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of Long Memory Bernoulli Shifts**

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EMPIRICAL CLT FOR A CLASS OF LONG MEMORY BERNOULLI SHIFTS

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Abstract. We prove a functional central limit theorem for the empirical process of a stationary process $X_t = Y_t + V_t$, where Y_t is a long memory moving average in i.i.d. r.v.'s $\{\zeta_s, s \leq t\}$, and $V_t = V(\zeta_t, \zeta_{t-1}, \dots)$ is a weakly dependent nonlinear Bernoulli shift. Conditions of weak dependence of V_t are written in terms of L^2 -norms of shift-cut differences $V(\zeta_t, \dots, \zeta_{t-n}, 0, \dots) - V(\zeta_t, \dots, \zeta_{t-n+1}, 0, \dots)$. Examples of Bernoulli shifts are discussed. The limit empirical process is a degenerated process of the form $f(x)Z$, where f is the marginal p.d.f. of X_0 and Z is a standard normal r.v. The proof is based on a uniform reduction principle for the empirical process.

Théorème de limite centrale fonctionnelle pour le processus empirique d'une classe de schémas de Bernoulli à longue mémoire

Résumé. Nous prouvons un théorème de limite centrale fonctionnelle pour le processus empirique d'un processus stationnaire de la forme $X_t = Y_t + V_t$, pour une moyenne mobile à longue mémoire Y_t d'une suite i.i.d. $\{\zeta_s, s \leq t\}$, et pour un schéma de Bernoulli $V_t = V(\zeta_t, \zeta_{t-1}, \dots)$ faiblement dépendant. Des conditions dépendance faible de V_t sont écrites en termes de la norme L^2 des accroissements $\Delta_{n,t}$ du schéma de Bernoulli à bruit coupé:

$$\Delta_{n,t} = V(\zeta_t, \dots, \zeta_{t-n}, 0, \dots) - V(\zeta_t, \dots, \zeta_{t-n+1}, 0, \dots).$$

Des exemples de tels schémas de Bernoulli sont aussi discutés. Le processus empirique limite est dégénéré sous la forme $f(x)Z$, où f désigne la densité marginale de X_0 et Z est une variable normale standard. La preuve est fondée sur un principe de réduction uniforme du processus empirique.

Key words. Empirical Central Limit theorem, Long range dependence, Stationary processes, non-linear model.

AMS subject classifications. 60F17, 60F99, 62M99, 60G10, 60G99

FUNCTIONAL LIMIT THEOREM FOR THE EMPIRICAL PROCESS OF A CLASS OF BERNOULLI SHIFTS WITH LONG MEMORY

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1 Introduction

Time series analysis has important statistical applications in various fields. For example, nonlinear times series are used to model crashes in financial markets.

The main object of times series analysis is the study of short-range dependent random sequences for which the usual Donsker and the Empirical Functional Limit Theorems (EFLT) hold with appropriate modifications. Rosenblatt (1961), in his seminal work, and afterwards, Taqqu (1975), Dobrushin and Major (1979) and other authors found that alternative limit behaviors may happen, in particular, non- \sqrt{N} rates and non-central limits were exhibited. Most of these authors studied Gaussian subordinated case, or partial sums of nonlinear functions of a stationary Gaussian process with long-range dependence. The EFLT (with normalization $\ll N^{1/2}$) for such Gaussian subordinated processes was proved in Dehling and Taqqu (1989). An important feature of their EFLT is the fact that the limiting empirical process is degenerated, i.e. it has the form $g(x)Z$, with some deterministic function $g(x)$ and a random variable Z . Similar results for linear processes with long-range dependence were obtained in Giraitis, Koul and Surgailis (1996), Ho and Hsing (1996), Giraitis and Surgailis (1999) and other papers.

It is clear that Gaussian subordination or linearity are very restrictive structural assumptions which might be hard to justify in practice. In this paper we discuss the EFLT for a class of long-range dependent processes which are neither linear nor Gaussian subordinated. These are strictly stationary processes with discrete time $t \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ which can be represented as the sum

$$X_t = Y_t + V_t \tag{1.1}$$

of a *linear long memory* process Y_t and (*nonlinear*) *short memory* process V_t of a rather general form. More precisely, we assume that

$$Y_t := \sum_{i=0}^{\infty} b_i \zeta_{t-i} \quad (1.2)$$

is a moving average process in i.i.d. random variables $\zeta_i, i \in \mathbb{Z}$ with zero mean and unit variance, with hyperbolically decaying coefficients

$$b_i \sim c_0 i^{d-1} \quad (\exists 0 < d < 1/2, c_0 \neq 0). \quad (1.3)$$

The short memory process V_t in (1.1) is the so-called *Bernoulli shift*:

$$V_t := V(\zeta_t, \zeta_{t-1}, \dots), \quad (1.4)$$

where $V(z_0, z_1, \dots)$ is a Borel function on $\mathbb{R}^{\mathbb{Z}_+}, \mathbb{Z}_+ = \{0, 1, \dots\}$. The short memory property of V_t roughly means that the dependence of the function $V(z_0, z_1, \dots)$ on coordinates z_n is negligible with $n \rightarrow \infty$. Rigorous definition of the short memory property of V_t involves L^2 -norms of the shift-cut differences $V(\zeta_0, \dots, \zeta_{-n}, 0, \dots) - V(\zeta_0, \dots, \zeta_{-n+1}, 0, \dots)$, see Section 2, which must decrease sufficiently fast (e.g. subexponentially or hyperbolically) with n .

Section 2 contains the main result of the paper (the EFLT for the empirical process of X_t (1.1)-(1.4)). Concrete examples of Bernoulli shifts V_t (1.4) are presented in Section 3. The remaining Sections 4-7 are given to the proof of the EFLT. It uses martingale techniques introduced in Ho and Hsing (1996) which were later applied by several authors.

2 Main result

Let X_1, \dots, X_N be the observed sample from the stationary process X_t of (1.1). The empirical c.d.f. (empirical process)

$$\hat{F}_N(x) := N^{-1} \sum_{t=1}^N I(X_t \leq x), \quad x \in \mathbb{R}, \quad (2.1)$$

is a consistent estimator of the marginal c.d.f. $F(x) = P[X_0 \leq x]$. In fact, from ergodicity of the Bernoulli shift and the Glivenko-Cantelli theorem it follows that $\hat{F}_N(x) \rightarrow F(x)$ ($N \rightarrow \infty$) uniformly in $x \in \mathbb{R}$ a.s. Write $f(x) := F'(x)$ for the marginal p.d.f. of X_t , provided it exists. In addition to (1.3), we shall assume the following conditions on the innovations:

$$|Ee^{iu\zeta_0}| \leq C(1 + u^2)^{-\delta} \quad (\exists C, 0 < \delta \leq 1/4, \forall u \in \mathbb{R}) \quad (2.2)$$

and

$$E|\zeta_0|^3 < \infty. \quad (2.3)$$

We do not consider the 'best' (*i.e.* the larger) δ available here. If the inequality in (2.2) is satisfied for $\delta > 1/4$ it is also satisfied for $\delta = 1/4$. Note that condition (2.2) is very general because it does not imply that the density of ζ exists. But it excludes discrete distributions. If the inequality in (2.2) is satisfied with $\delta > 1/4$, then the density exists in L^2 , and if $\delta > 1/2$ the density is bounded.

Put $\bar{Y}_N := N^{-1} \sum_{t=1}^N Y_t$, and write $\Longrightarrow_{D(\bar{\mathbb{R}})}$ for weak convergence of random processes in the Skorohod space $D(\bar{\mathbb{R}})$, $\bar{\mathbb{R}} := [-\infty, \infty]$ with the sup-norm topology. Let

$$\gamma_n := E^{1/2}(V_0^n - V_0^{n-1})^2, \quad (2.4)$$

where

$$V_t^n := V(\zeta_t, \dots, \zeta_{t-n}, 0, 0, \dots) \quad (2.5)$$

is the truncated Bernoulli shift (1.4) which is $(n+1)$ -dependent stationary process.

Theorem 1 *Assume conditions (1.2), (1.3), (2.2), (2.3). Moreover, let*

$$\gamma_n \leq Cn^{-\rho}, \quad (2.6)$$

where

$$\rho > \max \left\{ 24 - 22d, 13 - 11d + \frac{3(1-2d)}{4d} \right\}. \quad (2.7)$$

Then

$$\sup_{x \in \mathbb{R}} N^{1/2-d} \left| \hat{F}_N(x) - F(x) + f(x)\bar{Y}_N \right| = o_P(1). \quad (2.8)$$

Theorem 1 is the uniform reduction principle for the empirical process (2.1) which extends the reduction principle of Dehling and Taqqu (1989) to Bernoulli shifts of the form (1.1)-(1.4). From this theorem and well-known facts about the limit distribution of \bar{Y}_N , it easily follows the EFLT below. Put $\tilde{c} := (c_0^2 B(d, 2-2d)/d(1+2d))^{1/2}$, where $B(\cdot, \cdot)$ is the beta function.

Corollary 2 *Under conditions of Theorem 1,*

$$N^{1/2-d}(\hat{F}_N(x) - F(x)) \Longrightarrow_{D(\bar{\mathbb{R}})} \tilde{c}f(x)Z, \quad (2.9)$$

where $Z \sim N(0, 1)$.

