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# Coping with Imprecise Information : A Decision Theoretic Approach*

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# Coping with Imprecise Information: a Decision Theoretic Approach [1]

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**Abstract**

We provide a model of decision making under uncertainty in which the decision maker reacts to imprecision of the available data. Data is represented by a set of probability distributions. We axiomatize a decision criterion of the maxmin expected utility type, in which the revealed set of priors explicitly depends on the available data. We then characterize notions of comparative aversion to imprecision of the data as well as traditional notions of risk aversion. Interestingly, the study of comparative aversion to imprecision can be done independently of the utility function, which embeds risk attitudes. We also give a more specific result, in which the functional representing the decision maker's preferences is the convex combination of the minimum expected utility with respect to the available data and expected utility with respect to a subjective probability distribution, interpreted as a reference prior. This particular form is shown to be equivalent to some form of constant aversion to imprecision. We finally provide examples of applications first to unanimity rankings of imprecision and risk and then to optimal risk sharing arrangements.

**Keywords:** Imprecision, Ambiguity, Uncertainty, Decision, Multiple Priors.

**JEL Number:** D81.

**Résumé**

Nous proposons un modèle de décision dans l'incertain qui permet au décideur de tenir compte de l'imprécision des données dont il dispose, ces dernières étant représentées par un ensemble de distributions de probabilité. Nous caractérisons axiomatiquement un critère de décision de la forme "maxmin d'espérance d'utilité", dans lequel l'ensemble révélé de distributions de probabilité dépend explicitement des données disponibles. Nous caractérisons également les notions d'aversion à l'égard de l'imprécision des données, et d'aversion à l'égard du risque compatibles avec ce modèle de décision. En particulier, nous montrons que l'étude de l'aversion à l'égard de l'imprécision de l'information peut être faite indépendamment de la fonction d'utilité du décideur, tandis que l'attitude à l'égard du risque du décideur dépend uniquement de la forme de sa fonction d'utilité. Nous caractérisons également un modèle de décision plus spécifique, dans lequel la fonctionnelle représentant les préférences du décideur est une combinaison convexe du minimum de l'espérance d'utilité calculé sur l'ensemble des données disponibles, et de l'espérance d'utilité par rapport à une distribution de probabilité subjective, interprétée comme une distribution de référence. Cette forme particulière se révèle être équivalente à une certaine forme d'aversion constante à l'égard de l'imprécision. Enfin, nous proposons des exemples d'applications, d'une part au classement unanime de l'imprécision et du risque, et d'autre part au partage optimal du risque entre agents.

**Mots clés:** Imprécision, Ambiguïté, Incertitude, Décision, Probabilités a priori multiples.

**Numéros JEL:** D81.

# 1 Introduction

In many problems of choice under uncertainty, some information is available to the decision maker. Yet, this information is often far from being sufficiently precise to allow the decision maker to come up with an estimate of a probability distribution over the relevant states of nature. The archetypical example of such a situation is the so-called Ellsberg paradox (Ellsberg (1961)), in which subjects are given some imprecise information concerning the composition of an urn and are then asked to choose among various bets on the color of a ball drawn from that urn. Less anecdotal is the issue of climate change (see Intergovernmental Panel on Climatic Change (2001) for a thorough exposition): there, uncertainty is ubiquitous, as long recognized by experts. In this issue of utmost importance, recommendations to the authors for assessing and reporting uncertainties (Moss and Schneider (2000)) are particularly interesting to look at. While acknowledging at the outset that "the Bayesian paradigm is a formal and rigorous language to communicate uncertainties", the authors insist on carefully linking the prior belief with quantitative distributions when possible. Actually, most of the recommendations have to do with justifying the probability judgement made, if available. In particular, in case of disagreement among experts, the report insists on the necessity to communicate all the information available and not an aggregate of it:

> "In developing a best estimate, authors need to guard against aggregation of results (...) if it hides important regional or inter-temporal differences. It is important not to combine automatically different distributions into one summary distribution. For example, most participants or available studies might believe that the possible outcomes are normally distributed, but one group might cluster its mean far from the mean of another group, resulting in a bimodal aggregated distribution. In this case, it is inappropriate to combine these into one summary distribution, unless it is also indicated that there are two (or more) "schools of thought"." Moss and Schneider (2000), p.42.

This creates a tension with the general recommendation to use the Bayesian paradigm. Indeed, the usual argument in favor of aggregating the data is that decision makers cannot deal with sets of probability distributions.[1] Finally, even if experts agree, it may well be the case that available data only allows to identify a set of probability distributions (see, e.g., Manski (2003) and Walley (1991)); this is indeed what happens in Ellsberg's experiments. In that case, it might be difficult to subjectively assess a single prior, and there is no reason, besides tractability, to require experts or decision makers to come up with such a probabilistic assessment.[2] One may

[1] For an interesting methodological discussion of these issues in a practical case –the problem of sea level rise–, see Titus and Narayanan (1996).

[2] Moss and Schneider (2000) give the following argument: if experts did not give a probability distribution to the policy maker, the latter will do it by himself, because he is unable to process something else than probabilistic information.

however consider the problem the other way around: what tools can we provide to the decision maker so that he can deal with the brute information that experts deliver or with situations in which data only identify a set of distributions? In other words, is there a way to keep a formal and rigorous language to communicate uncertainties, that is still tractable but does not assume that probabilistic prior beliefs can always be formed?

In this paper, we develop a formal model of decision making under uncertainty which may provide a partial answer to this question. More precisely, we model a decision maker who reacts to imprecision of the available data in a given choice problem. We do so assuming that data can be represented by sets of probability distributions. Thus, we define preferences as a binary relationship on the cross product of acts (mappings from states of the world to –probability distributions over– outcomes) and available information (sets of probability distributions over the state space). Compared to the approach developed in Gilboa and Schmeidler (1989), we enrich the space on which preferences are defined. Denoting $\mathcal{P}$ the set of probability distributions over the state space that represents the information available to the decision maker, preferences bear on couples $(f, \mathcal{P})$ where $f$ is an act in the usual sense. This means that, at least conceptually, we allow agents to compare the same acts in different informational settings. In Gilboa and Schmeidler (1989), the (un-modelled) prior information that the decision maker has is fixed. Our general representation theorem axiomatizes a class of functional of the maxmin expected utility type à la Gilboa and Schmeidler (1989), where the revealed set of priors is a subset of the available information. For each set of probability distributions representing available information, we get a revealed set of priors. We prove that under our axioms, the revealed sets of priors satisfy a certain number of consistency requirements when information is changed. More precisely, the general decision criterion we axiomatize takes the following form: for two sets of probability distributions $\mathcal{P}$ and $\mathcal{Q}$ and two acts $f$ and $g$, $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$ if, and only if,

$$\min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ f dp \geq \min_{p \in \mathcal{F}(\mathcal{Q})} \int u \circ g dp.$$

In this expression $\mathcal{F}(\mathcal{P})$ is the revealed set of priors when information is given by $\mathcal{P}$. Our representation result imposes some consistency conditions on how $\mathcal{F}(\mathcal{P})$ is related to $\mathcal{P}$. These conditions (e.g., $\mathcal{F}(\mathcal{P}) \subset \mathcal{P}$) are obtained with an economy of axioms. The main axiom in this construction is one of aversion to imprecision which, loosely speaking, states that the decision maker always prefers to act in a setting in which he possesses more information. An advantage of having defined preferences on pairs (act,information) is to be able to simply capture imprecision aversion as aversion towards a "garbling" of the information at hand. At this stage, we simply remark that the notion we adopt of what it means for a set of probability distributions to be more imprecise than another one is rather weak and partial in the sense that it does not enable one to compare many sets (this will be discussed at length and illustrated via an example when we introduce our axiom); yet this axiom has clear behavioral content and can easily be tested.

Based on this representation theorem, one can characterize a comparative notion of aversion towards imprecision, with the nice feature that it can be completely separated from risk attitudes. We say that a decision maker $b$ is more imprecision averse than a decision maker $a$ if whenever $a$ prefers to *bet* on an event when the information is given by a (precise) probability distribution rather than some imprecise information, $b$ prefers the bet with the precise information as well. This notion captures in rather natural terms a preference for precise information, which does not require the two decision makers that are compared to have the same risk attitudes, the latter being captured, as we show, by the concavity of the utility function.[3] Our result states that two decision makers can be compared according to that notion if and only if the revealed set of priors of one of them is included in the other's. A Bayesian decision maker (that is, a subjective expected utility maximizer) will have a revealed set of priors reduced to a singleton, while an extremely imprecision averse decision maker will have a revealed set of priors that exactly correspond to the set compatible with available information. Whenever the sets of probability distributions representing the information have some underlying symmetric structure (which we'll define precisely), it is possible to define absolute and relative imprecision premia that characterize this notion of aversion towards imprecision.

The representation theorem described above does not pin down a functional form but rather a class of functional forms compatible with aversion towards imprecision. If one is willing to assume extra properties of the preference relation, one can come up with more precise functional forms. For instance, a convenient one consists of taking the convex combination of the minimum expected utility with respect to all the probability distributions compatible with prior information, with the expected utility with respect to a particular probability distribution in this set. The coefficient in the convex combination has then a direct interpretation in terms of attitude towards imprecision. As it turns out, this functional form can be axiomatized in a natural way for a large class of sets of probability distributions (including notably cores of beliefs functions), the extra axiom being one of constant relative imprecision premium. This gives rise to the following representation: $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$ if, and only if,

$$\theta \min_{p \in \mathcal{P}} \int u \circ f dp + (1-\theta) \int u \circ f dc_{\mathcal{P}} \geq \theta \min_{p \in \mathcal{Q}} \int u \circ g dp + (1-\theta) \int u \circ f dc_{\mathcal{Q}}.$$

where $c_{\mathcal{P}}$ is the, suitably defined, center of the family $\mathcal{P}$ and $\theta$ is the value of the constant relative imprecision aversion premium that allows all the attitudes towards imprecision from paranoic pessimism ($\theta = 1$) to Bayesianism ($\theta = 0$).

Our approach, we believe, might be of interest for economic applications where imprecision about the data is the rule. First, our theory can provide orderings that permit to develop results of comparative statics in the spirit of comparative statics for risk based on second order stochastic dominance. Here, two kinds of ordering matter: first, when can we say that a situation is more

[3]Whenever information is precise in the sense that it is compatible with only one probability distribution, our axioms imply that the decision maker is maximizing expected utility.

imprecise? and second, when can we say that a prospect is more risky? We propose some first insights about unanimity ordering of imprecision (defined by saying that $\mathcal{P}$ is unanimously more imprecise than $\mathcal{Q}$ if all decision makers that satisfy our axioms are such that $(f, \mathcal{P}) \succeq (f, \mathcal{Q})$ for any act $f$) that can be used to rank interesting classes of sets of probability distributions. For instance, we give an ordering for cores of beliefs functions. We also give a more complete ordering for a restricted class of preferences which display increasing absolute imprecision premium. We also consider how the second order stochastic dominance conditions can be extended in our framework.

Second, the separation of attitude towards risk and attitude towards imprecision that we achieve, paves the way for exploring the respective consequences of these two features in economic situations. Take for instance contracting models. These models usually assume that the principal and the agent have expected utility preferences with the same probabilistic beliefs and different attitudes towards risk. The shape of the optimal contract is crucially dependent on these two features. Our model could be used to study the equivalent of contractual "risk sharing" optimal arrangements in an environment where not only risk but also imprecision is present. More generally, one could reconsider risk/uncertainty-sharing optimal arrangements in the light of this model. The specification we develop in this paper could be meaningfully used to assess the effect of different attitudes towards imprecision on the set of Pareto optimal allocations or on equilibrium allocations, something that is not feasible using the two decision models mentioned above (see e.g., Chateauneuf, Dana, and Tallon (2000), Mukerji and Tallon (2001).) Although we do not develop these ideas in full generality in this paper, we provide a simple example of risk-sharing analysis in an economy with imprecision averse decision makers. This example, although a little contrived, illustrates a third point of interest of our model, namely to clearly separate the information agents have from their revealed beliefs, which might be different even when they have the same information. Our example shows that when agents have no information (all probability distributions are compatible), Pareto Optimal allocations are comonotone, even though revealed beliefs can differ across agents.

**Organization of the paper**

The paper is organized as follows. The next subsection discusses some related literature. The following section describes the setup and establishes the notation. Section 3 is divided in three. In the first subsection we introduce and discuss our axioms. In particular, we provide a lengthy discussion of our axiom of aversion to imprecision. In the second subsection we state our representation theorem and discusses some of its implications; the last subsection defines and characterizes a notion of comparative imprecision aversion and subsequently the usual notion of comparative risk aversion transposed to our model. Section 4 contains an exploration of more specific functional forms based on constant relative imprecision premium. In Section 5, we provide economic applications, first to unanimity rankings of imprecision and risk and then to optimal risk sharing. All proofs are gathered in the Appendix.

**Comparison with the related literature**

We end this introduction by mentioning some related literature, whose precise relationship with our model and results will be discussed further in the text. We also make clear what are the main conceptual differences between our approach and much of the recent literature.

Our model incorporates explicitly information as an object on which the decision maker has well defined preferences. To the best of our knowledge, Jaffray (1989) is the first to axiomatize a decision criterion that takes into account "objective information" in a setting that is more general than risk. In his model, preferences are defined over belief functions. The criterion he axiomatizes is a weighted sum of the minimum and of the maximum expected utility. This criterion prevents a decision maker from behaving as an expected utility maximizer, contrary to ours, which obtains as a limit case the expected utility criterion. Interest in this approach has been renewed recently, in which object of choices are sets of lotteries (Ahn (2003), Olszewski (2002), Stinchcombe (2003)). More closely related to our analysis, and actually a point of inspiration of this paper, is Wang (2003). In his approach the available information is explicitly incorporated in the decision model. That information takes the form of a set of probability distributions together with an anchor, i.e., a probability distribution that has particular salience. As in our analysis, he assumes that decision makers have preferences over couples (act, information). However, his axiom of ambiguity aversion is much stronger than ours and forces the decision maker to be a maximizer of the minimum expected utility taken over the entire set of probability distributions. There is no scope in his model for less extreme attitude towards ambiguity. Following Wang's approach, we proposed in Gajdos, Tallon, and Vergnaud (2004) a weaker version of aversion towards imprecision still assuming that information was coming as a set of priors together with an anchor.

The notion of aversion towards imprecision that we develop here is based on the one analyzed in our previous work and is different from the one defined in Gilboa and Schmeidler (1989) and Schmeidler (1989) and the subsequent literature. There, aversion towards ambiguity is defined *via* a preference for hedging, while ours is defined *via* a preference for information precision. Thus, in Gilboa and Schmeidler (1989), uncertainty aversion is only indirectly revealed by a preference for hedging, while our approach is in some sense more direct. This is because we observe the preference for different "objective information". This point is of theoretical importance, as it allows us to define aversion towards ambiguity or imprecision as a reaction of the decision maker to a change in the information he possesses. Taking this view might also shed new light on the debate around the right notion of "ambiguity aversion"[4] which is symptomatic of a lack of "objective" definition of what really constitutes "ambiguity". Indeed, contrary to risk aversion, which was first defined as the existence of a risk premium – whose computation is based on a given probability distribution– and then defined in a purely subjective framework

[4]See Ghirardato and Marinacci (2002), Epstein (1999) and the thorough, conceptual discussion in Ghirardato (2004)

(Yaari (1969)), ambiguity aversion was directly defined in a subjective framework, without a clear view of what kind of data would constitute an instance of ambiguous data.

Our notion of comparative imprecision aversion could itself be compared to the one found in Epstein (1999) and Ghirardato and Marinacci (2002). The latter define comparative ambiguity aversion using constant acts. They therefore need to control for risk attitudes in a separate manner and in the end, can compare (with respect to their ambiguity attitudes) only decision makers that have the same utility functions.[5] Epstein (1999) uses in place of our bets in the definition of comparative uncertainty aversion, acts that are measurable with respect to an exogenously defined set of unambiguous events. As a consequence, in order to be compared, preferences of two decision makers have to coincide on the set of unambiguous events. If the latter is rich enough, utility functions then coincide. Our notion of comparative imprecision aversion, based on the comparison of bets under precise and imprecise information does not require utility functions to be the same when comparing two decision makers.

The functional form that we axiomatize appears in some previous work (Gajdos, Tallon, and Vergnaud (2004) and Tapking (2004)). In independent work, Hayashi (2003) provides, in the same set up as ours, a different axiomatization of essentially the same decision criterion. His approach is different in at least one important direction. Hayashi's axiomatization of the equivalent of our general decision criterion rests on a notion of imprecision aversion that is based on gains via hedging, much as in Gilboa and Schmeidler (1989). Thus, imprecision aversion is not defined in terms of properties of the preferences when comparing various informational settings. Hence, one could argue that this definition does not take advantage of the full strength of the general setting adopted. On the other hand, Hayashi's main theorem is concerned with the more specific functional form discussed above, i.e., the convex combination of the minimum expected utility with respect to all the probability distributions compatible with prior information, with the expected utility with respect to a particular probability distribution, which, in his approach, turns out to be the Steiner point of this set. His main extra axiom is a geometric axiom stating that the decision maker's preferences are "invariant to similarity reshuffles". The latter are a generalization of the notion of permutation. His axiom has little behavioral content in the sense that it is difficult to interpret it as reflecting some kind of aversion towards imprecision even though it mechanically gives this result in the functional form axiomatized. In particular, it is not clear what kind of conceptual information about a decision maker's preferences one could extract from the fact that he does not obey Invariance to Similarity Reshuffle. Indeed, we give in this paper an example of a decision maker whose preferences are compatible with the general representation but not with the specific functional form (thus violating Invariance to Similarity Reshuffles). As we argue then, it is not clear on what behavioral grounds such preferences should be ruled out. We believe our approach provides a deeper understanding of how imprecision of

[5]They actually mention that if one wants to compare two decision makers with different utility functions, one has first to completely elicit them.

the data might or might not affect a decision maker's behavior.

Finally, we compare our approach with Klibanoff, Marinacci, and Mukerji (2003). They provide a fully subjective model of ambiguity aversion, in which attitude towards ambiguity is captured by a smooth function over the expected utilities associated with a set of priors. The latter, as in Gilboa and Schmeidler (1989) is subjective. Hence, although their model allows for a flexible and explicit modelling of ambiguity attitudes, there is no link between the subjective set of priors and the available information. Interestingly, part of Klibanoff, Marinacci, and Mukerji (2003)'s motivation is similar to ours, that is disentangling ambiguity attitude from the information the decision maker has. Formally, however, this separation holds in their model only if one makes the extra assumption that subjective beliefs coincide with the objective information available. In particular, comparative statics is more transparent in our model, as information can be exogenously changed. At a more conceptual level, Klibanoff, Marinacci, and Mukerji (2003)'s approach assumes that all uncertainty is eventually reduced to subjective probabilities, although on two different levels: essentially, the decision maker has in mind a second order probability distribution, but does not perform reduction of lotteries. The criterion they obtained is smooth and appeals only to probabilistic tools, which should make it easy to use in economic applications. Besides the different specific modelling choices, our conceptual departure from their approach is that we do not assume that, even subjectively, imprecise information can be reduced to probabilities (even of a second or higher order). In that sense we are more in line with **?**'s view, that when a decision maker lacks a determinate probability distribution over states, "there will correspond [to any available option], in general, a set of expected utility values, among which he cannot discriminate in terms of definite probabilities".

