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M. CHAZAL¹
E. JOUINI²
R. TAHRAOUI³

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¹ CREST-LFA, 15 boulevard Gabriel Péri, 92245 Malakoff Cedex, France. Fax: +33 1 41 17 76 66 et CEREMADE, Université Paris Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France. *Corresponding author* - Email : chazal@ensae.fr

² CEREMADE, Université Paris Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France.

³ IUFM de Rouen, 2 rue du Tronquet, BP 18, Mont Saint-Aignan, 76131 Rouen, France et CEREMADE, Université Paris Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France.

Production Planning and Inventories Optimization: A Backward Approach in the Convex Storage Cost Case

Marie Chazal^{ab*}, Elyès Jouini^b, Rabah Tahraoui^{bc}

^a *CREST, 15 Boulevard Gabriel Péri, 92245 Malakoff Cedex, France*

^b *CEREMADE, Université Paris Dauphine, Place du Maréchal de Lattre de Tassigny,
75775 Paris Cedex 16, France*

^c *IUFM de Rouen, 2 rue du Tronquet, BP 18, Mont-Saint-Aignan, 76131 Rouen, France*

*Corresponding author. E-mail: chazal@ensae.fr. Fax: +33 1 41 17 76 66. Mailing address:
LFA-CREST, Timbre J320, 15 boulevard Gabriel Péri, 92245 Malakoff Cedex, France.

Abstract

We study the deterministic optimization problem of a profit-maximizing firm which plans its sales/production schedule. The firm controls both its production and sales rates and knows the revenue associated to a given level of sales, as well as its production and storage costs. The revenue and the production cost are assumed to be respectively concave and convex. In [4], we provide an existence result and derive some necessary conditions of optimality. Here, we further assume that the storage cost is convex. This allows us to relate the optimal planning problem to the study of a backward integro-differential equation, from which we obtain an explicit construction of the optimal plan.

Résumé

Nous étudions un problème de gestion optimale de production avec coût de stockage, dans un cadre purement déterministe. La firme peut contrôler à la fois son taux de vente et son taux de production. Les fonctions de revenu et de coût de production sont supposées être respectivement concave et convexe. Dans [4], nous obtenons un résultat d'existence et des conditions nécessaires d'optimalité pour une fonction de coût de stockage générale. Ici, supposant en outre que le coût de stockage est convexe, nous ramenons la résolution du problème de planification à l'étude d'une équation integro-différentielle rétrograde qui nous permet de déterminer explicitement le plan optimal de production/vente.

Key words: Production planning, inventory management, integro-differential backward equations.

Mots clés : Planification de production, gestion de stocks, équations integro-différentielles rétrogrades.

Introduction

We consider a firm which produces and sells a good which can be stored. The firm acts in continuous time on a finite period in order to maximize dynamically its profit. The instantaneous profit of the firm is the revenue entailed by the instantaneous sales, diminished by the cost of the instantaneous production, and by the cost of storage of the current inventories. Our approach of this production planning and inventory management problem is in the same vein as the one launched in 1958 by K. J. Arrow, S. Karlin and H. Scarf [1]. Many contributions to this theory have been brought from the 50's until now, with many different approaches. Authors generally consider firms that do not have any control on the level of their sales that is driven by the (possibly stochastic) demand. A considerable part of this literature is devoted to the production-smoothing models which were initiated by C. Holt, F. Modigliani, J. Muth and H. Simon [5]. Some prominent alternative is given by the (S, s) models of inventory behaviour. See notably Scarf [6] for a basic existence result and A. Sulem [7] for an explicit solution of a diffusion inventory problem.

In this paper, we work in a competitive and deterministic context. Following, L. Arvan and L. N. Moses [2], we assume that the firm controls not only its production rate but also its sales rate (through the choice of the price level). It knows the revenue associated to the selling of x units of goods, the cost of producing y units, and the cost of storing S units of the good. We assume that the marginal revenue is non-increasing and that the marginal cost of production is non-decreasing, i.e. the revenue function is concave and the production cost function is convex.

The sales/production planning problem of the profit-maximizing firm is formulated as an optimal control problem where the controls, namely the sales and production paths are integrable. In other words, the cumulative production and sales processes are assumed to be absolutely continuous.

This problem has been studied by the same authors in [4] with a general storage cost function. We proved an existence result for a relaxed problem in which the cumulative sales process is allowed to have a jump at time 0 (i.e. the firm is allowed to make a partial depletion of its eventual initial inventory at time 0). We also derived the first order conditions of optimality for both problems and thus provided a qualitative description of the optimal plans. In particular, we establish the following result: the optimal inventory level must decrease until it reaches 0. It is then kept null by producing for immediate selling.

In the present article, we go further in this analysis. For this purpose, we

assume that the storage cost is convex. In this context, we first see that the initial problem has a solution if and only if the relaxed one has a solution without jump at time 0. Second, the first order conditions of [4] allows us to characterize the (unique) optimal plan for the relaxed problem. Using this characterization, we relate the relaxed optimization problem to the study of a class of solutions of a backward integro-differential equation, indexed by a class of admissible terminal conditions. We establish existence and uniqueness of a maximal solution for this backward integro-differential equation. We then study the behavior of the solution with respect to the terminal condition. This allows us to provide a constructive description of the optimal plan. The optimal plan is determined by selecting the greatest terminal condition r such that a certain functional of the solution of the backward integro-differential equation, representing the inventory level at time $0+$, $S_0(r)$, remains lower than the exogenous initial inventory s_0 . The difference $\alpha = s_0 - S_0(r)$ corresponds to the size of the jump of the cumulative sales process at time 0. After 0, the sales and production rates are explicitly determined in function of the corresponding maximal solution.

We obtain that there exists an exogenous threshold $\bar{s}_0$ on the initial inventory above which the size of the jump, α , is positive. The economic interpretation is the following. If the initial inventory s_0 entails too high storage costs, i.e. s_0 is greater than $\bar{s}_0$, then, it is optimal for the firm to sell out immediately the quantity $s_0 - \bar{s}_0$, so as to reduce its initial inventory to $\bar{s}_0$. If the initial inventory is lower than $\bar{s}_0$, then the firm has no interest in selling out immediately. The level $\bar{s}_0$ appears as the maximal level that it can afford to hold. We also obtain that this threshold depends on the length of the planning period and we observe some qualitative differences between the optimal sales/production plans obtained on the short planning periods and on the long ones.

The paper is organized as follows. Section 1 provides a precise description of the model and recalls some basic results of [4]. Section 2 is devoted to the backward characterization of the optimal plan for the relaxed problem. The constructive resolution of the production planning problem is given in Section 3. The proofs of our main results are given in Section 4 by the way of the study of the above mentioned backward integro-differential equation.

1 The Model Formulation

The firm acts in continuous time on a finite period $[0, T]$. It is endowed with an initial inventory of $s_0 \in \mathbb{R}^+$ units of the good.

A sales/production plan is represented by a couple (x, y) of functions in $L_+^1[0, T]$, the set of nonnegative elements of $L^1[0, T]$, where $x(t)$ (resp. $y(t)$) is the sales (resp. production) rate in units of the good at time t . In other words $\int_0^t x(u)du$ (resp. $\int_0^t y(u)du$) is the cumulative quantity of the good sold out (resp. produced) up to time t . We shall say that $(x, y) \in L_+^1[0, T] \times L_+^1[0, T]$ is a sales/production plan if the induced inventory $S^{(x,y)}$ satisfies

$$S^{(x,y)}(t) \triangleq s_0 + \int_0^t y(u)du - \int_0^t x(u)du \geq 0, \quad \forall t \in [0, T].$$

This means that the company must never be out of stock. We denote by $\mathcal{A}$ the set of all sales/production plans:

$$\mathcal{A} \triangleq \{(x, y) \in L_+^1[0, T] \times L_+^1[0, T] \mid S^{(x,y)}(t) \geq 0, \forall t \in [0, T]\}.$$

When selling out at the rate $x(t)$ at time t , the firm has a revenue rate of $\pi(x(t))$. The cost of producing at the rate $y(t)$ at time t is $c(y(t))$. Both π and c are continuous, non-decreasing functions on $\mathbb{R}^+$. They satisfy $\pi(0) = 0$, $c(0) = 0$ and $\pi(x) > 0$, $c(x) > 0$, for all positive x . The function π (resp. c) is assumed to be concave (resp. convex).

The cost of storing an amount $S(t)$ of goods at time t is denoted by $s(S(t))$. The function s is assumed to be continuous, non-decreasing on $\mathbb{R}^+$ and to satisfy $s(0) = 0$ and $s(S) > 0$, for all positive S .

Given the discount rate $\lambda > 0$, the profit over time induced by $(x, y) \in \mathcal{A}$ is defined by

$$J(x, y) \triangleq \int_0^T e^{-\lambda t} [\pi(x(t)) - c(y(t)) - s(S^{(x,y)}(t))] dt.$$

Observe that by concavity of π and Jensen's inequality

$$\int_0^T e^{-\lambda t} \pi(x(t)) dt < \infty.$$

The functions c and s being nonnegative, it follows that J is well defined as a map from $\mathcal{A}$ into $\mathbb{R} \cup \{-\infty\}$.

The profit-maximizing company plans its sales/production schedule by solving the following optimization problem

$$\sup_{(x,y) \in \mathcal{A}} J(x, y). \tag{1}$$

It turns out that the function J may fail to have a maximum on $\mathcal{A}$. This is typically the case when s_0 is too high (see section 3). Nevertheless, it was proved in [4] that existence holds for some relaxed problem that we now describe.

The sales rate is no longer described by an integrable function, but by a non-negative finite Borel measure on $[0, T]$ which has its singular part positively proportional to the Dirac measure at 0. In this framework, for a sales path equal to $\alpha\delta_0 + x$, where $x \in L_+^1[0, T]$ represents the absolutely continuous part of the considered Borel measure, the cumulative sales process is given by

$$X(0) = 0 \text{ and } X(t) = \alpha + \int_0^t x(u)du, \quad \forall t \in (0, T].$$

This means that the firm is allowed to sell out, at time 0, a share α of its initial inventory.

The production path is still assumed to be integrable. A sales/production plan is now a triplet (α, x, y) in $\mathbb{R}^+ \times L_+^1[0, T] \times L_+^1[0, T]$ which satisfies the inventory constraint

$$S^{(\alpha, x, y)}(t) = s_0 + \int_0^t y(u)du - \alpha - \int_0^t x(u)du \geq 0, \quad \forall t \in (0, T].$$

For $t = 0$, we set $S^{(\alpha, x, y)}(0) = s_0$. The inventory level $S^{(\alpha, x, y)}$ can have a downward jump ($\alpha \geq 0$) at $0+$.

We denote by $\mathcal{B}$ the set of relaxed sales/production plans:

$$\mathcal{B} \triangleq \{(\alpha, x, y) \in \mathbb{R}^+ \times L_+^1[0, T] \times L_+^1[0, T] \mid S^{(\alpha, x, y)}(t) \geq 0, \forall t \in (0, T]\}.$$

The relaxed profit is defined on $\mathcal{B}$ by

$$\mathcal{F}(\alpha, x, y) = \alpha\dot{\pi}(\infty) + \int_0^T e^{-\lambda t} [\pi(x(t)) - c(y(t)) - s(S^{(\alpha, x, y)}(t))]dt$$

where, we have set

$$\dot{\pi}(\infty) \triangleq \lim_{x \rightarrow \infty} \pi(x)/x$$

which is well defined in $\mathbb{R}^+$, by concavity and nonnegativity of π . This is the price at which the firm can sell at an infinite rate. It is also the lowest price accessible for the company. However, since holding inventories has a cost, the firm may take advantage of an immediate depletion, even at this price. Observe that, when π is differentiable, we have

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{x} = \lim_{x \rightarrow \infty} \dot{\pi}(x).$$

The relaxed optimization problem is

$$\sup_{(\alpha, x, y) \in \mathcal{B}} \mathcal{F}(\alpha, x, y). \quad (2)$$

It was proved in [4] that

Theorem 1 $\mathcal{F}$ has a maximum in $\mathcal{B}$ and $\max_{(\alpha, x, y) \in \mathcal{B}} \mathcal{F}(\alpha, x, y) = \sup_{(x, y) \in \mathcal{A}} J(x, y)$.

Notice that in the context of [4], where the storage cost is not assumed to be convex, uniqueness is not guaranteed in Problem (2). In this article, we shall go further in the analysis of the set of solutions of both problems. For this purpose, we require some regularity conditions on π , c and s and we assume that s is convex. In particular, this will allow us to assert that Problem (2) has a unique solution and thus that Problem (1) has a (unique) solution if and only if the solution of Problem (2) is in $\{0\} \times \mathcal{A}$.

