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Limiting Dependence Structure for Credit Default

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Abstract

Dependence structures for bivariate extremal events are analyzed using particular types of copulas. Weak convergence results for copulas along the lines of the Pickands-Balkema-De Haan Theorem provide limiting dependence structures for bivariate tail events. A characterization of those limiting copulas is also provided by means of invariance properties. The results obtained are applied to the credit risk area where, for intensity-based default models, stress-scenario dependence structures for widely traded products such as Credit Default Swap baskets or First-to-Default contract types are proposed.

Résumé

L'étude de la structure de dépendance d'évènements, ou de risques, repose de plus en plus souvent sur l'utilisation des copules. En reprenant l'approche de Pickands-Balkema-De Hanna, ce papier présente des résultats de convergence (faible) pour les copules, donnant ainsi des comportements limites de la structure de dépendance dans les queues de distribution. L'utilisation de propriétés d'invariance (théorèmes de points fixes) permettent d'obtenir une caractérisation des copules limites. Ces résultats peuvent être appliqués en particulier pour des modèles de risque de crédit, tels que les modèles à intensité. On s'intéressera alors à la dynamique de la dépendance dans le cas de contrats de type Credit Default Swap baskets ou First-to-Default.

Keywords: copula, credit risk, dependent defaults, dependent risks, extreme value theory, regular variation, tail dependence.

AMS: 62E20; 62H20; 62P05.

1 Introduction

The reasons for studying and modelling dependencies in finance and insurance are of different type. One motivation is that independence assumptions, which are typical of many stochastic models, are often due more to convenience rather than to the nature of the problem at hand. Furthermore, there are situations where neglecting dependence effects may incur into a (dramatic) risk underestimation (see e.g. Bäuerle and Müller 1998 and Daul et al. 2003). Besides this, widely used scalar dependence or risk measures such as linear correlation, tail dependence coefficients and Value-at-Risk generally do not provide a satisfactory description of the underlying dependence structure and have severe limitations when used for measuring (portfolio) risk outside the Gaussian world (see e.g. Embrechts et al. 2002 and Juri and Wüthrich 2004 for counterexamples).

Taking care of dependencies becomes therefore important in order to extend standard models towards a more efficient risk management. However, relaxing the independence assumption yields much less tractable models. It is therefore not surprising that only recently, i.e. in the last ten years, the mathematical literature on the risk management of dependent risks showed significant developments. The main message sent by much of this research is the following (see e.g. Dhaene and Goovaerts 1996, Dhaene and Denuit 1999, Frees and Valdez 1999, Joe 1997, Schönbucher and Schubert 2001, Juri and Wüthrich 2002, 2004 among others). It is (intuitively) clear that the probabilistic mechanism governing the interactions between random variables is completely described by their joint distribution. On the other hand, in most applied situations, the joint distribution may be unknown or difficult to estimate such that only the marginals are known (estimated or fixed a priori). To tackle this problem a flexible and powerful approach consists in trying to model the joint distribution by means of copulas. The latter, which are often called *dependence structures*, can be viewed as marginal free versions of joint distribution functions capturing scale invariant dependence properties of the several random variables.

The reverse side of the medal of the copula approach is that it is usually difficult to chose or find the appropriate copula for the problem at hand. Often, the only possibility is to start with some guess such as a parametric family of copulas and then try to fit the parameters (as made e.g. in Daul et al. 2003). As a consequence, the models obtained may suffer a certain degree of arbitrariness. As shown by Juri and Wüthrich (2002, 2004), some remedy to this weakness of the copula approach is provided by dependence models for (bivariate) conditional joint extremes, where limiting results along the lines of the Pickands-Balkema-De Haan Theorem are obtained. Such "copula-convergence theorems" reflect a distributional approach to the modelling of dependencies in the tails and provide natural descriptions of multivariate extremal events. Moreover, they differ from classical bivariate extreme value results since the limits obtained are not bivariate extreme value distributions. A further advantage of this kind of results is that they may also allow to better face the problem of the lack of data which is typical for rare events. In fact, there are situations where the knowledge of the limiting dependence structure reduces the issue of modelling tail events to the estimation of one parameter solely (Juri and Wüthrich 2002).

1.1 Outline of the paper

The paper is structured as follows. In Section 2.1 we briefly recall the copula concept and all its properties that we will need throughout the rest of the paper. The idea of dependence structures for tail events is then formalized in Section 2.2, where the concept of *tail dependence copula* (LTDC) is introduced; the latter provides a natural description of conditional bivariate joint extremes. Sections 3 and 4 contain the main results, which extend part of the work of Juri and Wüthrich (2002, 2004). In particular, Theorem 3.4 identifies, under suitable regularity conditions, possible LTDC-limits, i.e. limit laws for bivariate joint extremes. Motivated by classical results such as the Central Limit Theorem and the Fisher-Tippett Theorem, we show in Section 4 that LTDC-limits are characterized by invariance properties (Theorems 4.6, 4.10 and Corollary 4.11). In Section 5, we show how the results of the preceding sections can be applied to the credit risk area, where, for intensity-based default models, dependence structures characterizing the behavior under stress scenarios of widely traded credit derivatives such as Credit Default Swap baskets or First-to-Default contract types are obtained. The proofs of the several statements are collected in Section 6.

2 Dependence structures for tail events

2.1 Preliminaries

As mentioned above, one of the main concepts used to describe scale invariant dependence properties of multivariate distributions is the copula one. In this work, we focus on bivariate continuous random vectors only and most of the following material can be found in Nelsen (1999) or Joe (1997).

Definition 2.1. A *two-dimensional copula* is a two-dimensional distribution function restricted to $[0,1]^2$ with standard uniform marginals.

Copulas can be equivalently defined as functions $C : [0,1]^2 \to [0,1]$ satisfying for $0 \le x \le 1$ and $(x_1, y_1), (x_2, y_2) \in [0,1]^2$ with $x_1 \le x_2$, $y_1 \le y_2$ the conditions

$$C(x,1) = C(1,x) = x, \quad C(x,0) = C(0,x) = 0, \tag{2.1}$$

$$C(x_2,y_2) - C(x_2,y_1) - C(x_1,y_2) + C(x_1,y_1) \ge 0. \tag{2.2}$$

In fact, it is easily seen that (2.1) translates into the uniformity of the marginals and that inequality (2.2), which is known as the 2-*increasing property*, can be interpreted as $\mathbb{P}[x_1 \le X \le x_2, y_1 \le Y \le y_2]$ for (X,Y) having distribution function C. Note that (2.2) neither implies nor is implied by the fact that C is increasing in each argument. However, (2.1) together with (2.2) imply that C increases in each variable as well that C is Lipschitz-continuous with Lipschitz constant one.

One of the most important and useful results about copulas is Sklar's Theorem stated below in its bivariate form. A proof of Theorem 2.2 can be found e.g. in Nelsen (1999) or in Sklar (1959).

Theorem 2.2 (Sklar).

1. *Let C be a copula and F_1, F_2 be univariate distribution functions. Then, for $(t_1,t_2) \in \mathbb{R}^2$,*
$$F(t_1,t_2) := C(F_1(t_1), F_2(t_2)) \tag{2.3}$$
defines a distribution function with marginals F_1, F_2.

2. *Conversely, for a two-dimensional distribution function F with marginals F_1, F_2 there is a copula C satisfying* (2.3)*. This copula is not necessarily unique, but it is if F_1 and F_2 are continuous, in which case for any $(x,y) \in [0,1]^2$,*
$$C(x,y) = F(F_1^{-1}(x), F_2^{-1}(y)), \tag{2.4}$$

where F_1^{-1}, F_2^{-1} denote the generalized left continuous inverses of F_1 and F_2.

Sklar's Theorem constitutes the motivation for calling copulas *dependence structures* that capture scale invariant dependence properties. In fact, we see from (2.3) that C couples the marginals F_1, F_2 to the joint distribution function F separating thus dependence and marginal behaviors. Further, it is easy to check that for X_1, X_2 with joint distribution function F, copula C (in the sense that C is a copula satisfying (2.3)) and strictly increasing $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, the variables $f_1(X_1), f_2(X_2)$ have copula C too.

An example of a copula is the following. Throughout the rest of the paper we will encounter other examples of copulas.

Example 2.3. The Marshall and Olkin copula with parameters $\alpha, \beta \in [0,1]$ is defined for $x, y \in [0,1]$ as

$$C_{\alpha,\beta}(x,y) := (x^{1-\alpha}y) \wedge (xy^{1-\beta}), \tag{2.5}$$

where for real numbers s, t, the expression $s \wedge t$ denotes the minimum of s and t.

2.2 Tail dependence copulas

A natural way to construct dependence structures (copulas) for bivariate (lower) tail events, is to consider first two-dimensional continuous conditional distribution functions, where the condition is that both variables fall below small thresholds. The second step is to get then the relative copula using the second part of Sklar's Theorem (Equation (2.4)).

Remark 2.4. In the sequel, we will assume that the considered copula C is such that $x \mapsto C(x,y)$ and $y \mapsto C(x,y)$ are strictly increasing for all $x, y \in (0,1]$. We denote by $\mathcal{C}$ the set of such copulas.

Let (U,V) be a random vector with distribution function $C \in \mathcal{C}$. For any $(u,v) \in (0,1]^2$, the conditional distribution of (U,V) given $U \le u, V \le v$, denoted by $F(C,u,v)$, is given, for $0 \le x \le u$ and $0 \le y \le v$, by

$$F(C,u,v)(x,y) = \mathbb{P}[U \le x, V \le y | U \le u, V \le v] = \frac{C(x,y)}{C(u,v)}. \tag{2.6}$$

The marginal distribution functions of $F(C,u,v)$ in (2.6) are given for $0 \le x \le u$ and $0 \le y \le v$ respectively by

$$F_U(C,u,v)(x) = \frac{C(x,v)}{C(u,v)} \quad \text{and} \quad F_V(C,u,v)(y) = \frac{C(u,y)}{C(u,v)}. \tag{2.7}$$

Since, $F_U(C,u,v), F_V(C,u,v)$ are continuous, the unique copula relative to $F(C,u,v)$ is obtained from (2.4) and equals

$$F(C,u,v)(F_U(C,u,v)^{-1}(x), F_V(C,u,v)^{-1}(y)) = \frac{C(F_U(C,u,v)^{-1}(x), F_V(C,u,v)^{-1}(y))}{C(u,v)}. \tag{2.8}$$

Definition 2.5. For $C \in \mathcal{C}$, we call the copula defined by (2.8) the *lower tail dependence copula* relative to C, LTDC for short, and we denote it by $\Phi(C,u,v)$.

Note that the assumption that $C \in \mathcal{C}$ implies that $\{(u,v) \in [0,1]^2 : C(u,v) > 0\} = (0,1]^2$, i.e. it ensures that the LTDC $\Phi(C,u,v)$ is well defined for all $u,v \in (0,1]$. Furthermore, $\lim_{u,v\to 0} \Phi(C,u,v)$ describes naturally the dependence structure underlying conditional bivariate random samples in the lower-tails.

Furthermore, starting with uniform marginals, i.e. with a copula C, is not a restriction since the dependence structure that would be obtained with different marginals is again of the type $\Phi(C,u,v)$. In fact, let X_1, X_2 have joint distribution function G, strictly increasing continuous marginals G_1, G_2 and copula C. Analogously to the above, consider for appropriate (i.e. such that the following expressions are well defined) $z_1, z_2 \in \mathbb{R}$ the conditional distribution function

$$G^{z_1,z_2}(x_1,x_2) := \mathbb{P}[X_1 \le x_1, X_2 \le x_2 | X_1 \le z_1, X_2 \le z_2]. \tag{2.9}$$

Further, let $G_1^{z_1,z_2}(x_1) := G^{z_1,z_2}(x_1,z_2)$ and $G_2^{z_1,z_2}(x_2) := G^{z_1,z_2}(z_1,x_2)$, respectively. Because of Sklar's Theorem, we have that the copula relative to G^{z_1,z_2} is given by

$$\Phi(G,z_1,z_2)(u_1,u_2) := G^{z_1,z_2}((G_1^{z_1,z_2})^{-1}(u_1), (G_2^{z_1,z_2})^{-1}(u_2)). \tag{2.10}$$

Proposition 2.6. *In the above setting holds* $\Phi(C, G_1(z_1), G_2(z_2)) = \Phi(G, z_1, z_2)$.