The fact that the functional dependence of the limiting empirical process in (2.9) reduces to marginal p.d.f. of observable time series X_t and does not involve probability densities or any other characteristics of unobservable components Y_t and V_t , is rather surprising. It appears that many results in statistical inference of long memory processes which rely on the empirical process can be extended from linear or Gaussian models to the much more general class (1.1), and that "short memory perturbation" V_t has no effect on large-sample behavior of inferential procedures. It also seems that Theorem 1 can be extended to the asymptotic expansion of the empirical process similar to that given by Ho and Hsing (1996) in the case of linear process. This problem is closely related to the study of partial sums' processes of nonlinear functions of X_t (1.1) and the characterization of the limiting behavior. In the case of Gaussian innovations ζ_t and finitely dependent V_t this problem was studied in Surgailis (2000). Further possibilities involve nonadditive generalizations of (1.1) of the form $X_t = H(Y_t; \zeta_t, \zeta_{t-1}, \dots)$ and will be studied in another paper.

3 Examples of weakly dependent Bernoulli shifts

3.1 Causal Bernoulli shifts

Causal Bernoulli shift is a stationary sequence V_t of the form (1.4), where the function $V(z_0, z_1, \dots)$ is tacitly assumed known. For m -dependent Bernoulli shifts (i.e. such that $V(z_0, z_1, \dots) = V(z_0, z_1, \dots, z_{m-1})$ depends on coordinates z_0, \dots, z_{m-1} only), condition (2.6) is satisfied provided γ_n 's are finite. The most simple case of infinitely dependent causal Bernoulli shift is a moving average process

$$V_t = \sum_{i=0}^{\infty} a_i \zeta_{t-i}, \quad (3.1)$$

where $\sum_{i=0}^{\infty} a_i^2 < \infty$. In this case, $\gamma_n = |a_n|$ and condition (2.6) implies $a_n = O(n^{-\rho})$ with $\rho > 1/3$, in other words, a_n must decay rather fast. Of course, the linear case (3.1) is not very interesting as it reduces to $V_t \equiv 0$, with $Y_t = \sum_{i=0}^{\infty} (a_t + b_t) \zeta_{t-i}$.

3.1.1 Volterra processes

A Volterra process is a stationary process defined through a convergent Volterra expansion $V_t = \sum_{k=1}^{\infty} V_{k;t}$, where

$$V_{k;t} := \sum_{0 \leq i_1 < \dots < i_k} a_{k;i_1, \dots, i_k} \zeta_{t-i_1} \dots \zeta_{t-i_k}$$

converges in L^2 provided the weights are square summable: $\sum_{0 \leq i_1 < \dots < i_k} a_{k;i_1, \dots, i_k}^2 < \infty$. For a k -th order homogeneous chaotic process $V_t = V_{k;t}$, the coefficient $\gamma_n = \gamma_{k;n}$ (2.4) equals

$$\gamma_{k;n} = \left\{ \sum_{0 \leq i_1 < \dots < i_{k-1} < n} a_{i_1, \dots, i_{k-1}, n}^2 \right\}^{1/2}$$

if $n \geq k$, and $\gamma_{k;n} = 0$ otherwise. For general infinite order Volterra series $V_t = \sum_{k=1}^{\infty} V_{k;t}$, the corresponding coefficients (2.4) are trivially related by orthogonality:

$$\gamma_n = \left\{ \sum_{k=1}^{\infty} \gamma_{k;n}^2 \right\}^{1/2}.$$

3.1.2 ARCH(∞) processes

A particular case of (non-Markovian) Bernoulli shifts is the ARCH(∞) process (see Robinson (1991), Giraitis, Kokoszka and Leipus (2000), Giraitis and Surgailis (2002)). It is subject to the recursion equation

$$V_t = \left(a_0 + \sum_{j=1}^{\infty} a_j V_{t-j} \right) \zeta_t^2, \quad (3.2)$$

where $\zeta_t, t \in \mathbb{Z}$ are zero mean i.i.d. r.v.'s, as above, and $a_j \geq 0, j = 0, 1, \dots$ are nonnegative coefficients. Put $\mu_i := E\zeta_0^{2i}, i \geq 1$. Under the simple condition

$$\mu_2^{1/2} \sum_{i=1}^{\infty} a_i < 1, \quad (3.3)$$

equation (3.2) is known to have a unique stationary solution with finite variance and given by a convergent (although nonorthogonal) Volterra expansion

$$V_t = a_0 \zeta_t^2 \left(1 + \sum_{\ell=1}^{\infty} \sum_{s_1 < \dots < s_{\ell} < t} a_{t-s_1} \dots a_{s_{\ell-1}-s_{\ell}} \zeta_{s_1}^2 \dots \zeta_{s_{\ell}}^2 \right). \quad (3.4)$$

Below we assume (3.3) satisfied. Put

$$g_n := \sum_{\ell=1}^n \sum_{j_1 + \dots + j_{\ell} = n} \alpha_{j_1} \dots \alpha_{j_{\ell}}, \quad (3.5)$$

where $\alpha_n := \mu_2^{1/2} a_n$ and where the second sum is taken over all integers $j_1, \dots, j_\ell \geq 1$ such that $j_1 + \dots + j_\ell = n$.

Proposition 1 For the ARCH(∞) process of (3.4),

$$\gamma_n = E^{1/2}(V_0^n - V_0^{n-1})^2 \leq |a_0| \mu_2 g_n. \quad (3.6)$$

Proof. We shall assume $a_0 = 1$ for simplicity. From (3.4) we obtain

$$V_0^n = \zeta_0^2 \left(1 + \sum_{\ell=1}^n \sum_{-n \leq s_\ell < \dots < s_1 < 0} a_{-s_1} \dots a_{s_{\ell-1}-s_\ell} \zeta_{s_1}^2 \dots \zeta_{s_\ell}^2 \right)$$

and therefore

$$V_0^n - V_0^{n-1} = \zeta_0^2 \zeta_{-n}^2 \sum_{\ell=1}^n \sum_{-n < s_{\ell-1} < \dots < s_1 < 0} a_{-s_1} \dots a_{s_{\ell-1}+n} \zeta_{s_1}^2 \dots \zeta_{s_{\ell-1}}^2.$$

Therefore by Minkowski inequality,

$$\begin{aligned} \gamma_n &= \mu_2 E^{1/2} \left\{ \sum_{\ell=1}^n \sum_{-n < s_{\ell-1} < \dots < s_1 < 0} a_{-s_1} \dots a_{s_{\ell-1}+n} \zeta_{s_1}^2 \dots \zeta_{s_{\ell-1}}^2 \right\}^2 \\ &\leq \mu_2 \sum_{\ell=1}^n \sum_{-n < s_{\ell-1} < \dots < s_1 < 0} a_{-s_1} \dots a_{s_{\ell-1}+n} E^{1/2} \zeta_{s_1}^4 \dots E^{1/2} \zeta_{s_{\ell-1}}^4 \\ &= \mu_2 \sum_{\ell=1}^n \sum_{0 < s_1 < \dots < s_{\ell-1} < n} \alpha_{s_1} \dots \alpha_{n-s_{\ell-1}} = \mu_2 g_n. \quad \square \end{aligned}$$

Inequality (3.6) allows to compare decay rates of γ_n to those of a_n , or α_n . Note the following relation between the generating series $G(z) := \sum_{n=1}^{\infty} g_n z^n$, $A(z) := \sum_{n=1}^{\infty} \alpha_n z^n$:

$$G(z) = \frac{A(z)}{1 - A(z)}. \quad (3.7)$$

Proposition 2

(i) Let $a_n = O(p_0^n)$ for some $0 < p_0 < 1$. Then there exists $0 < p < 1$ such that $\gamma_n = O(p^n)$.

(ii) Let $a_n = O(n^{-\lambda})$ with some $\lambda > 1$. Then $\gamma_n = O(n^{-\lambda})$.

Proof. (i) Note $a_n = O(p_0^n)$ implies that $A(z)$ is analytic on $\{z \in \mathbb{C} : |z| < p_0^{-1}\}$. Moreover, $|A(z)| \leq \sum_{n=1}^{\infty} \alpha_n |z|^n \leq \sum_{n=1}^{\infty} \alpha_n < 1$ for $|z| \leq 1$ and therefore $1 - A(z) \neq 0$ for $|z| < c$ and some $c > 1$. Therefore $G(z)$ is analytic on a disc of the complex plane of radius > 1 , which implies $\gamma_n = O(g_n) = O(p^n)$ for some $p < 1$.

(ii) Follows again by (3.7) and the argument as in (i). □

3.2 Stable Markov chains

Any (homogeneous) Markov chain V_t may also be represented as solution of a recurrence equation

$$V_t = M(V_{t-1}, \zeta_t) \quad (3.8)$$

where $M(u, z)$ is a (measurable) kernel and $\{\zeta_t\}$ is an i.i.d. sequence, see e.g. Kallenberg (1997, Proposition 7.6). Below we show how iterative models (3.8) can be related to Bernoulli shifts (1.4) and condition (2.6) verified. We shall discuss a more general situation when V_t and ζ_t take values in Euclidean spaces \mathbb{R}^d and \mathbb{R}^D , respectively, $d, D \geq 1$.

