INSTITUT NATIONAL DE LA STATISTIQUE ET DES ETUDES ECONOMIQUES
Série des Documents de Travail du CREST
(Centre de Recherche en Economie et Statistique)

n° 2003-14

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Market Equilibrium with Insider Trading and Bid-Ask Prices

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Abstract. Many articles deal with the problem of asymmetric information on financial markets. Kyle (1985) studied the case of a strategic agent who knows the law of the prices at the end of a period. He shows the existence of an equilibrium composed by an optimal strategy for the agent and a efficient pricing rule. Glosten and Milgrom (1985) pointed out the rule played by the asymmetric information in the formation of the bid ask spread. The object of the paper is to exhibit an equilibrium in a one period model when the insider is strategic and where we study the effect of asymmetric information on the bid ask spread.

Résumé. Une large littérature traite des problèmes d'asymétrie d'information dans les marchés financiers. Kyle (1985) a étudié l'influence d'un agent stratégique qui connait la réalisation d'un prix futur. Il a établi l'existence d'un équilibre composé d'une stratégie optimisante et d'une règle de prix rationnelle. Glosten et Milgrom (1985) se sont focalisé sur le rôle stratégique du teneur de marché qui fixe un prix d'achat et un prix de vente afin de se couvrir contre l'asymétrie d'information. L'objet de cet article est de trouver un équilibre dans un modèle à une période quand l'agent informé et le teneur de marché ont un comportement stratégique.

Key words: equilibrium theory ; portfolio optimization ; asymmetric information ; pricing in discrete time .

JEL Classification: G11, G12

Mathematics Subject Classification (1991): 49K45, 90H12, 90H60

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[1]The author would like to thank Pr. Huyen Pham for useful comments and discussions.

1 Introduction

What are the effects of asymmetric information on financial markets? Numerous authors have tackled this subject. Many models have been built to study influence of private information on the bid ask spread, the price formation or the collapse of financial markets. The purpose of this paper is to show the existence of an equilibrium where both market maker and insider are strategic agents and where securities are traded at two different prices depending if it is a buy or a sell order.

The influence in price formation by a strategic informed trader has been studied by Kyle (1985). He shows that in a one period model where trades are organized as auctions, there exists an equilibrium composed by a market price and an optimal strategy when an agent has private information on the future liquidative value of a risky asset. In fact, extra information is incorporated into price through the maximization problem of the insider. However, the equilibrium price is unique and then he assumes that there is no difference between the ask and the bid price. One of the main advantage of Kyle's model is that it is robust when time becomes continuous. Back (1992) shows that the equilibrium has the same kind of properties when agents can trade in continuous time.

An other approach with a different kind of models has been proposed by Glosten and Milgrom (1985). The market is differently structured. Their goal is to show that private information is a fundamental component of the bid ask spread. In fact, they focus on the strategic behaviour of the market maker. The main point of their analysis is the fact that the bid ask spread is a consequence of a hedging strategy against private information. They assume that the market maker knows that there exists an informed trader and he fixes the prices in such a way that his expected gain is zero. However, the insider's wealth maximization problem is underlying since they suppose that agents can only trade a fixed quantity of the risky asset and that trades arrive sequentially. Thus Easley, Kiefer, O'Hara and Paperman (1996) extend this framework using two independent Poisson processes to model time arrivals of the informed trader and the noise trader. It turns out that the insider does not really choose his intervention dates. Hence, his strategy only consists in the decision to buy or sell. His strategic behaviour is reduced to the simplest one. Back (2002) shows that this kind of models converges to Kyle's model types in continuous time when the frequency of trades goes to zero and when the private signal can be understood as the knowledge of good or bad news concerning the future liquidative value of a risky asset. One of the results of this paper is the cancellation of the bid ask spread at the equilibrium. An other kind of models is the one developed by Battacharya and Spiegel (1991). The main difference comes from the absence of the noise trader. They show that the market may collapse if the market maker thinks that he has a severe informational disadvantage.

In this paper, we present a model where we want to capture several effects of asymmetric information. Our goal is to study how the strategic behaviour of an informed trader is reflected into prices when the market maker hedges himself against private information by fixing a bid and an ask price. In fact, we construct a model where Kyle's and Glosten Milgrom's conclusions live together. Thus, we consider a one period model where two kinds of agents trade : insider and noise trader and an other type fixes the prices by clearing the market. One of the main points of this paper is the choice of equilibrium characteristics. We define the equilibrium as the realization of two strategic behaviours. We replace the usual rational prices condition often called market efficiency, by the fact that the risk neutral market maker has an expected profit reduced to zero. The second condition, insider's optimality, is the fact that the informed agent maximizes his expected wealth knowing his private information. It turns out, that in general, there does not exist a linear equilibrium (where the insider's strategy is linear in the signal) with these two conditions. However, taking weaker conditions, we are able to show the existence of a linear equilibrium defined by an insider strategy, an ask and a bid price. The optimal strategy seems very close to the one obtained by Kyle (1985). On one hand, we keep the idea that its mean has to be zero which corresponds to the fact that the informed trader wants to hide his strategy in order to not reveal his private information. On the other hand, its variance is greater, hence his strategy is more risky which is natural since by fixing a bid ask spread, the market maker is more aggressive against private information.

The plan of the rest of this paper is as follows : In section 2, we define our model with classical equilibrium conditions, and we show that there does not exist a linear equilibrium. In section 3, we construct a linear equilibrium by weakening our conditions and we study its properties. Section 4 makes some concluding comments. Proofs of results are in the Appendix.

2 The Model

We consider a one period model with a single security and a riskless asset which is normalized to one. The market is defined by the presence of three different kinds of agent as in Kyle (1985) and Glosten and Milgrom (1985). The first one is called the insider, he is strategic and has private information. We assume that he observes the ex post liquidation value of the risky asset, denoted by V, which is normally distributed with mean p_0 and variance Σ^2. The second type of agent is the noise trader. His demand, denoted by Z, is exogeneous, independent of V and normally distributed with mean zero and variance σ^2. Finally there exists a third kind of agent called market makers. We assume that his behaviour is closer to the one given in Glosten and Milgrom than the one in Kyle. Indeed, we consider a market maker who is strategic in the sense that he knows that there exists private information on that market, but he does not know which agent has it. Hence, he fixes the prices (an ask and a bid price) in such a way that he does not loose in average. All agents have no initial endowments.

The trading protocol is as follows : the exogeneous values of V and Z are realized and the insider who observes V, chooses the quantity $X(V)$ he trades. In step two, the market maker only observes the aggregate demand $Y = X + Z$ and determines an ask price,$a(Y)$, and a bid price, $b(Y)$, at which he trades to clear the market. He can not recognize the source of the private information. Finally, the quantities are traded at the market prices and we denoted by $\pi(X, a, b)$ the insider's final wealth and by $G(Y, a, b)$ the market maker one. This framework differs from both Kyle and Glosten-Milgrom. Kyle uses an auction structure as well, but does not give strategic rule to the market maker. On the other hand, we take care to the fact that asymmetric information creates a spread as in Glosten Milgrom but we consider an aggregate demand instead of a protocol where demands arrive one after an other. This last change comes from the fact that our protocol is more consistent when we pass to continuous time.