# 2 Setup and notation

Let $S$ be a countably infinite set of states of nature, that we will identify to $\mathbb{N}$. We assume that prior information in any given decision problem comes as a set of probability distributions over that state space. We restrict attention to sets of probability distributions with finite support. Let $\mathbb{P}$ be the set of non-empty, closed (in the weak convergence topology) sets of priors with finite support, and $\mathbb{P}_C$ the set of convex elements of $\mathbb{P}$. Denote $\mathcal{P}$ a generic element of $\mathbb{P}$, and $S(\mathcal{P}) = \cup_{p \in \mathcal{P}} Supp(p)$ the finite support of $\mathcal{P}$. For any subset $E$ of $S$, let $\Delta(E)$ be the simplex on $E$, that is the set of probability distributions $p$ with $Supp(p) \subset E$.

As Gilboa and Schmeidler (1989) among others we use the framework of Anscombe and Aumann (1963). This is mostly for sake of simplicity, as it enables us to formally build on their representation theorem. Let $X$ be a set (the set of outcomes) and let $Y$ be the set of distributions over $X$ with finite supports (roulette lotteries). An act $f$ is a mapping from $S$ to $Y$. We denote by $\mathcal{A}$ the set of acts (horse lotteries) and $\mathcal{A}^c$ the set of constant acts. The decision maker's preferences is a binary relation $\succeq$ over $\mathcal{A} \times \mathbb{P}$, that is, on couples $(f, \mathcal{P})$. As usual, $\succ$ and $\sim$

denote the asymmetric and symmetric parts, respectively, of $\succeq$.

Before stating the axioms we impose on the decision maker's preferences, we need to introduce some further notation. For $E \subset S$, let $f_E g$ be the act giving $f(s)$ if $s \in E$ and $g(s)$ otherwise. Define the mixture of two acts $\alpha f + (1-\alpha) g$ as usual, i.e., it is the act giving $f(s)$ with probability $\alpha$ and $g(s)$ with probability $(1-\alpha)$ in state $s$.

For any $\varphi$ onto mapping from $S$ to $S$, for any $f \in \mathcal{A}$, we say that $f$ is $\varphi$-measurable if $f(s) = f(s')$ for all $s, s' \in S$ such that $\varphi(s) = \varphi(s')$. For a $\varphi$-measurable act $f$, define the act $f^\varphi$ on $\varphi(S)$ by $f^\varphi(s) = f(s')$ where $s' \in \varphi^{-1}(s)$ for all $s \in S$. $f^\varphi$ is the act $f$ "translated" on a different part of the state space. A similar operation for the available information can be defined as follows. For any $p \in \Delta(S)$ and $\mathcal{P} \in \mathbb{P}$ and $\varphi$ onto mapping from $S$ to $S$, $p^\varphi$ is defined by $p^\varphi(s) = p(\varphi^{-1}(s))$ for all $s \in \varphi(S)$ and $\mathcal{P}^\varphi$ is defined by $\mathcal{P}^\varphi = \{p^\varphi | p \in \mathcal{P}\}$. If $\varphi$ is a bijection, note that for all $p \in \Delta(S)$, there is a unique $q \in \Delta(S)$ such that $q^\varphi = p$. Let $(f^\varphi, \mathcal{P}^\varphi)$ be denoted for short $(f, \mathcal{P})^\varphi$.

For any $\mathcal{P}, \mathcal{P}' \in \mathbb{P}$, define $\alpha\mathcal{P} + (1-\alpha)\mathcal{P}'$, the convex combination of $\mathcal{P}$ and $\mathcal{P}'$, to be the set:

$$\left\{ q \,\middle|\, q = \alpha p + (1-\alpha) p',\, p \in \mathcal{P},\, p' \in \mathcal{P}' \right\}$$

For any $\mathcal{P} \in \mathbb{P}$, any $\alpha \in [0,1]$, and any bijection $\varphi$ from $S$ to $S$ such that $\varphi(S(\mathcal{P})) \cap S(\mathcal{P}) = \emptyset$, let $R[\mathcal{P}, \alpha, \varphi]$ be defined by

$$\{q | q = \alpha p + (1-\alpha) p^\varphi, p \in \mathcal{P}\}$$

We now state a technical axiom, which could be dispensed with at the cost of greatly complicating notation, without gaining much conceptual insights.

**Axiom 1** *(Act mixture) For all $\mathcal{P} \in \mathbb{P}$, for all $\alpha \in [0,1]$ and bijection $\varphi : S \to S$, for all $f, g \in \mathcal{A}$ such that $f(s) = g(s)$ for all $s \in S \setminus S(\mathcal{P})$,*

$$(\alpha f + (1-\alpha) g, \mathcal{P}) \sim (f_{S(\mathcal{P})} g^\varphi, R[\mathcal{P}, \alpha, \varphi])$$

Note that in this axiom, $f_{S(\mathcal{P})} g^\varphi(s) = f(s) = g(s)$ for all $s \in S \setminus (S(\mathcal{P}) \cup \varphi(S(\mathcal{P})))$.

The role of this axiom is simply to use the full strength of Gilboa and Schmeidler's technical analysis, which is cast in the Anscombe-Aumann setting and rests on a mixture operation. Formally, we do not need to cast our analysis in the Anscombe-Aumann setting since ours has a probabilistic structure already built in. However, doing so simplifies the proofs and the notation.

# 3 Axioms and Representation Theorem

In this section, we introduce our axioms, provide an extensive discussion of our main axiom (aversion towards imprecision) and give our representation theorem. We next provide a definition and a characterization of a notion of comparative imprecision aversion and introduce a notion of imprecision premium. We finally provide a characterization of risk aversion in our setting.

### 3.1 Axioms

We begin by assuming that the preference relation is a weak order.

**Axiom 2** *(Weak order)* $\succeq$ *is complete and transitive.*

The second axiom states that preferences are invariant to some intuitive changes: changing the outcome of an act on part of the state space that has zero probability according to all distributions in $\mathcal{P}$ does not alter preferences; relabeling the states of the world is also immaterial as far as preferences are concerned.

**Axiom 3** *(Equivalence indifference) For all* $f, g \in \mathcal{A}$, $\mathcal{P} \in \mathbb{P}$,

- $(f, \mathcal{P}) \sim (f_{S(\mathcal{P})}g, \mathcal{P})$
- *if* $\varphi$ *is an onto mapping from* $S$ *to* $S$ *such that*
  - $f$ *is* $\varphi$*-measurable,*
  - *whenever* $\left|\varphi^{-1}(s)\right| \geq 2$, $p(\varphi^{-1}(s)) = p'(\varphi^{-1}(s))$ *for all* $p, p' \in \mathcal{P}$,

  *then* $(f, \mathcal{P}) \sim (f, \mathcal{P})^{\varphi}$.

Axiom 3 implies in particular that for all constant act $f \in \mathcal{A}^c$, for all $\mathcal{P}, \mathcal{P}' \in \mathbb{P}$, $(f, \mathcal{P}) \sim (f, \mathcal{P}')$. To illustrate the second part of the axiom, take $\mathcal{P} = \{(p, \frac{1}{2} - p, \frac{p}{2}, \frac{1-p}{2}, 0, ...)|p \in [0, 1/2]\}$ and define $\varphi : S \to S$ by $\varphi(1) = \varphi(2) = 1$ and $\varphi(i) = i - 1$, $i \geq 3$. Then, the axiom states that, for any $f$ such that $f(1) = f(2)$, $(f, \mathcal{P}) \sim (f, \mathcal{P})^{\varphi}$. Note also that if $\varphi$ is a permutation of the states that leaves $\mathcal{P}$ unchanged (i.e., $\mathcal{P} = \mathcal{P}^{\varphi}$) then $(f, \mathcal{P}) \sim (f^{\varphi}, \mathcal{P})$. If, for instance, $\mathcal{P}$ is a simplex, then any permutation of the states in its support will continue to yield the simplex, and therefore $(f, \mathcal{P}) \sim (f^{\varphi}, \mathcal{P})$ for any act $f$.

Note that Axiom 3 *does not* force the decision maker to be indifferent between betting on drawing a red ball from an urn in which there are red, blue and green balls in unknown proportion and betting on red from an urn in which there are red and blue balls in unknown proportion. In particular, our model does not assume that the decision maker evaluates a couple $(f, \mathcal{P})$ only through the probability distributions induced over outcomes.[6]

The next axiom is an independence axiom in which the mixing operation bears on the sets of priors.

**Axiom 4** *(Independence) For all* $\mathcal{P}_1, \mathcal{Q}_1, \mathcal{P}_2, \mathcal{Q}_2 \in \mathbb{P}$, *and for all* $f, g \in \mathcal{A}$,

$$\left.\begin{array}{l}(f, \mathcal{P}_1) \succeq (\succ)(g, \mathcal{Q}_1) \\ (f, \mathcal{P}_2) \succeq (g, \mathcal{Q}_2)\end{array}\right\} \Rightarrow (f, \alpha\mathcal{P}_1 + (1-\alpha)\mathcal{P}_2) \succeq (\succ)(g, \alpha\mathcal{Q}_1 + (1-\alpha)\mathcal{Q}_2)$$

[6] Our model thus cannot be reduced to a model in which objects of choice are sets of lotteries.

When sets of priors are reduced to singletons this is the usual independence axiom. Its interpretation is the usual one: the set $\alpha\mathcal{P}_1 + (1-\alpha)\mathcal{P}_2$ can be seen as the outcome of a process in which nature chooses the "true" probability distribution over $S$ with probability $\alpha$ from $\mathcal{P}_1$ and $(1-\alpha)$ from $\mathcal{P}_2$.

The next axiom is a continuity axiom on acts, keeping information constant.

**Axiom 5** *(Continuity) For all $f$, $g$, $h \in \mathcal{A}$, and all $\mathcal{P} \in \mathbb{P}$, if $(f, \mathcal{P}) \succ (g, \mathcal{P}) \succ (h, \mathcal{P})$, then there exist $\alpha$ and $\beta$ in $(0, 1)$ such that :*

$$(\alpha f + (1-\alpha)h, \mathcal{P}) \succ (g, \mathcal{P}) \succ (\beta f + (1-\beta)h, \mathcal{P}).$$

Note that $\succeq$ induces a preference relation $\succeq^{\ell}_{\mathcal{P}}$ on $Y$ which is simply the restriction of $\succeq$ on $\mathcal{A}^c \times \mathcal{P}$. The next axiom states that the order $\succeq$ should be monotonic when comparing couples $(f, \mathcal{P})$ and $(g, \mathcal{P})$ in which the act-component is ranked according to the $\succeq^{\ell}_{\mathcal{P}}$ order.

**Axiom 6** *(Monotonicity) For all $f, g \in \mathcal{A}$, and all $\mathcal{P} \in \mathbb{P}$, if $f(s) \succeq^{\ell}_{\mathcal{P}} g(s)$ for all $s \in S(\mathcal{P})$, then $(f, \mathcal{P}) \succeq (g, \mathcal{P})$.*

The next axiom simply requires that no matter what the available information is, there exists a pair of acts that are not indifferent.

**Axiom 7** *(Non-degeneracy) For all $\mathcal{P} \in \mathbb{P}$, there exist $f$,$g \in \mathcal{A}$ such that $(f, \mathcal{P}) \succ (g, \mathcal{P})$.*

The next axiom is a Pareto axiom that states that if $f$ is judged better than $g$ according to any distribution $p \in \mathcal{P}$, then $f$ is judged better according to the whole set $\mathcal{P}$.

**Axiom 8** *(Pareto) For all $\mathcal{P} \in \mathbb{P}$, if for all $p \in \mathcal{P}$, we have $(f, \{p\}) \succeq (g, \{p\})$, then $(f, \mathcal{P}) \succeq (g, \mathcal{P})$.*

Our main axiom is an axiom of aversion towards imprecision. Compared to Gilboa and Schmeidler (1989)'s Uncertainty Aversion axiom and Hayashi (2003)'s Gains via Hedging axiom, ours deal with the problem in a more direct manner. According to their axiom, uncertainty aversion is revealed whenever the mixture of two indifferent acts is preferred to any of these acts. Our axiom of aversion towards imprecision directly points what kind of information the decision maker values to reduce imprecision of a set of probability distributions. This is in line with our view that aversion towards imprecision should be based on a notion of imprecision that has some content independently of the decision maker's preferences.

**Axiom 9** *(Aversion towards imprecision) For all $f \in \mathcal{A}$, $\mathcal{P} \in \mathbb{P}$, $\alpha \in [0, 1]$, and for all one-to-one function $\varphi : S \to S$ such that $\varphi(S(\mathcal{P})) \cap S(\mathcal{P}) = \emptyset$,*

$$(f, R[\mathcal{P}, \alpha, \varphi]) \succeq (f, \alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi})$$

In this axiom, the decision maker compares the same act in two different informational settings. The axiom states that he prefers acting with the information given by $R[\mathcal{P}, \alpha, \varphi]$ rather than with the information given by $\alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi}$. This suggests to define a (very partial) order on families of probability distribution as follows. Say that $\mathcal{P}'$ is more imprecise than $\mathcal{P}$ if there exist a set $\mathcal{Q}$ of probability distributions, a scalar $\alpha$ and a bijection $\varphi$ such that $\varphi(S(\mathcal{Q})) \cap S(\mathcal{Q}) = \emptyset$, such that $\mathcal{P} = \{\alpha q + (1-\alpha)q^{\varphi} | q \in \mathcal{Q}\}$ and $\mathcal{P}' = \alpha\mathcal{Q} + (1-\alpha)\mathcal{Q}^{\varphi}$.

Let us illustrate this notion by means of an example. Consider two urns made of a hundred balls, that could be black, red, white or yellow. The available information for urn I is the following: there are 50 balls that are black or white in unknown proportion, and 50 balls that are red or yellow in unknown proportion. The available information for urn II is the following: there are 50 balls that are black or white in unknown proportion, and 50 balls that are red or yellow in unknown proportion, **and** the number of black balls is the same as the number of red balls, and the number of white balls is the same as the number of yellow balls. Our partial order would assess that the urn II is more precise than urn I. Indeed, all the information available for urn I is also available for urn II, and there is some extra information for urn II. This extra information has a symmetric flavor: the ratio between black and red is the same as the ratio between white and yellow. Formally, if one encodes the color as follows: Black is state 1, White state 2, Red state 3 and Yellow state 4, the two urns can be described as follows. Let $\mathcal{Q} = \Delta(\{1,2\})$ and define $\varphi$ by $\varphi(1) = 3$, $\varphi(2) = 4$, $\varphi(3) = 1$, $\varphi(4) = 2$, and $\varphi(s) = s$, $s > 4$. Then, Urn I is described by $\mathcal{P}' = \frac{1}{2}\mathcal{Q} + \frac{1}{2}\mathcal{Q}^{\varphi}$, while Urn II is described by $\mathcal{P} = \{\frac{1}{2}q + \frac{1}{2}q^{\varphi} | q \in \mathcal{Q}\} = R[\mathcal{Q}, \frac{1}{2}, \varphi]$.

Graphically, the (projection on $p_R$ and $p_B$ of the) set of priors describing urn I is given by the square in figure 1, while it consists only of the diagonal in urn II.

Fig. 1: Possible distributions in urn I (left figure) and urn II (right figure)

This symmetry is important for the axiom, as the decision maker is required to prefer any act in the more precise situation compared to the less precise situation. On the other hand, the initial symmetry in the composition of the urn in this example is not necessary. Consider urn III in which there are 60 balls that are black or white in unknown proportion, and 40 balls that are red or yellow in unknown proportion and compare it with urn IV in which there are 60 balls that are black or white in unknown proportion, and 40 balls that are red or yellow in unknown proportion, **and** the ratio of black balls to red balls is the same as the ratio of white balls to yellow balls. Then, our partial order is able to rank these two urns and say that urn IV is more precise than urn III.

This order on sets of probability distributions is admittedly very partial. For instance singletons are not necessarily "more precise" than the description of urn I. To illustrate this point, consider the singleton set $\{(p_B, p_W, p_R, p_Y) = (2/3, 0, 1/6, 1/6)\}$; the order defined does not rank this set with respect to urn I (or any urn introduced above for that matter). From a behavioral viewpoint this makes sense as it seems obvious in that case that the decision maker will prefer to bet on white in urn I rather than on white in this urn.

Taken with the other axioms, our aversion towards imprecision has some extra implications that we will discuss in more details after the representation theorem. For instance, although we cannot deduce from that axiom alone that the decision maker would prefer any act when he is told that the composition of the urn is given by $\{(p_B, p_W, p_R, p_Y) = (1/4, 1/4, 1/4, 1/4)\}$ to the same act with urn I, this will be implied by the representation theorem. Note finally that the examples developed above can be used to test Axiom 9. According to that axiom the decision maker will prefer any bet involving urn II (resp. urn IV) to the same bet in urn I (resp. urn III). This provides an easy and direct way to test the axiom, which has a clear behavioral content.

### 3.2 Representation theorem and discussion

The following theorem provides a characterization of our set of axioms, in which the notion of uncertainty aversion is captured by aversion towards information imprecision.

**Theorem 1** *Axioms 1 to 9 hold if, and only if, there exists an unique (up to a positive linear transformation) affine function $u : Y \to \mathbb{R}$, and a unique function $\mathcal{F} : \mathbb{P} \to \mathbb{P}_C$ satisfying, for all $\mathcal{P}, \mathcal{Q} \in \mathbb{P}$:*

1. *$\mathcal{F}(\mathcal{P}) \subseteq co(\mathcal{P})$*
2. *For all onto mapping $\varphi$ from $S$ to $S$ such that $\left|\varphi^{-1}(s)\right| \geq 2$ implies $p(\varphi^{-1}(s)) = p'(\varphi^{-1}(s))$ for all $p, p' \in \mathcal{P}$, $\mathcal{F}(\mathcal{P}^{\varphi}) = (\mathcal{F}(\mathcal{P}))^{\varphi}$*
3. *For all $\alpha \in [0, 1]$, $\mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) = \alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$*
4. *For all $R[\mathcal{P}, \alpha, \varphi]$, $\mathcal{F}(R[\mathcal{P}, \alpha, \varphi]) = R[\mathcal{F}(\mathcal{P}), \alpha, \varphi]$*

*such that for all $(f, \mathcal{P}), (g, \mathcal{Q}) \in \mathcal{A} \times \mathbb{P}$, $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$ if, and only if,*

$$\min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ f dp \geq \min_{p \in \mathcal{F}(\mathcal{Q})} \int u \circ g dp.$$

The general functional form axiomatized in this theorem is of the multiple prior class, with restrictions on admissible sets of revealed priors. Conditions 1 to 4 provide the link between the revealed set of priors and prior information. The first condition states that the available information constitutes an "upper bound" as to which revealed set of priors is admissible. To illustrate the second condition take $\varphi$ to be a permutation of states in the support of $\mathcal{P}$. Then, the revealed set of priors of the permutation is the permutation of the revealed set of priors. Thus, if one starts with say the simplex on states 1, 2, and 3, the only admissible sets of revealed priors will be sets that are invariant to a permutation of states 1, 2, and 3. In particular, they have to include the point $(1/3, 1/3, 1/3)$. Actually, an implication of this condition is that, on that example, the only singleton that is admissible as a set of prior is that point of equiprobability. This is in fact a more general consequence of Condition 2. Say that $\mathcal{P}$ is symmetric if for any permutation $\varphi$ on the support of $\mathcal{P}$, $\mathcal{P} = \mathcal{P}^{\varphi}$. Let $c_{\mathcal{P}}$, the center of $\mathcal{P}$, be the probability distribution in $\mathcal{P}$ that has the property that $c_{\mathcal{P}} = c_{\mathcal{P}}^{\varphi} = c_{\mathcal{P}^{\varphi}}$ for any permutation $\varphi$ on states in the support of $\mathcal{P}$; it is the probability distribution putting weight $1/|S(\mathcal{P})|$ on any $s \in S(\mathcal{P})$ and 0 on any other state. Then, Condition 2 implies that the only singleton set that is admissible as a revealed set of priors of a symmetric $\mathcal{P}$ is its center $c_{\mathcal{P}}$. In Ellsberg three-color urn example, this condition implies that Bayesian decision makers would have beliefs $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, as intuition suggests.

