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Multidimensional Generalized Gini Indices

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Abstract. The axioms used to characterize the generalized Gini social evaluation orderings for one-dimensional distributions are extended to the multidimensional attributes case. A social evaluation ordering is shown to have a two-stage aggregation representation if these axioms and a separability assumption are satisfied. In the first stage, the distributions of each attribute are aggregated using generalized Gini social evaluation functions. The functional form of the second-stage aggregator depends on the number of attributes and on which version of a comonotonic additivity axiom is used. The implications of these results for the corresponding multidimensional indices of relative and absolute inequality are also considered.

Keywords and Phrases: Generalized Gini, multidimensional inequality.

JEL Classification Number: D63.

Résumé. Les axiomes qui caractérisent l'ordre de Gini généralisé sur des distributions unidimensionnelles sont étendus à des distributions portant sur plusieurs attributs. On montre que, si l'on ajoute à ces axiomes une hypothèse de séparabilité, les distributions peuvent être évaluées selon une procédure en deux étapes. Dans une première étape, la distribution de chaque attribut est agrégée par une fonction d'évaluation sociale de Gini généralisée. La forme fonctionnelle utilisée dans un second temps pour agréger les évaluations portant sur chaque attribut dépend du nombre d'attributs et de la version de l'axiome d'additivité comonotone retenue. Nous considérons également les conséquences de ces résultats en matière d'indices multidimensionnels (relatifs et absolus) d'inégalités.

Mots clefs: Indices de Gini généralisés, inégalités multidimensionnelles.

Numéro de classification JEL: D63.

1. Introduction

An individual's well-being depends on many factors, such as income, life expectancy, and health status. Because of the multidimensional nature of well-being, univariate indices of income inequality may give a misleading picture of the extent of inequality within a given population or between different groups of individuals. The formal analysis of multidimensional inequality has its origins in a pioneering article by Kolm (1977).[1] In Kolm's article and in the later work of Atkinson and Bourguignon (1982), the focus is on developing dominance criteria that can be used to determine when one multidimensional distribution exhibits more inequality than another. Kolm provided a number of multi-attribute generalizations of the Pigou-Dalton transfer principle, whereas Atkinson and Bourguignon proposed dominance principles that take account of the correlation between the distributions of the different components of well-being.

The rankings generated by these dominance principles are incomplete. In many circumstances, it is useful to have an index of inequality that can be used to compare any pair of distributions. In the normative approach to inequality measurement, an inequality index is constructed from a social evaluation ordering (or its representation, a social evaluation function) of the possible distributions.[2] For univariate distributions, the most commonly-used procedure for deriving a relative (i.e., scale invariant) inequality index from a social evaluation ordering was independently proposed by Atkinson (1970) and Kolm (1969) and popularized by Sen (1973). Kolm (1969) also proposed a procedure for deriving an absolute (i.e., translation invariant) inequality index from a social evaluation ordering. Multi-attribute generalizations of the univariate Atkinson-Kolm-Sen and Kolm methodologies have been proposed by Kolm (1977) and Tsui (1995), respectively.

This approach has been used by Tsui (1995) to develop new classes of multi-attribute inequality indices. Tsui first axiomatically characterized a class of social evaluation functions that are multi-attribute generalizations of a class of social evaluation functions for univariate distributions introduced by Blackorby, Donaldson, and Auersperg (1981) and then he used the methodology described above to determine the corresponding classes of relative

1. For surveys of the literature on multidimensional inequality, see Maasoumi (1999) and Savaglio (2002).

2. A social evaluation ordering is sometimes called a social welfare ordering.

and absolute inequality indices.[3]

The social evaluation functions identified by Tsui have the feature that they can be constructed in two steps. A utility function is first used to determine the utility of each person's allocation and then these utilities are aggregated (using simple summation) to provide the overall evaluation. Maasoumi (1986) had earlier suggested constructing a multi-attribute inequality index by first using a utility function to generate a distribution of utilities and then applying a univariate index of inequality to this distribution to obtain the multi-attribute index's value. However, unlike Tsui, the functional forms of the aggregator functions used at each stage are simply assumed by Maasoumi, whereas Tsui derives them from his axioms. List (1999) has proposed a hybrid approach. His procedure for constructing a multi-atrribute inequality index is similar to that of Maasoumi, but the functional forms of the aggregators at each stage are chosen in such a way that the resulting index is necessarily consistent with a number of inequality dominance criteria.

In this article, we follow the general approach of Tsui (1995), but consider a different set of axioms. We extend the axioms used by Weymark (1981) to characterize the generalized Gini social evaluation orderings for one-dimensional distributions to the multi-attributes case. We show that a social evaluation ordering has a two-stage aggregation representation if these axioms and a separability assumption are satisfied. In the first stage, the distributions of each attribute are aggregated using univariate generalized Gini social evaluation functions. The functional form of the second-stage aggregator depends on the number of attributes and on which version of a comonotonic additivity axiom is used. Note that the order in which individuals and attributes are aggregated is the reverse of the order used by Maasoumi (1986), Tsui (1995), and List (1999). We also determine which of our orderings satisfy the invariance assumptions needed to generate relative and absolute indices of inequality. The implications of these results for the corresponding multidimensional indices of relative and absolute inequality are also considered. Finally, we show that our separability axioms are inconsistent with a correlation increasing majorization axiom proposed by Tsui (1999) when individuals are treated symmetrically.

3. Tsui (1999) has also axiomatized a class of multi-attribute generalized entropy relative inequality indices. However, these inequality indices are characterized directly, rather than indirectly using social evaluation functions.

2. Preliminaries

The set of individuals is $N = \{1,\ldots,n\}$, with $n \geq 2$. There are q attributes of well-being, with $q \geq 2$. The set of attributes is $Q = \{1,\ldots,q\}$. In addition to income, examples of possible attributes are measures of educational attainment, health status, and longevity. The q attributes could be incomes in different states of the world, in which case we are concerned with inequality under uncertainty, as in Ben-Porath, Gilboa, and Schmeidler (1997) and Gajdos and Maurin (2002). Alternatively, the attributes could be incomes in different time periods.

An *allocation* of attributes among the population is an $n \times q$ real-valued matrix. A generic element of an allocation matrix X is x_{ij}, the quantity of attribute j allocated to individual i. The i^{th} row of a matrix X is denoted $x_{i\cdot}$, whereas its j^{th} column is denoted $x_{\cdot j}$. For all $Q_0 \subseteq Q$ for which $Q_0 \neq \varnothing$, let X_{Q_0} be the sub-allocation matrix of the attributes in Q_0. For any $j \in Q$, if $Q_0 = \{j\}$, we write $x_{\cdot j}$ instead of $X_{\{j\}}$, and if $Q_0 = \{1,\ldots,j-1,j+1,\ldots,q\}$, we write X_{-j} instead of X_{Q_0}. Let $\mathcal{Q}$ be the set of bi-partitions of Q. Formally,

$$\mathcal{Q} = \{(Q_1,Q_2) \subseteq Q \times Q | Q_1 \cup Q_2 = Q, Q_1 \cap Q_2 = \varnothing, Q_1 \neq \varnothing, Q_2 \neq \varnothing\}.$$

For all $(Q_1,Q_2) \in \mathcal{Q}$, it is sometimes convenient to let (X_{Q_1},X_{Q_2}) denote the matrix X.

The set of all allocation matrices is $\mathcal{M}$ and the set of all allocation matrices whose elements are nonnegative is $\mathcal{M}_+$. Let $\mathcal{M}_+^*$ denote the set of allocation matrices in $\mathcal{M}_+$ that have at least one positive element in each column. These three classes of allocation matrices are the multidimensional analogues of the standard domains used for univariate distributions of incomes. We also need to consider nonnegative allocation matrices that only differ from the null matrix in one column. Let $\mathcal{M}_+^{**}$ denote this class of allocation matrices. The allocation matrix whose elements are all equal to 0 (resp. 1) is $\mathbf{0}$ (resp. $\mathbf{1}$).

A *social evaluation* is a binary relation $\succeq$ on a set of allocation matrices $\mathcal{D}$. The relation $\succeq$ is interpreted as meaning "weakly socially preferred to". The symmetric and asymmetric factors of $\succeq$ are $\sim$ and $\succ$, respectively. We assume that $\mathcal{D} \in \left\{\mathcal{M},\mathcal{M}_+,\mathcal{M}_+^*\right\}$. If not stated explicitly, $\mathcal{D}$ can be any one of these three sets. An allocation matrix X is *nonincreasing comonotonic* if $x_{1j} \geq x_{2j} \geq \cdots \geq x_{nj}$ for all $j \in Q$. Let $\mathcal{D}^{\text{D}}$ denote the set of nonincreasing comonotonic matrices in $\mathcal{D}$.

For any $x \in \mathbb{R}^n$, $\tilde{x}$ is the permutation of x for which $\tilde{x}_1 \geq \tilde{x}_2 \geq \cdots \geq \tilde{x}_n$.[4] Let $\mathbb{R}^{n*}_+ = \mathbb{R}^n_+ \setminus \{(0,\ldots,0)\}$. For $\mathbb{D} \in \{\mathbb{R}^n, \mathbb{R}^n_+, \mathbb{R}^{n*}_+\}$, a *generalized Gini social evaluation function* is a function $g\colon \mathbb{D} \to \mathbb{R}$ for which

$$g(x) = \sum_{i=1}^{n} a_i \tilde{x}_i, \, \forall x \in \mathbb{D}, \tag{1}$$

where $0 < a_1 \leq a_2 \leq \cdots \leq a_n$. The *Gini social evaluation function* is the special case of (1) in which $a_i = (2i-1)/n^2$ for all $i \in N$.

