ where $L_{x_1}(b) = \text{Inf}_S \left\{ \int_S x_1 \mid \mu(S) = b \right\}$) has no point in common with the graph C of $f : [1/2; 1] \to [0; 1]$ with $f(b) = 1 - 1/(2b)$; for $b > 1/2$; or equivalently if and only if $L_{x_1}(b) > 1 - 1/(2b)$ for all $b > 1/2$.

The proof of this fact is given in Appendix A. The stricter requirement for making counterobjections does reduce significantly here the solution set, since it eliminates as "unstable" all income distributions with a Lorenz curve having a non-empty intersection with the closed shaded area in Fig. 1. Still, the set of stable income distributions remains rather large, and one might consider strengthening the logic of objections and counterobjections, in the spirit of Mas-Colell (1989, Section 5) and Dutta, Debraj, Sengupta and Vohra (1989), by allowing a majority S who wishes to make an objection $(S; x_2)$ to $x_1(a)$; to look forward one step further by taking into account the possible reply it could make to possible counterobjections, under the condition that it should maintain when doing so the income levels of all its members.

Fig. 1

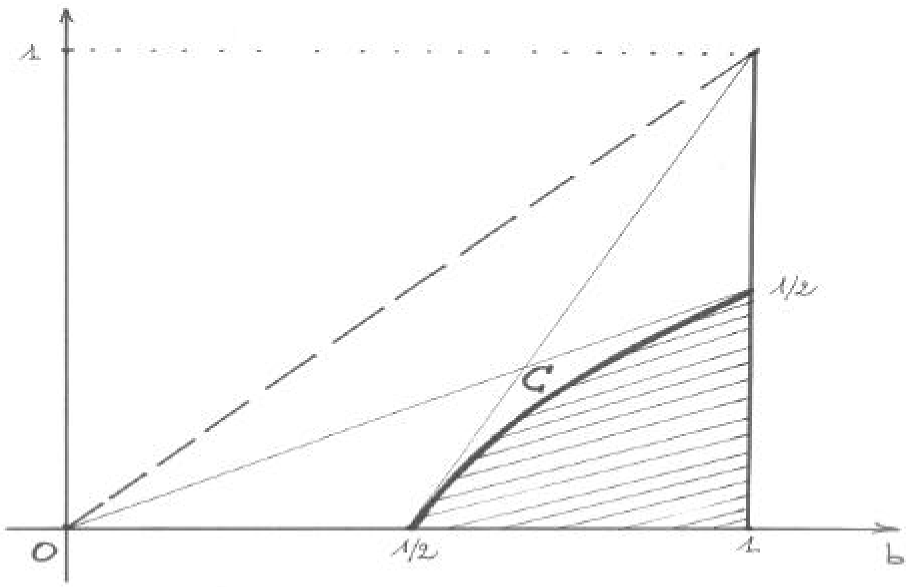

Figure 1:

De…nition 2. An objection $(S; x_2)$ against the minority $T = A\setminus S$ at the income distribution $x_1(a)$ is weakly justi…ed (noted $x_1 D^2 x_1$) [4] if for every counterobjection $(U; x_3)$ in the sense of De…nition 1 above, there is a majority winning reply by the majority S that guarantees their income levels to all its members, i.e. there is an income distribution $x_4(a)$ such that 1. $x_4 P_M x_3$ and 2. $x_4(a) = \mathrm{Max}\{x_3(a); x_2(a)\}$ for all a in S:

An income distribution $x_1(a)$ is strongly stable whenever there is no weakly justi…ed objection to it.

Allowing objecting coalitions to look ahead one step further in this way makes objections easier and should lead therefore to a solution set that is smaller than the one found in Proposition 1 (an income distribution that is "strongly stable" is a fortiori "stable"), but that still contains the core. We note in passing that, as a consequence, the notion of strong stability embodied in Definition 2 above, when transposed to the context of large market economies without and with atoms as in Mas-Colell (1989) and Shitovitz (1989), should lead there again to the equality of the set of "strongly stable" allocations and the core. [5] The following result shows that allowing majorities to plan one step further when making objections makes a significant difference in reducing the solution set in the context of the simple majority game considered here.