More generally, consider the family $\mathbb{S}$ which is the closure of all the symmetric sets in $\mathbb{P}$ under the two operations we defined overs sets of probabilities, namely the convex combination and the $R$-operation. Note that this family includes cores of belief functions. For any symmetric sets $\mathcal{P}_1$ and $\mathcal{P}_2$ and coefficient $\alpha$, define the center of $\alpha\mathcal{P}_1 + (1-\alpha)\mathcal{P}_2$ as $c_{\alpha\mathcal{P}_1+(1-\alpha)\mathcal{P}_2} = \alpha c_{\mathcal{P}_1} + (1-\alpha)c_{\mathcal{P}_2}$. For a symmetric set $\mathcal{P}$, define the center of $R[\mathcal{P}, \alpha, \varphi]$, $c_{R[\mathcal{P},\alpha,\varphi]}$ to be equal to $\alpha c_{\mathcal{P}} + (1-\alpha)\alpha c_{\mathcal{P}^{\varphi}}$. Then, the center of a set of probability distributions made of finite convex combinations and $R$-operation over symmetric sets is well defined and Condition 2 asserts that it will always belong to the revealed set of priors. For Bayesian decision makers, this center is their subjective probabilistic beliefs. For cores of beliefs functions this is nothing but the Shapley value.

Condition 3 is a direct consequence of the independence axiom. For sets of probability distributions that can be decomposed in the convex combination of other sets, it allows to recover the revealed priors from the revealed priors of these other sets. This is useful to extend properties from "well behaved" sets, like simplices, to sets that can be decomposed in these nicely behaved sets, like cores of beliefs functions. Condition 4 also enables one to recover the revealed set of priors for sets that are linked through the $R$ operation, i.e., if $\mathcal{Q} = R[\mathcal{P}, \alpha, \varphi]$, then the revealed set of prior for $\mathcal{Q}$ can be deduced from the revealed set for $\mathcal{P}$. Condition 4 together with Condition 3 yields that $\mathcal{F}(R[\mathcal{P}, \alpha, \varphi]) \subseteq \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi})$, thus establishing that the order implicit in Axiom 9 is preserved when one looks at the revealed sets of priors.

Note finally that, if one were to assume imprecision seeking instead of imprecision aversion, then the representation theorem would be the same with the min operator replaced by the max operator. A Bayesian decision maker would also reduce any set of priors to a singleton, while an extreme imprecision seeker would evaluate an act by taking the maximum expected utility with respect to the entire set of probability distributions representing prior information.

### 3.3 Comparative imprecision aversion and risk aversion

Based on this representation theorem, one can study how imprecision and risk attitudes are captured by the revealed set of priors. For sake of simplicity and in line with the literature on risk attitude, we will restrict our definitions to Savagian type of acts, that is the set $\mathcal{A}_X$ of acts $f$ for which the outcomes $f(s)$ are degenerate lotteries for all $s$. With a slight abuse of notation, we will consider that these acts are mappings from $S$ to $X$. Furthermore, we will denote $x$ a constant act that gives consequence $x \in X$ in all states.

#### 3.3.1 Comparative imprecision aversion

For $\bar{x}$ and $\underline{x}$ two prizes in $X$ and for the event $E \subset S$, $\bar{x}_E\underline{x}$ denotes the act $f$ in $\mathcal{A}_X$ that gives $\bar{x}$ for all $s$ in $E$ and $\underline{x}$ otherwise.

**Definition 1** *Let $\succeq_a$ and $\succeq_b$ be two preference relations defined on $\mathcal{A} \times \mathbb{P}$. Suppose there exist two prizes, $\bar{x}$ and $\underline{x}$ in $X$ such that both $a$ and $b$ strictly prefer $\bar{x}$ to $\underline{x}$.*[7] *We say that $\succeq_b$ is* more averse to imprecision than $\succeq_a$ *if for all $E \subset S$, $\mathcal{P} \in \mathbb{P}$, and $\{p\} \in \mathbb{P}$,*

$$(\bar{x}_E\underline{x}, \{p\}) \succeq_a [\succ_a](\bar{x}_E\underline{x}, \mathcal{P}) \Rightarrow (\bar{x}_E\underline{x}, \{p\}) \succeq_b [\succ_b](\bar{x}_E\underline{x}, \mathcal{P})$$

That is, $b$ is *more averse to imprecision than $a$* if whenever $a$ prefers to bet on $E$ with a precise probabilistic information rather than an imprecise one, $b$ does as well. Note that this definition differs from definitions of comparative aversion to ambiguity that can be found in Ghirardato and Marinacci (2002), Epstein (1999), and subsequently in Klibanoff, Marinacci, and Mukerji (2003), or Hayashi (2003) for instance, in that we restrict attention to binary acts. This is essential to characterize this notion independently of risk attitudes, which are captured by the shape of the utility function.

**Theorem 2** *Let $\succeq_a$ and $\succeq_b$ be two preference relations defined on $\mathcal{A} \times \mathbb{P}$, satisfying all axioms of Theorem 1. Then, the following assertions are equivalent:*
*(i) $\succeq_b$ is more averse to imprecision than $\succeq_a$*
*(ii) for all $\mathcal{P} \in \mathbb{P}$, $\mathcal{F}^a(\mathcal{P}) \subset \mathcal{F}^b(\mathcal{P})$.*

An interesting feature of this notion of aversion to imprecision is that it ranks preferences that do not necessarily have the same attitudes towards risk. This is of particular interest in applications if one wants to study the effects of risk aversion and imprecision aversion separately.

[7] In the sense that $\bar{x} \succ^{\ell}_{\mathcal{P}} \underline{x}$ for the two agents and all $\mathcal{P}$.

For instance, one might want to compare portfolio choices of two agents, one being less risk averse but more imprecision averse than the other. This type of comparison cannot be done if imprecision attitudes can be compared only among preferences that have the same risk attitude, represented by the utility function. To the best of our knowledge, there is no available result in the literature that achieves this separation of the characterization of comparative ambiguity or imprecision attitudes from risk attitudes.

We end that section by characterizing further this notion of comparative imprecision aversion when the sets of probability distributions are in the set $\mathbb{S}$ defined in the previous section. First define a notion of *imprecision premium* which captures how much an agent is "willing to lose" when betting on an event in order to be in a probabilistically precise situation. More precisely, consider a preference relation $\succeq$ and let $\bar{x}$ and $\underline{x}$ be two prizes in $X$ such that $\bar{x} \succ \underline{x}$. For any event $E \subset S$, let $q$ be the probability distribution such that $(\bar{x}_E\underline{x}, \mathcal{P}) \sim (\bar{x}_E\underline{x}, \{q\})$. Under our set of axioms, such a probability distribution exists and is independent of $\bar{x}$ and $\underline{x}$, since $(\bar{x}_E\underline{x}, \mathcal{P}) \sim (\bar{x}_E\underline{x}, \{q\})$ if, and only if, $q(E) = \min_{p \in \mathcal{F}(\mathcal{P})} p(E)$. Thus, the absolute imprecision premium, $c_{\mathcal{P}}(E) - q(E)$, can be interpreted as the mass of probability on the good event $E$ that the agent is willing to forego (compared to the center of $\mathcal{P}$) in order to act on a precise information rather than on the imprecise $\mathcal{P}$. An analogy with the risk premium can be drawn as follows: $c_{\mathcal{P}}$ plays the role of the expectation of the risky prospect while $q(E)$ plays the role of the certainty equivalent. The relative imprecision premium is defined to be the quantity $c_{\mathcal{P}}(E) - q(E)$ normalized by $c_{\mathcal{P}}(E) - Min_{p \in \mathcal{P}} p(E)$.

**Definition 2** *For any $\mathcal{P} \in \mathbb{S}$ and for any event $E \subset S$ such that $c_{\mathcal{P}}(E) > 0$.*

*The absolute imprecision premium, $\pi^A(E, \mathcal{P})$ is defined by $c_{\mathcal{P}}(E) - q(E)$ where $q$ is such that $(\bar{x}_E\underline{x}, \mathcal{P}) \sim (\bar{x}_E\underline{x}, \{q\})$*

*The relative imprecision premium, $\pi^R(E, \mathcal{P})$ is defined by $\frac{\pi^A(E,\mathcal{P})}{c_{\mathcal{P}}(E) - Min_{p \in \mathcal{P}} p(E)}$.*

An imprecision averse agent always exhibits positive imprecision premia. The relative premium is equal to zero for a Bayesian agent, and to one for an extremely averse agent. Note that the definition of the imprecision premia for any sets in $\mathbb{P}$ would require to fix a benchmark probability which would be the one used by Bayesian decision makers. Theorem 1 does not allow to identify uniquely such a benchmark outside of sets in $\mathbb{S}$. Restricting our attention to $\mathbb{S}$, we can now complete the previous result:

**Theorem 3** *Let $\succeq_a$ and $\succeq_b$ be two preference relations defined on $\mathcal{A} \times \mathbb{S}$, satisfying all axioms of Theorem 1. Then, the following assertions are equivalent:*

*(i) $\succeq_b$ is more averse to imprecision than $\succeq_a$*

*(ii) for all $\mathcal{P} \in \mathbb{S}$, $\mathcal{F}^a(\mathcal{P}) \subset \mathcal{F}^b(\mathcal{P})$*

*(iii) for all $\mathcal{P} \in \mathbb{S}$, for all event $E \subset S$ such that $c_{\mathcal{P}}(E) > 0$ , $\pi^A_b(E, \mathcal{P}) \geq \pi^A_a(E, \mathcal{P})$*

### 3.3.2 Risk aversion

In this subsection, we take $X$ to be equal to $[0, M] \subset \mathbb{R}$. For any act $f \in \mathcal{A}_X$ and $\{p\} \in \mathbb{P}$, denote $E_p f$ the expected value of act $f$.

**Definition 3** *Let $\succeq$ be a preference relation defined on $\mathcal{A} \times \mathbb{P}$. Say that $\succeq$ is* risk averse *if for all $f \in \mathcal{A}_X$ and $\{p\} \in \mathbb{P}$, $(E_p f, \{p\}) \succeq (f, \{p\})$.*

In our setting, risk aversion is characterized through the restriction of preferences to situations in which the information is probabilistic (the set of probability distributions representing prior information is reduced to a singleton).

**Definition 4** *Let $\succeq_a$ and $\succeq_b$ be two preference relations defined on $\mathcal{A} \times \mathbb{P}$. Say that $\succeq_b$ is* more risk averse than *$\succeq_a$ if for all $f \in \mathcal{A}_X, x \in X$ and $\{p\} \in \mathbb{P}$,*

$$(x, \{p\}) \succeq_a [\succ_a](f, \{p\}) \Rightarrow (x, \{p\}) \succeq_b [\succ_b](f, \{p\})$$

We obtain the classical characterization:

**Theorem 4** *Let $\succeq_a$ and $\succeq_b$ be two preference relations defined on $\mathcal{A} \times \mathbb{P}$, satisfying all axioms of Theorem 1.*

*(i) $\succeq_a$ is risk averse if, and only if, $u^a$ is concave on $X = [0, M]$,*

*(ii) $\succeq_b$ is more risk averse than $\succeq_a$ if, and only if, $u^b$ is more concave than $u^a$ on $X = [0, M]$.*

Taken with Theorem 2, we thus obtain a clear cut separation of attitudes towards risk and imprecision, in which one can, for instance compare imprecision attitudes of two decision makers, one being risk seeking the other being risk averse. In our model, a decision maker is an expected utility maximizer whenever confronted to a situation of risk. Hence, there is no scope for for probabilistic risk aversion as captured say by the Rank Dependent Utility model.

# 4 Functional forms

The representation theorem we gave does not pin down a very specific functional form, as we did not establish a univocal mapping from prior information to the revealed set of priors. This general approach can be further specified to yield functional forms that are more “user friendly” for economic applications.

We start by providing here some examples of how a simplex over a finite set $N = \{1, \ldots, n\}$ could be transformed. Essentially, Theorem 1 states that, in this case, any revealed family is possible provided it is symmetric around the center of the simplex, that is the point of equiprobability. In particular, consider the family whose extreme points are all the possible permutations of the probability distribution $(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$. An act $f$ such that $f(1) \leq f(2) \leq$

$\cdots \leq f(n)$ together with the simplex is then evaluated by $\frac{1}{2}u(f(1)) + \frac{1}{2}u(f(2))$. This of course generalizes to any permutation of the probability distribution $(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}, 0, \ldots, 0)$ for any $m \leq n$, yielding that that same act $f$ is evaluated by $\sum_{i=1}^{m} \frac{1}{m}u(f(i))$. This type of functional would hence correspond to first truncating the act in its upper part (that is cutting its best consequences out) and then applying an expected utility computation with equal weights on the remaining states.

Another obvious instance of a possible way to construct the revealed set of priors when starting from a simplex is to consider the homothetic reduction (or contraction) of that simplex around the point of equiprobability. Actually, this intuition could be extended to any symmetric set of probability distribution, and more generally, to arbitrary sets in $\mathbb{S}$. We now turn in more details to this possibility and give an axiomatic foundation for a decision criterion in which the revealed set of priors is the contraction of the set of priors around its center. The general approach we take here parallels the usual approach in expected utility theory, in which specific classes of utility functions are defined by characterizing some properties of the risk premium. Thus, we specify our general functional form by imposing the following property, called constant relative imprecision premium.

**Definition 5** *An agent is said to have constant relative imprecision premium $\theta$ if for any $\mathcal{P} \in \mathbb{S}$ and for any event $E \subset S$ such that $c_{\mathcal{P}}(E) > 0$, $\pi^R(E, \mathcal{P}) = \theta$.*

**Proposition 1** *Consider an agent satisfying axioms 1 to 9. The following two assertions are equivalent:*

*(i) the agent has constant relative imprecision premium $\theta$*

*(ii) for all $\mathcal{P} \in \mathbb{S}$, $\mathcal{F}(\mathcal{P}) = \theta\mathcal{P} + (1-\theta)\{c_{\mathcal{P}}\}$*

Therefore, if an agent has constant relative imprecision premium $\theta$, then the representation theorem takes the form: $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$ if, and only if,

$$\theta \min_{p \in \mathcal{P}} \int u \circ f dp + (1-\theta) \int u \circ f dc_{\mathcal{P}} \geq \theta \min_{p \in \mathcal{Q}} \int u \circ g dp + (1-\theta) \int u \circ g dc_{\mathcal{Q}}.$$

Thus, in this setting, $\succeq_b$ is more averse towards imprecision than $\succeq_a$ if, and only if, $\theta_a \leq \theta_b$. This parametrization of imprecision aversion is hence extremely simple and convenient to do comparative static exercises in applications. This functional form was axiomatized, although in a different manner, in Gajdos, Tallon, and Vergnaud (2004) and subsequently in Hayashi (2003). The latter actually is more general in the sense that he considers any possible set of priors and not only those in $\mathbb{S}$. We could also extend our result by defining our imprecision premium for any set of prior taking the Steiner point as the benchmark probability (note that it reduces to the center for sets in $\mathbb{S}$.)

Although convenient, this functional form does not have an axiomatic justification of the same nature as the general form of Theorem 1. Relative constant imprecision premium is not an axiom that have strong normative content. It should be viewed merely as a testable property that preferences might or might not satisfy. Similarly, Hayashi (2003)'s geometric axiom is not very telling as to what behavior it is meant to capture.

To better understand what is implied by this functional form and the underlying axioms that is not implied by the previous ones, consider the following example. Take $\mathcal{P} = \Delta(\{1,2,3\})$ and consider $f$ and $g$ such that $u(f(1)) = u(g(1)) = 0$, $u(f(2)) = 1$, $u(g(2)) = 3/2$, and $u(f(3)) = 2$, $u(g(3)) = 3/2$. Under the representation of Proposition 1, one has

$$\theta \min_{\mathcal{P}} \int u \circ f dp + (1-\theta) \int u \circ f dc_{\mathcal{P}} = \theta \min_{\mathcal{P}} \int u \circ g dp + (1-\theta) \int u \circ g dc_{\mathcal{P}} = 1-\theta$$

Consider now preferences that do not satisfy constant risk premium, giving rise to the following revealed set of priors:

$$\mathcal{F}(\Delta_{123}) = co\left( (\frac{1}{2}, \frac{1}{2}, 0), (0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2}) \right)$$

According to the functional form of Theorem 1, one has $\min_{\mathcal{F}(\mathcal{P})} \int u \circ f dp = \frac{1}{2}u(f(1)) + \frac{1}{2}u(f(2)) = \frac{1}{2}$, while $g$ is strictly better than $f$ since $\min_{\mathcal{F}(\mathcal{P})} \int u(g)\, dp = \frac{3}{4}$. Note that in this case, the relative imprecision premium is not constant since $\pi^R(\{1\}, \Delta_{123}) = 1$ while $\pi^R(\{1,2\}, \Delta_{123}) = 1/4$. Both preferences seem reasonable. Hence, although the functional form has the nice feature of summarizing the attitude towards imprecision in a single parameter, the underlying axiom reflects an attitude towards imprecision that is not to be expected to hold for all decision makers.

# 5 Applications

We consider two applications of our approach. One is concerned with defining a partial ordering on sets of probability distributions on which all decision makers satisfying our set of axioms agree. Another partial ordering can be defined based on unanimity of risk averters. The second application deals with optimal risk sharing in an economy with imprecision.

## 5.1 Unanimity ranking for imprecision

In view of Theorem 1, one can define a ranking of sets of probability distributions based on unanimity of decision makers: say that $\mathcal{Q}$ is unanimously more imprecise than $\mathcal{P}$ if for all preference relations that satisfy Axioms 1 to 9 and for all act $f$, $(f, \mathcal{P}) \succeq (f, \mathcal{Q})$. This order can be characterized for specific classes of sets of probability distributions, such as the cores of belief functions $\mathbb{B}$. Let $\Sigma$ be the set of simplices with finite support on $S$ and denote generically a simplex by $\Delta_i$, then

$$\mathbb{B} = \{\mathcal{P} \in \mathbb{P} | \exists (\alpha_i)_{i=1,\dots,n} \in [0,1] \text{ and } (\Delta_i)_{i=1,\dots,n} \in \Sigma \text{ s.th. } \sum_i \alpha_i = 1, \mathcal{P} = \sum_i \alpha_i \Delta_i\}$$

The following is a direct consequence of Theorem 1: for any $\mathcal{P} \in \mathbb{B}$, $\{c_{\mathcal{P}}\}$, which is the Shapley value, is unanimously more precise than $\mathcal{P}$. This follows from the fact that the center of a simplex has to belong to the revealed set of priors associated to that simplex (by convexity of $\mathcal{F}(\Delta_i)$ and Condition 2), and therefore, $c_{\mathcal{P}} \in \mathcal{F}(\mathcal{P})$ by Condition 3. This establishes that, for all act $f$, $\int u \circ f dc_{\mathcal{P}} \geq \min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ g dp$.

