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Preferencing and Dealer Inventory[1]

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Abstract

Preferencing and Dealer Inventory

This paper examines how preferencing practice affects the quote-setting behavior of dealers who differ in their inventory. In dealership markets, retail trades are generally placed with brokers, who often direct them to a specific dealer regardless of his quotes. In return to this preferenced and captive order flow, this dealer has agreed in advance to match the inside spread. Depending on the market structure (centralized vs. fragmented market), this paper shows how preferencing alters dealers' incentives to narrow market spreads. In a centralized market, preferencing impedes price-competition between dealers. Typically, preferencing leads to wider market spreads and generates higher profits for dealers. In a fragmented market, the impact of preferencing is more ambiguous since it may cause preferred dealers to earn profits, but also to lose money. Actually, preferencing creates risks for the designated dealer in terms of inventory imbalance and price impact. However this market practice generally generates rents for dealers and surprisingly also for the unpreferred dealer, who competes less aggressively given his greater chance to post the best price at equilibrium.

Keywords : Dealership market, preferencing, inventory. **JEL Classification** : D44, G15.

Accords de Préférence et Risque de Position

Ce papier analyse les stratégies de cotation de deux teneurs de marché averses au risque qui doivent contrôler leur position en actif risqué sur un marché où sont permis les 'accords de préférence' (*preferencing agreements*). Selon les modalités régissant ces accords, un courtier peut diriger son flux d'ordres vers un teneur de marché privilégié, indépendamment des cotations affichées par ce dernier. En contrepartie de ce flux d'ordres captif, le teneur de marché rémunère le courtier et lui garantit de s'aligner sur les meilleurs prix du marché pour l'exécution. Ce papier montre que cette pratique altère les incitations des teneurs de marché à coter des prix agressifs, quelle que soit la structure de marché considérée. Ainsi, sur un marché centralisé, ce papier montre que les accords de préférence détériorent les meilleurs prix du marché et génèrent des revenus plus élevés pour les teneurs de marché lors des échanges. Sur un marché fragmenté, l'impact des accords de préférence est plus ambigu puisque le teneur de marché en bénéficiant court un risque d'exécution en prix, qui peut engendrer des pertes lors de l'échange. Cependant, en dépit de ce risque, le teneur de marché privilégié réalise plus de profits en moyenne. En outre, la fourchette de marché se dégrade et le teneur de marché ne bénéficiant pas de flux d'ordres préférentiels peut, malgré tout, profiter indirectement de ceux reçus par son concurrent, étant donné qu'il a désormais une plus grande chance d'exécuter les flux d'ordres publics (i.e. non préférentiels).

1 Introduction

In equity markets, the duty of best execution refers to the fiduciary responsability of broker to execute customers' orders 'in the best market'. This obligation has been interpreted to mean that market orders are to be traded 'at the best available price'[1]. In fragmented markets, the publicly displayed best offer and best bid are an important indicator of the prices that dealers should provide to their customers, especially in case of preferenced retail order flow. Under preferencing arrangements, brokers may send their retail order flow to a *preferred* dealer who has guaranteed in advance to execute orders at the best price, even when that dealer is not quoting it. Orders are actually not exposed to the market. This price matching-like practice is widely used on Securities markets (around 71% of total trade is preferenced on the London Stock Echange (LSE)[2] and 76% on the Nasdaq[3]) and it is suspected to sustain anticompetitive prices. For this reason, preferencing receives much attention from regulators.

Opponents argue that preferencing constitutes a captive order flow that impairs dealers' incentives to narrow market spreads on Nasdaq, leading to inferior executions for retail investors. However, on the LSE, Hansh, Naik and Viswanathan (1998, 1999) find that preferenced order flows receive worse execution than their nonpreferenced counterpart. They also find that the trading profits of preferred dealers are not significantly different from zero. Moreover, the authors show that the dealers on the LSE who post the best market price accommodate a significative greater share of public trades volume, which constitutes a proof of their competitiveness despite preferencing agreements. Klock and McCormick (2002) also find that more aggressive quoting on Nasdaq does indeed result in more business. Consequently, it is still an open question whether this market practice impedes competition between dealers and whether it has some deleterious effects on the market performance.

To our knowledge, there exists no theoretical paper that explores systematically the link between preferencing, inventory costs and the quoting behavior of dealers. This paper constitutes a first attempt to model what impact preferencing has on the quote placement strategy of risk-averse dealers. More explicitely, we seek to answer the following questions :

- How do preferencing arrangements alter dealers' incentives to compete for unpreferenced order flows?
- How such a market practice affect the market spreads? Does it benefit to the dealers or to the investors ?

[1] SEC, 1996, p.14

[2] Figures originate from Hansh, Naik and Viswanathan (1998).

[3] Chung, Chuwonganant and McCormick (2002) found that 'on average, 79.92% of trades were preferenced before the decimalization of April, 2001, whereas 75,64 % of trades are now preferenced.'

Dealers have an obligation to supply liquidity on their own inventory, regardless of their position which may be far away from the desired level. Each inventory imbalance represents a cost for a dealer, which is reflected in his spread as a compensation for the liquidity service. The effect of inventory on quotes is the main consideration of 'inventory' models (see Stoll (1978) or Ho and Stoll (1981, 1983)). The pure inventory 'paradigm' predicts that (i) dealers with extreme inventory position should post the best quotes ; (ii) an increase in the inventory after a buy trade leads to a decrease in the selling quote to attract trades in the opposite direction (the so called 'inventory' control effect). While numerous empirical studies have proved the relevance of the inventory control effect[4], the empirical significance of the link between inventories and dealers' quoting behavior is less obvious. Hansh et al. (1998) suggest that inventory models should reflect some additional market features such as preferencing to test more accurately the link between quotes and inventories. Our paper analyses also the link between quotes, inventories and preferenced order flows.

To answer the previous questions, we consider two dealers with different inventory positions. We assume that the incoming order flow is partly pre-assignated to one of the dealers, regardless of his quotes. Consequently, dealers still compete to accommodate the unpreferenced part, while the preferenced part is exclusively executed by the preferred dealer. The preferenced order flow clears at the best price in accordance with best execution standards. We model price-competition among preferred and unpreferred dealers in two settings : a centralized market and a fragmented one. In both settings we characterize how dealers alter their quote placement strategies and how the market spreads are affected by the existence of preferencing arrangements. Whether the market is transparent or not, we find that :

- Preferencing has an impact on the reservation price of the preferred dealer, which may impede dealers' incentives to narrow quoted spreads.
- Preferencing leads to wider market spreads.

The intuition for the alteration of the reservation price is a reminiscence of the dilemma faced by a monopolist between the cost of providing more liquidity and the profit to execute more shares. For instance, on the sell side, under a certain price, it is more profitable to execute only the preferenced order flow rather than the total order flow. In fact, preferencing deteriorates the marginal benefit to execute the unpreferenced shares and rises the reservation sell price of the preferred dealer. Since preferencing changes the reservation price of the preferred dealer, it changes his possibility to narrow quoted spreads, which is fully anticipated by his opponent. As a result, it may finally soften price competition between both dealers

[4] See Lyons (1995), Hansh et al. (1998).

If the market is supposed to be transparent, we cast our analysis in the Ho and Stoll (1983)'s framework where dealers are supposed to observe each other inventory position. This framework allows us to model explicitely how the link between quotes and inventories may be altered by preferenced order flows. Indeed, we show that, under preferencing it is not necessarily dealers with extreme inventory position that post the best quote. In other words, the link between inventories and quotes predicted by Ho and Stoll (1983) may sometimes be invalidated under preferencing. It may explain the lack of significance of the empirical findings by Hansh et al. (1998) when they test the pure Ho and Stoll's prediction on the LSE whose total order flow is preferenced at 70%. Moreover, we show that in a centralized market, preferencing enlarges market spreads and leads to higher profits for both dealers. However, market spreads may narrow when a third unpreferred dealer enters the market. This additional competitive force restores a price-competition similar to a competition which would prevail without preferencing arrangements.

Then, we study what impact preferencing has on quotes in a market where dealers cannot observe each other's inventory position, as in a fragmented market such as the Nasdaq or the LSE. This alternative analysis is based on the Biais (1993)'s model. Even if opponents' reservation prices are uncertain, dealers observe which agent receives a preferenced demand and the scale of this demand. Despite the simplicity of the economic problem, the prices posted by dealers at equilibrium are quite complex. Actually the selling quotes correspond to those arising in a first price auction where bidders are asymmetric. The main problem of asymmetries is the lack of analytical solutions. First, this paper completely characterizes the Pareto-dominant equilibrium for dealers. Then, we adopt a numerical approach and we find that :

- In a fragmented market, preferencing reduces the incentives of the unpreferred dealer to compete aggressively using quoted spreads. The unpreferred dealer has indeed more chance to post the best price at equilibrium given the impact of preferencing on the reservation price of the preferred dealer.
- Surprisingly, the preferred dealer may incur *losses* in accommodating his captive order flow for some levels of his initial inventory. He faces indeed a risk in price execution whenever the market price he matches is below (resp. upper) his selling (resp. buying) reservation price.

The preferred dealer faces a greater inventory risk in so far as he has less control over his trades. In a fragmented market, the numerical approach allows to find that that dealer may even face some losses on his captive demand, which could explain the zero profit of the preferred dealers on the LSE (see Hansh et al (1998)). However, preferencing has a negative impact on the market performance since it widens in average market spreads.

Our paper contributes to a growing literature on the effects of preferenced order flow. Some early papers (Chordia and Subrahmanyam, 1995 or Kandel and Marx, 1999) focus on the link between the development of preferencing and price discreteness. They argue that preferencing would disappear as price grids went finer. However, consistent with the prediction of Battalio and Holden (2001a), preferencing has not been eliminated despite decimalization. Our results too do not depend on price discreteness.

Much of the literature indirectly deals with the impact of preferencing, since it concentrates on the effect of payment for order flow obtained by brokers in return for directing preferenced orders (see Chordia and Subrahmanyam, 1995, Battalio and Holden, 2001b among others). Our paper does not consider specific preferencing arrangements (internalization, noncash or cash arrangement) since it analyzes preferencing as a price matching-like practice. Such a practice is suspected to reduce incentive to compete in price and to facilitate coordination between competitors (see Salop (1986)). This intuition is corroborated by several theoretical papers in market microstructure including Godek (1995), Dutta and Madhavan (1997) or Parlour and Rajan (2002). Bloomfield and O'Hara (1998) demonstrate that the negative effect of preferencing can also be found in laboratory financial markets. Our work differs from the standard assumptions of these models since it mixes two kinds of demands : a preferenced order flow (analyzed as a captive demand submitted to price-matching) and an unpreferenced order flow for which preferred and unpreferred dealers compete. Then it analyzes the impact of preferencing on the price-competition for the unpreferenced demand. We show that preferencing enlarges market spreads and may increase dealers' rents as suspected. However, it can also constitute a competitive disadvantage for the preferred dealer since he may face some losses in a fragmented market.

Our work complements the models of Kandel & Marx (1997), Battalio & Holden (2001) who argue that preferencing reduces dealers' incentives to cut prices because dealers who undercut other preferred competitors cannot attract their preferenced order flow. Their models do not consider neither the inventory effect nor the existence of an unpreferenced order flow. As a result, our model shows that the unpreferred dealer has indeed less incentives to post aggressive quotes, but the reason comes from his greater chance at equilibrium to execute the unpreferenced demand compared with the preferred dealer. The unpreferred dealer anticipates indeed the less favorable position of the preferred dealer who faces an inventory imbalance caused by the preferenced demand.

Our work complements also the model of Rhodes-Kropf (1999) who studies the impact on spreads of price improvement in a similar framework. Price improvement is a market practice which consists of filling the order inside the spreads. It is showed that dealers offer price improvement to mid-size and large trades because of the negociation power of these customers (generally institutional traders). Whereas his works deals with a market practice concerning institutional trading, we focus on preferencing which is dedicated to retail trading. Our conclusion is similar to his : preferencing too is a market practice that widens market spreads. However we are not able to conclude anything concerning the overall brokerage service since price-matching also allows retail traders to benefit from speed execution and price guarantee (almost no price disimprovement under such a practice) even if it enlarges market spreads.

This paper is organized as follows. The next section describes the different types of preferencing and the institutional concerns that this market practice raises. Section 2 describes the institutional framework and the model. Section 3 shows which impact preferencing has on the link between quotes and inventories in a centralized market, whereas section 4 is dedicated to the analysis in a fragmented market. Section 5 explores some possible extensions and section 5 concludes. Proofs are in the Appendix.

2 Framework

2.1 Preferencing Practice and Institutional Concerns

Many securities[5] are traded in more than one markets : for instance, New York-and American Stock-Exchange-listed stocks are frequently traded on regional stock exchanges such as Cincinnati Stock Exchange. In the Nasdaq Stock Market, multiple dealers are in competition for the same securities. On average, twelve dealers trade the same stock. In this competing environment, as the 1991 report of the NASD Board of Governors underline, 'order flow is a valuable commodity and the competition to attract retail order flow is intense'. To encourage brokers to send them aggregated retail orders, dealers use inducements of various kinds known as preferencing arrangements which allow them to capture order flow. Retail order flow preferencing principally happens through three business arrangements : internalization, payment for order flow and payments in services (clearing, execution or research services, for instance).

Internalization which is allowed in United States or in United Kingdom is considered as self-preferencing. It leads to similar orders' execution : a firm (doing brokerage and market-making within a single entity) can 'internalize' its trades by executing them in-house against its own dealer inventory, provided that trades are executed at a price no worse than the consolidated best bid and offer (the NBBO[6]) in accordance with regulatory best execution standards. Internalization is cost-effective since it allows integrated firms to save costs related to transaction fees and clearing charges.

Concerning 'external' forms of preferencing, they vary according to the market. On the London Stock Exchange, cash payment to purchase order flow is not allowed. London preferencing agreements are 'soft-dollar' (i.e. noncash) arrangements, whereas, on Nasdaq, external preferencing principally happens through payment for order flow. Competition for retail order flow is more intense on Nasdaq than in London, since there exist more than 500 dealers on the Nasdaq and less than 20 dealers on the

[5] This work focuses on preferencing in equity markets. But preferencing is also found in options markets.

[6] The best market prices are also known as the National Best Bid or Offer : the so-called NBBO.

LSE[7]. Attracting retail order flow is a critical issue on Nasdaq, especially for dealers whose sole business is market-making and whose revenue comes from the pure market spread. These dealers known as 'wholesalers'[8] generally do not have retail sales forces to gather customers orders and they cannot rely on alternative income sources such as investment banking fees or brokerage commissions. Preferencing arrangements with brokerage firms[9] constitute a survival response for them. That is the reason why, even if the decimalization[10] have dramatically affected spread revenues, wholesalers still guarantee up to a certain number of shares an execution at the NBBO for preferenced order flow. The decimalization has however considerably affected cash inducements : the 2001 Nasdaq report to the SEC concerning the impact of decimalization [18] estimates that, for instance, a cash payment for a 500 shares market order in Microsoft has decreased by 80% : the offered rebate goes down from $2.50 to $0.50. But preferencing still abounds on Nasdaq. Notice that preferencing agreements are also found in US equity Exchanges - the NYSE and the five regional Exchanges - that are linked together via the Intermarket Trading System (ITS). For instance, the Boston Stock Exchange and the Cincinatti Stock Exchange which make continuous market in NYSE-listed securities have preferencing plans : brokers may route an order to a particular specialist on a regional exchange even if this specialist is not quoting the best prices, either because both broker and specialist are affiliated (as, for instance, the Fidelity's unit on the BSE) or for cash payments.

Preferencing is largely widespread and raises institutional and academics concerns. This practice indeed violates the principle of time priority which stipulates that orders have to be executed by the first dealer quoting the best price. Then, such arrangements forgo the opportunity of orders to interact and transact between the best bid and the best ask (to benefit from any price-improvement). As a result, preferencing is argued to increase market spreads and to lead to higher execution costs for investors. For instance, Huang and Stoll (1996) argue that higher execution costs on Nasdaq are closely linked to preferencing[11], while Dutta and Madhavan (1997) show that preferencing is indeed a way to facilitate implicit collusion among dealers. The Securities and Exchange Commission stresses[12] that preferencing should be cause for concern since, as a price matching-like practice, it is suspected to reduce dealers' incentives to undercut and to be anti-competitive. In response to this concern, the Commission issued two new rules in November 2000 that require market centers and brokers dealers to disclose execution

[7] The London Stock Exchange has changed to an hybrid market since 1997 : it combines few market-makers and a limit order book (SETS).

[8] Pure wholesalers are, for instance, Knight Securities, Madoff Securities or Schwab.

[9] Many online discount brokers (E*Trade, Quick and Reuilly, ...) are not owned by any market-making firms.

[10] The full decimalization on Nasdaq happened on April 9, 2001.

[11] 'We believe that preferencing has increased over time consistent with the increase in spreads, although we do not have direct evidence on this', Huang and Stoll (1996).

[12] The Commision underlines that internalization and payment for order flow 'interfere with order interaction and discourage the display of aggressively-priced quotations.' *Proposed Rule 11Acl-6.*

and order routing practices[13]. Our model focuses on the impact of preferencing on risk-averse dealers' incentives to narrow spreads. It analyzes then the consequence of this new competitive terms on the expected market spreads (a measure of market performance).

2.2 The Basic Setting

Consider the market for a risky asset, whose final cash flow is a normal random variable $\tilde{v}$ characterized by an expected value μ and a variance σ_v^2. There are two types of agents : (i) investors who demand liquidity ($|Q|$)[14] and (ii) dealers, who supply liquidity, standing ready to execute incoming market orders ($\pm Q$) at their bid or ask quote against their own inventory.

Dealers' reservation price and inventory cost

For ease of exposition, we focus on the *sell* side of the market and on the behavior of *two* strategic dealers who compete to post the lowest selling price (or *ask* price) so as to execute the incoming buy order ($+Q$). Dealers, denoted by D_1 and D_2, are identically risk-adverse but differ in their inventory position. In other words, the divergence in dealers' reservation prices is caused by the risk aversion of dealers facing each a more or less unbalanced position, as shown in a seminal paper of Stoll (1978). Adding inventory increases risks in moving the position away from the dealer's preferred level and alters his reservation price.

The reservation price to sell Q shares when a dealer holds an inventory position I_i is denoted by $a_r(I_i, Q)$, $i = 1, 2$. We use the result of Ho and Stoll (1983) to give a simple expression of $a_r(I_i, Q)$[15],

$$a_r(I_i, Q) = \mu + \frac{\rho\sigma_v^2}{2}(Q - 2I_i), i = 1, 2$$

where ρ is the coefficient of risk aversion of dealer D_i ($i = 1, 2$), $+Q$ is the incoming buy order to accommodate and I_i is dealer D_i's initial inventory. It is common knowledge that I_i is a realization of the random variable $\tilde{I}_i$ that is assumed to be distributed uniformly on $[I_d, I_u]$. We will, equivalently, consider (when it is more convenient) that the reservation prices $a_r(I_i, Q)$ are random variables that are *independently* distributed according to a uniform distribution on $[a_r(I_u, Q), a_r(I_d, Q)]$.

The reservation price may also be interpreted as the marginal cost of the dealer to produce liquidity. A dealer provides indeed liquidity against his own inventory, bearing risks that entail costs from which the dealer has to be compensated.

[13] SEC Rule 11Ac1-5, requires market centers to disclose monthly data about the quality of their trade executions.

SEC Rule 11Ac1-6 requires broker/dealers that route orders in equity and option securities to make publicly available quarterly reports that identify the venues to which client orders are routed for execution. The SEC's primary purpose for adopting this rule is to assure public disclosure of the significant venues to which a broker/dealer routes its client's orders and to facilitate an evaluation of potential conflicts of interest between the broker/dealer and its clients.

[14] 'liquidity traders'

[15] This expression can be obtained in a mean-variance framework, as in Biais (1993).

Preferenced vs. nonpreferenced Order flows

In this paper we analyze how a liquidity supplier deals with quotes, inventory uncertainty and preferencing agreements. We make a distinction between two types of order flows : (i) the preferenced order ($+\kappa$) which is pre-assigned to dealer D_2, and (ii) the unpreferenced order ($+Q$) which is not assigned to any dealer. While the preferenced order is routed exclusively to dealer D_2, the unpreferenced buy order is attributed to the dealer who quotes the best price (dealer D_1 or D_2).