2.1 Equilibrium

Our goal is to study price formation and optimal trading in an equilibrium theory. However, the definition which follows differs from Kyle's definition since his equilibrium is only based on the agent's behaviour.

Definition 1 *An equilibrium is composed by a triplet $(X(V), a(Y), b(Y))$ which satisfies the two following conditions :*
(C1) Profit maximization : For any insider's strategy X' and for any realization v of V :

$$E[\pi(X, a, b)|V = v] \geq E[\pi(X', a, b)|V = v]$$

(C2) Market efficiency : For any realization y of Y $E[G(Y, a, b)|Y = y] = 0$

We no longer define the market efficiency with respect to stock prices, saying that the prices are martingale with respect to the market maker filtration under a certain probability. In fact, our approach is close to the game theory, and generalizes Glosten and Milgrom's paper in the sense that we take care of the strategic behaviour of the insider and we allow him to influence the prices.

Definition 2 *An equilibrium (X, a, b) is linear if and only if*

$$\exists(\alpha, \beta) \in \mathbb{R}^2, \quad X(V) = \alpha + \beta V$$

2.2 Insider's Characteristics

We have seen that knowing the realization of the liquidative value of the risky asset, the insider submit an order, which will be executed, to the market maker. As there exist two different prices, if we want to compute his profit, we have to differentiate the case of a sell or a buy order.
We recall that the demand level of the insider is denoted by the random variable X and his final wealth by π. It follows that :

$$\pi(X, a, b) = (V - a)X^+ + (b - V)X^- \tag{1}$$

where $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$.
We remark that condition (C1) means that the insider is risk neutral with a linear utility function. For (X, a, b) given, we denote by $\Pi(X, v)$ the quantity the insider wants to maximize :

$$\Pi(X, v) = E[\pi(X, a, b)|V = v] = (v - E[a|V = v])X^+ + (E[b|V = v] - v)X^- \tag{2}$$

Remark 3 The random variable X is adapted to the filtration generated by V since the insider submit his order after the realization of V. Hence, $E[X|V=v]$ is no longer a random variable.

2.3 Consistency and Equilibrium Prices

We now show that condition (C2) implies some conditions on prices. We recall that the market maker only observes Y on the market. His rule is to fix the prices in such a way that he first clears the market and secondly his expected gain is zero. Even if he can only see the aggregate demand, he knows the existence of asymmetric information and he will fix the bid ask spread to hedge his position. Hence, his benefit G after he has fixed the prices is :

$$G(Y,a,b) = (a-V)Y^+ + (V-b)Y^-.$$

Condition (C2) tells us that at the equilibrium, we may have :

$$E[(a-V)Y^+ + (V-b)Y^-|Y=y] = 0.$$

Then, using the fact that a, b and Y are adapted to the filtration generated by Y, we get :

$$(a - E[V|Y=y])y^+ = (b - E[V|Y=y])y^-. \tag{3}$$

It appears that a necessary condition is :

$$\begin{cases} a(Y) &= E[V|Y=y] \text{ on } \{Y>0\} \\ b(Y) &= E[V|Y=y] \text{ on } \{Y<0\} \end{cases}$$

In fact, the random variables which really play a rule in price formation are Y^+ and Y^-. Thus, we restrict condition (C2) to :

Definition 4 *(C2') At the equilibrium, the prices have the following form :*

$$\begin{cases} a(Y) &= E[V|Y=y] \textit{ on } \{Y>0\} \\ a(Y) &= E[V|Y\le 0] \textit{ on } \{Y\le 0\} \end{cases}$$
$$\begin{cases} b(Y) &= E[V|Y\ge 0] \textit{ on } \{Y\ge 0\} \\ b(Y) &= E[V|Y=y] \textit{ on } \{Y<0\} \end{cases}$$

Remark 5 (i) This definition defines a “martingale price” p which can be understood as the reference price in the model with transaction costs :

$$p = a\mathbf{1}_{\{Y\ge 0\}} + b\mathbf{1}_{\{Y<0\}}$$

(ii) Condition (C2') is equivalent to $a(Y) = E[V|Y^+]$ and $b(Y) = E[V|Y^-]$.
(iii) According to (3), we have $(C2') \Rightarrow (C2)$.

We call ϕ the standard Gaussian density and Φ the associated cumulative function. Now, we are able to give the form of the bid and the ask prices in a linear equilibrium.

Proposition 6 *Under (C2') and if X is linear in V $(X(V) = \alpha + \beta V)$, then*

$$a(Y) = p_0 - \frac{\beta\Sigma^2}{\delta}\lambda(\frac{-m}{\delta})\mathbf{1}_{Y<0} + \frac{\beta\Sigma^2}{\delta^2}(Y-m)\mathbf{1}_{Y\geq 0} \tag{4}$$

$$b(Y) = p_0 + \frac{\beta\Sigma^2}{\delta}\lambda(\frac{m}{\delta})\mathbf{1}_{Y>0} + \frac{\beta\Sigma^2}{\delta^2}(Y-m)\mathbf{1}_{Y\leq 0} \tag{5}$$

where $\delta = \sqrt{\sigma^2 + \beta^2\Sigma^2}$, $m = \alpha + \beta p_0$ and $\lambda(.) = \frac{\phi(.)}{\Phi(.)}$ is the Mill's ratio.

This proposition tells us that the ask price changes linearly with respect to the market demand when the market is buyer and is completely non elastic when it is seller. One may think that this constitutes a weakness of this modeling, however it is quite natural. In fact, when the market is buyer, the ask price becomes the market price, and the bid price is just a reference price which does not have vocation to be used.
Using the definition (C2') of equilibrium bid ask prices, we get the following result

Proposition 7 *In a linear equilibrium (C1)-(C2'), we have :*

$$\forall (x,w) \in \mathbb{R}^2 \quad \Pi(x, p_0 + w) = \Pi(-x, p_0 - w) \tag{6}$$

This result is not surprising since it says that there exists a symmetry in our market. In fact, for the insider, his gain will be the same if he has a good information on the price ($v > p_0$) and he buys or if he has a bad information ($v < p_0$) and he sells. This is natural in a market where there is no short selling constraints.
By the way, from the previous proposition, we have

Corollary 8 *If $X(V) = \alpha + \beta V$ is optimal, and if prices satisfy (C2'), then* $\alpha + \beta p_0 = 0$

This condition is quite similar to the one in Kyle (1985). This condition means that in average, the strategy of the insider is zero ($E[X] = \alpha + \beta p_0$). In fact, the insider wants to hide his strategy and does not want to reveal information to the market maker. If this mean was positive, then the market maker would anticipate good information for the prices, and then adapts the prices in consequence.
However, the next remark tells us that our equilibrium conditions seem to be too strong.