For any $x \in \mathbb{R}^n$, $\mu(x)$ is the mean of x. For any $X \in \mathcal{D}$, we let X_μ denote the matrix for which every entry in the j^{th} column is equal to $\mu(x_{\cdot j})$.

3. Axioms

We now introduce our axioms. The first axiom requires $\succeq$ to be a complete preorder.

Ordering (ORD). The binary relation $\succeq$ is reflexive, complete, and transitive on $\mathcal{D}$.

The next axiom says that a strict ranking of two allocation matrices X and Y is invariant to small perturbations of these matrices.

Continuity (CONT). The sets $\{Y \in \mathcal{D} | Y \succ X\}$ and $\{Y \in \mathcal{D} | X \succ Y\}$ are open for all $X \in \mathcal{D}$.[5]

The monotonicity axiom says that if X is obtained from Y by increasing at least one person's allocation of some attribute without decreasing anyone's allocation of any attribute, then X is strictly preferred to Y.

Monotonicity (MON). For all X,$Y \in \mathcal{D}$,

$$[(x_{ij} \geq y_{ij}, \forall (i,j) \in N \times Q) \,\&\, (X \neq Y)] \Rightarrow X \succ Y.$$

The anonymity axiom requires $\succeq$ to treat individuals symmetrically. It says that it is a matter of social indifference if the individual allocations are permuted.

4. $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ denote the set of real numbers, nonnegative real numbers, and positive real numbers, respectively.

5. A matrix in $\mathcal{D}$ can be thought of as a vector in $\mathbb{R}^{nq}$. A subset of $\mathcal{D}$ is open if the corresponding set of vectors is open in $\mathbb{R}^{nq}$.

Anonymity (ANON). For all $n \times n$ permutation matrices Π and all $X \in \mathcal{D}$, $X \sim \Pi X$.

A number of different partial orders have been suggested for formalizing the idea that one allocation matrix exhibits less inequality than another when the mean value for each attribute is the same in both matrices. See Marshall and Olkin (1979) and Savaglio (2002). We consider three of these partial orders and, in each case, state an axiom that requires $\succeq$ to regard one allocation matrix to be weakly preferred to a second if the former exhibits less inequality according to this partial order than the latter.

When there is only one attribute, say income, a Pigou-Dalton transfer is a transfer of income from a richer to a poorer individual that diminishes the absolute value of the difference between their incomes. After the transfer, the incomes of these two individuals are convex combinations of their original incomes. In the multi-attribute case, this kind of transfer can be applied to each attribute. When, for a given pair of individuals, the same convex combinations are used for each attribute, we have a uniform Pigou-Dalton transfer. The partial order of allocation matrices in this case is defined by considering finite sequences of such transfers.

Definition. For all X,$Y \in \mathcal{D}$, X is obtained from Y by a *uniform Pigou-Dalton transfer* if $X \neq Y$ and there exist i_1 and i_2 in N and $\lambda \in (0,1)$ such that (i) $x_{i_1 j} = \lambda y_{i_1 j} + (1-\lambda) y_{i_2 j}$ for all $j \in Q$, (ii) $x_{i_2 j} = (1-\lambda) y_{i_1 j} + \lambda y_{i_2 j}$ for all $j \in Q$, and (iii) $x_{i\cdot} = y_{i\cdot}$ for all $i \notin \{i_1, i_2\}$.

Definition. For all X,$Y \in \mathcal{D}$, X *uniformly Pigou-Dalton majorizes* Y, denoted $X \succ_{\mathrm{U}} Y$, whenever X can be obtained from Y by a finite sequence of uniform Pigou-Dalton transfers.

Weak Uniform Pigou-Dalton Majorization (WUPM). For all X,$Y \in \mathcal{D}$,

$$X \succ_{\mathrm{U}} Y \Rightarrow X \succeq Y.$$

In the one-dimensional case, one distribution can be obtained from a second by a finite sequence of Pigou-Dalton transfers if and only if the former is obtained by multiplying the latter by a bistochastic matrix that is not a permutation matrix.[6] Uniform majorization is the multi-attribute analogue of this construction.

6. A nonnegative square matrix is *bistochastic* if all of its row and column sums are equal to 1.

Definition. For all X,$Y \in \mathcal{D}$, X *uniformly majorizes* Y, denoted $X \succ_{\mathrm{B}} Y$, if there exists an $n \times n$ bistochastic matrix B such that $X = BY$ and X is not a permutation of the rows of Y.

Weak Uniform Majorization (WUM). For all X,$Y \in \mathcal{D}$,

$$X \succ_{\mathrm{B}} Y \Rightarrow X \succeq Y.$$

If there are only two attributes, the partial orders $\succ_{\mathrm{U}}$ and $\succ_{\mathrm{B}}$ are equivalent. If $q > 2$, $X \succ_{\mathrm{U}} Y$ implies $X \succ_{\mathrm{B}} Y$, but the reverse implication need not hold. As a consequence, Weak Uniform Majorization is a stronger condition than Weak Uniform Pigou-Dalton Majorization. See Kolm (1977) and Marshall and Olkin (1979).

Another way in which an allocation matrix can be made more equal is to reduce the correlation between its rows. One way to increase the correlation between the rows of an allocation matrix is to rearrange two individuals' allocations in such a way that one of these individuals receives at least as much of every attribute as the other. A sequence of this kind of correlation-increasing transfer increases inequality. These ideas are made precise in the following definitions.

Definition. For all X,$Y \in \mathcal{D}$, Y is obtained from X by a *correlation-increasing transfer* if $X \neq Y$ and there exist i_1,$i_2 \in N$ such that (i) $y_{i_1 j} = \min\{x_{i_1 j}, x_{i_2 j}\}$ for all $j \in Q$, (ii) $y_{i_2 j} = \max\{x_{i_1 j}, x_{i_2 j}\}$ for all $j \in Q$, and (iii) $y_{i\cdot} = x_{i\cdot}$ for all $i \notin \{i_1, i_2\}$.

Definition. For all X,$Y \in \mathcal{D}$, Y is *more correlated* than X, denoted $Y \succ_{\mathrm{C}} X$, whenever Y can be obtained from X by a finite sequence of correlation-increasing transfers.

Correlation Increasing Majorization (CIM). For all X,$Y \in \mathcal{D}$,

$$Y \succ_{\mathrm{C}} X \Rightarrow X \succ Y.$$

CIM was introduced into the inequality literature by Tsui (1999). It is based on the concept of a multivariate arrangement increasing function due to Boland and Proschan (1988). See Tsui (1999) for further discussion of this axiom and its relationship to similar principles used in statistics and in the measurement of risk.

Note that CIM is stated in terms of the strict social evaluation relation $\succ$, whereas WUPM and WUM use the weak relation $\succeq$. Strong versions of the latter axioms can be obtained by replacing $\succeq$ with $\succ$ in their definitions.

Weymark (1981) introduced a comonotonic additivity axiom for the ranking of one-dimensional income distributions. It requires the ranking of two comonotonic distributions to be invariant to the addition of a common distribution to both of these distributions provided that the third distribution is also comonotonic with the original distributions. The rationale offered for this axiom is that each person's income may come from a number of different sources (wages, interest, etc.) and if the incomes from all but once source of income are the same in two distributions of total income, then the ranking of these distributions should only depend on the distributions of income from the variable source.

We consider two multi-attribute extensions of this axiom. In both cases, we require the ranking of two nonincreasing comonotonic allocation matrices X and Y by $\succeq$ to be invariant to the common addition a third allocation matrix Z that is nonincreasing comonotonic with respect to both X and Y. Our Weak Comonotonic Additivity axiom applies if X and Y differ in only one attribute and Z only has non-zero values for this attribute. In our Strong Comonotonic Additivity axiom, the distributions of any attribute in the three allocation matrices are permitted to be different from one another.

Weak Comonotonic Additivity (WCA). For all X,$Y \in \mathcal{D}^{\mathrm{D}}$ such that $x_{\cdot j} = y_{\cdot j}$ for all $j \neq j_0$ and all $Z \in \mathcal{D}^{\mathrm{D}} \cup \mathcal{M}_{+}^{**\mathrm{D}}$ such that $z_{ij} = 0$ for all $i \in N$ and all $j \neq j_0$,

$$X \succeq Y \Leftrightarrow X + Z \succeq Y + Z.$$

Strong Comonotonic Additivity (SCA). For all X,$Y \in \mathcal{D}^{\mathrm{D}}$ and all $Z \in \mathcal{D}^{\mathrm{D}} \cup \mathcal{M}_{+}^{**\mathrm{D}}$,

$$X \succeq Y \Leftrightarrow X + Z \succeq Y + Z.$$

Note that if the domain is $\mathcal{M}_{+}^{*}$, the allocation matrix Z that is added to both X and Y is not in the domain if $Z \in \mathcal{M}_{+}^{**\mathrm{D}}$. However, it is nevertheless the case that $X + Z$ and $Y + Z$ are in the domain, which is all that is needed for our comonotonic additivity principles to apply.