Proposition 2. An income distribution $x_1(a)$ is strongly stable in the sense of Definition 2 if and only if it is equal to the egalitarian one, i.e. $x_1(a) \equiv 1$ for all a in A:

The proof of this fact is given in Appendix B.

3 Concluding Remarks

We argued in this paper that in order to understand better why tax schedules appear to be most of the time fairly stable in democratic societies despite the fact that no Condorcet majority winner exists typically under majority voting on income distributions (the core is empty), it should be fruitful to represent the behavior of majorities who consider voting for an income tax redistribution away from the status quo as less shortsighted and make them more realistically take into account possible counterobjections from the minority who stands to lose from the move. The two variants of the Bargaining Set that we introduced along this line generated instead the remarkable prediction that there were indeed income distributions that were "stable" whenever majorities voting on tax schedules were assumed to be more or less forward looking in such a way. Even though the "empirical" test is admittedly quite loose, the solution sets exhibited in Proposition 1 and 2 provide upper and lower bounds for sets of stable income distributions that appear to be not completely irrelevant, suggesting that the approach may be fruitful to pursue (compare the Lorenz curves of industrialized countries in Fig. 1.a, b, c, d in Gottschalk and Smeeding (1999) and Fig. 1 above). As noted in the text, the variants of the Bargaining Set employed here are strengthenings of the requirements imposed in Mas-Colell (1989), Shitovitz (1989) in large market economies. While these stricter definitions of the Bargaining Set would not change the conclusion they reached that it must be equal to the core in their frameworks, the results of the present paper show by way of example that introducing these stricter requirements can make important di¤erences in other contexts such as ours, and that pursuing the analysis of our variants of the Bargaining Set in general cooperative games may be of independent interest.

APPENDIX

A. Proof of Proposition 1

Remark 1. In Definition 1, one can require without loss of generality that a counterobjection $(U; x_3)$ to the objection $(S; x_2)$ satisfies $x_3(a) = x_1(a) > x_2(a)$ on T; $x_3(a) = 0 < x_2(a)$ on $V = S \setminus U$ and $x_3(a) > x_2(a)$ on $W = U \cap S = S \setminus V$: Indeed, let $(U; x_3)$ be an arbitrary counterobjection as in Definition 1, with $x_3(a) \geq x_1(a)$ on T; $x_3(a) < x_2(a)$ on $V = S \setminus U$ and $x_3(a) \geq x_2(a)$ on $W = U \cap S$: Clearly, one can assume without loss of generality that $x_3(a) = x_1(a)$ on T; $x_3(a) = 0$ on V (otherwise, define another counterobjection $(U; x_3')$ by setting $x_3'(a) = x_1(a)$ on T, $x_3'(a) = 0$ on V and distributing the surplus $\int_{T \cup V}(x_3 - x_3') > 0$ to all agents in W to guarantee $x_3'(a) > x_3(a) \geq x_2(a)$ to all of them). On the other hand, if $x_3(a) > x_2(a)$ on a subset Z of positive measure of W; one can define another counterobjection $(U; x_3')$ by setting $x_3'(a) = x_3(a)$ on $T \cup V$ but with $x_3'(a) > x_2(a)$ everywhere on W by diminishing slightly the incomes of Z to ensure $x_2(a) < x_3'(a) < x_3(a)$ everywhere there and distributing the surplus $\int_Z(x_3 - x_3') > 0$ to all agents of $W \setminus Z$: In all these cases, the counterobjection $(U; x_3')$ satisfies $x_3(a) = x_1(a)$ on T; $x_3(a) = 0$ on V, $x_3(a) > x_2(a)$ on $W = S \setminus V$:

The only remaining case is when the counterobjection $(U; x_3)$ satisfies $x_3(a) = x_1(a) > x_2(a)$ on T; $x_3(a) = 0 < x_2(a)$ on $V = S \setminus U$ and $x_3(a) = x_2(a) \geq x_1(a)$ on $W = U \cap S$: Since $x_3 P_M x_2$; one has $\mu(T) > \mu(V)$; which implies that the set $Z = W \cap S^*$; where S^* is the set of agents of S who actually gain in the objection, i.e. for whom $x_2(a) > x_1(a) \geq 0$; has positive measure (otherwise S^* would be a subset of V; and one would get $\mu(S^*) < \mu(T)$; hence $x_1 P_M x_2$; a contradiction). Then $x_2(a) > 0$ on Z: It is therefore possible to choose a subset X of Z with a small enough positive measure so that $\mu(V \cup X) < \mu(T) + \mu(W \setminus X)$ and define a new counterobjection (U', x_3') with $U' = U \setminus X$; by setting $x_3'(a) = x_1(a)$ on T; $x_3'(a) = 0 < x_2(a)$ on $V' = V \cup X = S \setminus U'$; and distribute the surplus $\int_X x_2 > 0$ to all agents in $W' = W \setminus X$ to ensure $x_3'(a) > x_3(a)$ to each of them. Again, $(U'; x_3')$ is a counterobjection that satisfies the conditions stated in Remark 1.

It follows from Remark 1 that the existence of a counterobjection $(U; x_3)$ to an objection $(S; x_2)$ is equivalent to the existence of a subset W of S; with $\mu(T) + \mu(W) > 1/2$; or equivalently $\mu(W) > \mu(S) - 1/2 > 0$; such that

$1 = \int_T x_3 + \int_W x_3 > \int_T x_1 + \int_W x_2$; which is equivalent to $\int_S x_1 > \int_W x_2$. Therefore

Lemma A.1. The income distribution (status quo) $x_1(a)$ is stable if and only if for every objection $(S; x_2)$ to it, there exists a subset W of S with $\mu(W) > \mu(S) - 1/2 > 0$ and $\int_S x_1 > \int_W x_2$.

Remark 2. In Definition 1, one can also require without loss of generality that a justified objection $(S; x_2)$ against the minority $T = A \setminus S$ satisfies $x_2(a) = 0$ on T. Indeed, let $(S; x_2)$ be an objection against the minority T at $x_1(a)$. If it is justified, $x_2 D^1 x_1$; then for every income distribution $x_3(a)$ such that $x_3(a) = x_1(a)$ on T; one has $x_2 R_M x_3$. The objection $(S; x_2')$ will be also justified if one sets $x_2'(a) = 0$ on T and distribute the surplus $\int_T (x_2 - x_2') > 0$ to all agents in S on top of what they already get so as to ensure $x_2'(a) > x_2(a)$ on S.

Since for an objection $(S; x_2)$ satisfying $x_2(a) = 0$ on $T = A \setminus S$; one has obviously $\int_S x_2 = \int_V x_2 + \int_W x_2 = 1$; an equivalent formulation of Lemma A.1. is the following one.

Corollary A.2. The income distribution (status quo) $x_1(a)$ is stable if and only if for every objection $(S; x_2)$ to it satisfying $x_2(a) = 0$ on $T = A \setminus S$; there exists a subset V of S with $\mu(V) < 1/2$ and $\int_S x_1 + \int_V x_2 > 1$.

Lemma A.1. suggests that the income distribution $x_1(a)$ will be stable in the sense of Definition 1 if it displays enough equality, i.e. if the share $\int_S x_1$ of total income of every majority S with $\mu(S) > 1/2$, is large enough, or in other words if $L_{x_1}(b) = \mathrm{Inf}_S \left\{ \int_S x_1 \mid \mu(S) = b \right\}$ is large enough for $b > 1/2$. Indeed

Corollary A.3. The income distribution (status quo) $x_1(a)$ is stable in the sense of Definition 1 if $L_{x_1}(b) = \mathrm{Inf}_S \left\{ \int_S x_1 \mid \mu(S) = b \right\} > 1 - 1/(2b)$ for $b > 1/2$; i.e. if its Lorenz curve, the graph of $L_{x_1} : [0;1] \to [0;1]$, has no point in common with the graph C of $f : [1/2;1] \to [0;1]$, with $f(b) = 1 - 1/(2b)$, for $b > 1/2$.