Remark 2.7. Sometimes it may be more natural to look at dependencies in the upper-tails rather than in the lower-tails as e.g. in any situation where one is interested in the joint behavior of random variables conditional on high thresholds. To such an extent, one could consider in (2.6) the expression $\mathbb{P}[U > x, V > y | U > u, V > v]$ instead of $\mathbb{P}[U \leq x, V \leq y | U \leq u, V \leq v]$ yielding, through the analogous to (2.8), a dependence structure for upper-tail events. Such dependence structures can be also obtained replacing C in Definition 2.5 by the relative survival copula $\widehat{C}(x, y) := x + y - 1 + C(1 - x, 1 - y)$, $x, y \in [0, 1]^2$. Indeed, it is easily seen that for (X, Y) with distribution function F, marginals F_1, F_2 and copula C, the copula of $(-X, -Y)$ is precisely $\widehat{C}$ and that for $(x, y) \in \mathbb{R}^2$

$$\mathbb{P}[X > x, Y > y] = \widehat{C}(1 - F_1(x), 1 - F_2(y)). \tag{2.11}$$

3 A limit theorem

The main result of this section is given by Theorem 3.4, where limits of the type $\lim_{t \to 0} \Phi(C, r(t), s(t))$ are considered. An explicit form for the limit is provided under the assumption that the functions r, s defining the direction under which the limit is taken satisfy suitable regularity conditions. Further, an example of a non-symmetric LTDC-limit, i.e. a limit obtained under a direction (r, s) with $r \neq s$, is given in Proposition 3.10 where we show that a dependence model in the lower-tails may be given by the Marshall and Olkin copula of Example 2.3. As we will see in Section 5, this copula turns out to be a natural model for some credit derivatives.

For our purposes, the concept of regular variation appears to be the appropriate one. A standard reference to the topic of regular variation is Bingham et al. (1987) and results for the multivariate case can also be found in De Haan et al. (1984).

Definition 3.1. A measurable function $f : (0, \infty) \to (0, \infty)$ is called *regularly varying at* 0 *with index* $\rho \in \mathbb{R}$, if for any $x > 0$,

$$\lim_{t \to 0} \frac{f(tx)}{f(t)} = x^{\rho}. \tag{3.1}$$

We write $f \in \mathcal{R}_{\rho}^{0}$. In the case where $\rho = 0$, the function is said to be *slow varying at* 0.

Definition 3.2. A measurable function $f : (0,\infty)^2 \to (0,\infty)$ is called *regularly varying at* 0 *with auxiliary functions* $r, s : (0,\infty) \to (0,\infty)$ if $\lim_{t\to 0} r(t) = \lim_{t\to 0} s(t) = 0$ and there is a positive measurable function $\phi : (0,\infty)^2 \to (0,\infty)$ such that

$$\lim_{t\to 0} \frac{f(r(t)x, s(t)y)}{f(r(t), s(t))} = \phi(x,y) \quad \text{for all } x, y > 0. \tag{3.2}$$

We write $f \in \mathcal{R}(r,s)$ and we call ϕ the *limiting function under the direction* (r,s).

Remark 3.3. Definition 3.2 can be easily modified to include functions, such as copulas, having a domain different from $(0,\infty)^2$. This ensures in particular that the left hand side of (3.3) below is well-defined.

Theorem 3.4. *Let* $C \in \mathcal{C} \cap \mathcal{R}(r,s)$ *with limiting function* ϕ *and assume that* r, s *are strictly increasing continuous functions such that* $r \in \mathcal{R}^0_\alpha$ *and* $s \in \mathcal{R}^0_\beta$ *for some* $\alpha, \beta > 0$. *Then, for any* $(x,y) \in [0,1]^2$,

$$\lim_{t\to 0} \Phi(C, r(t), s(t))(x,y) = \phi(\phi_X^{-1}(x), \phi_Y^{-1}(y)), \tag{3.3}$$

where $\phi_X(x) := \phi(x,1)$ *and* $\phi_Y(y) := \phi(1,y)$. *Moreover, there is a constant* $\theta > 0$ *such that* $\phi(x,y) = x^{\theta/\alpha} h(yx^{-\beta/\alpha})$ *for* $x > 0$, *where*

$$h(t) := \begin{cases} \phi_Y(t) & \text{for } t \in [0,1] \\ t^{\theta/\beta}\phi_X(t^{-\alpha/\beta}) & \text{for } t \in (1,\infty) \end{cases}. \tag{3.4}$$

Remark 3.5. Note that the limiting function ϕ in (3.2) is obtained from a pointwise convergence. Because the domain of a copula is the compact set $[0,1]^2$, it follows that the assumption $C \in \mathcal{C} \cap \mathcal{R}(r,s)$ implies that the convergence in (3.3) is also uniform, i.e. we have that $\lim_{t\to 0} \|\Phi(C, r(t), s(t)) - \phi(\phi_X^{-1}(\cdot), \phi_Y^{-1}(\cdot))\|_\infty = 0$.

Remark 3.6. Observe that the hypothesis that r, s are continuous functions is necessary, otherwise counterexamples such as *copulas with fractal support* as considered in Fredricks et al. (2004) can be constructed. Let $T = (t_{ij})$ be a square matrix with non-negative entries whose sum equals to one determining the following subdivision of the unit square $[0,1]^2$ into rectangles: let c_i, $i = 0, \dots, n$ the sum of the entries of the first i columns of T with $c_0 := 0$ and let r_j, $j = 0, \dots, n$ be the sum of the entries in the first j rows of

T with $r_0 := 0$. Then, the vectors $r := (r_0, \ldots, r_n)$ and $c := (c_0, \ldots, c_n)$ define partitions of $[0,1]$, whence $[0,1]^2$ is partitioned into the rectangles $R_{ij} := [c_{i-1}, c_i] \times [r_{i-1}, r_i]$. Further, for a given copula C and $(x,y) \in R_{ij}$, consider the new copula $T(C)$ defined by

$$T(C)(x,y) := \sum_{u<i,v<j} t_{uv} + \frac{x - c_{i-1}}{c_i - c_{i-1}} \sum_{v<j} t_{iv} + \frac{y - r_{j-1}}{r_j - r_{j-1}} \sum_{u<i} t_{uj} + C\left(\frac{x - c_i}{c_i - c_{i-1}}, \frac{y - r_j}{r_j - r_{j-1}}\right) t_{ij}, \tag{3.5}$$

where empty sums are defined as zero. Fredricks et al. (2004) show that for any copula C and any $T \neq 1$ there is a unique copula C_T that depends only on T such that $T(C_T) = C_T$. Moreover, they show that $C_T = \lim_{n\to\infty} T^n C$, where $T^n C := T(T^{n-1}C)$, $n \geq 1$, $T^1 C := T(C)$ and $T^0 C := C$. Consider now the case where the starting copula C is the independent copula, i.e. $C(x,y) = C^{\perp}(x,y) := xy$ and the transformation matrix T is given by

$$T = \begin{pmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.6 & 0 \\ 0.1 & 0 & 0.1 \end{pmatrix}, \tag{3.6}$$

whence $c = r = (0, 0.2, 0.8, 1)$. Then, we have for $t_k = 0.2^k$, $k \geq 1$ that

$$\Phi(C_T, t_k, t_k) = C_T = \lim_{n\to\infty} T^n C, \quad \text{any } k \geq 1. \tag{3.7}$$

The fact that $\Phi(C_T, t_k, t_k) = C_T$ can be explained with the help of Figure 3.1, where the support of $T^n(C)$ is plotted for $n = 1, 2, 3, 4$ and the colored regions are the ones where the measure relative to $T^n(C)$ concentrates its mass (indeed, we see from (3.5) that the support of $T^n C$ is given by the rectangles corresponding to the non-zero elements of T). Observe that since C is the independent copula, the measure relative to $T^n C$ spreads its mass uniformly on the colored squares. Taking for example the upper right picture in Figure 3.1, we see that restricting ourselves to $[0, t_1]^2 = [0, 0.2]^2$ we have exactly the same picture as in the upper left of Figure 3.1. This means that if (U,V) has copula $T^n C$ for some $n \geq 1$, then $(U,V)|U,V \leq t_1$ has c.d.f. $T^{n-1}C(xt_1, yt_1)$, $x, y \in [0,1]$. It follows that the copula of $(U,V)|U,V \leq t_1$ is exactly $T^{n-1}C$, i.e. $\Phi(T^n C, t_1, t_1) = T^{n-1}C$. Using the same arguments, we have in general that $\Phi(T^n C, t_k, t_k) = T^{n-k}C$. Finally, because $\Phi(\cdot, t_k, t_k)$ is continuous (see Lemma 4.13), it follows that $\Phi(C_T, t_k, t_k) = \lim_{n\to\infty} \Phi(T^n C, t_k, t_k) = \lim_{n\to\infty} T^{n-k} C = C_T$.

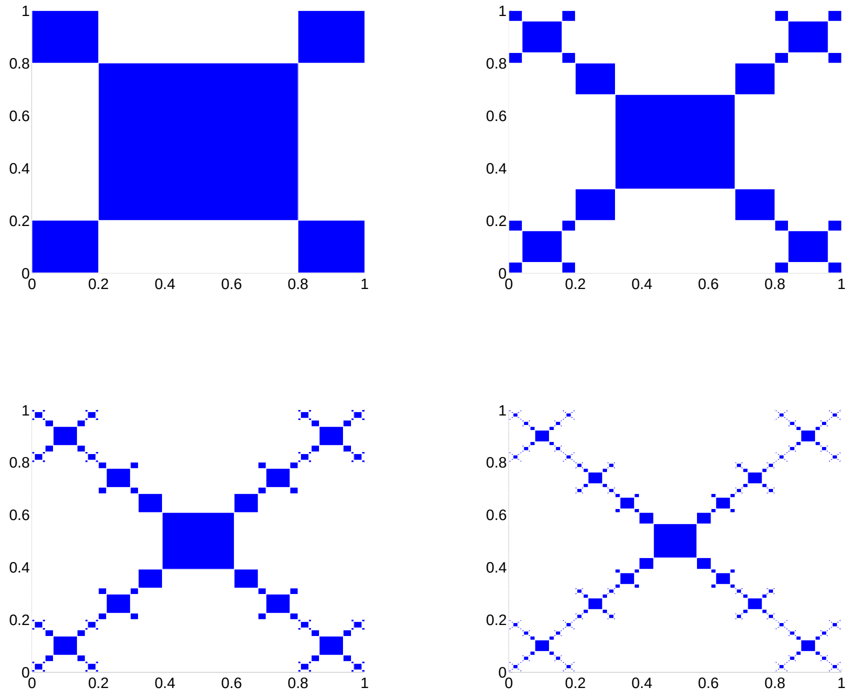

Figure 3.1: Support of $T^n(C)$ for $n = 1, 2, 3, 4$

Remark 3.7. Letting $\alpha = \beta = 1$, we have that Theorem 3.4 generalizes Theorem 2.4 in Juri and Wüthrich (2004), the latter stating that

$$\lim_{u\to 0} \Phi(C, u, u)(x, y) = G(g^{-1}(x), g^{-1}(y)), \tag{3.8}$$

where $g : [0, \infty) \to [0, \infty)$ is the strictly increasing continuous function defined by $g(x) := \lim_{u\to 0} C(xu, u)/C(u, u)$, $G(x, y) = y^\theta g(x/y)$ for $(x, y) \in (0, 1]^2$ and 0 elsewhere and θ is a positive constant. In particular, Theorem 2.4 in Juri and Wüthrich (2004) applies to archimedean copulas having regularly varying generators in which case the LTDC-limit is the Clayton copula of Example 4.9 of the next section with parameter equal to minus the regular variation parameter (Theorem 3.4 in Juri and Wüthrich 2004 and Theorem 3.3 in Juri and Wüthrich 2002).

Remark 3.8. Following the previous remark, the analytical expression (3.8) for the limiting copula is due to the fact that homogeneous functions of order

θ (in our case $G(x,y) = y^\theta g(x/y)$) have closed form expressions. Analogously, the closed form (3.4) comes from the fact that *generalized homogeneous functions* such as ϕ in Theorem 3.4 also have closed form representations (see the proof of Theorem 3.4 in Section 6 and Aczél (1966) for more details). Unfortunately, this is not the case in higher dimensions, so that, assuming that Theorem 3.4 could be extended case along the same lines to the multivariate, the limiting copula would not have a closed form expression.