With the exception of CIM, all of the preceding axioms are multi-attribute extensions of axioms for one-dimensional distributions. Their one-dimensional counterparts characterize the class of generalized Gini social evaluation functions with positive weights. See Weymark (1981).[7]

7. Weymark (1981) axiomatized the class of generalized Gini absolute inequality indices. It is straightforward to modify his analysis in order to obtain an axiomatization of the

If the conditional ordering of some subset of the variables obtained by fixing the values of the remaining variables is independent of the values of the conditioning variables, then the first set of variables is separable from the second. The next two axioms are concerned with the separability of $\succeq$ across attributes. In the subsequent discussion, when we say that the set of attributes $Q_1 \subseteq Q$ is separable from the complementary set of attributes, we mean that the set of all variables ij for which $i \in N$ and $j \in Q_1$ is separable from the variables associated with the other attributes. Weak Attribute Separability requires that there exist some attribute that is separable from the other attributes. Strong Attribute Separability strengthens this condition by requiring any subset of the attributes to be separable from the other attributes.[8]

Weak Attribute Separability (WSEP). There exists $j_0 \in Q$ such that for all $x_{\cdot j_0}$, $y_{\cdot j_0}$, $\bar{X}_{-j_0}$, and $\bar{Z}_{-j_0}$,

$$\left(x_{\cdot j_0},\bar{X}_{-j_0}\right) \succeq \left(y_{\cdot j_0},\bar{X}_{-j_0}\right) \Leftrightarrow \left(x_{\cdot j_0},\bar{Z}_{-j_0}\right) \succeq \left(y_{\cdot j_0},\bar{Z}_{-j_0}\right).$$

Strong Attribute Separability (SSEP). For all $(Q_1,Q_2) \in \mathcal{Q}$ and all X_{Q_1}, Y_{Q_1}, $\bar{X}_{Q_2}$, and $\bar{Z}_{Q_2}$,

$$\left(X_{Q_1},\bar{X}_{Q_2}\right) \succeq \left(Y_{Q_1},\bar{X}_{Q_2}\right) \Leftrightarrow \left(X_{Q_1},\bar{Z}_{Q_2}\right) \succeq \left(Y_{Q_1},\bar{Z}_{Q_2}\right).$$

In the normative approach to inequality measurement for one-dimensional distributions (see Atkinson (1970), Kolm (1969), and Sen (1973)), an inequality index is derived from a social evaluation function. A relative inequality index is invariant to a proportional change in all incomes, whereas an absolute inequality index is invariant if a common amount is added to or subtracted from all incomes. In order for an inequality index to be a relative (resp. absolute) index, the underlying social evaluation ordering must be homothetic (resp. translatable). Our final axioms are concerned with multi-attribute generalizations of these invariance properties for the social evaluation ordering $\succeq$.

class of generalized Gini social evaluation functions. He did not employ a monotonicity assumption. As a consequence, the weights in his representation theorem need not be positive.

8. In order to avoid introducing even more notation, we do not explicitly state the domains for the various variables that appear in the statement of the two separability axioms. Each of the allocation matrices considered in these axioms must be in $\mathcal{D}$. See Blackorby, Primont, and Russell (1978) for further discussion of these separability axioms.

Weak Homotheticity simply extends the requirement that the social evaluation ordering be homothetic to the multi-attribute case.

Weak Homotheticity (WHOM). For all X,$Y \in \mathcal{D}$ and all $\lambda > 0$,

$$X \succeq Y \Leftrightarrow \lambda X \succeq \lambda Y.$$

If there is only one attribute, homotheticity of the social evaluation ordering is equivalent to requiring that the ordering be invariant to any change in the units in which the attribute is measured. Tsui (1995) has suggested that the same invariance property should hold in the multi-attribute case. In other words, independent changes in the units in which different attributes are measured should not affect the social evaluation ordering, a property we call Strong Homotheticity.

Strong Homotheticity (SHOM). For all X,$Y \in \mathcal{D}$ and all $q \times q$ diagonal matrices Λ for which $\lambda_{jj} > 0$ for all $j \in Q$,

$$X \succeq Y \Leftrightarrow X\Lambda \succeq Y\Lambda.$$

SHOM is a natural assumption if the attributes are different kinds of goods, such as income and life expectancy. However, if different attributes are incomes in different states or time periods, then the attributes should be measured in the same units, in which case SHOM is inappropriate.

The multi-attribute analogue of the requirement that the social evaluation ordering be translatable is Weak Translatability.

Weak Translatability (WTRA). For all X,$Y \in \mathcal{D}$ and all $\lambda \in \mathbb{R}$ for which $X + \lambda\mathbf{1} \in \mathcal{D}$ and $Y + \lambda\mathbf{1} \in \mathcal{D}$,

$$X \succeq Y \Leftrightarrow X + \lambda\mathbf{1} \succeq Y + \lambda\mathbf{1}.$$

In the one-attribute case, translatability of the social evaluation ordering is equivalent to requiring the ordering to be invariant to any change in the origin from which the quantity of the attribute is measured. Strong Translatability, an axiom proposed by Tsui (1995), extends this condition by requiring that the social evaluation ordering be invariant to independent changes in the origins from which the quantities of the various attributes are measured. As with SHOM, this condition is inappropriate if there are attributes that should be measured using the same scale.

Strong Translatability (STRA). For all X,$Y \in \mathcal{D}$ and all $q \times q$ diagonal matrices Λ for which $X + \mathbf{1}\Lambda \in \mathcal{D}$ and $Y + \mathbf{1}\Lambda \in \mathcal{D}$,

$$X \succeq Y \Leftrightarrow X + \mathbf{1}\Lambda \succeq Y + \mathbf{1}\Lambda.$$

These four invariance axioms are closely related to axioms used in the literature on social choice with interpersonal comparisons of utility.[9] In this literature, a social welfare ordering is defined on distributions of utilities, one for each person. A social welfare ordering is *ratio-scale measurable and fully comparable* if the ranking of any two utility vectors is invariant to a proportional scaling of all utilities and it is *ratio-scale measurable* if the factor of proportionality can be person-specific. Similarly, a social welfare ordering is *translation-scale measurable and fully comparable* if the ranking of any two utility vectors is invariant when a common amount is added to or subtracted from all utilities and it is *translation-scale measurable* when the amounts added or subtracted can be person-specific. Although social welfare orderings are defined on vectors of utilities and the social evaluation orderings considered here are defined on allocation matrices, we are nevertheless able to exploit social choice theorems that use these ratio-scale and translation-scale axioms in Sections 5 and 6 to help characterize classes of multidimensional generalised Gini relative and absolute inequality indices.

4. Multidimensional Generalized Gini Social Evaluation Orderings

In this section, we show that if multi-attribute versions of the axioms that characterize the class of generalized Gini social evaluation orderings for univarate distributions are combined with SSEP, then the social evaluation ordering can be represented by a two-stage aggregator function. In the first stage, the distributions of each attribute are aggregated using a univariate generalized Gini social evaluation function. The weights in these functions can be attribute specific. In the second stage, the values of these generalized Ginis are aggregated. The functional form of the second-stage aggregator depends on which version of our comonotonic additivity axiom is used and on the number of attributes.

Our characterization theorems do not depend on whether our multi-attribute generalization of the Pigou-Dalton transfer principle is WUPM or

9. For an introduction to this literature, see Bossert and Weymark (1996).

WUM. The differences between these axioms are only manifested if $q > 2$. However, in the presence of ANON and SSEP, the implications of WUPM and WUM can be determined one attribute at a time. Consequently, as we show in Theorem 1, WUPM and WUM place equivalent restrictions on a social evaluation ordering if it satisfies ANON and SSEP.

Theorem 1. *Suppose that* $\mathcal{D} \in \left\{\mathcal{M},\mathcal{M}_{+},\mathcal{M}_{+}^{*}\right\}$. *If the binary relation* $\succeq$ *on* $\mathcal{D}$ *satisfies SSEP and ANON, then* $\succeq$ *satisfies WUM if and only if it satisfies WUPM.*

Démonstration. Because $X \succ_{\mathrm{U}} Y$ implies $X \succ_{\mathrm{B}} Y$ for all $X,Y \in \mathcal{D}$, if $\succeq$ satisfies WUM, it also satisfies WUPM. Hence, we only need to show that the converse implication holds.

Suppose that $\succeq$ satisfies SSEP, ANON, and WUPM. Let $X,Y \in \mathcal{D}$ be such that $X \succ_{\mathrm{B}} Y$.

Let $\hat{X}^1 = (x_{\cdot 1},\mathbf{1}_{-1})$ and $\hat{Y}^1 = (y_{\cdot 1},\mathbf{1}_{-1})$. Two cases may arise. In the first case, $x_{\cdot 1}$ is a permutation of $y_{\cdot 1}$.[10] By ANON, it then follows that $\hat{X}^1 \sim \hat{Y}^1$. In the second case, $x_{\cdot 1}$ is not a permutation of $y_{\cdot 1}$. Then, $\hat{X}^1$ is obtained from $\hat{Y}^1$ by a finite sequence of uniform Pigou-Dalton transfers. Therefore, because $\succeq$ satisfies WUPM, we have $\hat{X}^1 \succeq \hat{Y}^1$. Hence, in both cases, $\hat{X}^1 \succeq \hat{Y}^1$. But this implies, by SSEP, that $X^1 \succeq Y$, where $X^1 = (x_{\cdot 1},Y_{-1})$.

By the same reasoning as above, we have $(x_{\cdot 2},\mathbf{1}_{-2}) \succeq (y_{\cdot 2},\mathbf{1}_{-2})$, and therefore, by SSEP, that $X^2 \succeq X^1$, where $X^2 = (x_{\cdot 2},X^1_{-2})$. Because $X^1 \succeq Y$, transitivity of $\succeq$ then implies that $X^2 \succeq Y$. Note that $X^2 = (X_{\{1,2\}},Y_{Q\setminus\{1,2\}})$.

By iterating this process on Q, we conclude that $X \succeq Y$. □

In view of Theorem 1, it doesn't matter which of WUPM and WUM we include in our set of axioms. For concreteness, we use WUPM. In all of the characterization theorems in this section, we suppose that the social evaluation ordering $\succeq$ satisfies ORD, CONT, MON, ANON, WUPM, and SSEP.

As is well-known, the functional structure implications of separability axioms that operate on all bi-partitions of a set of variables depend on whether the number of variables being partitioned is two or whether it is three or more. See, for example, Blackorby, Primont, and Russell (1978). With SSEP, the variables being partitioned are the attributes. We first consider the case in which there are three or more attributes. When this is the case, Theorem 2 shows that $\succeq$ can be represented by a two-stage aggregator function, as

10. Note that this case applies if $x_{\cdot 1} = y_{\cdot 1}$.

described above, where the second-stage aggregator is a continuous, increasing additive function of the generalized Gini aggregators used in the first stage if WCA is added to the six axioms listed in the preceding paragraph.