Proof. Let $x_1(a)$ be the status quo. We remark that for every objection $(S; x_2)$ to it satisfying $x_2(a) = 0$ on $T = A \setminus S$; hence $\int_S x_2 = 1$; with $\mu(S) > 1/2$; if we define as usual

$$L_{x_2}(b) = \mathrm{Inf}_B \left\{ \int_B x_2 \mid B \subset S; \mu(B) > b \right\};$$

we have $L_{x_2}(\mu(S) - 1/2) \le \frac{\mu(S) - 1/2}{\mu(S)} \int_S x_2 = 1 - 1/(2\mu(S))$, with strict inequality if and only if $x_2(a)$ differs from the egalitarian income distribution on S, $x_2(a) \equiv \mu(S)$ for all a in S (this follows from the fact that $L_{x_2}(\beta)$ is convex with $L_{x_2}(0) = 0$ so that $L_{x_2}(\beta)/\beta$ is non-decreasing for $\beta > 0$, as the reader will easily verify).

It is then clear from Lemma A.1 and Remark 2 that $x_1(a)$ is stable in the sense of Definition 1 if for every majority S with $\mu(S) > 1/2$, on has $\int_S x_1 > 1 - 1/(2\mu(S))$. Indeed in such a case, for every objection (S, x_2) with $x_2(a) = 0$ on $T = A \setminus S$, one has then

$$\operatorname{Inf}_B \left\{ \int_B x_2 \mid B \subset S, \mu(B) > \mu(S) - 1/2 \right\} < \int_S x_1.$$

Therefore there is $W \subset S$ with $\mu(W) > \mu(S) - 1/2$ and $\int_W x_2 < \int_S x_1$. From Lemma A.1 and Remark 2, $x_1(a)$ is stable.

Corollary A.3 follows then from the fact the condition $L_{x_1}(\beta) > 1 - 1/(2\beta)$ for $\beta > 1/2$ implies by definition of $L_{x_1}(\beta)$

$$\int_S x_1 \ge L_{x_1}(\mu(S)) > 1 - 1/(2\mu(S))$$

for every majority S with $\mu(S) > 1/2$. Q.E.D.

The converse is established in the following fact, that will complete the proof of Proposition 1.

Corollary A.4. Consider the income distribution (status quo) $x_1(a)$. Assume that its Lorenz curve, i.e. the graph of $L_{x_1} : [0, 1] \to [0, 1]$, where $L_{x_1}(\beta) = \operatorname{Inf}_B \left\{ \int_B x_1 \mid \mu(B) \ge \beta \right\}$, has a point in common with the graph C of $f : [1/2, 1] \to [0, 1]$ where $f(\beta) = 1 - 1/(2\beta)$, for $\beta > 1/2$. Then $x_1(a)$ is unstable in the sense of Definition 1, i.e. there exists a justified objection (S, x_2) against it.

Proof. Without loss of generality assume that $x_1(a)$ is non-decreasing (relabel agents if necessary). Then $L_{x_1}(\beta) = \int_0^\beta x_1(a)\mu(da)$ for every β in $[0, 1]$. For every $0 < a < 1$, let $x_1^-(a) = \operatorname{Sup}_\beta \{x_1(\beta) \mid \beta < a\}$, $x_1^+(a) = \operatorname{Inf}_\beta \{x_1(\beta) \mid a < \beta\}$. The function $L_{x_1}(\beta)$ is continuous, convex, with for every $0 < \beta < 1$ a left (resp. right) derivative equal to $x_1^-(\beta)$ (resp. $x_1^+(\beta)$). Since L_{x_1} is convex while $f(\beta) = 1 - 1/(2\beta)$ is strictly concave, there are at most two points of intersection of the graph of L_{x_1} with the curve C on $[1/2; 1]$.