Obligation of execution by a preferred dealer

The brokerage industry is well-known to be very competitive. As a result, a retail broker who preferences his order flow against payment in cash or in services must offer a superior combination of price and service to attract customers away from his opponents. Preferencing agreements allow them to offer a quality of execution in terms of price certainty[16] and speed of execution[17]. When the preferred dealer faces an unwanted inventory position, she could send her preferenced order flow to the best-quoting dealer to control her inventory risk. She must, however, still pay her retail broker for receiving this order flow. Moreover, with fast-moving, narrower spreads due to decimalization, re-routing preferenced orders increases the risk not to provide the service of price certainty to her broker (price-disimprovement) and, then, the risk to lose this business relationship. Consequently, as the 2001 Nasdaq report underlines, preferred dealers 'rarely act in an agency capacity'. In this model, we do not model the business relationship between the discount broker and his preferred dealer, we simply assume that the potential costs to act as an agent are higher than the costs to act as principal. Consequently, the preferred dealer will not decline the order in re-routing it to the best-quoting dealer (D_1), but she will execute it instead.

The Best Offer

According to the usual standards of the Best Execution duty for retail order flow, the preferred dealer has to execute the preferenced order flows at the best available price (i.e. the best offer in our model) even when she does not quote it. We define the best market price by $\underline{a} = \min(a_1, a_2)$.

The timing of the game and the payoffs of the dealers

We present a time line of events (see Figure 1 below).

[16] Preferencing is argued to lower incidences of price disimprovement experienced by retail customers.

[17] On Nasdaq, most wholesalers or dealers involved in preferencing provide guaranteed auto-execution for preferenced retail orders which enhances a fast execution.

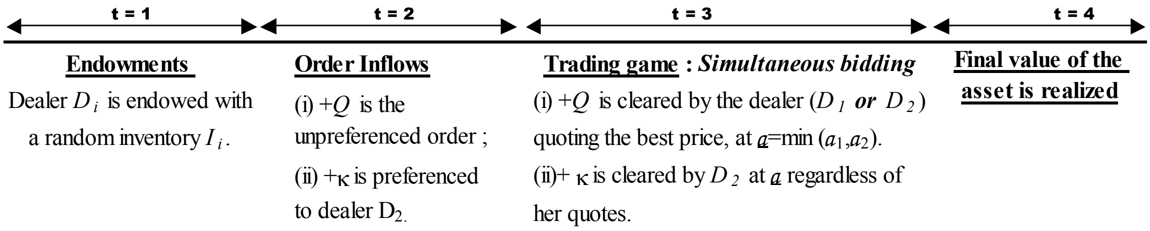


FIGURE 1 : Sequence of events

At $t = 2$ we suppose that an investor arrives and expresses his desire to buy Q shares. At the same time a broker sends a preferenced order flow ($\kappa > 0$) to dealer D_2. Dealer D_1 knows that D_2 is committed to accommodate a preferenced order flow of κ shares. At $t = 3$ dealers bid simultaneously to execute the public order flow Q. The dealer who posts the lowest selling price executes Q unpreferenced shares. Moreover, D_2 executes κ shares at the lowest selling price. Dealers are supposed to have linear preferences over the surplus from trade, i.e. they behave as *risk-neutral* bidders. Doing so, we limit the impact of risk aversion to the determination of reservation prices[18].

Given that dealer D_1 does not receive any preferenced order flow, his payoff is given by the following function :

$$A_1(a_1, a_2, I_1) = \begin{cases} (a_1 - a_r(I_1, Q)) \times Q & \text{if } a_1 < a_2 \\ 0 & \text{if } a_1 > a_2 \end{cases}$$

For the preferred dealer D_2, her payoff differs from dealer D_1 since she executes at least κ preferenced shares. Then, her payoff is given by the following function :

$$A_2(a_2, a_1, I_2) = \begin{cases} (a_2 - a_r(I_2, Q + \kappa)) \times (Q + \kappa) & \text{if } a_2 < a_1 \\ (\underbrace{a_1}_{\text{The Best Offer}} - a_r(I_2, \kappa)) \times \kappa & \text{if } a_2 > a_1 \end{cases}$$

If dealer D_2 quotes the lowest selling quote ($a_2 < a_1$), she accommodates the total order flow ($Q + \kappa$) at this price a_2, given that her own quote constitutes the best offer ($\underline{a} = a_2$). In the opposite case ($a_2 > a_1$), dealer D_2 executes only the preferenced trade at the best offer which is the quote posted by her opponent D_1. Because dealer D_2 does not execute the same volume in both alternatives (whether she posts the best price or not), it is natural to consider two reservation prices, corresponding each to the quantity to supply : the reservation price to accommodate only the preferenced order flow is $a_r(I_2, \kappa)$ whereas $a_r(I_2, Q + \kappa)$ is the reservation price to execute ($Q + \kappa$) shares.

[18]Dealers' reservation prices depend on the risk-aversion coefficient, which would affect their quoting behavior in the second-price auction or the first-price auction that are analyzed below. For simplicity, however, we remove the effect of risk aversion on preferences in using the first order linear approximation proposed by Biais (1993) and used by Rhodes-Kropft (1999).

Note also that, because dealer D_2 is compelled to execute the preferenced order flow κ, she might incur losses (as soon as $\underline{a} = a_1 < a_r(I_2, \kappa)$). As underlined by Kandel and Marx (1999), 'under preferenced arrangements, a dealer has less control over the trades she has to accommodate because she cannot withdraw from the market by adjusting quotes'.

Let us introduce a specific price termed as the *cutoff* price which leaves the preferred dealer indifferent between the trading profit earned from the execution of the total order flow $(Q + \kappa)$ and the one earned in executing only κ preferenced shares.

Definition 1 *Let $a_r^\kappa(I_2, Q, \kappa) \overset{Def}{=} a_{r,2}^\kappa$ be the value of the posted price $\underline{a}$ at which the preferred dealer is indifferent between trading κ shares or $(Q + \kappa)$ shares. That cutoff price $a_{r,2}^\kappa$ is defined as the solution of the following equation :*

$$(\underline{a} - a_r(I_2, \kappa)) \times \kappa = (\underline{a} - a_r(I_2, Q + \kappa)) \times (Q + \kappa)$$

For any posted price below the cutoff price ($\underline{a} < a_{r,2}^\kappa$), the trading profit is higher in executing only κ preferenced shares than the total order flow $(Q + \kappa)$. In the opposite case ($\underline{a} > a_{r,2}^\kappa$), the preferred dealer prefers to post the best price. Since κ is a captive order flow, the preferred dealer faces a simple monopolist tradeoff between cost and volume. As we mentionned earlier, reservation prices represent the cost to produce liquidity and are increasing with the quantity to supply : naturally, $a_r(I_2, \kappa) < a_r(I_2, Q + \kappa)$. The preferred dealer faces the classic dilemma : accommodating only κ shares at a small cost or supplying more $(Q + \kappa)$ at a greater cost. Below the cutoff price, the cost to accommodate the total order flow is not offset by the increase in the revenue.

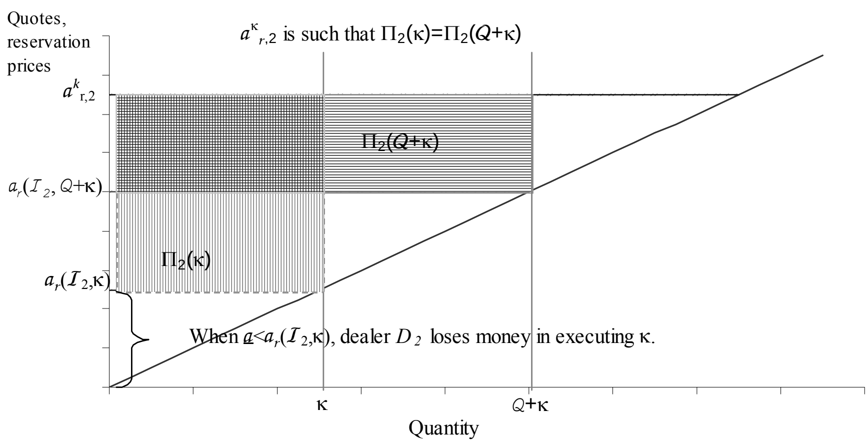


FIGURE 2 : Cutoff price and Reservation price

where $\Pi_2(\kappa)$: Profit from trading κ shares,

$\Pi_2(Q + \kappa)$:Profit from trading $(Q + \kappa)$ shares.

Remark 1 *The cutoff price is equal to* $a_{r,2}^{\kappa} = a_r(I_2, Q+\kappa) + \rho\sigma_v^2\kappa/2$ *where*

$$a_{r,2}^{\kappa} > a_r(I_2, Q+\kappa) > a_r(I_2, \kappa),\ \forall\kappa.$$

The ranking is consistent with the monopolistic situation of the preferred dealer on her captive demand. The cutoff price at which she is indifferent between posting or not the best offer must be strictly greater than her marginal costs to produce liquidity in either cases : wether she supplies liquidity for κ shares at a cost equal to $a_r(I_2, \kappa)$ or for $(Q+\kappa)$ shares at a greater cost $a_r(I_2, Q+\kappa)$.

A benchmark : the competitive case

We next introduce the 'competitive' case (or the No Preferencing case[19]) in which no order flow cannot be preferenced. Consequently, the order flow κ is now executed by the dealer quoting the best price. Then, the global quantity to be accommodated is $(Q+\kappa)$, and dealers' payoffs are such that :

$$A_i^{NP}(a_i, a_{-i}, I_i) = \begin{cases} \left(a_i^{NP} - a_r(I_i, Q+\kappa)\right) \times (Q+\kappa) & \text{if } a_i^{NP} < a_{-i}^{NP} \\ 0 & \text{if } a_i^{NP} > a_{-i}^{NP} \end{cases} \quad i = 1, 2.$$

The best price (or the best offer) is $\underline{a}^{NP} = \min\left(a_1^{NP}, a_2^{NP}\right)$. For the ease of the exposition of the results, we adopt the same notation as Biais (1993) and we note :

$$a_r(I_i, Q) \overset{Def}{=} a_{r,i}.$$

Let us give now some intuitions on these basic assumptions.

2.3 Discussion

2.3.1 Preferencing and Dealers' competition

We give some intuitions on the impact of preferencing on reservation prices of dealers and on the way to compete. Then, we discuss preferencing regards to economic concerns on price matching-like practices.

(i) Risk aversion, inventory and preferencing

Positions of dealers' quotes will depend on the relative ranking of their reservation prices since dealers do not quote under their reservation price. For a given inventory, dealers' reservation prices are increasing with the size of the transaction. The cost to trade is consequently bigger for the preferred dealer who may execute $(Q+\kappa)$ shares than for the unpreferred dealer who may trades Q shares. Moreover, the preferred dealer has guaranteed in advance to execute her preferenced trade even when she is with an extreme disadvantageous inventory position. Preferenced order flow induces a greater inventory risk

[19] We use the subscript NP to identify this 'competitive' case.

than unpreferenced trades which may be declined in posting an unfavorable quote. As a result, the risk premium required for executing κ preferenced shares should be greater than the premium for executing κ unpreferenced shares[20]. Then, preferenced trades create *asymmetric* marginal costs (or, equivalently, reservation prices) between both dealers.

(ii) Preferencing as a price-matching practice : advantages and disadvantages

Preferencing may be interpreted as a price matching-like practice. Opponents argue that such practices facilitate cartel pricing by removing the incentive to undercut (see Salop (1986) for industrial organization or Dutta and Madhavan (1998) concerning dealer markets). However, in our setting, undercutting the other competitor does increase revenue since the unpreferenced part of the demand is still at stake[21]. Dealers have still incentives to narrow market spreads despite preferencing. Moreover, the price-matching guarantee may create risks for the preferred dealer. In case of the best offer posted by her opponent is lower than her reservation price, the compulsory execution of the preferenced trade at that best price might cause the preferred dealer to lose money. We refer to this risk as a risk in *price execution.*

However, preferencing raises two concerns :

1) The preferenced order flow is a captive demand[22]. There are some cases when it is more profitable for the preferred dealer to execute only κ preferenced shares at a smaller liquidity cost than a larger order flow $(Q+\kappa)$ at a greater cost. As a result, this *monopolistic* situation may diminish her incentives to compete for the unpreferenced order flow, which should lead to higher market prices.

2) The nonpreferenced and the preferenced order flows are distinct economic goods since the latest is a captive demand. However, the preferred dealer still accommodates both at a *uniform* price when she posts the best price. This uniform price-matching rule creates rents for the preferred dealer on the nonpreferenced trade[23] since that trade is cleared at a price that combine two different risk premia, the premium for clearing κ preferenced shares being larger than the one for κ equivalent unpreferenced shares. As a uniform price-matching rule, preferencing may be suspected to lead to higher market price.

Now, the questions raised are the following : may the competitive disadvantage of facing a risk in price-execution be more than offset by the effects of preferencing in favoring higher prices ? May dealers' incentives to execute the unpreferenced order flow be more than offset by the disincentives to face a preferenced demand ? The following paragraph exposes a way to study the impact of preferencing on the positioning of dealers' quotes.

[20] This intuition is also an assumption made by Kandel and Marx (1999).

[21] This assumption is corroborated by the empirical findings of Hansh et al. (1998) that we mentionned in introduction.

[22] This feature clearly changes from usual economics model where the agreggate demand is still exposed to the whole market and to all competitors.

[23] In other words, this uniform price rule creates a negative externality on unpreferenced trades.

2.3.2 How to capture the impact of preferencing on the bidding behavior of dealers ?

In order to gain some intuitions on how preferencing may affect the way dealers quote, we will analyze their bidding behavior using two measures : (i) the probability to execute the unpreferred order flow and (ii) the 'quoting' aggressiveness. The quoting aggressiveness relates to how close, on average, the selling quote posted by dealer i is to his own reservation price. Consequently, we define a coefficient θ_i which measures the distance between the selling price posted by dealer D_i $(i = 1, 2)$ to his reservation price, i.e. $\theta_i\left(a_{r,i}\right) = \left(a_i\left(a_{r,i}\right) - a_{r,i}\right)/a_{r,i}$. The interpretation of this coefficient is straightforward : the lower is the coefficient, the more aggressive is the selling quote posted by the dealer, i.e. the more competitive is the selling price posted by the dealer.

3 Inventory Paradigm and Preferenced Order Flow

In the following sections, we consider the relation between preferenced order flows, inventories and quotes in a market with strategic dealers. We analyze how preferencing interacts with dealers' bidding strategy in two different market settings : (i) in the canonical one-period model of Ho and Stoll (1983) where dealers are assumed to perfectly observe each other's inventory ; (ii) in a fragmented market, that does not allow a dealer to observe competitors' inventory position.

3.1 Preferencing and Equilibrium Quoting Strategy in a Transparent Market

We consider a fully transparent market (e.g. a centralized structure) where dealers are able to observe perfectly the inventory positions of their competitors. In this context, without any preferencing agreement, Ho and Stoll (1983) show that the dealer with the longest position posts the best price in equilibrium and the (Nash) equilibrium strategy results in setting the best offer to the second best price. Now, we analyze how preferencing affects this standard result.

In our setting, before trading the preferred dealer D_2 receives a preferenced order flow large of κ shares. If she posts the best price, then her inventory position will shorten of Q unpreferenced shares and κ preferenced shares. Otherwise her inventory necessary shortens of κ shares. In sum, the preferenced trade acts as an inventory shock arising at date 2 that definitely alters the initial inventory of dealer D_2 from a position at I_2 to $(I_2 - \kappa)$.

Lemma 1 *In equilibrium, the preferred dealer has no incentive to quote below the cutoff price. As a result, the cutoff price is the effective reservation price of a preferred dealer.*

Suppose that dealer D_2 posts a price below her cutoff price and quotes the best price. Then she executes the total order flow. However, she gets a lower profit in doing so than in executing only her preferenced order flow at this price. Consequently, a preferred dealer is not induced to post quotes below one's cutoff price. That price is a 'natural' reservation price for the preferred dealer since it combines the inventory risk premium for the potential unpreferenced trade Q and the premium required by the execution of the riskier compulsory preferenced trade κ. As a result, executing $(Q+\kappa)$ shares whose κ shares are preferenced requires a higher reservation price that in the case when the κ shares would be unpreferenced : $a_{r,2}^{\kappa} > a_r\left(I_2, Q+\kappa\right)$.

Note that the cutoff price may be expressed in order to be interpreted as an inventory shock occuring at date 2 : $a_{r,2}^{\kappa} = a_r\left(\left(I_2-\kappa\right), Q\right)$.

Under preferencing agreements, dealer D_2 alters her reservation price in order to take into account her new effective position $(I_2-\kappa)$, i.e. her new marginal cost to trade, i.e. $a_r\left(\left(I_2-\kappa\right), Q\right) = a_{r,2}^{\kappa}$. In the remaining section, we designate the cutoff price as the reservation price of the preferred dealer. Since dealer D_1 is assumed to observe the magnitude of the preferenced trade, he anticipates correctly how dealer D_2 will modify her reservation price and her bidding strategy under preferencing agreement.

Theorem 1 *At equilibrium, when both dealers have a chance to post the best price ($a_{r,u}^{\kappa} \leq a_{r,d}$), then the dealer with the lowest reservation price ($\min\left(a_{r,1}, a_{r,2}^{\kappa}\right)$) posts a sell quote just below the second lowest reservation price. In other words, the Nash equilibrium consists of each dealer using the following pure strategy*[24] *:*

$$
\begin{aligned}
a_1^c &= \begin{cases} a_{r,2}^{\kappa} - \varepsilon & \textit{if } a_{r,1} < a_{r,2}^{\kappa} \\ a_{r,1} & \textit{otherwise.} \end{cases}, \\
a_2^c &= \begin{cases} a_{r,1} - \varepsilon & \textit{if } a_{r,2}^{\kappa} < a_{r,1} \\ a_{r,2}^{\kappa} & \textit{otherwise.} \end{cases}.
\end{aligned}
$$

where $\varepsilon > 0$ but ε is arbitrarily small.

At equilibrium, when the preferred dealer has no chance to post the best price ($a_{r,d} < a_{r,u}^{\kappa}$), then she quotes her reservation price, i.e. $a_2^c = a_{r,2}^{\kappa}$. Dealer D_1 quotes $a_1^c = a_{r,2}^{\kappa} - \varepsilon$.

The direct impact of preferencing is essentially to raise the preferred dealer's reservation price which may soften price competition between dealers. Observe that if preferencing was not allowed, the κ shares would be directed with the Q public shares to dealer D_1 if and only if his inventory position is the longest[25] as in Ho and Stoll (1983). Under preferencing, the unpreferred dealer executes the public trade

[24] We use the subscript c to identify dealers' quotes arising in a fully transparent market, in reference to Biais (1993)'s model that qualifies this transparent market structure as 'centralized'.

[25] $I_1^* = \max\left(I_1, I_2\right) \Leftrightarrow a_r\left(I_1^*, Q+\kappa\right) = \min\left(a_r\left(I_1, Q+\kappa\right), a_r\left(I_2, Q+\kappa\right)\right)$

even if he is not initially the longest ($I_1 < I_2$). In fact, in case of the inventory position of dealer D_1 is longest than $(I_2 - \kappa)$ (i.e. $a_{r,1} < a^{\kappa}_{r,2}$), then the preferred dealer will not undercut dealer D_1, letting him quoting the best price. As a result, there is no competition (compared to the benchmark) in case of the reservation price of the unpreferred dealer $a_{r,1}$ belongs to the interval $\left[a_r\left(I_2, Q+\kappa\right), a^{\kappa}_{r,2}\right]$.