Following Duflo (1990), we call (3.8) a Lipschitz Markov model if the kernel $M(u, z)$ satisfies

$$E\|M(u, \zeta_0) - M(v, \zeta_0)\|^2 \leq a\|u - v\|^2 \quad (3.9)$$

for all $u, v \in \mathbb{R}^d$ and some $a < 1$, where $\|\cdot\|$ is a norm on \mathbb{R}^d . We shall also assume that $\{\zeta_t\}$ are i.i.d. random vectors with values in \mathbb{R}^D and zero mean $E\zeta_0 = 0$ and that, for some $\theta \in \mathbb{R}^D$, the function $u \mapsto M(u, \theta)$ admits a fixed point which we denote u_0 ; there is no loss of generality to suppose that $\theta = 0$.

Let now $U_t^{(n)}, t \in \mathbb{Z}$ be a Markov chain defined by

$$U_t^{(n)} := \begin{cases} u_0, & \text{if } t \leq -n, \\ M(U_{t-1}^{(n)}, \zeta_t), & \text{if } t > -n. \end{cases}$$

The Lipschitz condition (3.9) implies $E\|U_0^{(n)} - U_0^{(n+1)}\|^2 \leq aE\|U_0^{(n-1)} - U_0^{(n)}\|^2$. An easy recursion implies

$$E\|U_0^{(n)} - U_0^{(n+1)}\|^2 \leq a^n E\|M(u_0, \zeta_0) - u_0\|^2.$$

Hence $U_0^{(n)} \rightarrow U_0$ ($n \rightarrow \infty$) in L^2 for some square integrable random variable U_0 . It is also clear that $U_0^{(n)}$ being measurable w.r.t. the σ -algebra generated by $\zeta_t, t \leq 0$, this is also the case for U_0 which thus can be represented as a measurable function $U_0 = V(\zeta_0, \zeta_{-1}, \dots)$. Then the sequence $V_t := V(\zeta_t, \zeta_{t-1}, \zeta_{t-2}, \dots)$ is a stationary solution of the recurrence equation (3.8).

Consider now the sequences $(U_t^{(0)})_t$ and $(U_t^{(1)})_t$, then $U_0^{(0)} = u_0$, $U_1^{(0)} = M(u_0, \zeta_1) = V(\zeta_1, 0, 0, \dots)$, $U_2^{(0)} = M(M(u_0, \zeta_1), \zeta_2) = V(\zeta_2, \zeta_1, 0, 0, \dots)$ and recursively, for any $t > 0$,

$$U_t^{(0)} = V(\zeta_t, \zeta_{t-1}, \dots, \zeta_1, 0, 0, 0, \dots).$$

Analogously,

$$U_t^{(1)} = V(\zeta_t, \zeta_{t-1}, \dots, \zeta_1, \zeta_0, 0, 0, \dots).$$

It follows that $\gamma_n = E^{1/2} \left\| U_n^{(0)} - U_n^{(1)} \right\|^2 \leq a^{1/2} \gamma_{n-1}$ and thus

$$\gamma_n \leq a^{n/2} \gamma_0 = a^{n/2} E^{1/2} \|M(u_0, \zeta_0) - u_0\|^2$$

decay exponentially, due to $a < 1$. Diaconis and Friedmann (1999) provide a wide variety of examples for which the previous techniques apply. See also Doukhan (1994, 2002).

3.2.1 ARCH-type processes

Assume for simplicity that $d = D = 1$, and let

$$M(u, z) := A(u) + B(u)z \tag{3.10}$$

for suitable Lipschitz functions $A(u), B(u), u \in \mathbb{R}$. Set $\text{Lip}(A) = \sup_{u \neq v} |A(u) - A(v)|/|u - v|$.

The corresponding iterative model (3.8) satisfies (3.9) if $E\zeta_t = 0$ and

$$a := (\text{Lip}(A))^2 + E\zeta_0^2 (\text{Lip}(B))^2 < 1.$$

Some examples of iterative Markov processes given by kernels (3.10) are:

- nonlinear AR(1) processes (case $B \equiv 1$);
- stochastic volatility models (case $A \equiv 0$);
- classical ARCH(1) models (case $A(u) = \alpha u, B(u) = \sqrt{\beta + \gamma u^2}, \alpha, \beta, \gamma \geq 0$).

In the last example, the Lipschitz constant a in (3.9) equals $a = \alpha^2 + E\zeta_0^2 \gamma$, which follows by calculation of the derivatives $A'(u), B'(u)$.

3.2.2 Branching type models

Here $d = 1, D \geq 2$. Set $\zeta_t = (\zeta_t^{(1)}, \dots, \zeta_t^{(D)})$ and assume that $E\zeta_t^{(i)}\zeta_t^{(j)} = 0$ if $i \neq j$. Let now $A_1(u), \dots, A_D(u), u \in \mathbb{R}$ be Lipschitz functions, and let

$$M\left(u, \left(z^{(1)}, \dots, z^{(D)}\right)\right) := \sum_{j=1}^D A_j(u)z^{(j)},$$

$(z^{(1)}, \dots, z^{(D)}) \in \mathbb{R}^D$. For such kernel M , relation (3.9) holds with the Euclidean norm $\|\cdot\|$ if

$$a := \sum_{j=1}^D (\text{Lip}(A_j))^2 E\left|\zeta_0^{(j)}\right|^2 < 1.$$

Some examples of this situation are

- If $D = 2$, and $\zeta_0^{(1)} \sim b(p)$ is a Bernoulli variable independent of a centered variable $\zeta_0^{(2)} \in L^2$ and $A_1(u) = u, A_2(u) = 1$ then the previous relations hold if $p < 1$.
- If $D = 3$, $\zeta_0^{(1)} = 1 - \zeta_0^{(2)} \sim b(p)$ is independent of a centered variable $\zeta_0^{(3)} \in L^2$ then one obtains usual threshold models if $A_3 \equiv 1$ and the conditions write $E \left| \zeta_0^{(3)} \right|^2 < \infty$ and $a = p (\text{Lip}(A_1))^2 + (1 - p) (\text{Lip}(A_2))^2 < 1$.

3.2.3 Nonlinear AR(p) models

Our last example is a nonlinear autoregressive process of order d :

$$V_t = r(V_{t-1}, \dots, V_{t-d}) + \zeta_t, \quad (3.11)$$

taking real values. Then the sequence $U_n := (V_n, V_{n-1}, \dots, V_{n-d+1})$ is a Markov process with values in \mathbb{R}^d . In this case, $D = 1$ and

$$M(u_1, \dots, u_d, z) = A(u_1, \dots, u_d) + (1, 0, \dots, 0)z,$$

where $A(u_1, \dots, u_d) := (r(u_1, \dots, u_d), u_1, \dots, u_{d-1})$. We assume $E\zeta_0^2 < \infty$ and

$$|r(u_1, \dots, u_d) - r(v_1, \dots, v_d)| \leq \sum_{i=1}^d a_i |u_i - v_i|$$

for some $a_1, \dots, a_d \geq 0$ such that

$$\alpha := \left(\sum_{i=1}^d a_i \right)^{1/d} < 1.$$

Introduce a norm on \mathbb{R}^d by

$$\|(u_1, \dots, u_d)\| := \max\{|u_1|, \alpha|u_2|, \dots, \alpha^{d-1}|u_d|\}.$$

Let $u = (u_1, \dots, u_d), (v_1, \dots, v_d) \in \mathbb{R}^d$ and let $w_j := |u_j - v_j|, j = 1, \dots, d$, then

$$\|A(u) - A(v)\| \leq \max \left\{ \alpha^d \max\{w_1, \dots, w_d\}, \alpha w_1, \dots, \alpha^{d-1} w_{d-1} \right\} \leq \alpha \|u - v\|.$$

This implies condition (3.9) with $a = \alpha^2 < 1$. Hence for nonlinear AR(p) process (3.11), we obtain $\gamma_n \leq C\alpha^n$. Therefore the γ_n 's decay geometrically under the conditions given above.

4 Plan of the proof of Theorem 1

We first note that this theorem is known in the linear case $X_t = Y_t$, for $V_t = 0$; see e.g. Ho and Hsing (1996) and Giraitis and Surgailis (1999). However, the proofs in our paper are essentially self-contained. Put $X_t^n := Y_t + V_t^n$,

$$\hat{F}_N^n(x) := N^{-1} \sum_{t=1}^N I(X_t^n \leq x), \quad F^n(x) := E\hat{F}_N^n(x) = P[X_0^n \leq x].$$

Let

$$n(N) := N^\lambda, \quad \lambda \in (0, 1),$$

where $\lambda \in (0, 1)$ will be specified below. The uniform reduction principle of (2.8) clearly follows from Lemmas 1 and 2 below.

Lemma 1 *If $\lambda < 2d$, then*

$$\sup_{x \in \mathbb{R}} N^{1/2-d} \left| \hat{F}_N^{n(N)}(x) - F^{n(N)}(x) + f(x)\bar{Y}_N \right| = o_P(1). \quad (4.1)$$

Lemma 2 *If $\lambda > (3/2)(1 - 2d)/(\rho + 11d - 13)$, then*

$$\sup_{x \in \mathbb{R}} N^{1/2-d} \left(\left| \hat{F}_N^{n(N)}(x) - \hat{F}_N(x) \right| + \left| F^{n(N)}(x) - F(x) \right| \right) = o_P(1). \quad (4.2)$$

Remark 1 Note condition (2.7) of Theorem 1 ensures that there exists λ satisfying conditions of Lemmas 1 and 2.