Consider next sets that can be written as the combination of a simplex with its center, that is $\mathcal{P} = \alpha \Delta_i + (1-\alpha)\{c_{\Delta_i}\}$. These are sets that are contractions of the simplex around its center. By Condition 3, $\mathcal{F}(\mathcal{P}) = \alpha \mathcal{F}(\Delta_i) + (1-\alpha)\{c_{\Delta_i}\}$. Since $c_{\Delta_i} \in \mathcal{F}(\Delta_i)$, this establishes that $\mathcal{P}$ is unanimously more imprecise than $\{c_{\Delta_i}\}$ but unanimously less precise than $\Delta_i$, i.e.,

$$\int u \circ f dc_{\mathcal{P}} \geq \min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ g dp \geq \min_{p \in \mathcal{F}(\Delta_i)} \int u \circ g dp$$

Furthermore, it is possible to compare two contractions of the same simplex: if $\mathcal{P} = \alpha \Delta_i + (1-\alpha)\{c_{\Delta_i}\}$ and $\mathcal{Q} = \beta \Delta_i + (1-\beta)\{c_{\Delta_i}\}$ with $\alpha \leq \beta$, then $\mathcal{P}$ is unanimously more precise than $\mathcal{Q}$. This discussion extends to cores of beliefs functions in a straightforward manner, using Condition 3: the linear combination of contractions of simplices is unanimously more precise than the linear combination of the simplices themselves, and unanimously less precise than the Shapley value. Similarly, if two sets can be decomposed as a linear combination of the contractions of the same simplices, the contractions for the first set being larger than for the second set, then the latter is unanimously less precise. The following proposition gives a way to check whether two cores of belief functions can be compared according to that order.

**Proposition 2** *Assume unanimity is based on preferences satisfying Axioms 1 to 9. Let $\mathcal{P}, \mathcal{Q} \in \mathbb{B}$. Then, $\mathcal{P}$ is unanimously more imprecise than $\mathcal{Q}$ if, and only if,*

*(i) $c_{\mathcal{P}} = c_{\mathcal{Q}}$*

*(ii) There exists a collection $(\Delta_i)_{i=1,\ldots,n}$ in $\Sigma$, and positive coefficients $\alpha_i, \beta_i$ such that $\mathcal{P} = \sum_i \alpha_i \Delta_i$, $\mathcal{Q} = \sum_i \beta_i \Delta_i$, and for all $i$ such that $|S(\Delta_i)| \geq 2$, $\alpha_i \geq \beta_i$.*

Much of the analysis we have done in this subsection for cores of belief functions extends to more general combinations of symmetric sets of probability distributions, i.e., for sets in $\mathbb{S}$. Take a symmetric set $\mathcal{P}$. Condition 2 in Theorem 1 implies that $c_{\mathcal{P}} \in \mathcal{F}_{\mathcal{P}}$. Then, sets that can be written as $\beta \mathcal{P} + (1-\beta)\{c_{\mathcal{P}}\}$, i.e., contractions of $\mathcal{P}$ around its center, are unanimously more precise than $\mathcal{P}$ and unanimously less precise than $c_{\mathcal{P}}$. Using Condition 3 allows to generalize the analysis to any combination of symmetric sets.

We now give a characterization of unanimity ranking for decision makers that satisfy an extra property, dubbed *increasing absolute imprecision premium.*

**Definition 6** *An agent is said to have* increasing absolute imprecision premium *if for any $\mathcal{P}, \mathcal{Q} \in \mathbb{S}$ such that $c_{\mathcal{P}} = c_{\mathcal{Q}}$ and $\mathcal{Q} \subset \mathcal{P}$, for any event $E \subset S$ such that $c_{\mathcal{P}}(E) > 0$, $\pi^A(E, \mathcal{P}) \geq \pi^A(E, \mathcal{Q})$.*

An equivalent way of saying that an agent has increasing absolute imprecision premium is to say that whenever $\mathcal{P}$ and $\mathcal{Q}$ are such that $c_{\mathcal{P}} = c_{\mathcal{Q}}$ and $\mathcal{Q} \subset \mathcal{P}$, he finds $\mathcal{P}$ more imprecise than $\mathcal{Q}$.

**Proposition 3** *Assume that unanimity is based on preferences satisfying increasing absolute imprecision premium, and let $\mathcal{P}, \mathcal{Q} \in \mathbb{S}$. Then, $\mathcal{P}$ is unanimously more imprecise than $\mathcal{Q}$ if, and only if,*

*(i) $c_{\mathcal{P}} = c_{\mathcal{Q}}$*

*(ii) $\mathcal{Q} \subset \mathcal{P}$.*

## 5.2 Unanimity ranking for risk

Let us consider here $X$ to be equal to $[0, M] \subset \mathbb{R}$. In risky situations, second order stochastic dominance was proven by Rothschild and Stiglitz (1970) to be the unanimity order for risk averse expected utility maximizers. These results were extended by Scarsini (1992) for Choquet Expected Utility maximizers. He showed for instance that the classical integral condition for second order stochastic dominance must hold with respect to the Choquet integral. Inspired by these results, can we answer the following question: when can we say that act $g$ is more risky than act $f$ and hence when can we say that $f$ is unanimously preferred to $g$ by all risk averse decision makers? More formally, we will say that $(g, \mathcal{P})$ is unanimously more risky than $(f, \mathcal{P})$ if for all risk averse preference relations that satisfy Axioms 1 to 9,[8] $(f, \mathcal{P}) \succeq (g, \mathcal{P})$.

In the particular case where $\mathcal{P}$ is a simplex, we have the following result:

**Proposition 4** *Assume unanimity is based on preferences satisfying Axioms 1 to 9. Let $\Delta \in \Sigma$. Then, $(g, \Delta)$ is unanimously more risky than $(f, \Delta)$ if, and only if, act $f$ second order stochastically dominates $g$ with respect to the probability distribution $c_\Delta$, that is, for all $t \in [0, M]$, $\int_t^M c_\Delta(\{s | f(s) \geqslant x\})dx \geq \int_t^M c_\Delta(\{s | g(s) \geqslant x\})dx$.*

For the particular case of simplices, one has just to check whether second order stochastic dominance holds for the equiprobability distribution. For more general information, the characterization is not so simple. Below, we give a characterization restricting our attention to the particular class of preferences satisfying the supplementary property of constant relative imprecision premium.

**Proposition 5** *Let $\mathcal{P} \in \mathbb{B}$. Assume that unanimity is based on preferences satisfying Axioms 1 to 9 and constant relative imprecision premium. Then, $(g, \mathcal{P})$ is unanimously more risky than $(f, \mathcal{P})$ if, and only if,*

*(i) $f$ second order stochastically dominates $g$ with respect to the probability distribution $c_{\mathcal{P}}$*

[8] We also assume that the preference are increasing in $X$, that is, agents prefers the certainty of a bigger $x$.

*(ii) $f$ second order stochastically dominates $g$ with respect to the family $\mathcal{P}$, that is, $\forall t \in [0, M]$,*

$$\min_{p\in\mathcal{P}} \int_t^M p(\{s|f(s) \geqslant x\})dx \geq \min_{p\in\mathcal{P}} \int_t^M p(\{s|g(s) \geqslant x\})dx$$

### 5.3 Optimal risk sharing

We study, in a specific economy, how risk sharing is affected by the presence of imprecision when agents behave according to the criterion axiomatized in the previous section. Consider an economy made of $H$ agents, $h = 1, \ldots, H$ and one good. There are $n$ states, $s \in N = \{1, \ldots, n\}$. There is no available information, i.e., $\mathcal{P} = \Delta_N$. Denote $C_h^s$ household $h$'s consumption in state $s$, and $C_h = (C_h^1, \ldots, C_h^n)$. Household $h$ has preferences represented by $V_h(C_h) = \theta_h \min_s u_h(C_h^s) + (1-\theta_h)E_\pi u_h(\widetilde{C}_h)$, where $\theta_h \in (0,1)$ is $h$'s aversion towards imprecision, $p$ is the equiprobable distribution on $N$, i.e., $p = (1/n, \ldots, 1/n)$, $u_h$ is $h$'s utility function, that is assumed smooth, strictly increasing and strictly concave. Let $e^s$ denote aggregate endowment in state $s$ and assume that $e^1 < e^2 < \cdots < e^n$.

**Proposition 6** *Any Pareto optimal allocation of the economy described is comonotone, that is, $C_h^1 \leq C_h^2 \leq \cdots \leq C_h^n$ for all $h = 1, \ldots, H$.*

Note the result does not fall into known results in the literature. In particular, it is not a special case of Chateauneuf, Dana, and Tallon (2000) since here, the risk-sharing analysis is done in a model which amounts to a (particular and well structured case of) model with maxmin expected utility maximizers with different sets of revealed priors. To the best of our knowledge there is no available results for this class of economies.[9] In a sense, the result of Proposition 6 provides a natural extension of what is known in the von Neumann-Morgenstern case: when there is a given probability, Pareto optimal allocations are comonotone, regardless of the degrees of risk aversion of the agents. Here, when the same information is known to all, Pareto optimal allocations are also comonotone, regardless of the degrees of imprecision aversion of the agents. The result however does not extend straightforwardly if one moves away from total absence of information (i.e. if the information is not represented by the simplex).

We pursue the analysis of risk sharing in this economy by looking at only two agents with CARA utility functions. Thus, $u_h(C_h) = \frac{e^{-a_h C_h}}{-a_h}$ where $a_h$ is the constant degree of risk aversion of agent $h = 1, 2$. The Pareto optimal allocations in this case are the following, where $\lambda$ is a parameter in $(0,1)$ giving the weight put on agent 1 and $K = log\frac{1-\lambda}{\lambda}$:

$$\begin{cases} C_1^1 = \frac{1}{a_1+a_2}\left[a_2 e^1 - K - log\frac{\theta_2+\frac{1-\theta_2}{S}}{\theta_1+\frac{1-\theta_1}{S}}\right] & C_2^1 = \frac{1}{a_1+a_2}\left[a_1 e^1 + K + log\frac{\theta_2+\frac{1-\theta_2}{S}}{\theta_1+\frac{1-\theta_1}{S}}\right] \\ C_1^s = \frac{1}{a_1+a_2}\left[a_2 e^s - K - log\frac{1-\theta_2}{1-\theta_1}\right] & C_2^s = \frac{1}{a_1+a_2}\left[a_1 e^s + K + log\frac{1-\theta_2}{1-\theta_1}\right] \quad s = 2, \ldots, n \end{cases}$$

[9] Wakai (2004) shows that Pareto optimal allocations are comonotone when agents have homogeneous multiple priors, when their utility function is of the HARA type.

If $\theta_1 = \theta_2$, risk sharing proceeds as in the von Neumann Morgenstern case. For instance, if one looks at the egalitarian case, $\lambda = 1/2$ and $K = 0$, the optimal risk sharing arrangement is to split the aggregate endowment state by state according to a fixed rule; give $a_2/(a_1 + a_2)$ percent to agent 1 and $a_1/(a_1 + a_2)$ percent to agent 2. In presence of differential imprecision aversion, this is not the optimal arrangement any longer. Assume for instance $\theta_1 > \theta_2$, i.e., agent 1 is strictly more imprecision averse than agent 2. In this case, $log\frac{1-\theta_2}{1-\theta_1} > 0$ while $log\frac{\theta_2+\frac{1-\theta_2}{S}}{\theta_1+\frac{1-\theta_1}{S}} < 0$. Then, agent 1's consumption is higher (compared to the vNM case) in state 1, but lower in all the other states. It is as if 1 were subscribing a contract, with premium $log\frac{1-\theta_2}{1-\theta_1}$, that pays off only if state 1 occurs (it then pays $log\frac{1-\theta_2}{1-\theta_1} - log\frac{\theta_2+\frac{1-\theta_2}{S}}{\theta_1+\frac{1-\theta_1}{S}}$). The less imprecise averse agent is providing some insurance against the worst possible realization. This "contract" is of course very much dependent of the fact that there is no available information; the example simply illustrating that new optimal risk sharing arrangement can emerge in our setting. Note however, that this risk sharing arrangement could not be reproduced in the von Neumann Morgenstern nor in the Rank Dependent Utility setting (unless one assumes different beliefs to begin with). A more thorough investigation of risk sharing in our setting is left for further research.

## Appendix: proofs

We start with an extension of the multiple prior model of Gilboa and Schmeidler (1989) taking into account the information given to the decision maker (Theorem 5 below) and then use it to prove Theorem 1. Gilboa and Schmeidler (1989) have six axioms, four of which are already included in our construction (Axioms 2, 5, 6, and 7). The other two are:

**Axiom 10** *(Certainty-Independence) For all $f, g \in \mathcal{A}$, $h \in \mathcal{A}^c$, $\mathcal{P} \in \mathbb{P}$, $\alpha \in ]0,1[$,*

$$(f, \mathcal{P}) \succ (g, \mathcal{P}) \Leftrightarrow (\alpha f + (1-\alpha)h, \mathcal{P}) \succ (\alpha g + (1-\alpha)h, \mathcal{P})$$

**Axiom 11** *(Uncertainty aversion) For all $f, g \in \mathcal{A}$, $\mathcal{P} \in \mathbb{P}$, and all $\alpha \in ]0,1[$,*

$$(f, \mathcal{P}) \sim (g, \mathcal{P}) \Rightarrow (\alpha f + (1-\alpha)g, \mathcal{P}) \succeq (f, \mathcal{P})$$

Gilboa and Schmeidler have proved that Axioms 2, 5, 6, 7, 10, 11 hold if, and only if, for all $\mathcal{P} \in \mathbb{P}$ there exist an unique (up to a positive linear transformation) affine function $u_{\mathcal{P}} : Y \to \mathbb{R}$, and an unique, non-empty, closed and convex set $\mathcal{F}(\mathcal{P})$ of probability measures on $2^S$, such that for all $f, g \in \mathcal{A}$, $(f, \mathcal{P}) \succeq (g, \mathcal{P})$ if and only if: $\min_{p \in \mathcal{F}(\mathcal{P})} \int u_{\mathcal{P}} \circ f dp \geq \min_{p \in \mathcal{F}(\mathcal{P})} \int u_{\mathcal{P}} \circ g dp$. In the next theorem, we extend the representation for variable $\mathcal{P}$.

**Theorem 5** *Axioms 2, 3, 5, 6, 7, 10, and 11 hold iff there exists an unique (up to a positive linear transformation) affine function $u : Y \to \mathbb{R}$, and a unique function $\mathcal{F} : \mathbb{P} \to \mathbb{P}_C$ satisfying, for all $\mathcal{P} \in \mathbb{P}$:*

1. *$\forall p \in \mathcal{F}(\mathcal{P})$, $p(S(\mathcal{P})) = 1$,*
2. *For all onto mapping $\varphi$ from $S$ to $S$ such that $\left|\varphi^{-1}(s)\right| \geq 2$ implies that $p(\varphi^{-1}(s)) = p'(\varphi^{-1}(s))$ for all $p, p' \in \mathcal{P}$, $\mathcal{F}(\mathcal{P}^{\varphi}) = (\mathcal{F}(\mathcal{P}))^{\varphi}$,*

*such that $\forall (f, \mathcal{P}), (g, \mathcal{P}) \in \mathcal{A} \times \mathbb{P}$, $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$ iff $\min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ f dp \geq \min_{p \in \mathcal{F}(\mathcal{Q})} \int u \circ g dp$.*

**Proof.** **[Theorem 5]** The necessity part of the theorem is straightforward to verify. We only prove sufficiency. Let $\mathcal{P}, \mathcal{Q} \in \mathbb{P}$. Gilboa and Schmeidler's theorem implies that the decision maker is an expected utility maximizer over constant acts. Axiom 3 implies that $u_{\mathcal{P}}$ and $u_{\mathcal{Q}}$ represent the same expected utility over constant acts (which implies that $\succeq^{\ell}_{\mathcal{P}} = \succeq^{\ell}_{\mathcal{Q}} = \succeq^{\ell}$). Hence, they can be taken so that $u_{\mathcal{P}} = u_{\mathcal{Q}} = u$.

To show that the representation can be extended to compare acts associated to different sets $\mathcal{P}$, let $(f, \mathcal{P}) \succeq (g, \mathcal{Q})$. Since $S(\mathcal{P})$ and $S(\mathcal{Q})$ are finite and $f(s)$ and $g(s)$ have finite support, there exist $\overline{x}$ and $\underline{x}$ in $X$ such that for all $s \in S(\mathcal{P}) \cup S(\mathcal{Q})$, for all $x \in Supp(f(s)) \cup Supp(g(s))$, $\delta_{\overline{x}} \succeq^{\ell} \delta_x \succeq^{\ell} \delta_{\underline{x}}$. Hence, by Axioms 3 and 6 we know that $(k_{\overline{x}}, \mathcal{P}) \succeq (f, \mathcal{P}) \succeq \left(k_{\underline{x}}, \mathcal{P}\right)$ and $(k_{\overline{x}}, \mathcal{Q}) \succeq (g, \mathcal{Q}) \succeq \left(k_{\underline{x}}, \mathcal{Q}\right)$ where $k_{\overline{x}}$ (resp. $k_{\underline{x}}$) is the constant act giving $\delta_{\overline{x}}$ (resp. $\delta_{\underline{x}}$) in all states. By Axioms 2 and 5, there exists $\lambda$ such that $(f,\mathcal{P}) \sim (\lambda k_{\overline{x}} + (1-\lambda)k_{\underline{x}}, \mathcal{P})$. Similarly, there exists $\mu$ such that $(g,\mathcal{Q}) \sim (\mu k_{\overline{x}} + (1-\mu)k_{\underline{x}}, \mathcal{Q})$. Thus,

$$\begin{aligned}(f,\mathcal{P}) \succeq (g,\mathcal{Q}) &\Leftrightarrow (\lambda k_{\overline{x}} + (1-\lambda)k_{\underline{x}}, \mathcal{P}) \succeq (\mu k_{\overline{x}} + (1-\mu)k_{\underline{x}}, \mathcal{Q}) \\ &\Leftrightarrow \lambda k_{\overline{x}} + (1-\lambda)k_{\underline{x}} \succeq^{\ell} \mu k_{\overline{x}} + (1-\mu)k_{\underline{x}}\end{aligned}$$

Now, $(f,\mathcal{P}) \sim (\lambda k_{\overline{x}}+(1-\lambda)k_{\underline{x}}, \mathcal{P})$ implies that $\min_{p\in\mathcal{F}(\mathcal{P})} \int u\circ f dp = u(\lambda\delta_{\overline{x}}+(1-\lambda)\delta_{\underline{x}})$. We also have that $\min_{p\in\mathcal{F}(\mathcal{Q})} \int u\circ g dp = u(\mu\delta_{\overline{x}}+(1-\mu)\delta_{\underline{x}})$ and $u(\lambda\delta_{\overline{x}}+(1-\lambda)\delta_{\underline{x}}) \geq u(\mu\delta_{\overline{x}}+(1-\mu)\delta_{\underline{x}})$, which implies that

$$\min_{p\in\mathcal{F}(\mathcal{P})} \int u\circ f dp \geq \min_{p\in\mathcal{F}(\mathcal{Q})} \int u\circ g dp$$

We now turn to Condition 1. Let $p^* \in \mathcal{F}(\mathcal{P})$ and suppose that $p^*(S(\mathcal{P})) = q \neq 1$. Consider $\overline{x}$ and $\underline{x}$ in $X$ such that $u(\delta_{\overline{x}}) > u(\delta_{\underline{x}})$ and let $f$ be defined by $f(s) = \delta_{\overline{x}}$ for all $s \in S(\mathcal{P})$, $f(s) = \delta_{\underline{x}}$ for all $s \in S\backslash S(\mathcal{P})$, and $g$ by $g(s) = \delta_{\overline{x}}$ for all $s \in S$. Then,

$$\min_{p\in\mathcal{F}(\mathcal{P})} \int u\circ f dp \leq \int u\circ f dp^* = qu(\overline{x}) + (1-q)u(\underline{x}) < u(\overline{x}) = \min_{p\in\mathcal{F}(\mathcal{P})} \int u\circ g dp$$

Hence, $(g,\mathcal{P}) \succ (f,\mathcal{P})$, a violation of Axiom 3 since $g = f_{S(\mathcal{P})}g$. Thus, $\forall p \in \mathcal{F}(\mathcal{P}),\ p(S(\mathcal{P})) = 1$.