A numerical example

To offer a numerical support, suppose that, at date 2, there is an incoming *public* buy order of 6500 shares to accommodate and no preferenced order. Dealer D_2 with an inventory long of 19000 shares knows that her reservation price is the lowest at 98.30. She is able to undercut the second-longest dealer with a reservation price of, say, 98.45 ($I_1 = 17500$). Any price up to 98.45 is a winning quote, the optimal quotation strategy of the longest dealer is then to quote $98.45 - \varepsilon$. Doing so, dealer D_2 maximizes the profit earned from the trade $(98.45 - 98.30) \times 6500 = 975$ without changing her chance to accommodate the order flow. Anticipating the strategy of the longest dealer, the best reply of the other dealer is to quote his own reservation price.

$$\begin{array}{l|lll} & \text{Reservation prices} & \text{Quotes} & \text{Trading profit} \\ \text{Dealer } D_1 & 98.45 = a_r\left(I_1, 6500\right) & a_1^{NP} = 98.45 & \Pi_1^{c,NP} = 0 \\ \text{Best-quoting dealer : } D_2 & 98.30 = a_r\left(I_2, 6500\right) & \underbrace{a_2^{NP}}_{\text{Best Offer}=\underline{a}^{c,NP}} = 98.45 - \varepsilon & \Pi_2^{c,NP} = 975 \end{array}$$

Now, suppose that, at date 2, the incoming order flow is partly preferenced for $\kappa = 3500$ shares to dealer D_2. The remaining unpreferenced order flow amounts to $Q = 3000$ shares. Then, dealer D_2's cost to trade is altered by the preferenced demand and she rises her reservation price from 98.30[26] to $98.425 = a^{\kappa}_{r,2}$:

$$\begin{array}{l|l} & \text{Reservation prices} \\ \text{Dealer } D_1 & 98.275 = a_r\left(I_1, 3000\right) \\ \text{Preferred dealer } D_2 & 98.475 = a_r\left(I_2 - 3500, 3000\right) = a^{\kappa}_{r,2} \end{array}$$

Given that dealer D_2 does not to quote under $a^{\kappa}_{r,2}$, competition leads dealers to post the following quotes :

$$\begin{array}{l|ll} & \text{Quotes} & \text{Trading profit} \\ \text{Preferred dealer } D_2 & 98.475 = a_2^c & \Pi_2^c = 1050 \\ \text{Best-quoting dealer : } D_1 & 98.475 - \varepsilon = a_1^c = \underline{a}^c & \Pi_1^c = 1137.5 \end{array}$$

Preferencing may be prejudiciable to the unpreferred dealer D_1 since he cannot compete on 3500 preferenced shares in comparison with the competitive situation described above. However, notice that even if he is suffering from a loss in volume, this dealer posts now the best price and he improves his

[26] This price is the reservation price that prevails indeed in the competitive case described below, since $98,30 = a_r\left(I_2, Q+\kappa\right)$ where $(Q+\kappa) = 6500$.

profit ($\Pi_1^c = 1137.5 > 0 = \Pi_1^{c,NP}$). Note also that the modification of dealer D_2's reservation price alters competition between dealers and, in this case, it causes the best offer to worsen from 98.45 to 98.475. It illustrates how preferencing may soften price competition between dealers. In order to gain some intuitions on the impact of preferencing on the way to compete, let us turn to a more detailed analysis of the bidding behavior of dealers.

Preferenced order flow and dealers' quoting behavior

Preferencing alters the reservation price of the preferred dealer and also her quoting behavior. Under preferencing, dealer D_2 is less likely to post the best price at equilibrium. Moreover as the volume of preferenced shares rises, she is induced to post quotes closer to her reservation price $a_{r,2}^{\kappa}$ than in the competitive case, i.e. she competes in average 'more' aggressively due to the impact of preferencing on her reservation price[27]. This result could be rather counter-intuitive compared with the arguments of Kandel and Marx (1997) or Dutta and Madhavan (1997) previously mentionned but it has to be moderated by the initial rising of her reservation price.

Preferencing alters also the bidding behavior of the *unpreferred* dealer : dealer D_1 posts higher selling prices. In other words, dealer D_1 quotes in average *less* aggressively which is associated with his greater chance to accommodate the unpreferenced order flow. In sum, preferencing is a disincentive to improve the quoted prices for the unpreferred dealer.

To sum up, in a centralized market, preferencing alters definitely the incentives of dealers to narrow quoted spreads because of the alteration of the reservation price of the preferred dealer. However the following questions are still open : what is the impact of such a practice on the expected market spreads, does preferencing necessarily lead to higher profits for dealer D_2 (remind that she faces a price-execution risk in matching the price of her opponent) ? Does it impair or not the expected profit of dealer D_1 who loses the opportunity to accommodate the preferenced trade compared with a 'competitive' situation ?

3.2 Market Performance and Preferencing

In order to analyze the impact of preferenced trade on the overall market performance, we use the competitive case, in which no preferencing is allowed, as a benchmark.

Best offer and preferenced order flow

In equilibrium, the Best Offer is : $\underline{a}^c = \max\left(a_{r,1}, a_{r,2}^{\kappa}\right)$. In order to measure the impact of preferencing agreement on execution costs, we turn to the analysis of the expected Best Offer.

[27] All the proofs are in the Appendix in the section dedicated to 'Bidding strategy characterization'.

Lemma 2 *The expected Best Offer denoted by $E(\underline{a}^c)$ worsens as the preferenced order flow increases ($\partial E(\underline{a}^c)/\partial\kappa > 0$). Moreover, the expected Best Offer under preferencing is larger than the one which would prevail in the competitive case (No Preferencing allowed) : $E(\underline{a}^c) > E(\underline{a}^{c,NP})$.*

Increasing the scale of preferenced order flow increases the best selling price. In a symmetric way, it will decrease the best bid price. Hence, preferencing widens the expected bid-ask spreads. Thus, preferencing in a fully transparent market leads to an increase in transaction costs for investors. This supports the point of view of Huang and Stoll (1997) who argue that the larger execution costs on Nasdaq relative to NYSE are at least partially due to preferencing.

Dealers' expected Profit and Preferencing

To gain some intuitions, preferencing may be decomposed into three effects in this model : (i) the price effect, (ii) the chance effect and (iii) the volume effect. The price effect is obviously linked to the previous lemma : preferencing increases the expected trading profit since it enlarges expected bid-ask spreads. Moreover, preferencing makes rising the ex ante probability to execute the unpreferenced order flow for the unpreferred dealer and decreasing the one for the preferred dealer, what we called the 'chance' effect. However, since the unpreferred dealer cannot compete on the captive order flow, he suffers from a loss in the total expected volume compared with the competitive case (the 'volume' effect).

Lemma 3 *1. The preferred dealer' s expected profit is always larger under preferencing arrangements, , i.e. $E(\Pi_2^c) > E(\Pi_2^{NP})$.*

2. Depending on the value of the parameters, there exist cases in which the unpreferred dealer surprisingly expects higher profits when his opponent is preferenced : $E(\Pi_1^c) \geq E(\Pi_1^{NP})$ when (i) $Q \geq (I_u - I_d)/3$ and (ii) when $Q < (I_u - I_d)/3$ and $\kappa \geq \underline{\kappa}(Q)$.

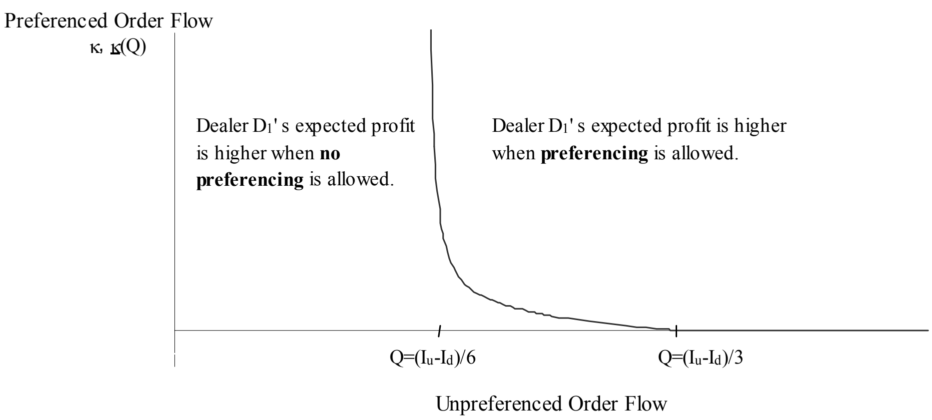


FIGURE 3 : A comparison of dealer D_1's expected profit

Preferencing increases the expected profit of the preferred dealer, even if she cannot control the price execution of the preferenced trade. In this centralized two-dealer market, there is no price-execution risk since the best offer is equal to the second best price and cannot be lower than the reservation price of the preferred dealer[28]. Dealer D_2 takes fully advantage of the price-matching rule as a source of rents.

Surprisingly, the expected profit of the unpreferred dealer may also be larger in the preferencing case than in the competitive case (see the previous numerical example for an illustration). Even if he is suffering from a truncated competition and a loss in trading volume, dealer D_1 may benefit from the increase in spreads (the price effect) and from a larger chance to execute the unpreferenced order flow (the chance effect). So, preferencing may create rents for all dealers.

These results show that preferencing can significantly affect (i) the market performance since it enlarges market spreads at investors' expense, (ii) it increases the preferred dealer's profit. These results provide a theoretical support to the experimental findings of Bloomfield and O'Hara (1998). Using laboratory financial markets, their research demonstrates that in a two-dealer market, increasing preferencing increases dramatically market spreads and enriches dealers at the expense of investors. However, they find also that these deleterious effects may be avoided when more than one dealer does not receive preferenced orders. We study whether this is the case in our framework in the next subsection. We first generalize the previous theorem to N dealers. Then we compute the best offer when one *unpreferred* dealer enters the two-dealer market ($N = 3$). Finally we turn to a study of the empirical implications of this model.

3.3 Extension

The previous setting at two dealers can easily be extended to N dealers.

Suppose that N dealers compete to execute a public (i.e. unpreferenced) order flow. Among the N dealers, M dealers have preferencing arrangements where $M \leq N$. It means that each of the M dealers receives a preferenced order flow large of κ_i shares where $\kappa_i \in [0, +\infty[\,, i = 1, ..., M$.

Following Lemma 1, each preferred dealer will not quote below one's cutoff price. The reservation price of a preferred dealer is given by $a^{\kappa}_{r,i} = \mu + \rho\sigma^2_v \left(Q - 2\left(I_i - \kappa_i\right)\right)/2$, $i = 1, ..., M$. The remaining $(M - N)$ dealers who do not get any preferenced order flow are characterized by the Ho and Stoll (1983)'s reservation price : $a_{r,i} = \mu + \rho\sigma^2_v \left(Q - 2I_i\right)/2$, $i = M + 1, ..., N$. Observe that the reservation price of an unpreferred dealer is simply equal to the reservation price of a preferred dealer whose preferenced order flow is zero since $a_{r,i} = a^{\kappa}_{r,i}$ when $\kappa_i = 0$ for $i = M + 1, ..., N$. Consequently, to ease the exposition of the results, we denote by $a^{\kappa}_{r,i}$ the reservation price of any dealer D_i for $i = 1, ..., N$[29].

[28] Analytically, the preferred dealer matches $\underline{a}^c = a^c_1 = a^{\kappa}_{r,2} - \varepsilon > a_r\left(I_2, \kappa\right)$.Q.E.D.

[29] Implicitly, when dealer D_i is not preferred, his reservation price is equal to the Ho & Stoll's one, i.e. $a^{\kappa}_{r,i} = a_{r,i}$ for

Corollary 1 *In a transparent market where a part of the total order flow is preferenced to $M \leq N$ dealers, the dealer with the lowest reservation price $\left(\min_{i\in[1;N]} a^{\kappa}_{r,i}\right)$, denoted by D_T, posts the best price and executes the unpreferenced part of the order flow. At equilibrium, the best-quoting dealer undercuts the second-lowest reservation price and the $(N-1)$ other dealers quote their own reservation price, i.e.*

$$\begin{aligned} a_T &= \min_{i\in[1;N]\setminus\{T\}} a^{\kappa}_{r,i} - \varepsilon \\ a_i &= a^{\kappa}_{r,i} \end{aligned}$$

for $i \in [1;N]\setminus\{T\}$.

Notice that the best-quoting dealer is not necessarily the dealer with the most extreme inventory position. Remind indeed that preferenced order flows may be interpreted as inventory shocks occuring at date 2. In other words, at that time, the ranking of the *effective* inventory position $(I_i - \kappa_i)_{i\in N}$ of the dealers determine the ranking of dealers' reservation prices $\left(a^{\kappa}_{r,i}\right)_{i\in N}$ which yields the outcome of the quote-competition between dealers at date 3. Hence, the best-quoting dealer is the dealer with the following inventory position $(I_T - \kappa_T) = \max_{i\in N}(I_i - \kappa_i)$ which is not necessarily the dealer who was the longest at date 1.

Even if this Corollary is a straightforward generalization of Theorem 1, it allows us to examine how preferencing affects the market competitiveness when more than one dealer is unpreferred. Secondly, this theorem is useful to make a prediction about the relationship between inventories, quotes and preferenced order flow.

3.3.1 The Expected Best Offer in a Three-dealer Market

Now, we assume that the number of dealers in the market is $N = 3$. In this setting, dealer D_2 receives a preferenced trade ($\kappa_2 > 0$) whereas the two remaining dealers have no preferencing agreements ($\kappa_1 = \kappa_3 = 0$).

Lemma 4 *When the number of unpreferred dealers goes from one to two, expected market spreads narrow.*

In a three-dealer market, the additionnal dealer without preferenced order flow reinforces competition for the public order flow Q among unpreferred dealers. This competition effect decreases the best ask price on average. Symmetrically, it would increase the best bid price on average. Thus, expected market spreads narrow. In fact, the additional unpreferred dealer D_3 provides a competitive force that restores unpreferred dealers' incentives to narrow market spreads in order to attract the unpreferenced order flow. This result is consistent with the experimental finding of Bloomfield and O'Hara (1998) described above

$i = M+1, ..., N$.

and with the intuition of Kandel and Marx (1997). The latter state that preferencing should not change market spreads as long as the marginal dealer has no preferenced order flow.

Figure 4 displays how the expected best offer is improved (lowered) when the number of unpreferred dealers goes from one to two.

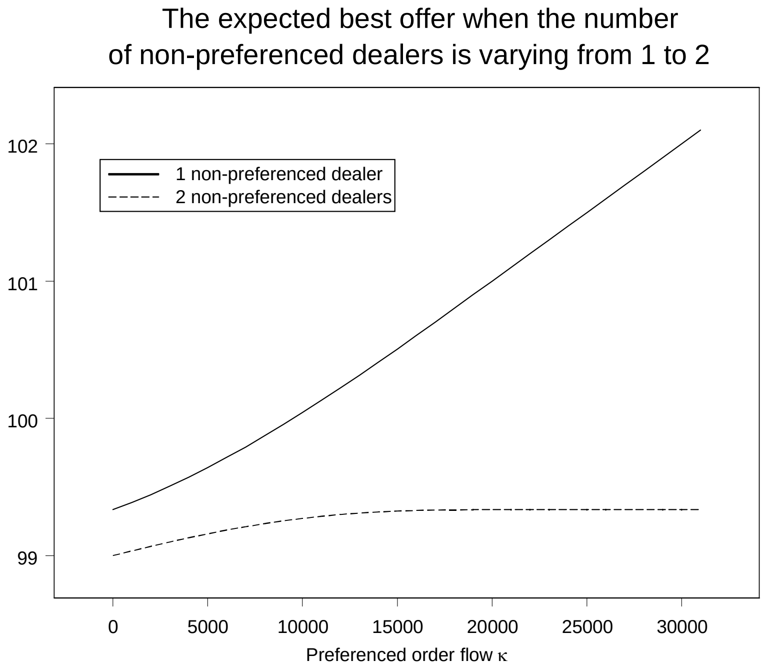


FIGURE 4 : Expected best offers, κ varying

Parameters are such that :

$\mu = 100\$$, $\sigma_v^2 = \frac{1}{10{,}000}$; $Q = 2,500$ shares; $I_d = 0$ and $I_u = 20,000$ shares (i.e. $a_{r,u} = 98\$$ and $a_{r,d} = 100\$$).

3.3.2 Empirical Implications

Theorem 1 predicts that in presence of preferenced trades, it is not necessarily the *longest* dealer who posts the best quote. This result invalidates partially the literal prediction of Ho and Stoll (1983)'s model. The aim of this paragraph is to investigate the link between quoted prices, inventories and preferenced order flow.

The link between inventories and best quotes

As we mention at the beginning of this section, Ho and Stoll (1983) show that the dealer with the most extreme inventory posts the best price and should consequently execute the *public* trades. In Ho and Stoll (1983), dealers' quotes can be expressed as a monotone function of their initial inventory positions. Hansh et al. (1998) deduce that there exists a simple relationship between the relative positionning of dealers' quotes and their relative inventory level. They express this link as follows

$$a_i - \underline{a}^c = \mathcal{F}\left(I_i - I_T\right) \qquad (E1)$$

where the position of the quote a_i posted by dealer D_i relative to the best market price ($\underline{a}^c$) quoted by the longest dealer D_T depends monotonically (though the decreasing function $\mathcal{F}$) on the difference between the level of his inventory I_i relative to that of the best-quoting dealer I_T .

Testing the previous equation on a dataset from the London Stock Exchange, Hansh et al. found that the dealers with extreme inventory position execute only 59% of the incoming public orders and not 100% as predicted by Ho and Stoll (1983). They conclude that this deviation from the Ho and Stoll's prediction could be explained by the practice of order flow preferencing. Preferencing arrangements abound on the LSE (71 % of all trades) and the impact of this market practice is not taken into account in the Equation $E1$.

Is there a link between preferenced order flows, quotes and inventories ?

As showed in the Theorem 1, unpreferred trades may be executed by dealers with an inventory position at some distance from the longest inventory because of preferencing ($(I_T - \kappa_T)$ is the effective inventory position to consider for the best-quoting dealer). More explicitly, given the Theorem 1, the link between inventories, best quotes and preferencing could be expressed as follows :

$$a_i - \underline{a}^c = \mathcal{F}\left((I_i - \kappa_i) - (I_T - \kappa_T)\right) \qquad (E2)$$

where κ_i and κ_T are respectively the preferenced trade executed by dealer D_i and that executed by the best-quoting dealer D_T. Our model suggests indeed that inventories should be shortened by the scale of the preferenced trades in order to test a relation between the positioning of quotes, the level of dealers' inventories and the preferencing practice.

4 Preferencing in a Fragmented Market

In a fragmented market as the Nasdaq or the London Stock Exchange, dealers' bidding behavior will differ from the quoting behavior they would adopt in a centralized market since the information available is not the same. In this market structure, dealers cannot observe the inventory positions of their opponent. Actually, the preferred dealer only forms an expectation on the best price at which she could be constrained to execute the preferenced trade in case of she does not post the best price. Does the preferred dealer take advantage of this lack of transparency ? What is the impact on the bidding behavior of her opponent ?

First we characterize the equilibrium bidding behaviors of dealers in fragmented market. Then, we analyze the expected bid-ask spreads and the expected dealers' profit and we compare them to those get in a centralized market.

In this section, we analyze how a preferenced order flow interacts with the quotation strategies of the two dealers in a market where competitors' reservation prices are not observed. A Bayes-Nash equilibrium is a couple $(a_i(.), a_{-i}(.))$ such that the quote function $a_i(.)$ is a best reply to the bidding strategy of the opponent $a_{-i}(.)$ where $-i$ denotes the opponent. This means that dealer D_1 sets a price y so as to maximize his expected profit Π_1 given that the best reply of his opponent is a_2 :

$$\Pi_1(y, a_{r,1}) = \Pr(y < a_2) \times (y - a_{r,1}) \times Q \tag{1}$$

and given that dealer D_1's best response is a_1, dealer D_2 sets y so as to maximize :

$$\begin{aligned}\Pi_2(y, a_r(I_2, Q+\kappa), a_r(I_2,\kappa)) &= \Pr(y < a_1) \times (y - a_r(I_2, Q+\kappa)) \times (Q+\kappa) \\ &\quad + \Pr(y > a_1) \times (E(a_1 \mid y > a_1) - a_r(I_2,\kappa)) \times \kappa\end{aligned}$$

The latter expression can be written :

$$\begin{aligned}\Pi_2\left(y, a_{r,2}^{\kappa}\right) &= \Pr(y < a_1) \times \left(y - a_{r,2}^{\kappa}\right) \times (Q+\kappa) + \Pr(y > a_1) \times \left(E(a_1 \mid y > a_1) - a_{r,2}^{\kappa}\right) \times \kappa \\ &\quad + \frac{\rho\sigma_v^2}{2}\kappa \times (Q+\kappa)\end{aligned} \tag{2}$$

Consequently the expected profit of the preferred dealer may be written as some function of the unique cutoff price $a_{r,2}^{\kappa}$, without any explicit reference to other reservation prices : $a_r(I_2, Q+\kappa)$ or $a_r(I_2,\kappa)$. Once dealer D_2 knows her inventory and her preferenced trade, she never posts a selling price below her cutoff price. As in a centralized market structure (see Section 3), the cutoff price plays the role of the effective reservation price of dealer D_2. The optimal ask price posted by the preferred dealer should increase with the volume of the preferenced order flow $+\kappa$, since the cutoff price is doing so. Because the optimal quote submitted by dealer D_1 is a best reply to the bidding strategy of dealer D_2, selling prices should intuitively rise with the magnitude of the preferenced trade κ.