Remark 9 In general, there is no linear equilibrium satisfying (C1) and (C2). We give a quick argument for this. From Corollary 8, we know that if X is linear, then its mean must be zero. Hence we have to maximize the quantity :

$$\Pi(x,v)\mathbf{1}_{\{x>0\}} = x\left(v - p_0 + \delta\sqrt{\frac{2}{\pi}}\left(\Phi(-\frac{x}{\sigma}) - \sqrt{\frac{\pi\sigma^2}{2(\sigma^2+\beta^2\Sigma^2)}}W(\frac{x}{\sigma})\right)\right)$$

where $W(t) = \int_{-\infty}^{t} \Phi(u)du = t\Phi(t) + \phi(t)$ and $x = \beta(v - p_0)$. Hence we have to find the maximum of $\widetilde{\Pi}(\beta) = \Pi(\beta w, v)$ $(w = v - p_0)$ independently of v. For fixed $w > 0$, and $\Sigma = \sigma = 1$, we get

$$\widetilde{\Pi}(\beta)\mathbf{1}_{\{\beta w > 0\}} = w\beta \left(w + \sqrt{\frac{2}{\pi}} \frac{\beta}{\sqrt{\beta^2+1}} \Phi(-w\beta) - \frac{\beta}{\beta^2+1} W(w\beta) \right)$$

Using Mathematica®, we can compute the argument maximum of $\widetilde{\Pi}$. Thus, we get $\arg\max(\widetilde{\Pi}(\beta)|_{w=1}) \simeq 1.06$ and $\arg\max(\widetilde{\Pi}(\beta)|_{w=0.1}) \simeq 8.17$. The fact that we used a software to compute the maximum is just for simplicity. Indeed the form of $\widetilde{\Pi}$ is quite complicated, and the maximization of this function is not really the point, here.

It seems that the definition of our equilibrium, condition $(C1)$ and $(C2)$ (or (C2')) is not well adapted to our modeling. In fact, contrary to Kyle's model, the optimal strategy is not linear in the signal V. Its dependence seems more complex. The next section will give us a linear equilibrium when we change condition $(C1)$.

3 A Weak Linear Equilibrium

We define a slightly different profit maximization condition for the insider. We introduce :

$$\begin{aligned} F: \quad \mathcal{C}(\mathbb{R},\mathbb{R}) &\to \mathcal{C}(\mathbb{R},\mathbb{R}) \\ X(.) &\mapsto \Pi(X(.),.) \end{aligned}$$

Let us define $\mathcal{A} = \left\{ X \in \mathcal{C}(\mathbb{R},\mathbb{R}), \forall t \in \mathbb{R} \quad X : t \to \alpha + \beta t,\ (\alpha,\beta) \in \mathbb{R}^2 \right\}$. Hence, for $x \in \mathcal{A}$ and $v \in \mathbb{R}$, we have $F(x)(v) = \Pi(\alpha + \beta v, v)$. Thus, we give the new profit maximization condition.

Definition 10 *(C1') $X^*(V)$ is optimal if $X^* \in \mathcal{A}$ and for any $X \in \mathcal{A}$*

$$E[F(X^*(V))(V)] \geq E[F(X(V))(V)]$$

Remark 11 We consider now a two dimensions optimization problem while (C1) was an infinite dimensions optimization. Moreover, according to Remark 9, if there exists a solution to (C1'), it will not satisfy (C1).

Now, we define a new kind of equilibrium.

Definition 12 *A triplet $(X(V), a(Y), b(Y))$ is called a weak linear equilibrium (WLE) iff it satisfies (C1') and (C2')*

Remark 13 First, the statement of Corollary 8 is still valid under conditions $(C1')$ and $(C2')$. Hence, we only need to compute the quantity $F(\beta(v - p_0), v)$. Secondly, remarking that $(\beta(v - p_0))^+ = -\beta(v - p_0)^-$ for negative β, we get when $\beta < 0$

$$F(\beta(v - p_0))(v) = -F(-\beta(v - p_0))(v)$$

Using Proposition 6, Corollary 8 and (2), we get for positive β

$$\begin{aligned} F(\beta(v-p_0))(v) &= \beta w^2 + \beta A(w^-\Phi(\frac{w}{\sigma}\beta) + w^+\Phi(-\frac{w}{\sigma}\beta)) \\ & -\beta B(\beta w w^+\Phi(\frac{w}{\sigma}\beta) - \beta w w^-\Phi(-\frac{w}{\sigma}\beta) + \sigma|w|\phi(\frac{w}{\sigma}\beta)) \end{aligned}$$

where $w = v - p_0$, $A = \sqrt{\frac{2}{\pi}}\frac{\Sigma^2\beta}{\sqrt{\Sigma^2\beta^2+\sigma^2}}$ and $B = \frac{\Sigma^2\beta}{\Sigma^2\beta^2+\sigma^2}$. Thus, we can compute the desired quantity :

$$\begin{aligned} \frac{\Sigma}{\beta}E[F(\beta(V-p_0)(V)] &= \int_{\mathbb{R}} w^2\phi(\frac{w}{\Sigma})dw - B\sigma\int_{\mathbb{R}}|w|\phi(\frac{w}{\Sigma})\phi(\frac{w}{\sigma}\beta)dw \\ & -B\int_{\mathbb{R}}(\beta w w^+\Phi(\frac{w}{\sigma}\beta) - \beta w w^-\Phi(-\frac{w}{\sigma}\beta))\phi(\frac{w}{\Sigma})dw \\ & +A\int_{\mathbb{R}}(w^-\Phi(\frac{w}{\sigma}\beta) + w^+\Phi(-\frac{w}{\sigma}\beta))\phi(\frac{w}{\Sigma})dw \\ &= I - B\sigma J - BK + AL \end{aligned}$$

where I, J, K and L are the corresponding integrals.

Lemma 14 *Using the previous notations, we get*

$$\begin{aligned} I &= \Sigma^3 \\ J &= \frac{1}{\pi}\frac{\Sigma^2\sigma^2}{\Sigma^2\beta^2+\sigma^2} \\ K &= \frac{1}{\pi}\frac{\Sigma^4\sigma\beta}{\Sigma^2\beta^2+\sigma^2} + \frac{1}{2}\Sigma^3 + \frac{1}{\pi}\Sigma^3\arctan(\frac{\beta\Sigma}{\sigma}) \\ L &= \frac{1}{\sqrt{2\pi}}\Sigma^2(1 - \frac{\beta\Sigma}{\sqrt{\sigma^2+\beta^2\Sigma^2}}) \end{aligned}$$

We have now a closed formula for $E[F(\beta(V-p_0)(V)]$. Let us define the parameter $d = \frac{\Sigma}{\sigma}$. After some computations, using the previous lemma, we get

$$E[F(\beta(V-p_0)(V)] = \Sigma^2 f_d(\beta) \quad (7)$$

where

$$f_d(\beta) = \frac{\beta}{\pi(1+d^2\beta^2)}(\pi - d\beta(1-\sqrt{1+d^2\beta^2}) + d^2\beta^2(\arctan(\frac{1}{d\beta}) - 1)) \quad (8)$$

Remark 15 The first properties of f_d are
(i) $f_d(0) = 0$
(ii) $f_d(\beta) = O(\frac{1}{\beta})$
(iii) $f_d(\beta) = \beta g(\beta)$. Hence, we have $f'(\beta) = g(0) = 1$.
Those three properties tell us that f_d has a global maximum on $\mathbb{R}^+$

Proposition 16 *f_d has a unique maximum for $\beta^* = \frac{\zeta}{d}$ where ζ is the unique solution of $f_1'(x) = 0$.*