To be more precise, we shall work under the following conditions.

Standing assumption (H).

- (i) $\text{Argmax}(\pi - c) = \{a\}$ for some $a > 0$.
- (ii) π is differentiable on $(0, \infty)$ and $\dot{\pi}$ is continuous and one to one on $[a, \infty)$.
- (iii) c is differentiable on $\mathbb{R}^+$ and $\dot{c}$ is continuous and one to one on $[0, a]$.
- (iv) s is continuously differentiable on $\mathbb{R}^+$.
- (v) s is convex.

Assumptions (i) to (iv) were already required in [4] in order to obtain a precise characterization of the set of plans which satisfy the first order conditions of optimality in Problem (2). The existence result has been obtained under a weaker assumption: the functions π , c and s are continuous, and $\pi - c$ admits a, possibly not unique, maximum.

Theorem 2 Problem (2) has a unique solution (α, x, y) . Problem (1) has a solution if and only if $\alpha = 0$. If $\alpha = 0$ then, (x, y) is the unique solution of (1).

Proof. Existence for Problem (2) is stated in Theorem 1 and was proved in [4]. Moreover, Theorem 7 in [4] states that, if (α, x, y) is a solution then $x \geq a \geq y$ a.e. Since π and c are respectively strictly concave on $[a, \infty)$ and strictly convex on $[0, a]$, and since s is convex, uniqueness in Problem (2) is straightforward. By noticing that

$$(0, x, y) \in \mathcal{B} \Leftrightarrow (x, y) \in \mathcal{A} \text{ and } \mathcal{F}(0, \cdot)|_{\mathcal{A}} \equiv J$$

and recalling that by Theorem 1

$$\max_{(\alpha, x, y) \in \mathcal{B}} \mathcal{F}(\alpha, x, y) = \sup_{(x, y) \in \mathcal{A}} J(x, y)$$

the proof of Theorem 2 is easy to complete. $\square$

In light of Theorem 2, we see that planning the optimal sales/production schedule can be done by solving the relaxed problem. Indeed, if the initial problem (1) has a solution (x, y) then, $(\alpha = 0, x, y)$ is the solution of the relaxed problem (2). Since, we further exhibit situations where the solution of Problem (2) is not regular i.e. $\alpha > 0$ (see section 3), meaning that Problem (1) has no solution, we must in actual fact consider Problem (2) to solve the planning problem. We shall see that for some revenue and costs functions π , c and s and some initial inventory s_0 , the firm can not act in an optimal way without getting rid at time 0 of a certain share of s_0 . From now on, we focus our attention on the relaxed problem and what we call the optimal plan is the unique solution (α, x, y) of Problem (2).

2 Characterization of the Optimal Plan

In this section we begin with recalling some characterization and thus some qualitative description of the optimal plan derived from the first order conditions obtained in [4]. We then introduce some backward formulation of these optimality conditions in order to reach our main goal here: to provide a constructive resolution of the planning problem.

2.1 The first order conditions

It was shown in [4] that the optimal plan must be such that there is no inventory accumulation. In particular, if the firm has no starting inventories ($s_0 = 0$) then it adopts a static strategy consisting in producing for immediate sales. Moreover, the concavity of the revenue and the convexity of the cost of production imposes to the firm to minimize the variations of its sales and production rates, so that it must produce and sell at the same constant rate a which maximizes $\pi - c$. To sum up, when $s_0 = 0$, the optimal plan is $(0, a, a)$. Let us now turn to the case where s_0 is positive.

Standing assumption. *The initial inventory s_0 is positive.*

As a direct consequence of Theorem 7 in [4], we have the following characterization of the solution of Problem (2).

Proposition 3 $(\alpha, x, y) \in \mathcal{B}$ is the solution of (2) if and only if it satisfies:

(I) x (resp. y) is non-increasing (resp. non-decreasing) and $x \geq a \geq y$ on $[0, T]$,

(II) $S^{(\alpha, x, y)}(T) = 0$ and $T_0 \triangleq \inf\{t \in (0, T] \mid S^{(\alpha, x, y)}(t) = 0\} > 0$,

(III) if $T_0 < T$ then,

a) $S^{(\alpha, x, y)} = 0$ on $[T_0, T]$,

b) $x = y = a$ on $(T_0, T]$,

c) $x(T_0-) = a$.

(IV) $S^{(\alpha, x, y)}$ is decreasing on $(0, T_0]$,

(V) x and y are both continuous on $(0, T_0)$ and satisfy

$$e^{-\lambda t} \dot{\pi}(x(t)) = \dot{\pi}(x(0+)) + \int_0^t e^{-\lambda u} \dot{S}^{(\alpha, x, y)}(u) du, \quad \forall t \in (0, T_0) \quad (3)$$

$$y(t) = g(x(t)), \quad \forall t \in (0, T_0) \quad (4)$$

$$\text{where } g(r) \triangleq \begin{cases} \dot{c}^{-1}(\dot{\pi}(r)) & \text{if } \dot{\pi}(r) > \dot{c}(0) \\ 0 & \text{elsewhere} \end{cases}, \quad \forall r \in [a, \infty). \quad (5)$$

(VI) If $\alpha > 0$ then $x(0+) = \infty$.

Proof. By Theorem 7 in [4], items (I) to (VI) are equivalent to:

$$\limsup_{\substack{\varepsilon > 0 \\ \varepsilon \rightarrow 0}} \left\{ \frac{\mathcal{F}((\alpha, x, y) + \varepsilon(\beta, h, k)) - \mathcal{F}(\alpha, x, y)}{\varepsilon} \right\} \leq 0,$$

for all $(\beta, h, k) \in \mathbb{R} \times L^1[0, T] \times L^1[0, T]$ such that $(\alpha, x, y) + \varepsilon_0(\beta, h, k) \in \mathcal{B}$ for some $\varepsilon_0 > 0$. Then, Proposition 3 holds by concavity of $\mathcal{F}$ and uniqueness in Problem (2). $\square$

Proposition 3 furnishes some qualitative properties of the optimal plan which were obtained in [4] without assuming that the cost of storage is convex. The main property is that the inventory level is non-increasing and must be null at the end of the period. This phenomenon arises directly from the concavity of the revenue, the convexity of the cost of production and the fact that holding inventories has a cost.

Let us now put in light the qualitative description provided by Proposition 3. The optimal way to deplete the initial inventory is in two phases. This leads

to a three phases sales/production plan. The first phase is devoted to the selling activity. The firm begins with possibly depleting a share $\alpha \geq 0$ of its initial inventory at time 0, selling at an infinite rate. The sales rate is then non-increasing or equivalently the marginal revenue $t \rightarrow \dot{\pi}(x(t))$ is non-decreasing. If the marginal revenue overtakes the lowest marginal cost of production $\dot{c}(0)$ then, the production activity actually starts (see (4) and (5)). During this second phase, the sales rate and the production rate are such that the marginal revenue and the marginal production cost remain equal: $\dot{c}(y(t)) = \dot{\pi}(x(t))$. The production rate is non-decreasing. During this destocking stage the sales rate is greater than a and the production rate lower. If inventories are all cleared before the end of the period ($T_0 < T$) then, the third phase starts : production and sales are at the same constant rate, a , maximizing the instantaneous profit.

Remark that by (II), even if the firm is allowed to get rid of its whole initial inventory at time 0, it does not. This is mainly due to the fact that α is sold out at time 0 at an infinite rate and hence at the lowest price $\dot{\pi}(\infty)$.

2.2 The Backward Characterization

We now turn to our first step on the way to our constructive resolution of the planning problem. This first step consists in providing a backward procedure to find the optimal plan (α, x, y) . This is Proposition 4 below which states some sufficient conditions of optimality which are expressed by means of some integro-differential backward equation that must be satisfied by the (possibly translated) sales rate x .

For sake of readability we introduce a notation which will be useful to express the difference between the optimal sales rate and the optimal production rate. We shall put:

$$\delta(r) \triangleq r - g(r) \quad , \quad \forall r \in [a, \infty).$$

The following remark embodies some useful properties of g and δ .

Remark 1 1. Under assumption (H) and by construction the function g is continuous and decreasing on $[a, \infty)$. Moreover, it satisfies $g(a) = \dot{c}^{-1}(\dot{\pi}(a)) = a$ and takes values in $[0, a]$.

2. The function δ is continuous and increasing on $[a, \infty)$. It satisfies $\delta(a) = 0$ and hence it is positive on (a, ∞) . Moreover $\lim_{r \rightarrow \infty} \delta(r) = \infty$.

We can now state our backward procedure to find the optimal plan.

Proposition 4 Consider some function w and some couple $(r, \tau_0) \in [a, \infty) \times [0, T)$ such that:

(i) $w(T) = r$, w is continuous and decreasing on $(\tau_0, T]$,

(ii)

$$\text{either } (r, \tau_0) \in (a, \infty) \times \{0\}, \text{ or } (r, \tau_0) \in \{a\} \times [0, T), \quad (6)$$

$$e^{-\lambda t} \dot{\pi}(w(t)) = e^{-\lambda T} \dot{\pi}(r) - \int_t^T e^{-\lambda u} \dot{s} \left(\int_u^T \delta(w(s)) ds \right) du, \quad \forall t \in (\tau_0, T], \quad (7)$$

$$\text{either } \int_{\tau_0}^T \delta(w(u)) du = s_0, \text{ or } \int_{\tau_0}^T \delta(w(u)) du < s_0 \text{ and } w(\tau_0+) = \infty. \quad (8)$$

Then

$$\begin{pmatrix} \alpha \\ x \\ y \end{pmatrix} = \begin{pmatrix} s_0 - \int_{\tau_0}^T \delta(w(u)) du \\ w(\cdot + \tau_0) \mathbf{1}_{(0, T-\tau_0]} + a \mathbf{1}_{(T-\tau_0, T]} \\ g(w(\cdot + \tau_0) \mathbf{1}_{(0, T-\tau_0]} + a \mathbf{1}_{(T-\tau_0, T]}) \end{pmatrix} \quad (9)$$

Proof. See Appendix. $\square$

First, Proposition 4 tells us that the optimal production rate can be obtained as an exogenous function of the optimal sales rate: we have $y = g(x)$ on the whole planning period. Therefore, in order to find the optimal plan, it suffices to determine the optimal sales policy (α, x) . Second, we see that the optimal sales rate x shall be obtained in terms of some solution of equation (7), characterized by the boundary conditions (6) and (8).

In the next section, by using the study of the set of solutions of the integro-differential equation involved in Proposition 4:

$$BW(r) : e^{-\lambda t} \dot{\pi}(w(t)) = e^{-\lambda T} \dot{\pi}(r) - \int_t^T e^{-\lambda u} \dot{s} \left(\int_u^T \delta(w(s)) ds \right) du, \quad t \leq T$$

when r varies in $[a, \infty)$ we shall achieve our main aim: to provide a constructive resolution of the production planning problem.

3 Constructive resolution of the production planning problem

In addition to the explicit determination of the optimal plan, we shall give necessary and sufficient conditions for this optimal plan to be *regular* ($\alpha = 0$) i.e. without depletion at time 0. Given the revenue and costs functions and the length of the planning period, the regularity of the optimal plan depends only on the level of the initial inventory s_0 . More precisely, it appears that there is a positive

(possibly infinite) level $\bar{s}_0$, depending on π , c , s and T such that: if $s_0 \leq \bar{s}_0$ (resp. $s_0 > \bar{s}_0$) then the solution is (not) regular. We shall also exhibit some qualitative distinctions between the optimal schedule obtained on a long planning period or on a short one. All proofs are reported to Section 4.

We start with the existence and uniqueness of a maximal solution for $BW(r)$, given any terminal condition $r \in [a, \infty)$. From this maximal feature we deduce a necessary and sufficient condition for the solution to explode at some given time. This property may be needed when applying Proposition 4 in order to determine the optimal plan (see 8). We also check that the solution has the continuity and monotonicity properties required by Proposition 4. For this purpose we work under the following technical assumptions.

Standing Assumptions

- (H π) The inverse of $\dot{\pi}|_{[a, \infty)}$ is locally Lipschitz on $(\dot{\pi}(\infty), \dot{\pi}(a)]$.
- (H c) The inverse of $\dot{c}|_{[0, a]}$ is locally Lipschitz on $[\dot{c}(0), \dot{c}(a)]$.
- (H s) The function s is strictly convex and $\dot{s}$ is locally Lipschitz on $\mathbb{R}^+$.