We finally prove Condition 2. Let $\mathcal{P} \in \mathbb{P}$ and $\varphi$ be an onto mapping from $S$ to $S$ such that $\left|\varphi^{-1}(s)\right| \geq 2$ implies $p(\varphi^{-1}(s)) = p'(\varphi^{-1}(s))$ for all $p, p' \in \mathcal{P}$.

We first prove that $\mathcal{F}(\mathcal{P}^{\varphi}) \subseteq (\mathcal{F}(\mathcal{P}))^{\varphi}$. Assume there exists $p^* \in \mathcal{F}(\mathcal{P}^{\varphi})$ such that $p^* \notin (\mathcal{F}(\mathcal{P}))^{\varphi}$. Since $\mathcal{F}(\mathcal{P})$ is a convex set, $(\mathcal{F}(\mathcal{P}))^{\varphi}$ is also convex. Hence, using a separation argument, we know that there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \min_{p\in(\mathcal{F}(\mathcal{P}))^{\varphi}} \int \phi dp$. Since $S(\mathcal{P}^{\varphi})$ is a finite set, there exist numbers $a, b$ with $a > 0$, such that $\forall s \in S(\mathcal{P}^{\varphi})$, $(a\phi(s) + b) \in u(Y)$. Then, for all $s \in S(\mathcal{P}^{\varphi})$ there exists $y(s) \in Y$ such that $u(y(s)) = a\phi(s)+b$. Define $f$ by $f(s) = y(s)$ for all $s \in S(\mathcal{P}^{\varphi})$, $f(s) = \delta_x$ for all $s \in S\backslash S(\mathcal{P}^{\varphi})$, where $x \in X$. Define $g$ by $g^{\varphi} = f$. Since for all $p \in \mathcal{F}(\mathcal{P})$, $\int u\circ g dp = \int u\circ g^{\varphi} dp^{\varphi}$, we have:

$$\min_{p\in\mathcal{F}(\mathcal{P})} \int u\circ g dp = \min_{p\in(\mathcal{F}(\mathcal{P}))^{\varphi}} \int u\circ g^{\varphi} dp = \min_{p\in(\mathcal{F}(\mathcal{P}))^{\varphi}} \int u\circ f dp$$

Condition 1 implies that $\min_{p\in(\mathcal{F}(\mathcal{P}))^{\varphi}} \int u\circ f dp = \min_{p\in(\mathcal{F}(\mathcal{P}))^{\varphi}} \int (a\phi + b)dp$. But:

$$\min_{p\in(\mathcal{F}(\mathcal{P}))^{\varphi}} \int (a\phi + b)dp > \int (a\phi + b)dp^* \geq \min_{p\in\mathcal{F}(\mathcal{P}^{\varphi})} \int u\circ f dp$$

and therefore $(g,\mathcal{P}) \succ (f,\mathcal{P}^{\varphi})$ which is a violation of Axiom 3.

A similar argument may be used to prove that $\mathcal{F}(\mathcal{P}^{\varphi}) \supseteq (\mathcal{F}(\mathcal{P}))^{\varphi}$. ■

**Lemma 1** *Assume that Axioms 3, 4, 5 and 6 hold. Let $\mathcal{P} \in \mathbb{P}$, $f, g \in \mathcal{A}$ and $\varphi$ a bijection from $S$ to $S$ such that $\varphi(S(\mathcal{P}) \cap S(\mathcal{P}) = \emptyset$. Then,*

$$\forall \alpha \in [0,1],\quad (f,\mathcal{P}) \sim (g,\mathcal{P}) \Rightarrow (f,\mathcal{P}) \sim (g,\mathcal{P}) \sim \left(f_{S(\mathcal{P})}g^{\varphi}, \alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi}\right)$$

**Proof. [Lemma 1]** Let $(f,\mathcal{P})$ and $(g,\mathcal{P})$ be given, such that $(f,\mathcal{P})\sim(g,\mathcal{P})$. Let $\varphi$ be a bijective function from $S$ to $S$ such that $(\varphi(S(\mathcal{P})))\cap S(\mathcal{P})=\emptyset$. By Axiom 3, since $\varphi$ is a bijection, $(g^{\varphi},\mathcal{P}^{\varphi})\sim(g,\mathcal{P})$. Therefore, applying Axiom 3 again, we have: $(f_{S(\mathcal{P})}g^{\varphi},\mathcal{P}^{\varphi})\sim(g,\mathcal{P})\sim(f,\mathcal{P})$. On the other hand, we also have (still applying Axiom 3) $(f_{S(\mathcal{P})}g^{\varphi},\mathcal{P})\sim(f,\mathcal{P})$. Therefore:

$$\begin{cases}(f_{S(\mathcal{P})}g^{\varphi},\mathcal{P}^{\varphi})\sim(f,\mathcal{P})\\(f_{S(\mathcal{P})}g^{\varphi},\mathcal{P})\sim(f,\mathcal{P})\end{cases}$$

Therefore Axiom 4 implies for all $\alpha\in[0,1]$, $(f_{S(\mathcal{P})}g^{\varphi},\alpha\mathcal{P}+(1-\alpha)\mathcal{P}^{\varphi})\sim(f,\alpha\mathcal{P}+(1-\alpha)\mathcal{P})$. Finally, since $S(\mathcal{P})$ is finite and $f(s)$ has finite support, there exist $\overline{x}$ and $\underline{x}$ in $X$ such that for all $s\in S(\mathcal{P}))$, for all $x\in Supp(f(s))$, $\delta_{\overline{x}}\succeq^{\ell}\delta_x\succeq^{\ell}\delta_{\underline{x}}$. Hence, by Axioms 3 and 6 we know that $(k_{\overline{x}},\mathcal{P})\succeq(f,\mathcal{P})\succeq\left(k_{\underline{x}},\mathcal{P}\right)$ where $k_{\overline{x}}$ (resp. $k_{\underline{x}}$) is the constant act giving $\delta_{\overline{x}}$ (resp. $\delta_{\underline{x}}$) in all states. By Axioms 2 and 5, there exists $\lambda$ such that $(f,\mathcal{P})\sim(\lambda k_{\overline{x}}+(1-\lambda)k_{\underline{x}},\mathcal{P})$. Furthermore, by Axiom 3, $(\lambda k_{\overline{x}}+(1-\lambda)k_{\underline{x}},\mathcal{P})\sim(\lambda k_{\overline{x}}+(1-\lambda)k_{\underline{x}},\{p\})$ for all $p\in\mathcal{P}$. Therefore, by Axiom 4, $(f,\alpha\mathcal{P}+(1-\alpha)\mathcal{P})\sim(\lambda k_{\overline{x}}+(1-\lambda)k_{\underline{x}},\alpha\{p\}+(1-\alpha)\{p\})\sim(\lambda k_{\overline{x}}+(1-\lambda)k_{\underline{x}},\{p\})$, from which it follows that $(f,\mathcal{P})\sim(f,\alpha\mathcal{P}+(1-\alpha)\mathcal{P})$. Hence, $(f_{S(\mathcal{P})}g^{\varphi},\alpha\mathcal{P}+(1-\alpha)\mathcal{P}^{\varphi})\sim(f,\mathcal{P})$, the desired result. ■

**Lemma 2** *Axioms 1, 3, 4, 5, 6 and 9 imply Axioms 10 and 11.*

**Proof. [Lemma 2]** We first check Axiom 10. Let $(f,\mathcal{P})\succ(g,\mathcal{P})$ and $h\in\mathcal{A}^c$ (i.e., $h$ is a constant act). Let $\varphi:S\to S$ be a bijection such that $S(\mathcal{P}^{\varphi})\cap S(\mathcal{P})=\emptyset$ and $\psi:S\to S$ be an onto mapping such that $\psi(s)=s$ for all $s\in S(\mathcal{P})$, and $(\psi\circ\varphi)(S(\mathcal{P}))=\{s^*\}$, with $s^*\in S\setminus S(\mathcal{P})$. Finally, let $p^*$ be the probability distribution defined by $p^*(s^*)=1$.

By Axiom 3, $(\alpha f+(1-\alpha)h,\mathcal{P})\sim(\alpha f_{S(\mathcal{P})}h+(1-\alpha)h,\mathcal{P})$. Since $(\alpha f_{S(\mathcal{P})}h)(s)=h(s)$ for all $s\in S\setminus S(\mathcal{P})$, Axiom 1 implies:

$$(\alpha f_{S(\mathcal{P})}h+(1-\alpha)h,\mathcal{P})\sim\left(f_{S(\mathcal{P})}h^{\varphi},R[\mathcal{P},\alpha,\varphi]\right)$$

Therefore, $(\alpha f+(1-\alpha)h,\mathcal{P})\sim\left(f_{S(\mathcal{P})}h^{\varphi},R[\mathcal{P},\alpha,\varphi]\right)$

Since $f_{S(\mathcal{P})}h^{\varphi}$ is $\psi-$measurable, Axiom 3 implies:

$$\left(f_{S(\mathcal{P})}h^{\varphi},R[\mathcal{P},\alpha,\varphi]\right)\sim\left(f_{S(\mathcal{P})}h^{\varphi},R[\mathcal{P},\alpha,\varphi]\right)^{\psi}$$

and therefore:

$$(\alpha f+(1-\alpha)h,\mathcal{P})\sim\left(f_{S(\mathcal{P})}h^{\varphi},R[\mathcal{P},\alpha,\varphi]\right)^{\psi}\tag{1}$$

The same reasoning holds with $g$ instead of $f$, and therefore we also have:

$$(\alpha g+(1-\alpha)h,\mathcal{P})\sim\left(g_{S(\mathcal{P})}h^{\varphi},R[\mathcal{P},\alpha,\varphi]\right)^{\psi}\tag{2}$$

On the other hand, by Axiom 3,

$$(f,\mathcal{P})\sim\left((f_{S(\mathcal{P})}h^{\varphi})^{\psi},\mathcal{P}\right)\text{ and }(g,\mathcal{P})\sim\left((g_{S(\mathcal{P})}h^{\varphi})^{\psi},\mathcal{P}\right)$$

Therefore, $(f,\mathcal{P}) \succ (g,\mathcal{P})$ if and only if $\left((f_{S(\mathcal{P})}h^{\varphi})^{\psi},\mathcal{P}\right) \succ \left((g_{S(\mathcal{P})}h^{\varphi})^{\psi},\mathcal{P}\right)$
Since $\left(f_{S(\mathcal{P})}h\right)^{\psi}(s) = \left(g_{S(\mathcal{P})}h^{\varphi}\right)^{\psi}(s)$ for all $s \in S \setminus S(\mathcal{P})$ and $s^* \in S \setminus S(\mathcal{P})$, Axiom 4 implies that $(f,\mathcal{P}) \succ (g,\mathcal{P})$ if and only if:

$$\left((f_{S(\mathcal{P})}h^{\varphi})^{\psi}, \alpha\mathcal{P} + (1-\alpha)\{p^*\}\right) \succ \left((g_{S(\mathcal{P})}h^{\varphi})^{\psi}, \alpha\mathcal{P} + (1-\alpha)\{p^*\}\right) \tag{3}$$

But observe that $(R[\mathcal{P},\alpha,\varphi])^{\psi} = \alpha\mathcal{P} + (1-\alpha)\{p^*\}$. Therefore, equation (3) is equivalent to:

$$\left(f_{S(\mathcal{P})}h^{\varphi}, R[\mathcal{P},\alpha,\varphi]\right)^{\psi} \succ \left(g_{S(\mathcal{P})}h^{\varphi}, R[\mathcal{P},\alpha,\varphi]\right)^{\psi} \tag{4}$$

Finally, substituting equations (1) and (2) in (4), we obtain that $(f,\mathcal{P}) \succ (g,\mathcal{P})$ if and only if:

$$(\alpha f + (1-\alpha)h, \mathcal{P}) \succ (\alpha g + (1-\alpha)h, \mathcal{P})$$

thus proving Axiom 10.

We now check that Axiom 11 holds as well. Let $(f,\mathcal{P})$ and $(g,\mathcal{P})$ be given, such that $(f,\mathcal{P}) \sim (g,\mathcal{P})$. According to Lemma 1, we have that

$$(f,\mathcal{P}) \sim (g,\mathcal{P}) \sim \left(f_{S(\mathcal{P})}g^{\varphi}, \alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi}\right)$$

Axiom 9 implies that $\left(f_{S(\mathcal{P})}g^{\varphi}, R[\mathcal{P},\alpha,\varphi]\right) \succeq \left(f_{S(\mathcal{P})}g^{\varphi}, \alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi}\right)$
Since by Axiom 1, we have that $(\alpha f + (1-\alpha)g, \mathcal{P}) \sim \left(f_{S(\mathcal{P})}g^{\varphi}, R[\mathcal{P},\alpha,\varphi]\right)$, it follows that

$$(\alpha f + (1-\alpha)g, \mathcal{P}) \succeq (f,\mathcal{P}) \sim (g,\mathcal{P})$$

Hence, Axiom 11 is satisfied. ■

We are now in a position to prove Theorem 1.

**Proof. [Theorem 1]**
**1. Sufficiency**. By Lemma 2, we know that Axioms 1, 3, 4, 5, 6 and 9 imply Axioms 10 and 11. Hence, we can invoke Theorem 5 to prove that there exists an unique (up to a positive affine transformation) affine function $u : Y \to \mathbb{R}$, and for a unique function $\mathcal{F} : \mathbb{P} \to \mathbb{P}_C$ such that for all $(f,\mathcal{P}),(g,\mathcal{Q}) \in \mathcal{A} \times \mathbb{P}$, $(f,\mathcal{P}) \succeq (g,\mathcal{Q})$ if, and only if:

$$\min_{p\in\mathcal{F}(\mathcal{P})} \int u \circ f dp \geq \min_{p\in\mathcal{F}(\mathcal{Q})} \int u \circ g dp$$

Furthermore, for all $\varphi$ onto mapping from $S$ to $S$, $\mathcal{F}(\mathcal{P}_i^{\varphi}) = (\mathcal{F}(\mathcal{P}))^{\varphi}$. We will now show that the axioms imply Conditions 1 to 4 of the theorem.

<u>Condition 1</u>: Suppose Condition 1 does not hold, that is there exists $\mathcal{P} \in \mathbb{P}$ such that $\mathcal{F}(\mathcal{P}) \nsubseteq co(\mathcal{P})$. Then, there exists $p^* \in \mathcal{F}(\mathcal{P})$ such that $p^* \notin co(\mathcal{P})$. Since $co(\mathcal{P})$ is a convex set, using a separation argument, we know there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \min_{p\in co(\mathcal{P})} \int \phi dp$. Note that since Axiom 8 implies that $(f,\mathcal{P}) \sim (f_{S(\mathcal{P})}g,\mathcal{P})$, we have that

Condition 1 in Theorem 5 holds and thus, for all $p \in \mathcal{F}(\mathcal{P})$, $p(S(\mathcal{P})) = 1$. Thus $Supp(p^*) \subseteq S(\mathcal{P})$ and since $S(\mathcal{P})$ is a finite set, there exist numbers $a, b$ with $a > 0$, such that $\forall s \in S(\mathcal{P})$, $(a\phi(s) + b) \in u(Y)$. Then, for all $s \in S(\mathcal{P})$ there exists $y(s) \in Y$ such that $u(y(s)) = a\phi(s) + b$. Define $f$ by $f(s) = y(s)$ for all $s \in S(\mathcal{P})$, $f(s) = \delta_x$ for all $s \in S \backslash S(\mathcal{P})$ where $x \in X$. Note that $\min_{p \in co(\mathcal{P})} \int (a\phi + b)dp \in Y$ and thus there exists $y^*$ such that $u(y^*) = \min_{p \in co(\mathcal{P})} \int (a\phi + b)dp$. Define $g$ by $g(s) = y^*$ for all $s \in S$. Observe that for all $p \in \Delta(S)$ such that $p \in \mathcal{P}$, $p \in co\,(\mathcal{P})$ and thus

$$\int u \circ f dp \geq \min_{p \in co(\mathcal{P})} \int u \circ f dp = \min_{p \in co(\mathcal{P})} \int (a\phi + b)dp = u(y^*) = \int u \circ g dp$$

So for all $p \in \Delta(S)$ such that $p \in \mathcal{P}$, $(f, \{p\}) \succeq (g, \{p\})$. Yet

$$\min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ f dp \leq \int u \circ f dp^* = \int (a\phi + b)dp < \min_{p \in co(\mathcal{P})} \int (a\phi + b)dp = u(y^*) = \min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ g dp$$

and thus $(f, \mathcal{P}) \prec (g, \mathcal{P})$ which is a violation of Axiom 8.

<u>Condition 2</u> was proved in Theorem 5.

<u>Condition 3</u>: Consider $\mathcal{P}, \mathcal{Q} \in \mathbb{P}$ and $\alpha \in [0, 1]$.

*Step 1.* $\mathcal{F}(\alpha\mathcal{P} + (1 - \alpha)\mathcal{Q}) \supseteq \alpha\mathcal{F}(\mathcal{P}) + (1 - \alpha)\mathcal{F}(\mathcal{Q})$

Suppose that there exist $p^* \in \mathcal{F}(\mathcal{P})$ and $q^* \in \mathcal{F}(\mathcal{Q})$ such that $r^* = \alpha p^* + (1 - \alpha)q^* \notin \mathcal{F}(\alpha\mathcal{P} + (1 - \alpha)\mathcal{Q})$. Since $\mathcal{F}(\alpha\mathcal{P} + (1 - \alpha)\mathcal{Q})$ is a convex set, using a separation argument, we know there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dr^* < \min_{p \in \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q})} \int \phi dp$. Since $S(\mathcal{P})$ and $S(\mathcal{Q})$ are finite sets, there exist numbers $a, b$ with $a > 0$, such that $\forall s \in S(\mathcal{P}) \cup S(\mathcal{Q})$, $(a\phi(s) + b) \in u(Y)$.[10] Then, for all $s \in S(\mathcal{P}) \cup S(\mathcal{Q})$ there exists $y(s) \in Y$ such that $u(y(s)) = a\phi(s) + b$. Define $f$ by $f(s) = y(s)$ for all $s \in S(\mathcal{P}) \cup S(\mathcal{Q})$, $f(s) = \delta_x$ for all $s \in S \setminus (S(\mathcal{P}) \cup S(\mathcal{Q}))$, where $x \in X$. Since for all $p \in \mathcal{F}(\alpha\mathcal{P} + (1 - \alpha)\mathcal{Q})$, $p\,(S(\alpha\mathcal{P} + (1 - \alpha)\mathcal{Q})) = p\,(S(\mathcal{P}) \cup S(\mathcal{Q})) = 1$,

$$\begin{aligned} \min_{p \in \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q})} \int u \circ f dp &= \min_{p \in \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q})} \int (a\phi + b)dp \\ &> \int (a\phi + b)dr^* = \alpha \int u \circ f dp^* + (1 - \alpha) \int u \circ f dq^* \end{aligned} \tag{5}$$

Since $\int u \circ f dp^* \in u(Y)$ and $\int u \circ f dq^* \in u(Y)$ there exist $y_1, y_2 \in Y$ such that $u(y_1) = \int u \circ f dp^*$ and $u(y_2) = \int u \circ f dq^*$. Let $\varphi : S \to S$ a bijective mapping such that $\varphi(S(\mathcal{P}) \cap S(\mathcal{Q})) = \emptyset$. Define $g$ by $g(s) = y_1$ for all $s \in \varphi(S(\mathcal{P}))$, $g(s) = y_2$ for all $s \in S(\mathcal{Q})$ and $g(s) = \delta_x$ for all $s \in S \setminus (S(\mathcal{P}) \cup \varphi(S(\mathcal{Q})))$, with $x \in X$. We have:

$$\min_{p \in \mathcal{F}(\mathcal{P}^\varphi)} \int u \circ g dp = u(y_1) = \int u \circ f dp^* \geq \min_{p \in \mathcal{F}(\mathcal{Q})} \int u \circ f dp$$

and

$$\min_{p \in \mathcal{F}(\mathcal{Q})} \int u \circ g dp = u(y_2) = \int u \circ f dq^* \geq \min_{p \in \mathcal{F}(\mathcal{Q})} \int u \circ f dp$$

[10] Completeness and continuity imply that $u(Y)$ is convex.