We are able to have a numerical value of ζ using Mathematica®. It turns out that $\zeta \in [1.05, 1.06]$. Hence, it seems that the equilibrium value of β is not exactly the same as in Kyle's paper even if it is very close, $\zeta\frac{\sigma}{\Sigma}$ instead of $\frac{\sigma}{\Sigma}$. Besides, we can formulate the main result :

Theorem 17 *There exists a unique weak linear equilibrium defined by :*

$$\begin{aligned} X(V) &= \frac{\sigma\zeta}{\Sigma}(V - p_0) \\ a(Y) &= p_0 - \sqrt{\frac{2}{\pi}}\frac{\Sigma\zeta}{\sqrt{1+\zeta^2}}\mathbf{1}_{\{Y\leq 0\}} + \frac{\Sigma}{\sigma}\frac{\zeta}{1+\zeta^2}Y\mathbf{1}_{\{Y>0\}} \\ b(Y) &= p_0 + \sqrt{\frac{2}{\pi}}\frac{\Sigma\zeta}{\sqrt{1+\zeta^2}}\mathbf{1}_{\{Y\geq 0\}} + \frac{\Sigma}{\sigma}\frac{\zeta}{1+\zeta^2}Y\mathbf{1}_{\{Y<0\}} \end{aligned}$$

where ζ is defined in Proposition 16.

Proof: In fact, we have constructed a strategy X which realizes the criterion $(C1')$ from Proposition 16. However, also by construction, the prices satisfy $(C2')$, then the triplet (X, a, b) given in the theorem is an equilibrium. By the way, the unicity of the equilibrium comes from the fact that the function f_d admits a unique maximum. □

According to the weaker assumption $(C1')$, we are able to construct a linear equilibrium. Actually, the strategy of Theorem 17 does not satisfy the criterion $(C1)$, it is not even sure that this strategy is the best linear approximation of the optimal strategy satisfying $(C1)$ in a certain sense. We study the properties of the weak linear equilibrium in the next section.

4 Properties of the Weak Linear Equilibrium

4.1 Influence of Parameters

There are three essential exogeneous data in this model which are the mean of the signal p_0, its variance Σ^2 and the variance of noise trader's demand σ^2. First, we observe that the optimal strategy of WLE is very closed to the one of the Kyle equilibrium. In fact, the volatility of the strategy of the insider is 6% higher in our model, since $\zeta \simeq 1.06$. Moreover, we still see the importance of the ratio of the standard deviations $\frac{\sigma}{\Sigma}$ for the strategy and the prices. However, even if the market price (the ask price when $Y > 0$ and the bid price when $Y < 0$) remains the same, it turns out that the reference price (a when $Y < 0$ and b when $Y > 0$) is only proportional to Σ.

It appears that this modeling advantages the insider in terms of expected profits. Indeed, in the WLE, we have $E[\pi(X, a, b)] = \sigma\Sigma f_1(\zeta)$ while in the equilibrium of Kyle, we have $E[\pi(X, p)] = \frac{1}{2}\sigma\Sigma$. Using Mathematica®, we get the following approximation $f_1(\zeta) \simeq 0.53 > \frac{1}{2}$. Hence, even if this strategy is more risky, its gain will be larger in this modeling. This shows that the strategic behaviour of the market maker does not affect the profit of the insider. Hence, the existence of a bid ask spread due to extra information does not cancel the possible benefits generated by this asymmetry.

4.2 Bid Ask Spread

A natural question is how the spread is influenced by the exogeneous data. First, we have to be careful with the usual commentaries on the bid ask spread, since we have an order driven market. Thus, results of Jouini-Kallal's paper (1995) do not apply to our case. In fact, in such models, as the orders are made without regrets, one can not construct an arbitrage when for example the ask price is below the bid price because one can not order anymore.
In Jouini-Kallal (1995), a necessary condition for no free lunch is the fact that the ask price has to be greater than the bid price. It turns out that this is not the case in our model with a probability strictly between 0 and 1. This result is a direct consequence of the auction framework. We may hope that in a multi period model or if the quote are in continuous time, the no free lunch hypothesis will force the ask to be greater than the bid. However, we can define a martingale price p which reflects the rationality of the market $p(Y) = a\mathbf{1}_{\{Y>0\}} + b\mathbf{1}_{\{Y<0\}}$. If we denote the bid ask spread by $s = a - b$, at the WLE, we get

$$s(Y) = \frac{\Sigma\zeta}{1+\zeta^2}\left(\left(\frac{Y}{\sigma} - \sqrt{\frac{2}{\pi}}\sqrt{1+\zeta^2}\right)\mathbf{1}_{\{Y\geq 0\}} + \left(\frac{-Y}{\sigma} - \sqrt{\frac{2}{\pi}}\sqrt{1+\zeta^2}\right)\mathbf{1}_{\{Y<0\}}\right)$$

Hence, it becomes

$$s(Y) = \frac{\Sigma}{\sigma}\frac{\zeta}{1+\zeta^2}(Y^+ + Y^-) - \sqrt{\frac{2}{\pi}}\frac{\Sigma\zeta}{\sqrt{1+\zeta^2}}$$

Finally, we get the form of the bid ask spread when the WLE is realized :

$$s(Y) = \frac{\Sigma}{\sigma}\frac{\zeta}{1+\zeta^2}|Y| - \sqrt{\frac{2}{\pi}}\frac{\Sigma\zeta}{\sqrt{1+\zeta^2}} \tag{9}$$

Thus we see that for small values of Y, the spread may be negative. But, small values of Y means that the noise trader demand and the insider demand are opposed, hence the market maker knows that the insider is against the trend, hence as he does not know if he is buyer or seller, his hedging becomes extreme and the spread negative. One may say that in this case, the market collapses as in Battacharya and Spiegel and the orders are not realized at all.
At the first look, the bid ask spread seems to depend on the ratio $\frac{\Sigma}{\sigma}$. In fact, the spread only depends on the volatility of the signal V. Indeed, we have $Y = X + Z = \sigma(\frac{\zeta}{\Sigma}(V - p_0) + \frac{Z}{\sigma})$. But $\frac{Z}{\sigma} \sim \mathcal{N}(0,1)$, hence $\frac{1}{\sigma}|Y|$ does not depend on σ. It turns out that the spread is proportional to Σ, which means that a part of the extra information is incorporated into the prices.

5 Conclusion

This model tells us that the asymmetric information on financial markets influences all parameters of the equilibrium. We keep the ideas of Kyle and Glosten Milgrom and we have shown that they can live together.

As in Kyle's paper, a natural extension of this model is the multi period case. We consider, in fact, that there are N trading dates before the reveal of information. A quick study shows up that it is not as simple as in the classical auction's case. The construction is quite similar, however to show the existence of an equilibrium using the dynamic programming, we need to conjecture a form for the wealth process (quadratic in V in Kyle's case). It turns out that this conjecture is problematic in our model. In fact, it suffices to see the form of the wealth after the first step of the regression, passing to step N to $N-1$, to understand the difficulty to prove the existence of a maximum (see proof of Proposition 16) at each step.
Continuous time seems to offer better chance to determine an equilibrium. On one hand, tools associated to the maximization of utility as Hamilton-Jacobi-Bellamn equations, are easier to use. On the other hand, to compute the prices, we should need results on reflected Brownian Motion or Bessel processes to generalize the notions of Y^+, Y^- and $|Y|$ present in our model.