Theorem 2. *Suppose that $\mathcal{D} = \mathcal{M}$ (resp. $\mathcal{D} = \mathcal{M}_+$, resp. $\mathcal{D} = \mathcal{M}_+^*$). If $q > 2$, then the binary relation $\succeq$ on $\mathcal{D}$ satisfies ORD, CONT, MON, ANON, WUPM, SSEP, and WCA if and only if there exists an $n \times q$ matrix A of positive coefficients with $a_{\cdot j}$ nondecreasing and $\sum_{i=1}^{n} a_{ij} = 1$ for all $j \in Q$ and there exist q continuous increasing functions $v_j \colon \mathbb{R} \to \mathbb{R}$ (resp. $v_j \colon \mathbb{R}_+ \to \mathbb{R}$, resp. $v_j \colon \mathbb{R}_{++} \to \mathbb{R}$) such that*

$$X \succeq Y \Leftrightarrow \sum_{j=1}^{q} v_j \left(\sum_{i=1}^{n} a_{ij} \tilde{x}_{ij} \right) \geq \sum_{j=1}^{q} v_j \left(\sum_{i=1}^{n} a_{ij} \tilde{y}_{ij} \right), \forall X,Y \in \mathcal{D}. \tag{2}$$

Furthermore, the functions v_j are unique up to a common increasing affine transformation and the matrix of coefficients A is unique.

Démonstration. The necessity part of the theorem is straightforward to verify. We therefore only prove sufficiency. We establish the theorem for the domain $\mathcal{D} = \mathcal{M}$.[11]

By Theorem I in Debreu (1954), we know that if $\succeq$ satisfies ORD, CONT, and MON, then there exists a continuous, increasing function $f \colon \mathcal{D} \to \mathbb{R}$ that represents $\succeq$. Furthermore, for all $j \in Q$, SSEP implies that the j^{th} attribute is separable from the the complementary set of attributes. Note that MON implies that every attribute is essential.[12] Because $q > 2$, Theorem 3 in Debreu (1960) applies. Hence, there exist q continuous and increasing functions $U_j \colon \mathbb{R}^n \to \mathbb{R}$ such that

$$X \succeq Y \Leftrightarrow \sum_{j=1}^{q} U_j(x_{\cdot j}) \geq \sum_{j=1}^{q} U_j(y_{\cdot j}), \forall X,Y \in \mathcal{D}. \tag{3}$$

Furthermore, the functions U_j are unique up to a common increasing affine transformation.

Consider any $X \in \mathcal{D}$. For all $j \in Q$, we define the binary relation $\succeq_j$ on $\mathbb{R}^n$ by setting

$$y_{\cdot j} \succeq_j z_{\cdot j} \Leftrightarrow (y_{\cdot j}, X_{-j}) \succeq (z_{\cdot j}, X_{-j}), \forall y_{\cdot j}, z_{\cdot j} \in \mathbb{R}^n. \tag{4}$$

11. For the domain $\mathcal{M}_+$ (resp. $\mathcal{M}_+^*$), $\mathbb{R}^n$ must be changed to $\mathbb{R}_+^n$ (resp. $\mathbb{R}_{++}^n$) throughout the proof, but otherwise the argument is identical.

12. The j^{th} attribute is *essential* if there exist values for the allocations of the other attributes such that the conditional ordering of the allocations of the j^{th} attribute is not the trivial one in which all allocations are indifferent to each other.

Because of SSEP, $\succeq_j$ does not depend on the choice of the matrix X. For all $j \in Q$, (3) implies that $\succeq_j$ can be represented by U_j.

Consider any $j \in Q$. Because $\succeq$ satisfies CONT and ANON, $\succeq_j$ is a continuous, symmetric ordering of $\mathbb{R}^n$. Now, consider any $y_{\cdot j}, z_{\cdot j}, t_{\cdot j} \in \mathbb{R}^n$ and any $X \in \mathcal{D}^{\mathrm{D}}$. By (4), $\tilde{y}_{\cdot j} \succeq_j \tilde{t}_{\cdot j}$ implies $(X_{-j}, \tilde{y}_{\cdot j}) \succeq (X_{-j}, \tilde{z}_{\cdot j})$. Note that $(X_{-j}, \tilde{y}_{\cdot j})$ and $(X_{-j}, \tilde{z}_{\cdot j})$ both belong to $\mathcal{D}^{\mathrm{D}}$. Applying WCA, we obtain

$$(X_{-j}, \tilde{y}_{\cdot j}) \succeq (X_{-j}, \tilde{z}_{\cdot j}) \Leftrightarrow (X_{-j}, \tilde{y}_{\cdot j}) + (\mathbf{0}_{-j}, \tilde{t}_{\cdot j}) \succeq (X_{-j}, \tilde{z}_{\cdot j}) + (\mathbf{0}_{-j}, \tilde{t}_{\cdot j}).$$

Equivalently,

$$(X_{-j}, \tilde{y}_{\cdot j}) \succeq (X_{-j}, \tilde{z}_{\cdot j}) \Leftrightarrow (X_{-j}, \tilde{y}_{\cdot j} + \tilde{t}_{\cdot j}) \succeq (X_{-j}, \tilde{z}_{\cdot j} + \tilde{t}_{\cdot j}).$$

We have thus shown that

$$\tilde{y}_{\cdot j} \succeq_j \tilde{z}_{\cdot j} \Leftrightarrow \tilde{y}_{\cdot j} + \tilde{t}_{\cdot j} \succeq_j \tilde{z}_{\cdot j} + \tilde{t}_{\cdot j}, \, \forall y_{\cdot j}, z_{\cdot j}, t_{\cdot j} \in \mathbb{R}^n.$$

Therefore, $\succeq_j$ satisfies Axiom 4 in Weymark (1981). It then follows from MON and Theorem 3 in Weymark (1981) that there exist $a_{ij} > 0$, $i \in N$, such that

$$\tilde{x}_{\cdot j} \succeq_j \tilde{y}_{\cdot j} \Leftrightarrow \sum_{i=1}^{n} a_{ij} \tilde{x}_{ij} \geq \sum_{i=1}^{n} a_{ij} \tilde{y}_{ij}, \, \forall x_{\cdot j}, y_{\cdot j} \in \mathbb{R}^n.^{13} \tag{5}$$

We adopt the normalization $\sum_i a_{ij} = 1$, which implies that the sequence of weights $(a_{ij})_i$ is unique. WUPM implies that $\succeq_j$ satisfies the weak form of the unidimensional Pigou-Dalton transfer principle, which in turn implies that $a_{1j} \leq a_{2j} \leq \cdots \leq a_{nj}$.[14]

Define the function $G_j \colon \mathbb{R}^n \to \mathbb{R}$ by setting

$$G_j(x_{\cdot j}) = \sum_{i=1}^{n} a_{ij} \tilde{x}_{ij}, \, \forall x_{\cdot j} \in \mathbb{R}^n. \tag{6}$$

Because $\succeq_j$ is symmetric, (5) implies that G_j is a continuous representation of $\succeq_j$. Because U_j is also a continuous representation of $\succeq_j$, there exists a continuous, increasing function $v_j \colon \mathbb{R} \to \mathbb{R}$ such that $U_j = v_j \circ G_j$.

13. Weymark's theorem is for vectors in $\mathbb{R}^n_+$, but, as he notes, it also holds for vectors in $\mathbb{R}^n$.

14. The weak form of the unidimensional Pigou-Dalton transfer principle is simply the one attribute analogue of WUPM. See, for example, Hardy, Littlewood, and Pólya (1934) or Marshall and Olkin (1979).

The preceding argument holds for all $j \in Q$. It then follows that (3) can be rewritten as (2). Because the functions U_j are unique up to a common increasing affine transformation, so are the functions v_j. □

SSEP and ORD imply that the conditional ordering of the distributions of any attribute are independent of the values of the other variables. The proof of Theorem 2 shows that the properties of these orderings that are inherited from the axioms of the theorem are exactly those properties that are used to characterize the univariate generalized Ginis. This accounts for the functional structure of the first-stage aggregators. The additive structure of the second-stage aggregator follows from a standard separability theorem for three or more variables.

In Theorem 3, we consider the same set of axioms as in Theorem 2, but now suppose that there are only two attributes. Reasoning as in the proof of Theorem 2, the first-stage aggregators are generalized Ginis. However, because the separability axiom only operates on bi-partitions of two attributes, we can no longer conclude that the second-stage aggregator is additive.

Theorem 3. *Suppose that $\mathcal{D} = \mathcal{M}$ (resp. $\mathcal{D} = \mathcal{M}_+$, resp. $\mathcal{D} = \mathcal{M}_+^*$). If $q = 2$, then the binary relation $\succeq$ on $\mathcal{D}$ satisfies ORD, CONT, MON, ANON, WUPM, SSEP, and WCA if and only if there exists an $n \times 2$ matrix A of positive coefficients with $a_{\cdot j}$ nondecreasing and $\sum_{i=1}^n a_{ij} = 1$ for all $j \in \{1,2\}$ and there exists a continuous increasing function $V\colon \mathbb{R}^2 \to \mathbb{R}$ (resp. $V\colon \mathbb{R}_+^2 \to \mathbb{R}$, resp $V\colon \mathbb{R}_{++}^2 \to \mathbb{R}$) such that*

$$X \succeq Y \Leftrightarrow V\left(\sum_{i=1}^n a_{i1}\tilde{x}_{i1}, \sum_{i=1}^n a_{i2}\tilde{x}_{i2}\right) \geq V\left(\sum_{i=1}^n a_{i1}\tilde{y}_{i1}, \sum_{i=1}^n a_{i2}\tilde{y}_{i2}\right), \quad \forall X,Y \in \mathcal{D}. \quad (7)$$

Furthermore, the function V is unique up to an increasing transformation and the matrix of coefficients A is unique.