The proofs of the above lemmas require some bounds of marginal densities and their derivatives of the stationary processes Y_t and X_t and their approximations. These bounds are discussed in Section 5. Everywhere below we assume that conditions of Theorem 1 are satisfied. Note that the short memory process V_t need not have a density and its marginal distribution can be discrete. On the other hand, the long memory component Y_t is known to have a smooth density under the hypotheses of Theorem 1. As V_t and Y_t are dependent, the fact that the marginal density $f(x)$ of the sum $X_t = V_t + Y_t$ exists, is not trivial.

Notation. Put, for any $0 \leq n < m < \infty$,

$$Y_t^{n,m} := \sum_{i=n}^m b_i \zeta_{t-i}, \quad G^{n,m}(x) := P[Y_0^{n,m} \leq x],$$

$$Y_t^{n,\infty} := \sum_{i=n}^{\infty} b_i \zeta_{t-i}, \quad G^{n,\infty}(x) := P[Y_0^{n,\infty} \leq x].$$

Let $G(x) := P[Y_0 \leq x]$. Note $Y_t = Y_t^{0,n} + Y_t^{n+1,\infty}$, where $Y_t^{0,n}$ and $Y_t^{n+1,\infty}$ are independent for each t, n . Also, for $0 < n \leq m < \infty$, put

$$\begin{aligned} X_t^n &:= Y_t + V_t^n, & F^n(x) &:= P[X_0^n \leq x], \\ X_t^{n,m} &:= Y_t^{0,m} + V_t^n, & F^{n,m}(x) &:= P[X_0^{n,m} \leq x]. \end{aligned}$$

We also write $g(x) := G'(x)$, $g^{n,m}(x) := (G^{n,m}(x))'$, $g^{n,\infty}(x) := (G^{n,\infty}(x))'$, $f^n(x) := (F^n(x))'$, $f^{n,m}(x) := (F^{n,m}(x))'$ for the corresponding probability densities, provided they exist, and

$$\begin{aligned} \hat{g}(u) &:= Ee^{iuY_0}, & \hat{g}^{n,m}(u) &:= Ee^{iuY_0^{n,m}}, & \hat{g}^{n,\infty}(u) &:= Ee^{iuY_0^{n,\infty}}, \\ \hat{f}(u) &:= Ee^{iuX_0}, & \hat{f}^{n,m}(u) &:= Ee^{iuX_0^{n,m}}, & \hat{f}^n(u) &:= Ee^{iuX_0^n} \end{aligned}$$

for the characteristic functions. Also, put

$$\Phi(x) := P[\zeta_0 \leq x], \quad \hat{\phi}(u) := Ee^{iu\zeta_0}.$$

In the sequel, C stands for generic constant which may change from line to line.

5 Bounds of marginal densities

Put

$$B_{n,j} := \prod_{i=n}^{n+j} b_i^2.$$

Note that, by (1.3), for each $j = 0, 1, \dots$

$$B_{n,j} \sim c_0^{2(1+j)} n^{2(d-1)(1+j)}, \quad n \rightarrow \infty.$$

In particular, for any fixed $j \geq 0$ there exists a constant $C > 0$ such that for all sufficiently large n ,

$$B_{n,j}^{-1} \leq Cn^{2(1-d)(1+j)}. \quad (5.1)$$

Lemma 3 *For any $p = 0, 1, \dots$ there exist an integer $j_0 = j_0(p) \geq 1$ and a constant $C = C(p) < \infty$ such that for all $0 \leq n < m$, $m - n \geq j_0$ and any $k = 0, 1, 2$,*

$$|(u^p \hat{g}^{n,m}(u))^{(k)}| + |(u^p \hat{g}^{n,\infty}(u))^{(k)}| \leq Cn^\theta (1+u^2)^{-3}, \quad (5.2)$$

and

$$|(u^p(\hat{g}^{n,m}(u) - \hat{g}^{n,m-1}(u)))^{(k)}| \leq C b_m^2 n^\theta (1+u^2)^{-2}, \quad (5.3)$$

where $\theta := (1-d)(9+p)$.

Proof. Let us prove (5.2) for $k = 0$. Without loss of generality, assume $|b_i| \leq 1 \forall i$. By (2.2),

$$\begin{aligned} |\hat{g}^{n,m}(u)| &\leq \prod_{i=n}^{n+j_0} |\hat{\phi}(ub_i)| \leq C \left(\prod_{i=n}^{n+j_0} (1+u^2 b_i^2)^{-1} \right)^\delta \\ &\leq C \left(\prod_{i=n}^{n+j_0} b_i^{-2} (1+u^2)^{-1} \right)^\delta \leq C B_{n,j_0}^{-\delta} (1+u^2)^{-(j_0+1)\delta}. \end{aligned}$$

Hence

$$|u^p \hat{g}^{n,m}(u)| \leq C B_{n,j_0}^{-\delta} (1+u^2)^{p/2-(j_0+1)\delta}.$$

Taking $j_0 = [(6+p)/2\delta]$ leads to $p/2 - (j_0+1)\delta \leq -3$ and, by (5.1),

$$B_{n,j_0}^{-\delta} \leq C n^{(1-d)(6+p+2\delta)},$$

thereby proving (5.2) for $k = 0$.

In a similar way, let us prove (5.2) for $k = 1$ and $k = 2$. We have

$$\begin{aligned} (\hat{g}^{n,m}(u))' &= \sum_{j=n}^m i b_j \hat{\phi}'(ub_j) \prod_{i=n, i \neq j}^m \hat{\phi}(ub_i), \\ (\hat{g}^{n,m}(u))'' &= - \sum_{j=n}^m b_j^2 \hat{\phi}''(ub_j) \prod_{i=n, i \neq j}^m \hat{\phi}(ub_i) \\ &\quad - \sum_{j_1, j_2=n, j_1 \neq j_2}^m b_{j_1} b_{j_2} \hat{\phi}'(ub_{j_1}) \hat{\phi}'(ub_{j_2}) \prod_{i=n, i \neq j_1, j_2}^m \hat{\phi}(ub_i). \end{aligned}$$

Hence, using $|\hat{\phi}(u)| \leq 1$, $|\hat{\phi}'(u)| \leq |u|$, $|\hat{\phi}''(u)| \leq 1$, as well as (1.3) and (2.2) we obtain

$$|(\hat{g}^{n,m}(u))'| \leq \sum_{j=n}^{\infty} b_j^2 |u| \prod_{i=n, i \neq j}^{n+j_0} |\hat{\phi}(ub_i)| \leq C B_{n,j_0-1}^{-\delta} |u| (1+u^2)^{-j_0\delta}.$$

By taking $j_0 = [(7+p)/2\delta] + 1$ and noting that $B_{n,j_0-1}^{-\delta} \leq C n^{(1-d)(7+p+2\delta)}$, we obtain (5.2)

for $k = 1$. Similarly,

$$\begin{aligned} |(\hat{g}^{n,m}(u))''| &\leq \sum_{j=n}^{\infty} b_j^2 \prod_{i=n, i \neq j}^{n+j_0} |\hat{\phi}(ub_i)| + \sum_{j_1, j_2=n, j_1 \neq j_2}^{\infty} u^2 b_{j_1}^2 b_{j_2}^2 \prod_{i=n, i \neq j_1, j_2}^{n+j_0} |\hat{\phi}(ub_i)| \\ &\leq C \left(B_{n,j_0-1}^{-\delta} (1+u^2)^{-j_0\delta} + B_{n,j_0-2}^{-\delta} u^2 (1+u^2)^{-(j_0-1)\delta} \right) \\ &\leq C B_{n,j_0-1}^{-\delta} (1+u^2)^{1-(j_0-1)\delta}. \end{aligned}$$

Taking $j_0 = [(8+p)/2\delta] + 2$ and noting that $B_{n,j_0-1}^{-\delta} \leq Cn^{2(1-d)\delta j_0} \leq Cn^{(1-d)(8+p+4\delta)}$, we obtain (5.2) for $k = 2$.

It remains to prove (5.3). Let $k = 0$. Since

$$u^p (\hat{g}^{n,m}(u) - \hat{g}^{n,m-1}(u)) = (u^p \hat{g}^{n,m-1}(u)) (\hat{\phi}(b_m u) - 1), \quad (5.4)$$

inequality (5.3) follows from (5.2) and the bound $|\hat{\phi}(b_m u) - 1| \leq b_m^2 u^2$. Cases $k = 1, 2$ follow similarly. For instance, among the three terms appearing in the case $k = 2$, $(u^p \hat{g}^{n,m-1}(u))'' (\hat{\phi}(b_m u) - 1)$ leads to the value of θ given in the lemma. Indeed, using (5.2),

$$\left| (u^p \hat{g}^{n,m-1}(u))'' (\hat{\phi}(b_m u) - 1) \right| \leq u^2 b_m^2 |(u^p \hat{g}^{n,m-1}(u))''| \leq C b_m^2 n^\theta (1+u^2)^{-2}.$$

The lemma is proved. \square

Lemma 4 *For any $p = 0, 1, \dots$ there exist an integer $j_0 \geq 1$ and a constant $C = C_p < \infty$ such that for all $0 \leq n < m$, $m - n \geq j_0$, $x \in \mathbb{R}$,*

$$|(g^{n,m}(x))^{(p)}| + |(g^{n,\infty}(x))^{(p)}| \leq C n^\theta (1+x^2)^{-1} \quad (5.5)$$

and

$$|(g^{n,m}(x) - g^{n,m-1}(x))^{(p)}| \leq C b_m^2 n^\theta (1+x^2)^{-1}, \quad (5.6)$$

where θ is the same as in the previous lemma.