Appendix

Proof of Proposition 6:
To compute $a = E(V \mid Y^+)$, we need to determine $P^{V|Y^+=y^+}(v)$. We recall that $V \sim \mathcal{N}(p_0, \Sigma^2)$ and $Y = \alpha + \beta V + Z$ where $Z \sim \mathcal{N}(0, \sigma^2)$. Hence, we can compute $P^{V,Y}(v,y)$ the density of the couple (V,Y).

$$P^{V,Y}(v,y) = \frac{1}{2\pi\sigma\Sigma}\exp\left(-\frac{1}{2\sigma^2\Sigma^2}\begin{pmatrix}\widetilde{v}\\ \widetilde{y}\end{pmatrix}^T\begin{pmatrix}\beta^2\Sigma^2+\sigma^2 & -\beta\Sigma^2\\ -\beta\Sigma^2 & \Sigma^2\end{pmatrix}\begin{pmatrix}\widetilde{v}\\ \widetilde{y}\end{pmatrix}\right)$$

where $\begin{pmatrix}\widetilde{v}\\ \widetilde{y}\end{pmatrix} = \begin{pmatrix}v-p_0\\ y-\alpha-\beta p_0\end{pmatrix}$
Now, we need to find the law of (V, Y^+). First, we know that :

$$P^{Y^+}(y) = \Phi\left(-\frac{\alpha+\beta p_0}{\sqrt{\sigma^2+\beta^2\Sigma^2}}\right)\delta_0(y) + \frac{1}{\sqrt{\sigma^2+\beta^2\Sigma^2}}\phi\left(\frac{y-\alpha-\beta p_0}{\sqrt{\sigma^2+\beta^2\Sigma^2}}\right)\mathbf{1}_{\mathbb{R}_+^*}(y)$$

where ϕ is the density of $\mathcal{N}(0,1)$, Φ is the associated cumulative function and δ_0 is the Dirac mass in 0. Hence, we can see that Y^+ has no density with respect to the Lebesgues measure.
We can compute the law of couple (V, Y^+). After some computations, we get :

$$P^{V,Y^+}(v,y) = \mathbf{1}_{\mathbb{R}\times\mathbb{R}_+^*}(v,y)P^{V,Y}(v,y) + \frac{1}{\Sigma}\phi\left(\frac{v-p_0}{\Sigma}\right)\Phi\left(-\frac{\alpha+\beta v}{\sigma}\right)\delta_0(y)$$

Now, obviously, we get what we needed :

$$P^{V|Y^+=y}(v,y) = \frac{P^{V,Y^+}(v,y)}{P^{Y^+}(y)}$$

We are able to compute the ask and the bid prices :

$$a = E(V \mid Y^+ = y) = \int_{\mathbb{R}} vP^{V|Y^+=y}(v,y)dv$$

For simplicity, we look to two different cases.
(i) y=0

$$P^{V|Y^+=0}(v,0) = \frac{1}{\Phi\left(-\frac{\alpha+\beta p_0}{\sqrt{\sigma^2+\beta^2\Sigma^2}}\right)} \frac{1}{\Sigma}\phi\left(\frac{v-p_0}{\Sigma}\right)\Phi\left(-\frac{\alpha+\beta v}{\sigma}\right)$$

For simplicity, we define $C = \Sigma\Phi\left(-\frac{\alpha+\beta p_0}{\sqrt{\sigma^2+\beta^2\Sigma^2}}\right)$.

$$\begin{aligned}
\sqrt{2\pi}C\int_{\mathbb{R}} vP^{V|Y^+=0}(v,0)dv &= \int_{\mathbb{R}} xe^{-\frac{1}{2\Sigma^2}(x-p_0)^2}\Phi\left(-\frac{\alpha+\beta x}{\sigma}\right)dx \\
&= -\Sigma^2\int_{\mathbb{R}}(\frac{-x+p_0}{\Sigma^2})e^{-\frac{1}{2\Sigma^2}(x-p_0)^2}\Phi\left(-\frac{\alpha+\beta x}{\sigma}\right)dx \\
&\quad +p_0\int_{\mathbb{R}} e^{-\frac{1}{2\Sigma^2}(x-p_0)^2}\Phi\left(-\frac{\alpha+\beta x}{\sigma}\right)dx
\end{aligned}$$

Using an IPP for the first term on the right hand side, we get :

$$\begin{aligned}
\sqrt{2\pi}C\int_{\mathbb{R}} vP^{V|Y^+=0}(v,0)dv &= -\Sigma^2\left[e^{-\frac{1}{2\Sigma^2}(x-p_0)^2}\Phi\left(-\frac{\alpha+\beta x}{\sigma}\right)\right]_{-\infty}^{+\infty} \\
&\quad +\Sigma^2\int_{\mathbb{R}}\left(\frac{-\beta}{\sqrt{2\pi}\sigma}\right)e^{-\frac{1}{2\Sigma^2}(x-p_0)^2}e^{-\frac{1}{2\sigma^2}(\beta x+\alpha)^2}dx \\
&\quad +p_0\int_{\mathbb{R}} e^{-\frac{1}{2\Sigma^2}(x-p_0)^2}\Phi\left(-\frac{\alpha+\beta x}{\sigma}\right)dx \\
&= -\frac{\beta\Sigma^2}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(\frac{p_0^2}{\Sigma^2}+\frac{\alpha^2}{\sigma^2})}\int_{\mathbb{R}} e^{-\frac{\sigma^2+\beta^2\Sigma^2}{2\sigma^2\Sigma^2}\left(x^2+2\frac{\alpha\beta\Sigma^2-p_0\sigma^2}{\sigma^2+\beta^2\Sigma^2}\right)}dx \\
&\quad +\sqrt{2\pi}Cp_0
\end{aligned}$$

The first term of the right hand side of the first equality is zero since Φ is bounded. We remark that the third one is just the marginal probability of Y^+ in zero, hence finally, we try to compute the second term by building a square in the exponential. Then, after some usual computations, we get

$$\int_{\mathbb{R}} vP^{V|Y^+=0}(v,0)dv = p_0 - \frac{\beta\Sigma^2}{2\pi C\sigma}e^{-\frac{1}{2}\frac{(\alpha+\beta p_0)^2}{\sigma^2+\beta^2\Sigma^2}}\int_{\mathbb{R}} e^{-\frac{\sigma^2+\beta^2\Sigma^2}{2\sigma^2\Sigma^2}\left(x-\frac{p_0\sigma^2-\alpha\beta\Sigma^2}{\sigma^2+\beta^2\Sigma^2}\right)^2}dx$$