Technically, prices arising in this context correspond to those arising in a Dutch auction or, equivalently, in a first-price auction (FPA). In our set up (unknown reservation prices and preferencing), the equilibrium quotation strategies are quite complex, because dealers' expected profits from trade are different and because the supports of dealers' reservation prices are not identical. Indeed, the distribution'support for the reservation price of the preferred dealer is 'shifted' to the right compared with the distribution'support for dealer D_1's reservation price. That is, the reservation price of the unpreferred dealer D_1 is distributed uniformly on $[a_{r,u}, a_{r,d}]$ whereas the preferred dealer's cutoff price is distributed uniformly on $\left[a^{\kappa}_{r,u}, a^{\kappa}_{r,d}\right] = \left[a_{r,u} + \rho\sigma^2_v\kappa, a_{r,d} + \rho\sigma^2_v\kappa\right]$. Consequently, we distinguish F the uniform cumulative distribution function (c.d.f.) of dealer D_1' s reservation price on $[a_{r,u}, a_{r,d}]$ from F_κ the uniform c.d.f. of dealer D_2' s cutoff price on $\left[a^{\kappa}_{r,u}, a^{\kappa}_{r,d}\right]$. To sum up, preferencing creates a double asymmetry : (i) reservation prices are asymmetrically distributed ; (ii) expected profit functions are also asymmetric (see Equations (1) and (2)). It is well-known that these asymmetries preclude analytical solutions for FPA (see Lebrun (1999), Castillon (2000) and Maskin and Riley (2000)). Thus we use a numerical approach to derive equilibrium bidding strategies. In our setting, it is, however, possible to characterize analytical equilibrium strategies in two cases : (i) when the preferenced order flow is so large that the preferred dealer has no chance to port the best price at equilibrium ($\kappa \geq 2(I_u - I_d)$) and (ii) in the competitive benchmark where no preferencing is allowed (NP).

We begin by characterizing the general case where the equilibrium bidding strategies are numerically investigated, then we turn to the determination of the analytical equilibrium solutions obtained when preferencing is large and when it is not allowed.

Note that the numerical results illustrated below are computed under the following values for parameters : $\rho = 1$, $\mu = 100\$$, $\sigma^2_v = \frac{1}{10{,}000}$; $Q = 2{,}500$ shares; $I_d = 0$ and $I_u = 20{,}000$ shares. Hence, $a_{r,u} = 98\$$ and $a_{r,d} = 100\$$.

4.1 Preferencing and Equilibrium Quotes

We now turn to the detailed analysis of the Bayes-Nash equilibrium that consists of a pair of selling quote functions : $a_1 : [a_{r,u}, a_{r,d}] \to \mathbb{R}$, $a_2 : \left[a^{\kappa}_{r,u}, a^{\kappa}_{r,d}\right] \longrightarrow \mathbb{R}$. We assume that a_i are strictly increasing functions (see Lebrun (1999) for formal proofs). Then we can define the inverse bidding functions, which are more convenient to analyze. Consequently, we denote $v_1(y)$ and $v_2(y)$ the reservation prices drawn respectively by dealer D_1 and dealer D_2, that lead them to quote y. Note that $v_1 = (a_1)^{-1}$ and $v_2 = (a_2)^{-1}$.

Using the inverse functions, the dealers' profit expressions given by equations (1) and (2) write also :

$$\Pi_1(y, a_{r,1}) = \bar{F}_\kappa(v_2(y)) \times (y - a_{r,1}) \times Q \tag{3}$$

where $\bar{F}_\kappa$ is the survivor function[30] : $\bar{F}_\kappa = 1 - F_\kappa$; and

$$\begin{aligned}\Pi_2\left(y, a^{\kappa}_{r,2}\right) &= \bar{F}(v_1(y)) \times \left(y - a^{\kappa}_{r,2}\right) \times (Q + \kappa) \\ &\quad + \left(1 - \bar{F}(v_1(y))\right) \times \left(E(a_1(a_{r,1}) \mid y > a_1(a_{r,1})) - a^{\kappa}_{r,2}\right) \times \kappa \\ &\quad + \frac{\rho\sigma^2_v}{2}\kappa \times (Q + \kappa).\end{aligned} \tag{4}$$

where $\bar{F} = 1 - F$.

[30] $\bar{F}_\kappa(y)$ is the probability that dealer D_2 bids at least y.

4.1.1 Case 1 : the Equilibrium when Preferencing is Small $\kappa < 2(I_u - I_d)$

Dealers' bidding strategies have the same support $\left[a^{\inf}, a^{\sup}\right]$. Notice that, on this support, both dealers have a strictly positive probability to execute the nonpreferenced order flow.

The lower bound $a^{\inf}$ is the smallest possible ask price quoted by a dealer and it is defined such that $\bar{F}_\kappa\left(v_2\left(a^{\inf}\right)\right) \times \bar{F}\left(v_1\left(a^{\inf}\right)\right) = 1$. Intuitively, if dealer D_1 should post a smaller lower price than dealer D_2 ($a_1^{\inf} < a_2^{\inf}$), then he could quote any price $a_1 \in \left[a_1^{\inf}, a_2^{\inf}\right[$ and be sure to post the best price. However, this strategy is strictly dominated by $\left(a_1 + a_2^{\inf}\right)/2$. Hence, it cannot be an equilibrium by elimination of iterated dominated strategy (the same holds in case when $a_1^{\inf} > a_2^{\inf}$). As a result, dealers' best reply must have the same lower bound $a^{\inf}$.

The upper bound $a^{\sup}$ is the largest possible ask price quoted by a dealer who has a strictly positive probability to execute the unpreferenced order flow. This upper bound is defined such that $\bar{F}_\kappa\left(v_2\left(a^{\sup}\right)\right) \times \bar{F}\left(v_1\left(a^{\sup}\right)\right) = 0$. Using the same argument as before, we conclude that dealers must quote no more than the largest possible ask price to get a chance to execute the unpreferenced order flow.

Theorem 2 *Assume that both dealers have a chance to post the best price ($\kappa < 2(I_u - I_d)$).*
(i) The equilibrium inverse bidding functions v_1 and v_2 are solutions to the following pair of differential equations :

$$\begin{aligned} \frac{-\bar{F}_\kappa'\left(v_2(y)\right)}{\bar{F}_\kappa\left(v_2(y)\right)} \times v_2'(y) &= \frac{1}{y - v_1(y)} \qquad (5) \\ \frac{-\bar{F}'\left(v_1(y)\right)}{\bar{F}\left(v_1(y)\right)} \times v_1'(y) &= \frac{(1+\kappa/Q)}{y - v_2(y)} \qquad (6) \end{aligned}$$

(ii) If $a_{r,d} \leq a^{\sup} \leq \frac{a_{r,d} + a_{r,d}^\kappa}{2}$, there exists an equilibrium.
When $a_{r,2}^\kappa > a^{\sup}$ dealer D_2 can never post the best price and she quotes her cutoff price : $a_2 = a_{r,2}^\kappa$.

Notice that (i) the equilibrium is not necessarily unique and that (ii) the lower bound $a^{\inf}$ is endogenously determined by the upper bound $a^{\sup}$ and $\bar{F}_\kappa\left(v_2\left(a^{\inf}\right)\right) = \bar{F}\left(v_1\left(a^{\inf}\right)\right) = 1$. Observe also that, at bounds, the bidding functions must be equal to :

$$\begin{aligned} a_1\left(a_{r,u}\right) &= a^{\inf}, \quad a_1\left(a_{r,d}\right) = a^{\sup}, \\ a_2\left(a_{r,u}^\kappa\right) &= a^{\inf}, \quad a_2\left(a^{\sup}\right) = a^{\sup}. \end{aligned}$$

Indeed, given that $a_{r,d}^\kappa > a_{r,d}$, then the upper bound is defined such that $\Pr\left(a_1 > a^{\sup}\right) = 0$, or equivalently, $\bar{F}\left(v_1\left(a^{\sup}\right)\right) = 0$. Then, if v_2 is the best reply of dealer D_2, it must verify the equation (5) $-\bar{F}'\left(v_1\left(a^{\sup}\right)\right) \times v_1'\left(a^{\sup}\right)\left(a^{\sup} - v_2\left(a^{\sup}\right)\right) = 0$, or $a^{\sup} = v_2\left(a^{\sup}\right)$.

Among the multiplicity of equilibria, we use the Pareto-dominance criterion to select one of them. This criterion is defined as follows : the equilibrium denoted by the subscript $^{(2)}$ is Pareto-dominant under the initial conditions$^{(2)}$ if for each equilibrium $^{(1)}$ under other initial conditions$^{(1)}$, both following inequalities hold :

$$\begin{aligned}
\Pi_1^{(1)}\left(a_1\left(a_{r,1}\right), a_{r,1}\right) &< \Pi_1^{(2)}\left(a_1\left(a_{r,1}\right), a_{r,1}\right) \text{ for each } a_{r,1} \in \left[a_{r,u}, a_{r,d}\right], \\
\Pi_2^{(1)}\left(a_2\left(a_{r,2}^{\kappa}\right), a_{r,2}^{\kappa}\right) &< \Pi_2^{(2)}\left(a_2\left(a_{r,2}^{\kappa}\right), a_{r,2}^{\kappa}\right) \text{ for each } a_{r,2}^{\kappa} \in \left[a_{r,u}^{\kappa}, a_{r,d}^{\kappa}\right].
\end{aligned}$$

Proposition 1 *The unique Pareto-Dominant equilibrium is obtained when the initial condition is such that* $a^{\text{sup}} = (a_{r,d} + a_{r,d}^{\kappa})/2$.

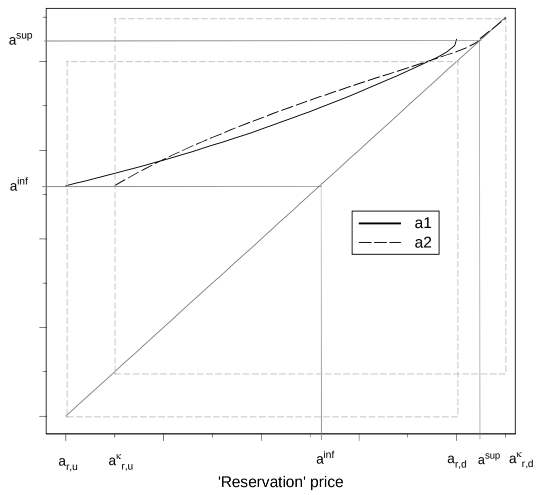


FIGURE 5 : An illustration of quotes at equilibrium

It is worth stressing two facts about the equilibrium described by the system of the ordinary differential equations (4) and (5) and by the initial condition of Proposition 1. It is impossible to get an analytical solution to this asymmetric equilibrium (at least we have not been able to find one). Second, given that preferencing makes dealers asymmetric, they will in general have different bidding strategies. We will further analyze *numerical* solutions of the ODE system. However, there are two cases in which we can dispense from numerical solutions (i) when preferencing is so large that the preferred dealer cannot post the best price $(\kappa \geq 2(I_u - I_d))$ (ii) when no preferenced order flow is allowed as in a competitive situation.

4.1.2 Case 2 : the Equilibrium when Preferencing is Large $(\kappa \geq 2(I_u - I_d))$

When $\kappa \geq 2(I_u - I_d)$, Theorem 2 does not apply. However, we can characterize the equilibrium in closed form.

Proposition 2 *Assume that dealer D_2 can never post the best price at equilibrium ($a^{\kappa}_{r,u} \geq a^{\sup}$ or $\kappa \geq 2(I_u - I_d)$). In this case, she quotes a selling price equal to her reservation price : $a_2 = a^{\kappa}_{r,2}$, and dealer D_1 posts $a_1 = a^{\kappa}_{r,u}$.*

In this case, the portion of the captive order flow is so large that it precludes any kind of competition between dealers. When the unpreferred dealer does not quote more than the lowest price posted by the preferred dealer, he is sure to post the best price. In this case, dealer D_1 posts the best offer with probability 1. In other words, he is not in competition with anyone to execute the unpreferenced trade. Thus, he has no incentive to improve the best offer, as predicted by Stoll (2001) : 'if all order flow were preferenced, no market-maker would have an incentive to narrow the spread to attract orders'.

A link between the equilibrium when preferencing is small (Case 1) and the equilibrium when preferencing is large (Case 2)

As it is proved in the appendix (Theorem 2 and Proposition 1), the initial condition on the upper bound $a^{\sup}$ determines the equilibrium (the lower bound $a^{\inf}$ is indeed some function of $a^{\sup}$). Given that the upper bound $a^{\sup}$ increases when preferencing increases, $a^{\inf}$ is also varying with preferencing as Figure 6 depicts. When $\kappa = 2(I_u - I_d)$, then the equilibrium defined in Case 1 degenerates ($a^{\inf} = a^{\sup} = a^{\kappa}_{r,u}$) and it is now characterized analytically by Proposition 2 (Case 2).

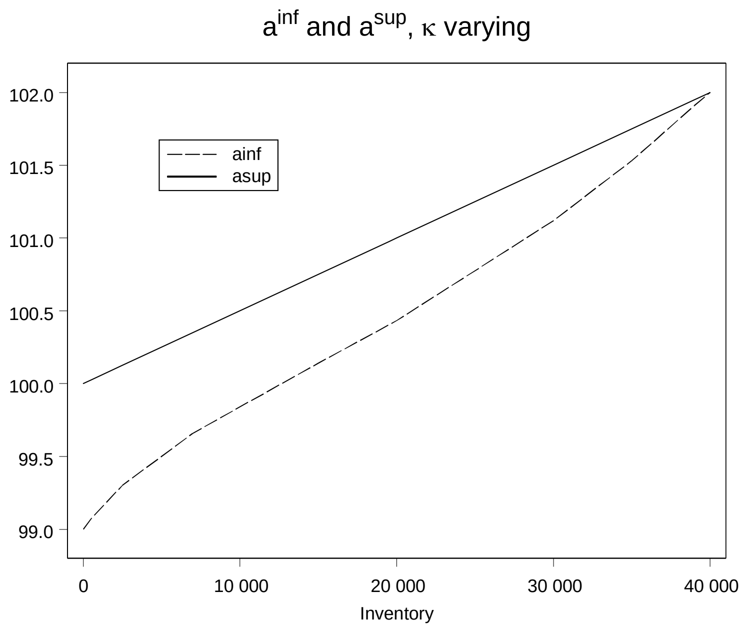


FIGURE 6 : Evolution of the quotes' support $[a^{\inf}, a^{\sup}]$, κ varying.

4.1.3 The Competitive Case (Biais, 1993)

Now, we turn to the characterization of the competitive equilibrium. In this case when dealers are not allowed to receive any preferenced order flow, the dealers draw their reservation price, $a_r(I_i, Q+\kappa)$, from the same probability distribution F on the common support $[a_r(I_u, Q+\kappa), a_r(I_d, Q+\kappa)]$. Consequently, dealers are symmetric when there is no preferencing, that is $v_1 = v_2 = v$ and $a_1 = a_2 = a_{NP}$.

Then, the system of ODE described in Theorem 2 (equations (5) and (6)) simply writes :

$$\frac{-\bar{F}'(v(y))}{\bar{F}(v(y))} \times v'(y) = \frac{1}{y - v(y)}, \tag{7}$$

subject to the following boundary conditions :

$$a_{NP}^{\inf} = \frac{a_r(I_u, Q+\kappa) + a_r(I_d, Q+\kappa)}{2} \quad \text{and} \quad a_{NP}^{\sup} = a_r(I_d, Q+\kappa). \tag{8}$$

It is easy to verify that the symmetric equilibrium characterized by the ordinary differential equation (7) and by the initial conditions (8) is unique. Furthermore, there exists an analytical solution, which is identical to the equilibrium described in Biais (1993, Corollary 1). Dealers post sell quotes which are equal to the sum of their reservation price and a mark-up : $a_{NP}(a_r(I_i, Q+\kappa)) = a_r(I_i, Q+\kappa) + \gamma(a_r(I_i, Q+\kappa)), i = 1, 2$. This quoting strategy shows that dealers post an ask price strictly above their reservation price. The mark-up $\gamma(a_r(I_i, Q+\kappa))$ allows them to make non zero profit.

In this symmetric case, the sell quotes and the mark-up are *linear* in the reservation price, as follows :

$$\begin{aligned} a_{NP}(a_r(I_i, Q+\kappa)) &= \frac{a_r(I_i, Q+\kappa) + a_r(I_d, Q+\kappa)}{2}, \\ \gamma(a_r(I_i, Q+\kappa)) &= \frac{a_r(I_d, Q+\kappa) - a_r(I_i, Q+\kappa)}{2} \geq 0. \end{aligned}$$

This mark up also writes :

$$\gamma(a_r(I_i, Q+\kappa)) = E\left[a_r(I_{-i}, Q+\kappa) - a_r(I_i, Q+\kappa) \mid a_r(I_{-i}, Q+\kappa) - a_r(I_i, Q+\kappa) > 0\right].$$

Actually, dealer D_i estimates how far upper his own reservation price the opponent's reservation price is on average and he submits a selling price equal to this amount. The dealer who executes the incoming order flow is the agent with the most extreme inventory. Note that this result is consistent with the prediction of Ho and Stoll (1983)'s model since, without preferencing, the dealer who has the lowest reservation price is also the longest and he posts the best price at equilibrium.

Because the equilibrium is symmetric, both dealers quote with an equal aggressiveness :

$$\theta_i = \left(a_r(I_d, Q+\kappa) - a_r(I_i, Q+\kappa)\right) / 2a_r(I_i, Q+\kappa).$$

This result is consistent with the fact that when both dealers post a selling quote equal to y they have the same probability to execute the incoming order flow $+(Q+\kappa)$, which is

$$\Pr(y < a_{-i}) = 2(a_r(I_d, Q+\kappa) - y) / (a_r(I_d, Q+\kappa) - a_r(I_u, Q+\kappa))\ (i = 1, 2).$$

4.2 The Impact of Preferencing on the Quotes Placement

In order to analyze how preferencing agreements alter the way to bid of dealers, we present first a numerical investigation on (i) the probability to post the best price and (ii) the quoting aggressiveness. Then, we explain qualitatively the numerical results obtained.

4.2.1 Preferencing and Bidding Strategy of the Unpreferred Dealer

The primary concern raised by opponents of preferencing (see Dutta and Madhavan 1994 or Kandel and Marx 1997) is that this practice reduces price competition because the preferred dealer does not have enough incentives to narrow spreads given her captive order flow. This model shows that preferencing reduces the incentives of the *unpreferred* dealer to compete aggressively to attract the unpreferenced order flow.

As numerical results illustrate, the probability that the unpreferred dealer executes the unpreferenced order flow increases with the magnitude of the preferenced order flow (Figure 4) since it insulates more the preferred dealer from competition (as κ rises, dealer D_2 has less chance to draw a low cutoff price and less chance to post the best price). As the probability that dealer D_1 executes the unpreferenced trade rises, D_1 bids less aggressively (Figure 7). As a result, the highest selling price posted by dealer D_1 is obtained when the preferenced trade is qualified as 'large' (Case $2: \kappa \geq 2(I_u - I_d)$)[31], whereas the lowest quote that dealer D_1 may post is obtained when there is no preferenced order flow ($\kappa = 0$).

[31] Note that, in this case, the selling quote posted by dealer D_1 is also the market price.