Proof. Relation (5.5) follows from (5.2) and from

$$(g^{n,m}(x))^{(p)} = C \int e^{-iux} u^p \hat{g}^{n,m}(u) du, \quad x^2 (g^{n,m}(x))^{(p)} = C \int e^{-iux} (u^p \hat{g}^{n,m}(u))'' du.$$

Similarly, (5.6) follows from (5.3). \square

Next we consider bounds for p.d.f. $f(x)$, $f^n(x)$, $f^{n,m}(x)$ of X_t , $X_t^n = V_t^n + Y_t$, $X_t^{n,m} = V_t^n + Y_t^{0,m}$, respectively. To that end, we need elementary Lemma 5 below, whose proof can be found in Doukhan, Lang and Surgailis (2002). Let

$$\varphi_r(x) := (1 + |x|)^{-r}, \quad \mu_r(x, y) := \int_x^y \varphi_r(z) dz, \quad x < y, \quad r > 1, \quad x, y \in \mathbb{R}.$$

Note μ_r is a finite measure on \mathbb{R} . For any integrable function $\psi = \psi(x)$, $x \in \mathbb{R}$, $\hat{\psi}(u) := \int_{\mathbb{R}} e^{iux} \psi(x) dx$, $u \in \mathbb{R}$ denotes its Fourier transform.

Lemma 5 Let $h(x), x \in \mathbb{R}$ be a real valued function such that

$$|h(x)| \leq C\varphi_r(x), \quad |h(x) - h(y)| \leq C|x - y|\varphi_r(x), \quad (5.7)$$

hold for any $x, y \in \mathbb{R}, |x - y| \leq 1$ and some $C < \infty, 1 < r \leq 2$. Then there exists a constant C_r depending only on r and C in (5.7), such that for any $x, y, v, z \in \mathbb{R}$

$$|h(x + y)| \leq C\varphi_r(x)(1 \vee |y|^r), \quad (5.8)$$

$$\left| \int_0^y h(x + w)dw \right| \leq C_r\varphi_r(x)(|y| \vee |y|^r), \quad (5.9)$$

$$\left| \int_x^y (h(\xi + v) - h(\xi))d\xi \right| \leq C_r\mu_r(x, y)(|v| \vee |v|^r), \quad (5.10)$$

$$\left| \int_0^v dw \int_x^y (h(\xi + w - z) - h(\xi - z))d\xi \right| \leq C_r\mu_r(x, y)|v|^r(1 \vee |z|^r). \quad (5.11)$$

Lemma 6 Let p_0 be a non negative integer and $\theta = (1 - d)(9 + p_0)$. If

$$\sum_{n=1}^{\infty} n^\theta \gamma_n^{1/2} < \infty, \quad (5.12)$$

then for any $p \in \{0, \dots, p_0\}$ and any $1 < r < 3/2$, there exist $j_0 \geq 1$ and a constant $C = C_{p,r} < \infty$ such that for all $0 \leq n < m, m - n \geq j_0, x \in \mathbb{R}$

$$|(f(x))^{(p)}| + |(f^n(x))^{(p)}| + |(f^{n,m}(x))^{(p)}| \leq C(1 + |x|)^{-r} \quad (5.13)$$

and, moreover,

$$|(f(x))^{(p)} - (f^n(x))^{(p)}| \leq C\alpha_n(1 + |x|)^{-r} \quad (5.14)$$

and

$$|(f^n(x))^{(p)} - (f^{n,m}(x))^{(p)}| \leq C\beta_m^{1/2}(1 + |x|)^{-r}, \quad (5.15)$$

where $\alpha_n := \sum_{j=n}^{\infty} j^\theta \gamma_j^{1/2}$, $\beta_n := \sum_{j=n}^{\infty} b_j^2$.

Proof. We shall prove (5.13) for $f^n(x)$ and $p = 0$ only, as the remaining inequalities can be proved analogously. Let

$$\psi^n(x) := f^n(x) - f^{n-1}(x),$$

$n \geq 0, f^{-1}(x) := g(x)$. Clearly (5.13) (for $f^n(x)$ and $p = 0$) follows from

$$(1 + |x|)^r |\psi^n(x)| \leq Cn_+^\theta \gamma_n^{1/2}, \quad (5.16)$$

where $n_+ := n \vee 1$, $\gamma_0 := 1$. To show (5.16), consider the Fourier transform

$$\begin{aligned}
|\hat{\psi}^n(u)| &= |Ee^{iuY_0^{n+1,\infty}} (Ee^{iu(Y_0^{0,n}+V_0^n)} - Ee^{iu(Y_0^{0,n}+V_0^{n-1})})| \\
&= |\hat{g}^{n+1,\infty}(u)| |Ee^{iu(Y_0^{0,n}+V_0^{n-1})} (e^{iu(V_0^n-V_0^{n-1})} - 1)| \\
&\leq |\hat{g}^{n+1,\infty}(u)| |E(e^{iu(V_0^n-V_0^{n-1})} - 1)| \\
&\leq |\hat{g}^{n+1,\infty}(u)| |u| E^{1/2}(V_0^n - V_0^{n-1})^2 \\
&\leq Cn^\theta \gamma_n (1+u^2)^{-2}, \quad n \geq 1,
\end{aligned}$$

where in the last line we used (5.2) with $p = 1, k = 0$. Also, $|\hat{\psi}^0(u)| = |\hat{f}^0(u) - \hat{f}^{-1}(u)| = |Ee^{iuY_0^{1,\infty}}| |Ee^{iu\zeta_0} (e^{iuV_0^0} - 1)| \leq 2|\hat{g}^{1,\infty}(u)| \leq C(1+u^2)^{-3}$ according to (5.2). This proves $|\psi^n(x)| \leq Cn_+^\theta \gamma_n, n \geq 0$. To show (5.16), it remains to prove that there exist constants $C, c > 0$ such that

$$|x|^r |\psi^n(x)| \leq Cn_+^\theta \gamma_n^{1/2}, \quad |x| \geq c. \quad (5.17)$$

The proof of (5.17) is more complicated and uses fractional differentiation (c.f. Doukhan, Lang and Surgailis (2002)). To that end, we will show that there exists a (complex-valued) function $q(x)$ satisfying

$$|q(x)| \geq c_1 |x|^{r-1}, \quad |x| \geq c, \quad (5.18)$$

where $c, c_1 > 0$ are some constants, and such that

$$|q(x)x\psi^n(x)| \leq Cn_+^\theta \gamma_n^{1/2}, \quad x \in \mathbb{R}, n \geq 0. \quad (5.19)$$

Similarly as in the above mentioned paper, take

$$\begin{aligned}
q(x) &:= \int_0^\infty (1 - e^{-ixz}) z^{-r} dz - \int_1^\infty (1 - e^{-ixz}) z^{-r} dz \\
&= |x|^{r-1} e^{i \operatorname{sgn}(x) \pi(r-1)/2} - \tilde{q}(x),
\end{aligned}$$

where $\tilde{q}(x) := \int_1^\infty (1 - e^{-ixz}) z^{-r} dz$ is a bounded function on the real line. Therefore $q(x)$ satisfies (5.18). To show (5.19), note that by Parseval's identity,

$$q(x)\phi(x) = C \int_{\mathbb{R}} e^{-iux} du \int_0^1 (\hat{\phi}(u) - \hat{\phi}(u - \xi)) \xi^{-r} d\xi,$$

for any smooth and integrable test function ϕ . Consequently,

$$q(x)(ix)\psi^n(x) = C \int_{\mathbb{R}} e^{-iux} du \int_0^1 ((\hat{\psi}^n)'(u) - (\hat{\psi}^n)'(u - \xi)) \xi^{-r} d\xi.$$

Here,

$$\hat{\psi}^n(u) = \hat{g}^{n+1,\infty}(u) E \left(e^{iuX_0^{n-1,n}} (e^{iu(V_0^n - V_0^{n-1})} - 1) \right) \equiv h(u)a(u),$$

where

$$h(u) := \hat{g}^{n+1,\infty}(u), \quad a(u) := E e^{iuX} (e^{iu\Delta V} - 1), \quad X := X_0^{n-1,n}, \quad \Delta V := V_0^n - V_0^{n-1}.$$

Next,

$$\begin{aligned} (\hat{\psi}^n)'(u) - (\hat{\psi}^n)'(u - \xi) &= h'(u)a(u) + h(u)a'(u) - h'(u - \xi)a(u - \xi) - h(u - \xi)a'(u - \xi) \\ &= (h'(u) - h'(u - \xi))a(u) + (h(u) - h(u - \xi))a'(u) \\ &\quad + h'(u - \xi)(a(u) - a(u - \xi)) + h(u - \xi)(a'(u) - a'(u - \xi)). \end{aligned}$$