Therefore, $(g, \mathcal{P}^\varphi) \succeq (f, \mathcal{P})$ and $(g, \mathcal{Q}) \succeq (f, \mathcal{Q})$. Therefore, by Axiom 4,

$$(g, \alpha\mathcal{P}^\varphi + (1-\alpha)\mathcal{Q}) \succeq (f, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \tag{6}$$

On the other hand,

$$\alpha \int u \circ f dp^* + (1-\alpha) \int u \circ f dq^* = \alpha u(y_1) + (1-\alpha) u(y_2) = \min_{p \in \mathcal{F}(\alpha\mathcal{P}^\varphi + (1-\alpha)\mathcal{Q})} \int u \circ g dp$$

where the last equality follows from Condition 1.

Therefore, (5) implies $(f, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \succ (g, \alpha\mathcal{P}^\varphi + (1-\alpha)\mathcal{Q})$, which contradicts (6).

*Step 2.* Assume that $S(\mathcal{P}) \cap S(\mathcal{Q}) = \emptyset$. Then, $\mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \subseteq \alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$

Suppose that there exists $r^* \in \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q})$ such that $r^* \notin \alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$. By Condition 1, there exist $p^* \in \mathcal{P}$ and $q^* \in \mathcal{Q}$ such that $r^* = \alpha p^* + (1-\alpha)q^*$. Assume, for instance, that $p^* \notin \mathcal{F}(\mathcal{P})$. Since $\mathcal{F}(\mathcal{P})$ is a convex set, using a separation argument, we know there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \min_{p \in \mathcal{F}(\mathcal{P})} \int \phi dp$. Since $S(\mathcal{P})$ is a finite set, there exist numbers $a, b$ with $a > 0$ such that $(a\phi(s) + b) \in u(Y)$ for all $s \in S(\mathcal{P})$. Then, for all $s \in S(\mathcal{P})$, there exists $y(s) \in Y$ such that $u(y(s)) = a\phi(s) + b$. There also exists $y^* \in Y$ such that $u(y^*) = \min_{p \in \mathcal{P}} \int a\phi + bdp$. Define $f$ by $f(s) = y(s)$ for all $s \in S(\mathcal{P})$, $f(s) = y^*$ for all $s \in S \setminus S(\mathcal{P})$, and define $g$ by $g(s) = y^*$ for all $s \in S$. Since Condition 1 applies, we have:

$$\begin{aligned}
\min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ f dp &= \min_{p \in \mathcal{F}(\mathcal{Q})} \int u \circ f dp \\
\min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ g dp &= \min_{p \in \mathcal{F}(\mathcal{Q})} \int u \circ g dp = u(y^*)
\end{aligned}$$

Thus, $(f, \mathcal{P}) \sim (g, \mathcal{P}) \sim (f, \mathcal{Q}) \sim (g, \mathcal{Q})$. By Axiom 3, $(f, \mathcal{P}) \sim (f_{S(\mathcal{P})}g, \mathcal{P})$. By Axiom 4, $(f_{S(\mathcal{P})}g, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \sim (f, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q})$ and $(f_{S(\mathcal{P})}g, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \sim (g, \alpha\mathcal{P} + (1-\alpha)\mathcal{P})$, establishing that:

$$(f, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \sim (g, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \tag{7}$$

Since $g$ is a constant act, we have $\min_{p \in \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q})} \int u \circ g dp = u(y^*)$. Yet,

$$\begin{aligned}
\min_{p \in \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q})} \int u \circ f dp &\leq \int u \circ f dr^* = \alpha \int u \circ f dp^* + (1-\alpha) \int u \circ f dq^* \\
&= \alpha \int (a\phi + b) dp^* + (1-\alpha) u(y^*) \\
&< \alpha \min_{p \in \mathcal{F}(\mathcal{P})} \int (a\phi + b) dp + (1-\alpha) u(y^*) = u(y^*)
\end{aligned}$$

which contradicts equation (7).

*Step 3.* $\mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \subseteq \alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$

Suppose that there exists $r^* \in \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{Q})$ such that $r^* \notin \alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$. By Condition 1, there exist $p^* \in \mathcal{P}$ and $q^* \in \mathcal{Q}$ such that $r^* = \alpha p^* + (1-\alpha)q^*$. Since

$\alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{Q})$ is a convex set, using a separation argument, we know there exists a function $\phi : S \to \mathbb{R}$ such that

$$\begin{aligned}\int \phi dr^* &= \alpha \int \phi dp^* + (1-\alpha)\int \phi dq^* \\ &< \min_{p\in\alpha\mathcal{F}(\mathcal{P})+(1-\alpha)\mathcal{F}(\mathcal{Q})} \int \phi dp \\ &= \alpha \min_{p\in\mathcal{F}(\mathcal{P})} \int \phi dp + (1-\alpha)\min_{p\in\mathcal{F}(\mathcal{Q})}\int \phi dp\end{aligned} \tag{8}$$

Since $S(\mathcal{P}) \cup S(\mathcal{Q})$ is a finite set, there exist numbers $a, b$ with $a > 0$, such that $\forall s \in S(\mathcal{P}) \cup S(\mathcal{Q})$, $(a\phi(s) + b) \in u(Y)$. Then, for all $s \in S(\mathcal{P}) \cup S(\mathcal{Q})$ there exists $y(s) \in Y$ such that $u(y(s)) = a\phi(s) + b$. Let $f$ be defined by $f(s) = y(s)$ for all $s \in S(\mathcal{P}) \cup S(\mathcal{Q})$, and $f(s) = \delta_x$, with $x \in X$, for all $s \notin S(\mathcal{P}) \cup S(\mathcal{Q})$.

Let $\varphi$ and $\psi$ be two bijective mappings on $S$, such that: $\varphi(S(\mathcal{P})) \cap (S(\mathcal{P}) \cup S(\mathcal{Q}) \cup \psi(S(\mathcal{Q}))) = \emptyset$ and $\psi(S(\mathcal{Q})) \cap (S(\mathcal{P}) \cup S(\mathcal{Q}) \cup \varphi(S(\mathcal{P}))) = \emptyset$

Define $g$ by $g(s) = f(\varphi^{-1}(s))$ if $s \in \varphi(S(\mathcal{P}))$, $g(s) = f(\psi^{-1}(s))$ if $s \in \psi(S(\mathcal{Q}))$, and $g(s) = \delta_x$, with $x \in X$ otherwise. By Axiom 2, we have: $(g, \mathcal{P}^\varphi) \sim (f, \mathcal{P})$ and $(g, \mathcal{Q}^\psi) \sim (f, \mathcal{Q})$. Therefore, Axiom 3 implies:

$$(f, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q}) \sim (g, \mathcal{P}^\varphi + (1-\alpha)\mathcal{Q}^\psi) \tag{9}$$

On the other hand, since $S(\mathcal{P}^\varphi) \cap S(\mathcal{Q}^\psi) = \emptyset$, Steps 1 and 2 imply:

$$\mathcal{F}(\alpha\mathcal{P}^\varphi + (1-\alpha)\mathcal{Q}^\psi) = \alpha\mathcal{F}(\mathcal{P}^\varphi) + (1-\alpha)\mathcal{F}(\mathcal{Q}^\psi) = \alpha(\mathcal{F}(\mathcal{P}))^\varphi + (1-\alpha)(\mathcal{F}(\mathcal{Q}))^\psi$$

where the last equality follows from condition 2. Therefore:

$$\begin{aligned}\min_{p\in\mathcal{F}(\alpha\mathcal{P}^\varphi+(1-\alpha)\mathcal{Q}^\psi)} \int g dp &= \min_{p\in\alpha\mathcal{F}(\mathcal{P}^\varphi)+(1-\alpha)\mathcal{F}(\mathcal{Q})^\psi} \int u\circ g dp \\ &= \alpha \min_{p\in(\mathcal{F}(\mathcal{P}))^\varphi} \int u\circ g dp + (1-\alpha)\min_{p\in(\mathcal{F}(\mathcal{Q}))^\psi} \int u\circ g dp \\ &= \alpha \min_{p\in\mathcal{F}(\mathcal{P})} \int u\circ f dp + (1-\alpha)\min_{p\in\mathcal{F}(\mathcal{Q})} \int u\circ f dp \\ &> \alpha \int (a\phi + b) dp^* + (1-\alpha)\int (a\phi+b) dq^* \\ &= \alpha \int u\circ f dp^* + (1-\alpha)\int u\circ f dq^* \\ &\geq \min_{p\in\mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{Q})} \int u\circ f dp\end{aligned}$$

where the strict inequality follows from equation (8). Therefore, $(g, \alpha\mathcal{P}^\varphi + (1-\alpha)\mathcal{Q}^\psi) \succ (f, \alpha\mathcal{P} + (1-\alpha)\mathcal{Q})$, which contradicts equation (9).

<u>Condition 4</u>: Before proving Condition 4, we establish that for all $\mathcal{P} \in \mathbb{P}$, $\alpha \in [0,1]$, and all one-to-one function $\varphi : S \to S$ such that $S(\mathcal{P}) \cap \varphi(S(\mathcal{P})) = \emptyset$, $\mathcal{F}(R[\mathcal{P}, \alpha, \varphi]) \subseteq \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{P}^\varphi)$

Let $\mathcal{P} \in \mathbb{P}$, $\alpha \in [0,1]$, and $\varphi : S \to S$ a one-to-one function such that $S(\mathcal{P}) \cup \varphi(S(\mathcal{P})) = \emptyset$. Assume that $\mathcal{F}(R[\mathcal{P},\alpha,\varphi]) \nsubseteq \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi})$. This implies that there exists $p^* \in \mathcal{F}(R[\mathcal{P},\alpha,\varphi])$ such that $p^* \notin \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi})$. Since $\mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi})$ is a convex set, using a separation argument, we know that there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \min_{p \in \mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{P}^{\varphi})} \int \phi dp$. Since $S(\alpha\mathcal{P}+(1-\alpha)\mathcal{P}^{\varphi})$ is a finite set, there exist numbers $a, b$ with $a > 0$ such that $(a\phi(s) + b) \in u(Y)$ for all $s \in S(\alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi})$. Then, for all $s \in S(\alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi})$, there exists $y(s) \in Y$ such that $u(y(s)) = a\phi(s) + b$. Define $f$ by $f(s) = y(s)$ for all $s \in S(\alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi})$, and $f(s) = \delta_x$ for all $s \in S \setminus S(\alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi})$, where $x \in X$. Note that $S(\alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi}) = (R[\mathcal{P},\alpha,\varphi])$. We thus have:

$$\begin{aligned}
\min_{p \in \mathcal{F}(R[\mathcal{P},\alpha,\varphi])} \int u \circ f dp &\leq \int u \circ f dp^* = \int (a\phi + b) dp^* \\
&< \min_{p \in \mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{P}^{\varphi})} \int (a\phi + b) dp = \min_{p \in \mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{P}^{\varphi})} \int u \circ f dp
\end{aligned}$$

which implies that $(f, \alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi}) \succ (f, R[\mathcal{P},\alpha,\varphi])$, which is a violation of Axiom 9.

We can now proceed to the proof of Condition 4.

*Step 1.* $\mathcal{F}(R[\mathcal{P},\alpha,\varphi]) \subseteq R[\mathcal{F}(\mathcal{P}),\alpha,\varphi]$

Let $\mathcal{P} \in \mathbb{P}$ , and $R[\mathcal{P},\alpha,\varphi]$ with $\alpha \in ]0,1[$.[11] Suppose that there exists $p^* \in \mathcal{F}(R[\mathcal{P},\alpha,\varphi])$ such that $p^* \notin R[\mathcal{F}(\mathcal{P}),\alpha,\varphi]$. By Condition 3 and the result we just established, we have that $\mathcal{F}(R[\mathcal{P},\alpha,\varphi]) \subseteq \mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{P}^{\varphi})$, $\mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{P}^{\varphi}) = \alpha\mathcal{F}(\mathcal{P}) + (1-\alpha)\mathcal{F}(\mathcal{P}^{\varphi})$. Therefore, there exist $p_1^*, p_2^* \in \mathcal{F}(\mathcal{P})$ such that $p^* = \alpha p_1^* + (1-\alpha)p_2^{*\varphi}$. Note that $p_1^* \neq p_2^*$, since otherwise $p^* \in R[\mathcal{F}(\mathcal{P}),\alpha,\varphi]$.

Thus, there exist $E_1, E_2 \subset S(\mathcal{P})$ such that $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2 = S(\mathcal{P})$, $p_1^*(E_1) > p_2^*(E_1)$ (and thus $p_1^*(E_2) = (1 - p_1^*(E_1)) < p_2^*(E_2) = (1 - p_1^*(E_2)))$. There also exist $\overline{x}$ and $\underline{x}$ such that $u(\delta_{\overline{x}}) > u(\delta_{\underline{x}})$.

Assume first that $\alpha \geq \frac{1}{2}$. Define $f$ by $f(s) = \left(\frac{2\alpha-1}{\alpha}\right)\delta_{\overline{x}} + \left(\frac{1-\alpha}{\alpha}\right)\delta_{\underline{x}}$ for all $s \in E_1$, $f(s) = \delta_{\underline{x}}$ for all $s \in E_2$, $f(s) = \alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}}$ for all $s \in S \backslash S(\mathcal{P})$ and define $g$ by $g(s) = \delta_{\overline{x}}$ for all $s \in E_1$, $g(s) = \delta_{\underline{x}}$ for all $s \in E_2$, $g(s) = \alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}}$ for all $s \in S \backslash \{s_1, s_2\}$. One can easily check that $(\alpha f + (1-\alpha)g)(s) = \alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}}$ for all $s \in S(\mathcal{P})$. And thus

$$\min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ (\alpha f + (1-\alpha)g)\, dp = u\left(\alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}}\right) = \alpha u(\delta_{\overline{x}}) + (1-\alpha)u(\delta_{\underline{x}})$$

[11] For $\alpha = 0$ we have trivially $R[\mathcal{P},0,\varphi] = \mathcal{P}$ and for $\alpha = 1$, Condition 4 can be deduced from Condition 2.

Consider now $f_{S(\mathcal{P})}g^{\varphi}$. We have:

$$\begin{aligned}
\min_{p\in\mathcal{F}(R[\mathcal{P},\alpha,\varphi])}\int u\circ f_{S(\mathcal{P})}g^{\varphi}dp &\leq \int u\circ f_{S(\mathcal{P})}g^{\varphi}dp^{*}=\alpha\int u\circ fdp_{1}^{*}+(1-\alpha)\int u\circ gdp_{2}^{*}\\
&= \alpha\left[p_{1}^{*}(E_{1})u(\left(\frac{2\alpha-1}{\alpha}\right)\delta_{\overline{x}}+\left(\frac{1-\alpha}{\alpha}\right)\delta_{\underline{x}})+p_{1}^{*}(E_{2})u(\delta_{\overline{x}})\right]\\
&+ (1-\alpha)\left[p_{2}^{*}(E_{1})u(\delta_{\overline{x}})+p_{2}^{*}(E_{2})u(\delta_{\underline{x}})\right]\\
&= \alpha u(\delta_{\overline{x}})+(1-\alpha)u(\delta_{\underline{x}})\\
&+ (1-\alpha)\left(p_{2}^{*}(E_{1})-p_{1}^{*}(E_{1})\right)\left(u(\delta_{\overline{x}})-u(\delta_{\underline{x}})\right)\\
&< \alpha u(\delta_{\overline{x}})+(1-\alpha)u(\delta_{\underline{x}})
\end{aligned}$$

and thus $(\alpha f+(1-\alpha)g,\mathcal{P})\succ\left(f_{S(\mathcal{P})}g^{\varphi},R[\mathcal{P},\alpha,\varphi]\right)$ which is a violation of Axiom 1.

A similar reasoning applies in the case $\alpha\leq\frac{1}{2}$.

*Step 2.* $\mathcal{F}(R[\mathcal{P},\alpha,\varphi])\supseteq R[\mathcal{F}(\mathcal{P}),\alpha,\varphi]$

Suppose that there exists $p^{*}\in\mathcal{F}(\mathcal{P})$ such that $\alpha p^{*}+(1-\alpha)p^{*\varphi}\notin\mathcal{F}(R[\mathcal{P},\alpha,\varphi])$. Since we just proved that $\mathcal{F}(R[\mathcal{P},\alpha,\varphi])\subseteq R[\mathcal{F}(\mathcal{P}),\alpha,\varphi]$ for all $p\in\mathcal{F}(R[\mathcal{P},\alpha,\varphi])$, there exists $p_{\varphi^{-1}}\in\mathcal{F}(\mathcal{P})$ such that $p=\alpha p_{\varphi^{-1}}+(1-\alpha)\left(p_{\varphi^{-1}}\right)^{\varphi}$. Consider $\mathcal{Q}=\left\{p_{\varphi^{-1}}|p\in\mathcal{F}(R[\mathcal{P},\alpha,\varphi])\right\}$. Since $\mathcal{F}(R[\mathcal{P},\alpha,\varphi])$ is convex, $\mathcal{Q}$ is also convex and $p^{*}\notin\mathcal{Q}$. Hence, using a separation argument, we know there exists a function $\phi:S\to\mathbb{R}$ such that $\int\phi dp^{*}<\min_{p\in\mathcal{Q}}\int\phi dp$. Since $S(\mathcal{P})$ is a finite set, there exist numbers $a,b$ with $a>0$, such that $\forall s\in S(\mathcal{P})$, $(a\phi(s)+b)\in u(Y)$. Then, for all $s\in S(\mathcal{P})$ there exists $y(s)\in Y$ such that $u(y(s))=a\phi(s)+b$. Define $f$ by $f(s)=y(s)$ for all $s\in S(\mathcal{P})$, $f(s)=\delta_{x}$ for all $s\in S\backslash S(\mathcal{P}^{\varphi})$, where $x\in X$. Observe that for all $p\in\mathcal{F}(R[\mathcal{P},\alpha,\varphi])$,

$$\begin{aligned}
\int u\circ f_{S(\mathcal{P})}f^{\varphi}dp &= \int u\circ f_{S(\mathcal{P})}f^{\varphi}d\left(\alpha p_{\varphi^{-1}}+(1-\alpha)\left(p_{\varphi^{-1}}\right)^{\varphi}\right)\\
&= \alpha\int u\circ fdp_{\varphi^{-1}}+(1-\alpha)\int u\circ f^{\varphi}d\left(p_{\varphi^{-1}}\right)^{\varphi}\\
&= \int u\circ fdp_{\varphi^{-1}}=\int(a\phi+b)dp_{\varphi^{-1}}
\end{aligned}$$

Thus

$$\begin{aligned}
\min_{p\in\mathcal{F}(\mathcal{P})}\int u\circ fdp &\leq \int u\circ fdp^{*}=\int(a\phi+b)dp^{*}<\min_{p\in\mathcal{Q}}\int(a\phi+b)dp\\
&= \min_{p\in\mathcal{Q}}\int u\circ fdp=\min_{p\in\mathcal{F}(R[\mathcal{P},\alpha,\varphi])}\int u\circ f_{S(\mathcal{P})}f^{\varphi}dp
\end{aligned}$$

which shows that $\left(f_{S(\mathcal{P})}f^{\varphi},R[\mathcal{P},\alpha,\varphi]\right)\succ(f,\mathcal{P})$, a violation of Axiom 1.