Theorem 5 For every $r \in [a, \infty)$, there exists a unique couple $(\tau(r), w)$, where $\tau(r) \in [-\infty, T)$ and w is a continuous function on $(\tau(r), T]$, such that:

$$e^{-\lambda t} \dot{\pi}(w(t)) = e^{-\lambda T} \dot{\pi}(r) - \int_t^T e^{-\lambda u} \dot{s} \left(\int_u^T \delta(w(s)) ds \right) du, \quad \forall t \in (\tau(r), T],$$

and

$$\text{either } \tau(r) = -\infty, \quad \text{or } \tau(r) > -\infty \text{ and } w(\tau(r)+) = \infty. \quad (10)$$

In addition, w is decreasing and takes values in $[a, \infty)$ and $\delta \circ w$ is continuous, decreasing and nonnegative on $(\tau(r), T]$.

By using changes of variable, it is easy to deduce from Theorem 5 the following time homogeneity property for the equation $BW(r)$ (invariance by translation on the terminal time).

Remark 2 Let $r \in [a, \infty)$ and $T_1, T_2 \in [0, \infty)$. If $(w_1, \tau_1(r))$ (resp. $(w_2, \tau_2(r))$) is the maximal solution of $BW(r)$ with terminal time $T = T_1$ (resp. T_2) then we have: $T_1 - \tau_1(r) = T_2 - \tau_2(r)$ and $w_1(t) = w_2(t + T_2 - T_1)$, for any $t \in (\tau_1(r), T_1]$.

Therefore the length of the domain of the maximal solution of $BW(a)$, $T_a \triangleq T - \tau(a)$, depends only on π , c and s and not on the length of the planning period T . In the sequel, we shall say that the planning period is *long* (resp. *short*) if $\tau(a) > 0$ (resp. $\tau(a) \leq 0$).

Notation Given $r \in [a, \infty)$, if $(\tau(r), w)$ is associated to r by Theorem 5 we shall write in short w^r and say that w^r is the maximal solution of $BW(r)$.

We denote by τ the map from $[a, \infty)$ into $[-\infty, T)$ defined by Theorem 5.

3.1 The Long Planning Period Case

Theorem 6 Assume that $\tau(a) > 0$. The limit $s^a \triangleq \int_{\tau(a)}^T \delta(w^a(u))du$ exists in $(0, \infty]$.

1. If $s_0 \leq s^a$ then there exists a unique $\tau_0 \in [0, T)$ such that $\int_{\tau_0}^T \delta(w^a(u))du = s_0$ and the optimal plan is regular, given by

$$\begin{pmatrix} \alpha \\ x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ w^a(\cdot + \tau_0)\mathbf{1}_{(0, T-\tau_0]} + a\mathbf{1}_{(T-\tau_0, T]} \\ g(w^a(\cdot + \tau_0)\mathbf{1}_{(0, T-\tau_0]} + a\mathbf{1}_{(T-\tau_0, T]}) \end{pmatrix}$$

2. If s^a is finite and $s_0 > s^a$ then the optimal plan is not regular, given by

$$\begin{pmatrix} \alpha \\ x \\ y \end{pmatrix} = \begin{pmatrix} s_0 - \int_{\tau(a)}^T \delta(w^a(u))du \\ w^a(\cdot + \tau(a))\mathbf{1}_{(0, T-\tau(a)]} + a\mathbf{1}_{(T-\tau(a), T]} \\ g(w^a(\cdot + \tau(a))\mathbf{1}_{(0, T-\tau(a)]} + a\mathbf{1}_{(T-\tau(a), T]}) \end{pmatrix}$$

From Theorem 6, we see that when the planning period is long ($\tau(a) > 0$), the optimal plan is regular if and only if the initial inventory is below the exogenous level

$$\overline{s_0} = s^a = \int_{\tau(a)}^T \delta(w^a(s))ds.$$

Notice that it may be infinite. In that case, whatever the initial inventory is, the optimal plan is regular. Also remark that, by the time homogeneity property of $BW(a)$, the level s^a does not depend on the length of the planning period.

From an economic point of view, $\overline{s_0}$ acts as a threshold. If the initial inventory s_0 is greater than $\overline{s_0}$, then holding such a (high) amount of stock is too expensive. It is then optimal for the firm to sell out immediately the quantity $\alpha = s_0 - \overline{s_0}$, so as to reduce its initial inventory to $\overline{s_0}$. This is the maximal level that it can afford to hold. Recall that the quantity α is sold at an infinite rate and hence at the lowest price $\dot{\pi}(\infty)$. Therefore the firm has no interest in selling more than necessary. This explain why it sells exactly $s_0 - \overline{s_0}$.

For every $s_0 > \overline{s_0}$, the regular part (x, y) of the optimal plan is the same, it corresponds to the optimal plan obtained for an initial inventory level equal to $\overline{s_0}$.

Only the size of the jump on the sales $\alpha = s_0 - \bar{s}_0$ changes with s_0 . If $s_0 \leq \bar{s}_0$ then, the optimal plan is regular and actually depends on s_0 since it is obtain from the translation of w^a on the interval $(0, T - \tau_0]$ where τ_0 solves $s_0 = \int_{\tau_0}^T \delta(w^a(u))du$.

Remark that the existence of such a threshold and the corresponding properties of the optimal plan will also hold in the case of a short planning period. We shall determine this threshold but not reproduce the discussion.

To the contrary what follows is specific to the long planning periods. First, the threshold $\bar{s}_0$ does not depend on T . Second, the plan that the firm adopts is in two (non trivial) phases. The first phase consists in selling out the initial inventory in an optimal way. During the second phase, the firm follows the just-in-time strategy, producing and selling at the same constant rate a , until the end of the period. The depletion phase does not depend on T . It is the one associated to the optimal plan obtained on the period $[0, T_a]$. In view of item 2, this property is straightforward when $s_0 > \bar{s}_0 = \int_{\tau(a)}^T \delta(w^a(u))du$. When $s_0 \leq \bar{s}_0$, it follows from the time homogeneity property of $BW(a)$ (see Remark 2).

We end up this first step of the constructive resolution by stressing the fact that, in the case of a long planning period, nothing more than the computation of w^a is necessary to solve the planning problem. Let us now turn to the short planning period case.

3.2 The Short Planning Period Case

Theorem 7 *Assume that $\tau(a) \leq 0$. The integral $\int_0^T \delta(w^a(u))du$ exists in $(0, \infty]$.*

If $s_0 \leq \int_0^T \delta(w^a(u))du$ then there exists a unique $\tau_0 \in [0, T)$ such that $\int_{\tau_0}^T \delta(w^a(u))du = s_0$ and the optimal plan is regular given by:

$$\begin{pmatrix} \alpha \\ x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ w^a(\cdot + \tau_0)\mathbf{1}_{(0, T-\tau_0]} + a\mathbf{1}_{(T-\tau_0, T]} \\ g(w^a(\cdot + \tau_0)\mathbf{1}_{(0, T-\tau_0]} + a\mathbf{1}_{(T-\tau_0, T]}) \end{pmatrix}$$

We see that even if the planning period is short, when the initial inventory s_0 is lower than $\int_0^T \delta(w^a(u))du$, the firm has time to clear out its stock before the horizon T , selling out at the rate $x = w^a(\cdot + \tau_0)$ on $(0, T - \tau_0]$ and it can afford a static management final phase: $x = y = a$ on $(\tau_0, T]$. We shall find out that this is not the case when the initial inventory is greater than $\int_0^T \delta(w^a(u))du$.

Notice that if $\tau(a) = 0$, the quantity $\int_0^T \delta(w^a(u))du$ may be infinite and in that case, Theorem 7 put an end to the constructive resolution of the planning

problem. From now on we assume that

$$\int_0^T \delta(w^a(s))ds < \infty.$$

When $s_0 > \int_0^T \delta(w^a(s))ds$, we shall prove that the optimal sales rate is given by the solution $w^{\hat{r}}$ of $BW(\hat{r})$, for some $\hat{r} > a$, on the whole planning period. The difference between the optimal sales rate and the optimal production rate which is given by $\delta \circ w^{\hat{r}}$ shall therefore be positive (see Theorem 5) on the whole planning period.

The terminal condition $\hat{r}$ characterizing the optimal sales rate shall be the greatest one for which the domain of $w^{\hat{r}}$ is at least $(0, T]$ and such that the induced backward cumulative inventory level at time 0+, $\int_0^T \delta(w^{\hat{r}}(u))du$, remains lower or equal to s_0 . To see this we investigate the monotonicity and continuity with respect to r of the following map

$$\begin{aligned} S_0 : \{r \in [a, \infty) \mid \tau(r) \leq 0\} &\longrightarrow \mathbb{R}^+ \cup \{\infty\} \\ r &\longmapsto \int_0^T \delta(w^r(s))ds. \end{aligned}$$

In order to prove the existence of a threshold on the initial inventory below (above) which the optimal plan is (not) regular and in order to determine this threshold we also investigate the boundness of S_0 : we seek for the greatest (finite) value that it can reach when the terminal condition r varies in $\{r \in [a, \infty) \mid \tau(r) \leq 0\}$. It appears that there are two cases to keep distinct. Theorem 8 below provides the optimal plan in each of these two cases. The study of the map S_0 can be red in detail in Section 4.

Theorem 8 *Assume that $\tau(a) \leq 0$ and that $\int_0^T \delta(w^a(u))du < \infty$.*

The set $D_0 \triangleq \{r \in [a, \infty) \mid \tau(r) \leq 0 \text{ and } \int_0^T \delta(w^r(u))du < \infty\}$ is a bounded interval with minimum equal to a . Let us denote $d \triangleq \sup D_0$.

Consider $s_0 > \int_0^T \delta(w^a(u))du$.

1. *If $\sup_{r \in D_0} \int_0^T \delta(w^r(u))du = \infty$ then there exists a unique $\hat{r} \in (a, d)$ such that $\int_0^T \delta(w^{\hat{r}}(u))du = s_0$ and the optimal plan is regular, given by:*

$$(\alpha, x, y) = (0, w^{\hat{r}}, g(w^{\hat{r}})).$$

2. *If $\sup_{r \in D_0} \int_0^T \delta(w^r(u))du < \infty$ then $\tau(d) = 0$, $\int_0^T \delta(w^d(u))du < \infty$ and*

(a) if $s_0 \leq \int_0^T \delta(w^d(u))du$ then there exists a unique $\hat{r} \in (a, d]$ such that $\int_0^T \delta(w^{\hat{r}}(u))du = s_0$ and the optimal plan is regular, given by:

$$(\alpha, x, y) = (0, w^{\hat{r}}, g(w^{\hat{r}})) ;$$

(b) if $s_0 > \int_0^T \delta(w^d(u))du$ then the optimal plan is not regular, given by:

$$(\alpha, x, y) = (s_0 - \int_0^T \delta(w^d(u))du, w^d, g(w^d)) .$$

We have seen from Theorem 7 that, when $s_0 \leq \int_0^T \delta(w^a(u))du$, the firm can afford to sell out its inventory at some rate which ranges a part of the path of w^a . From Theorem 8 we see that, if the initial inventory is higher than $\int_0^T \delta(w^a(u))du$ then the firm must move out its stock faster: the optimal sales rate is given by $x = w^{\hat{r}}$ on $(0, T]$ with $\hat{r} > a$ so that it is greater than w^a on the whole planning period (see Proposition 14 in Section 4 for the increasing feature of w^r with respect to r). In this context, the production may never start. Typically, if $\dot{c}(0) > \dot{\pi}(\infty)$ and $\hat{r} > \dot{\pi}^{-1}(\dot{c}(0))$ then $\dot{\pi}(x) = \dot{\pi}(w^{\hat{r}}) \leq \dot{\pi}(\hat{r}) < \dot{c}(0)$ on $(0, T]$ and hence, by definition of g , $y = g(x) = 0$ on $(0, T]$. In any case, the inventory is totally depleted right in T , there is no phase where the firm produces for immediate sales at the constant rate a . This makes qualitatively different the optimal plan obtained on a short planning period and the one obtained on a long period.

From Theorems 7 and 8 we check that there is positive threshold above (below) which the optimal plan is with (without) depletion at time 0. It is given by

$$\bar{s}_0 = \sup_{\{r \in [a, \infty) | \tau(r) \leq 0, \int_0^T \delta(w^r(u))du < \infty\}} \int_0^T \delta(w^r(u))du .$$

Recall that, by time homogeneity of BW (see Remark 2), the length T_a of the domain of the maximal solution of $BW(a)$ and the threshold $\bar{s}_0$ obtained for long planning periods do not depend on T . To the contrary, when the planning period is short, the threshold may depend on T . Let us denote $(w_{T_a}^r, \tau_{T_a}(r))$ the maximal solution of $BW(r)$ with terminal time T_a , for any $r \in [a, \infty)$. By using Remark 2 we see that, given π, c, s and hence T_a , the dependence of $\bar{s}_0$ with respect to T is given by:

$$\bar{s}_0(T) = \sup_{\{r \in [a, \infty) | \tau_{T_a}(r) \leq 0, \int_{T_a-T}^{T_a} \delta(w_{T_a}^r(u))du < \infty\}} \int_{T_a-T}^{T_a} \delta(w_{T_a}^r(u))du$$

It would be of interest to study the monotonicity of $\bar{s}_0$ with respect to T and it would be economically funded to obtain that the shorter the period is, the lower

the threshold is. Also, notice that, while we indeed prove the existence of such a threshold, we do not provide some conditions for deciding whether it is finite or infinite. These are directions for future research.