Assume that the graph of L_{x_1} has a point in common with C for $b > 1/2$. We claim that there exists $b^* > 1/2$ such that $L_{x_1}(b^*) \geq 1 - 1/(2b^*)$ and $x_1(a) < 1/b^*$ for all $a < b^*$. We distinguish three cases. Case 1 : $L_{x_1}(1/2) > 0$. In that case, take $b^* > 1/2$ as the smallest value of b for which $L_{x_1}(b) = f(b) = 1 - 1/(2b)$. One has $L_{x_1}(b^*) = 1 - 1/(2b^*)$ and the left derivative of L_{x_1} at b^*, i.e. $x_1^-(b^*)$, does not exceed $f'(b^*)$, which is itself less than $1/(b^*)$. So $x_1(a) < 1/b^*$ for all $a < b^*$. Case 2 : $L_{x_1}(b) = 0$ on a whole interval $[0; b^*]$ with $b^* > 1/2$, and $L_{x_1}(b) > 0$ for $b > b^*$. Then clearly, $L_{x_1}(b^*) = 0 < 1 - 1/(2b^*)$ and $x_1(a) = 0 < 1/b^*$ for all $a < b^*$. Case 3 : $L_{x_1}(1/2) = 0$ but $L_{x_1}(b) > 0$ for $b > 1/2$. The graph of L_{x_1} and C have an intersection for $b = 1/2$ and another one for $b > 1/2$, so one has $L_{x_1}(b^*) < 1 - 1/(2b^*)$ for every $b^* > 1/2$ close enough to 1/2. Furthermore, the right derivative of $L_{x_1}(b)$ at $b = 1/2$ must be less than $f'(1/2) = 2$. So $x_1^+(1/2) < 2 = (1/b)_{b=1/2}$, and thus for every $b^* > 1/2$ close enough to 1/2, one will also get $x_1(a) < 1/b^*$ for all $a < b^*$.

Given the choice of such a $b^* > 1/2$, take $S = [0; b^*)$ and consider $x_2(a) = 1/b^*$ on S, $x_2(a) = 0$ on $T = A \setminus S$. It is easy to see that $(S; x_2)$ is a justified objection against T at $x_1(a)$. By construction, $x_1(a) < 1/b^* = x_2(a)$ on S; $x_2(a) = 0 < x_1(a)$ on T with $\mu(S) = b^* > 1/2$, so $(S; x_2)$ is an objection to $x_1(a)$. Moreover it is justified because for every subset $W \subset S$ with $\mu(W) > \mu(S) - 1/2 = b^* - 1/2$, one has $\int_S x_1 = L_{x_1}(b^*) \geq 1 - 1/(2b^*) < \mu(W)/b^* = \int_W x_2$. From Lemma A.1, $x_1(a)$ is unstable in the sense of Definition 1. Q.E.D.

B. Proof of Proposition 2

By definition, the income distribution (status quo) $x_1(a)$ is strongly stable in the sense of Definition 2 if and only if for every objection (S, x_2) against the minority $T = A \backslash S$ at $x_1(a)$, there is a counterobjection (U, x_3) by a new majority U that contains T, with $x_3(a) \geq x_1(a) > x_2(a)$ on T, $x_3(a) \geq x_2(a) \geq x_1(a)$ on $W = S \cap U$, $x_3(a) < x_2(a)$ on $V = S \backslash U$, to which S cannot reply while maintaining the income levels of its members, i.e. such that $x_3 R_M x_4$ for all income distributions $x_4(a)$ satisfying $x_4(a) \geq \mathrm{Max}\{x_2(a), x_3(a)\}$ on S.