From Lemma 3, (5.2) and Lemma 5, (5.8), it easily follows that

$$\begin{aligned} |h(u - \xi)| &\leq Cn^\theta(1 + u^2)^{-3}, \\ |h'(u - \xi)| &\leq Cn^\theta(1 + u^2)^{-3}, \\ |h(u) - h(u - \xi)| &\leq Cn^\theta|\xi|(1 + u^2)^{-3}, \\ |h'(u) - h'(u - \xi)| &\leq Cn^\theta|\xi|(1 + u^2)^{-3}, \end{aligned}$$

where the constant C does not depend on n , $u \in \mathbb{R}$, $\xi \in (0, 1)$. Next, consider $a(u)$, $a'(u)$, $a(u) - a(u - \xi)$, $a'(u) - a'(u - \xi)$. We have

$$|a(u)| = |E e^{iuX} (e^{iu\Delta V} - 1)| \leq |u|E|\Delta V| \leq |u|\gamma_n.$$

Similarly,

$$|a'(u)| \leq |E e^{iuX} X (e^{iu\Delta V} - 1)| + |E e^{iuX} e^{iu\Delta V} \Delta V| \leq |u|E|X\Delta V| + E|\Delta V| \leq C(1 + |u|)\gamma_n,$$

where we used Cauchy-Schwarz inequality and the fact that

$$EX^2 \equiv E \left(X_0^{n-1,n} \right)^2 \leq C.$$

Next,

$$\begin{aligned} |a(u) - a(u - \xi)| &= |E e^{iuX} \{ (e^{iu\Delta V} - 1)(1 - e^{-i\xi(X + \Delta V)}) + e^{-i\xi X} (1 - e^{-i\xi\Delta V}) \}| \\ &\leq |u\xi|E|\Delta V|(|X| + |\Delta V|) + |\xi|E|\Delta V| \leq C|\xi|(1 + |u|)\gamma_n, \end{aligned}$$

where we used $\gamma_n^2 \leq C\gamma_n$. Finally, consider the most delicate term

$$\begin{aligned}
|a'(u) - a'(u - \xi)| &= |Ee^{iuX}(iX)\{(e^{iu\Delta V} - 1)(1 - e^{-i\xi(X+\Delta V)}) + e^{-i\xi X}(1 - e^{-i\xi\Delta V})\} \\
&\quad - Ee^{iu(X+\Delta V)}(i\Delta V)(e^{-i\xi(X+\Delta V)} - 1)| \\
&\leq E|X|(2 \wedge |u\Delta V|)(2 \wedge |\xi(X + \Delta V)|) + |\xi|E|X||\Delta V| \\
&\quad + |\xi|E|\Delta V|(|X| + |\Delta V|).
\end{aligned}$$

To evaluate the first expectation on the r.h.s., use the inequality $2 \wedge x \leq (2x)^{1/2}$, then by Cauchy-Schwartz inequality,

$$\begin{aligned}
E|X|(2 \wedge |u\Delta V|)(2 \wedge |\xi X|) &\leq 2|u\xi|^{1/2}E(|X|^{3/2}|\Delta V|^{1/2}) \\
&\leq 2|u\xi|^{1/2}(E|X|^2)^{3/4}(E|\Delta V|^2)^{1/4} \\
&\leq C|u\xi|^{1/2}\gamma_n^{1/2}.
\end{aligned}$$

Similarly, $E|X|(2 \wedge |u\Delta V|)(2 \wedge |\xi\Delta V|) \leq 2|u\xi|^{1/2}E|X||\Delta V| \leq C|u\xi|^{1/2}\gamma_n$ and we obtain

$$|a'(u) - a'(u - \xi)| \leq C|\xi|^{1/2}(1 + |u|)\gamma_n^{1/2}.$$

Combining the above bounds(recall that $|\xi| < 1$) results in

$$|(\hat{\psi}^n)'(u) - (\hat{\psi}^n)'(u - \xi)| \leq C|\xi|^{1/2}n^\theta\gamma_n^{1/2}(1 + u^2)^{-2},$$

yielding

$$|q(x)(ix)\psi^n(x)| \leq Cn^\theta\gamma_n^{1/2} \int_{\mathbb{R}} (1 + u^2)^{-2} du \int_0^1 \xi^{1/2-r} d\xi \leq Cn^\theta\gamma_n^{1/2},$$

for any $1 < r < 3/2$. This proves (5.17) and (5.16), hence also (5.13) for $f^n(x)$ and $p = 0$, with any $1 < r < 3/2$. Clearly, (5.16) implies (5.14) as well.

It remains to prove (5.15), where we again restrict ourselves to the case $p = 0$, as the case $p \geq 1$ is analogous. We have

$$f^n(x) - f^{n,m}(x) = \int_{\mathbb{R}} (f^{n,m}(x - y) - f^{n,m}(x))g^{m+1,\infty}(y)dy.$$

By (5.13),

$$\begin{aligned}
&\int_{|y|\leq 1} |f^{n,m}(x - y) - f^{n,m}(x)|g^{m+1,\infty}(y)dy \leq C(1 + |x|)^{-r} \int_{|y|\leq 1} |y|g^{m+1,\infty}(y)dy \\
&= C(1 + |x|)^{-r} E|Y_0^{m+1,\infty}| \leq C(1 + |x|)^{-r} E^{1/2}|Y_0^{m+1,\infty}|^2 = C(1 + |x|)^{-r}\beta_{m+1}^{1/2}.
\end{aligned}$$

Similarly, by (5.13) and Lemma 5 (5.8),

$$\begin{aligned} \int_{|y|>1} f^{n,m}(x-y)g^{m+1,\infty}(y)dy &\leq C(1+|x|)^{-r} \int_{|y|>1} |y|^r g^{m+1,\infty}(y)dy \\ &\leq C(1+|x|)^{-r} E|Y_0^{m+1,\infty}|^r \leq C(1+|x|)^{-r} (E|Y_0^{m+1,\infty}|^2)^{r/2} \\ &= C(1+|x|)^{-r} \beta_{m+1}^{r/2}. \end{aligned}$$

A similar estimate holds for the integral $\int_{|y|>1} f^{n,m}(x)g^{m+1,\infty}(y)dy$. As $\beta_{m+1}^{r/2} \leq C\beta_{m+1}^{1/2}$ for $r > 1$, this proves the bound (5.15) for $p = 0$. \square

Remark 2 In the sequel, we use Lemma 6 with $p_0 = 2$ and $\theta = 11(1-d)$ only.

6 Proof of Lemma 1

Lemma 1 follows from Lemma 7 below combined with a standard chaining argument as in Dehling and Taqqu (1989). For any function $\psi(x)$, and any $x < y$, put $\psi(x, y) := \psi(y) - \psi(x)$.

Lemma 7 Assume condition (2.6), where

$$\rho > 24 - 22d. \quad (6.1)$$

Let $n(N) := N^\lambda$, $0 < \lambda < 2d$. Define

$$\kappa := \min\{2d - \lambda, 1 - 2d, \lambda(\rho + 22d - 24)\}.$$

There exists a finite measure $\mu = \mu_r$ ($1 < r < 3/2$) and a constant $C < \infty$ such that for all $N \geq 1, x < y$

$$E \left| \hat{F}_N^{n(N)}(x, y) - F^{n(N)}(x, y) + f(x, y)\bar{Y}_N \right|^2 \leq C\mu(x, y)N^{2d-1-\kappa}, \quad (6.2)$$

Proof. Put

$$S_N^n(x) := N \left(\hat{F}_N^n(x) - F^n(x) + f(x)\bar{Y}_N \right) = \sum_{t=1}^N R_t^n(x),$$

where $R_t^n(x) := I(X_t^n \leq x) - F^n(x) + f(x)Y_t$. By the telescoping identity due to Ho and Hsing (1996), for any $m_0 > 0$,

$$R_t^n(x) = \sum_{m \geq m_0} U_t^{n,m}(x),$$

where

$$\begin{aligned} U_t^{n,m_0}(x) &:= I(X_t^n \leq x) - P[X_t^n \leq x | \mathcal{B}_{t-m_0-1}] + f(x)Y_t^{0,m_0}, \\ U_t^{n,m}(x) &:= P[X_t^n \leq x | \mathcal{B}_{t-m}] - P[X_t^n \leq x | \mathcal{B}_{t-m-1}] + f(x)b_m \zeta_{t-m}, \quad m > m_0, \end{aligned}$$

where $\mathcal{B}_s := \sigma\{\zeta_u, u \leq s\}$ is the history σ -field. Clearly, $E(S_{N_0}^n(x, y))^2 \leq 2(E(S_{N_0}^n(x, y))^2 + E(S_{N_1}^n(x, y))^2)$, where

$$S_{N_0}^n(x, y) := \sum_{t=1}^N U_t^{n,m_0}(x, y), \quad S_{N_1}^n(x, y) := \sum_{t=1}^N \sum_{m>m_0} U_t^{n,m}(x, y).$$

We claim that there exists $1 < r < 3/2, j_0 \geq 1$ and a constant $C < \infty$ such that for any $0 \leq n < m_0, m_0 - n \geq j_0$, and any $x < y$

$$E(S_{N_0}^n(x, y))^2 \leq C\mu_r(x, y)Nm_0, \quad (6.3)$$

$$E(S_{N_1}^n(x, y))^2 \leq C\mu_r(x, y)(\Theta(N) + N^{1+2d}\alpha_n^2), \quad (6.4)$$

where α_n is defined in Lemma 6 and

$$\Theta(N) := \begin{cases} N, & \text{if } d < 1/4, \\ N \log N, & \text{if } d = 1/4, \\ N^{4d}, & \text{if } d > 1/4. \end{cases} \quad (6.5)$$

The claim (6.3), (6.4) is proved below. To conclude the statement of the lemma, note $\alpha_n^2 = O(n^{24-22d-\rho})$ as $\theta = 11(1-d)$, see Remark 2. Hence $\alpha_{n(N)}^2 = O(N^{-\lambda(\rho+22d-24)})$ and $m_0(N) = O(n(N)) = O(N^\lambda)$, implying $E(S_{N_1}^{n(N)}(x, y))^2 \leq C\mu_r(x, y)N^{1+2d-\kappa}$, or (6.2), with κ given in Lemma 7.