**2. Necessity.** The axioms to check are Axioms 1, 4, 8, and 9 since the others hold by Theorem 5.

Axiom 1: Consider $\mathcal{P} \in \mathbb{P}$, $R[\mathcal{P}, \alpha, \varphi]$, and $f, g \in \mathcal{A}$ such that $f(s) = g(s)$ for all $s \in S \backslash S(\mathcal{P})$. Since Conditions 3 and 4 hold,

$$\min_{p \in \mathcal{F}(R[\mathcal{P},\alpha,\varphi])} \int u \circ f_{S(\mathcal{P})} g^{\varphi} dp = \min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ f_{S(\mathcal{P})} g^{\varphi} d(\alpha p + (1-\alpha) p^{\varphi})$$

For all $p \in \mathcal{F}(\mathcal{P})$,

$$\begin{aligned} \int u \circ f_{S(\mathcal{P})} g^{\varphi} d(\alpha p + (1-\alpha) p^{\varphi}) &= \alpha \int u \circ f_{S(\mathcal{P})} g^{\varphi} dp + (1-\alpha) \int u \circ f_{S(\mathcal{P})} g^{\varphi} dp^{\varphi} \\ &= \alpha \int u \circ f dp + (1-\alpha) \int u \circ g dp \\ &= \int u \circ (\alpha f + (1-\alpha) g) dp \end{aligned}$$

and thus $\min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ f_{S(\mathcal{P})} g^{\varphi} d(\alpha p + (1-\alpha)p^{\varphi}) = \min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ (\alpha f + (1-\alpha) g) dp$ which shows that $(\alpha f + (1-\alpha) g, \mathcal{P}) \sim \left(f_{S(\mathcal{P})} g^{\varphi}, R[\mathcal{P}, \alpha, \varphi]\right)$.

Axiom 4: Let $\mathcal{P}_1, \mathcal{Q}_1, \mathcal{P}_2, \mathcal{Q}_2 \in \mathbb{P}$, $\alpha \in [0,1]$, and $f, g \in \mathcal{A}$ such that $(f, \mathcal{P}_1) \succeq (g, \mathcal{Q}_1)$ and $(f, \mathcal{P}_2) \succeq (g, \mathcal{Q}_2)$. By Condition 3, $\mathcal{F}(\alpha \mathcal{P}_1 + (1-\alpha)\mathcal{P}_2) = \alpha \mathcal{F}(\mathcal{P}_1) + (1-\alpha)\mathcal{F}(\mathcal{P}_2)$ and $\mathcal{F}(\alpha \mathcal{Q}_1 + (1-\alpha)\mathcal{Q}_2) = \alpha \mathcal{F}(\mathcal{Q}_1) + (1-\alpha)\mathcal{F}(\mathcal{Q}_2)$. Therefore,

$$\begin{aligned} \min_{p \in \mathcal{F}(\alpha \mathcal{P}_1 + (1-\alpha)\mathcal{P}_2)} \int u \circ f dp &= \min_{p \in \alpha \mathcal{F}(\mathcal{P}_1) + (1-\alpha)\mathcal{F}(\mathcal{P}_2)} \int u \circ f dp \\ &= \alpha \min_{p \in \mathcal{F}(\mathcal{P}_1)} \int u \circ f dp + (1-\alpha) \min_{p \in \mathcal{F}(\mathcal{P}_2)} \int u \circ f dp \end{aligned}$$

Similarly,

$$\min_{p \in \mathcal{F}(\alpha \mathcal{Q}_1 + (1-\alpha)\mathcal{Q}_2)} \int u \circ g dp = \alpha \min_{p \in \mathcal{F}(\mathcal{Q}_1)} \int u \circ g dp + (1-\alpha) \min_{p \in \mathcal{F}(\mathcal{Q}_2)} \int u \circ g dp$$

Therefore, $(f, \alpha \mathcal{P}_1 + (1-\alpha)\mathcal{P}_2) \succeq (g, \alpha \mathcal{Q}_1 + (1-\alpha)\mathcal{Q}_2)$. Moreover, if $(f, \mathcal{P}_1) \succ (g, \mathcal{Q}_1)$, then $(f, \alpha \mathcal{P}_1 + (1-\alpha)\mathcal{P}_2) \succ (g, \alpha \mathcal{Q}_1 + (1-\alpha)\mathcal{Q}_2)$. Hence, Axiom 4 is satisfied.

Axiom 8: Consider $f, g \in \mathcal{A}$ and $\mathcal{P} \in \mathbb{P}$ such that $(f, \{p\}) \succeq (g, \{p\})$ for all $p \in \mathcal{P}$. Remark that for all $p \in co(\mathcal{P})$ there exist $p_1, p_2 \in \Delta(S)$ and $\alpha \in [0,1]$ such that $p = \alpha p_1 + (1-\alpha) p_2$. Thus,

$$\begin{aligned} \int u \circ f d(\alpha p_1 + (1-\alpha) p_2) &= \alpha \int u \circ f dp_1 + (1-\alpha) \int u \circ f dp_2 \\ &\geq \alpha \int u \circ g dp_1 + (1-\alpha) \int u \circ g dp_2 \\ &= \int u \circ g d(\alpha p_1 + (1-\alpha) p_2) \end{aligned}$$

and hence $\int u \circ f dp \geq \int u \circ g dp$. Since by Condition 1, $\mathcal{F}(\mathcal{P}) \subseteq \mathcal{P}$, we get that $\int u \circ f dp \geq \int u \circ g dp$ for all $p \in \mathcal{F}(\mathcal{P})$, and thus $\min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ f dp \geq \min_{p \in \mathcal{F}(\mathcal{P})} \int u \circ g dp$, which implies that $(f, \mathcal{P}) \succeq (g, \mathcal{P})$. ■

Axiom 9: By Conditions 3 and 4, for all $\mathcal{P} \in \mathbb{P}$, $\alpha \in [0,1]$, and all one-to-one function $\varphi : S \to S$, such that $\varphi(S(\mathcal{P})) \cap S(\mathcal{P}) = \emptyset$, $\mathcal{F}(R[\mathcal{P}, \alpha, \varphi]) \subseteq \mathcal{F}(\alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi})$. Therefore, for all $f \in \mathcal{A}$,

$$\min_{p \in \mathcal{F}(\alpha\mathcal{P}+(1-\alpha)\mathcal{P}^{\varphi})} \int u \circ f dp \leq \min_{p \in \mathcal{F}(R[\mathcal{P},\alpha,\varphi])} \int u \circ f dp$$

and therefore $(f, R[\mathcal{P}, \alpha, \varphi]) \succeq (f, \alpha\mathcal{P} + (1-\alpha)\mathcal{P}^{\varphi})$, which proves Axiom 9.

**Proof. [Theorem 2]** $[(i) \Rightarrow (ii)]$ Let $\mathcal{P} \in \mathbb{P}$ and assume that $\mathcal{F}^a(\mathcal{P}) \not\subset \mathcal{F}^b(\mathcal{P})$, i.e., there exists $p^* \in \mathcal{F}^a(\mathcal{P})$ such that $p^* \notin \mathcal{F}^b(\mathcal{P})$. Using a separation argument, there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \underset{p \in \mathcal{F}^b(\mathcal{P})}{min} \int \phi dp$. Note that we can choose by normalization $u_a$ and $u_b$ such that $u_a(\bar{x}) = u_b(\bar{x}) > u_a(\underline{x}) = u_b(\underline{x})$. Since $S(\mathcal{P})$ is a finite set, there exist numbers $k > 0$ and $\ell$, such that for all $s \in S(\mathcal{P})$, $k\phi(s) + \ell \in [u_a(\underline{x}), u_a(\bar{x})]$. W.l.o.g, suppose that $S(\mathcal{P}) = \{1, .., n\}$. Consider the event

$$E = \bigcup_{k=1,..,n} \left[ \bigcup_{i=1,..,2^{(k-1)}} \left[ \bigcup_{j=0,..,2^{n-k}-1} \left\{ k + (i-1)\, n + 2^k jn \right\} \right] \right]$$

Let $\alpha_i = \frac{k\phi(i)+\ell-u_a(\underline{x})}{u_a(\bar{x})-u_a(\underline{x})}$. Let $\psi^k$ for $k = 1, .., n$ be the following one to one function: $\psi^k(s) = s + 2^{(k-1)}n$ for all $s \in \left\{1, .., 2^{(k-1)}n\right\}$, $\psi^k(s) = s - kn$ for all $s \in \left\{1 + 2^{(k-1)}n, .., 2^k n\right\}$, $\psi^k(s) = s$ otherwise. Define $p^{n*}$ recursively: $p^{1*} = \alpha_1 p^* + (1-\alpha_1)(p^*)^{\psi^1}$ and for $n \geq k \geq 2$, $p^{k*} = \alpha_k p^{(k-1)*} + (1-\alpha_k)\left(p^{(k-1)*}\right)^{\psi^k}$. Next, define $\mathcal{P}^n$ recursively: $\mathcal{P}^1 = R[\mathcal{P}, \alpha_1, \psi^1]$, and for $n \geq k \geq 2$, $\mathcal{P}^k = R[\mathcal{P}^{(k-1)}, \alpha_k, \psi^k]$

Using Axiom 1, one can check that $(\bar{x}_E\underline{x}, \mathcal{P}^n) \sim_a (f, \mathcal{P})$ and $(\bar{x}_E\underline{x}, \mathcal{P}^n) \sim_b (f, \mathcal{P})$ where $f(i) = \alpha_i\delta_{\bar{x}} + (1-\alpha_i)\delta_{\underline{x}}$ for $i = 1, .., n$, (therefore $u_a(f(i)) = u_b(f(i)) = k\phi(i) + \ell$) and $f(i) = \delta_{\underline{x}}$ otherwise. One can also check that $(\bar{x}_E\underline{x}, p^{n*}) \sim_a (f, p^*)$ and $(\bar{x}_E\underline{x}, p^{n*}) \sim_b (f, p^*)$. Since by assumption we have that $(f, p^*) \prec_b (f, \mathcal{P})$ and $(f, p^*) \succeq_a (f, \mathcal{P})$, then $(\bar{x}_E\underline{x}, p^{n*}) \prec_b (\bar{x}_E\underline{x}, \mathcal{P}^n)$ while $(\bar{x}_E\underline{x}, p^{n*}) \succeq_a (\bar{x}_E\underline{x}, \mathcal{P}^n)$ which is a contradiction with the fact that $\succeq_b$ is more averse to imprecision than $\succeq_a$.

$[(ii) \Rightarrow (i)]$ Straightforward. ■

**Proof. [Theorem 3]** Equivalence between $(i)$ and $(ii)$ was proved in theorem 2.

$[(ii) \Rightarrow (iii)]$ Consider $\mathcal{P} \in \mathbb{S}$ and $E \subset S$ such that $c_{\mathcal{P}}(E) > 0$.
Since $\pi_a^A(E, \mathcal{P}) = c_{\mathcal{P}}(E) - Min_{p \in \mathcal{F}^a(\mathcal{P})} p(E)$ and $\pi_b^A(E, \mathcal{P}) = c_{\mathcal{P}}(E) - Min_{p \in \mathcal{F}^b(\mathcal{P})} p(E)$, $\mathcal{F}^a(\mathcal{P}) \subset \mathcal{F}^b(\mathcal{P})$ implies that $\pi_b^A(E, \mathcal{P}) \geq \pi_a^A(E, \mathcal{P})$.

$[(iii) \Rightarrow (i)]$ Consider prizes $\bar{x}$ and $\underline{x}$ in $X$ such that both $a$ and $b$ strictly prefer $\bar{x}$ to $\underline{x}$ and let $\mathcal{P} \in \mathbb{S}$ and $E \subset S$.

Fist consider the case where $c_{\mathcal{P}}(E) = 0$. Since $\mathcal{P} \in \mathbb{S}$, this implies that for all $q \in \mathcal{P}$, $q(E) = 0$ and therefore by axioms 3 and 8, for agent $i = a, b$, $(\bar{x}_E\underline{x}, \{p\}) \succeq_i [\succ_i](\bar{x}_E\underline{x}, \mathcal{P})$ if and only if $p(E) \geq [>]0$. We therefore trivially have

$$(\bar{x}_E\underline{x}, \{p\}) \succeq_a [\succ_a](\bar{x}_E\underline{x}, \mathcal{P}) \Rightarrow (\bar{x}_E\underline{x}, \{p\}) \succeq_b [\succ_b](\bar{x}_E\underline{x}, \mathcal{P})$$

Let us suppose now that $c_{\mathcal{P}}(E) > 0$. Then for any $p \in \mathbb{S}$, for any agent $i = a, b$, $(\bar{x}_E \underline{x}, \{p\}) \succeq_i [\succ_i](\bar{x}_E \underline{x}, \mathcal{P})$ if, and only if, $\pi_i^A(E, \mathcal{P}) \geq [\succ] c_{\mathcal{P}}(E) - p(E)$. Therefore since $\pi_b^A(E, \mathcal{P}) \geq \pi_a^A(E, \mathcal{P})$, it implies that we have

$$(\bar{x}_E \underline{x}, \{p\}) \succeq_a [\succ_a](\bar{x}_E \underline{x}, \mathcal{P}) \Rightarrow (\bar{x}_E \underline{x}, \{p\}) \succeq_b [\succ_b](\bar{x}_E \underline{x}, \mathcal{P})$$

which complete the proof that $\succeq_b$ is more averse to imprecision than $\succeq_a$. ■

**Proof. [Proposition 1]** $[(i) \Rightarrow (ii)]$ Let $\mathcal{P} \in \mathbb{S}$, let $p$ be a boundary point $p$ of $co(\mathcal{P})$ and consider

$$\overline{\theta} = Sup \left\{ \theta' | \theta' \in [0,1] \text{ s.th. } \left( \theta' p + (1-\theta) c_{\mathcal{P}} \right) \in \mathcal{F}(\mathcal{P}) \right\}$$

Then $\overline{p} = \overline{\theta} p + (1-\overline{\theta}) c_{\mathcal{P}}$ is a boundary point of $\mathcal{F}(\mathcal{P})$ since $\mathcal{F}(\mathcal{P})$ is closed. Since it is convex as well, there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi d\overline{p} = \underset{p \in \mathcal{F}(\mathcal{P})}{min} \int \phi dp$.

Using the notation and definitions introduced in the proof of Theorem 2 in order to define the act $f$, the probabilities $\overline{p}^n, c_{\mathcal{P}}^n$, $p^n$ and the sets $\mathcal{P}^n$ and $\mathcal{F}(\mathcal{P})^n$, we have, by Condition 4 of Theorem 1, that $\overline{p}^n \in \mathcal{F}(\mathcal{P})^n = \mathcal{F}(\mathcal{P}^n)$. We have $(f, \{\overline{p}\}) \sim (f, \mathcal{P})$.

Using Axiom 1, one can check that $(\bar{x}_E \underline{x}, \{\overline{p}^n\}) \sim (f, \{\overline{p}\})$ and $(\bar{x}_E \underline{x}, \mathcal{P}^n) \sim (f, \mathcal{P})$. Therefore $(\bar{x}_E \underline{x}, \{\overline{p}^n\}) \sim (\bar{x}_E \underline{x}, \mathcal{P}^n)$ and

$$\pi^R(E, \mathcal{P}^n) = \frac{c_{\mathcal{P}^n}(E) - \overline{p}^n(E)}{c_{\mathcal{P}^n}(E) - Min_{q \in \mathcal{P}^n} q(E)} \leq \frac{c_{\mathcal{P}^n}(E) - \overline{p}^n(E)}{c_{\mathcal{P}^n}(E) - p^n(E)} = \overline{\theta}$$

If $\theta > \overline{\theta}$ we get a contradiction with the fact that $\pi^R(E, \mathcal{P}^n) = \theta$. Therefore, for any boundary point $q$ of $co(\mathcal{P})$, $\overline{\theta}\,(q) = Sup \{\theta' | \theta' \in [0,1] \text{ s.th. } (\theta' q + (1-\theta) c_{\mathcal{P}}) \in \mathcal{F}(\mathcal{P})\}$ is such that $\overline{\theta}\,(q) \geq \theta$. Let $p$ be a boundary point of $co(\mathcal{P})$ such that $\overline{\theta}\,(p) \geq \overline{\theta}\,(q)$ for all $q$ boundary point of $co(\mathcal{P})$. Then, there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp = \underset{q \in \mathcal{P}}{min} \int \phi dq$. Define $\overline{p} = \overline{\theta}(p) p + (1-\overline{\theta}(p)) c_{\mathcal{P}}$ and consider now $q' \in \mathcal{F}(\mathcal{P})$. There exists a boundary point $q$ of $co(\mathcal{P})$ and $\theta' < \overline{\theta}(p)$ such that $q' = \theta' q + (1-\theta') c_{\mathcal{P}}$.

Let us use again the notation and definition introduced in the proof of Theorem 2. Since $\int uofdp \leq \int uofdq$ and $\int uofdp \leq \int uofdc_{\mathcal{P}}$, we have that $\int uofd\overline{p} \leq \int uofdq'$. Thus $\int uofd\overline{p} = \underset{q' \in \mathcal{F}(\mathcal{P})}{min} \int uofdq'$ while $\int uofdp = \underset{q \in \mathcal{P}}{min} \int uofdq$. Therefore

$$\pi^R(E, \mathcal{P}^n) = \frac{c_{\mathcal{P}^n}(E) - \overline{p}^n(E)}{c_{\mathcal{P}^n}(E) - Min_{q \in \mathcal{P}^n} q(E)} = \overline{\theta}\,(p)$$

and thus $\overline{\theta}\,(p) = \theta$. Thus, for all boundary point $q$ of $co(\mathcal{P})$, $\overline{\theta}\,(q) = \theta$ which proves that $\mathcal{F}(\mathcal{P}) = \theta \mathcal{P} + (1-\theta) \{c_{\mathcal{P}}\}$.