Remark 3.9. There are many papers in the literature concerning multivariate extremes. In particular, Bivariate Extreme Value (BEV) distributions are obtained as limit laws of suitably normalized componentwise maxima as it can be found e.g. in De Haan and Resnick (1977), Resnick (1987), Coles and Tawn (1991, 1994) and Joe (1997). It can be shown that the copula C of any BEV distribution satisfies the max-stability property

$$C^t(u,v) = C(u^t, v^t) \quad \text{for all } (u,v) \in [0,1]^2 \text{ and any } t > 0. \tag{3.9}$$

As mentioned in Juri and Wüthrich (2004), BEV copulas differ from LTDC-limits, the difference being similar to the one between the univariate Generalized Extreme Value (GEV) distributions and the Generalized Pareto Distribution (GPD). In fact, the GPD lives on the log-scaled compared to GEV distributions (Theorem 4.2 in Juri and Wüthrich 2004). For instance, the Gumbel copula satisfies (3.9), but is not an LTDC-limit. For a more detailed discussion about relations with other results from the area of multivariate extremes we refer to Juri and Wüthrich (2004).

We finish this section with an example of an LTDC-limit which is not of the form (3.8). We will see in Section 4 that Theorem 4.6 provides a whole family of other examples of this type.

Proposition 3.10. *Let* $a, b : [0,1] \to [0,1]$ *be two increasing functions with* $a(0) = b(0) = 0$, $a(1) = b(1) = 1$ *and such that* $t \mapsto a(t)/t$, $t \mapsto b(t)/t$ *are decreasing on* $(0,1]$. *Then,*

$$C(x,y) := (a(x)y) \wedge (xb(y)) \tag{3.10}$$

defines a copula. Additionally, if $a \in \mathcal{R}_\alpha^0$, $b \in \mathcal{R}_\beta^0$, *where* $(\alpha,\beta) \in [0,1]^2 \setminus \{(0,0)\}$ *and for* $r \in \mathcal{R}_\gamma^0$ *and* $s \in \mathcal{R}_\delta^0$ *with* $\gamma, \delta \geq 0$ *such that* $\alpha\gamma + \delta = \beta\delta + \alpha$, *we have that*

$$\lim_{t \to 0} \Phi(C, r(t), s(t))(x,y) = (x^\alpha y) \wedge (xy^\beta), \tag{3.11}$$

which is the Marshall and Olkin copula with parameters $1-\alpha$ *and* $1-\beta$*. If* $\alpha\gamma+\delta\neq\beta\delta+\alpha$*, then*

$$\lim_{t\to 0}\Phi(C,r(t),s(t))(x,y)=xy, \tag{3.12}$$

which is the independent copula.

4 Invariant copulas

There are many examples of (functional) limit theorems where the limit obtained is invariant under some kind of transformation. This is the case of the Central Limit Theorem, where stable laws (which coincide with the class of possible limit laws for sums of iid random variables) are invariant under the sum operator. A similar result holds for the GEV distribution, which is the limit of maxima of iid random variables as stated in the Fisher-Tippett Theorem (Embrechts et al. 1997, Theorem 3.2.3).

In our context, we have that equation (2.8) can be seen as the result of a copula transformation mapping a copula $C\in\mathcal{C}$ to its LTDC $\Phi(C,u,v)$. Motivated by the above classical results, it seems therefore natural to look at copulas which are invariant under the LTDC-transformation (2.8).

Definition 4.1. We say that $C\in\mathcal{C}$ is *invariant on the unit square* if $\Phi(C,u,v)=C$ for all $(u,v)\in(0,1]^2$.

Lemma 4.2. *Let* (U,V) *have distribution function* $C\in\mathcal{C}$ *and* $(u,v)\in(0,1]^2$*. Then,* $\Phi(C,u,v)$ *satisfies for* $(x,y)\in[0,u]\times[0,v]$ *the identity*

$$\frac{C(x,y)}{C(u,v)}=\Phi(C,u,v)\left(\frac{C(x,v)}{C(u,v)},\frac{C(u,y)}{C(u,v)}\right). \tag{4.1}$$

From Lemma 4.2, we have that C is invariant on the unit square if and only if for any $(u,v)\in(0,1]^2$

$$\frac{C(x,y)}{C(u,v)}=C\left(\frac{C(x,v)}{C(u,v)},\frac{C(u,y)}{C(u,v)}\right)\quad\text{for all }(x,y)\in[0,u]\times[0,v]. \tag{4.2}$$

A weaker type of invariance than the one of Definition 4.1, is given by copulas C such that $\Phi(C,u,v)=C$ holds only for a particular set of parameters $(u,v)\in(0,1]^2$.

Definition 4.3. A copula $C \in \mathcal{C}$ is said to be *invariant on the diagonal* if $\Phi(C, u, u) = C$ for all $u \in (0, 1]$. Similarly, $C \in \mathcal{C}$ is called *invariant on the curve* $\mathcal{D} = \{(r(t), s(t)) \,|\, t \in T\}$, $T \subset \mathbb{R}$ where $r, s : T \to (0, 1]$, whenever

$$\Phi(C, r(t), s(t)) = C \quad \text{for all } t \in T. \tag{4.3}$$

Invariant copulas on the diagonal have been considered by Juri and Wüthrich (2002, 2004) and examples of such a copulas are given in Examples 4.4 and 4.9 below.

Example 4.4. For $\alpha \in [0, 1]$ consider the Cuadras-Augé copula

$$C_\alpha(x, y) := (x^{1-\alpha} y) \wedge (x y^{1-\alpha}). \tag{4.4}$$

The copula C_α can be seen as a particular case of a Marshall and Olkin copula of Example 2.3 with identical parameters and is a geometric mixture with weights α and $1 - \alpha$ of the upper Fréchet bound $C^+(x, y) := x \wedge y$ and of the independent copula $C^\perp(x, y) = xy$. In fact,

$$C_\alpha(x, y) = C^+(x, y)^\alpha C^\perp(x, y)^{1-\alpha}. \tag{4.5}$$

For U, V with joint distribution function C_α, we have for $0 \leq x, y \leq u$ that

$$\begin{aligned} F_U(C_\alpha, u, u)(x) = F_V(C_\alpha, u, u)(x) &= \frac{C_\alpha(x, u)}{C_\alpha(u, u)} = \frac{x}{u}, \\ F(C_\alpha, u, u)(x, y) = \frac{C_\alpha(x, y)}{C_\alpha(u, u)} &= C_\alpha\left(\frac{x}{u}, \frac{y}{u}\right). \end{aligned} \tag{4.6}$$

Thus, we immediately get from (2.8) that C_α is an invariant copula on the diagonal.

A particular family of curve-invariant copulas is the one of Definition 4.5 below. We will see in Corollary 4.11 that this family of copulas coincides with the LTDC-limits obtained in Theorem 3.4.

Definition 4.5. Let α, β, θ be positive constants and P, Q be increasing continuous univariate distribution functions on $[0, 1]$. We denote by $\mathcal{H}(\alpha, \beta, \theta)$ the set of two-dimensional distribution functions H on $[0, 1]^2$ that can be expressed as

$$H(x, y) = x^{\theta/\alpha} h(y x^{-\beta/\alpha}), \quad \text{where} \quad h(t) := \begin{cases} Q(t) & \text{if } t \in [0, 1] \\ t^{\theta/\beta} P(t^{-\alpha/\beta}) & \text{if } t \in (1, \infty) \end{cases}. \tag{4.7}$$

Theorem 4.6. *Let $\alpha, \beta, \theta > 0$ and $H \in \mathcal{H}(\alpha, \beta, \theta)$. Then,*

$$\Gamma(P, Q, \alpha, \beta, \theta)(u, v) := \begin{cases} Q^{-1}(v)^{\theta/\beta} P(P^{-1}(u) Q^{-1}(v)^{-\alpha/\beta}), & P^{-1}(u)^{\beta} \leq Q^{-1}(v)^{\alpha} \\ P^{-1}(u)^{\theta/\alpha} Q(P^{-1}(u)^{-\beta/\alpha} Q^{-1}(v)), & P^{-1}(u)^{\beta} > Q^{-1}(v)^{\alpha} \end{cases} \tag{4.8}$$

defines an invariant copula on $\mathcal{D} = \{(P(t^{\alpha}), Q(t^{\beta})) \mid t \in (0, 1]\}$.

Remark 4.7. From Theorem 4.6, we get that $\lim_{t \to 0} \Phi(\Gamma(P, Q, \alpha, \beta, \theta), P(t^{\alpha}), Q(t^{\beta})) = \Gamma(P, Q, \alpha, \beta, \theta)$, i.e. that $\Gamma(P, Q, \alpha, \beta, \theta)$ is a LTDC-limit. Further, note that $\Gamma(g, g, 1, 1, \theta)$ is precisely the copula in (3.8).

Example 4.8. The copula $\Gamma(\mathrm{Id}, \mathrm{Id}, \beta(\alpha + \beta - \alpha\beta)^{-1}, \alpha(\alpha + \beta - \alpha\beta)^{-1}, 1)$ is the Marshall and Olkin copula which, because of Theorem 4.6, is invariant on

$$\mathcal{D} = \{(t^{\beta/(\alpha+\beta-\alpha\beta)}, t^{\alpha/(\alpha+\beta-\alpha\beta)}) \mid t \in (0, 1]\} = \{(t^{\beta}, t^{\alpha}), t \in (0, 1]\}. \tag{4.9}$$

Similarly, $\Gamma(\mathrm{Id}, \mathrm{Id}, \alpha, \beta, 1)$ is also the Marshall and Olkin copula with parameters $(\alpha + \beta - 1/\alpha$ and $(\alpha + \beta - 1)/\beta$.

Example 4.9. For $P(x) = 2^{1/\theta}(1 + x^{-\theta})^{-1/\theta}$ with $\theta = \alpha + \beta$, the copula $\Gamma(P, P, \alpha, \beta, \theta)$ is the Clayton copula with parameter θ, i.e. for $(x, y) \in [0, 1]^2$

$$\Gamma(P, P, \alpha, \beta, \theta)(x, y) = (x^{-\theta} + y^{-\theta} - 1)^{-1/\theta}. \tag{4.10}$$

From Theorem 4.6, one has that this copula is invariant on $\mathcal{D} = \{(t^{\alpha}, t^{\beta}) \mid t \in (0, 1]\}$ for all α, β, i.e. $\Gamma(P, P, \alpha, \beta, \alpha + \beta)$ is invariant on $(0, 1]^2$.

Theorem 4.10 below characterizes the possible LTDC-limits stating that they coincide with the set of invariant copulas on $(0, 1]^2$. In particular, the family $\mathcal{H}(\alpha, \beta, \theta)$ characterizes LTDC-limits on curves $\mathcal{D} = \{(r(t), s(t)) \mid t \in T\}$ provided that the starting copula C belongs to $\mathcal{C} \cap \mathcal{R}(r, s)$ and that r, s are strictly increasing continuous and regularly varying at 0 (Corollary 4.11).

Theorem 4.10. *If $C \in \mathcal{C}$ and C_0 are copulas such that $\lim_{u,v \to 0} \|\Phi(C, u, v) - C_0\|_{\infty} = 0$, then C_0 is invariant on the unit square.*

Corollary 4.11. *Assume that C satisfies the hypothesis of Theorem 3.4 and let $C_0 = \lim_{t \to 0} \Phi(C, r(t), s(t))$. Then, there is a constant $\theta > 0$ such that $C_0 = \Gamma(\phi_X, \phi_Y, \alpha, \beta, \theta)$ according to (4.8). As a consequence, C_0 is invariant on $\mathcal{D} = \{(\phi_X(t^{\alpha}), \phi_Y(t^{\beta})) \mid t \in (0, 1]\}$.*

The proof of Theorem 4.10 is based on the fact that $\Phi(C, u', v')$ can be seen as the LTDC obtained from another LTDC $\Phi(C, u, v)$, where $u \geq u'$ and $v \geq v$ (Lemma 4.12). The second ingredient in the proof is the continuity of $\Phi(\cdot, u, v)$ (Lemma 4.13). We state these preliminary results below and not only in the Proof Section 6 since we believe they are interesting in their own.