Démonstration. The necessity part of the theorem is straightforward to verify. We therefore only prove sufficiency. As in the proof of Theorem 2, we only consider the domain $\mathcal{M}$ as it is trivial to modify the proof so that it applies to the other two domains.

We know from the proof of Theorem 2 that ORD, CONT, MON, and SSEP imply that $\succeq$ can be represented by a continuous, increasing function $f\colon \mathcal{D} \to \mathbb{R}$. Furthermore, for all $j \in Q$, the j^{th} attribute is essential and separable from the the complementary set of attributes. Hence, by

Lemma 1 in Gorman (1968), there exist three continuous increasing functions $U_0\colon U_1(\mathbb{R}^2) \times U_2(\mathbb{R}^2) \to \mathbb{R}$, $U_1\colon \mathbb{R}^2 \to \mathbb{R}$, and $U_2\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$X \succeq Y \Leftrightarrow U_0(U_1(x_{\cdot 1}),U_2(x_{\cdot 2})) \geq U_0(U_1(y_{\cdot 1}),U_2(y_{\cdot 2})),\ \forall X,Y \in \mathcal{D}. \quad (8)$$

For $j = 1,2$, defining $\succeq_j$ as in (3), the same argument as the one used in the proof of Theorem 2 implies that $\succeq_j$ can be represented by U_j and that there exist two nondecreasing series of positive weights $(a_{ij})_i$ for which $\sum_{i=1}^n a_{ij} = 1$ such that

$$x_{\cdot j} \succeq_j y_{\cdot j} \Leftrightarrow G_j(x_{\cdot j}) \geq G_j(y_{\cdot j}),\ \forall x_{\cdot j},y_{\cdot j} \in \mathbb{R}^n,$$

where G_j is defined in (6). Furthermore, each of these series of weights is unique. Therefore, for $j = 1,2$, there exists a nondecreasing continuous function $v_j\colon \mathbb{R} \to \mathbb{R}$ such that $U_j = v_j \circ G_j$. Substituting in (6), we obtain

$$X \succeq Y \Leftrightarrow U_0(v_1(G_j(x_{\cdot 1})),v_2(G_j(x_{\cdot 2})) \geq U_0(v_1(G_j(y_{\cdot 1})),v_2(G_j(y_{\cdot 2}))),\ \forall X,Y \in \mathcal{D}. \quad (9)$$

Defining the function $V\colon \mathbb{R}^2 \to \mathbb{R}$ by setting $V(\alpha,\beta) = U_0(v_1(\alpha),v_2(\beta))$ for all $(\alpha,\beta) \in \mathbb{R}^2$ and substituting V into (9), we obtain (7). Clearly, (7) is also satisfied if V is subjected to an increasing transformation. □

The final characterization theorem in this section strengthens WCA to SCA. Theorem 4 shows that the second-stage aggregator in this case must be linear, and this is true for any number of attributes (greater than or equal to 2). As in Theorems 2 and 3, the first-stage aggregators are generalized Ginis.

Theorem 4. *Suppose that $\mathcal{D} \in \left\{\mathcal{M},\mathcal{M}_+,\mathcal{M}_+^*\right\}$. The binary relation $\succeq$ on $\mathcal{D}$ satisfies ORD, CONT, MON, ANON, WUPM, SSEP, and SCA if and only if there exists an $n \times q$ matrix A of positive coefficients with $a_{\cdot j}$ nondecreasing and $\sum_{i=1}^n a_{ij} = 1$ for all $j \in Q$ and a vector $\gamma \in \mathbb{R}_{++}^q$ with $\sum_{i=1}^q \gamma_j = 1$ such that*

$$X \succeq Y \Leftrightarrow \sum_{j=1}^q \left(\gamma_j \sum_{i=1}^n a_{ij}\tilde{x}_{i1}\right) \geq \sum_{j=1}^q \left(\gamma_j \sum_{i=1}^n a_{ij}\tilde{y}_{i1}\right),\ \forall X,Y \in \mathcal{D}. \quad (10)$$

Furthermore, the matrix of coefficients A and the vector γ are unique.

Démonstration. It is straightforward to show necessity, so we only establish the sufficiency part of the theorem.

We first show that ORD, CONT, and SCA imply that $\succeq$ can be represented by a linear functional on $\mathcal{D}^{\mathrm{D}}$, the set of nonincreasing comonotonic matrices in $\mathcal{D}$. For $X \in \mathcal{D}^{\mathrm{D}}$, let $E(X) = \{Y \in \mathcal{D} | Y \sim X\}$ and $B(X) = \{Y \in \mathcal{D} | Y \succ X\}$. By applying the argument used by Weymark (1981) in the proof of his Theorem 3, it follows that $E(X)$ is a convex set.[15] By interpreting elements of $\mathcal{D}^{\mathrm{D}}$ as vectors in $\mathbb{R}^{nq}$, MON then implies that $E(X)$ is the restriction of an $(nq-1)-$dimensional hyperplane in $\mathbb{R}^{nq}$ to $\mathcal{D}^{\mathrm{D}}$. Hence, the indifference contours of $\succeq$ in $\mathcal{D}^{\mathrm{D}}$ are parallel $(qn-1)-$dimensional hyperplanes. For an arbitrary $Z \in B(X)$, we choose a matrix $B \in \mathcal{M}$ in the subspace of normals to $E(X)$ so that $\sum_{i=1}^{n}\sum_{j=1}^{q} b_{ij}z_{ij} > \sum_{i=1}^{n}\sum_{j=1}^{q} b_{ij}x_{ij}$. Thus,

$$X \succeq Y \Leftrightarrow \sum_{i=1}^{n}\sum_{j=1}^{q} b_{ij}x_{ij} \geq \sum_{i=1}^{n}\sum_{j=1}^{q} b_{ij}y_{ij}, \forall X,Y \in \mathcal{D}^{\mathrm{D}}.$$

Because SCA implies WCA, Theorem 2 applies if $q > 2$ and Theorem 3 applies if $q = 2$. Consider any $X,X' \in \mathcal{D}$ for which each column of X' can be obtained by a permutation of the corresponding column of X. It then follows from either (2) or (7) that $X \sim X'$. Hence,

$$X \succeq Y \Leftrightarrow \sum_{i=1}^{n}\sum_{j=1}^{q} b_{ij}\tilde{x}_{ij} \geq \sum_{i=1}^{n}\sum_{j=1}^{q} b_{ij}\tilde{y}_{ij}, \forall X,Y \in \mathcal{D}. \tag{11}$$

In order for (11) to be consistent with (2) and (7), the functions v_j, $j \in N$, in (2) and the function V in (7) must be linear. Thus, (10) holds. The uniqueness of A and γ then follow from the corresponding uniqueness results in Theorems 2 and 3. □

The key insight underlying the proof of Theorem 4 is that ORD, CONT, and SCA imply that $\succeq$ can be represented by a linear functional on the set of nonincreasing comonotonic matrices in $\mathcal{D}$. This is only consistent with what has been established in Theorems 2 and 3 if the second-stage aggregator function is linear.

5. Homothetic Social Evaluation Orderings

Normative indices of relative inequality are constructed using homothetic social evaluation orderings. In this section, for each of the theorems in the

15. Weymark's argument is for the set of nonincreasing vectors in $\mathbb{R}^{n}_{+}$, but his argument applies equally well to the domain $\mathcal{D}^{\mathrm{D}}$.

preceding section, we consider the implications of also requiring the social evaluation function to satisfy one of our homotheticity axioms. When defining relative inequality indices, it is customary to assume that there is a positive amount of each attribute. Accordingly, in this section, we suppose that the domain of the social evaluation ordering is $\mathcal{M}_+^*$.

We begin by assuming that there are at least three attributes and supplement the axioms in Theorem 2 with WHOM. In this case, WHOM places strong restrictions on the functional form of the second-stage aggregator—it must be a mean of order r function. In other words, the function that aggregates the values of the generalized Ginis for the q attributes must be a constant-elasticity-of-substitution function. When $r = 0$, this aggregator is a Cobb-Douglas function.

Theorem 5. *If $q > 2$, then the binary relation $\succeq$ on $\mathcal{M}_+^*$ satisfies ORD, CONT, MON, ANON, WUPM, SSEP, WCA, and WHOM if and only if there exists an $n \times q$ matrix A of positive coefficients with $a_{\cdot j}$ nondecreasing and $\sum_{i=1}^{n} a_{ij} = 1$ for all $j \in Q$, a vector $\gamma \in \mathbb{R}_{++}^q$ with $\sum_{j=1}^{q} \gamma_j = 1$, and a scalar r such that*

$$X \succeq Y \Leftrightarrow \left[\sum_{j=1}^{q} \gamma_j \left(\sum_{i=1}^{n} a_{ij}\tilde{x}_{ij}\right)^r\right]^{\frac{1}{r}} \geq \left[\sum_{j=1}^{q} \gamma_j \left(\sum_{i=1}^{n} a_{ij}\tilde{y}_{ij}\right)^r\right]^{\frac{1}{r}}, \forall X,Y \in \mathcal{M}_+^*, \tag{12}$$

if $r \neq 0$ and

$$X \succeq Y \Leftrightarrow \prod_{j=1}^{q} \left(\sum_{i=1}^{n} a_{ij}\tilde{x}_{ij}\right)^{\gamma_j} \geq \prod_{j=1}^{q} \left(\sum_{i=1}^{n} a_{ij}\tilde{y}_{ij}\right)^{\gamma_j}, \forall X,Y \in \mathcal{M}_+^*, \tag{13}$$

if $r = 0$. Furthermore, the matrix of coefficients A, the vector γ, and the scalar r are unique.