To prove (6.3) and (6.4), note by the orthogonality property of conditional expectations, for any x, y

$$\begin{aligned} E[U_t^{n,m_0}(x)U_{t'}^{n,m_0}(y)] &= 0, \quad |t - t'| > m_0, \\ E[U_t^{n,m}(x)U_{t'}^{n,m'}(y)] &= 0, \quad t - m \neq t' - m', \quad m, m' > m_0. \end{aligned}$$

Then

$$\begin{aligned} E(S_{N_0}^n(x, y))^2 &= \sum_{1 \leq t, t' \leq N, |t-t| \leq m_0} EU_t^{n,m_0}(x, y)U_{t'}^{n,m_0}(x, y), \\ E(S_{N_1}^n(x, y))^2 &= \sum_{t, t'=1}^N \sum_{m>m_0, t'-t+m>m_0} EU_t^{n,m}(x, y)U_{t'}^{n, t'-t+m}(x, y) \\ &\leq \sum_{t, t'=1}^N \sum_{m>m_0, t'-t+m>m_0} E^{1/2}(U_t^{n,m}(x, y))^2 E^{1/2}(U_{t'}^{n, t'-t+m}(x, y))^2. \end{aligned}$$

Clearly, by stationarity,

$$\begin{aligned} E(S_{N_0}^n(x, y))^2 &\leq \sum_{1 \leq t, t' \leq N, |t' - t| \leq m_0} E^{1/2}(U_t^{n, m_0}(x, y))^2 E^{1/2}(U_{t'}^{n, m_0}(x, y))^2 \\ &\leq CNm_0 E(U_0^{n, m_0}(x, y))^2, \end{aligned}$$

where the constant C does not depend on n, m_0, N . Using definition of $U^{n, m_0}(x, y)$ together with Lemma 6 (5.13), we obtain

$$\begin{aligned} E(U^{n, m_0}(x, y))^2 &\leq 3(2P[x < X_0^n \leq y] + |f(x, y)|^2 E(Y_t^{0, m_0})^2) \\ &\leq C \int_x^y (f^n(u) + |f'(u)|) du \leq C\mu_r(x, y) \end{aligned}$$

and therefore

$$E(S_{N_0}^n(x, y))^2 \leq CNm_0 \mu_r(x, y),$$

where the constant C does not depend on N, m_0, n . This proves (6.3).

To prove (6.4), we need a convenient representation of $U_t^{n, m}(x, y)$. Note, for any $m > n$,

$$\begin{aligned} P[X_t^n \leq x | \mathcal{B}_{t-m}] &= P[Y_t^{0, m-1} + V(\zeta_t, \dots, \zeta_{t-n}, 0, \dots) + Y_t^{m, \infty} \leq x | \zeta_{t-m}, \zeta_{t-m-1}, \dots] \\ &= F^{n, m-1}(x - Y_t^{m, \infty}), \end{aligned}$$

where (recall notation) $F^{n, m}(x) := P[X_t^{n, m} \leq x]$, $X_t^{n, m} := Y_t^{0, m} + V_t^n = \sum_{i=0}^m b_i \zeta_{t-i} + V(\zeta_t, \dots, \zeta_{t-n}, 0, \dots)$. Then $U_t^{n, m}(x, y) = U_{t,1}^{n, m}(x, y) + U_{t,2}^{n, m}(x, y)$, where

$$\begin{aligned} U_{t,1}^{n, m}(x, y) &:= \int_x^y \int_{\mathbb{R}} \{f^{n, m-1}(u - b_m \zeta_{t-m} - Y_t^{m+1, \infty}) - f^{n, m-1}(u - b_m \tilde{z} - Y_t^{m+1, \infty}) \\ &\quad + b_m(\zeta_{t-m} - \tilde{z})(f^{n, m-1})'(u - Y_t^{m+1, \infty})\} d\Phi(\tilde{z}) du, \\ U_{t,2}^{n, m}(x, y) &:= b_m \zeta_{t-m} \int_x^y \{f'(u) - (f^{n, m-1})'(u - Y_t^{m+1, \infty})\} du, \end{aligned}$$

where we used $E\zeta_0 = \int_{\mathbb{R}} \tilde{z} d\Phi(\tilde{z}) = 0$. The term $U_{t,1}^{n, m}(x, y)$ can be further rewritten as

$$\begin{aligned} U_{t,1}^{n, m}(x, y) &= \int_{\mathbb{R}} \left[\int_x^y du \int_{b_m \tilde{z}}^{b_m \zeta_{t-m}} dv \left\{ (f^{n, m-1})'(u - Y_t^{m+1, \infty}) \right. \right. \\ &\quad \left. \left. - (f^{n, m-1})'(u - v - Y_t^{m+1, \infty}) \right\} \right] d\Phi(\tilde{z}). \end{aligned}$$

By Lemma 6 (5.13), the function $h(x) = (f^{n, m-1})'(x)$ satisfies conditions (5.7). Therefore by Lemma 5 (5.11),

$$\begin{aligned} &\left| \int_x^y du \int_{b_m \tilde{z}}^{b_m \zeta_{t-m}} dv \left\{ (f^{n, m-1})'(u - Y_t^{m+1, \infty}) - (f^{n, m-1})'(u - v - Y_t^{m+1, \infty}) \right\} \right| \\ &\leq C\mu_r(x, y) (|b_m \zeta_{t-m}|^r + |b_m \tilde{z}|^r) (1 \vee |Y_t^{m+1, \infty}|^r), \quad \text{a.s.}, \end{aligned}$$

implying

$$\left| U_{t,1}^{n,m}(x,y) \right| \leq C\mu_r(x,y)|b_m|^r (1 + |\zeta_{t-m}|^r) \left(1 + \left| Y_t^{m+1,\infty} \right|^r \right), \quad \text{a.s.}$$

By independence of ζ_{t-m} and $Y_t^{m+1,\infty}$, and using $E|\zeta_0|^3 < \infty, E|Y_t^{m+1,\infty}|^3 \leq C < \infty$, see (2.3), we obtain for any $r < 3/2$ and any $0 \leq n < m, m - n \geq j_0$,

$$E(U_{t,1}^{n,m}(x,y))^2 \leq C\mu_r(x,y)|b_m|^{2r}. \quad (6.6)$$

with C is independent of n, m .

Next, consider

$$\begin{aligned} U_{t,2}^{n,m}(x,y) &= b_m \zeta_{t-m} \int_x^y \{f'(u) - f'(u - Y_t^{m+1,\infty})\} du \\ &+ b_m \zeta_{t-m} \int_x^y \{f'(u - Y_t^{m+1,\infty}) - (f^{n,m-1})'(u - Y_t^{m+1,\infty})\} du \\ &=: \sum_{i=1}^2 W_{t,i}^{n,m}(x,y). \end{aligned}$$

By Lemma 6, f' satisfies (5.7), hence by Lemma 5 (5.10),

$$\left| W_{t,1}^{n,m}(x,y) \right| \leq C\mu_r(x,y)|b_m \zeta_{t-m}| \left(\left| Y_t^{m+1,\infty} \right| + \left| Y_t^{m+1,\infty} \right|^r \right), \quad \text{a.s.}$$

By Rosenthal inequality, for any $1 \leq r < 3/2$ we obtain $E|Y_t^{m+1,\infty}|^{2r} \leq C(\beta_{m+1}^r + \sum_{i=m+1}^{\infty} |b_i|^{2r}) \leq C\beta_{m+1}^r \leq C\beta_{m+1}$ and therefore

$$E\left(W_{t,1}^{n,m}(x,y)\right)^2 \leq C\mu_r(x,y)b_m^2\beta_m. \quad (6.7)$$

Finally, to estimate $W_{t,2}^{n,m}(x,y)$, use Lemma 6 (5.14)-(5.15) together with Lemma 5 (5.8).