Lemma 4.12. *Let $C \in \mathcal{C}$. For $0 \leq u' \leq u \leq 1$ and $0 \leq v' \leq v \leq 1$ we have that*

1. *$\Phi(C, u', v') = \Phi(\Phi(C, u, v), u^*, v^*)$, where u^* and v^* are given respectively by $u^* = C(u', v)/C(u, v)$ and $v^* = C(u, v')/C(u, v)$,*
2. *$\Phi(\Phi(C, u, v), u', v') = \Phi(C, u^*, v^*)$ where u^* and v^* satisfy respectively the relations $C(u^*, v) = u'C(u, v)$ and $C(u, v^*) = v'C(u, v)$.*

Lemma 4.13. *For any $u, v \in (0, 1]$, the map $\mathcal{C} \to \mathcal{C}$, $C \mapsto \Phi(C, u, v)$ is continuous with respect to the $\| \cdot \|_\infty$-norm.*

Remark 4.14. The parameters α, β of the LTDC-limit $\Gamma(P, Q, \alpha, \beta, \theta)$ can be interpreted as parameters describing the direction under which the limit is taken since, as stated in Theorem 4.6, $\Gamma(P, Q, \alpha, \beta, \theta)$ is invariant on $\mathcal{D} = \{(P(t^\alpha), Q(t^\beta)) \,|\, t \in (0, 1]\}$. However, such a distribution is not *identifiable*. In fact, α, β and θ are defined up to a positive multiplicative constant, thus $\Gamma(P, Q, \alpha, \beta, \theta)$ could be defined using two parameters solely. More precisely, for $\eta := \beta/\alpha$,

$$\Gamma(P, Q, \alpha, \beta, \theta) = \Gamma(P, Q, 1, \eta, \theta) =: \Gamma(P, Q, \eta, \theta). \tag{4.11}$$

Moreover, for all $k > 0$, we have that

$$\Gamma(P, Q, \eta, \theta) = \Gamma(P_k, Q_k, k\eta, k\theta), \tag{4.12}$$

where $P_k(x) := P(x^k)$ and $Q_k(x) := Q(x^k)$, $x \in [0, 1]$.

We finish this section with a Proposition stating that the only copula which is absolutely continuous and is also invariant on the unit square is the Clayton copula.

Proposition 4.15. *The only copula which is absolutely continuous and invariant on $[0, 1]^2$ is the Clayton copula.*

5 An application to credit risk

The main risk drivers of almost all credit derivatives such as e.g. *Credit Default Swap baskets* (CDS baskets) or *first-to-default* contract types are given by the relevant default times. Among the most popular (univariate) default time models we find intensity-based ones. As shown by Schönbucher and Schubert (2001) a copula approach allows to model naturally arbitrary dependence structures in such an intensity-based framework.

In this section we first review the setup of Schönbucher and Schubert (2001) and we then show how our LTDC-limits can be used as dependence structures for credit stress scenarios.

5.1 Intensity-based default models

For σ-algebras $\mathcal{A}, \mathcal{B}$ with $\mathcal{A} \subset \mathcal{B}$ and for a set $B \in \mathcal{B}$, we will use in the sequel the notation $\mathcal{A} \wedge B := \{A \cap B \mid A \in \mathcal{A}\}$. Further, all filtrations are supposed to satisfy the usual conditions, i.e. they are assumed to be right continuous and such that the smallest σ-filed of the filtration is trivial. Finally, for a review of point process intensities we refer to Brémaud (1981).

Schönbucher and Schubert (2001) propose the following intensity-based default model which we recall in the two-dimensional case. Let λ_i, $i = 1, 2$ be non-negative càdlàg processes adapted to a filtration $(\mathcal{G}_t)_{t \geq 0}$ representing the general market information except explicit information on the occurrence of defaults. For U_1, U_2 standard uniformly distributed random variables, which are assumed to be independent from $\mathcal{G}_\infty := \cup_{t \geq 0} \mathcal{G}_t$, we define the default times as the random variables

$$\tau_i := \inf\{t > 0 \mid \gamma_i(t) \leq U_i\}, \quad i = 1, 2, \tag{5.1}$$

where $\gamma_i(t) := \exp(-\Lambda_i(t))$ is called *countdown processes* and $\Lambda_i(t) := \int_0^t \lambda_i(s)\, ds$. Note that, conditioned on $\mathcal{G}_\infty$, we have that

$$\mathbb{P}[\tau_1 \leq t_1, \tau_2 \leq t_2 | \mathcal{G}_\infty] = \widehat{C}(\gamma_1(t_1), \gamma_2(t_2)), \tag{5.2}$$

where C is the distribution function of (U_1, U_2). Thus, we see that defining default times as in (5.1) implies that, given general market information, the default dependence mechanism is completely described by C.

Remark 5.1. The motivation behind (5.1) comes from the fact that, for a Cox process with intensity λ, the time τ of the first jump can be written

$$\tau = \inf\left\{ t > 0 \Big| \int_0^t \lambda(s)\, ds \geq Z \right\}, \tag{5.3}$$

where Z is exponentially distributed with parameter 1 (see Lando 1998).

In general, the intensity of a point process depends on the information which is conditioned on. Denoting by N_i the default counting process of counterparty $i = 1, 2$ and by $\mathcal{F}_t^i$ the augmented filtration of $\sigma(N_i(s); 0 \leq s \leq t)$, we have that λ_i is the $\mathcal{F}_t^i$-intensity of N_i. However it is in the spirit of any multivariate model also consists in considering the information relative to the other counterparties such as the one given by C and $\mathcal{H}_t := \vee_{i=1,2}(\mathcal{F}_t^i \vee \mathcal{G}_t)$, $t \geq 0$. Indeed, we find in Schönbucher and Schubert (2001) that the $\mathcal{H}_t$-intensity h_i of N_i equals to

$$h_i(t) := \lambda_i(t) \cdot \gamma_i(t) \cdot \partial_i \log(C(\gamma_1(t), \gamma_2(t))). \tag{5.4}$$

Because of the term $\partial_i \log(C(\gamma_1(t), \gamma_2(t)))$, the intensity of a single counterparty is also affected by the dependence structure of the several counterparties. In the case where U_1, U_2 are independent, i.e. whenever $C = C^{\perp}$, we have that the right hand side of (5.4) reduces to $\lambda_i(t)$, i.e. to the $\mathcal{F}_t^i$-intensity of N_i. Further, under the additional information that the other obligor has already defaulted, i.e. $\{\tau_j = t_j\}$, $j \neq i$, $t_j > 0$, the default intensity of the survived counterparty takes the form

$$h_i^{-j}(t) := \lambda_i(t) \cdot \gamma_i(t) \cdot \frac{\partial_{ij} C(\gamma_1(t), \gamma_2(t))}{\partial_j C(\gamma_1(t), \gamma_2(t))}. \tag{5.5}$$

A special case of (5.4) and (5.5) is given by C equal to the Clayton copula with parameter θ of Example 4.9. In that case,

$$h_i(t) = \left(\frac{C(\gamma_1(t), \gamma_2(t))}{\gamma_i(t)} \right)^{\theta} \lambda_i(t) \quad \text{and} \quad h_i^{-j}(t) = (1 + \theta) h_i(t). \tag{5.6}$$

As stated in Schönbucher and Schubert (2001), such a dependence structure reflects one of the main features of a model introduced by Davis and Lo (1999a, 1999b), where knowledge of one obligor's default determines a jump in the spread of the other obligor by a factor $(1 + \theta)$.

5.2 Dependence structures for stress scenarios

Stress scenarios for default times arise in many different situations. For example, pension funds have to invest only in investment grade bonds because of regulatory reasons. Thus, a default (or downgrade) of a bond in the pension fund's portfolio determines the replacement of that bond, whence a possible (large) losses due to the bonds's value decrease. Another example is given by first-to-default CDS baskets where in the case of an "early" default the protection seller receives the premium only for a short time but has to deliver the underlying very soon.

More generally, knowing or modelling the dependence structure of the several default times and in particular the joint behavior under averse market conditions, avoids risk underestimation allowing thus for a risk-adjusted pricing (for instance of credit derivatives). Such stress situations can be described by conditional distributions of the type

$$\mathbb{P}[\tau_1 \leq t_1, \tau_2 \leq t_2 | \mathcal{G}_\infty \wedge \{\tau_1 \leq T, \tau_2 \leq T\}], \tag{5.7}$$

as T tends to zero. Since the conditional distribution of τ_i given $\mathcal{H}_t^i$ equals $\gamma_i(t)$, it follows from Proposition 2.6 and Equation (5.2) that the copula relative to the conditional distribution in (5.7) is given by

$$\Phi(\widehat{C}, 1 - \gamma_1(T), 1 - \gamma_2(T)), \tag{5.8}$$

where $\widehat{C}$ is the survival copula of C.

Example 5.2 (First-to-default). The conditional distribution of the first-to-default time $\tau := \tau_1 \wedge \tau_2$ conditioned on $\mathcal{G}_\infty \wedge \{\tau_1 \leq T, \tau_2 \leq T\}$ is given for $t \leq T$ by

$$\begin{aligned} \mathbb{P}[\tau \leq t | \mathcal{G}_\infty \wedge \{\tau_1 \leq T, \tau_2 \leq T\}] &= 1 - \mathbb{P}[\tau_1 > t, \tau_2 > t | \mathcal{G}_\infty \wedge \{\tau_1 \leq T, \tau_2 \leq T\}] \\ &= 1 - C^*(1 - \gamma_1(t), 1 - \gamma_2(t)), \end{aligned} \tag{5.9}$$

where C^* is the survival copula of $\Phi(\widehat{C}, 1 - \gamma_1(T), 1 - \gamma_2(T))$.

Suppose now that λ_i is regularly varying at 0 with parameter $\delta_i \geq 0$ which, as it is easy to check, implies that $1-\gamma_i \in \mathcal{R}^0_{1+\delta_i}$. Further, assume $\widehat{\widehat{C}} \in \mathcal{C} \cap \mathcal{R}(1-\gamma_1, 1-\gamma_2)$ with limiting function ϕ. Then, because of Corollary 4.11, there is a constant $\theta > 0$ such that

$$\lim_{T \to 0} \Phi(\widehat{C}, 1 - \gamma_1(T), 1 - \gamma_2(T)) = \Gamma(\phi_X, \phi_Y, 1 + \delta_1, 1 + \delta_2, \theta). \tag{5.10}$$

As a special case, we have for $\gamma_1 = \gamma_2 =: \gamma$ and $\delta_1 = \delta_2 = 0$ that

$$\lim_{T\to 0} \Phi(\widehat{C}, 1-\gamma(T), 1-\gamma(T)) = \Gamma(g, g, 1, 1, \theta), \quad g := \phi_X, \tag{5.11}$$

which corresponds to the limiting copula (3.8) of Remark 3.7.

As we already mentioned at the end of Section 3, a special case of Theorem 3.4 is given by the situation where the starting copula is archimedean with a regularly varying generator. In this case, the LTDC-limit on the diagonal is the Clayton copula. Thus, the Davis-Lo-model can be seen as stress-scenario one.