4 Study of the Backward Integro-Differential Equation: Proofs of Theorems 5, 6, 7 and 8

4.1 Maximal solutions for BW : Proof of Theorem 5

Let us set

$$D \triangleq (\dot{\pi}(\infty), \dot{\pi}(a)].$$

By Assumption (H) we can define Δ on D by

$$\Delta(v) \triangleq \delta(\dot{\pi}^{-1}(v)) = \dot{\pi}^{-1}(v) - \dot{c}^{-1}(v)\mathbf{1}_{v > \dot{c}(0)}, \quad \forall v \in D.$$

For later purpose, we make the following remark.

Remark 3 *By Assumptions (H) , $(H\pi)$ and (Hc) , Δ is nonnegative, locally Lipschitz and bounded from above by $\dot{\pi}^{-1}$ on D .*

In the sequel, we use the following notation: for all interval I and all set $E \subset \mathbb{R}$, we denote by $C(I; E)$ the set of functions that are continuous on I with values in E . For $E = \mathbb{R}$, we simply write $C(I)$ for $C(I; E)$.

We shall obtain Theorem 5 as a Corollary of

Theorem 9 *For all $\theta \in D$, there exists a unique couple $(\tau, z) \in [-\infty, T) \times C((\tau, T]; D)$, such that z satisfies the following equation on $(\tau, T]$*

$$\widetilde{BW}(\theta) : z(t) = \theta - \int_t^T \left[\lambda z(u) + \dot{s} \left(\int_u^T \Delta(z(s)) ds \right) \right] du,$$

and such that,

$$\text{either } \tau = -\infty, \quad \text{or } \tau > -\infty \text{ and } z(\tau+) = \dot{\pi}(\infty). \quad (11)$$

Moreover, z is increasing.

If Theorem 9 holds then, for all $r \in [a, \infty)$, since $\dot{\pi}(r) \in D$, there exists a unique couple $(\tau, z) \in [-\infty, T) \times C((\tau, T]; D)$ such that

$$z(t) = \dot{\pi}(r) - \int_t^T \left[\lambda z(u) + \dot{s} \left(\int_u^T \Delta(z(s)) ds \right) \right] du, \quad \forall t \in (\tau, T] \quad (12)$$

and such that the boundary condition (11) holds. Since z is continuous on $(\tau, T]$, by Remark 3 and by Assumption (Hs) , we see that the function

$$u \longrightarrow \lambda z(u) + \dot{s} \left(\int_u^T \Delta(z(s)) ds \right)$$

is continuous on $(\tau, T]$. We then deduce from (12) that z is in actual fact continuously differentiable on $(\tau, T]$ with

$$\dot{z}(t) = \lambda z(t) + \dot{s} \left(\int_t^T \Delta(z(s)) ds \right), \quad \forall t \in (\tau, T]. \quad (13)$$

Then, it is easy to check that z satisfies:

$$e^{-\lambda t} z(t) = e^{-\lambda T} \dot{\pi}(r) - \int_t^T e^{-\lambda u} \dot{s} \left(\int_u^T \Delta(z(s)) ds \right) du, \quad \forall t \in (\tau, T]. \quad (14)$$

Under Assumption (H) , we can set $w \triangleq \dot{\pi}^{-1} \circ z$ and see that, since z is in $C((\tau, T]; D)$, w is in $C((\tau, T]; [a, \infty))$. From (14) and (11), we deduce that w solves $BW(r)$ on $(\tau, T]$ and satisfies the following boundary condition:

$$\text{either } \tau = -\infty, \quad \text{or } \tau > -\infty \text{ and } w(\tau+) = \infty.$$

Besides, since z is increasing, it follows from assumption (H) again, that w is decreasing. Since $w(T) = r \in [a, \infty)$, it follows that w takes values in $[a, \infty)$ on $(\tau, T]$. Recalling that, by Remark 1.2, the function δ is continuous, increasing and nonnegative on $[a, \infty)$, we finally obtain that $\delta \circ w$ is continuous, decreasing and nonnegative on $(\tau, T]$. This concludes the proof of Theorem 5.

We now turn to the proof of Theorem 9. It is based on Propositions 10, 11 and 12 below and an induction argument. More precisely, some regularity and Lipschitzianity properties of the right-hand side of the equation $\widetilde{BW}(\theta)$ are given in Proposition 10. Using this properties we shall prove that for every terminal condition $\theta \in D$, the equation $\widetilde{BW}(\theta)$ has at least one solution: there exists some continuous function z which satisfies $\widetilde{BW}(\theta)$ on some nonempty interval $(\gamma, T]$. We will also provide some result of continuous dependence of the solution with respect to the terminal condition. This is Proposition 11. With Proposition 12, we will see that a solution (γ, z) which does not satisfy the boundary condition (11) can be extended in another solution on some larger interval.

In the sequel, for $\phi \in C([a, b])$, we denote $\|\phi\|_{[a, b]} \triangleq \sup_{a \leq s \leq b} |\phi(s)|$.

The right-hand side of the backward equation is studied in terms of the following operator: for fixed $\eta \in [-\infty, T)$,

$$\begin{aligned} F_\eta : (\eta, T] \times C((\eta, T]; D) &\longrightarrow \mathbb{R} \\ (t, v) &\longmapsto \lambda v(t) + \dot{s} \left(\int_t^T \Delta(v(s)) ds \right). \end{aligned}$$

Proposition 10 *Let $\eta \in [-\infty, T)$ and let L be some compact subset of D .*

- (o) *The operator F_η is well-defined and takes values in $(0, \infty)$.*
- (i) *For all $v \in C((\eta, T]; D)$, the function $t \longmapsto F_\eta(t, v)$ is continuous on $(\eta, T]$.*
- (ii) *There exists some constant $K_{\eta, L} > 0$ such that, for all $t \in (\eta, T]$,*

$$v, \tilde{v} \in C([t, T]; L) \implies |F_\eta(t, v) - F_\eta(t, \tilde{v})| \leq K_{\eta, L} \|v - \tilde{v}\|_{[t, T]},$$

Proof. Let $v \in C((\eta, T]; D)$. First, since $D = (\dot{\pi}(\infty), \dot{\pi}(a)] \subset (0, \infty)$, v takes values in $(0, \infty)$. Second, from Remark 3, the function $\Delta \circ v$ is continuous and nonnegative on $(\eta, T]$. This implies that the function $t \longmapsto \int_t^T \Delta(v(s)) ds$ is well-defined, taking values in $\mathbb{R}^+$ and continuous on $(\eta, T]$. Since $\dot{s}$ is continuous on $\mathbb{R}^+$, (o) and (i) of Proposition 10 hold.

We now prove (ii). Let $t \in (\eta, T]$. Recall that, by Remark 3,

$$0 \leq \Delta(u) \leq \dot{\pi}^{-1}(u), \quad \forall u \in D.$$

Therefore, since $\dot{\pi}^{-1}$ is decreasing on D and v and $\tilde{v}$ both map $[t, T]$ into $L \subset D$, we have

$$\int_t^T \Delta(v(s)) ds, \int_t^T \Delta(\tilde{v}(s)) ds \in [0, (T - \eta) \dot{\pi}^{-1}(\min L)],$$

Since, by (Hs), the function $\dot{s}$ is locally lipschitz, it follows that there exists some constant $C_{\eta, L} > 0$ such that:

$$\begin{aligned} |F_\eta(t, v) - F_\eta(t, \tilde{v})| &\leq \lambda |v(t) - \tilde{v}(t)| + \left| \dot{s} \left(\int_t^T \Delta(v(s)) ds \right) - \dot{s} \left(\int_t^T \Delta(\tilde{v}(s)) ds \right) \right| \\ &\leq \lambda |v(t) - \tilde{v}(t)| + C_{\eta, L} \left| \int_t^T \Delta(v(s)) ds - \int_t^T \Delta(\tilde{v}(s)) ds \right|. \end{aligned}$$

Using the local lipschitzianity of the function Δ stated in Remark 3, we deduce from the last inequality that there exists some constant $K_L > 0$ such that

$$|F_\eta(t, v) - F_\eta(t, \tilde{v})| \leq \lambda |v(t) - \tilde{v}(t)| + K_L C_{\eta, L} \int_t^T |v(s) - \tilde{v}(s)| ds, \quad (15)$$

and hence, $|F_\eta(t, v) - F_\eta(t, \tilde{v})| \leq K_{\eta, L} \|v - \tilde{v}\|_{[t, T]}$, for some constant $K_{\eta, L}$. This ends the proof of Proposition 10. $\square$

Observe that if $\eta_1 \leq \eta_2$ then, $F_{\eta_1|(\eta_2, T] \times C((\eta_2, T]; D)} \equiv F_{\eta_2}$. Therefore, in the sequel we shall omit the index and always consider F with its maximal set of definition.

Definition 1 *Given $\theta \in D$, we shall say that (γ, z) is a solution of $\widetilde{BW}(\theta)$ if:*

$\gamma \in [-\infty, T)$, $z \in C((\gamma, T]; D)$ and z satisfies the equation $\widetilde{BW}(\theta)$ on $(\gamma, T]$.

Recall that the equation $\widetilde{BW}(\theta)$ reads as follows in terms of F :

$$\widetilde{BW}(\theta) : z(t) = \theta - \int_t^T F(u, z) du, \quad t \leq T.$$

Proposition 11

1. *For all $\theta \in D$, the equation $\widetilde{BW}(\theta)$ has at least one solution. If (γ, z) is such a solution then, z is increasing on $(\gamma, T]$.*
2. *Let $\eta \in (-\infty, T]$ and let L be a compact subset of D . Then, there exists some constant $K_{\eta, L} > 0$ such that: if (γ, z) is a solution of $\widetilde{BW}(\theta)$ and (γ', z') is a solution of $\widetilde{BW}(\theta')$ then, for all $t \in [\eta, T]$ with $t > \gamma, \gamma'$ and such that z and z' both map $[t, T]$ into L we have:*

$$\|z - z'\|_{[t, T]} \leq K_{\eta, L} |\theta - \theta'|.$$

Proof. 1. The proof of the first item is decomposed in three steps. Let $\theta \in D$. First observe that the last assertion is a direct consequence of the nonnegativity of $\dot{s}$ and the positivity of elements of D . We now concentrate on the first assertion. We shall use a Picard's approximations argument.

Step 1. We start with the construction of a suitable set in which we will construct our Picard's approximations. For this purpose, let us fix some $\eta \in (-\infty, T)$ and choose some $b \geq 0$ such that the compact

$$L \triangleq [\theta - b, \min\{(\theta + b), \dot{\pi}(a)\}] \text{ is included in } D. \quad (16)$$

We claim that there exists some constant $M > 0$ for which

$$\begin{aligned} \forall t \in (\eta, T], \quad \forall v \in C([t, T]; D) : \\ \|v - \theta\|_{[t, T]} \leq b \implies |F(t, v)| \leq M. \end{aligned} \quad (17)$$

To see this, first notice that by Proposition 10 (o) and (i), the function $t \longrightarrow F(t, \theta)$ is continuous on $(-\infty, T]$, with positive values and therefore bounded on $[\eta, T]$ by some constant $M_0 > 0$. Let us now prove that (17) holds for $M = K_{\eta, L}b + M_0$ where $K_{\eta, L}$ is given by Proposition 10 (ii).

Let $t \in (\eta, T]$ and $v \in C([t, T]; D)$ with $\|v - \theta\|_{[t, T]} \leq b$. Then, by definition of L , v is in $C([t, T]; L)$. We therefore have by Proposition 10 (ii)

$$|F(t, v) - F(t, \theta)| \leq K_{\eta, L}\|v - \theta\|_{[t, T]} \leq K_{\eta, L}b$$

and hence, $|F(t, v)| \leq K_{\eta, L}b + M_0 \triangleq M$. This provides (17).

We are now in position to construct a suitable set. Recalling that $\eta < T$, we may choose some $\gamma \in (\eta, T)$ such that

$$0 < T - \gamma < \min\{T - \eta, \frac{b}{M}, \frac{1}{K_{\eta, L}}\}. \quad (18)$$

Our Picard's approximations will be constructed on the set

$$\mathcal{S} \triangleq \{ v \in C([\gamma, T]; L) \mid v(T) = \theta, |v(t) - v(u)| \leq M|t - u|, \forall t, u \in [\gamma, T] \}.$$

Observe that $\mathcal{S}$ is closed for the pointwise convergence on $[\gamma, T]$.