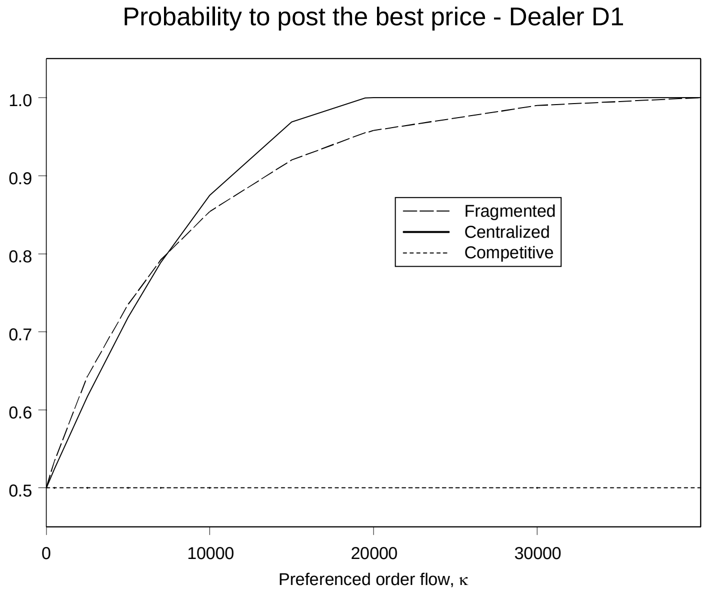


FIGURE 7

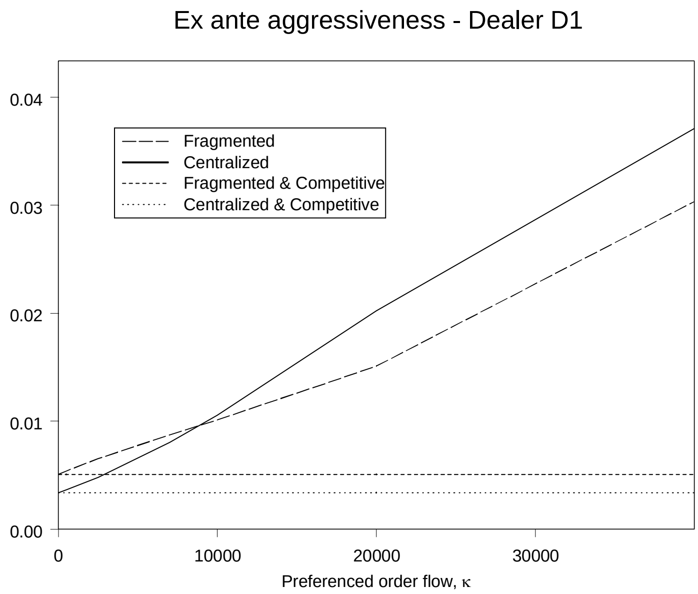


FIGURE 8

4.2.2 Preferencing and Bidding Strategy of the Preferred Dealer

Now, we turn to the analysis of the bidding behavior of the preferred dealer. As the scale of preferencing enlarges, dealer D_2 is less likely to draw a low reservation price and she has less chance to post the market price (Figure 9). Intuitively, because of her lower probability to execute the unpreferenced trade, dealer D_2 should post more aggressive prices. Surprisingly, the quoting aggressiveness of dealer D_2 is *not monotonous* with the scale of preferenced order flow. For instance, let us suppose that her inventory position is 15,000 shares, then her quoting aggressiveness is $\theta_2\left(a^{\kappa}_{r,2}\right)=0.75$ when $\kappa=0$, $\theta_2\left(a^{\kappa}_{r,2}\right)=0.79$ when $\kappa=500$ and $\theta_2\left(a^{\kappa}_{r,2}\right)=0.83$ when $\kappa=2,500$. However, θ_2 decreases to 0.72 when $\kappa=7,000$ (Figure 10).

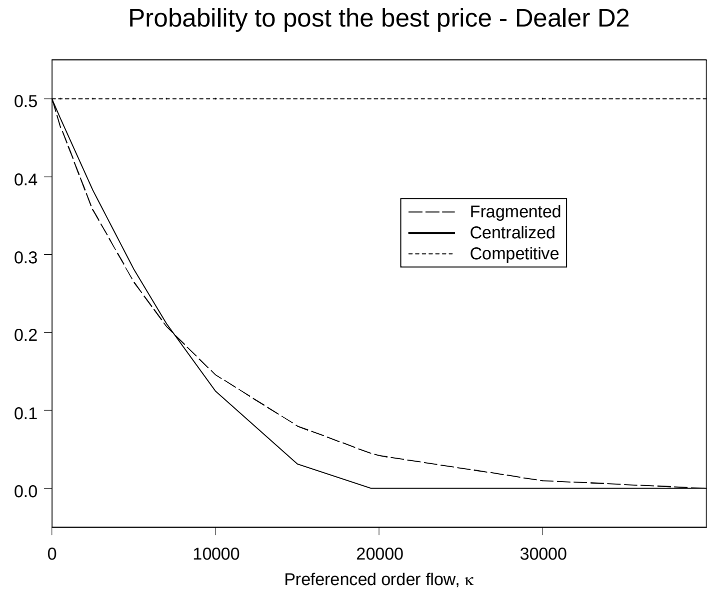


FIGURE 9

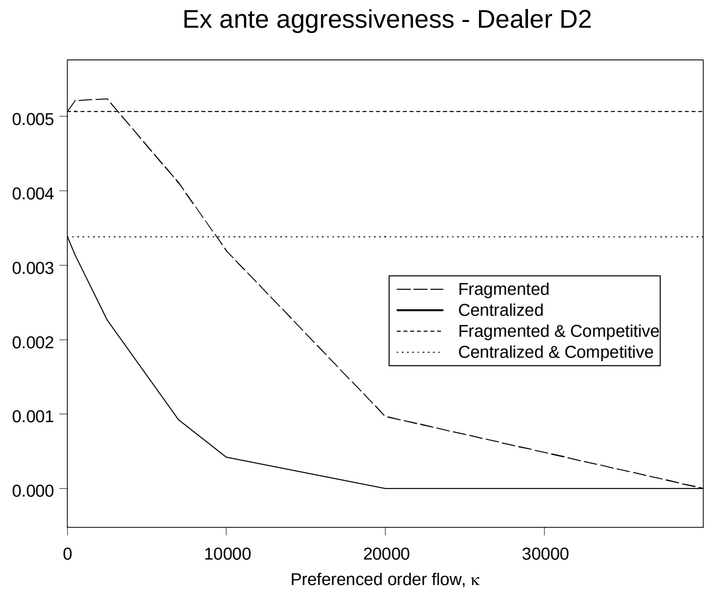


FIGURE 10

Actually, the preferenced order flow creates two types of asymmetry which generate opposite bidding behaviors of the preferred dealer :

(i) on one side, it forces her to bid more aggressively due to her lower probability to win the non preferenced order flow (her reservation price is distributed over an interval which is shifted to the right : dealer D_2' s reservation price is most likely higher than one's opponent) ;

(ii) on the other side, the price matching practice creates a rent for the preferred dealer that destroys her incentive to compete in prices.

To analyze these opposite forces, we make a distinction between the asymmetry in the supports of their reservation prices and the asymmetry in terms of payoff created by the preferenced trade.

EFFECT 1 : Let us suppose that we remove the payoff generated by the execution of the captive order flow and restrict the expected profit of dealer D_2 to be : $\Pi_2\left(y, a_{r,2}^{\kappa}\right) = \bar{F}\left(v_1\left(y\right)\right) \times \left(y - a_{r,2}^{\kappa}\right) \times Q$.

In this one-kind asymmetric situation, only the supports of reservation prices would differ (they are distributed over different intervals). Remind that the reservation price of dealer D_2 is the cutoff price that is distributed on $\left[a_{r,u} + \rho\sigma_v^2\kappa, a_{r,d} + \rho\sigma_v^2\kappa\right]$. As mentionned, this support is 'shifted' to the right compared with the distribution support of dealer D_1 's reservation price on $[a_{r,u}, a_{r,d}]$. Consequently, dealer D_2 has less chances to draw a low reservation price and less chance to execute the unpreferenced order flow than dealer D_1. In this type of asymmetry, the condition of Conditional Stochastic Dominance[32] used by Maskin and Riley (2000) applies and it proves[33] that dealer D_2 competes *more* aggressively than dealer D_1. That dealer quotes indeed less aggressively since he is the most likely supplier of the unpreferenced order flow.

EFFECT 2 : Now, let us analyze the asymmetry created exclusively by the payoff function. To do so, we analyze the ODE system in restricting the distributions' supports of $a_{r,1}$ and $a_{r,2}^{\kappa}$ to be on the same interval $[a_{r,u}, a_{r,d}]$. Actually, the payoff of dealer D_2, which still takes into account the execution of the preferenced order flow creates an asymmetric situation that is equivalent to a preferred dealer who would be *risk-lover*, facing a risk-neutral dealer D_1. The guarantee to execute a captive order flow in matching the best price induces dealer D_2 to post *less aggressive* quotes. Hence, her bidding aggressiveness decreases as the volume of preferenced shares becomes larger (see proofs and an illustration of this phenomenon (Figure A1) in Appendix 7.10 'Comments on Effect 2').

To sum up, the preferenced order flow changes the supports of dealers' reservation prices (EFFECT 1) and also the distribution of the probability function, so it changes the degree of price-competition

[32] $\bar{F}'/\bar{F} > \bar{F}_{\kappa}'/\bar{F}_{\kappa}$

[33] See Maskin and Riley (2000) for the case of two dealers (equivalently, two *bidders*).

between dealers (EFFECT 2). Unlike the previous works related to asymmetric auctions, this paper mixes two kinds of asymmetry which generate ambiguous bidding behavior for the preferred dealer. Moreover, the combination of both asymmetries invalidates any condition related to the conditional stochastic dominance. Then it is not easy to compare analytically dealers' bidding behavior as in Maskin and Riley (2000). It explains however the puzzling quoting behavior of dealer D_2, whose quoting aggressiveness is not monotonous with the preferenced order flow. In conclusion, numerical examples indicate that even if the preferenced order flow has no clear impact on dealers D_2 's incentive to compete on the quoted prices (Effect 2 numerically dominates Effect 1 only for small preferenced order flow), it deletes however her competitor's incentive to set narrower spreads (actually, we do not find numerical examples that invalidate this result).

4.3 Comparisons with a Centralized Market

Now, we analyze how preferencing alters market spreads in a fragmented market compared with spreads in a centralized market, before turning to the analysis of dealers' profit in both market structures.

4.3.1 The Expected Best Offer

Result 1 *Under preferencing, the expected best offer in a centralized market differs from the best offer arising in a fragmented market. When preferencing is large ($\kappa \geq 2\left(I_u - I_d\right)$), a fragmented market offers better market prices than a centralized market, i.e.*

$$E\left(\underline{a}\right) < \frac{a^{\kappa}_{r,d} + a^{\kappa}_{r,u}}{2} = E\left(\underline{a}^{c}\right)$$

where $E\left(\underline{a}\right) = E\left(a_1\right) = a^{\kappa}_{r,u}$.

When no preferencing is allowed (the competitive benchmark of this model), it is shown that the expected best offer in a centralized market and in a fragmented market are the same[34]. However, under preferencing agreement, the rising of asymmetries invalidates this result as illustrated below (Figure 11). This result is also a well-known result in auction theory : asymmetries prevent the 'revenue-equivalence theorem' to prevail and the equality of best offers accross the different market structures cannot hold any more in our model.

[34] This result comes from the well-known 'revenue-equivalence theorem' obtained in the theory of auction as Biais (1993) explains in the Proposition 4 of his model.

A comparison of best offers

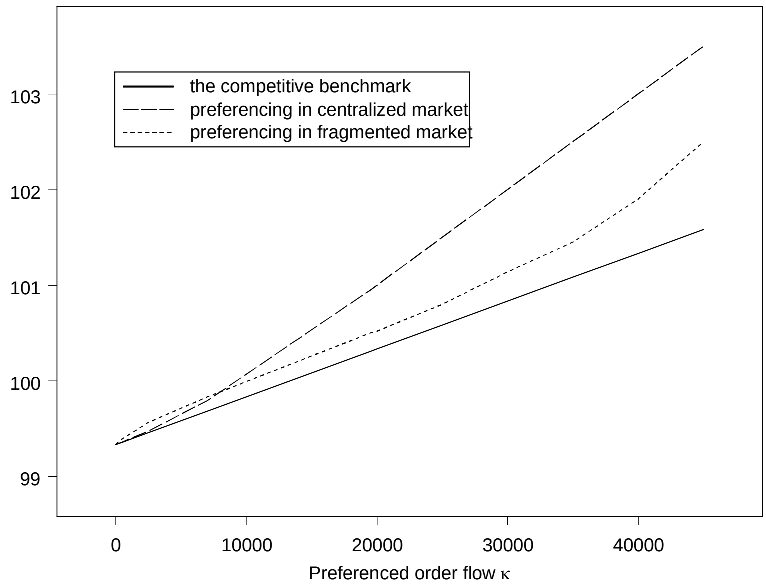


FIGURE 11[35]

Remind that preferencing deteriorates the best offer in a centralized market. This result is also verified numerically in a fragmented market. Numerically, we can show that the impact of large preferenced order flow harms more centralized market than fragmented market. The impact of small preferenced order flow is more ambiguous. For small preferenced order flow, the distribution effect (EFFECT 2) dominates the support effect (EFFECT 1), dealer D_2 competes *less* aggressively even if she has less chance to execute the unpreferenced order. Since it is expected by dealer D_1, he has also less incentives to narrow spreads. This intuition could explain why a fragmented market suffers more from small preferenced order flow than a centralized market.

4.3.2 Preferencing, Market Structure and Dealers' expected profit

A. The unpreferred dealer's expected profit

Result 2 *In a two-dealer market when preferencing is large ($\kappa \geq 2(I_u - I_d)$) the expected profit of the unpreferred dealer is higher in a centralized structure than in a fragmented market,*

$$E(\Pi_1) = \left(\rho\sigma_v^2\kappa - \frac{(a_{r,d} - a_{r,u})}{2}\right) \times Q \leq \rho\sigma_v^2\kappa \times Q = E(\Pi_1^c).$$

When preferencing is small ($\kappa < 2(I_u - I_d)$), it is still numerically validated.

[35] Whether the 'competitive' market is centralized or fragmented, remind that the expected best offers are equal : $E(\underline{a}_{NP}) = E(\underline{a}_{NP}^c)$ (Revenue-equivalence theorem).

In a fragmented market, the unpreferred dealer takes less advantage of the widening of spreads since market spreads are expected to be smaller than those in a centralized market since, apart from small preferenced order flow (Result 1). Moreover, when preferencing is large ($\kappa \geq 2(I_u - I_d)$), the unpreferred dealer posts the best price which, in a centralized market, is such that $a^{\kappa}_{r,u} \leq \underline{a}^c \leq a^{\kappa}_{r,d}$, whereas, in a fragmented market, he too posts the best offer $\underline{a} = a_1 = a^{\kappa}_{r,u}$, which is however lower than in a centralized market. Consequently, in this case, a centralized market generates a higher expected profit for the unpreferred dealer than a fragmented setting.

Moreover, even if the unpreferred dealer cannot get any preferenced shares, there still exist some parameters values for which the expected profit of the unpreferred dealer is higher under preferencing than in a competitive situation. However, in a fragmented market, the positive effects of preferencing (chance and price effects) only dominate the disadvantage of the loss in volume when the preferenced order flow is large and it has to be larger than in a centralized market, i.e. $E(\Pi_1) \geq E\left(\Pi_1^{NP}\right)$ for $\kappa \geq \underline{\kappa}^{\text{fr}}(Q)$ where it is numerically showed that $\underline{\kappa}^{\text{fr}}(Q) > \underline{\kappa}(Q)$.(Remind Lemma 3 and Figure 3).

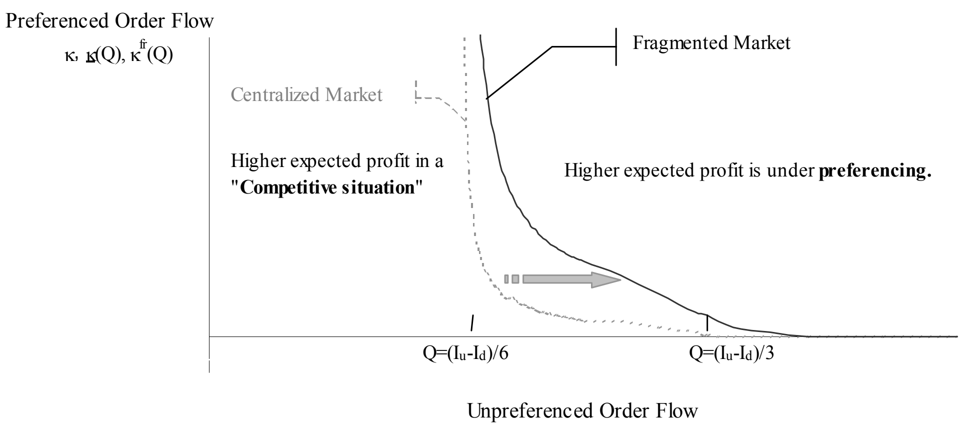


FIGURE 12 : The unpreferred dealer's expected profit under preferencing vs. no preferencing allowed.

B. The preferred dealer

As discussed in Section 2, under preferencing agreements, the preferred dealer faces two additionnal risks. First, there is an inventory risk since the preferenced trade must be executed whatever her inventory position is. For that risk, she is compensated by an additional risk premium since her effective reservation price is higher under preferencing than under no preferencing (competitive situation) : $a^{\kappa}_{r,2} > a_r(I_2, Q+\kappa)$. Second, there is also a risk caused by the price matching rule. When she is not able to post the best price, dealer D_2 matches the best price posted by her opponent which may be lower than her reservation price for clearing κ shares ($a_r(I_2,\kappa) < \underline{a} \in \left[a^{\inf}, a^{\sup}\right]$).

Remind that there is no price execution risk in a centralized two-dealer market and that the price-matching rule generates rents for the preferred dealer (see Lemma 3). In a fragmented market, this assertion is not any more verified.

Result 3 *In a fragmented market, the preferred dealer may incurs losses in executing her captive order flow.*

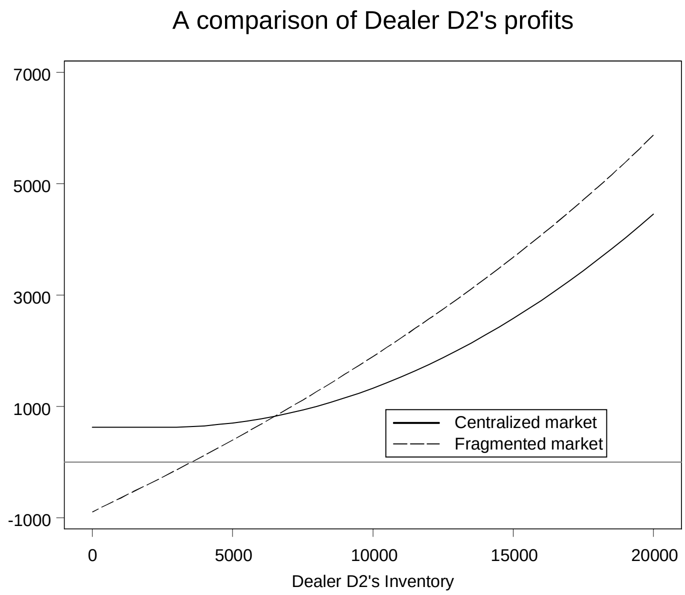


FIGURE 13

This result is consistent with the empirical evidence of Hansh, Naik and Viswanathan (1999) that preferred dealers on the LSE make zero profits over all trades. Losses could even be bigger if we now assume that the unpreferred dealer cannot observe whether a preferenced order flow is received or not by her opponent. Then, the unpreferred dealer quotes more aggressively and the price to match is more competitive which makes the price execution risk rising for the preferred dealer (see Lescourret and Robert (2002)).

In which market structure is preferencing the more profitable for the preferred dealer ?