6 Proofs

Proof of Proposition 2.6. For $w_i := G_i(z_i)$, $i = 1, 2$, we have by definition that

$$\Phi(C, G_1(z_1), G_2(z_2))(u_1, u_2) = \frac{C(F_{U_1}(C, w_1, w_2)^{-1}(u_1), F_{U_2}(C, w_1, w_2)^{-1}(u_2))}{C(G_1(z_1), G_2(z_2))}. \tag{6.1}$$

Further,

$$\begin{aligned} F_{U_1}(C, w_1, w_2)(v_1) &= \frac{C(v_1, w_2)}{C(w_1, w_2)} = \frac{C(v_1, G_2(z_2))}{C(G_1(z_1), G_2(z_2))} = \frac{G(G_1^{-1}(v_1), z_2)}{G(z_1, z_2)} \\ &= G_1^{z_1,z_2}(G_1^{-1}(v_1)), \end{aligned} \tag{6.2}$$

whence $F_{U_1}(C, w_1, w_2)^{-1}(u_1) = G_1((G_1^{z_1,z_2})^{-1}(u_1))$. Similarly, we have that $F_{U_2}(C, w_1, w_2)^{-1}(u_2) = G_2((G_2^{z_1,z_2})^{-1}(u_2))$. Thus,

$$\begin{aligned} \Phi(C, G_1(z_1), G_2(z_2))(u_1, u_2) &= \frac{C(G_1((G_1^{z_1,z_2})^{-1}(u_1)), G_2((G_2^{z_1,z_2})^{-1}(u_2)))}{G(z_1, z_2)} \\ &= \frac{G((G_1^{z_1,z_2})^{-1}(u_1), (G_2^{z_1,z_2})^{-1}(u_2))}{G(z_1, z_2)} \\ &= G^{z_1,z_2}((G_1^{z_1,z_2})^{-1}(u_1), (G_2^{z_1,z_2})^{-1}(u_2)) \\ &= \Phi(G, z_1, z_2)(u_1, u_2), \end{aligned} \tag{6.3}$$

which finishes the proof of Proposition 2.6. □

Proof of Theorem 3.4. The proof of this theorem is based on the following lemma.

Lemma 6.1. *Suppose that (X_n, Y_n) have continuous strictly increasing marginals and are such that $\lim_{n\to\infty}(X_n, Y_n) = (X, Y)$ in distribution for some (X, Y). Then,*

$$\lim_{n\to\infty} \|C_n - C\|_\infty = 0, \tag{6.4}$$

where C_n and C denote the copulas of (X_n, Y_n) and (X, Y), respectively.

Proof of Lemma 6.1. Denote by $F_{X_n}, F_{Y_n}, F_X, F_Y, F_n$ and F the distribution functions of X_n, Y_n, X, Y, (X_n, Y_n) and (X, Y), respectively. Then, for $u, v \in [0, 1]$,

$$\begin{aligned} |C_n(u,v) - C(u,v)| &= |F_n(F_{X_n}^{-1}(u), F_{Y_n}^{-1}(v)) - F(F_X^{-1}(u), F_Y^{-1}(v))| \\ &\leq |F_n(F_{X_n}^{-1}(u), F_{Y_n}^{-1}(v)) - F_n(F_X^{-1}(u), F_Y^{-1}(v))| \\ &\quad + |F_n(F_X^{-1}(u), F_Y^{-1}(v)) - F(F_X^{-1}(u), F_Y^{-1}(v))|. \end{aligned} \tag{6.5}$$

Because F_n is continuous and since F_{X_n} and F_{Y_n} are strictly increasing, $F_{X_n}^{-1}(u) \to F_X^{-1}(u)$ and $F_{Y_n}^{-1}(v) \to F_Y^{-1}(v)$ as $n \to \infty$ for any $u, v \in [0, 1]$. So, for any $\varepsilon > 0$ there is some positive integer N_1 such that for any $n \geq N_1$

$$|F_n(F_{X_n}^{-1}(u), F_{Y_n}^{-1}(v)) - F_n(F_X^{-1}(u), F_Y^{-1}(v))| \leq \varepsilon/2. \tag{6.6}$$

Similarly, because $\lim_{n\to\infty} F_n(x, y) = F(x, y)$, there is N_2 such that for any $n \geq N_2$

$$|F_n(F_X^{-1}(u), F_Y^{-1}(v)) - F(F_X^{-1}(u), F_Y^{-1}(v))| \leq \varepsilon/2. \tag{6.7}$$

Thus, for any $u, v \in [0, 1]$ and any $n \geq N := \max\{N_1, N_2\}$, we have that $|C_n(u, v) - C(u, v)| \leq \varepsilon$, i.e. $\lim_{n\to\infty} C_n = C$ pointwise. Because $[0, 1]^2$ is compact and both C_n and C are continuous, this convergence is also uniform. This finishes the proof of Lemma 6.1. □

Let now (U, V) have distribution function C. Note that

$$\frac{C(r(t)x, s(t))}{C(r(t), s(t))} = \mathbb{P}[U \leq r(t)x | U \leq r(t), V \leq s(t)], \tag{6.8}$$

$$\frac{C(r(t), s(t)y)}{C(r(t), s(t))} = \mathbb{P}[V \leq s(t)y | U \leq r(t), V \leq s(t)], \tag{6.9}$$

$$\frac{C(r(t)x, s(t)y)}{C(r(t), s(t))} = \mathbb{P}[U \leq r(t)x, V \leq s(t)y | U \leq r(t), V \leq s(t)], \tag{6.10}$$

i.e. the distributions in (6.8)–(6.10) are respectively the conditional distributions of $U/r(t)$, $V/s(t)$ and $(U/r(t), V/s(t))$ given $U \leq r(t), V \leq s(t)$. Since copulas are invariant under strictly increasing transformations of the underlying variables, it follows that we can take the conditional distributions in (6.8)–(6.10) instead of $F_U(C, r(t), s(t))$, $F_V(C, r(t), s(t))$ and $F(C, r(t), s(t))$ in order to construct $\Phi(C, r(t), s(t))$. Further, since $C \in \mathcal{C}$ and because r, s are strictly increasing and continuous, we have that the distributions in (6.8)–(6.10) are continuous too and strictly increasing. By hypothesis $C \in \mathcal{R}(r, s)$, i.e.

$$\lim_{t \to 0} \frac{C(r(t)x, s(t)y)}{C(r(t), s(t))} = \phi(x, y) \quad \text{for all } x, y \in [0, 1], \tag{6.11}$$

so that the expressions in (6.10) converge to ϕ_X, ϕ_Y and ϕ as $t \to 0$ respectively. Thus, applying Lemma 6.1, we get

$$\lim_{t \to 0} \Phi(C, r(t), s(t))(x, y) = \phi(\phi_X^{-1}(x), \phi_Y^{-1}(y)), \tag{6.12}$$

whence (3.3) has been proved. Since $r \in \mathcal{R}_\alpha^0$, $s \in \mathcal{R}_\beta^0$, we have according to Theorem 2.1 in De Haan et al. (1984) that there is $\theta > 0$ such that for all $t, x, y > 0$

$$\phi(t^\alpha x, t^\beta y) = t^\theta \phi(x, y). \tag{6.13}$$

Further, according to Aczél (1966) the most general solution to the functional equation (6.13) is given by

$$\phi(x, y) = \begin{cases} x^{\theta/\alpha} h(y x^{-\beta/\alpha}) & \text{if } x \neq 0 \\ c y^{\theta/\beta} & \text{if } x = 0 \text{ and } y \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}, \tag{6.14}$$

where c is a constant and h is function of one variable. Because $\phi(0, y) = 0$ and $\phi_Y(y) = \phi(1, y) = h(y)$, it follows that $c = 0$ and that the restriction of h on $[0, 1]$ equals ϕ_Y, respectively. Further, we have for $x \in (0, 1]$ that

$$\phi_X(x^{\alpha/\beta}) = \phi(x^{\alpha/\beta}, 1) = x^{\theta/\beta} h(1/x), \tag{6.15}$$

whence for $t = 1/x > 1$ we obtain $h(t) = h(1/x) = x^{-\theta/\beta}\phi_X(x^{\alpha/\beta}) = t^{\theta/\beta}\phi_X(t^{-\alpha/\beta})$, which shows (3.4) and finishes therefore the proof of Theorem 3.4. □

Proof of Proposition 3.10. In order to prove that (3.10) defines a copula, we have to show (2.1) and (2.2). For $x \in [0, 1]$, the conditions $C(x, 0) =$

$C(0,x) = 0$ are satisfied because $a(0) = b(0) = 0$. Further, since $x \mapsto a(x)/x$ is decreasing with $a(1) = 1$, we have that $a(x) \geq x$ for any $x \in [0,1]$. Thus, because $b(1) = 1$, we get $C(x,1) = a(x) \wedge x = x$. Similarly, $C(1,x) = x$, $x \in [0,1]$, which shows (2.1). Consider now $0 < x_1 \leq x_2 \leq 1$ and $0 < y_1 \leq y_2 \leq 1$. Then,

$$\begin{aligned}\Delta &:= C(x_2,y_2) - C(x_1,y_2) - C(x_2,y_1) + C(x_1,y_1)\\ &= x_2 y_2 \left(\frac{a(x_2)}{x_2} \wedge \frac{b(y_2)}{y_2}\right) - x_1 y_2 \left(\frac{a(x_1)}{x_1} \wedge \frac{b(y_2)}{y_2}\right) - x_2 y_1 \left(\frac{a(x_2)}{x_2} \wedge \frac{b(y_1)}{y_1}\right)\\ &\quad + x_1 y_1 \left(\frac{a(x_1)}{x_1} \wedge \frac{b(y_1)}{y_1}\right).\end{aligned} \tag{6.16}$$

Since $x \mapsto a(x)/x$ and $x \mapsto b(x)/x$ are decreasing, six different cases have to be considered:

1. Assume that $a(x_2)/x_2 \leq a(x_1)/x_1 \leq b(y_2)/y_2 \leq b(y_1)/y_1$. Then, $\Delta = (y_2 - y_1)(a(x_2) - a(x_1)) \geq 0$ since a is increasing.

2. If $b(y_2)/y_2 \leq b(y_1)/y_1 \leq a(x_2)/x_2 \leq a(x_1)/x_1$, then $\Delta = (x_2 - x_1)(b(y_2) - b(y_1))$, which is of course non-negative.

3. Suppose now that $a(x_2)/x_2 \leq b(y_2)/y_2 \leq b(y_1)/y_1 \leq a(x_1)/x_1$. Then, $\Delta = x_1(b(y_1) - b(y_2)) + a(x_2)(y_2 - y_1)$ is non-negative if and only if

$$\frac{b(y_2) - b(y_1)}{y_2 - y_1} \leq \frac{a(x_2)}{x_1}. \tag{6.17}$$

   Since $x \mapsto b(x)/x$ is decreasing, the left hand side of (6.17) can be bounded as follows:

$$\frac{b(y_2) - b(y_1)}{y_2 - y_1} = \frac{b(y_2)}{y_2}\frac{y_2}{y_2 - y_1} - \frac{b(y_1)}{y_1}\frac{y_1}{y_2 - y_1} \leq \frac{b(y_1)}{y_1}. \tag{6.18}$$

   By hypothesis and since a is increasing, we have $b(y_1)/y_1 \leq a(x_1)/x_1 \leq a(x_2)/x_1$, whence (6.17).

4. The case $b(y_2)/y_2 \leq a(x_2)/x_2 \leq a(x_1)/x_1 \leq b(y_1)/y_1$ yields $\Delta = (a(x_1) - a(x_2))y_1 + (x_2 - x_1)b(y_2)$, which can be shown to be non-negative using the same arguments as in (3).

5. If $a(x_2)/x_2 \leq b(y_2)/y_2 \leq a(x_1)/x_1 \leq b(y_1)/y_1$, then $\Delta = (y_2 - y_1)a(x_2) - x_1 b(y_2) + y_1 a(x_1)$. By hypothesis,

$$x_1 b(y_2) \leq a(x_1) y_2 = a(x_1) y_1 + a(x_1)(y_2 - y_1) \leq a(x_1) y_1 + a(x_2)(y_2 - y_1), \tag{6.19}$$

where the last inequality follows because a is increasing. This shows that $\Delta \geq 0$.

6. The last case is given by $b(y_2)/y_2 \leq a(x_2)/x_2 \leq b(y_1)/y_1 \leq a(x_1)/x_1$ and $\Delta = (x_2 - x_1)b(y_2) - y_1 a(x_2) + x_1 b(y_1)$. As in (5), it follows that $\Delta \geq 0$.