We now define an operator P on $\mathcal{S}$ by:

$$P(v)(t) \triangleq \theta - \int_t^T F(u, v)du, \quad \forall t \in [\gamma, T].$$

We shall prove in the following steps that P admits a fixed point in $\mathcal{S}$.

Step 2. We first prove that P maps $\mathcal{S}$ into $\mathcal{S}$. Let $v \in \mathcal{S}$. First recall that by Proposition 10 (o) we have

$$F(u, v) > 0, \quad \forall u \in [\gamma, T] \quad (19)$$

and notice that by (17), (18) and by definition of $\mathcal{S}$ we have

$$F(u, v) \leq M, \quad \forall u \in [\gamma, T]. \quad (20)$$

We now check that $P(v)$ takes values in L . Let $t \in [\gamma, T]$. By equations (19) and (20) and by definition of γ (see 18), we have

$$P(v)(t) = \theta - \int_t^T F(u, v)du \leq \theta \leq \min\{(\theta + b), \dot{\pi}(a)\},$$

and

$$P(v)(t) = \theta - \int_t^T F(u, v)du \geq \theta - M(T - \gamma) \geq \theta - b$$

which implies that $P(v)(t) \in L$. The proof of Step 2 is concluded by writing

$$|P(v)(t) - P(v)(u)| = \left| \int_u^t F(s, v) ds \right| \leq M|t - u|, \quad \forall t, u \in [\gamma, T],$$

where the last inequality follows from (20).

Step 3. We shall now construct a Cauchy sequence in $\mathcal{S}$ that converges to a fixed point of P . Let $(z_n)_n$ be the sequence defined by

$$z_0 = \theta \text{ on } [\gamma, T] \text{ and } z_{n+1} = P(z_n), \quad \forall n \in \mathbb{N}.$$

Since $\gamma \in (\eta, T]$, it follows from Proposition 10 (ii) that, for all $t \in [\gamma, T]$ we have

$$\begin{aligned} |z_{n+1}(t) - z_n(t)| &= \left| \int_t^T [F(u, z_n) - F(u, z_{n-1})] du \right| \\ &\leq |T - t| K_{\eta, L} \|z_n - z_{n-1}\|_{[t, T]} \\ &\leq ((T - \gamma) K_{\eta, L})^n \|z_1 - z_0\|_{[\gamma, T]}. \end{aligned}$$

By (18), $0 < (T - \gamma) K_{\eta, L} < 1$. Therefore, (z_n) is a Cauchy sequence. Since $\mathcal{S}$ is closed, its limit z is in $\mathcal{S}$. Arguing as above, we see that

$$\|P(z) - P(z_n)\|_{[\gamma, T]} \leq K_{\eta, L} (T - \gamma) \|z - z_n\|_{[\gamma, T]}.$$

It follows that

$$\begin{aligned} \|P(z) - z\|_{[\gamma, T]} &= \|P(z) - P(z_n)\|_{[\gamma, T]} + \|z_{n+1} - z\|_{[\gamma, T]} \\ &\leq K_{\eta, L} (T - \gamma) \|z - z_n\|_{[\gamma, T]} + \|z_{n+1} - z\|_{[\gamma, T]}, \end{aligned}$$

which shows, by passing to the limit, that $P(z) = z$ on $[\gamma, T]$. This completes the proof of the first item of Proposition 11.

2. Let $\eta \in (-\infty, T]$ and let L be a compact subset of D . Consider (γ, z) a solution of $\widetilde{BW}(\theta)$ and (γ', z') a solution of $\widetilde{BW}(\theta')$. Let $t \in [\eta, T]$ with $t > \gamma, \gamma'$ and such that z and z' both map $[t, T]$ into L . Then, for all $u \in [t, T]$, z and z' both map $[u, T]$ into L . Therefore, by (15) in the proof of Proposition 10, there exists some $C = C_{\eta, L} > 0$ such that

$$|F(u, z) - F(u, z')| \leq \lambda |z(u) - z'(u)| + C \int_u^T |z(v) - z'(v)| dv, \quad \forall u \in [t, T].$$

This implies that, for all s in $[t, T]$, we have:

$$|z(s) - z'(s)| \leq |\theta - \theta'| + \int_s^T |F(u, z) - F(u, z')| du$$

$$\begin{aligned}
&\leq |\theta - \theta'| + \int_s^T \left(\lambda |z(u) - z'(u)| + C \int_u^T |z(v) - z'(v)| dv \right) du \\
&\leq |\theta - \theta'| + (\lambda + C(T - \eta)) \int_s^T |z(u) - z'(u)| du \\
&\leq |\theta - \theta'| + C_{\eta, L} \int_s^T |z(u) - z'(u)| du.
\end{aligned}$$

By applying Gronwall's Lemma to the last inequality and since $\eta \leq s$ we obtain:

$$|z(s) - z'(s)| \leq |\theta - \theta'| e^{C_{\eta, L}(T-s)} \leq |\theta - \theta'| e^{C_{\eta, L}(T-\eta)},$$

which ends the proof of Proposition 11. $\square$

We now turn to the extension Proposition.

Proposition 12 *Let (γ, z) be a solution of $\widetilde{BW}(\theta)$ for some $\theta \in D$. If $\gamma > -\infty$ and if z satisfies*

$$z(\gamma+) > \dot{\pi}(\infty)$$

then $\widetilde{BW}(\theta)$ has a solution $(\hat{\gamma}, \hat{z})$ which satisfies: $\hat{\gamma} < \gamma$ and $\hat{z}|_{(\gamma, T]} \equiv z$.

Proof. Let (γ, z) be a solution of $\widetilde{BW}(\theta)$ satisfying $\gamma > -\infty$. Since z is increasing the limit $z(\gamma+) = \lim_{t \searrow \gamma} z(t)$ exists in $\overline{D} = [\dot{\pi}(\infty), \dot{\pi}(a)]$. Let us assume that $z(\gamma+) > \dot{\pi}(\infty)$. We have to prove that there exists some $\hat{\gamma} < \gamma$ and some $\hat{z} \in C((\hat{\gamma}, T]; D)$ such that $\hat{z}$ satisfies $\widetilde{BW}(\theta)$ on $(\hat{\gamma}, T]$ i.e.

$$\hat{z}(t) = \theta - \int_t^T \left[\lambda \hat{z}(u) + \dot{s} \left(\int_u^T \Delta(\hat{z}(s)) ds \right) \right] du, \quad \forall t \in (\hat{\gamma}, T],$$

and such that

$$\hat{z}|_{(\gamma, T]} \equiv z.$$

To do this, recall that z satisfies

$$z(t) = \theta - \int_t^T \left[\lambda z(u) + \dot{s} \left(\int_u^T \Delta(z(s)) ds \right) \right] du, \quad \forall t \in (\gamma, T],$$

and hence, by passing to the limit as t goes to γ

$$z(\gamma+) = \theta - \int_{\gamma}^T \left[\lambda z(u) + \dot{s} \left(\int_u^T \Delta(z(s)) ds \right) \right] du.$$

Therefore, it suffices to prove that

$$\left\{ \begin{array}{l} \text{there exists some } \hat{\gamma} < \gamma \text{ and some } \tilde{z} \in C((\hat{\gamma}, \gamma]; D) \text{ such that: } \forall t \in (\hat{\gamma}, \gamma], \\ \tilde{z}(t) = z(\gamma+) - \int_t^{\gamma} \left[\lambda \tilde{z}(u) + \dot{s} \left(\int_u^{\gamma} \Delta(\tilde{z}(s)) ds + \int_{\gamma}^T \Delta(z(s)) ds \right) \right] du. \end{array} \right. \quad (21)$$

Indeed, the result will then hold for $(\hat{\gamma}, \hat{z})$ where $\hat{z}$ is defined by setting $\hat{z}|_{(\hat{\gamma}, \gamma]} \equiv \tilde{z}$ and $\hat{z}|_{(\gamma, T]} \equiv z$.

Let us establish (21). For ease of notation, let us write $I \triangleq \int_{\gamma}^T \Delta(z(s))ds$. Now, for all $\eta \in [-\infty, \gamma]$ consider the operator defined by

$$\begin{aligned} \tilde{F}_{\eta} : (\eta, \gamma] \times C((\eta, \gamma]; D) &\longmapsto \mathbb{R} \\ (t, v) &\longrightarrow \lambda v(t) + \dot{s} \left(\int_t^T \Delta(v(s))ds + I \right) \end{aligned}$$

Since the translation $\dot{s}(\cdot + I)$ inherits of the increasing and Lipschitzian feature of $\dot{s}$, one should be convinced that $\tilde{F}_{\eta}$ satisfies the regularity and Lipschitzianity properties gathered in Proposition 10. This properties together with the fact that $z(\gamma+)$ is in D are sufficient to let the Picard's approximations argument be valid to prove the existence of some $\hat{\gamma} < \gamma$ and some $\tilde{z} \in C((\hat{\gamma}, \gamma]; D)$ such that $\tilde{z}$ satisfies (21) on $(\hat{\gamma}, \gamma]$. This ends the proof of Proposition 12. $\square$

We are now in position to give the

Proof of Theorem 9. Let us first establish uniqueness. Let (γ, z) and (γ', z') satisfying the requirements of Theorem 9. Fix some arbitrary $t \in (\max(\gamma, \gamma'), T]$. Since z and z' are in $C((\max\{\gamma, \gamma'\}, T]; D)$, there exists some compact $L \subset D$ such that z and z' both map $[t, T]$ into L . It follows from item 2 of Proposition 11 that $z \equiv z'$ on $[t, T]$. By arbitrariness of t in $(\max(\gamma, \gamma'), T]$, we then have $z \equiv z'$ on $(\max(\gamma, \gamma'), T]$, so that $\gamma, \gamma' \geq \max\{\gamma, \gamma'\}$. Therefore $\gamma = \gamma'$ and hence $(\gamma, z) = (\gamma, z')$. This provides uniqueness.

We now prove that existence holds. Define

$$E \triangleq \{(\gamma, z) \text{ solution of } \widetilde{BW}(\theta)\}.$$

By item 1 of Proposition 11, E is not empty. It is easy to check that E is inductive for the order defined by:

$$(\gamma_2, z_2) \succeq (\gamma_1, z_1) \Leftrightarrow \gamma_2 \leq \gamma_1 \text{ and } z_2 \equiv z_1 \text{ on } (\gamma_1, T].$$

Thus, by Zorn's lemma, it admits a maximal element (τ, z) . If $\tau = -\infty$, then the proof is concluded. We now assume that $\tau \neq -\infty$. We have to prove that $z(\tau+) = \dot{\pi}(\infty)$. Assume to the contrary that $z(\tau+) > \dot{\pi}(\infty)$. Then, by Proposition 12 there exists some $(\hat{\tau}, \hat{z})$ in E such that $\hat{\tau} < \tau$ and $\hat{z}|_{(\hat{\tau}, T]} \equiv z$. This contradicts the maximal feature of (τ, z) in E . The proof of Theorem 9 is completed. $\square$

4.2 Proofs of Theorems 6 and 7

Proof of Theorem 6. Recall that by Theorem 5 the function $\delta \circ w^a$ is continuous, decreasing and nonnegative on $(\tau(a), T]$. Therefore the function

$$t \in (\tau(a), T] \longrightarrow S(t, a) \triangleq \int_t^T \delta(w^a(s)) ds \in \mathbb{R}^+$$

is well-defined, decreasing and continuous on $(\tau(a), T]$. It has a limit as t goes to $\tau(a)$ which is:

$$s^a \triangleq \sup_{t \in (\tau(a), T]} \int_t^T \delta(w^a(u)) du = \int_{\tau(a)}^T \delta(w^a(u)) du \in \mathbb{R}^+ \cup \{\infty\}$$

Since $\tau(a) < T$, it is positive. Assume that $\tau(a) > 0$.

1. Recall that $s_0 > 0 = S(T, a)$. So that, if $s_0 \leq s^a = S(\tau(a)+, a)$ the existence of a unique $\tau_0 \in [\tau(a), T)$ such that $s_0 = \int_{\tau_0}^T \delta(w^a(u)) du$ is insured by the Mean-Value Theorem and the above stated properties of the function $S(\cdot, a)$. In addition, since $\tau(a) > 0$, we have $\tau_0 > 0$. At this point, it is easy to see that, by Theorem 5, w^a satisfies the requirements of Proposition 4 with the couple (a, τ_0) . We can therefore conclude that the optimal plan is given by (9) which reads:

$$\begin{pmatrix} \alpha \\ x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ w(\cdot + \tau_0) \mathbf{1}_{(0, T-\tau_0]} + a \mathbf{1}_{(T-\tau_0, T]} \\ g(w(\cdot + \tau_0) \mathbf{1}_{(0, T-\tau_0]} + a \mathbf{1}_{(T-\tau_0, T]}) \end{pmatrix}$$

It is regular.