Now, we are able to compute this integral using the density of a Gaussian variable and we get :

$$E(V \mid Y^+ = 0) = p_0 - \frac{\beta\Sigma^2}{\sqrt{\sigma^2 + \beta^2\Sigma^2}}\lambda\left(-\frac{\alpha + \beta p_0}{\sqrt{\sigma^2 + \beta^2\Sigma^2}}\right) \tag{10}$$

where $\lambda(.) = \frac{\phi(.)}{\Phi(.)}$ is usually called Mill's ratio.
(ii) $y > 0$
In this case, there is no difference with the Gaussian case. It turns out that the computation are exactly the same. Hence, we directly have :

$$E(V \mid Y^+ = y) = p_0 + \frac{\beta\Sigma^2}{\sigma^2 + \beta^2\Sigma^2}(y - (\alpha + \beta p_0)) \tag{11}$$

We are able to give the expression of $b = E(V \mid Y^-)$. In fact, let us define $W = -Y$. Hence, we remark that $Y^- = W^+$. No more computations are needed to compute the bid price b since

$$b = E(V \mid Y^-) = E(V \mid W^+)$$

where $W = -\alpha - \beta V - Z$. Hence, $W \sim \mathcal{N}(-\alpha - \beta p_0, \sigma^2 + \beta^2\Sigma^2)$. We remark that the computation are exactly the same except that the correlation between V and W is $-\beta\Sigma^2$ instead of $\beta\Sigma^2$.

$$\begin{cases} E(V \mid Y^- = 0) & = & p_0 + \frac{\beta\Sigma^2}{\sqrt{\sigma^2+\beta^2\Sigma^2}}\lambda\left(\frac{\alpha+\beta p_0}{\sqrt{\sigma^2+\beta^2\Sigma^2}}\right) \\ E(V \mid Y^- = y) & = & p_0 - \frac{\beta\Sigma^2}{\sigma^2+\beta^2\Sigma^2}(y + (\alpha + \beta p_0)) \end{cases}$$

Proof of Proposition 7:
Using (2) and definition (C2'), we have :

$$\begin{aligned} \Pi(x, p_0 + w) &= E\left[(V - E(V \mid Y^+))X^+ + (E(V \mid Y^-) - V)X^- \mid V = p_0 + w\right] \\ &= (p_0 + w - E(E(V \mid Y^+) \mid V = p_0 + w))x^+ \\ &\quad +(E(E(V \mid Y^+) \mid V = p_0 + w) - p_0 - w)x^- \\ &= (w - E(E(V - p_0 \mid Y^+) \mid V - p_0 = w))x^+ \\ &\quad +(E(E(V - p_0 \mid Y^+) \mid V - p_0 = w) - w)x^- \\ &= (w - E(E(W \mid Y^+) \mid W = w))x^+ \\ &\quad +(E(E(W \mid Y^+) \mid W = w) - w)x^- \end{aligned}$$

where $W \sim \mathcal{N}(0, \sigma^2)$. Using the fact that $(-x)^+ = x^-$ and $(-x)^- = x^+$ We can easily compute the other part of the equality of the lemma :

$$\begin{aligned} \Pi(-x, p_0 - w) &= (-w - E(E(W \mid Y^+) \mid W = -w))(-x)^+ \\ &\quad +(E(E(W \mid Y^+) \mid W = -w) + w)(-x)^- \\ &= (w - E(E(-W \mid Y^-) \mid -W = w))x^+ \\ &\quad +(E(E(-W \mid Y^+) \mid -W = w) - w)x^- \end{aligned}$$

Now, it remains to show :

$$E(E(W \mid Y^+) \mid W = w) = E(E(-W \mid Y^-) \mid -W = w)$$

Let us recall that we have a relation between x and w : $x = \alpha + \beta p_0 + \beta w$. Hence on $\{X = -(\alpha + \beta p_0 + \beta w)\} \cap \{W = -w\}$, we have the fact that :

$$\begin{aligned} Y^- = (X+Z)^- &= (-\alpha - \beta p_0 - \beta w + Z)^- \\ &= (\alpha + \beta p_0 + \beta w - Z)^+ \end{aligned}$$

But, we know that Z is Gaussian with mean zero, hence, Z and $-Z$ have the same law (same thing for W and $-W$) then the two quantities are exactly the same. We can have the same kind of arguments for the other part. Finally, we get : $\Pi(-x, p_0 - w) = \Pi(x, p_0 + w)$. $\square$

Proof of Proposition 8:
First, we need to compute explicitly the quantity $\Pi(x, v)$. Hence using (2), we have :

$$\Pi = (v - E(a \mid V = v))x^+ + (E(b \mid V = v) - v)x^-$$

Thus we need to compute $E[a|V = v]$. Using (C2') and the fact that $Z \sim \mathcal{N}(0, \sigma^2)$, we get after a few computations :

$$E(a \mid V = v) = M\Phi\left(-\frac{x}{\sigma}\right) + (m' + px)\Phi\left(\frac{x}{\sigma}\right) + p\sigma\phi\left(\frac{x}{\sigma}\right)$$

where $M = p_0 - \frac{\beta\Sigma^2}{\delta}\lambda(-\frac{m}{\delta})$, $m' = p_0 - \frac{\beta\Sigma^2}{\delta^2}m$ and $p = \frac{\beta\Sigma^2}{\delta^2}$. Using the same kind of computation we get :

$$E(b \mid V = v) = M'\Phi\left(\frac{x}{\sigma}\right) + (m' + px)\Phi\left(-\frac{x}{\sigma}\right) - p\sigma\phi\left(\frac{x}{\sigma}\right)$$

where $M' = p_0 + \frac{\beta\Sigma^2}{\delta}\lambda(\frac{m}{\delta})$.
Let us assume for convenience that $x > 0$:

$$\begin{aligned} D_3 &= \Pi(x, p_0 + w) - \Pi(-x, p_0 - w) \\ &= 2xp_0 - x\Phi(-\frac{x}{\sigma})(M + M') - 2m'x\Phi(\frac{x}{\sigma}) \\ &= 2xp_0 - x\Phi(-\frac{x}{\sigma})(p_0 - \frac{\beta\Sigma^2}{\delta}\lambda(-\frac{m}{\delta}) + p_0 + \frac{\beta\Sigma^2}{\delta}\lambda(\frac{m}{\delta})) \\ &\quad -2x\Phi(\frac{x}{\sigma})(p_0 - \frac{\beta\Sigma^2}{\delta}\frac{m}{\delta}) \\ &= 2xp_0 - x\Phi(-\frac{x}{\sigma})(p_0 - \frac{\beta\Sigma^2}{\delta}\lambda(-\frac{m}{\delta}) + p_0 + \frac{\beta\Sigma^2}{\delta}\lambda(\frac{m}{\delta})) \\ &\quad -2x(1 - \Phi(-\frac{x}{\sigma}))(p_0 - \frac{\beta\Sigma^2}{\delta}\frac{m}{\delta}) \\ &= x\frac{\beta\Sigma^2}{\delta}(\Phi(-\frac{x}{\sigma})(\lambda(-\frac{m}{\delta}) - \lambda(\frac{m}{\delta}) - 2\frac{m}{\delta}) + 2\frac{m}{\delta}) \end{aligned}$$

Using Proposition 7, an equilibrium condition is $D_3 = 0$, hence, this will give us a condition on m and so on α and β. We know that $0 \leq \Phi(-\frac{x}{\sigma}) \leq 1$. We study the function f defined by $f(\gamma) = q(\lambda(-\gamma) - \lambda(\gamma) - 2\gamma) + 2\gamma$ where $q \in [0,1]$ is a parameter.