Therefore, (3.10) defines a copula.
Let us prove this result for Marshall and Olkin copulas. Let C denote Marshall and Olkin's copula with parameters α and β, i.e.

$$C(x, y) = \min\{x^{1-\alpha} y, x y^{1-\beta}\}, \ \forall x, y \in [0,1], \tag{6.20}$$

where α and β are real parameters, in $(0,1)$, then, the limiting copulae for Marshall and Olkin's copula C under some power function directions are

$$\lim_{t \to 0} \Phi\left(C, t, t^\theta\right)(x, y) = \begin{cases} C(x, y) & \text{if } \theta = \alpha/\beta \\ C^\perp(x, y) & \text{if } \theta \neq \alpha/\beta. \end{cases} \tag{6.21}$$

On the one hand, if $\theta = \alpha/\beta$, the result is trivial since those copulas are invariant under direction

$$\mathcal{D} = \left\{\left(t^\beta, t^\alpha\right), t \in [0,1]\right\} = \left\{\left(t, t^{\alpha/\beta}\right), t \in [0,1]\right\}. \tag{6.22}$$

On the other hand, if $\theta \neq \alpha/\beta$, assume that $0 \leq \theta < \alpha/\beta$. Let (U, V) be a random pair with c.d.f. C, the joint distribution of (U, V) given $U \leq t, V \leq t^\theta$, is

$$\mathbb{P}\left(U \leq x, V \leq y | U \leq t, V \leq t^\theta\right) = \frac{\min\{x^{1-\alpha} y, x y^{1-\beta}\}}{\min\{t^{1-\alpha} t^\theta, t t^{\theta(1-\beta)}\}} = \frac{\min\{x^{1-\alpha} y, x y^{1-\beta}\}}{t^{\theta(1-\beta)+1}}, \tag{6.23}$$

where $x \in [0, t]$ and $y \in \left[0, t^\theta\right]$: since $\beta\theta < \alpha$, and $t \in (0,1]$, $t^{-\alpha} > t^{-\theta\beta}$, and so $t^{1+\theta-\alpha} > t^{1+\theta-\theta\beta}$. And the marginal distributions are

$$\mathbb{P}\left(U \leq x | U \leq t, V \leq t^\theta\right) = \frac{\min\{x^{1-\alpha} t^\theta, x t^{\theta(1-\beta)}\}}{t^{\theta(1-\beta)+1}} = \min\{x^{1-\alpha} t^{\theta\beta - 1}, x t^{-1}\}, \tag{6.24}$$

and analogously,

$$\mathbb{P}\left(V \leq y | U \leq t, V \leq t^{\theta}\right) = \frac{\min\{t^{1-\alpha}y, ty^{1-\beta}\}}{t^{\theta(1-\beta)+1}} = \min\{t^{-\alpha-\theta(1-\beta)}y, t^{-\theta(1-\beta)}y^{1-\beta}\}. \tag{6.25}$$

One can notice that $x^{1-\alpha}t^{\theta\beta-1} \leq xt^{-1}$ if and only if $x \geq t^{\theta\beta/\alpha}$; but since $\theta < \alpha/\beta$, $\theta\beta/\alpha < 1$, and so $x^{1-\alpha}t^{\theta\beta-1} \leq xt^{-1}$ if and only if $x \geq t^{\theta\beta/\alpha} > t$ which can not occur since $x \in [0, t]$. It come that, necessarily,

$$\mathbb{P}\left(U \leq x | U \leq t, V \leq t^{\theta}\right) = \frac{x}{t},\ x \in [0, t]. \tag{6.26}$$

Analogously $t^{-\alpha-\theta(1-\beta)}y \geq t^{-\theta(1-\beta)}y^{1-\beta}$ if and only if $t^{-\alpha} \geq y^{-\beta}$ i.e. $t^{\alpha/\beta} \geq y$; but since $\theta < \alpha/\beta$ and $t \in [0, 1]$, $t^{\alpha/\beta} < t^{\theta}$, and so $t^{-\alpha-\theta(1-\beta)}y \geq t^{-\theta(1-\beta)}y^{1-\beta}$ if and only if $y \leq t^{\alpha/\beta} < t^{\theta}$ which always occurs since $y \in \left[0, t^{\theta}\right]$. It come that, necessarily,

$$\mathbb{P}\left(V \leq y | U \leq t, V \leq t^{\theta}\right) = t^{-\theta(1-\beta)}y^{1-\beta}. \tag{6.27}$$

So finally, the generalized inverse of those marginal c.d.f.'s are respectively

$$x \mapsto xt \text{ and } y \mapsto y^{1/(1-\beta)}t^{\theta}. \tag{6.28}$$

The copula of (U, V) given $U \leq t, V \leq t^{\theta}$ is then given by

$$\begin{aligned} \Phi\left(C, t, t^{\theta}\right)(x, y) &= \frac{1}{t^{\theta(1-\beta)+1}} \min\{(xt)^{1-\alpha}\left(y^{1/(1-\beta)}t^{\theta}\right), (xt)\left(y^{1/(1-\beta)}t^{\theta}\right)^{1-\beta}\}, \\ &= \frac{1}{t^{\theta(1-\beta)+1}} \min\{x^{1-\alpha}y^{1/(1-\beta)}t^{\theta+1-\alpha}, xyt^{1+\theta(1-\beta)}\}, \\ &= xy\min\{x^{-\alpha}y^{\beta/(1-\beta)}t^{-\alpha+\theta\beta}, 1\}. \end{aligned}$$

Since $\theta < \alpha/\beta$. then $\alpha > \beta\theta$, or equivalently $\theta\beta - a < 0$, and so, for all $x, y > 0$, there is t_0 such that, if $t > t_0$, $x^{-\alpha}y^{\beta/(1-\beta)}t^{-\alpha+\theta\beta} \geq 1$.
So finally, for all x, y in $(0, 1]$ there exists t_0 such that, for any $t > t_0$, $\Phi\left(C, t, t^{\theta}\right)(x, y) = xy$. It comes that

$$\lim_{t\to 0} \Phi\left(C, t, t^{\theta}\right)(x, y) = xy = C^{\perp}(x, y). \tag{6.29}$$

(one gets the uniform convergence since copula functions are defined on the compact set $[0, 1] \times [0, 1]$).

(iii) assume now that $\alpha/\beta < \theta$. Let (U,V) be a random pair with c.d.f. C, the joint distribution of (U,V) given $U \leq t, V \leq t^\theta$, is

$$\mathbb{P}\left(U \leq x, V \leq y | U \leq t, V \leq t^\theta\right) = \frac{\min\{x^{1-\alpha}y, xy^{1-\beta}\}}{\min\{t^{1-\alpha}t^\theta, tt^{\theta(1-\beta)}\}} = \frac{\min\{x^{1-\alpha}y, xy^{1-\beta}\}}{t^{1+\theta-\alpha}}, \tag{6.30}$$

where $x \in [0,t]$ and $y \in \left[0, t^\theta\right]$: since $\alpha < \beta\theta$, and $t \in (0,1]$, $t^{-\alpha} < t^{-\theta\beta}$, and so $t^{1+\theta-\alpha} < t^{1+\theta-\theta\beta}$. And the marginal distributions are

$$\mathbb{P}\left(U \leq x | U \leq t, V \leq t^\theta\right) = \frac{\min\{x^{1-\alpha}t^\theta, xt^{\theta(1-\beta)}\}}{t^{1+\theta-\alpha}} = \min\{x^{1-\alpha}t^{\alpha-1}, xt^{\alpha-1-\theta\beta}\}, \tag{6.31}$$

and analogously,

$$\mathbb{P}\left(V \leq y | U \leq t, V \leq t^\theta\right) = \frac{\min\{t^{1-\alpha}y, ty^{1-\beta}\}}{t^{1+\theta-\alpha}} = \min\{t^{-\theta}y, t^{\alpha-\theta}y^{1-\beta}\}. \tag{6.32}$$

One can notice that $x^{1-\alpha}t^{\alpha-1} \leq xt^{\alpha-\theta\beta-1}$ if and only if $x^{-\alpha} \leq t^{-\theta\beta}$, or equivalently, $x \leq t^{\theta\beta/\alpha}$; but since $\theta > \alpha/\beta$, or $\theta\beta/\alpha > 1$, and $t \in [0,1]$, then $x^{1-\alpha}t^{\alpha-1} \leq xt^{\alpha-\theta\beta-1}$ if and only if $x \leq t^{\theta\beta/\alpha} < t$ which always occurs since $x \in [0,t]$. It come that, necessarily,

$$\mathbb{P}\left(U \leq x | U \leq t, V \leq t^\theta\right) = x^{1-\alpha}t^{\alpha-1} = \left(\frac{x}{t}\right)^{1-\alpha}, \; x \in [0,t]. \tag{6.33}$$

Analogously $t^{-\theta}y \geq t^{\alpha-\theta}y^{1-\beta}$ if and only if $t^{-\alpha} \geq y^{-\beta}$ i.e. $t^{\alpha/\beta} \leq y$; but since $\theta > \alpha/\beta$ and $t \in [0,1]$, $t^{\alpha/\beta} > t^\theta$, and so $t^{-\theta}y \leq t^{\alpha-\theta}y^{1-\beta}$ if and only if $y \geq t^{\alpha/\beta} > t^\theta$ which can not occurs since $y \in \left[0, t^\theta\right]$. It come that, necessarily,

$$\mathbb{P}\left(V \leq y | U \leq t, V \leq t^\theta\right) = t^{-\theta}y. \tag{6.34}$$

So finally, the generalized inverse of those marginal c.d.f.'s are respectively

$$x \mapsto tx^{1/(1-a)} \text{ and } y \mapsto t^\theta y. \tag{6.35}$$

The copula of (U,V) given $U \leq t, V \leq t^\theta$ is then given by

$$\begin{aligned}
\Phi\left(C,t,t^\theta\right)(x,y) &= \frac{1}{t^{1+\theta-\alpha}}\min\{\left(tx^{1/(1-a)}\right)^{1-\alpha}t^\theta y, tx^{1/(1-a)}\left(t^\theta y\right)^{1-\beta}\} \\
&= \frac{1}{t^{1+\theta-\alpha}}\min\{xyt^{\theta+1-\alpha}, x^{1/(1-a)}y^{1-\beta}t^{1+\theta-\theta\beta}\}, \\
&= \min\{xy, x^{1/(1-a)}y^{1-\beta}t^{-\theta\beta-a}\} = xy\min\{1, x^{\alpha/(1-a)}y^{-\beta}t^{-\theta\beta-a}\}.
\end{aligned}$$

Since $\theta > \alpha/\beta$. then $a - \theta\beta < 0$, and so, for all $x, y > 0$, there is t_0 such that, if $t > t_0$, $x^{\alpha/(1-a)}y^{-\beta}t^{\alpha-\theta\beta} \geq 1$.

So finally, for all x, y in $(0, 1]$ there exists t_0 such that, for any $t > t_0$, $\Phi\left(C, t, t^{\theta}\right)(x, y) = xy$. It comes that

$$\lim_{t\to 0} \Phi\left(C, t, t^{\theta}\right)(x, y) = xy = C^{\perp}(x, y). \tag{6.36}$$

(one gets the uniform convergence since copula functions are defined on the compact set $[0, 1] \times [0, 1]$). Moreover, one can notice that $\theta < \alpha/\beta$ and $\theta > \alpha/\beta$ are symmetric cases.