2. Assume that s^a is finite and that $s_0 > s^a$. Since $\tau(a) > 0 > -\infty$, by Theorem 5, we have $w^a(\tau(a)+) = \infty$. Therefore w^a satisfies the requirements of Proposition 4 for the couple $(a, \tau(a))$ and the optimal plan is not regular, given by

$$\begin{pmatrix} \alpha \\ x \\ y \end{pmatrix} = \begin{pmatrix} s_0 - \int_{\tau(a)}^T \delta(w^a(u)) du \\ w^a(\cdot + \tau(a)) \mathbf{1}_{(0, T-\tau(a)]} + a \mathbf{1}_{(T-\tau(a), T]} \\ g(w^a(\cdot + \tau(a)) \mathbf{1}_{(0, T-\tau(a)]} + a \mathbf{1}_{(T-\tau(a), T]}) \end{pmatrix}$$

This completes the proof of Theorem 6. The proof of Theorem 7 is very similar to item 1 above. It is therefore omitted. $\square$

4.3 Study of the Backward Cumulative Inventory Level at time 0+: Proof of Theorem 8

As mentioned in Section 3, Theorem 8 relies on the monotonicity, continuity and boundness properties of the map S_0 which associates to each terminal condition

r such that $\tau(r) \leq 0$ the backward cumulative inventory level at time $0+$ induced by w^r :

$$\begin{aligned} S_0 : \{r \in [a, \infty) \mid \tau(r) \leq 0\} &\longrightarrow \mathbb{R}^+ \cup \{\infty\} \\ r &\longmapsto \int_0^T \delta(w^r(s)) ds. \end{aligned}$$

On key result for the study of the regularity properties of S_0 is the following corollary of item 2 of Proposition 11, which obviously holds under Assumption $(H\pi)$ and Remark 1.2.

Corollary 13 *Let $\eta \in (-\infty, T]$ and let L be a compact subset of $[a, \infty)$. Then, there exists some constant $K_{\eta, L} > 0$ such that: if for some $r, r' \in [a, \infty)$ and some $t \in [\eta, T]$ the functions w^r and $w^{r'}$ both map $[t, T]$ into L then,*

$$\|w^r - w^{r'}\|_{[t, T]} + \|\delta(w^r) - \delta(w^{r'})\|_{[t, T]} \leq K_{\eta, L} |r - r'|.$$

We will also use the following result on the increasing feature of the maximal solution of $BW(r)$ with respect to r .

Proposition 14 *Fix r and r' in $[a, \infty)$ such that $r' < r$. Then, for all $\bar{t} \in [\max\{\tau(r), \tau(r')\}, T]$, we have: $w^{r'} \leq w^r$ on $(\bar{t}, T]$.*

Proof. For ease of notation, we shall write w (resp. w') for w^r (resp. $w^{r'}$). Assume to the contrary that the set $\{t \in (\bar{t}, T] \mid w(t) < w'(t)\}$ is not empty. Then, by continuity of w and w' , and since $w(T) = r > r' = w'(T)$, we have:

$$\sigma \triangleq \sup\{t \in (\bar{t}, T] \mid w(t) < w'(t)\} \in (\bar{t}, T) \text{ and } w'(\sigma) = w(\sigma).$$

By definition of w and w' , we see that, for all $t \in (\bar{t}, \sigma]$,

$$e^{-\lambda t} \dot{\pi}(w(t)) = e^{-\lambda \sigma} \dot{\pi}(w(\sigma)) - \int_t^\sigma e^{-\lambda u} \dot{s} \left(\int_u^T \delta(w(s)) ds \right) du$$

and

$$e^{-\lambda t} \dot{\pi}(w'(t)) = e^{-\lambda \sigma} \dot{\pi}(w'(\sigma)) - \int_t^\sigma e^{-\lambda u} \dot{s} \left(\int_u^T \delta(w'(s)) ds \right) du.$$

Since $w'(\sigma) = w(\sigma)$, this implies that, for all $t \in (\bar{t}, \sigma)$,

$$\frac{\dot{\pi}(w(t)) - \dot{\pi}(w'(t))}{\sigma - t} = \frac{e^{\lambda t}}{\sigma - t} \int_t^\sigma e^{-\lambda u} \left[\dot{s} \left(\int_u^T \delta(w'(s)) ds \right) - \dot{s} \left(\int_u^T \delta(w(s)) ds \right) \right] du.$$

Since the function

$$u \longmapsto e^{-\lambda u} \left[\dot{s} \left(\int_u^T \delta(w'(s)) ds \right) - \dot{s} \left(\int_u^T \delta(w(s)) ds \right) \right]$$

is continuous on $(\bar{t}, T]$, we can let t tend to σ in the above equality to obtain

$$\lim_{t \nearrow \sigma} \left\{ \frac{[\dot{\pi}(w(t)) - \dot{\pi}(w'(t))]}{\sigma - t} \right\} = \dot{s} \left(\int_{\sigma}^T \delta(w'(s)) ds \right) - \dot{s} \left(\int_{\sigma}^T \delta(w(s)) ds \right) \quad (22)$$

We claim that the right-hand side is negative, so that, for some $\eta > 0$, $\dot{\pi}(w) - \dot{\pi}(w') < 0$, on $(\sigma - \eta, \sigma)$, and hence, by the increasing feature of $\dot{\pi}$ on $[a, \infty)$, $w > w'$ on $(\sigma - \eta, \sigma)$. This will contradict the definition of σ and finally proves Proposition 14.

To see that the above claim holds, recall that δ is increasing on D . Since $\dot{s}$ is increasing, the claim follows from the definition of σ and the fact that the continuous functions w and w' satisfy $w(T) = r > r' = w'(T)$. $\square$

From now on, we assume that

$$\tau(a) \leq 0 \quad \text{and} \quad \int_0^T \delta(w^a(u)) du < \infty,$$

and we switch back to the study of S_0 which will provide the proof of Theorem 8. We begin with examining the set

$$\tau^{-1}(\mathbb{R}^-) = \{r \in [a, \infty) \mid \tau(r) \leq 0\}$$

on which S_0 is defined.

Proposition 15 *The function $\tau : r \in [a, \infty) \mapsto \tau(r) \in [-\infty, T]$ is non-decreasing, right-continuous and satisfies $\lim_{r \rightarrow \infty} \tau(r) = T$. Therefore, the set $\tau^{-1}(\mathbb{R}^-)$ is a bounded interval with $\min \tau^{-1}(\mathbb{R}^-) = a$.*

Proof. We split it in three steps.

Step 1. We first prove that $\tau : r \in [a, \infty) \mapsto \tau(r) \in [-\infty, T]$ is non-decreasing. Fix r and r' in $[a, \infty)$ such that $r' < r$ and denote w' (resp. w) for $w^{r'}$ (resp. w^r). By Proposition 14, we have:

$$w'(t) \leq w(t), \quad \forall t \in (\max\{\tau(r'), \tau(r)\}, T]. \quad (23)$$

Assume that $\tau(r') > \tau(r)$. Then $w'(\tau(r')+) = \infty$ and $w(\tau(r')) < \infty$. By letting t tend to $\tau(r')$ in (23) we obtain a contradiction. Therefore $\tau(r') \leq \tau(r)$.

Step 2. We now prove that τ is right-continuous as a map from $[a, \infty)$ into $[-\infty, T)$. Since τ is non-decreasing it suffices to prove that

$$\inf_{r > r'} \tau(r) = \tau(r') \text{ for all } r' \in [a, \infty).$$

Assume to the contrary that there exists some $r' \in [a, \infty)$ such that:

$$\ell \triangleq \inf_{r > r'} \tau(r) > \tau(r') \in [-\infty, T) \quad (24)$$

For ease of notation, we write w' for $w^{r'}$. Since $\ell > \tau(r')$, $w'(\ell)$ is finite and we can fix some real number M such that:

$$M > w'(\ell). \quad (25)$$

Since, by Theorem 5, w' is decreasing, we have $M > w'(\ell) > r'$. Therefore, there exists some $r \in [a, M)$ such that

$$0 < r - r' < \frac{M - w'(\ell)}{K_{\ell, [a, M]}}, \quad (26)$$

where $K_{\ell, [a, M]}$ is given by Corollary 13. Moreover, since τ is non-decreasing and by (24), we have:

$$\tau(r) \geq \inf_{z > r'} \tau(z) = \ell > -\infty.$$

It follows from Theorem 5 that $w^r(\tau(r)+) = \infty$. Now, since w^r is continuous, decreasing and satisfies $w^r(T) = r < M < \infty$, by the Mean-Value Theorem there exists some $t_M \in (\tau(r), T)$ such that $w^r(t_M) = M$. Observing that

$$\ell \leq \tau(r) < t_M, \quad (27)$$

we deduce from the decreasing feature of w' together with (26) that

$$w^r(t_M) - w'(t_M) > M - w'(\ell) > K_{\ell, [a, M]}(r - r'). \quad (28)$$

We shall now put in light the required contradiction. Since w^r and w' are decreasing, it follows from the equality $w^r(t_M) = M$, (27) and (25), that they both map the interval $[t_M, T]$ into $[a, M]$. Therefore, applying Corollary 13 with $\eta = \ell$ and $L = [a, M]$ we have

$$0 \leq w^r(t_M) - w'(t_M) \leq K_{\ell, [a, M]}(r - r'),$$

which combined with (28) yields the required contradiction.

Step 3. We finally prove that $\lim_{r \rightarrow \infty} \tau(r) = T$. Assume to the contrary that

$$\bar{T} \triangleq \lim_{r \rightarrow \infty} \tau(r) < T. \quad (29)$$

Considering the sequence $(w^n)_{n \geq a}$, we see that

$$e^{-\lambda \bar{T}} \dot{\pi}(w^n(\bar{T})) - e^{-\lambda T} \dot{\pi}(n) = - \int_{\bar{T}}^T e^{-\lambda t} \dot{s} \left(\int_t^T \delta(w^n(s)) ds \right) dt.$$

Since, by Theorem 5, w^n is decreasing, we have $w^n(\bar{T}) \geq n$, $\forall n \geq a$. Then letting n tend to ∞ in the previous inequality shows that

$$0 \leq (e^{-\lambda \bar{T}} - e^{-\lambda T}) \dot{\pi}(\infty) = \lim_{n \rightarrow \infty} \left\{ - \int_{\bar{T}}^T e^{-\lambda t} \dot{s} \left(\int_t^T \delta(w^n(s)) ds \right) dt \right\} \quad (30)$$

where the inequality holds because $\bar{T} < T$ and $\dot{\pi}(\infty) \geq 0$. In order to obtain a contradiction, we shall prove that the right-hand side is negative. Observe that, since $\dot{s}$ is nonnegative,

$$- \int_{\bar{T}}^T e^{-\lambda t} \dot{s} \left(\int_t^T \delta(w^n(s)) ds \right) dt \leq - \int_{\bar{T}}^{(\bar{T}+T)/2} e^{-\lambda t} \dot{s} \left(\int_t^T \delta(w^n(s)) ds \right) dt. \quad (31)$$

Recall that w^n is decreasing and satisfies $w^n(T) = n$. By Remark 2, we deduce that there exists some $n_0 \geq a$ such that for all $n \geq n_0$,

$$\delta(w^n(s)) \geq \delta(w^n(T)) \geq 2, \quad \forall s \in [(\bar{T} + T)/2, T].$$

Therefore, for all $n \geq n_0$ and $t \in [\bar{T}, (\bar{T} + T)/2]$

$$\int_t^T \delta(w^n(s)) ds \geq \int_{(\bar{T}+T)/2}^T \delta(w^n(s)) ds \geq 2 \left(T - \frac{\bar{T} + T}{2} \right) \geq T - \bar{T},$$

where we used the nonnegativity of δ . By the increasing feature and nonnegativity of $\dot{s}$, this implies that for all $n \geq n_0$,

$$- \int_{\bar{T}}^{(\bar{T}+T)/2} e^{-\lambda t} \dot{s} \left(\int_t^T \delta(w^n(s)) ds \right) dt \leq -\dot{s}(T - \bar{T}) \int_{\bar{T}}^{(\bar{T}+T)/2} e^{-\lambda t} dt. \quad (32)$$

From (31) and (32) we deduce that

$$\lim_{n \rightarrow \infty} \left\{ - \int_{\bar{T}}^T e^{-\lambda t} \dot{s} \left(\int_t^T \delta(w^n(s)) ds \right) dt \right\} \leq -\dot{s}(T - \bar{T}) \int_{\bar{T}}^{(\bar{T}+T)/2} e^{-\lambda t} dt < 0, \quad (33)$$

where the last inequality holds because $\bar{T} < T$ and $\dot{s}$ is increasing, nonnegative and therefore positive on $(0, \infty)$. The required contradiction follows from (30) and (33).