$[(ii) \Rightarrow (i)]$ Let consider $\mathcal{P} \in \mathbb{S}$ and $E \subset S$ such that $c_{\mathcal{P}}(E) > 0$. We have that

$$\min_{p \in \mathcal{F}(\mathcal{P})} p(E) = \theta \min_{p \in \mathcal{P}} p(E) + (1-\theta) c_{\mathcal{P}}(E)$$

and therefore

$$\pi^R(E,\mathcal{P}) = \frac{c_{\mathcal{P}}(E) - Min_{p\in\mathcal{F}(\mathcal{P})}p(E)}{c_{\mathcal{P}}(E) - Min_{p\in\mathcal{P}}p(E)} = \theta$$

■

**Proof. [Proposition 2]**

**1. Necessity**. Let $\mathcal{P},\mathcal{Q}\in\mathbb{B}$ satisfy condition (i) and (ii). Note that condition (i) means that

$$\sum_{i/|S(\Delta_i)|=1}\alpha_i c_{\Delta_i} + \sum_{i/|S(\Delta_i)|\geq 2}\alpha_i c_{\Delta_i} = \sum_{i/|S(\Delta_i)|=1}\beta_i c_{\Delta_i} + \sum_{i/|S(\Delta_i)|\geq 2}\beta_i c_{\Delta_i}$$

and therefore $\sum_{i/|S(\Delta_i)|=1}\alpha_i c_{\Delta_i} = \sum_{i/|S(\Delta_i)|=1}\beta_i c_{\Delta_i} - \sum_{i/|S(\Delta_i)|\geq 2}\alpha_i(1-\frac{\beta_i}{\alpha_i})c_{\Delta_i}$. Hence, we have that $\mathcal{Q} = \sum_{i/|S(\Delta_i)|=1}\alpha_i\Delta_i + \sum_{i/|S(\Delta_i)|\geq 2}\alpha_i\left[\frac{\beta_i}{\alpha_i}\Delta_i + (1-\frac{\beta_i}{\alpha_i})\{c_{\Delta_i}\}\right]$ since for all $i$ such that $|S(\Delta_i)|\geq 2$, $\alpha_i\geq\beta_i$ while $\mathcal{P} = \sum_{i/|S(\Delta_i)|=1}\alpha_i\Delta_i + \sum_{i/|S(\Delta_i)|\geq 2}\alpha_i\Delta_i$. Since for all all $i$ such that $|S(\Delta_i)|\geq 2$, $\Delta_i$ is unanimously more imprecise than $\frac{\beta_i}{\alpha_i}\Delta_i + (1-\frac{\beta_i}{\alpha_i})\{c_{\Delta_i}\}$, Axiom 4 implies that $\mathcal{P}$ is unanimously more imprecise than $\mathcal{Q}$.

**2. Sufficiency.** Let $\mathcal{P},\mathcal{Q}\in\mathbb{B}$ and $\mathcal{P}$ be unanimously more imprecise than $\mathcal{Q}$. Given that a Bayesian decision maker for whom $\mathcal{F}(\Delta_i)=\{c_{\Delta_i}\}$ for all simplex $\Delta_i$ and $\mathcal{F}(\mathcal{P}^*)=\{c_{\mathcal{P}^*}\}$ for all $\mathcal{P}^*\in\mathbb{B}$ satisfy Axioms 1 to 9, condition (i) has to hold since otherwise, we could find an act $f$ for which such a Bayesian decision maker would strictly prefer $(f,\{c_{\mathcal{P}}\})$ to $(f,\{c_{\mathcal{Q}}\})$.

Let us suppose now that condition (ii) does not hold, that is, $\mathcal{P}=\sum_i\alpha_i\Delta_i$, $\mathcal{Q}=\sum_i\beta_i\Delta_i$ with positive or null coefficients $\alpha_i,\beta_i$, and there exists $j$ such that $|S(\Delta_j)|\geq 2$, $\alpha_j<\beta_j$. Then consider a decision maker for whom $\mathcal{F}(\Delta_i)=\{c_{\Delta_i}\}$ for all simplex $\Delta_i$ such that $|S(\Delta_i)|\neq|S(\Delta_j)|$, $\mathcal{F}(\Delta_i)=\Delta_i$ for all simplex $\Delta_i$ such that $|S(\Delta_i)|=|S(\Delta_j)|$ and which satisfies Axioms 1 to 9. Therefore, $\mathcal{F}(\mathcal{P}) = \sum_{i/|S(\Delta_i)|\neq|S(\Delta_j)|}\alpha_i\{c_{\Delta_i}\} + \sum_{i/|S(\Delta_i)|=|S(\Delta_j)|}\alpha_i\Delta_i$ while $\mathcal{F}(\mathcal{Q}) = \sum_{i/|S(\Delta_i)|\neq|S(\Delta_j)|}\beta_i\{c_{\Delta_i}\} + \sum_{i/|S(\Delta_i)|=|S(\Delta_j)|}\beta_i\Delta_i$. Since $\alpha_j<\beta_j$, $\mathcal{F}(\mathcal{Q})\nsubseteq\mathcal{F}(\mathcal{P})$ and we could find an act $f$ for which such a decision maker would strictly prefer $(f,\mathcal{P})$ to $(f,\mathcal{Q})$. ■

**Proof. [Proposition 3]**

**1. Necessity**.

Let $\mathcal{P},\mathcal{Q}\in\mathbb{S}$ satisfy condition (i) and (ii). We show that for any agent satisfying increasing absolute imprecision premium, $\mathcal{F}(\mathcal{Q})\subset\mathcal{F}(\mathcal{P})$. Consider an agent such that $\mathcal{F}(\mathcal{Q})\nsubseteq\mathcal{F}(\mathcal{P})$. Let $\overline{p}\in\mathcal{F}(\mathcal{Q})\backslash\mathcal{F}(\mathcal{P})$ . There exists a function $\phi: S\to\mathbb{R}$ such that $\underset{p\in\mathcal{F}(\mathcal{Q})}{min}\int\phi dp\leq\int\phi d\overline{p} < \underset{p\in\mathcal{F}(\mathcal{P})}{min}\int\phi dp$. Using the notation and definitions introduced in the proof of Theorem 2, we have that $(f,\mathcal{P})\succ(f,\{\overline{p}\})\succsim(f,\mathcal{Q})$ and $(\bar{x}_E\underline{x},\mathcal{P}^n)\succ(\bar{x}_E\underline{x},\{\overline{p}^n\})\succsim(\bar{x}_E\underline{x},\mathcal{Q}^n)$.

Therefore,

$$\pi^A(E,\mathcal{P}^n) = c_{\mathcal{P}^n}(E) - \min_{p\in\mathcal{F}(\mathcal{P}^n)}p(E) < c_{\mathcal{P}^n}(E) - \overline{p}^n(E)$$

while

$$\pi^A(E,\mathcal{Q}^n) = c_{\mathcal{Q}^n}(E) - \min_{p\in\mathcal{F}(\mathcal{Q}^n)}p(E) \geq c_{\mathcal{Q}^n}(E) - \overline{p}^n(E)$$

Note that $c_{\mathcal{P}^n}(E) = c_{\mathcal{Q}^n}(E)$ and therefore

$$c_{\mathcal{Q}^n}(E) - \overline{p}^n(E) = c_{\mathcal{P}^n}(E) - \overline{p}^n(E)$$

which proves that $\pi^A(E, \mathcal{P}^n) < \pi^A(E, \mathcal{Q}^n)$. Therefore, such an agent does not satisfy increasing absolute imprecision premium.

**2. Sufficiency.** Given that a Bayesian decision maker satisfies increasing absolute imprecision premium, condition (i) must apply. Note also that an extremely imprecision averse decision maker, that is, for whom $\mathcal{F}(\mathcal{P}) = \mathcal{P}$ for all $\mathcal{P}$, also satisfies increasing absolute imprecision premium. Therefore, condition (ii) must also apply. ■

**Proof.** **[Proposition 4]** $[(i) \Rightarrow (ii)]$ Since expected utility maximizers w.r.t. the probability distribution $c_\Delta$ constitute a subclass of the agents we consider, second order stochastic dominance w.r.t. this equiprobability distribution is necessary.

$[(ii) \Rightarrow (i)]$ W.l.o.g, suppose that $S(\Delta) = \{1, .., n\}$ and let us consider an agent with a revealed set $\mathcal{F}(\Delta)$ and an increasing and concave $u$. Let us consider two comonotone acts $f$ and $g$, such that $\exists i, j \in \{1, .., n\}$ such that $f(i) = g(i) + \epsilon$, $f(j) = g(j) - \epsilon'$, $f(s) = g(s)$ otherwise, $g(i) < g(j)$ and $\epsilon > \epsilon'$. There exists $p^*$ such that $\int u \circ f dp^* = \min_{p \in \mathcal{F}(\Delta)} \int u \circ f dp^*$. We have that $p^*(i) \leq p^*(j)$. Indeed, consider otherwise the permutation $\varphi$ that exchange $i$ and $j$ and left all other $s$ unmodified. Then we would have $\int u \circ f dp^* > \int u \circ f d\,(p^*)^\varphi$ which would be a contradiction with the fact that $(p^*)^\varphi \in \mathcal{F}(\Delta)$ since $\mathcal{F}(\Delta)$ is invariant to permutation. Therefore $\int u \circ f dp^* \geq \int u \circ g dp^* \geq \min_{p \in \mathcal{F}(\Delta)} \int u \circ f dp^*$ which proves that the agent prefers $f$ to $g$. Since for all comonotone acts $f$ and $g$ such that $f$ second order stochastically dominates $g$ w.r.t. $c_\Delta$, $f$ differs from $g$ by a finite series of such simple transfers, the agent prefers $f$ to $g$. Since $\mathcal{F}(\Delta)$ is invariant to permutation, this holds also when $f$ and $g$ are not comonotone. ■

**Proof.** **[Proposition 5]** Consider $\mathcal{P} \in \mathbb{B}$ where $\mathcal{P} = \sum_i \alpha_i \Delta_i$. Condition $(i)$ is a necessary condition since expected utility maximizers w.r.t. $c_\mathcal{P}$ are a subclass of the agents we consider, second order stochastic dominance w.r.t. this equiprobability distribution is necessary. A second subclass of agents are those for which $\mathcal{F}(\mathcal{P}) = \mathcal{P}$. Since $\mathcal{P}$ is the core of a belief function, the multi-prior model is equivalent to the Choquet Expected utility model and we can apply Scarsini's results, that is, $f$ must second order stochastically dominates $g$ w.r.t. the Choquet capacity that corresponds to $\mathcal{P}$. The second order dominance conditions can be written as follows: $\forall t \in [0, M]$,

$$\int_t^M p_f(\{s | f(s) \geqslant x\}) dx \geq \int_t^M p_g(\{s | g(s) \geqslant x\}) dx$$

where $p_f$ and $p_g$ are the probability distribution such that $\forall s \in S(\mathcal{P})$, $p_f(s) = \sum_{i \in J_f(s)} \alpha_i$ and $p_g(s) = \sum_{i \in J_g(s)} \alpha_i$ where $J_f(s) = \{i | \nexists s' \in S(\Delta_i) \text{ s.th.} f(s') < f(s) \text{ or s.th.} f(s') =$

$f(s)$ and $s' < s\}$ and $J_g(s) = \{i | \nexists s' \in S(\Delta_i)$ s.th. $g(s') < g(s)$ or s.th. $g(s') = g(s)$ and $s' < s\}$.

Clearly, for all $t \in [0, M]$, $\min_{p \in \mathcal{P}} \int_t^M p(\{s | f(s) \geqslant x\}) dx = \int_t^M p_f(\{s | f(s) \geqslant x\}) dx$, and $\min_{p \in \mathcal{P}} \int_t^M p(\{s | g(s) \geqslant x\}) dx = \int_t^M p_g(\{s | g(s) \geqslant x\}) dx$, which shows that condition $(ii)$ is a restatement of Scarsini's results.

Since we focus on functionals of the form $\theta \min_{p \in \mathcal{P}} \int u \circ f dp + (1-\theta) \int u \circ f dc_{\mathcal{P}}$, conditions $(i)$ and $(ii)$ are clearly sufficient. ■

**Proof.** [**Proposition 6**] We first show that $C_h^1 \leq C_h^s$ for all $h$ and all $s \neq 1$.

Assume this is not the case. W.l.o.g. assume that $C_1^1 > C_1^s$ for some $s \neq 1$. It must then be the case that there exists $h \neq 1$, say $h = 2$, such that $C_2^1 < C_1^1$, since otherwise one cannot have that $\sum_h C_h^1 = e^1 < \sum_h C_h^s = e^s$. Consider the allocation $\bar{C}$ equal to $C$, except for the following changes $\bar{C}_1^1 = C_1^1 - \varepsilon, \quad \bar{C}_2^1 = C_2^1 + \varepsilon, \quad \bar{C}_1^s = C_1^1 + \varepsilon', \quad \bar{C}_2^s = C_2^s - \varepsilon'$, where $\varepsilon, \varepsilon'$ are positive real numbers small enough so that the ranking is preserved. Taking Taylor expansion, one has:

$$V_1(\bar{C}_1) - V_1(C_1) = \left(\theta_1 + \frac{1-\theta_1}{S}\right) \varepsilon' u_1'(C_1^s) - \frac{1-\theta_1}{S} \varepsilon u_1'(C_1^1) + \varepsilon' \alpha(\varepsilon') - \varepsilon \beta(\varepsilon)$$

where $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ tend to zero when $\varepsilon \to 0$.

For $h = 2$, two cases have to be considered, depending on whether $C_2^s$ is the lowest consumption across all states or not. In the first case,

$$V_2(\bar{C}_2) - V_2(C_2) = \left(\theta_2 + \frac{1-\theta_2}{S}\right) \varepsilon u_2'(C_2^1) - \frac{1-\theta_2}{S} \varepsilon' u_2'(C_2^s) + \varepsilon \alpha(\varepsilon) - \varepsilon' \beta(\varepsilon')$$

In the second case,

$$V_2(\bar{C}_2) - V_2(C_2) = \frac{1-\theta_2}{S} \varepsilon u_2'(C_2^1) - \frac{1-\theta_2}{S} \varepsilon' u_2'(C_2^s) + \varepsilon \alpha(\varepsilon) - \varepsilon' \beta(\varepsilon')$$

We simply report the argument in the first case, but a similar argument holds for the second case. The allocation $\bar{C}$ Pareto dominates $C$ if it is possible to find $\varepsilon$, $\varepsilon'$ such that

$$\begin{cases} \left(\theta_1 + \frac{1-\theta_1}{S}\right) \varepsilon' u_1'(C_1^s) - \frac{1-\theta_1}{S} \varepsilon u_1'(C_1^1) & > & 0 \\ \left(\theta_2 + \frac{1-\theta_2}{S}\right) \varepsilon u_2'(C_2^1) - \frac{1-\theta_2}{S} \varepsilon' u_2'(C_2^s) & > & 0 \end{cases}$$

or, equivalently,

$$\begin{cases} \frac{\varepsilon'}{\varepsilon} \frac{\theta_1 + \frac{1-\theta_1}{S}}{\frac{1-\theta_1}{S}} & > & \frac{u_1'(C_1^1)}{u_1'(C_1^s)} \\ \frac{\varepsilon}{\varepsilon'} \frac{\theta_2 + \frac{1-\theta_2}{S}}{\frac{1-\theta_2}{S}} & > & \frac{u_2'(C_2^s)}{u_2'(C_2^1)} \end{cases}$$

The right hand side of the two inequalities are less than 1 by construction and concavity of the $u_h$s. Hence a sufficient condition for these inequalities to be satisfied is that the left hand side terms are greater than 1. These conditions can be written

$$\frac{\theta_2 + \frac{1-\theta_2}{S}}{\frac{1-\theta_2}{S}} > \frac{\varepsilon'}{\varepsilon} > \frac{\frac{1-\theta_1}{S}}{\theta_1 + \frac{1-\theta_1}{S}}$$

Hence, the existence of $\varepsilon, \varepsilon'$ such that this holds is always ensured since $\theta_1 \neq 0$. This establishes that $C_h^1 \leq C_h^s$ for all $h$ and all $s \neq 1$.

Next, we show that Pareto optimal allocations are such that $C_h^2 \leq C_h^3 \leq \cdots \leq C_h^n$. This can be done directly (for a similar argument see the proof of Proposition 4.1. in Chateauneuf, Dana, and Tallon (2000)), or noticing that agents respect second-order stochastic dominance (w.r.t. $p$ conditioned on $\{2, \ldots, S\}$) and invoke Landsberger and Meilijson (1994) results. ■

# References


Ahn, D. (2003): "Ambiguity without a state space," mimeo, Stanford University.

Chateauneuf, A., R.-A. Dana, and J.-M. Tallon (2000): "Optimal risk-sharing rules and equilibria with Choquet expected utility," *Journal of Mathematical Economics*, 34, 191–214.

Ellsberg, D. (1961): "Risk, ambiguity, and the Savage axioms," *Quarterly Journal of Economics*, 75, 643–669.

Epstein, L. (1999): "A definition of uncertainty aversion," *Review of Economic Studies*, 66, 579–608.

Gajdos, T., J.-M. Tallon, and J.-C. Vergnaud (2004): "Decision making with imprecise probabilistic information," *Journal of Mathematical Economics*, Forthcoming.

Ghirardato, P. (2004): "Defining Ambiguity and Ambiguity Attitude," in *Uncertainty in Economic Theory: A collection of essays in honor of David Schmeidler's 65th birthday*, ed. by I. Gilboa. Routledge Publishers.

Ghirardato, P., and M. Marinacci (2002): "Ambiguity aversion made precise: a comparative foundation and some implications," *Journal of Economic Theory*, 102, 251–282.

Gilboa, I., and D. Schmeidler (1989): "Maxmin expected utility with a non-unique prior," *Journal of Mathematical Economics*, 18, 141–153.

Hayashi, T. (2003): "Information, Subjective Belief and Preference," mimeo, University of Rochester.

Intergovernmental Panel on Climatic Change (2001): *Climate change 2001: the scientific basis*. Cambridge University Press, Cambridge, UK.

Jaffray, J.-Y. (1989): "Linear utility for belief functions," *Operations Research Letters*, 8, 107–112.

Klibanoff, P., M. Marinacci, and S. Mukerji (2003): "A Smooth Model of Decision Making Under Uncertainy," mimeo 11/2003, ICER.

Landsberger, M., and I. Meilijson (1994): “Co-monotone allocations, Bickel-Lehmann dispersion and the Arrow-Pratt measure of risk aversion,” *Annals of Operations Research*, 52, 97–106.

Manski, C. (2003): *Partial Identification of Probability Distributions*. Springer Verlag, New York.

Moss, R., and S. Schneider (2000): “Uncertainties in the IPCC TAR: Recommendations to lead authors for more consistent assessment and reporting,” in *Guidance Papers on the Cross Cutting Issues of the Third Assessment of the IPCC*, ed. by T. Pachauri, T. Taniguchi, and K. Tanaka, pp. 33–51. World Meteorological Organization, Geneva.

Mukerji, S., and J.-M. Tallon (2001): “Ambiguity aversion and incompleteness of financial markets,” *Review of Economic Studies*, (68), 883–904.

Olszewski, W. (2002): “Preferences Over Sets of Lotteries,” mimeo, Northwestern University.

Rothschild, M., and J. Stiglitz (1970): “Increasing risk I: a definition,” *Journal of Economic Theory*, 2, 225–243.

Scarsini, M. (1992): “Dominance conditions in non-additive expected utility theory,” *Journal of Mathematical Economics*, 21(2), 173–184.

Schmeidler, D. (1989): “Subjective probability and expected utility without additivity,” *Econometrica*, 57(3), 571–587.

Stinchcombe, M. (2003): “Choice and games with ambiguity as sets of probabilities,” mimeo, University of Texas, Austin.

Tapking, J. (2004): “Axioms for preferences revealing subjective uncertainty and uncertainty aversion,” *Journal of Mathematical Economics*, forthcoming.

Titus, J. G., and V. Narayanan (1996): “The risk of sea level rise,” *Climate Change*, 33, 151–212.

Wakai, K. (2004): “Two-Fund Separation under Homogeneous Ambiguity,” mimeo, SUNY at Buffalo.

Walley, P. (1991): *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London.

Wang, T. (2003): “A class of multi-prior preferences,” Discussion paper, University British Columbia.

Yaari, M. (1969): “Some Remarks on Measures of Risk Aversion and on Their Uses,” *Journal of Economic Theory*, 1, 315–329.