This finishes the proof if C is Marshall and Olkin copula. An analogous proof, but slightly more technical, leads to Proposition 3.10. □

Proof of Lemma 4.2. Because $C \in \mathcal{C}$, we have that $F_U(C, u, v)$ and $F_V(C, u, v)$ are strictly increasing. Because of Sklar's Theorem and using (2.6), (2.7), we get

$$\begin{aligned}\Phi(C, u, v)\left(\frac{C(x, v)}{C(u, v)}, \frac{C(u, y)}{C(u, v)}\right) &= \Phi(C, u, v)(F_U(C, u, v)(x), F_V(C, u, v)(y)) \\ &= F(C, u, v)(x, y) = \frac{C(x, y)}{C(u, v)}.\end{aligned} \tag{6.37}$$

This finishes the proof of Lemma 4.2. □

Proof of Theorem 4.6. We will first prove that $\Gamma(\alpha, \beta, \theta)$ defined by (4.8) is a copula and then show the invariance property. The function H defined by (4.7) can be rewritten as

$$\begin{aligned} H(x, y) &= \begin{cases} x^{\theta/\alpha}[yx^{-\beta/\alpha}]^{\theta/\beta} P([yx^{-\beta/\alpha}]^{-\alpha/\beta}) & \text{if } x^{\beta} < y^{\alpha} \\ x^{\theta/\alpha} Q(yx^{-\beta/\alpha}) & \text{if } x^{\beta} \geq y^{\alpha} \end{cases} \\ &= \begin{cases} y^{\theta/\beta} P(y^{-\alpha/\beta}x) & \text{if } x^{\beta} < y^{\alpha} \\ x^{\theta/\alpha} Q(yx^{-\beta/\alpha}) & \text{if } x^{\beta} \geq y^{\alpha} \end{cases}. \end{aligned} \tag{6.38}$$

By hypothesis, the marginals P, Q of H are strictly increasing continuous functions, whence it follows from Sklar's Theorem that the copula associated to H equals

$$H(P^{-1}(u), Q^{-1}(v)) = \begin{cases} Q^{-1}(v)^{\theta/\beta} P(P^{-1}(u) Q^{-1}(v)^{-\alpha/\beta}), & \text{if } P^{-1}(u)^{\beta} < Q^{-1}(v)^{\alpha} \\ P^{-1}(u)^{\theta/\alpha} Q(P^{-1}(u)^{-\beta/\alpha} Q^{-1}(v)), & \text{if } P^{-1}(u)^{\beta} \geq Q^{-1}(v)^{\alpha} \end{cases}, \tag{6.39}$$

which is precisely $\Gamma(P,Q,\alpha,\beta,\theta)$. We show now that $\Gamma(P,Q,\alpha,\beta,\theta)$ is invariant on the curve $\mathcal{D}=\{(P(t^\alpha),Q(t^\beta))\,|\,t\in(0,1]\}$. For notational convenience we denote $\Gamma(P,Q,\alpha,\beta,\theta)$ by C. In order to derive the LTDC associated to C, we first notice that from (6.39) it follows

$$\begin{aligned} &C(P(t^\alpha),Q(t^\beta)) = t^\theta, \\ &\qquad C(x,Q(t^\beta)) = \begin{cases} t^\theta P(P^{-1}(x)t^{-\alpha}) & \text{if } P^{-1}(x)<t^\alpha \\ P^{-1}(x)^{\theta/\alpha}Q(P^{-1}(x)^{-\beta/\alpha}t^\beta) & \text{if } P^{-1}(x)\geq t^\alpha \end{cases}, \\ &\qquad C(P(t^\alpha),y) = \begin{cases} Q^{-1}(y)^{\theta/\beta}P(t^\alpha Q^{-1}(y)^{-\alpha/\beta}) & \text{if } t^\beta<Q^{-1}(y) \\ t^\theta Q(P^{-1}(t^\alpha)^{-\beta/\alpha}Q^{-1}(y)) & \text{if } t^\beta\geq Q^{-1}(y) \end{cases}. \end{aligned} \tag{6.40}$$

Let now $(x,y)\in[0,P(t^\alpha)]\times[0,Q(t^\beta)]$. Because of (2.7) and (6.40), we have that the marginals of $F(C,P(t^\alpha),Q(t^\beta))$ are given respectively by

$$F_U(C,P(t^\alpha),Q(t^\beta))(x) = \frac{C(x,Q(t^\beta))}{C(P(t^\alpha),Q(t^\beta))} = \frac{t^\theta P(P^{-1}(x)t^{-\alpha})}{t^\theta} = P(P^{-1}(x)t^{-\alpha}), \tag{6.41}$$

$$F_V(C,P(t^\alpha),Q(t^\beta))(y) = \frac{C(P(t^\alpha),y)}{C(P(t^\alpha),Q(t^\beta))} = Q(t^{-\beta}Q^{-1}(y)). \tag{6.42}$$

Their inverses equal

$$\begin{aligned} F_U(C,P(t^\alpha),Q(t^\beta))^{-1}(x) &= P(P^{-1}(x)t^\alpha), \\ F_V(C,P(t^\alpha),Q(t^\beta))^{-1}(y) &= Q(t^\beta Q^{-1}(y)). \end{aligned} \tag{6.43}$$

Assume now that x,y are such that $P^{-1}(x)^\beta < Q^{-1}(y)^\alpha$. From (6.40) we obtain that

$$F(C,P(t^\alpha),Q(t^\beta))(x,y) = \frac{C(x,y)}{C(P(t^\alpha),Q(t^\beta))} = \frac{Q^{-1}(y)^{\theta/\beta}P(P^{-1}(x)Q^{-1}(y)^{-\alpha/\beta})}{t^\theta}. \tag{6.44}$$

Thus, for any $(x,y)\in(0,1]^2$ such that $P^{-1}(F_U^{-1}(x))^\beta \leq Q^{-1}(F_V^{-1}(y))^\alpha$, i.e. $P^{-1}(x)^\beta\leq Q^{-1}(y)^\alpha$, we have that

$$\begin{aligned} &\Phi(C,P(t^\alpha),Q(t^\beta))(x,y) \\ &= F(C,P(t^\alpha),Q(t^\beta))(F_U(C,P(t^\alpha),Q(t^\beta))^{-1}(x),F_V(C,P(t^\alpha),Q(t^\beta))^{-1}(y)) \\ &= t^{-\theta}(t^\beta Q^{-1}(y))^{\theta/\beta}P(P^{-1}(x)t^\alpha(t^\beta Q^{-1}(y))^{-\alpha/\beta}) \\ &= Q^{-1}(y)^{\theta/\beta}P(P^{-1}(x)Q^{-1}(y)^{-\alpha/\beta}) = C(x,y). \end{aligned} \tag{6.45}$$

Similarly, if $(x, y) \in [0, P(t^\alpha)] \times [0, Q(t^\beta)]$ are such that $P^{-1}(x)^\beta \geq Q^{-1}(y)^\alpha$, then

$$F(C, P(t^\alpha), Q(t^\beta))(x, y) = \frac{P^{-1}(x)^{\theta/\alpha} Q(P^{-1}(x)^{-\beta/\alpha} Q^{-1}(y))}{t^\theta}. \qquad (6.46)$$

Thus,

$$\begin{aligned} &\Phi(C, P(t^\alpha), Q(t^\beta))(x, y) \\ &= F(C, P(t^\alpha), Q(t^\beta))(F_U(C, P(t^\alpha), Q(t^\beta))^{-1}(x), F_V(C, P(t^\alpha), Q(t^\beta))^{-1}(y)) \\ &= t^{-\theta} P^{-1}(x)^{\theta/\alpha} Q(P^{-1}(y)^{-\beta/\alpha} Q^{-1}(x)) = C(x, y). \end{aligned} \qquad (6.47)$$

Hence, for all $(x, y) \in [0, 1]^2$, $\Phi(C, P(t^\alpha), Q(t^\beta))(x, y) = C(x, y)$, i.e. C is invariant on $\mathcal{D} = \{(P(t^\alpha), Q(t^\beta)) \,|\, t \in (0, 1]\}$. This finishes the proof of Theorem 4.6. □

Proof of Lemma 4.12. (i) Let $C^* = \Phi(\Phi(C, u, v), u^*, v^*)$. Because of Lemma 4.2, we have for $0 \leq x \leq u^*$ and $0 \leq y \leq v^*$ that

$$\frac{\Phi(C, u, v)(x, y)}{\Phi(C, u, v)(u^*, v^*)} = C^* \left(\frac{\Phi(C, u, v)(x, v^*)}{\Phi(C, u, v)(u^*, v^*)}, \frac{\Phi(C, u, v)(u^*, y)}{\Phi(C, u, v)(u^*, v^*)} \right). \qquad (6.48)$$

On the other hand, we have, again using Lemma 4.2, that $\Phi(C, u, v)(u^*, v^*)$ equals

$$\Phi(C, u, v)(u^*, v^*) = \Phi(C, u, v) \left(\frac{C(u', v)}{C(u, v)}, \frac{C(u, v')}{C(u, v)} \right) = \frac{C(u', v')}{C(u, v)}. \qquad (6.49)$$

Further, $F_U(C, u, v)^{-1}(u^*) = u'$ and $F_V(C, u, v)^{-1}(v^*) = v'$ by definition of u^* and v^*. Because,

$$\Phi(C, u, v)(x, y) = \frac{C(F_U(C, u, v)^{-1}(x), F_V(C, u, v)^{-1}(y))}{C(u, v)}, \qquad (6.50)$$

it follows multiplying (6.49) with (6.50) that

$$\frac{\Phi(C, u, v)(x, y)}{\Phi(C, u, v)(u^*, v^*)} = \frac{C(F_U(C, u, v)^{-1}(x), F_V(C, u, v)^{-1}(y))}{C(u', v')}. \qquad (6.51)$$

Let $s = F_U(C,u,v)^{-1}(x)$ and $t = F_V(C,u,v)^{-1}(y)$, then, substituting into (6.48), we have

$$\frac{C(s,t)}{C(u',v')} = C^* \left(\frac{C(s,v')}{C(u',v')}, \frac{C(u',t)}{C(u',v')} \right) \tag{6.52}$$

for all x, y in $[0,u^*] \times [0,v^*]$. Because C is continuous, $F_U(C,u,v)$ and $F_V(C,u,v)$ are also continuous on $[0,u]$ and $[0,v]$ respectively. Hence, (6.52) holds for all s,t in $[0,u'] \times [0,v']$ because $F_U(C,u,v)^{-1}(u^*) = u'$ and $F_V(C,u,v)^{-1}(v^*) = v'$.

Finally, if $0 < u' \leq u \leq 1$ and $0 < v' \leq v \leq 0$, then $\Phi(C,u',v') = \Phi(\Phi(C,u,v),u^*,v^*)$, where u^* are v^* satisfy respectively $u^* = C(u',v)/C(u,v)$ and $v^* = C(u,v')/C(u,v)$.

(ii) Conversely, $C^* = \Phi(\Phi(C,u,v),u',v')$ satisfies, for $0 \leq x \leq u'$ and $0 \leq y \leq v'$

$$\frac{\Phi(C,u,v)(x,y)}{\Phi(C,u,v)(u',v')} = C^* \left(\frac{\Phi(C,u,v)(x,v')}{\Phi(C,u,v)(u',v')}, \frac{\Phi(C,u,v)(u',y)}{\Phi(C,u,v)(u',v')} \right). \tag{6.53}$$

Since

$$\frac{C(x,y)}{C(u,v)} = \Phi(C,u,v) \left(\frac{C(x,v)}{C(u,v)}, \frac{C(u,y)}{C(u,v)} \right), \tag{6.54}$$

we get that for all $x \leq u'$ and $y \leq v'$ that

$$\frac{C(F_U(C,u,v)^{-1}(x), F_V(C,u,v)^{-1}(y))}{C(F_U(C,u,v)^{-1}(u'), F_V(C,u,v)^{-1}(v'))}$$
$$= C^* \left(\frac{C(F_U(C,u,v)^{-1}(x), F_V(C,u,v)^{-1}(v'))}{C(F_U(C,u,v)^{-1}(u'), F_V(C,u,v)^{-1}(v'))}, \frac{C(F_U(C,u,v)^{-1}(u'), F_V(C,u,v)^{-1}(y))}{C(F_U(C,u,v)^{-1}(u'), F_V(C,u,v)^{-1}(v'))} \right) \tag{6.55}$$

Let $u^* = F_U(C,u,v)^{-1}(u')$ and $v^* = F_V(C,u,v)^{-1}(v')$, i.e. u^* and v^* satisfy respectively $C(u^*,v) = u'C(u,v)$ and $C(u,v^*) = v'C(u,v)$. Then, for all $x \leq u^*$ and $y \leq v^*$

$$\frac{C(x,y)}{C(u^*,v^*)} = C^* \left(\frac{C(x,v^*)}{C(u^*,v^*)}, \frac{C(u^*,y)}{C(u^*,v^*)} \right), \tag{6.56}$$

i.e. $C^* = \Phi(C,u^*,v^*)$ from Sklar's Theorem since the functions $x \mapsto C(x,v^*)/C(u^*,v^*)$ and $y \mapsto C(u^*,y)/C(u^*,v^*)$ are continuous.