The second assertion Proposition 15 is a direct consequence of the above steps. The proof is completed. $\square$

Let us now turn to the monotonicity, continuity and boundness properties of S_0 .

Proposition 16 1. The function S_0 is well-defined, non-decreasing and left-continuous as a map from $\tau^{-1}(\mathbb{R}^-)$ into $\mathbb{R}^+ \cup \{\infty\}$.

2. The set $D_0 = \{r \in \tau^{-1}(\mathbb{R}^-) \mid S_0(r) \in \mathbb{R}^+\}$ is a nonempty bounded interval with minimum equal to a . Let $d \triangleq \sup D_0$. We have $\tau(d) \geq 0$.

3. The function S_0 is continuous and increasing on its domain D_0 .

4. If $\tau(d) > 0$ then $\sup_{D_0} S_0 = \infty$.

Proof. We shall make use of the following

Remark 4 By Theorem 5, for any $r \in [a, \infty)$, the function $\delta \circ w^r$ is continuous, decreasing and nonnegative on $(\tau(r), T]$. Therefore, for any $r \in [a, \infty)$, the function

$$t \in (\tau(r), T] \longrightarrow S(t, r) \triangleq \int_t^T \delta(w^r(s)) ds \in \mathbb{R}^+$$

is well-defined, decreasing and continuous on $(\tau(r), T]$. In addition, it has a limit as t goes to $\tau(r)$ which is:

$$\sup_{t \in (\tau(r), T]} \int_t^T \delta(w^r(u)) du = \int_{\tau(r)}^T \delta(w^r(u)) du \in \mathbb{R}^+ \cup \{\infty\}.$$

1. Thus, for any $r \in [a, \infty)$ such that $\tau(r) \leq 0$ we have: $S_0(r) = \int_0^T \delta(w^r(u)) du \in \mathbb{R}^+ \cup \{\infty\}$. The function S_0 is thus well-defined as a map from $\tau^{-1}(\mathbb{R}^-)$ into $\mathbb{R}^+ \cup \{\infty\}$. Besides, from Proposition 14 it follows that it is non-decreasing.

Let us prove that it is left-continuous. Since S_0 is non-decreasing, the limit $\lim_{r \nearrow r'} S_0(r)$ exists in $\mathbb{R}^+ \cup \{\infty\}$ for all $r' \in \tau^{-1}(\mathbb{R}^-)$. We shall prove that

$$\lim_{r \nearrow r'} S_0(r) = S_0(r') \text{ in } \mathbb{R}^+ \cup \{\infty\}, \quad \forall r' \in \tau^{-1}(\mathbb{R}^-) \cap (a, \infty). \quad (34)$$

Fix $r' \in \tau^{-1}(\mathbb{R}^-) \cap (a, \infty)$.

We first consider the case $S_0(r') \in \mathbb{R}^+$. Fix $\varepsilon > 0$. By Remark 4, there exists some $t_\varepsilon \in (0, T]$ such that

$$0 \leq S_0(r') - S(t_\varepsilon, r') < \frac{\varepsilon}{3}. \quad (35)$$

Let $r \in \tau^{-1}(\mathbb{R}^-)$ such that $r < r'$. Then, by Proposition 14 we have: $w^r \leq w^{r'}$ on $[t_\varepsilon, T]$. Therefore, w^r and $w^{r'}$ both map $[t_\varepsilon, T]$ into $[a, w^{r'}(t_\varepsilon)]$. Then, by Corollary 13, for $\eta = 0$ and $L = [a, w^{r'}(t_\varepsilon)]$, there exists some $K > 0$ such that

$$0 \leq \delta(w^{r'}) - \delta(w^r) \leq K(r' - r) \quad \text{on } [t_\varepsilon, T], \quad (36)$$

where the first inequality follows from the increasing feature of δ . Since S_0 is non-decreasing, we also have

$$0 \leq S_0(r) - S(t_\varepsilon, r) \leq S_0(r') - S(t_\varepsilon, r) < \frac{\varepsilon}{3}. \quad (37)$$

Now, defining

$$\zeta \triangleq \frac{\varepsilon}{3(T - t_\varepsilon)K}$$

and using (35), (36) and (37), we see that for all $r \in [r' - \zeta, r']$,

$$\begin{aligned} |S_0(r') - S_0(r)| &\leq |S_0(r') - S(t_\varepsilon, r')| + |S(t_\varepsilon, r') - S(t_\varepsilon, r)| + |S(t_\varepsilon, r) - S_0(r)| \\ &\leq \frac{2\varepsilon}{3} + \int_{t_\varepsilon}^T [\delta(w^{r'}(s)) - \delta(w^r(s))] ds \\ &\leq \frac{2\varepsilon}{3} + (T - t_\varepsilon)K(r' - r) \\ &\leq \varepsilon. \end{aligned}$$

This proves (34) when $S_0(r') < \infty$.

We now consider the case where $S_0(r') = \infty$. Assume to the contrary that $M \triangleq \lim_{r \nearrow r'} S_0(r) < \infty$. Since S_0 is non-decreasing, we have

$$M = \sup_{r < r'} S_0(r).$$

In order to obtain a contradiction, we shall find some $r \in \tau^{-1}(\mathbb{R}^-)$ such that

$$r < r' \quad \text{and} \quad S_0(r) > M. \quad (38)$$

First notice that since $S_0(r') = \infty$ we necessarily have $\tau(r') = 0$. Then, by Remark 4, there exists some $t_M \in (0, T]$ such that $S(t_M, r') > 2M$. Let $K \triangleq K_{0, [a, w(t_M)]}$ be given by Corollary 13. We set:

$$r \triangleq \max\left\{r' - \frac{M}{K(T - t_M)}, a\right\}.$$

By construction, $r < r'$ because by assumption $r' > a$. Then, by the same arguments as in the previous case, we can apply Corollary 13 for $\eta = 0$ and $L =$

$[a, w^{r'}(t_M)]$ on $[t_M, T]$ to obtain:

$$\begin{aligned}
S(t_M, r) &= S(t_M, r) - S(t_M, r') + S(t_M, r') \\
&\geq \int_{t_M}^T [\delta(w^r(s)) - \delta(w^{r'}(s))] ds + S(t_M, r') \\
&\geq (T - t_M)K(r - r') + S(t_M, r') \\
&> -M + 2M = M.
\end{aligned}$$

This proves (38) and therefore (34) when $S_0(r') = \infty$. The proof of item 1 is completed.

2. Since S_0 is non-decreasing, its domain D_0 is an interval. Recall that we have assumed that $\tau(a) \leq 0$ and $S_0(a) = \int_0^T \delta(w^a(u)) du < \infty$. Therefore D_0 is nonempty and $\min D_0 = a$. Since D_0 is a subset of $\tau^{-1}(\mathbb{R}^-)$ which is bounded from above, we can define $d \triangleq \sup D_0 \in \mathbb{R}^+$. Assume that $\tau(d) < 0$ then, since by Proposition 15, the function τ is right-continuous, there exists some $d' > d$ such that $\tau(d') < 0$ and hence such that $\int_0^T \delta(w^{d'}(u)) du < \infty$, so that d' is in D_0 and $d' > \sup D_0$. This is a contradiction. Therefore $\tau(d) \geq 0$.

3. We now prove that S_0 is continuous on its domain D_0 . Recall that D_0 is an interval. Since S_0 is left-continuous, we only have to prove that

$$\text{if } r' \in D_0 \text{ and } r' < \sup D_0 \text{ then } \lim_{r \searrow r'} S_0(r) = S_0(r').$$

Let $r' \in D_0$, $r' < \sup D_0$. Then there exists $\bar{r} > r'$ such that $\tau(\bar{r}) \leq 0$ and $S_0(\bar{r}) < \infty$. Fix $\varepsilon > 0$. By Remark 4, there exists some $t_\varepsilon \in (0, T]$ such that:

$$0 \leq S_0(\bar{r}) - S(t_\varepsilon, \bar{r}) < \frac{\varepsilon}{3}.$$

Now, for all $r \in (r', \bar{r}]$, by Proposition 14 and by the increasing feature of δ we have:

$$0 \leq S_0(r') - S(t_\varepsilon, r') \leq S_0(r) - S(t_\varepsilon, r) \leq S_0(\bar{r}) - S(t_\varepsilon, \bar{r}) < \frac{\varepsilon}{3}.$$

Notice that by Proposition 14 again, for all $r \in [r', \bar{r}]$, w^r maps $[t_\varepsilon, T]$ into $[a, w^{\bar{r}}(t_\varepsilon)]$. The proof can therefore be completed by applying Corollary 13, with $\eta = 0$ and $L = [a, w^{\bar{r}}(t_\varepsilon)]$, to $w^{r'}$ and w^r on $[t_\varepsilon, T]$. We have proved that S_0 is continuous on D_0 .

Let us now prove that S_0 is increasing on D_0 . Fix some $r, r' \in D_0$ such that $r < r'$. Since $w^r(T) = r < r' = w^{r'}(T)$, by continuity we have : $w^r < w^{r'}$ on a

subset of $(0, T]$ which has a positive Lebesgue measure. Besides, by Proposition 14, $w^r \leq w^{r'}$ on $(0, T]$. Therefore, since the function δ is increasing, we have:

$$S_0(r) - S_0(r') = \int_0^T [\delta(w^r(u)) - \delta(w^{r'}(u))] du > 0.$$

This ends the proof of item 3.

4. We now prove that if $\tau(d) > 0$ then $\sup_{D_0} S_0 = \infty$. We claim that

$$\lim_{r \nearrow d} w^r(\tau(d)) = \infty. \quad (39)$$

Therefore, there exists some sequence (r_n) in D_0 increasing to d such that

$$\lim_{n \rightarrow \infty} w^{r_n}(\tau(d)) = \infty. \quad (40)$$

Using the fact that $\delta \circ w^{r_n}$ is nonnegative and decreasing, we see that

$$S_0(r_n) = \int_0^T \delta(w^{r_n}(s)) ds \geq \int_0^{\tau(d)} \delta(w^{r_n}(s)) ds \geq \tau(d) \delta(w^{r_n}(\tau(d))).$$

The proof is completed by using (40) and recalling that $\tau(d) > 0$ and that $\lim_{r \rightarrow \infty} \delta(r) = \infty$ (see Remark 2).

We now prove (39). Observe that, by Proposition 14, the function $r \mapsto w^r(\tau(d))$ is non-decreasing on $[a, d)$. We therefore have

$$M \triangleq \lim_{r \nearrow d} w^r(\tau(d)) = \sup_{r < d} w^r(\tau(d)) \quad \text{and} \quad M \in [a, \infty].$$

We shall prove that $M = \infty$. Assume to the contrary that

$$M = \sup_{r < d} w^r(\tau(d)) < \infty.$$

In order to have a contradiction, we shall find some $r' \in [a, d)$ such that

$$w^{r'}(\tau(d)) > M. \quad (41)$$

First notice that, since $\tau(d) > 0 > -\infty$, we have by Theorem 5, $w^d(\tau(d)+) = \infty$. Therefore, since $w^d(T) = d$, by continuity, there exists some $t_M \in (\tau(d), T]$ such that

$$w^d(t_M) = 2 \max\{M, d\}. \quad (42)$$

Let $K \triangleq K_{\tau(d), [a, 2 \max\{M, d\}]}$ be given by Corollary 13 and define

$$r' \triangleq \max \left\{ \left(d - \frac{M}{2K} \right), a \right\}. \quad (43)$$

By definition, $r' \in [a, \infty)$. Moreover, since by assumption $\tau(d) > 0 \geq \tau(a)$ and since τ is non-decreasing, we have $d > a$. Therefore,

$$r' \in [a, d).$$

We now prove (41). Since w^d is decreasing and $w^d(t_M) = 2 \max\{M, d\}$, w^d maps $[t_M, T]$ into $[a, 2 \max\{M, d\}]$. Besides, since $r' < d$, it follows from Proposition 14 that $w^{r'} \leq w^d$ on $[t_M, T]$. Hence, $w^{r'}$ also maps $[t_M, T]$ into $[a, 2 \max\{M, d\}]$. Therefore, by Corollary 13 with $\eta = \tau(d)$ and $L = [a, 2 \max\{M, d\}]$, applied to w^d and $w^{r'}$ we have

$$|w^{r'}(t_M) - w^d(t_M)| \leq K|r' - d|,$$

which reads

$$w^{r'}(t_M) - w^d(t_M) \geq K(r' - d).$$

By (42), we therefore have: $w^{r'}(t_M) \geq 2 \max\{M, d\} + K(r' - d)$. Noticing that, by (43), $K(r' - d) \geq -M/2$, we obtain: $w^{r'}(t_M) \geq 2M - M/2 > M$. Since $t_M > \tau(d)$ and $w^{r'}$ is decreasing, this implies that

$$w^{r'}(\tau(d)) > M,$$

which ends the proof of (41) and therefore proves (39). The proof of Proposition 16 is completed. $\square$

We are now in position to complete the

Proof of Theorem 8. Recall that $\tau(a) \leq 0$ and $\int_0^T \delta(w^a(u))du < \infty$.