Démonstration. The necessity of the axioms is easy to check, so we only establish the sufficiency part of the theorem.

By Theorem 2, (2) must be satisfied. Let $W\colon \mathbb{R}_{++}^q \to \mathbb{R}$ be defined by setting

$$W(g) = \sum_{j=1}^{q} v_j(g_j), \forall g \in \mathbb{R}_{++}^q, \tag{14}$$

where the functions v_j are the functions that appear in (2). Because these functions are continuous and increasing, so is W. Because $\succeq$ satisfies WHOM,

(2) implies that W is ratio-scale measurable and fully comparable. Hence, by Theorem 2 in Blackorby and Donaldson (1982), W must be a mean of order r function. That is, there exists a vector $\gamma \in \mathbb{R}^q_{++}$ with $\sum_{j=1}^q \gamma_j = 1$ and a scalar r such that

$$W(g) = \left[\sum_{j=1}^{q} \gamma_j (g_j)^r\right]^{\frac{1}{r}}, \forall g \in \mathbb{R}^q_{++}, \tag{15}$$

if $r \neq 0$ and

$$W(g) = \prod_{j=1}^{q} (g_j)^{\gamma_j}, \forall g \in \mathbb{R}^q_{++}, \tag{16}$$

if $r = 0$. It then follows from (2), (14), (15), and (16) that (12) and (13) hold. In view of the normalization adopted for γ, the parameters A, γ, and r are unique. □

The sufficiency part of the proof of Theorem 5 exploits the fact that we know from Theorem 2 that the social evaluation ordering can be represented by a two-stage aggregator function and that the first-stage aggregators are generalized Ginis. WHOM implies that the second-stage aggregator function is ratio-scale measurable and fully comparable and this permits us to use a result from the social choice literature due to Blackorby and Donaldson (1982) to characterize the set of admissible second-stage aggregators.

The characterization in Theorem 5 makes essential use of our earlier result that the second-stage aggregator is additively separable. If there are only two attributes, the second-stage aggregator need not be separable. As a consequence, when the axioms in Theorem 3 are supplemented with WHOM, the only additional structure placed on the function V in (7) is that it is homothetic. [16]

Our next theorem demonstrates that the second-stage aggregator in Theorem 5 must be a Cobb-Douglas function if WHOM is strengthened to SHOM. Further, this conclusion holds if there are two or more attributes, not just if there are at least three.

Theorem 6. *If $q \geq 2$, then the binary relation $\succeq$ on $\mathcal{M}^*_+$ satisfies ORD, CONT, MON, ANON, WUPM, SSEP, WCA, and SHOM if and only if there exists an $n \times q$ matrix A of positive coefficients with $a_{\cdot j}$ nondecreasing and $\sum_{i=1}^n a_{ij} = 1$ for all $j \in Q$ and a vector $\gamma \in \mathbb{R}^q_{++}$ with $\sum_{j=1}^q \gamma_j = 1$ such*

16. To economize on space, we do not state this result formally.

that (13) *holds. Furthermore, the matrix of coefficients* A *and the vector* γ *are unique.*

Démonstration. We only establish the sufficiency part of the theorem as necessity is straightforward to verify.

It follows from Theorems 2 and 3 that there exists an $n \times q$ matrix A of positive coefficients with $a_{\cdot j}$ nondecreasing and $\sum_{i=1}^{n} a_{ij} = 1$ for all $j \in Q$ and there exists a continuous increasing function $V\colon \mathbb{R}^q_{++} \to \mathbb{R}$ such that

$$X \succeq Y \Leftrightarrow V\left(\sum_{i=1}^{n} a_{i1}\tilde{x}_{i1}, \ldots, \sum_{i=1}^{n} a_{iq}\tilde{x}_{iq}\right) \geq V\left(\sum_{i=1}^{n} a_{i1}\tilde{y}_{i1}, \ldots, \sum_{i=1}^{n} a_{iq}\tilde{y}_{iq}\right), \quad \forall X,Y \in \mathcal{M}^*_+.^{17} \tag{17}$$

By SHOM, V is ratio-scale measurable. Hence, by Theorem 4 in Tsui and Weymark (1997), V must be a continuous increasing transform of a Cobb-Douglas function with positive coefficients. That is, V must be a continuous increasing transform of a function of the form given in (16). The uniqueness of A and γ (given the normalization rule for γ) follows from Theorems 2 and 3. □

When there are at least three attributes, Theorem 6 is a corollary to Theorem 5. SHOM implies that the second-stage aggregator function is separable, and this is enough separability for the $q > 2$ characterization to also hold when there are only two attributes.

The second-stage aggregator function in (10) is linear. Consequently, the social evaluation orderings identified in Theorem 4 also satisfy WHOM. Note that, when $q > 2$, this linear aggregator is obtained by setting $r = 1$ in Theorem 5. However, if SHOM is added to the axioms in Theorem 4, an impossibility theorem is obtained because having a linear second-stage aggregator is inconsistent with Theorem 6.

Theorem 7. *If* $q \geq 2$*, there is no binary relation* $\succeq$ *on* $\mathcal{M}^*_+$ *that satisfies ORD, CONT, MON, ANON, WUPM, SSEP, SCA, and SHOM.*

Démonstration. On the contrary, suppose that there exists a $\succeq$ that satisfies these eight axioms. Then, by Theorem 4, (10) must hold and by Theorem 6, (13) must hold. However, (10) and (13) are inconsistent. □

17. When $q > 2$, we also know from Theorem 2 that V is additive.

6. Translatable Social Evaluation Orderings

Normative indices of absolute inequality are constructed from translatable social evaluation orderings. In this section, we provide the corresponding results for translatable social evaluation orderings to those established in the preceding section for homothetic social evaluation orderings. For simplicity, we now suppose that the domain of the social evaluation ordering is $\mathcal{M}$.

If WTRA is used instead of WHOM in Theorem 5 and the domain is changed from $\mathcal{M}_+^*$ to $\mathcal{M}$, then the second-stage aggregator must be a Kolm-Pollak function. Of particular note is that linear aggregation functions are members of this class.

Theorem 8. *If $q > 2$, then the binary relation $\succeq$ on $\mathcal{M}$ satisfies ORD, CONT, MON, ANON, WUPM, SSEP, WCA, and WTRA if and only if there exists an $n \times q$ matrix A of positive coefficients with $a_{\cdot j}$ nondecreasing and $\sum_{i=1}^{n} a_{ij} = 1$ for all $j \in Q$, a vector $\gamma \in \mathbb{R}^q_{++}$, and a scalar r such that*

$$X \succeq Y \Leftrightarrow \frac{1}{r} \ln \left[\sum_{j=1}^{q} \gamma_j \exp \left(r \sum_{i=1}^{n} a_{ij} \tilde{x}_{ij} \right) \right] \geq \frac{1}{r} \ln \left[\sum_{j=1}^{q} \gamma_j \exp \left(r \sum_{i=1}^{n} a_{ij} \tilde{y}_{ij} \right) \right], \quad \forall X,Y \in \mathcal{M}, \quad (18)$$

if $r \neq 0$ and

$$X \succeq Y \Leftrightarrow \sum_{j=1}^{q} \gamma_j \left(\sum_{i=1}^{n} a_{ij} \tilde{x}_{ij} \right) \geq \sum_{j=1}^{q} \gamma_j \left(\sum_{i=1}^{n} a_{ij} \tilde{y}_{ij} \right), \forall X,Y \in \mathcal{M}, \quad (19)$$

if $r = 0$. Furthermore, the matrix of coefficients A, the vector γ, and the scalar r are unique.

Démonstration. With the following modifications, the proof of this theorem is the same as the proof ofTheorem 5. Because $\succeq$ satisfies WTRA instead of WHOM, the function W (now defined on $\mathbb{R}^q$) is translation-scale measurable and fully comparable. Hence, by Theorem 3 in Blackorby and Donaldson (1982), W must be a Kolm-Pollak function. That is, there exists a vector $\gamma \in \mathbb{R}^q_{++}$ with $\sum_{j=1}^{q} \gamma_j = 1$ and a scalar r such that

$$W(g) = \frac{1}{r} \ln \left[\sum_{j=1}^{q} \gamma_j \exp(r g_j) \right], \forall g \in \mathbb{R}^q, \quad (20)$$

if $r \neq 0$ and

$$W(g) = \sum_{j=1}^{q} \gamma_j g_j, \forall g \in \mathbb{R}^q, \tag{21}$$

if $r = 0$. □

If $q = 2$, the axioms in Theorem 8 characterize the subset of the social evaluation orderings characterized in Theorem 3 for which the function V in (7) is translatable.

Theorem 9 shows that if STRA is substituted for SHOM in Theorem 6 and the domain is changed from $\mathcal{M}_+^*$ to $\mathcal{M}$, then the second-stage aggregator must be linear.