Finally, if $0 < u', u \leq 1$ and $0 < v', v \leq 0$, then $\Phi(\Phi(C, u, v), u', v') = \Phi(C, u^*, v^*)$ where u^* are v^* satisfy respectively $C(u^*, v) = u'C(u, v)$. Moreover, because $C(u^*, v) = u'C(u, v) \leq C(u, v)$ and because $x \mapsto C(x, v)/C(u, v)$ is an increasing function, it follows that $u^* \leq u$. Similarly, $v^* \leq v$, which completes the proof of Lemma 4.12. □

Proof of Lemma 4.13. In order to show the continuity of $\Phi(\cdot, u, v)$, we have to bound differences of the form

$$|\Phi(C', u, v)(s, t) - \Phi(C, u, v)(s, t)|, \tag{6.57}$$

where $C, C' \in \mathcal{C}$ and $s, t \in [0, 1]$. Since the functions $C(\cdot, v)/C(u, v)$ and $C(u, \cdot)/C(u, v)$ are continuous and take the values 0 and 1 at u, respectively v, we may assume without loss of generality that $s = C(x, v)/C(u, v)$ and $t = C(u, y)/C(u, v)$ for some $(x, y) \in [0, u] \times [0, v]$. Applying Lemma 4.2, it follows then

$$\Phi(C, u, v)(s, t) = \frac{C(x, y)}{C(u, v)}. \tag{6.58}$$

Let now $\Delta := C' - C$ and consider

$$\alpha_C(x, y) := \frac{C(x, y)}{C(u, v) + \Delta(u, v)} \quad \text{and} \quad \delta_\Delta(x, y) := \frac{\Delta(x, y)}{C(u, v) + \Delta(u, v)}. \tag{6.59}$$

We obtain that

$$\begin{aligned} \frac{C'(x, v)}{C'(u, v)} &= \frac{C(x, y) + \Delta(x, y)}{C(u, v) + \Delta(u, v)} = \alpha_C(u, v)s + \delta_\Delta(x, v), \\ \frac{C'(u, y)}{C'(u, v)} &= \alpha_C(u, v)t + \delta_\Delta(u, y). \end{aligned} \tag{6.60}$$

Thus, using again Lemma 4.2, we get

$$\Phi(C', u, v)(\alpha_C(u, v)s + \delta_\Delta(x, v), \alpha_C(u, v)t + \delta_\Delta(u, y)) = \frac{C'(x, y)}{C'(u, v)}. \tag{6.61}$$

Now, the expression in (6.57) can be bounded as follows:

$$\begin{aligned} &|\Phi(C', u, v)(s, t) - \Phi(C, u, v)(s, t)| \\ &\leq |\Phi(C', u, v)(s, t) - \Phi(C', u, v)(\alpha_C(u, v)s + \delta_\Delta(x, v), \alpha_C(u, v)t + \delta_\Delta(u, y))| \\ &\quad + |\Phi(C', u, v)(\alpha_C(u, v)s + \delta_\Delta(x, v), \alpha_C(u, v)t + \delta_\Delta(u, y)) - \Phi(C, u, v)(s, t)| \\ &\leq |\alpha_C(u, v)s + \delta_\Delta(x, v) - s| + |\alpha_C(u, v)t + \delta_\Delta(u, y) - t| + \left| \frac{C'(x, y)}{C'(u, v)} - \frac{C(x, y)}{C(u, v)} \right|, \end{aligned} \tag{6.62}$$

where the last inequality follows because any copula is Lipschitz-continuous with Lipschitz constant 1 and because of (6.59), (6.61). Further, from the definition of α_C, δ_Δ and because $x \leq u, s \leq 1$, we have that

$$|\alpha_C(u,v)s + \delta_\Delta(x,v) - s| \leq \left|\frac{-\Delta(u,v)s}{C(u,v)+\Delta(u,v)} + \frac{\Delta(x,v)}{C(u,v)+\Delta(u,v)}\right| \leq \frac{2|\Delta(u,v)|}{C(u,v)+\Delta(u,v)} \leq \frac{2\|\Delta\|_\infty}{C(u,v)-\|\Delta\|_\infty}. \tag{6.63}$$

Similarly,

$$|\alpha_C(u,v)t + \delta_\Delta(u,y) - t| \leq \frac{2\|\Delta\|_\infty}{C(u,v)-\|\Delta\|_\infty}. \tag{6.64}$$

Further, since $x \leq u, y \leq v$, we have that

$$\left|\frac{C'(x,y)}{C'(u,v)} - \frac{C(x,y)}{C(u,v)}\right| = \frac{|\Delta(x,y)C(u,v) - C(x,y)\Delta(u,v)|}{C'(u,v)C(u,v)} \leq \frac{2|C(u,v)\Delta(u,v)|}{C'(u,v)C(u,v)} \leq \frac{2\|\Delta\|_\infty}{C'(u,v)}. \tag{6.65}$$

From (6.62), (6.63), (6.64) and (6.65), we get

$$|\Phi(C',u,v)(s,t) - \Phi(C,u,v)(s,t)| \leq \frac{4\|\Delta\|_\infty}{C(u,v)-\|\Delta\|_\infty} + \frac{2\|\Delta\|_\infty}{C'(u,v)}, \tag{6.66}$$

where the right hand side is independent from s, t and can be made arbitrarily small as $\|\Delta\|_\infty$ becomes small. This finishes the proof of proof of Lemma 4.13 □

Proof of Theorem 4.10. Let (u_n) and (v_n) be the two sequences defined recursively by the following relationship: Let α and β be two constants in $(0,1]$ with $(\alpha,\beta) \neq (1,1)$ so that, given u_n and v_n strictly positive, $C(u_{n+1},v_n)/C(u_n,v_n) = \alpha$ and $C(u_n,v_{n+1})/C(u_n,v_n) = \beta$ for all $n \geq 1$. Given u_n and v_n, u_{n+1} and v_{n+1}, we have from the continuity of C that are well defined (but not necessarily unique). Those sequences can be defined starting in $(1,1)$ so that $u_1 = \alpha$ and $v_1 = \beta$.
Because $\alpha, \beta \in (0,1]$, we have that $0 \leq u_{n+1} \leq u_n$ and $0 \leq v_{n+1} \leq v_n$. Let $u = \lim_{n\to\infty} u_n$ and $v = \lim_{n\to\infty} v_n$. If $u > 0$ and $v > 0$, then $C(u,v)/C(u,v) = \alpha = \beta$, i.e. $\alpha = \beta = 1$ contradicting the hypothesis $(\alpha,\beta) \neq (1,1)$ meaning that either $u = 0$ or $v = 0$.
Let $C_n = \Phi(C,u_n,v_n)$. From Lemma 4.12, we have that $\Phi(C,u_{n+1},v_{n+1}) = \Phi(\Phi(C,u_n,v_n),u^*_{n+1},v^*_{n+1})$ where u^*_{n+1} and v^*_{n+1} are given respectively by

$u^*_{n+1} = C(u_{n+1}, v_n)/C(u_n, v_n)$ and $v^*_{n+1} = C(u_n, v_{n+1})/C(u_n, v_n)$, i.e. $u^*_{n+1} = \alpha$ and $v^*_{n+1} = \beta$, and so, $\Phi(C, u_{n+1}, v_{n+1}) = \Phi(C_n, \alpha, \beta) = C_{n+1}$. Because $C_n = \Phi(C, u_n, v_n)$, then, as soon as either $u_n \to 0$ or $v_n \to 0$ when $n \to \infty$, C_n converges towards C_0 when $n \to \infty$. And so, because, given α and β, $\Phi(., \alpha, \beta)$ is a continuous function, from Lemma 4.13, then necessarily, C_0 satisfies $\Phi(C_0, \alpha, \beta) = C_0$. This finishes the proof of Theorem 4.10. □

Proof of Corollary 4.11. Since $C_0 = \lim_{t\to 0} \Phi(C, r(t), s(t))$, then it follows from Theorem 3.4 that $C_0(x, y) = \phi(\phi_X^{-1}(x), \phi_Y^{-1}(y))$, where for $x > 0$

$$\phi(x, y) = x^{\theta/\alpha} h(yx^{-\beta/\alpha}), \quad h(x) = \begin{cases} \phi_Y(x) & \text{if } x \in [0, 1] \\ x^{\theta/\beta}\phi_X(x^{-\alpha/\beta}) & \text{if } x \in (1, \infty) \end{cases}. \tag{6.67}$$

In other words,

$$\begin{aligned} C_0(x, y) =& \phi(\phi_X^{-1}(x), \phi_Y^{-1}(y)) \\ =& \begin{cases} \phi_Y^{-1}(y)^{\theta/\beta}\phi_X(\phi_Y^{-1}(y)^{-\alpha/\beta}\phi_X^{-1}(x)) & \text{if } \phi_X^{-1}(x)^\beta < \phi_Y^{-1}(y)^\alpha \\ \phi_X^{-1}(x)^{\theta/\alpha}\phi_Y(\phi_Y^{-1}(y)\phi_X^{-1}(x)^{-\beta/\alpha}) & \text{if } \phi_X^{-1}(x)^\beta \geq \phi_Y^{-1}(y)^\alpha \end{cases}, \end{aligned} \tag{6.68}$$

i.e. $C_0 = \Gamma(\phi_X, \phi_Y, \alpha, \beta, \theta)$ according to (4.8). This finishes the proof of Corollary 4.11. □

Proof of Proposition 4.15. Let C be an absolutely continuous and invariant copula on the unit square. Because of Lemma 4.2, we have for all $x, y, u, v \in]0, 1]$ that

$$\frac{C(xu, yv)}{C(u, v)} = C\left(\frac{C(xu, v)}{C(u, v)}, \frac{C(u, yv)}{C(u, v)}\right). \tag{6.69}$$

Since C is absolutely continuous, then derivating with respect to x and y yields

$$\frac{uvC_{12}(xu, yv)}{C(u, v)} = \frac{vC_2(u, yv)}{C(u, v)}\frac{uC_1(xu, v)}{C(u, v)}C_{12}\left(\frac{C(xu, v)}{C(u, v)}, \frac{C(u, yv)}{C(u, v)}\right), \tag{6.70}$$

where C_1, C_2 and C_{12} denote the partial derivatives of C with respect to the relative variables. The latter equation can be written as

$$\frac{C(u, v)C_{12}(xu, yv)}{C_2(u, yv)C_1(xu, v)} = C_{12}\left(\frac{C(xu, v)}{C(u, v)}, \frac{C(u, yv)}{C(u, v)}\right). \tag{6.71}$$

Inserting $x = y = 1$, we obtain that

$$\frac{C(u,v)C_{12}(u,v)}{C_2(u,v)C_1(u,v)} = C_{12}(1,1) =: \theta - 1. \tag{6.72}$$

The latter equation can be rewritten as

$$\frac{C_{12}(u,v)}{C_1(u,v)} = (\theta - 1)\frac{C_2(u,v)}{C(u,v)}. \tag{6.73}$$

Integrating with respect to v leads to

$$\log C_1(u,v) = (\theta - 1)\log C(u,v) + \kappa(u) \tag{6.74}$$

for some function κ of u. In order to determine the function κ, notice that $\log C_1(u,1) = (\theta - 1)\log u + \kappa(u)$. Substituting into equation (6.74) yields

$$\log \frac{C_1(u,v)}{C_1(u,1)} = (\theta - 1)\log \frac{C(u,v)}{u}, \tag{6.75}$$

Taking the exponential on both sides produces the identity

$$\frac{C_1(u,v)}{C(u,v)^{\theta-1}} = \frac{C_1(u,1)}{u^{\theta-1}}. \tag{6.76}$$

Integrating with respect to u, we obtain

$$\frac{C(u,v)^{-\theta}}{-\theta} = \frac{C(u,1)^{-\theta}}{-\theta} + \lambda(v) = \frac{u^{-\theta}}{-\theta} + \lambda(v) \tag{6.77}$$

for some function λ of v. Because of symmetry, it follows that λ does not depend on v, i.e. that

$$\frac{C(u,v)^{-\theta}}{-\theta} = \frac{u^{-\theta}}{-\theta} + \frac{v^{-\theta}}{-\theta} + \text{constant}, \tag{6.78}$$

which can also be written as

$$C(u,v)^{-\theta} = u^{-\theta} + v^{-\theta} + \text{constant} \quad 0 \le u, v \le 1, \tag{6.79}$$

where c is some constant. Finally, because C is a copula, it must be $C(1,1) = 1$, whence the constant in the equation above must be -1, i.e. C is the Clayton copula with parameter θ. Conversely, since the Clayton copula is absolutely continuous and also invariant on $[0,1]^2$, it follows that it is the only copula with this properties. □

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