$$f'(\gamma) = -q(\lambda'(-\gamma) + \lambda'(\gamma)) + 2(1-q)$$

We recall that λ is decreasing, hence knowing that $q \in [0,1]$, we easily get that f is increasing. Let us remark that $f(0) = 0$. Thus we get :

$$D_3 = 0 \Leftrightarrow m = 0 \Leftrightarrow \alpha + \beta p_0 = 0 \tag{12}$$

□

Proof of Lemma 14:
I is just the second moment of a Gaussian variable up to a multiplicative constant, hence $\frac{1}{\Sigma}\int_{\mathbb{R}} w^2\phi(\frac{w}{\Sigma})dw = \Sigma^2 = \frac{I}{\Sigma}$. J can be compute easily :

$$\begin{aligned}
\int_{\mathbb{R}} |w|\phi(\frac{w}{\Sigma})\phi(\frac{w}{\sigma}\beta)dw &= 2\int_0^{+\infty} w\phi(\frac{w}{\Sigma})\phi(\frac{w}{\sigma}\beta)dw \\
&= \frac{2}{2\pi}\int_0^{+\infty} we^{-\frac{1}{2}w^2\frac{\Sigma^2\beta^2+\sigma^2}{\Sigma^2\sigma^2}}dw \\
&= \left[-\frac{1}{\pi}\frac{\Sigma^2\sigma^2}{\Sigma^2\beta^2+\sigma^2}e^{-\frac{1}{2}w^2\frac{\Sigma^2\beta^2+\sigma^2}{\Sigma^2\sigma^2}}\right]_0^{+\infty} = \frac{1}{\pi}\frac{\Sigma^2\sigma^2}{\Sigma^2\beta^2+\sigma^2}
\end{aligned}$$

The computation of K is not straightforward :

$$\begin{aligned}
K &= \int_0^{+\infty} w^2\Phi(\frac{w}{\sigma}\beta)\phi(\frac{w}{\Sigma})dw + \int_{-\infty}^0 w^2\Phi(-\frac{w}{\sigma}\beta)\phi(\frac{w}{\Sigma})dw \\
&= 2\int_0^{+\infty} w^2\Phi(\frac{w}{\sigma}\beta)\phi(\frac{w}{\Sigma})dw
\end{aligned}$$

Using an IPP, we get :

$$\begin{aligned}
K &= 2\left[-w\Sigma^2\Phi(\frac{w}{\sigma}\beta)\phi(\frac{w}{\Sigma})\right]_0^{+\infty} + 2\Sigma^2\int_0^{+\infty}(\Phi(\frac{w}{\sigma}\beta) + w\frac{\beta}{\sigma}\phi(\frac{w}{\sigma}\beta))\phi(\frac{w}{\Sigma})dw \\
&= 0 + 2\Sigma^2\int_0^{+\infty}\Phi(\frac{w}{\sigma}\beta)\phi(\frac{w}{\Sigma})dw + 2\Sigma^2\int_0^{+\infty} w\frac{\beta}{\sigma}\phi(\frac{w}{\sigma}\beta)\phi(\frac{w}{\Sigma})dw \\
&= 2\Sigma^2\int_0^{+\infty}\Phi(\frac{w}{\sigma}\beta)\phi(\frac{w}{\Sigma})dw + \frac{\Sigma^2\beta}{\sigma\pi}\frac{\Sigma^2\sigma^2}{\Sigma^2\beta^2+\sigma^2}
\end{aligned}$$

The last equality comes from the computation of J. Now, we need the following lemma to go further :

Lemma 18 *Let $(a,b) \in (\mathbb{R}_+^*)^2$. If ϕ and Φ are the density and the cumulative function of $X \sim \mathcal{N}(0,1)$, then*

$$\int_0^{+\infty}\phi(ax)\Phi(bx)dx = \frac{1}{4\pi a}(\pi + 2\arctan(\frac{b}{a}))$$

Proof: First, we remark that with a change of variable we have :

$$\int_0^{+\infty} \phi(ax)\Phi(bx)dx = \frac{1}{b}\int_0^{+\infty} \phi(\frac{a}{b}x)\Phi(x)dx$$

Let $\alpha = \frac{b}{a}$ and f the following function $f(\alpha) = \int_0^{+\infty} \phi(\frac{x}{\alpha})\Phi(x)dx$. It is straightforward to show that the function f is continuous and differentiable on $\mathbb{R}_+^*$, using the dominated convergence theorem. Thus, we can compute f' :

$$f'(\alpha) = \int_0^{+\infty} (-\frac{x^2}{2})(-\frac{2}{\alpha^3})\phi(\frac{x}{\alpha})\Phi(x)dx$$

Hence, we get $\alpha^3 f'(\alpha) = \int_0^{+\infty} x^2\phi(\frac{x}{\alpha})\Phi(x)dx$. Using an IPP with $u = x\Phi(x)$ and $v' = x\phi(\frac{x}{\alpha})$, we have :

$$\begin{aligned}
\alpha^3 f'(\alpha) &= \left[-\alpha^2 x\Phi(x)\phi(\frac{x}{\alpha})\right]_0^{+\infty} + \alpha^2\int_0^{+\infty} \phi(\frac{x}{\alpha})(\Phi(x) + x\phi(x))dx \\
&= 0 + \alpha^2 f(\alpha) + \frac{\alpha^2}{2\pi}\int_0^{+\infty} xe^{-\frac{\alpha^2+1}{2\alpha^2}x^2}dx \\
&= \alpha^2 f(\alpha) + \frac{\alpha^2}{2\pi}\left[-\frac{\alpha^2}{\alpha^2+1}e^{-\frac{\alpha^2+1}{2\alpha^2}x^2}\right]_0^{+\infty}
\end{aligned}$$

Hence, f satisfies the following differential equation : $\alpha f'(\alpha) - f(\alpha) = \frac{1}{2\pi}\frac{\alpha^2}{\alpha^2+1}$. We are going to solve this equation. First, a solution to the homogeneous equation is $f_h(\alpha) = C\alpha$. Assuming that $f(\alpha) = C(\alpha)\alpha$, we get $\alpha^2 C'(\alpha) = \frac{1}{2\pi}\frac{\alpha^2}{\alpha^2+1}$. Hence, $C(\alpha) = C + \frac{1}{2\pi}\arctan(\alpha)$. Finally, we have to determine the constant C and with the previous formula, we have $f(1) = C + \frac{1}{8}$. We can also compute $f(1)$ with the general form of f :

$$f(1) = \int_0^{+\infty} \phi(x)\Phi(x)dx = \left[\frac{1}{2}\Phi^2(x)\right]_0^{+\infty} = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

Finally, we get $C = \frac{1}{4}$, and $f(\alpha) = \alpha(\frac{1}{4} + \frac{1}{2\pi}\arctan(\alpha))$. Replacing α by $\frac{b}{a}$, we get the result of the lemma. □
The computation of K is instantaneous with this lemma. Finally, it remains to compute L.