This yields

$$\left| W_{t,2}^{n,m}(x,y) \right| \leq C\mu_r(x,y) |b_m \zeta_{t-m}| (\alpha_n + \beta_m^{1/2}) \left(1 + \left| Y_t^{m+1,\infty} \right|^r \right), \quad \text{a.s.}$$

Consequently,

$$E(W_{t,2}^{n,m}(x,y))^2 \leq C\mu_r(x,y)b_m^2(\alpha_n^2 + \beta_m). \quad (6.8)$$

Note $|b_m|^{2r} = o(b_m^2\beta_m)$ for $r < 3/2$ sufficiently close to $3/2$. Hence and from (6.6) - (6.8) we obtain

$$E(U_t^{n,m}(x,y))^2 \leq C\mu_r(x,y)b_m^2(\beta_m + \alpha_n^2),$$

implying

$$E(S_{N_1}^n(x, y))^2 \leq C\mu_r(x, y) \sum_{1 \leq t \leq t' \leq N} \sum_{m=1}^{\infty} |b_m| |b_{t'-t+m}| (\beta_m^{1/2} + \alpha_n) (\beta_{t'-t+m}^{1/2} + \alpha_n). \quad (6.9)$$

Here,

$$\begin{aligned} \sum_{1 \leq t \leq t' \leq N} \sum_{m=1}^{\infty} |b_m| |b_{t'-t+m}| \beta_m^{1/2} \beta_{t'-t+m}^{1/2} &\leq \sum_{1 \leq t \leq t' \leq N} \sum_{m=1}^{\infty} m^{(4d-3)/2} (t' - t + m)^{(4d-3)/2} \\ &\leq C\Theta(N), \end{aligned}$$

where $\Theta(N)$ is defined in (6.5). The remaining sums on the r.h.s. of (6.9) can be similarly estimated. This proves the claim (6.4) and Lemma 7, too. \square

7 Proof of Lemma 2

We shall use a chaining argument together with the following bound: for any $N \geq 1, x_1 < x_2$

$$E \left(\hat{F}_N(x_1, x_2) - \hat{F}_N^{n(N)}(x_1, x_2) \right)^2 \leq CN^{-(\rho+11d-13)\lambda}. \quad (7.1)$$

Clearly, it suffices to show (7.1) for $x_1 = -\infty, x_2 = x$. Note $E(\hat{F}_N(x) - \hat{F}_N^n(x))^2 \leq E(I(X_0 \leq x) - I(X_0^n \leq x))^2 =: q(x)$ by Cauchy-Schwarz inequality. Put $X := X_0, X^n := X_0^n$. By Minkowski inequality,

$$q(x) = E \left(\sum_{k=n+1}^{\infty} (I(X^k \leq x) - I(X^{k-1} \leq x)) \right)^2 \leq \left(\sum_{k=n+1}^{\infty} q_k^{1/2}(x) \right)^2,$$

where

$$q_k(x) := E \left(I(X^k \leq x) - I(X^{k-1} \leq x) \right)^2.$$

Recall $X^k = Y^{0,k} + V^k + Y^{k+1, \infty}$, where $Y^{k+1, \infty} = \sum_{i=k+1}^{\infty} b_i \zeta_{-i}$ is independent of $Y^{0,k} + V^k$.

Then

$$q_k(x) = E \int_{\mathbb{R}} (I_{(-\infty, x-V^k]}(y) - I_{(-\infty, x-V^{k-1}]}(y))^2 g^{k+1, \infty}(y + Y^{0,k}) dy.$$

Note for any $x, a < b$

$$\int_{\mathbb{R}} (I_{(-\infty, x-a]}(y) - I_{(-\infty, x-b]}(y))^2 dy = |a - b|.$$

Therefore by $\sup_x g^{k+1, \infty}(x) \leq Ck^\theta$, see Lemma 2, we obtain

$$|q_k(x_1, x_2)| \leq Ck^\theta E |V^k - V^{k-1}| \leq Ck^\theta \gamma_k \leq Ck^{\theta-\rho}.$$

Thus, $q(x_1, x_2) \leq C(n^{1+(\theta-\rho)/2})^2 = Cn^{2+\theta-\rho}$, where $\theta = 11(1-d)$. This proves (7.1).

Next we describe chaining. For any integer $k \geq 1$ define the partition

$$-\infty =: \pi_{0,k} < \pi_{1,k} < \cdots < \pi_{2^k-1,k} < \pi_{2^k,k} =: +\infty,$$

such that

$$\mu_r(\pi_{j,k}, \pi_{j+1,k}) = \mu_r(\mathbb{R})2^{-k}, \quad j = 0, 1, \dots, 2^k - 1.$$

Here, μ_r is the measure in the formulation of Lemma 7. Let $K = K(N) = O(\log N)$ be an integer which will be specified below. For any $x \in \mathbb{R}$ and any $k = 0, 1, \dots, K$, define j_k^x by

$$\pi_{j_k^x,k} \leq x < \pi_{j_k^x+1,k}.$$

Put $V_N(x) := N^{1/2-d} \left(\hat{F}_N^{n(N)}(x) - \hat{F}^{n(N)}(x) \right)$. Then

$$V_N(x) = V_N(\pi_{j_K^x,K}) + V_N(\pi_{j_K^x,K}, x). \quad (7.2)$$

By definition of V_N , for any $y < w < z$,

$$\begin{aligned} V_N(y, w) &\leq N^{1/2-d} \hat{F}_N^{n(N)}(y, z) - N^{1/2-d} \hat{F}_N^{n(N)}(y, w) \\ &= N^{1/2-d} (\hat{F}_N^{n(N)}(y, z) - \hat{F}_N^{n(N)}(y, w)) + N^{1/2-d} \hat{F}_N^{n(N)}(w, z) \\ &\leq |V_N(y, z)| + 2 \sup_{x \in \mathbb{R}} N^{1/2-d} \left| \hat{F}_N^{n(N)}(x) - F^{n(N)}(x) + f(x) \bar{Y}_N \right| \\ &\quad + \sup_{y < w < z} N^{1/2-d} (F^{n(N)}(w, z) + f(w, z) |\bar{Y}_N|). \end{aligned}$$

In a similar way,

$$\begin{aligned} V_N(y, w) &\geq -N^{1/2-d} \hat{F}_N^{n(N)}(y, z) \\ &\geq -2 \sup_{x \in \mathbb{R}} N^{1/2-d} \left| \hat{F}_N^{n(N)}(x) - F^{n(N)}(x) + f(x) \bar{Y}_N \right| \\ &\quad - \sup_{y < w < z} N^{1/2-d} (F^{n(N)}(w, z) + f(w, z) |\bar{Y}_N|). \end{aligned}$$

The above bounds combine to

$$\sup_{y < w < z} |V_N(y, w)| \leq |V_N(y, z)| + 2R_N + W_N(y, z),$$

where

$$\begin{aligned} R_N &:= \sup_{x \in \mathbb{R}} N^{1/2-d} \left| \hat{F}_N^{n(N)}(x) - F^{n(N)}(x) + f(x) \bar{Y}_N \right|, \\ W_N(y, z) &:= N^{1/2-d} \left(F^{n(N)}(y, z) + |\bar{Y}_N| \int_y^z |f'(u)| du \right). \end{aligned}$$

Thus,

$$\sup_x |V_N(\pi_{j_K^x, K}, x)| \leq \max_{1 \leq j \leq 2^K} |V_N(\pi_{j-1, K}, \pi_{j, K})| + 2R_N + \max_{1 \leq j \leq 2^K} W_N(\pi_{j-1, K}, \pi_{j, K}).$$

Then from (7.2) we obtain

$$\begin{aligned} \sup_x |V_N(x)| &\leq 2R_N + \max_{0 \leq j \leq 2^K - 1} |V_N(\pi_{j, K})| + \max_{1 \leq j \leq 2^K} |V_N(\pi_{j-1, K}, \pi_{j, K})| \\ &\quad + \max_{1 \leq j \leq 2^K} W_N(\pi_{j-1, K}, \pi_{j, K}). \end{aligned} \quad (7.3)$$

Here, $R_N = o_P(1)$ by Lemma 7. Next, by (7.1),

$$\begin{aligned} P\left(\max_{0 \leq j \leq 2^K - 1} |V_N(\pi_{j, K})| > \delta\right) &\leq \sum_{j=0}^{2^K - 1} P(|V_N(\pi_{j, K})| > \delta) \\ &\leq \delta^{-2} \sum_{j=0}^{2^K - 1} E|V_N(\pi_{j, K})|^2 \\ &\leq C\delta^{-2}(2^K + 1)N^{1-2d-(\rho+11d-13)\lambda}. \end{aligned} \quad (7.4)$$

so that the second term is also $o_P(1)$. The third term is treated exactly the same way.

Consider the last term on the r.h.s. of (7.3). By Lemma 6, $W_N(x, y) \leq C\mu_r(x, y)N^{1/2-d}(1+|\bar{Y}_N|)$, implying

$$E \max_{1 \leq j \leq 2^K} W_N(\pi_{j-1, K}, \pi_{j, K}) \leq C2^{-K}N^{1/2-d}E(1+|\bar{Y}_N|) \leq C2^{-K}N^{1/2-d}. \quad (7.5)$$

Now choose $K = \lceil \log_2 N^{(1/2)-d+\epsilon} \rceil$, where $\epsilon > 0$ is small enough. Then the r.h.s. of (7.4) and (7.5) tend to 0 in view of inequality (2.7). This completes the proof of Lemma 2. \square

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