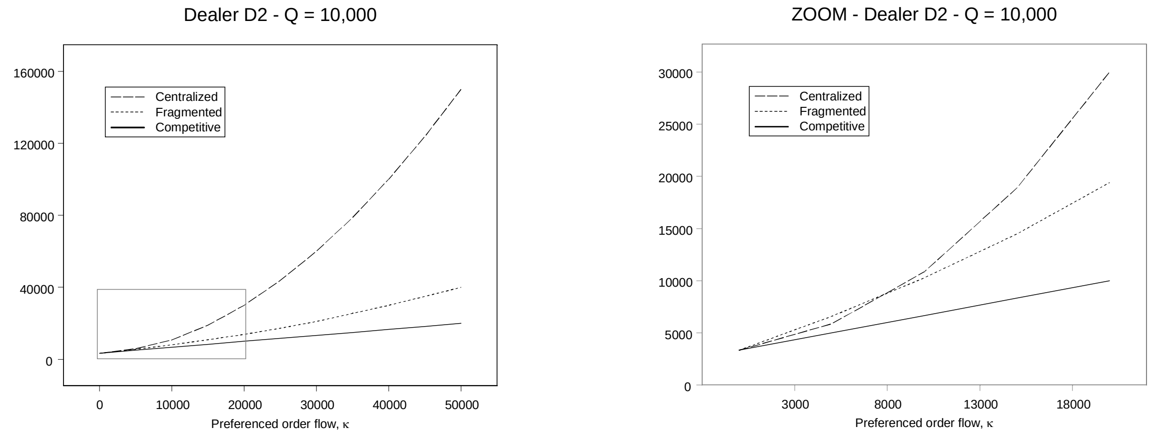
Result 4 *In a two-dealer market, when preferencing is large ($\kappa \geq 2(I_u - I_d)$), the preferred dealer expects higher profits in a centralized market than in a fragmented market :*

$$E(\Pi_2^c) > E(\Pi_2) = \frac{\rho\sigma_v^2(\kappa + Q) - (a_{r,u} - a_{r,d})}{2} \times \kappa > 0.$$

When preferencing is small, we numerically find that there exist some cases where the expected profit of *preferred dealer is higher in a fragmented market than in a centralized market even if she may face some losses.*

This ambiguous part of this result is explained by the distribution effect (EFFECT 2). Remind that when preferencing is small this effect dominates the effect on support (EFFECT 1) and it explains why dealer D_2 may quote less aggressively even when she has less chance to clear the unpreferenced trade, which yields higher expected best offers in a fragmented market .

Note that, in the case when preferencing is large ($\kappa \geq 2\,(I_u - I_d)$), even if the preferred dealer has no chance to accommodate the unpreferenced order flow due to a too large preferenced order, she secures however a positive expected profit thanks to the absence of competition that leads her opponent to post the highest quote (the absence of competition results in a disincentive to narrow the quoted spread).



FIGURES 14a, 14b : The expected profit of the preferred dealer in a preferencing vs. competitive case

Note that we numerically find that even if the preferred dealer may incur losses, she expects a higher profit when preferencing is allowed than when it is not allowed (competitive situation) as in a centralized market : $E\left(\Pi_2\right) > E\left(\Pi_2^{NP}\right)$.

5 Robustness

Payment for order flow

We extend our model to incorporate a payment for order flow, which is a practice quite often embedded in preferencing plans between dealers and brokers. Let us denote τ, the payment that dealer D_2 offers to the broker who sends a preferenced order flow to her. Then, we show[36] that in this model, the payment

[36] See section 7.11 in Appendix.

for order flow does not intervene in the determination of the cutoff price, which still writes as :

$$a_{r,2}^{\kappa} = \mu + \frac{\rho\sigma_v^2}{2}\left(Q - 2\left(I_2 - \kappa\right)\right)$$

Since the preferred dealer will not quote under this unchanged cutoff price, including a payment for order flow does not change our equilibrium bidding strategies (whether the reservation prices are commonly observed or not). It changes however the profit expected by dealer D_2.

Another trading process

Now, suppose that dealers D_1 and D_2 compete first in prices in order to accommodate an unpreferred trade large of Q shares. The best-quoting dealer executes the public trade. Then, suppose that at that time ($t = 3$), dealer D_2 receives a preferenced order flow with a probability α. Does this new trading process alter our results ?

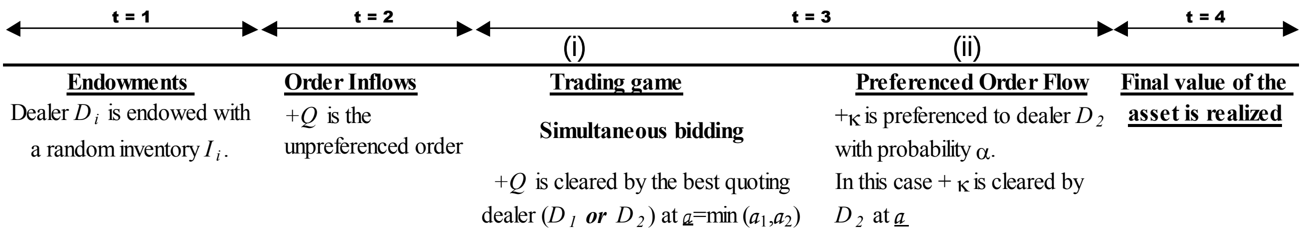


FIGURE 15 : A new time line of events

At $t = 3$ (i) when dealer D_2 posts her selling price, her expected payoff is then

$$A_2\left(a_2, a_1, I_2\right) = \begin{cases} \alpha \times \left(\ a_1 - a_r\left(I_2, \kappa\right)\right) \times \kappa & \text{if } a_2 > a_1 \\ \left(a_2 - a_r\left(I_2, Q\right)\right) \times Q + \alpha \times \left(\ a_2 - a_r\left(\left(I_2 - Q\right), \kappa\right)\right) \times \kappa & \text{if } a_2 < a_1 \end{cases}$$

Then the cutoff $a_{r,2}^{\kappa,\alpha}$ price is defined such as :

$$\alpha \times \left(a_{r,2}^{\kappa,\alpha} - a_r\left(I_2, \kappa\right)\right) \times \kappa = \left(a_{r,2}^{S,\alpha} - a_r\left(I_2, Q\right)\right) \times Q + \alpha \times \left(a_{r,2}^{\kappa,\alpha} - a_r\left(\left(I_2 - Q\right), \kappa\right)\right) \times \kappa$$

or,

$$a_{r,2}^{\kappa,\alpha} = \mu + \frac{\rho\sigma_v^2}{2}\left(Q - 2I_2\right) + \rho\sigma_v^2 \times \kappa_\alpha$$

where $\kappa_\alpha = \alpha \times \kappa$. Then, including an uncertainty on the reception of a preferenced order flow does not alter at last the equilibrium bidding strategies which are now some function of $a_{r,2}^{\kappa,\alpha}$ and $a_{r,1}$.

6 Conclusion

This paper investigates how preferencing alters the quoting behavior of two dealers with different inventory position. Dealers are supposed to undercut each other's quote to accommodate an incoming order flow. However, we assume that part of this order flow is already pre-assigned to one of the two dealers, regardless of his posted quotes. In accordance with best execution standards, that preferred dealer has guaranteed in advance to match the best price in executing the preferenced order flow. The best price to match results however from the price-competition with his opponent to attract the unpreferenced part of the order flow. In our framework, preferencing is analysed as a price-matching practice which generates inventory risks for the preferred dealer. We find that these risk may entail some losses for that agent. However, consistent with institutional concerns on price-matching like practices, we show that preferencing generates negative effects on the market performance since it widens market spreads despite dealers' incentives to undercut to attract the unpreferenced order flow. Preferencing softens indeed price-competition among dealers. Moreover, under preferencing, the market mechanism fails to allocate efficiently the order flow : the longest dealer is not necessarily the dealer who posts the best price, which partially invalidates the literal prediction of Ho an Stoll (1983)' model.

Finally, we mention that to determine whether preferencing is good or not for markets is much more complex. Preferencing results from long-term relationships between brokers and dealers (or specialists) from whose investors may benefit, especially because of the guarantee to be executed at the best price. Indeed brokers could direct their orders to another place but incur the risk to be price-disimproved when the time of execution is taken into account. Preferencing yields to supra-competitive prices, which could also represent the remuneration of this execution guarantee. However, it remains that the unpreferenced order flow suffers then from the widening of market spreads without benefiting from any guarantee.

`Note : The numerical approach has been performed under a special Mathematica 4.1 package (ODE.M). Runge-Kutta method.`

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7 Appendix

Let F be the uniform distribution function of the r.v. $a_{r,1}$, in the interval $[a_{r,u}, a_{r,d}]$ and let F_κ be the uniform distribution function of the r.v. $a^\kappa_{r,2}$, in the interval $\left[a^\kappa_{r,u}, a^\kappa_{r,d}\right]$.

7.1 Proof of Theorem 1

In Corollary 2, we show that dealer D_2 has no incentive to post a selling price below her cutoff price. It may be interesting to see why dealer D_2 modifies her reservation price. The natural reservation price would indeed be the reservation price that prevails in a competitive situation where the κ shares would not be executed by a preferred dealer but by the best-quoting dealer. This competitive reservation is defined in introduction by $a_r(I_2, Q+\kappa)$. To show that under preferencing, at equilibrium a preferred dealer raises one's reservation price from a competitive level to a preferenced level, we allow in the following proof dealer D_2 to quote as a function of her cutoff price or her competitive reservation price.

(i) Suppose that the ranking of reservation prices is such that $a_{r,1} > a^\kappa_{r,2} > a_r(I_2, Q+\kappa)$.

Then dealer D_2 posts the best price $a^c_2 = a_{r,1} - \varepsilon$ with probability 1. It is never optimal to quote lower than this price, given that the probability is still equal to 1, and the trading profit could only be lower.

Then, dealer D_1 quotes $a^c_1 = a_{r,1}$ since he cannot post the best price anyway.

(ii) Suppose that the ranking of reservation prices is such that $a^\kappa_{r,2} > a_{r,1} > a_r(I_2, Q+\kappa)$.

We suppose that dealer D_2 quotes $a^c_2 = a^\kappa_{r,2}$. Then, the best reply of dealer D_1 is to $a^c_1 = a^\kappa_{r,2} - \varepsilon$, which is the best price. If he quotes this price, it is indeed not optimal for dealer D_2 to undercut him, in posting $a^c_1 - \varepsilon$, till the competition yields to reach the reservation price of dealer D_1. Dealer D_2 would earn lower profit in this case than in not deviating from the quote equal to her cutoff price, since $(a_{r,1} - a_r(I_2, Q+\kappa)) \times (Q+\kappa) < \left(a^\kappa_{r,2} - a_r(I_2, \kappa)\right) \times \kappa$.

(iii) Suppose that the ranking of reservation prices is such that $a^\kappa_{r,2} > a_r(I_2, Q+\kappa) > a_{r,1}$.

Same than before. Dealer D_1 posts the best price $\underline{a}^c = a^c_1 = a^\kappa_{r,2} - \varepsilon$ and dealer D_2 quotes $a^c_2 = a^\kappa_{r,2}$. However since the competitive reservation price is bigger than the reservation price of her opponent, does dealer D_2 have any incentive to deviate from her strategy in undercutting her opponent ? Given that $a^c_1 = a^\kappa_{r,2} - \varepsilon > a_r(I_2, Q+\kappa)$, if dealer D_2 decides to undercut her opponent, she posts the best price equal to $\underline{a}^c = a^c_2 = a^c_1 - \varepsilon = a^\kappa_{r,2} - 2\varepsilon$. In this case she has to execute the total order flow at this price. However, the trading profit is lower in undercutting her opponent since, $(a^c_2 - a_r(I_2, Q+\kappa)) \times (Q+\kappa) < (\underline{a}^c - a_r(I_2, \kappa)) \times \kappa$.

Consequently, at equilibrium it is not optimal for dealer D_2 to post a price below her cutoff price. This price plays the role of the reservation price of a prefrenced dealer. It combines indeed the value of two different order flows : Q unpreferenced shares and κ preferenced shares. It follows that at equilibrium the dealer executing the total order flow has the lowest reservation price : $\min\left(a_{r,1}, a^{\kappa}_{r,2}\right)$.

■

7.2 Bidding strategy characterization

Step 1 : **Dealer D_1's probability to post the best price (ex ante)**

$$
\begin{aligned}
\Pr\left(D_1 \text{ posts the best price}\right) &= \int_{a_{r,u}}^{a^{\kappa}_{r,u}} 1 \times f(x)\,dx + \int_{a^{\kappa}_{r,u}}^{a_{r,d}} \frac{a^{\kappa}_{r,d} - x}{a_{r,d} - a_{r,u}} \times f(x)\,dx \\
&= \frac{\rho\sigma_v^2\kappa}{\left(a_{r,d} - a_{r,u}\right)} - \frac{\left[\left(a^{\kappa}_{r,d} - x\right)^2\right]_{a^{\kappa}_{r,u}}^{a_{r,d}}}{2\left(a_{r,d} - a_{r,u}\right)^2} \\
&= \frac{\rho\sigma_v^2\kappa}{\left(a_{r,d} - a_{r,u}\right)} + \frac{\left(a^{\kappa}_{r,d} - a^{\kappa}_{r,u}\right)^2}{2\left(a_{r,d} - a_{r,u}\right)^2} - \frac{\left(a^{\kappa}_{r,d} - a_{r,d}\right)^2}{2\left(a_{r,d} - a_{r,u}\right)^2} \\
&= \frac{1}{2} + \frac{\kappa}{\left(I_u - I_d\right)} - \frac{\kappa^2}{2\left(I_u - I_d\right)^2}
\end{aligned}
$$

Step 1Bis : **Dealer D_1's ex ante aggressiveness**

$$
\begin{aligned}
\theta_1\left(a_{r,1}\right) &= \frac{\left(E\left(a^{\kappa}_{r,2}\right) - a_{r,1}\right)}{a_{r,1}} \mathbb{1}_{a_{r,1} < a^{\kappa}_{r,u}} + \frac{\left(E\left(a^{\kappa}_{r,2} \mid a_{r,1} < a^{\kappa}_{r,2}\right) - a_{r,1}\right)}{a_{r,1}} \times \Pr\left(a_{r,1} < a^{\kappa}_{r,2}\right) \mathbb{1}_{a_{r,1} \geq a^{\kappa}_{r,u}} \\
&= \frac{\left(\frac{a^{\kappa}_{r,u} + a^{\kappa}_{r,d}}{2} - a_{r,1}\right)}{a_{r,1}} \mathbb{1}_{a_{r,1} < a^{\kappa}_{r,u}} + \frac{\left(\int_{a_{r,1}}^{a^{\kappa}_{r,d}} x f_{\kappa}(x)\,dx - a_{r,1}\bar{F}_{\kappa}\left(a_{r,1}\right)\right)}{a_{r,1}} \mathbb{1}_{a_{r,1} \geq a^{\kappa}_{r,u}} \\
&= \left(\frac{a^{\kappa}_{r,u} + a^{\kappa}_{r,d}}{2 \times a_{r,1}} - 1\right) \mathbb{1}_{a_{r,1} < a^{\kappa}_{r,u}} + \frac{1}{2} \times \frac{\left(a^{\kappa}_{r,d} - a_{r,1}\right)^2}{\left(a_{r,d} - a_{r,u}\right) \times a_{r,1}} \mathbb{1}_{a_{r,1} \geq a^{\kappa}_{r,u}}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
E\left(\theta_1\right) &= \int_{a_{r,u}}^{a_{r,u}^{\kappa}}\left(\frac{a_{r,u}^{\kappa}+a_{r,d}^{\kappa}}{2\times x}-1\right)f\left(x\right)dx+\int_{a_{r,u}^{\kappa}}^{a_{r,d}}\frac{\left(a_{r,d}^{\kappa}-x\right)^2}{2\left(a_{r,d}-a_{r,u}\right)\times x}f\left(x\right)dx \\
&= \left[\frac{\left(\frac{a_{r,u}^{\kappa}+a_{r,d}^{\kappa}}{2}\times\ln\left(x\right)-x\right)}{a_{r,d}-a_{r,u}}\right]_{a_{r,u}}^{a_{r,u}^{\kappa}}+\frac{\left[\left(a_{r,d}^{\kappa}\right)^2\times\ln\left(x\right)+\frac{1}{2}x^2-2xa_{r,d}^{\kappa}\right]_{a_{r,u}^{\kappa}}^{a_{r,d}}}{2\left(a_{r,d}-a_{r,u}\right)^2} \\
&= \frac{\frac{a_{r,u}^{\kappa}+a_{r,d}^{\kappa}}{2}\times\ln\left(\frac{a_{r,u}^{\kappa}}{a_{r,u}}\right)-\rho\sigma_v^2\kappa}{a_{r,d}-a_{r,u}}+\frac{\left(a_{r,d}^{\kappa}\right)^2\times\ln\left(\frac{a_{r,d}}{a_{r,u}^{\kappa}}\right)+2a_{r,d}^{\kappa}\left(a_{r,u}^{\kappa}-a_{r,d}\right)+\frac{1}{2}\left(a_{r,d}\right)^2-\frac{1}{2}\left(a_{r,u}^{\kappa}\right)^2}{2\left(a_{r,d}-a_{r,u}\right)^2}
\end{aligned}
$$

STEP 2 : **Dealer D_2's probability to post the best price (ex ante)**

$$
\begin{aligned}
\Pr\left(D_2\text{ posts the best price}\right) &= \int_{a_{r,d}}^{a_{r,d}^{\kappa}}0\times f_{\kappa}\left(x\right)dx+\int_{a_{r,u}^{\kappa}}^{a_{r,d}}\frac{a_{r,d}-x}{a_{r,d}-a_{r,u}}\times f_{\kappa}\left(x\right)dx \\
&= -\frac{\left[\left(a_{r,d}-x\right)^2\right]_{a_{r,u}^{\kappa}}^{a_{r,d}}}{2\left(a_{r,d}-a_{r,u}\right)^2} \\
&= \frac{\left(a_{r,d}-a_{r,u}^{\kappa}\right)^2}{2\left(a_{r,d}-a_{r,u}\right)^2} \\
&= \frac{\left(I_u-I_d-\kappa\right)^2}{2\left(I_u-I_d\right)^2} \\
&= \frac{1}{2}-\frac{\kappa}{\left(I_u-I_d\right)}+\frac{\kappa^2}{2\left(I_u-I_d\right)^2}
\end{aligned}
$$

STEP 2Bis : **Dealer D_2's ex ante aggressiveness**

$$
\begin{aligned}
\theta_2\left(a_{r,2}^{\kappa}\right) &= 0\mathbb{1}_{a_{r,d}\leq a_{r,2}^{\kappa}}+\frac{E\left(a_{r,1}\mid a_{r,2}^{\kappa}<a_{r,1}\right)-a_{r,2}^{\kappa}}{a_{r,2}^{\kappa}}\Pr\left(a_{r,2}^{\kappa}<a_{r,1}\right)\mathbb{1}_{a_{r,d}>a_{r,2}^{\kappa}} \\
&= \frac{\int_{a_{r,2}^{\kappa}}^{a_{r,d}}xf\left(x\right)dx-a_{r,2}^{\kappa}\times\bar{F}\left(a_{r,2}^{\kappa}\right)}{a_{r,2}^{\kappa}}\mathbb{1}_{a_{r,d}>a_{r,2}^{\kappa}} \\
&= \frac{1}{2}\times\frac{\left(a_{r,d}-a_{r,2}^{\kappa}\right)^2}{\left(a_{r,d}-a_{r,u}\right)\times a_{r,2}^{\kappa}}\mathbb{1}_{a_{r,d}>a_{r,2}^{\kappa}}
\end{aligned}
$$

$$
\begin{aligned}
E\left(\theta_2\right) &= \int_{a_{r,u}^{\kappa}}^{a_{r,d}}\frac{\left(a_{r,d}-x\right)^2}{2\left(a_{r,d}-a_{r,u}\right)\times x}f_{\kappa}\left(x\right)dx \\
&= \frac{\left[\left(a_{r,d}\right)^2\times\ln\left(x\right)+\frac{x^2}{2}-2a_{r,d}x\right]_{a_{r,u}^{\kappa}}^{a_{r,d}}}{2\left(a_{r,d}-a_{r,u}\right)^2} \\
&= \frac{\left(a_{r,d}\right)^2\times\ln\left(\frac{a_{r,d}}{a_{r,u}^{\kappa}}\right)+\frac{\left(a_{r,d}\right)^2}{2}-\frac{\left(a_{r,u}^{\kappa}\right)^2}{2}+2a_{r,d}\left(a_{r,u}^{\kappa}-a_{r,d}\right)}{2\left(a_{r,d}-a_{r,u}\right)^2}
\end{aligned}
$$

7.3 Proof of Lemma 2

Before proceeding to the computation of the expected best offer, it is worth noticing than when the preferenced order flow is large, then at equilibrium the preferred dealer is not able to post the best price anyway. Specifically, when $\kappa > (I_u - I_d)$, then $a^{\kappa}_{r,u} > a_{r,d}$.