Theorem 9. *If $q \geq 2$, then the binary relation $\succeq$ on $\mathcal{M}$ satisfies ORD, CONT, MON, ANON, WUPM, SSEP, WCA, and STRA if and only if there exists an $n \times q$ matrix A of positive coefficients with $a_{\cdot j}$ nondecreasing and $\sum_{i=1}^{n} a_{ij} = 1$ for all $j \in Q$ and a vector $\gamma \in \mathbb{R}_{++}^q$ such that (19) holds. Furthermore, the matrix of coefficients A and the vector γ are unique.*

Démonstration. With the following modifications, the proof of this theorem is the same as the proof of Theorem 6. Because $\succeq$ satisfies STRA instead of SHOM, the function V (now defined on $\mathbb{R}^q$) is translation-scale measurable. Hence, by Theorem 8.1 in Bossert and Weymark (1996), V must be a continuous increasing transform of a function of the form given in (21). □

In Theorems 5 and 6, the second-stage aggregation function represents a binary relation R on $\mathbb{R}_{++}^q$. We can define an ordering R^* on $\mathbb{R}$ by setting $uR^*v \Leftrightarrow (\exp(u_1), \ldots, \exp(u_q))R(\exp(v_1), \ldots, \exp(v_q))$ for all u,$v \in \mathbb{R}^q$. The ordering R is continuous, increasing, and ratio-scale measurable and fully comparable (resp. ratio-scale measurable) if and only if R^* is continuous, increasing, and translation-scale measurable and fully comparable (resp. translation-scale measurable). This observation accounts for why the functional forms of the second-stage aggregators in Theorems 8 and 9 can be obtained from those in Theorems 5 and 6 by a simple exponential change of variables.

Note that the social evaluation orderings identified in Theorem 4 also satisfy STRA (and, hence, WTRA). Thus, Theorem 9 also characterizes all of the social evaluation orderings on $\mathcal{M}$ that satisfy ORD, CONT, MON, ANON, WUPM, SSEP, SCA, and WTRA (resp. STRA). Because the second-stage aggregator is linear, the conflict we found in Theorem 7 with SHOM does not arise if STRA is used instead.

7. Multidimensional Inequality Indices

The Atkinson-Kolm-Sen (see Atkinson (1970), Kolm (1969), and Sen (1973)) and Kolm (1969) procedures for constructing univariate indices of relative and absolute inequality, respectively, both employ a particular representation of the social evaluation function known as the equally-distributed-equivalent income function. The *equally-distributed-equivalent income* associated with a given univariate income distribution is the per capita income which, if distributed equally, is indifferent to the actual income distribution according to the social evaluation ordering. The equally-distributed-equivalent income function is the mapping that assigns the equally-distributed-equivalent income to each income distribution in the domain. The Atkinson-Kolm-Sen inequality index measures inequality by computing the ratio of the equally-distributed-equivalent income to the mean income and subtracting this value from 1. This index is a relative index if the underlying social evaluation function is homothetic. The Kolm inequality index measures inequality by the difference between the mean income and the equally-distributed-equivalent income. This index is an absolute index if the underlying social evaluation function is translatable.

In this section, we describe how the multi-attribute generalizations of the Atkinson-Kolm-Sen and Kolm inequality indices due to Kolm (1977) and Tsui (1995), respectively, are constructed. We also derive the functional forms of these indices for some of the social evaluation orderings characterized in the preceding sections. For further discussion of this approach to the measurement of multidimensional inequality, see Tsui (1995) and Weymark (1999).

As a domain, we use $\mathcal{M}_+^*$ when we consider relative inequality indices and we use $\mathcal{M}$ when we consider absolute inequality indices. Throughout this section, we suppose that the social evaluation function $\succeq$ satisfies the following basic properties: ORD, CONT, MON, ANON, and WUPM.

We begin by considering relative indices of inequality. In the univariate case, the value of the Atkinson-Kolm-Sen inequality index for a given distribution has a natural interpetation. It is the fraction of the aggregate income that could be destroyed if incomes are equalized and the resulting distribution is socially indifferent to the original distribution. The Kolm (1977) multi-attribute generalization of this index measures the inequality of an allocation by the fraction of the aggregate amount of each attribute that could be destroyed if every attribute is equalized and the resulting allocation is

indifferent to the original allocation according to $\succeq$.

Formally, we first define the function $\Delta_{\mathrm{R}}\colon \mathcal{M}_+^* \to \mathbb{R}$ by setting, for each $X \in \mathcal{M}_+^*$, $\Delta_{\mathrm{R}}(X)$ equal to the scalar that solves

$$\Delta_{\mathrm{R}}(X) X_\mu \sim X.^{18} \tag{22}$$

Our assumptions on $\succeq$ ensure that Δ_{R} is well-defined. The *multi-attribute Kolm inequality index* associated with $\succeq$ is the function $I_{\mathrm{R}}\colon \mathcal{M}_+^* \to \mathbb{R}$ defined by setting

$$I_{\mathrm{R}}(X) = 1 - \Delta_{\mathrm{R}}(X), \forall X \in \mathcal{M}_+^*. \tag{23}$$

If $\succeq$ satisfies WHOM, then I_{R} is a relative index.

It is easy to determine the functional form of the multi-attribute Kolm inequality index that corresponds to each of the social evaluation orderings identified in Section 5. For example, when $\succeq$ is defined as in (12), then (22) is satisfied if

$$\left[\sum_{j=1}^{q} \gamma_j \left(\sum_{i=1}^{n} a_{ij} \Delta_{\mathrm{R}}(X) \mu(x_{\cdot j})\right)^r\right]^{\frac{1}{r}} = \left[\sum_{j=1}^{q} \gamma_j \left(\sum_{i=1}^{n} a_{ij} \tilde{x}_{ij}\right)^r\right]^{\frac{1}{r}}, \forall X \in \mathcal{M}_+^*.$$

Hence,

$$\Delta_{\mathrm{R}}(X) = \frac{\left[\sum_{j=1}^{q} \gamma_j \left(\sum_{i=1}^{n} a_{ij} \tilde{x}_{ij}\right)^r\right]^{\frac{1}{r}}}{\left[\sum_{j=1}^{q} \gamma_j \mu(x_{\cdot j})^r\right]^{\frac{1}{r}}}, \forall X \in \mathcal{M}_+^*.$$

Therefore,

$$I_{\mathrm{R}}(X) = 1 - \frac{\left[\sum_{j=1}^{q} \gamma_j \left(\sum_{i=1}^{n} a_{ij} \tilde{x}_{ij}\right)^r\right]^{\frac{1}{r}}}{\left[\sum_{j=1}^{q} \gamma_j \mu(x_{\cdot j})^r\right]^{\frac{1}{r}}}, \forall X \in \mathcal{M}_+^*, \tag{24}$$

is the multi-attribute Kolm inequality index corresponding to (12).

This index has a particularly simple form if $r = 1$, $\gamma_j = 1/q$ for all $j \in Q$ (so each attribute receives the same weight), and the Gini social evaluation function is used as the first-stage aggregator for each attribute. In this case, (24) becomes

$$I_{\mathrm{R}}(X) = \frac{\sum_{j=1}^{q} \mu(x_{\cdot j}) I_{\mathrm{RG}}(x_{\cdot j})}{\sum_{j=1}^{q} \mu(x_{\cdot j})}, \forall X \in \mathcal{M}_+^*, \tag{25}$$

18. Recall that X_μ is the allocation matrix for which every entry in the j^{th} column is equal to $\mu(x_{\cdot j})$.

where $I_{\mathrm{RG}}(x_{\cdot j})$ is the relative Gini inequality index evaluated at the distribution $x_{\cdot j}$. This index differs from the arithmetic mean of the relative Gini inequality indices for the individual attributes, which Koshevoy and Mosler (1997, p. 275) describe as being a "popular approach" to measuring multidimensional inequality, because the weights in (25) depend on the relative quantities of the attributes.

Similar calculations show that the multi-attribute Kolm inequality index corresponding to (13) is

$$I_{\mathrm{R}}(X) = 1 - \frac{\prod_{j=1}^{q} (\sum_{i=1}^{n} a_{ij} \tilde{x}_{ij})^{\gamma_j}}{\prod_{j=1}^{q} \mu(x_{\cdot j})^{\gamma_j}}, \forall X \in \mathcal{M}_{+}^{*}. \tag{26}$$

If $\gamma_j = 1/q$ for all $j \in Q$ and the Gini social evaluation function is used as the first-stage aggregator for each attribute, then (26) simplifies to

$$I_{\mathrm{R}}(X) = 1 - \prod_{j=1}^{q} (E_{\mathrm{RG}}(x_{\cdot j}))^{\frac{1}{q}}, \forall X \in \mathcal{M}_{+}^{*}, \tag{27}$$

where $E_{\mathrm{RG}}(x_{\cdot j}) = 1 - I_{\mathrm{RG}}(x_{\cdot j})$ is the relative Gini equality index evaluated at the distribution $x_{\cdot j}$.

We now consider absolute indices of inequality. The univariate Kolm inequality index is equal to the amount of income that could be taken away from every individual in order to obtain a distribution that is socially indifferent to the original distribution if incomes are equalized. The generalization of this index proposed by Tsui (1995) measures inequality by the amount of each attribute that could be taken away from every individual in order to obtain an allocation that is indifferent to the original allocation according to $\succeq$ if the distribution of each attribute is equalized.

To define Tsui's index formally, we first define the function $\Delta_{\mathrm{A}} \colon \mathcal{M} \to \mathbb{R}$ by setting, for each $X \in \mathcal{M}$, $\Delta_{\mathrm{A}}(X)$ equal to the scalar that solves

$$X_{\mu} - \Delta_{\mathrm{A}}(X)\mathbf{1} \sim X. \tag{28}$$

Our assumptions on $\succeq$ ensure that Δ_{A} is well-defined. The *multi-attribute Tsui inequality index* associated with $\succeq$ is the function $I_{\mathrm{A}} \colon \mathcal{M} \to \mathbb{R}$ defined by setting

$$I_{\mathrm{A}}(X) = \Delta_{\mathrm{A}}(X), \forall X \in \mathcal{M}. \tag{29}$$

If $\succeq$ satisfies WTRA, then I_{A} is an absolute index.