The property that $x_1(a)$ is strongly stable in this sense implies that for every objection (S, x_2) against it, there is a counterobjection (U, x_3) satisfying $\int_W x_3 + \int_V x_2 \geq 1$ (otherwise, there would exist an income distribution $x_4(a)$ with $x_4(a) = 0$ on T and $x_4(a) > \mathrm{Max}\{x_2(a), x_3(a)\}$ on $S = V \cup W$, implying $x_4 P_M x_3$, a contradiction). Conversely, suppose that the status quo $x_1(a)$ has the property that for every objection (S, x_2) against it, there is a counterobjection (U, x_3) satisfying $\int_W x_3 + \int_V x_2 > 1$. Then clearly $x_1(a)$ is strongly stable, because there is no income distribution $x_4(a)$ satisfying $x_4(a) \geq \mathrm{Max}\{x_2(a), x_3(a)\}$ on $S = V \cup W$.

Remark also that one can impose without loss of generality in the above definition of strong stability, that a counterobjection (U, x_3) satisfies $x_3(a) = x_1(a)$ on T and $x_3(a) = 0$ on V (otherwise, define another counterobjection (U, x_3') by setting $x_3'(a) = x_1(a)$ on T, $x_3'(a) = 0$ on V and by distributing the surplus $\int_T (x_3 - x_1) + \int_V x_3 > 0$ to members of W so as to ensure $x_3'(a) > x_3(a)$ to all of them. Then $\int_W x_3' + \int_V x_2 > \int_W x_3 + \int_V x_2 \geq 1$, and there is no income distribution $x_4(a)$ with $x_4 \geq \mathrm{Max}\{x_2(a), x_3'(a)\}$ on S). For any counterobjection satisfying these conditions, one has $\int_W x_3 = 1 - \int_T x_1 = \int_S x_1$. Therefore if the status quo $x_1(a)$ is strongly stable, for every objection (S, x_2) against it, there is a subset V of S with $\mu(V) < 1/2$ such that $\int_S x_1 + \int_V x_2 \geq 1$. Conversely, let the income distribution $x_1(a)$ have the property that for every objection (S, x_2) against it, there is a subset V of S with $\mu(V) < 1/2$ such that $\int_S x_1 + \int_V x_2 > 1$. If we consider an income distribution $x_3(a)$ with $x_3(a) = x_1(a)$ on T, $x_3(a) = 0$ on V, we have $\int_W x_3 = \int_{S_1} x_1 > 1 - \int_V x_2 \geq \int_W x_2$, so it is possible to design $x_3(a)$ so as to ensure $x_3(a) > x_2(a)$ on W. Such an income distribution $x_3(a)$ determines a counterobjection (U, x_3), where the new majority U is composed of the union of T, of W and of those members of V such that $x_2(a) = 0$ if there are any. That counterobjection satisfies by construction $\int_W x_3 + \int_V x_2 > 1$, hence $x_1(a)$ is strongly stable in the sense of Definition 2. The following fact summarizes this discussion.

Lemma B.1. Consider the income distribution (status quo) $x_1(a)$:

1) If $x_1(a)$ is strongly stable in the sense of Definition 2, then for every objection $(S; x_2)$ against the minority $T = A \backslash S$ at $x_1(a)$; there is a subset V of S with $\mu(V) < 1/2$ such that $\int_S x_1 + \int_V x_2 = 1$:

2) Conversely, assume that for every objection $(S; x_2)$ against $T = A \backslash S$ at $x_1(a)$; there is a subset V of S with $\mu(V) < 1/2$ such that $\int_S x_1 + \int_V x_2 > 1$: Then the income distribution $x_1(a)$ is strongly stable in the sense of Definition 2.

Lemma B.1. is the analogue of Corollary A.2 for stability. The basic difference is that for stability alone, objections $(S; x_2)$ could be assumed without loss of generality to assign $x_2(a) = 0$ to all members of the minority $T = A \backslash S$: This is not so when dealing with strong stability where members of the objecting majority S may need to ensure $\int_T x_2 > 0$ in order to be able to reply to counterobjections.

Corollary B.2. Assume that the income distribution (status quo) $x_1(a)$ differs from the egalitarian one. Then it is weakly unstable in the sense of Definition 2, i.e. there is a weakly justified objection against it.