Lemma 5 *When the preferenced order flow κ is so large that $\kappa > (I_u - I_d)$, then the preferred dealer can never post the best price at equilibrium. Dealer D_1 quotes $a^c_1 = a^{\kappa}_{r,2} - \varepsilon$, and then the expression of the expected Best Offer simply writes :*

$$E\left(\underline{a}^c\right) = \frac{a^{\kappa}_{r,d} + a^{\kappa}_{r,u}}{2}$$

In the equilibrium where $\kappa > (I_u - I_d)$, dealer D_1 posts the best price with probability 1 and quotes $a^c_1 = a^{\kappa}_{r,2} - \varepsilon$. Then, it is optimal for dealer D_2 to quote her cutoff price $a^c_2 = a^{\kappa}_{r,2}$. She would indeed earn lower profit in undercutting her opponent by a^c_1 $-\varepsilon$ since $(a^c_1 - \varepsilon - a_r\left(I_2, Q+\kappa\right)) \times (Q+\kappa) < (a^c_1 - a_r\left(I_2, \kappa\right)) \times \kappa$.

Moreover, when $\kappa > (I_u - I_d)$, the expected best offer is simply equal to

$$E\left(\underline{a}^c\right) = E\left(a^{\kappa}_{r,2}\right) = \frac{a^{\kappa}_{r,d} + a^{\kappa}_{r,u}}{2}.$$

Now we have to consider the case where $\kappa \leq (I_u - I_d)$.

STEP 1 : **Determination of the expected Best Offer when $\kappa \leq (I_u - I_d)$**

By definition, the best offer writes : $\underline{a}^c = \min\left(\left(a^c_1\right)^*, \left(a^c_2\right)^*\right)$. In this two-dealer transparent market, the best offer is simply equal to $\max\left(a_{r,1}, a^{\kappa}_{r,2}\right)$.

Let us denote F_M, the c.d.f. of $\max\left(a_{r,1}, a^{\kappa}_{r,2}\right)$. Then this satifies :

$$\begin{aligned} F_M(x) &= F(x)\, F_{\kappa}(x) \\ &= \frac{(x - a_{r,u})}{(a_{r,d} - a_{r,u})} \frac{\left(x - a^{\kappa}_{r,u}\right)}{\left(a^{\kappa}_{r,d} - a^{\kappa}_{r,u}\right)} \mathbb{1}_{\left[a^{\kappa}_{r,u}, a_{r,d}\right]} + \frac{\left(x - a^{\kappa}_{r,u}\right)}{\left(a^{\kappa}_{r,d} - a^{\kappa}_{r,u}\right)} \mathbb{1}_{\left[a_{r,d}, a^{\kappa}_{r,d}\right]} \\ \bar{F}_M(x) &= 1 - F_M(x) \end{aligned}$$

Consequently, we write the expected best offer as follows :

$$\begin{aligned} E\left(\underline{a}^c\right) &= E\left(\max\left(a_{r,1}, a^{\kappa}_{r,2}\right)\right) \\ &= \int_0^{+\infty} \bar{F}_M(x)\, dx \\ &= a^{\kappa}_{r,u} + \int_{a^{\kappa}_{r,u}}^{a^{\kappa}_{r,d}} \bar{F}_M(x)\, dx \end{aligned} \tag{9}$$

Notice that :

$$\begin{aligned}
\int_{a_{r,u}^{\kappa}}^{a_{r,d}^{\kappa}} \bar{F}_M(x)\,dx &= \int_{a_{r,u}^{\kappa}}^{a_{r,d}^{\kappa}} (1 - F(x)\,F_\kappa(x))\,dx \\
&= \left(a_{r,d}^{\kappa} - a_{r,u}^{\kappa}\right) - \int_{a_{r,u}^{\kappa}}^{a_{r,d}} \frac{(x - a_{r,u})}{(a_{r,d} - a_{r,u})} \frac{\left(x - a_{r,u}^{\kappa}\right)}{\left(a_{r,d}^{\kappa} - a_{r,u}^{\kappa}\right)} dx - \int_{a_{r,d}}^{a_{r,d}^{\kappa}} \frac{\left(x - a_{r,u}^{\kappa}\right)}{\left(a_{r,d}^{\kappa} - a_{r,u}^{\kappa}\right)} dx
\end{aligned}$$

After staightforward computations, this expression rewrites

$$\int_{a_{r,u}^{\kappa}}^{a_{r,d}^{\kappa}} \bar{F}_M(x)\,dx = \frac{(a_{r,d} - a_{r,u})}{2} + \frac{(a_{r,d} - a_{r,u})}{6} \left(1 - \frac{\rho\sigma_v^2\kappa}{(a_{r,d} - a_{r,u})}\right)^3$$

Substituting this expression in equation (9) yields :

$$\begin{aligned}
E\left(\underline{a}^c\right) &= a_{r,u}^{\kappa} + \frac{(a_{r,d} - a_{r,u})}{2} + \frac{(a_{r,d} - a_{r,u})}{6} \left(1 - \frac{\rho\sigma_v^2\kappa}{(a_{r,d} - a_{r,u})}\right)^3 \\
&= \rho\sigma_v^2\kappa + \frac{(a_{r,d} + a_{r,u})}{2} + \frac{(a_{r,d} - a_{r,u})}{6} \left(1 - \frac{\rho\sigma_v^2\kappa}{(a_{r,d} - a_{r,u})}\right)^3
\end{aligned}$$

Finally,

$$\begin{aligned}
E\left(\underline{a}^c\right) &= \left(\frac{a_{r,d}^{\kappa} + a_{r,u}^{\kappa}}{2} + \frac{(a_{r,d} - a_{r,u})}{6} \left(1 - \frac{\rho\sigma_v^2\kappa}{(a_{r,d} - a_{r,u})}\right)^3\right) \mathbb{1}_{a_{r,u}^{\kappa} \leq a_{r,d}} \\
&\quad + \frac{a_{r,d}^{\kappa} + a_{r,u}^{\kappa}}{2} \mathbb{1}_{a_{r,u}^{\kappa} > a_{r,d}}.
\end{aligned}$$

■

STEP 2 : **Determination of the expected Best Offer prevailing in a competitive situation.**

Remind that in a situation where no preferencing is allowed (Competitive Case), dealers are symmetric. In this case, the best offer is defined by $\underline{a}^{c,NP} = \max(a_{r,1}, a_{r,2})$. Then, the c.d.f. of $\max(a_{r,1}, a_{r,2})$ writes :

$$\begin{aligned}
F_M(x) &= F(x)\,F(x) \\
&= \frac{(x - a_{r,u})^2}{(a_{r,d} - a_{r,u})^2} \\
\bar{F}_M(x) &= 1 - F_M(x)
\end{aligned}$$

Then, the best offer in a competitive market is expected to be :

$$\begin{aligned}
E\left(\underline{a}^{c,NP}\right) &= E\left(\max(a_{r,1}, a_{r,2})\right) \\
&= \int_0^{+\infty} \bar{F}_M(x)\,dx \\
&= a_{r,u} + \int_{a_{r,u}}^{a_{r,d}} \bar{F}_M(x)\,dx
\end{aligned}$$

where

$$\int_{a_{r,u}}^{a_{r,d}} \bar{F}_M(x)\,dx = \int_{a_{r,u}}^{a_{r,d}} (1 - F(x)F(x))\,dx$$
$$= \frac{2(a_{r,d} - a_{r,u})}{3}$$

Finally,

$$E\left(\underline{a}^{c,NP}\right) = \frac{2a_{r,d} + a_{r,u}}{3}$$

Step 3 : **Comparison of the expected best offers (competitive vs preferenced case)**

- When $\kappa > (I_u - I_d)$, it is straightforward to show that

$$E(\underline{a}^c) = \frac{a^{\kappa}_{r,d} + a^{\kappa}_{r,u}}{2} > E\left(\underline{a}^{c,NP}\right) = \frac{2a_{r,d} + a_{r,u}}{3}$$

- When $\kappa \leq (I_u - I_d)$, we denote $\psi(\kappa)$ the following expression : $\psi(\kappa) = E(\underline{a}^c) - E(\underline{a}^c_{NP})$. Then for each $\kappa \leq (I_u - I_d)$,

$$\psi(\kappa) = \frac{a^{\kappa}_{r,d} + a^{\kappa}_{r,u}}{2} + \frac{(a_{r,d} - a_{r,u})}{6}\left(1 - \frac{\rho\sigma_v^2\kappa}{(a_{r,d} - a_{r,u})}\right)^3 - \frac{2a_{r,d}(Q+\kappa) + a_{r,u}(Q+\kappa)}{3}$$
$$= \frac{\rho\sigma_v^2}{2}\left(\frac{(I_u - I_d)}{3}\left(\left(1 - \frac{\kappa}{(I_u - I_d)}\right)^3 - 1\right) + \kappa\right)$$

We observe that :

$$\psi'(\kappa) = \frac{\rho\sigma_v^2}{2}\left(1 - \left(1 - \frac{\kappa}{(I_u - I_d)}\right)^2\right)$$

and

$$\psi''(\kappa) = \frac{\rho\sigma_v^2}{2(I_u - I_d)}\left(1 - \frac{\kappa}{(I_u - I_d)}\right)$$

Since $\kappa \leq (I_u - I_d)$, then $\psi''(\kappa) > 0$, $\psi'(0) = 0$, $\psi'(I_u - I_d) = \rho\sigma_v^2/2$, then $\psi'(\kappa) > 0$ for each $\kappa \leq (I_u - I_d)$ Notice that $\psi(0) = 0$ and $\psi(I_u - I_d) = \frac{\rho\sigma_v^2(I_u - I_d)}{3} > 0$, then we can conclude that $\psi(\kappa, (I_u - I_d))$ for each $\kappa \leq (I_u - I_d)$.

It follows that $E(\underline{a}^c) > E(\underline{a}^c_{NP})$.

■

7.4 Proof of Lemma 3

Step 1 : **The expected payoff of dealer D_1**

- When $\kappa > (I_u - I_d)$, then dealer D_1 posts the best price with probability 1, and his payoff is

$$\begin{aligned}\Pi_1^c(a_{r,1}) &= \left(E\left(a_{r,2}^{\kappa}\right) - a_{r,1}\right) \times Q \\ &= \left(\frac{a_{r,d}^{\kappa} + a_{r,u}^{\kappa}}{2} - a_{r,1}\right) \times Q\end{aligned}$$

Hence, at $t = 1$ (ex ante), dealer D_1 expects the following profit :

$$E\left(\Pi_1^c\right) = \left[\int_{a_{r,u}}^{a_{r,d}} \left(\frac{a_{r,d}^{\kappa} + a_{r,u}^{\kappa}}{2} - x\right) f(x)\, dx\right] \times Q = \rho\sigma_v^2\kappa \times Q.$$

- When $\kappa \leq (I_u - I_d)$, then

$$\Pi_1^c(a_{r,1}) = \Pr\left(a_{r,2}^{\kappa} > a_{r,1}\right) \times \left[E\left(a_{r,2}^{\kappa} \mid a_{r,2}^{\kappa} > a_{r,1}\right) - a_{r,1}\right] \times Q$$

The uniform distribution $F_{\kappa}(.)$ of the r.v. $a_{r,2}^{\kappa}$ is on the interval $\left[a_{r,u}^{\kappa}, a_{r,d}^{\kappa}\right]$, then

  – If $a_{r,u}^{\kappa} \leq a_{r,1}$

$$\begin{aligned}\Pi_1^c(a_{r,1}) &= \bar{F}_{\kappa}(a_{r,1}) \times \left(\frac{\int_{a_{r,1}}^{a_{r,d}^{\kappa}} x f_{\kappa}(x)\, dx}{\bar{F}_{\kappa}(a_{r,1})} - a_{r,1}\right) \times Q \\ &= \left(\frac{\left(a_{r,d}^{\kappa} - a_{r,1}\right)\left(a_{r,d}^{\kappa} + a_{r,1}\right)}{2\left(a_{r,d} - a_{r,u}\right)} - a_{r,1}\frac{\left(a_{r,d}^{\kappa} - a_{r,1}\right)}{\left(a_{r,d} - a_{r,u}\right)}\right) \times Q \\ &= \frac{1}{2} \times \frac{\left(a_{r,d}^{\kappa} - a_{r,1}\right)^2}{\left(a_{r,d} - a_{r,u}\right)} \times Q\end{aligned}$$

  – if $a_{r,u} \leq a_{r,1} < a_{r,u}^{\kappa}$

$$\begin{aligned}\Pi_1^c(a_{r,1}) &= \left(\int_{a_{r,u}^{\kappa}}^{a_{r,d}^{\kappa}} x f_{\kappa}(x)\, dx - a_{r,1}\right) \times Q \\ &= \left(\frac{a_{r,d}^{\kappa} + a_{r,u}^{\kappa}}{2} - a_{r,1}\right) \times Q\end{aligned}$$

  Consequently,

$$\begin{aligned}E\left(\Pi_1^c\right) &= \int_{a_{r,u}^{\kappa}}^{a_{r,d}} \frac{1}{2} \times \frac{\left(a_{r,d}^{\kappa} - x\right)^2}{\left(a_{r,d} - a_{r,u}\right)} \times Q \times f(x)\, dx + \int_{a_{r,u}}^{a_{r,u}^{\kappa}} \left(\frac{a_{r,d}^{\kappa} + a_{r,u}^{\kappa}}{2} - x\right) \times Q \times f(x)\, dx \\ &= \left(\frac{a_{r,d} - a_{r,u}}{6} - \frac{\left(\rho\sigma_v^2\kappa\right)^3}{6\left(a_{r,d} - a_{r,u}\right)^2} + \rho\sigma_v^2\frac{\left(a_{r,d} - a_{r,u} + \rho\sigma_v^2\kappa\right)}{2\left(a_{r,d} - a_{r,u}\right)} \times \kappa\right) \times Q\end{aligned}$$

STEP 2 : **The expected payoff of dealer D_2.**

- If $a_{r,d} < a^{\kappa}_{r,2}$. In this case dealer D_1 posts the best price with probability 1 and he quotes $a^c_1 = a^{\kappa}_{r,2} - \varepsilon$. If dealer D_1 behaves in this way, dealer D_2 quotes her cutoff price since she is not able to post the best price anyway. Then, her payoff is :

$$\begin{aligned}
\Pi^c_2\left(a^{\kappa}_{r,2}\right) &= \left(a^{\kappa}_{r,2} - a_r\left(I_2,\kappa\right)\right)\times\kappa \\
&= \left(a^{\kappa}_{r,2} - \left(a^{\kappa}_{r,2} - \frac{\rho\sigma^2_v}{2}\left(Q+\kappa\right)\right)\right)\times\kappa \\
&= \frac{\rho\sigma^2_v}{2}\left(Q+\kappa\right)\times\kappa
\end{aligned}$$

- If $a^{\kappa}_{r,2} \le a_{r,d}$. In this case, following Theorem 3, we get

$$\begin{aligned}
\Pi^c_2\left(a^{\kappa}_{r,2}\right) &= \Pr\left(a_{r,1} > a^{\kappa}_{r,2}\right)\times\left(E\left(a_{r,1}\mid a_{r,1} > a^{\kappa}_{r,2}\right) - a_r\left(I_2, Q+\kappa\right)\right)\times(\kappa+Q) \\
&\quad +\Pr\left(a_{r,1} < a^{\kappa}_{r,2}\right)\times\left(a^{\kappa}_{r,2} - a_r\left(I_2,\kappa\right)\right)\times\kappa \\
&= \bar{F}\left(a^{\kappa}_{r,2}\right)\times\left(E\left(a_{r,1}\mid a_{r,1} > a^{\kappa}_{r,2}\right) - \left(a^{\kappa}_{r,2} - \frac{\rho\sigma^2_v}{2}\times\kappa\right)\right)\times(Q+\kappa) \\
&\quad +F\left(a^{\kappa}_{r,2}\right)\times\left(\left(a^{\kappa}_{r,2} - \left(a^{\kappa}_{r,2} - \frac{\rho\sigma^2_v}{2}\left(Q+\kappa\right)\right)\right)\times\kappa\right) \\
&= \bar{F}\left(a^{\kappa}_{r,2}\right)\times\left(E\left(a_{r,1}\mid a_{r,1} > a^{\kappa}_{r,2}\right) - a^{\kappa}_{r,2}\right)\times(Q+\kappa) + \frac{\rho\sigma^2_v}{2}\times(Q+\kappa)\times\kappa
\end{aligned}$$

This expression is quite natural since we argue in Corollary 1 that dealer D_2 will not post selling prices below her cutoff price. The latter expression rewrites,

$$\begin{aligned}
\Pi^c_2\left(a^{\kappa}_{r,2}\right) &= \bar{F}\left(a^{\kappa}_{r,2}\right)\times\left(\frac{\int_{a^{\kappa}_{r,2}}^{a_{r,d}} xf(x)\,dx}{\bar{F}\left(a^{\kappa}_{r,2}\right)} - a^{\kappa}_{r,2}\right)\times(Q+\kappa) + \frac{\rho\sigma^2_v}{2}\times(Q+\kappa)\times\kappa \\
&= \left(\frac{\left(a_{r,d} - a^{\kappa}_{r,2}\right)^2}{a_{r,d} - a_{r,u}} + \rho\sigma^2_v\times\kappa\right)\times\frac{(Q+\kappa)}{2}
\end{aligned}$$

Then, at $t=1$, when $\kappa < (I_u - I_d)$, dealer D_2 expects the following profit :

$$\begin{aligned}
E\left(\Pi^c_2\right) &= \int_{a^{\kappa}_{r,u}}^{a_{r,d}}\left(\frac{\left(a_{r,d} - x\right)^2}{a_{r,d} - a_{r,u}} + \rho\sigma^2_v\times\kappa\right)\times\frac{(Q+\kappa)}{2}f_{\kappa}(x)\,dx + \int_{a_{r,d}}^{a^{\kappa}_{r,d}}\frac{\rho\sigma^2_v}{2}(Q+\kappa)\times\kappa f_{\kappa}(x)\,dx \\
&= \frac{(Q+\kappa)}{2}\left[\frac{\left(a_{r,d} - a_{r,u} - \rho\sigma^2_v\kappa\right)^3}{3\left(a_{r,d} - a_{r,u}\right)^2} + \rho\sigma^2_v\times\kappa\right]
\end{aligned}$$

Finally,

$$\begin{aligned}
E\left(\Pi^c_2\right) &= \frac{\rho\sigma^2_v\times(Q+\kappa)}{2}\left(\frac{\left(I_u - I_d - \kappa\right)^3}{3\left(I_u - I_d\right)^2} + \kappa\right)\mathbb{1}_{\kappa\le(I_u - I_d)} \\
&\quad + \frac{\rho\sigma^2_v}{2}(Q+\kappa)\times\kappa\mathbb{1}_{\kappa>(I_u - I_d)}.
\end{aligned}$$

Step 3 : **Comparison with the competitive case**

If the preferenced order flow was directed to the first dealer who quotes the best price then dealers'expected profits would be :

$$E\left(\Pi_1^{NP}\right) = E\left(\Pi_2^{NP}\right) = \left(\frac{a_{r,d} - a_{r,u}}{6}\right) \times (Q + \kappa)$$

then,

STEP 3.1 : **Comparison of dealer D_2's expected profit**

$$\begin{aligned}
& E\left(\Pi_2^c\right) - E\left(\Pi_2^{NP}\right) \\
= & \left(\frac{\left(a_{r,d} - a_{r,u} - \rho\sigma_v^2\kappa\right)^3}{3\left(a_{r,d} - a_{r,u}\right)^2} + \rho\sigma_v^2 \times \kappa\right) \times \frac{(Q+\kappa)}{2} - \left(\frac{\left(a_{r,d} - a_{r,u}\right)}{6}\right)(Q+\kappa) \\
= & \left(\frac{3\left(a_{r,d} - a_{r,u}\right) - \rho\sigma_v^2 \times \kappa}{3\left(a_{r,d} - a_{r,u}\right)^2}\right) \times \frac{(Q+\kappa)}{2}\left(\rho\sigma_v^2 \times \kappa\right)^2 > 0
\end{aligned}$$