1. By Proposition 16 the function S_0 is continuous and increasing on its domain D_0 . Assume that $\sup_{D_0} S_0 = \infty$. Recall that $\min D_0 = a$. Then for $s_0 > \int_0^T \delta(w^a(u))du$, by the Mean-Value Theorem and since S_0 is increasing on D_0 , there exists a unique $\hat{r} \in D_0$, $\hat{r} < \sup D_0 = d$ such that: $s_0 = \int_0^T \delta(w^{\hat{r}}(u))du$. It is then easy to see that $w^{\hat{r}}$ satisfies the requirements of Proposition 4 for the couple $(r, \tau_0) = (\hat{r}, 0)$ and hence that the optimal plan is regular given by:

$$(\alpha, x, y) = (0, w^{\hat{r}}, g(w^{\hat{r}})).$$

2. Assume that $\sup_{D_0} S_0 < \infty$. From item 4 of Proposition 16 it follows that $\tau(d) = 0$. Besides, since S_0 is non-decreasing and left-continuous on $\tau^{-1}(\mathbb{R}^-)$ we have:

$$\sup_{D_0} S_0 = \lim_{\substack{r \rightarrow d \\ r < d}} S_0(r) = S_0(d) = \int_0^T \delta(w^d(u))du$$

and hence, $\int_0^T \delta(w^d(u))du < \infty$ and $D_0 = [a, d]$. Recalling that S_0 is continuous and increasing on its D_0 we see that, if $s_0 \in [S_0(a), S_0(d)]$ then there exists a unique $\hat{r} \in (a, d]$ such that $s_0 = \int_0^T \delta(w^{\hat{r}}(u))du$. We can then conclude as in the previous case that the optimal plan is regular given by:

$$(\alpha, x, y) = (0, w^{\hat{r}}, g(w^{\hat{r}})) .$$

Consider now $s_0 > \int_0^T \delta(w^d(u))du$. Since $\tau(d) = 0 > -\infty$, we have by Theorem 5, $w^d(0+) = \infty$. Th function w^d thus satisfies the requirements of Proposition 4 for the couple $(d, 0)$. The optimal plan is then given by:

$$(\alpha, x, y) = (s_0 - \int_0^T \delta(w^d(u))du, w^d, g(w^d)) .$$

It is not regular. The proof of Theorem 8 is completed. $\square$

Appendix

Proof of Proposition 4. Let w and (r, τ_0) be as in the statement. We shall prove that the plan $(\tilde{\alpha}, \tilde{x}, \tilde{y})$ defined by

$$\tilde{\alpha} \triangleq s_0 - \int_{\tau_0}^T \delta(w(u))du \tag{44}$$

$$\tilde{x} \triangleq w(\cdot + \tau_0)\mathbf{1}_{(0, T-\tau_0]} + a\mathbf{1}_{(T-\tau_0, T]} \tag{45}$$

$$\tilde{y} \triangleq g(\tilde{x}) \tag{46}$$

is in $\mathcal{B}$ and satisfies conditions (I) to (VI) of Proposition 3 which characterize (α, x, y) .

Since w is decreasing on $(\tau_0, T]$ and $w(T) = r \in [a, \infty)$, by definition (see 45) the function $\tilde{x}$ is non-increasing and takes values in $[a, \infty)$ on $(0, T]$. Recall that by Remark 1.1, the function g is non-increasing, taking values in $[0, a]$ on $[a, \infty)$. Therefore, by (46) the function $\tilde{y}$ is non-decreasing and takes values in $[0, a]$ on $(0, T]$. Besides, by (8), we have:

$$\int_{\tau_0}^T \delta(w(u))du \leq s_0 \text{ i.e. } \int_{\tau_0}^T [w(u) - g(w(u))]du \leq s_0$$

and hence, since g takes values in $[0, a]$ and $\tau_0 \geq 0$ we have:

$$\int_{\tau_0}^T w(u)du \leq s_0 + Ta.$$

Since $w \geq 0$, this implies that $w \in L_+^1[\tau_0, T]$ and by construction that $\tilde{x} \in L_+^1[0, T]$. By (8) again, we have $\tilde{\alpha} \geq 0$. Thus $(\tilde{\alpha}, \tilde{x}, \tilde{y}) \in \mathbb{R}^+ \times L_+^1[0, T] \times L_+^1[0, T]$ and $\tilde{x}$ and $\tilde{y}$ satisfy (I).

Let us now prove that

$$S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})} \geq 0 \text{ on } (0, T] \text{ and that (II), (IIIa), (IIIb), and (IV) hold.} \quad (47)$$

We shall obtain that

$$\tilde{T}_0 \triangleq \inf\{t \in (0, T] \mid S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})}(t) = 0\} = T - \tau_0.$$

We begin with proving that $\tilde{x} = \tilde{y} = a$ and $S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})} = 0$ on $(T - \tau_0, T]$. By definition (see 45 and 46) and since $g(a) = a$ we have:

$$\tilde{x} = a = g(a) = g(\tilde{x}) = \tilde{y} \text{ on } (T - \tau_0, T], \quad (48)$$

so that

$$S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})}(t) = S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})}(T - \tau_0), \text{ for all } t \in (T - \tau_0, T]. \quad (49)$$

By (46), (44), (45) and after a change of variable, for any $t \in (0, T - \tau_0]$, we have:

$$\begin{aligned} S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})}(t) &= s_0 - \tilde{\alpha} + \int_0^t [\tilde{y}(u) - \tilde{x}(u)] du \\ &= s_0 - \tilde{\alpha} + \int_0^t [g(\tilde{x}(u)) - \tilde{x}(u)] du \\ &= \int_{\tau_0}^T \delta(w(u)) du + \int_0^t [g(w(u + \tau_0)) - w(u + \tau_0)] du \\ &= \int_{\tau_0}^T \delta(w(u)) du + \int_{\tau_0}^{t+\tau_0} [g(w(u)) - w(u)] du, \end{aligned}$$

that is, by definition of δ ,

$$S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})}(t) = \int_{t+\tau_0}^T \delta(w(u)) du. \quad (50)$$

Therefore $S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})}(T - \tau_0) = 0$ and by (49)

$$S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})}(t) = 0, \text{ for all } t \in [T - \tau_0, T]. \quad (51)$$

In order to complete the proof of (47) it suffices to check that

$$S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})} \text{ is decreasing on } (0, T - \tau_0]. \quad (52)$$

Indeed from (51) and since by assumption $\tau_0 < T$ we will therefore deduce that

$$S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})} \geq 0 \text{ on } (0, T], \quad \tilde{T}_0 \triangleq \inf\{t \in (0, T] \mid S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})}(t) = 0\} = T - \tau_0 \text{ and } \tilde{T}_0 > 0,$$

so that (II) and (IIIa) hold by (51) again, (IIIb) holds by (48) and (IV) holds by (52).

In order to prove that $S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})}$ is decreasing on $(0, T - \tau_0]$, recall that by Remark 1.2, δ takes positive values on (a, ∞) . Moreover, since by (i) the function w is decreasing on $(\tau_0, T]$ and satisfies $w(T) = r \geq a$, it takes values in (a, ∞) on (τ_0, T) . Consequently, $\delta \circ w$ is positive on (τ_0, T) . It then follows from (50) that $S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})}$ is decreasing on $(0, T - \tau_0]$. This concludes the proof of (47).

Let us now check that (IIIc) is satisfied i.e. that

$$\text{if } \tilde{T}_0 < T \text{ then } \tilde{x}(\tilde{T}_0 -) = a,$$

Observe that, by (i) and (45) we have: $\tilde{x}(\tilde{T}_0 -) = w(T -) = w(T) = r \in [a, \infty)$. Hence, since $[\tilde{T}_0 < T \Leftrightarrow \tau_0 > 0]$ and since $(r, \tau_0) \in \{(a, \infty) \times \{0\}\} \cup \{a\} \times [0, T)\}$ (see 6) condition (IIIc) holds.

Let us now turn to the proof of (v). By construction (see 46) $\tilde{y} = g(\tilde{x})$ on $[0, T]$ so that (4) holds. Since w is continuous on $(\tau_0, T]$, by construction $\tilde{x}$ is continuous on $(0, T - \tau_0] = (0, \tilde{T}_0]$ and so is $\tilde{y} = g(\tilde{x})$ (recall that g is continuous on $[a, \infty)$).

Let us establish equation (3). Since by (45) $w = x(\cdot - \tau_0)$ on $(\tau_0, T]$, equation (7) reads:

$$e^{-\lambda t} \dot{\pi}(\tilde{x}(t - \tau_0)) = e^{-\lambda T} \dot{\pi}(r) - \int_t^T e^{-\lambda u} \dot{s} \left(\int_u^T \delta(w(s)) ds \right) du, \quad \forall t \in (\tau_0, T],$$

which, after a change of variable from $(\tau_0, T]$ onto $(0, T - \tau_0] = (0, \tilde{T}_0]$ yields:

$$e^{-\lambda(t+\tau_0)} \dot{\pi}(\tilde{x}(t)) = e^{-\lambda T} \dot{\pi}(r) - \int_{t+\tau_0}^T e^{-\lambda u} \dot{s} \left(\int_u^T \delta(w(s)) ds \right) du, \quad \forall t \in (0, \tilde{T}_0].$$

Equivalently,

$$e^{-\lambda t} \dot{\pi}(\tilde{x}(t)) = e^{-\lambda(T-\tau_0)} \dot{\pi}(r) - \int_{t+\tau_0}^T e^{-\lambda(u-\tau_0)} \dot{s} \left(\int_u^T \delta(w(s)) ds \right) du, \quad \forall t \in (0, \tilde{T}_0].$$

Using a change of variable in the integral and since $T - \tau_0 = \tilde{T}_0$ we obtain:

$$e^{-\lambda t} \dot{\pi}(\tilde{x}(t)) = e^{-\lambda \tilde{T}_0} \dot{\pi}(r) - \int_t^{\tilde{T}_0} e^{-\lambda u} \dot{s} \left(\int_{u+\tau_0}^T \delta(w(s)) ds \right) du, \quad \forall t \in (0, \tilde{T}_0].$$

Plugging (50) in this last equation we get:

$$e^{-\lambda t} \dot{\pi}(\tilde{x}(t)) = e^{-\lambda \tilde{T}_0} \dot{\pi}(r) - \int_t^{\tilde{T}_0} e^{-\lambda u} \dot{s} \left(S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})}(u) \right) du, \quad \forall t \in (0, \tilde{T}_0]. \quad (53)$$

Observe that since $\tilde{x}$ is non-increasing and takes values in $[a, \infty)$ and since $\dot{\pi}$ is continuous and decreasing on $[a, \infty)$ and satisfies $\lim_{x \rightarrow \infty} \dot{\pi}(x) = \dot{\pi}(\infty)$ we have: $\lim_{t \searrow 0} \dot{\pi}(\tilde{x}(t)) = \dot{\pi}(\tilde{x}(0+)) \in [\dot{\pi}(\infty), \dot{\pi}(a)]$. Therefore, sending t to 0 in (53) we obtain:

$$\dot{\pi}(\tilde{x}(0+)) = e^{-\lambda \tilde{T}_0} \dot{\pi}(r) - \int_0^{\tilde{T}_0} e^{-\lambda u} \dot{s} \left(S^{(\tilde{\alpha}, \tilde{x}, \tilde{y})}(u) \right) du. \quad (54)$$

By computing the difference between (53) and (54), we obtain (3).

To complete the proof of Proposition 4, it remains to show (vi) is satisfied i.e. to check that:

$$\text{if } \tilde{\alpha} > 0 \text{ then } \tilde{x}(0+) = \infty.$$

By (45) we have $\tilde{x}(0+) = w(\tau_0+)$ and by (44) $\tilde{\alpha} = s_0 - \int_{\tau_0}^T \delta(w(u)) du$ so that the above assertion holds since it is equivalent to condition (8). This ends the proof of Proposition 4. $\square$

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