Proof. Assume without loss of generality (relabel agents if necessary) that $x_1(a)$ is non-decreasing, so that $L_{x_1}(b) = \int_0^b x_1(a)\,\mu(da)$: If $x_1(a)$ involves some degree of inequality, $L_{x_1}(b) < b$ for every $0 < b < 1$:

We know from Proposition 1 that if $L_{x_1}(b) = 0$ for some $b > 1/2$; there is a justified objection against $x_1(a)$; which is a fortiori weakly justified. Corollary B.2 would be verified trivially in that case, so we may focus on the other configurations where $L_{x_1}(b) > 0$; hence $x_1(b) > 0$; for all $b > 1/2$:

Choose $b^* > 1/2$ close to $1/2$ and define $S = [0; b^*]; T = (b^*; 1]$: Since $L_{x_1}(1/2) < 1/2$; one has $\int_S x_1 < 1/2 < \int_T x_1$ when b^* is close enough to 1/2. One can then define a weakly justified objection $(S; x_2)$ by setting $x_2(a)$ only very slightly less than $x_1(a)$ on T; and distributing the surplus $\int_T (x_1 - x_2) > 0$ to members of S on top of $x_1(a)$; so as to ensure $x_2(a) > x_1(a)$ to all of them. $(S; x_2)$ is an objection against T at $x_1(a)$: If $x_2(a)$ differs very little from $x_1(a)$ everywhere, one will get that for every subset V of S (in particular when $\mu(V) < 1/2$), $\int_S x_1 + \int_V x_2$ will not exceed $\int_S x_1 + \int_S x_2$; while the later will be close to $2\int_S x_1 < 1$; and thus also less than 1. By 1) of Lemma B.1, such an objection $(S; x_2)$ is weakly justified, $x_2 D^2 x_1$: Q.E.D.

The converse is established in the following fact, that will complete the proof of Proposition 2.

Corollary B.3. The income distribution (status quo) $x_1(a)$ is strongly stable in the sense of Definition 2 if it is equal to the egalitarian one, $x_1(a) \equiv 1$ for all a in $A = [0;1]$:

Proof. Let $x_1(a) \equiv 1$ for all a and consider any objection $(S; x_2)$ against $T = A \setminus S$ at $x_1(a)$: By definition, $x_2(a) \geq x_1(a)$ on S and $\int_S x_2 > \int_S x_1 = \mu(S) > 1/2$: So there must exist a subset V^* of S with $\mu(V^*) = 1/2$ and $\int_{V^*} x_2 > \int_{V^*} x_1 = 1/2$: By continuity, it is possible to find $V \subset V^*$ with $\mu(V)$ slightly less than $1/2$ such that $\int_V x_2 > 1/2$; with the consequence that $\int_S x_1 + \int_V x_2 > 1$: From 2) of Lemma B.1, the egalitarian income distribution $x_1(a) \equiv 1$ for all a in $A = [0;1]$ is strongly stable in the sense of Definition 2. Q.E.D.

Footnotes

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1. Similar results would hold if the cost of forming a coalition was assumed to be proportional to its size, so that $1 - \varepsilon\mu(S) = \int_S x_2 > \int_S x_1$ for any objection $(S; x_2)$ to $x_1(a)$. In that case, as the reader will easily verify, the income distribution $x_1(a)$ will be in the $\varepsilon -$ core if and only if its Lorenz curve satisfies $L_{x_1}(1/2) = 1 - (\varepsilon/2)$. That $\varepsilon -$ core is empty whenever the unit cost is less tan one, $\varepsilon < 1$.

2. Shitovitz (1989) proved that the equality of the Bargaining Set defined along this line with the core, still holds in the presence of atoms. The core, hence the Bargaining Set, is however larger than the set of competitive allocations in such a case.

3. The new Bargaining Set with the stricter requirement contains the core and should be a subset of their Bargaining Set, which they proved to be equal to the core in their contexts.

4. The superscript 2 is intended to mean that objecting coalitions look forward two rounds ahead here, in contrast with Definition 1.

5. Here again, the new solution set of “strongly stable” allocations must contain the core and should be a subset of their Bargaining Set, which they proved to be equal to the core in their contexts.

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