$$\begin{aligned}
L &= 2\int_0^{+\infty} w\Phi(-\frac{w}{\sigma}\beta)\phi(\frac{w}{\Sigma})dw \\
&= 2\left[-\Sigma^2\Phi(-\frac{w}{\sigma}\beta)\phi(\frac{w}{\Sigma})\right]_0^{+\infty} - \frac{2\beta\Sigma^2}{\sigma}\int_0^{+\infty} \phi(\frac{w}{\sigma}\beta)\phi(\frac{w}{\Sigma})dw \\
&= \frac{1}{\sqrt{2\pi}}\Sigma^2 - \frac{2\beta\Sigma^2}{\sqrt{2\pi}\sigma}\frac{\sigma\Sigma}{2\sqrt{\sigma^2+\beta^2\Sigma^2}} = \frac{1}{\sqrt{2\pi}}\Sigma^2(1 - \frac{\beta\Sigma}{\sqrt{\sigma^2+\beta^2\Sigma^2}})
\end{aligned}$$

This ends the proof of this technical lemma.□

Proof of Proposition 16
First, let us remark that $f_d(\frac{\beta}{d}) = df_1(\beta)$. Hence, we are going to study $f = \frac{1}{d}f_1$. We see that we can write $f(x) = \frac{x}{1+x^2}g(x)$ where

$$g(x) = \pi - x - x^2 + x\sqrt{1+x^2} + x^2\arctan(\frac{1}{x})$$

Hence, we have

$$f'(x) = \frac{1}{(1+x^2)^2}((1-x^2)g(x) + (1+x^2)g'(x)) \tag{13}$$

We are going to study two cases : first for $x \in [0,1]$, and than for $x \in [1,+\infty]$
We have $g(x) = \pi + x(x\arctan(\frac{1}{x}) - 1) + x^2(\sqrt{1+\frac{1}{x^2}} - 1)$. It is obvious that the third term is positive, and then for $x \in [0,1]$ we have $0 \leq x\arctan(\frac{1}{x}) \leq \frac{\pi}{4}$, hence we have $-1 \leq x(x\arctan(\frac{1}{x}) - 1) \leq \frac{\pi}{4} - 1$, and then, adding π to those inequalities we get $\pi + x(x\arctan(\frac{1}{x}) - 1) \geq 0$. We can conclude that $\forall x \in [0,1] \quad g(x) \geq 0$. Now, we want to know the sign of g' on $[0,1]$.

$$g'(x) = (\sqrt{1+x^2} - 1)\frac{1+2x^2}{1+x^2} + 2x(\arctan(\frac{1}{x}) - 1) = m(x) + l(x)$$

Hence, we have $m'(x) = \frac{x}{(1+x^2)^2}(3\sqrt{1+x^2} + 2x^2\sqrt{1+x^2} - 2)$, and thus we can compute the second derivative : $m''(x) = \frac{1}{(1+x^2)^3}(3\sqrt{1+x^2} - 2 + 6x^2)$. Comparing the two first terms in the parentheses, we can say that $m'' \geq 0$ hence m is convex, thus m is above its tangents. On the other side a quicker computation leads to $l''(x) = -\frac{2x^2}{(1+x^2)^2} \leq 0$. Hence, l is concave, and then is above its cords.
We can remark that $\forall x \in [0, \cot(1)] \quad g'(x) \geq 0$ as a sum of two positive terms. Now for $x \in [\cot(1), 1]$, we have the two following inequalities

$$m(x) \geq m'(\cot(1))(x - \cot(1)) + m(\cot(1)) \text{ and } l(x) \geq \frac{l(1) - l(\cot(1))}{1 - \cot(1)}(x - \cot(1))$$

Thus we have an inequality of the form $g'(x) \geq ax + b$ where $a\cot(1) + b = m(\cot(1)) > 0$ and $a * 1 + b \simeq 0.107 > 0$. Then we can say that

$$\forall x \in [\cot(1), 1] \quad g'(x) > 0$$

Finally, we can conclude that $\forall x \in]0,1] \quad g'(x) > 0$, hence looking at (13), we directly have :

$$\forall x \in [0,1] \quad f'(x) > 0 \tag{14}$$

Now, we study f on $[1, +\infty[$. From properties (1), (2), (3) and what we have just done, we know that f has at least one maximum on $[1, +\infty[$. Let us study the number of solutions of the equation :

$$f(x) = m \quad \text{where} \quad m \in [0, +\infty] \tag{15}$$

It turns out that we have to solve : $g(x) = m(x + \frac{1}{x}) = h_m(x)$. First, we see that $h''_m(x) = \frac{2m}{x^3} > 0$. Thus, h_m is a convex function.
Let us show that g is concave on $[1, +\infty[$. As we have already computed g', we have :

$$\begin{aligned} g''(x) &= \frac{x}{(1+x^2)^2}(3\sqrt{1+x^2} - 4 + 2x^2\sqrt{1+x^2}) + 2(\arctan(\frac{1}{x}) - 1) \\ g'''(x) &= \frac{1}{(1+x^2)^{\frac{7}{2}}}(3 + x^2(3 + 2\sqrt{1+x^2}) - 6\sqrt{1+x^2}) \end{aligned}$$

Using the conjugate quantity and showing that the polynomial part of the result is increasing, we show that the parenthesis is increasing and positive on $x = 1$, hence $g''' > 0$ on $[1, +\infty[$ and thus g'' is increasing on $[1, +\infty[$. Now, we remark that $\lim_{x \to +\infty} g''(x) = 2 - 2 = 0$, hence $\forall x \in [1, +\infty[\quad g''(x) < 0$. Finally, g is concave on $[1, +\infty[$.
We need to show that if on the interval I, we have to solve the equation $a(x) = b(x)$ with a strictly concave and $\mathcal{C}^2$, b strictly convex and $\mathcal{C}^2$, then this equation has at most two solutions.
Let us consider the function $c = a - b$. This is quite clear that c is strictly concave since $c'' = a'' - b'' < 0$. We need to show that a strictly concave function reaches any value at most two times. If there exists x_0 and x_1 in I such that $c(x_0) = c(x_1) = m$ and $x_0 < x_1$, then $\forall x \in]x_0, x_1[\ c(x) > m$ (this directly comes from the concavity). After that, we use the fact that a concave function is strictly below its tangents and that $c'(x_0) \neq 0$ and $c'(x_1) \neq 0$ (they cannot be zero if $x_0 < x_1$) to show that $c(x) < m$ on $I \setminus [x_0, x_1]$. Finally, we have shown that c reaches at most twice any $m \in \mathbb{R}$. Thus, this is the case for 0, and then we have our result.
To conclude, we need to show the following result : If a continuous function h admits two local maximums, then there exists a real number m such that the equation $h(x) = m$ has at least three solutions. We are not going to show this result which is a application of the intermediate values theorem. However, we can finally conclude that f has a unique maximum on $[0, +\infty[$ which is attained for a ζ in $[1, +\infty[$. Using Remark 13, it turns out that for negative β, the quantity we want to maximize is negative.
Hence we have $\frac{\sigma\zeta}{\Sigma} = \arg\max(E[F(\beta(V - p_0))(V)], \beta \in \mathbb{R})$ $\square$

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