We illustrate the construction of I_{A} using the social evaluation orderings in (18) and (19). In the case of (18), (28) is satisfied if

$$\frac{1}{r}\ln\left[\sum_{j=1}^{q}\gamma_j\exp\left(r\sum_{i=1}^{n}a_{ij}[\mu(x_{\cdot j})-\Delta_{\mathrm{A}}(X)]\right)\right]= \frac{1}{r}\ln\left[\sum_{j=1}^{q}\gamma_j\exp\left(r\sum_{i=1}^{n}a_{ij}\tilde{x}_{ij}\right)\right],\forall X\in\mathcal{M}.$$

Hence, by (29), the multi-attribute Tsui inequality index corresponding to (18) is given by

$$I_{\mathrm{A}}(X)=\Delta_{\mathrm{A}}(X)=\frac{1}{r}\ln\left[\frac{\sum_{j=1}^{q}\gamma_j\exp\left(r\mu(x_{\cdot j})\right)}{\sum_{j=1}^{q}\gamma_j\exp\left(r\sum_{i=1}^{n}a_{ij}\tilde{x}_{ij}\right)}\right],\ \forall X\in\mathcal{M}. \tag{30}$$

Similarly, the multi-attribute Tsui inequality index corresponding to (19) is

$$I_{\mathrm{A}}(X)=\Delta_{\mathrm{A}}(X)=\sum_{j=1}^{q}\gamma_j\left[\mu(x_{\cdot j})-\sum_{i=1}^{n}a_{ij}\tilde{x}_{ij}\right],\ \forall X\in\mathcal{M}. \tag{31}$$

If $\gamma_j=1/q$ for all $j\in Q$ and the Gini social evaluation function is used as the first-stage aggregator for each attribute, then (31) simplifies to

$$I_{\mathrm{A}}(X)=\sum_{j=1}^{q}\left[\frac{I_{\mathrm{AG}}(x_{\cdot j})}{q}\right],\ \forall X\in\mathcal{M}, \tag{32}$$

where $I_{\mathrm{AG}}(x_{\cdot j})$ is the absolute Gini inequality index evaluated at the distribution $x_{\cdot j}$. Thus, in contrast to the relative case, taking the arithmetic average of the absolute Gini inequality indices for the individual attributes yields a multi-attribute index of absolute inequality that is consistent with our approach.

8. Correlation Increasing Majorization

WUPM and WUM are multi-attribute generalizations of the Pigou-Dalton transfer principle. In contrast, CIM is a majorization axiom that has no univariate counterpart. CIM regards a correlation-increasing transfer as being socially undesirable. By its very definition, CIM takes account of the dependencies that exist between the distributions of different attributes. The

theorems in the preceding sections all employ SSEP as one of their axioms. SSEP requires the conditional distribution of any attribute to be independent of the distributions of the other attributes. This separability of the social evaluation ordering $\succeq$ across attributes creates a tension between CIM and SSEP. This tension is also present if SSEP is weakened to WSEP; i.e., if only one attribute is required to be separable from the other attributes. Theorem 10 shows that this conflict is fundamental: If $\succeq$ satisfies ANON, it is not possible to satisfy both CIM and WSEP.

Theorem 10. *Suppose that* $\mathcal{D} \in \left\{\mathcal{M}, \mathcal{M}_+, \mathcal{M}_+^*\right\}$. *Then, there does not exist a binary relation* $\succeq$ *on* $\mathcal{D}$ *that satisfies ANON, WSEP, and CIM.*

Démonstration. Let $\succeq$ be a binary relation on $\mathcal{D}$ satisfying WSEP and ANON. By WSEP, there exists an attribute j_0 such that j_0 is separable from the attributes in $Q \setminus \{j_0\}$. Let $X \in \mathcal{D}$ be defined by setting (i) $x_{ij} = 1$ for all $j \neq j_0$ and all $i \in N$ and (ii) $x_{ij_0} = i$ for all i. Define $Y \in \mathcal{D}$ by setting (i) $y_{ij} = x_{ij}$ for all $j \neq j_0$ and all $i \in N$ and (ii) $y_{ij_0} = n - i + 1$ for all $i \in N$. Note that Y is obtained from X by a permutation of the rows of X. Hence, by ANON, $X \sim Y$. Now, consider the matrices X' and Y' obtained from X and Y, respectively, by replacing, for all $j \neq j_0$ and all $i \in N$, x_{ij} and y_{ij_0} with i. Because j_0 is separable from $Q \setminus \{j_0\}$, $X \sim Y$ implies that $X' \sim Y'$. The columns of X' are all identical, whereas those of Y' are not. Consequently, $X' \succ_{\mathrm{C}} Y'$, which violates CIM. □

The argument used in the proof of Theorem 10 can be illustrated quite simply when $n = q = 3$. For concreteness, suppose that attribute 3 is separable from attributes 1 and 2. By ANON, we have

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = Y.$$

By WSEP, indifference is preserved if the first two columns of X and Y are replaced by any other entries, provided that the replacement entries are the same, component by component, in both matrices. In particular, this is the case if each of these columns is replaced with the third column of X. Thus,

$$X' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 3 & 3 & 1 \end{pmatrix} = Y'.$$

The columns of X' are perfectly correlated, whereas the last column of Y' differs from the first two. As a consequence, X' is more correlated than Y'. To satisfy CIM, we would have to have $Y' \succ X'$, which is not the case.

As noted in Section 2, the framework used here has also been employed to analyze the measurement of inequality under uncertainty. In this interpretation, the ij^{th} entry in an allocation matrix is the income (or the interpersonally comparable utility) of individual i in state j. An important feature of this model is that the units in which incomes in different states are measured are the same. If $\succeq$ is interpreted as being the preference relation of a social decision-maker, then Theorem 10 implies that CIM must be violated if $\succeq$ treats individuals symmetrically and it respects the axioms of expected utility theory applied to allocation matrices. This follows because the expected utility axioms include an independence assumption that implies SSEP and, hence, WSEP. This raises the question as to which, if any, model of decision-making under uncertainty is compatible with ANON and CIM.

Schmeidler (1989) has suggested representing a preference over uncertain outcomes by a Choquet integral with respect to a non-additive measure. This model does not employ the independence assumption of expected utility theory. Nevertheless, by adapting an example due to Ben-Porath, Gilboa, and Schmeidler (1997), we are able to show that Schmeidler's proposal is inconsistent with CIM if $\succeq$ satisfies ANON.

Suppose that $\succeq$ is represented by a Choquet functional V on $\mathcal{D}$. A subset $\mathcal{S}$ of $\mathcal{D}$ is *comonotonic* if for all X,$Y \in \mathcal{S}$, $x_{ij} > x_{i'j'}$ implies that $y_{ij} \geq y_{i'j'}$.[19] The most relevant feature here of a Choquet functional is that it is linear on any cone of comonotonic allocation matrices. Suppose that $n = q = 2$. By ANON, we have

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = Y.$$

If Z is comonotonic with both X and Y, then the linearity of V implies that $V(X + Z) = V(X) + V(Z)$ and $V(Y + Z) = V(Y) + V(Z)$, from which it follows that $V(X + Z) = V(Y + Z)$. The matrix

$$Z = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

19. Note that this definition implicitly assumes that the nq variables are all measured in the same units.

is such a matrix. Hence, $X \sim Y$ implies that

$$X' = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = Y',$$

where $X' = X + Z$ and $Y' = Y + Z$. Clearly, $X' \succ_{\mathrm{C}} Y'$. Therefore, provided that individuals are treated symmetrically, Schmeidler's model is not compatible with CIM.[20]

Ben-Porath, Gilboa, and Schmeidler (1997) have suggested that instead of using Schmeidler's model when measuring inequality under uncertainty, one should instead represent $\succeq$ by a multiple-priors (min-of-means) functional. A functional V on $\mathcal{D}$ is a *multiple-priors functional* if there exists a compact and convex set $\mathcal{C}$ of probability measures over the product space $N \times Q$ such that for all $X \in \mathcal{D}$, $V(X) = \min_{p \in \mathcal{C}} \sum_{i,j} p_{ij} x_{ij}$, where p_{ij} is the probability put by p on the ij^{th} entry in an allocation matrix. This kind of functional was introduced by Gilboa and Schmeidler (1989). The multiple-priors model is flexible enough to be compatible with both ANON and CIM. As a consequence, the more general model considered by Gajdos and Maurin (2002) is also compatible with these axioms.

9. Concluding Remarks

The social evaluation orderings axiomatized in this article all have representations that can be expressed in terms of a two-stage aggregation procedure. In the first stage, the distributions of each attribute are aggregated using generalized Gini social evaluation functions. The value of the representation function is then determined by aggregating these generalized Ginis. When an inequality index is derived from a social evaluation ordering having this structure, it is a simple matter to determine the contribution of each attribute to overall inequality. However, as Theorem 10 demonstrates, the cost of having this aggregation property is that it is not possible to satisfy CIM. How serious this drawback is depends on whether one thinks that an

20. Koshevoy and Mosler (1997) have proposed two multi-attribute extensions of the Gini index of relative inequality. One of these indices is based on the characterization of the Gini in terms of the expected relative mean difference of the incomes and the other is based on the characterization of the Gini in terms of the area between the Lorenz curve and the diagonal of a unit square. For each of their indices, it is readily verified that the allocation matrices X' and Y' are regarded as exhibiting the same degree of inequality even though X' is more correlated than Y'.

inequality index is simply a measure of the dispersion of an allocation matrix or whether one thinks that it should also take account of the correlation between its rows. For those that hold the latter view, our results provide a benchmark from which to judge the role that axioms like CIM play in determining the functional structure of an inequality index.

The work of List (1999) and Tsui (1999) sheds some light on the classes of inequality indices that satisfy the inequality counterparts of CIM and either WUPM or WUM. However, as List notes, Tsui employs a controversial decomposability axiom. On the other hand, List's indices are all constructed by using a utility function to reduce the problem to one of univariate inequality measurement and this, too, may be seen as being unduly restrictive. There is therefore much scope for further axiomatic investigations of multidimensional inequality.

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