STEP 3.2 : **Comparison of dealer D_1's expected profit**

- When $\kappa \leq (I_u - I_d)$, then

$$\begin{aligned}
E\left(\Pi_1^c\right) - E\left(\Pi_1^{NP}\right) &= \left(\frac{\left(a_{r,d} - a_{r,u}\right)}{6} - \frac{\left(\rho\sigma_v^2\kappa\right)^3}{6\left(a_{r,d} - a_{r,u}\right)^2} + \rho\sigma_v^2\frac{\left(a_{r,d} - a_{r,u} + \rho\sigma_v^2\kappa\right)}{2\left(a_{r,d} - a_{r,u}\right)} \times \kappa\right) \times Q \\
&\quad - \left(\frac{a_{r,d} - a_{r,u}}{6}\right) \times (Q+\kappa) \\
&= \left(-\frac{\left(\rho\sigma_v^2\kappa\right)^3}{6\left(a_{r,d} - a_{r,u}\right)^2} + \frac{\left(a_{r,d} - a_{r,u}\right) + \rho\sigma_v^2\kappa}{2\left(a_{r,d} - a_{r,u}\right)} \times \rho\sigma_v^2\kappa\right) \times Q - \left(\frac{a_{r,d} - a_{r,u}}{6}\right) \times \kappa \\
&= \rho\sigma_v^2\kappa Q \times \left(-\frac{\left(\rho\sigma_v^2\kappa\right)^2}{6\left(a_{r,d} - a_{r,u}\right)^2} + \frac{\rho\sigma_v^2\kappa}{2\left(a_{r,d} - a_{r,u}\right)} + \frac{1}{2} - \frac{a_{r,d} - a_{r,u}}{6\rho\sigma_v^2 Q}\right)
\end{aligned}$$

- When $\kappa > (I_u - I_d)$, then

$$\begin{aligned}
E\left(\Pi_1^c\right) - E\left(\Pi_1^{NP}\right) &= \rho\sigma_v^2\kappa Q - \left(\frac{a_{r,d} - a_{r,u}}{6}\right)(Q+\kappa) \\
&= \rho\sigma_v^2\kappa Q\left(1 - \left(\frac{a_{r,d} - a_{r,u}}{6\rho\sigma_v^2 Q}\right)\left(1 + \frac{1}{\frac{\kappa}{Q}}\right)\right)
\end{aligned}$$

In other words,

$$\begin{aligned}
& E\left(\Pi_1^c\right) - E\left(\Pi_1^{NP}\right) \\
= & \rho\sigma_v^2\kappa Q \\
& \times\left[\frac{1}{6}\left(-\frac{\kappa^2}{\left(I_u - I_d\right)^2} + 3\frac{\kappa}{\left(I_u - I_d\right)} + 3 - \frac{\left(I_u - I_d\right)}{Q}\right)\mathbb{1}_{\kappa \leq (I_u - I_d)} + \left(1 - \frac{\left(I_u - I_d\right)}{6Q}\left(1 + \frac{1}{\frac{\kappa}{Q}}\right)\right)\mathbb{1}_{\kappa > (I_u - I_d)}\right]
\end{aligned}$$

Let us now define the following function :

$$g(\kappa,Q) = \frac{1}{6}\left(-\frac{\kappa^2}{(I_u-I_d)^2}+3\frac{\kappa}{(I_u-I_d)}+3-\frac{(I_u-I_d)}{Q}\right)\mathbb{1}_{\kappa\le(I_u-I_d)} + \left(1-\frac{(I_u-I_d)}{6Q}\left(1+\frac{1}{\frac{\kappa}{Q}}\right)\right)\mathbb{1}_{\kappa>(I_u-I_d)}$$

$$g(0,Q)=\frac{1}{6}\left(3-\frac{(I_u-I_d)}{Q}\right),\quad g((I_u-I_d),Q)=\frac{1}{6}\left(5-\frac{(I_u-I_d)}{Q}\right),\quad \lim_{\kappa\to\infty} g(\kappa,Q)=1-\frac{(I_u-I_d)}{6Q}$$

$$\frac{\partial g(\kappa,Q)}{\partial\kappa} = \frac{1}{6(I_u-I_d)}\left(-\frac{2\kappa}{(I_u-I_d)}+3\right)\mathbb{1}_{\kappa\le(I_u-I_d)}+\frac{(I_u-I_d)}{6\kappa^2}\mathbb{1}_{\kappa>(I_u-I_d)}$$

g is an increasing function, the sign of this function depends on the initial condition. Then,

$Q\ge\frac{(I_u-I_d)}{3}$	$E(\Pi_1^c)\ge E(\Pi_1^{NP})$
$Q<\frac{(I_u-I_d)}{3}$	
$\frac{(I_u-I_d)}{5}\le Q<\frac{(I_u-I_d)}{3}$	$E(\Pi_1^c)\le E(\Pi_1^{NP})$, if $\kappa\le\kappa^*(Q)$ $E(\Pi_1^c)> E(\Pi_1^{NP})$, otherwise
$\frac{(I_u-I_d)}{6}< Q<\frac{(I_u-I_d)}{5}$	$E(\Pi_1^c)\le E(\Pi_1^{NP})$, if $\kappa\le\kappa^{**}(Q)$ $E(\Pi_1^c)> E(\Pi_1^{NP})$, otherwise
$Q<\frac{(I_u-I_d)}{6}$	$E(\Pi_1^c)< E(\Pi_1^{NP})$

where

$$\kappa^*(Q) = \frac{(I_u-I_d)}{2}\left(3-\sqrt{21-4\frac{(I_u-I_d)}{Q}}\right)$$
$$\kappa^{**}(Q) = \frac{(I_u-I_d)Q}{(6Q-(I_u-I_d))}$$

Now, we define $\underline{\kappa}$ such that

$$\underline{\kappa}(Q)=\kappa^*(Q)\,\mathbb{1}_{\frac{(I_u-I_d)}{5}\le Q<\frac{(I_u-I_d)}{3}}+\kappa^{**}(Q)\,\mathbb{1}_{\frac{(I_u-I_d)}{6}\le Q<\frac{(I_u-I_d)}{5}}$$

■

7.5 Proof of Lemma 4

In a three-dealer market, suppose that only one on three dealers (dealer D_2) is preferred where $a_{r,1}, a^\kappa_{r,2}, a_{r,3}$ are the respective reservation prices.

$$\begin{aligned} E(\underline{a}^c) &= E\left(a_{r,3}\mathbb{1}_{a_{r,1}<a_{r,3}<a^\kappa_{r,2}}\right)+E\left(a^\kappa_{r,2}\mathbb{1}_{a_{r,1}<a^\kappa_{r,2}<a_{r,3}}\right)+E\left(a_{r,1}\mathbb{1}_{a^\kappa_{r,2}<a_{r,1}<a_{r,3}}\right) \\ &\quad +E\left(a_{r,3}\mathbb{1}_{a^\kappa_{r,2}<a_{r,3}<a_{r,1}}\right)+E\left(a^\kappa_{r,2}\mathbb{1}_{a_{r,3}<a^\kappa_{r,2}<a_{r,1}}\right)+E\left(a_{r,1}\mathbb{1}_{a_{r,3}<a_{r,1}<a^\kappa_{r,2}}\right) \\ &= 2\left(E\left(a_{r,3}\mathbb{1}_{a_{r,1}<a_{r,3}<a^\kappa_{r,2}}\right)+E\left(a^\kappa_{r,2}\mathbb{1}_{a_{r,1}<a^\kappa_{r,2}<a_{r,3}}\right)+E\left(a_{r,1}\mathbb{1}_{a^\kappa_{r,2}<a_{r,1}<a_{r,3}}\right)\right)\end{aligned}$$

Let us denote $x = a_{r,1}$, $y = a^{\kappa}_{r,2}$ and $z = a_{r,3}$

$$E\left(a_{r,3}\mathbb{1}_{a_{r,1}<a_{r,3}<a^{\kappa}_{r,2}}\right) = \int_{a^{\kappa}_{r,u}}^{a_{r,d}} \frac{z}{(a_{r,d}-a_{r,u})}\left(\frac{z-a_{r,u}}{a_{r,d}-a_{r,u}}\right)\left(\frac{a^{\kappa}_{r,d}-z}{a_{r,d}-a_{r,u}}\right)dz + \int_{a_{r,u}}^{a^{\kappa}_{r,u}} \frac{z}{(a_{r,d}-a_{r,u})}\left(\frac{z-a_{r,u}}{a_{r,d}-a_{r,u}}\right)dz$$

$$E\left(a^{\kappa}_{r,2}\mathbb{1}_{a_{r,1}<a^{\kappa}_{r,2}<a_{r,3}}\right) = \int_{a^{\kappa}_{r,u}}^{a_{r,d}} \frac{y}{(a_{r,d}-a_{r,u})}\left(\frac{y-a_{r,u}}{a_{r,d}-a_{r,u}}\right)\left(\frac{a_{r,d}-y}{a_{r,d}-a_{r,u}}\right)dy$$

$$E\left(a_{r,1}\mathbb{1}_{a^{\kappa}_{r,2}<a_{r,1}<a_{r,3}}\right) = \int_{a^{\kappa}_{r,u}}^{a_{r,d}} \frac{x}{(a_{r,d}-a_{r,u})}\left(\frac{x-a^{\kappa}_{r,u}}{a_{r,d}-a_{r,u}}\right)\left(\frac{a_{r,d}-x}{a_{r,d}-a_{r,u}}\right)dx$$

$$\begin{aligned}
& E\left(a_{r,3}\mathbb{1}_{a_{r,1}<a_{r,3}<a^{\kappa}_{r,2}}\right) \\
= & \int_{a^{\kappa}_{r,u}}^{a_{r,d}} \frac{\left(z-a^{\kappa}_{r,u}+a^{\kappa}_{r,u}\right)}{(a_{r,d}-a_{r,u})}\left(\frac{z-a^{\kappa}_{r,u}+a^{\kappa}_{r,u}-a_{r,u}}{a_{r,d}-a_{r,u}}\right)\left(\frac{a^{\kappa}_{r,d}-\left(z-a^{\kappa}_{r,u}+a^{\kappa}_{r,u}\right)}{a_{r,d}-a_{r,u}}\right)dz \\
& + \int_{a_{r,u}}^{a^{\kappa}_{r,u}} \frac{z}{(a_{r,d}-a_{r,u})}\left(\frac{z-a_{r,u}}{a_{r,d}-a_{r,u}}\right)dz \\
= & \frac{1}{\left(a_{r,d}-a_{r,u}\right)^3}\int_0^{a_{r,d}-a^{\kappa}_{r,u}} \frac{\left(z+a^{\kappa}_{r,u}\right)}{(a_{r,d}-a_{r,u})}\left(\frac{z+a^{\kappa}_{r,u}-a_{r,u}}{a_{r,d}-a_{r,u}}\right)\left(\frac{a^{\kappa}_{r,d}-z-a^{\kappa}_{r,u}}{a_{r,d}-a_{r,u}}\right)dz \\
& + \frac{\left(\rho\sigma_v^2\kappa\right)^2\left(3a_{r,u}+2\rho\sigma_v^2\kappa\right)}{6\left(a_{r,d}-a_{r,u}\right)^2} \\
= & \frac{1}{\left(a_{r,d}-a_{r,u}\right)^3}\int_0^{a_{r,d}-a^{\kappa}_{r,u}} \left(z+a^{\kappa}_{r,u}\right)\left(z+\rho\sigma_v^2\kappa\right)\left(a^{\kappa}_{r,d}-z-a^{\kappa}_{r,u}\right)dz \\
& + \frac{\left(\rho\sigma_v^2\kappa\right)^2\left(3a_{r,u}+2\rho\sigma_v^2\kappa\right)}{6\left(a_{r,d}-a_{r,u}\right)^2} \\
= & \frac{1}{\left(a_{r,d}-a_{r,u}\right)^3}\int_0^{a_{r,d}-a^{\kappa}_{r,u}} \left(\begin{array}{c} -z^3+z^2\left(a^{\kappa}_{r,d}-2a^{\kappa}_{r,u}-\rho\sigma_v^2\kappa\right)+z\left(\left(\rho\sigma_v^2\kappa+a^{\kappa}_{r,u}\right)\left(a^{\kappa}_{r,d}-a^{\kappa}_{r,u}\right)-\rho\sigma_v^2\kappa a^{\kappa}_{r,u}\right) \\ +\rho\sigma_v^2\kappa a^{\kappa}_{r,u}\left(a^{\kappa}_{r,d}-a^{\kappa}_{r,u}\right) \end{array}\right)dz \\
& + \frac{\left(\rho\sigma_v^2\kappa\right)^2\left(3a_{r,u}+2\rho\sigma_v^2\kappa\right)}{6\left(a_{r,d}-a_{r,u}\right)^2} \\
= & \frac{1}{\left(a_{r,d}-a_{r,u}\right)^3}\left(\begin{array}{c} -\frac{\left(a_{r,d}-a^{\kappa}_{r,u}\right)^4}{4}+\frac{\left(a_{r,d}-a^{\kappa}_{r,u}\right)^3}{3}\left(a^{\kappa}_{r,d}-2a^{\kappa}_{r,u}-\rho\sigma_v^2\kappa\right) \\ +\frac{\left(a_{r,d}-a^{\kappa}_{r,u}\right)^2}{2}\left(\left(\rho\sigma_v^2\kappa+a^{\kappa}_{r,u}\right)\left(a^{\kappa}_{r,d}-a^{\kappa}_{r,u}\right)-\rho\sigma_v^2\kappa a^{\kappa}_{r,u}\right) \\ +\rho\sigma_v^2\kappa a^{\kappa}_{r,u}\left(a^{\kappa}_{r,d}-a^{\kappa}_{r,u}\right)\left(a_{r,d}-a^{\kappa}_{r,u}\right) \end{array}\right) \\
& + \frac{\left(\rho\sigma_v^2\kappa\right)^2\left(3a_{r,u}+2\rho\sigma_v^2\kappa\right)}{6\left(a_{r,d}-a_{r,u}\right)^2}
\end{aligned}$$

$$E\left(a^{\kappa}_{r,2}\mathbb{1}_{a_{r,1}<a^{\kappa}_{r,2}<a_{r,3}}\right) = \frac{1}{\left(a_{r,d}-a_{r,u}\right)^3}\left(\begin{array}{c} -\frac{\left(a_{r,d}-a^{\kappa}_{r,u}\right)^4}{4}+\frac{\left(a_{r,d}-a^{\kappa}_{r,u}\right)^3}{3}\left(a_{r,d}-2a^{\kappa}_{r,u}-\rho\sigma_v^2\kappa\right) \\ +\frac{\left(a_{r,d}-a^{\kappa}_{r,u}\right)^2}{2}\left(\left(a^{\kappa}_{r,u}+\rho\sigma_v^2\kappa\right)\left(a_{r,d}-a^{\kappa}_{r,u}\right)+a^{\kappa}_{r,u}\rho\sigma_v^2\kappa\right) \end{array}\right)$$

$$E\left(a_{r,1}\mathbb{1}_{a^{\kappa}_{r,2}<a_{r,1}<a_{r,3}}\right) = \frac{1}{\left(a_{r,d}-a_{r,u}\right)^3}\left(\begin{array}{c} -\frac{\left(a_{r,d}-a^{\kappa}_{r,u}\right)^4}{4}+\frac{\left(a_{r,d}-a^{\kappa}_{r,u}\right)^3}{3}\left(a_{r,d}-2a^{\kappa}_{r,u}\right) \\ +\frac{\left(a_{r,d}-a^{\kappa}_{r,u}\right)^3}{2}a^{\kappa}_{r,u} \end{array}\right)$$

$$
\begin{aligned}
&\left(a_{r,d}-a_{r,u}\right)^{3} E\left(\frac{\underline{a}^{c}}{2}\right) \\
=\ & \frac{\left(a_{r,d}-a_{r,u}^{\kappa}\right)^{4}}{4}-\frac{1}{6}\left(a_{r,d}-a_{r,u}^{\kappa}\right)^{3} \rho \sigma_{v}^{2} \kappa+\left(a_{r,d}-a_{r,u}\right) \frac{\left(a_{r,d}-a_{r,u}^{\kappa}\right)^{2}}{2}\left(2 \rho \sigma_{v}^{2} \kappa+a_{r,u}\right) \\
&+\rho \sigma_{v}^{2} \kappa a_{r,u}^{\kappa}\left(a_{r,d}-a_{r,u}\right)\left(a_{r,d}-a_{r,u}^{\kappa}\right)+\frac{\left(\rho \sigma_{v}^{2} \kappa\right)^{2}\left(3 a_{r,u}+2 \rho \sigma_{v}^{2} \kappa\right)}{6\left(a_{r,d}-a_{r,u}\right)^{2}}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
E\left(\underline{a}^{c}\right) =\ & \frac{\left(a_{r,d}-a_{r,u}\right)}{2}\left(1-\frac{\rho \sigma_{v}^{2} \kappa}{\left(a_{r,d}-a_{r,u}\right)}\right)^{4}-\frac{\rho \sigma_{v}^{2} \kappa}{3}\left(1-\frac{\rho \sigma_{v}^{2} \kappa}{\left(a_{r,d}-a_{r,u}\right)}\right)^{3} \\
&+\left(2 \rho \sigma_{v}^{2} \kappa+a_{r,u}\right)\left(1-\frac{\rho \sigma_{v}^{2} \kappa}{\left(a_{r,d}-a_{r,u}\right)}\right)^{2}+\frac{2 \rho \sigma_{v}^{2} \kappa a_{r,u}^{\kappa}}{\left(a_{r,d}-a_{r,u}\right)}\left(1-\frac{\rho \sigma_{v}^{2} \kappa}{\left(a_{r,d}-a_{r,u}\right)}\right) \\
&+\frac{\left(\rho \sigma_{v}^{2} \kappa\right)^{2}\left(3 a_{r,u}+2 \rho \sigma_{v}^{2} \kappa\right)}{3\left(a_{r,d}-a_{r,u}\right)^{2}}
\end{aligned}
$$

■

7.6 Proof of Theorem 2

STEP 1 : **Determination of the ordinary differential equations system**

Given the best reply of dealer D_2, dealer D_1 chooses y so as to maximize his profit,

$$\Pi_1\left(y, a_{r,1}\right)=\bar{F}_{\kappa}\left(v_2(y)\right) \times\left(y-a_{r,1}\right) \times Q .$$

Then the first order condition (FOC) yields

$$
\begin{aligned}
\frac{\partial \Pi_1\left(y, a_{r,1}\right)}{\partial y} &= 0, \text{ or} \\
\bar{F}_{\kappa}\left(v_2(y)\right)+v_2^{\prime}(y) \times \bar{F}_{\kappa}^{\prime}\left(v_2(y)\right) \times\left(y-a_{r,1}\right) &= 0 .
\end{aligned}
$$

At equilibrium, if a_1 is the optimal strategy ($a_1\left(a_{r,1}\right)=y$), then $v_1(y)$ must verify the FOC such that for each y :

$$\bar{F}_{\kappa}\left(v_2(y)\right)+v_2^{\prime}(y) \times \bar{F}_{\kappa}^{\prime}\left(v_2(y)\right) \times\left(y-v_1(y)\right)=0. \tag{10}$$

Now, given that dealer D_1 quotes $a_1=\left(v_1\right)^{-1}$, then dealer D_2 chooses y so as to maximize her profit Π_2, where

$$
\begin{aligned}
\Pi_2\left(y, a_{r,2}^{\kappa}\right) =\ & \bar{F}\left(v_1(y)\right) \times\left(y-a_{r,2}^{\kappa}\right) \times(Q+\kappa) \\
&+\left(1-\bar{F}\left(v_1(y)\right)\right) \times\left(E\left(a_1\left(a_{r,1}\right) \mid y>a_1\left(a_{r,1}\right)\right)-a_{r,2}^{\kappa}\right) \times \kappa \\
&+\frac{\rho \sigma_{v}^{2}}{2} \kappa \times(Q+\kappa) .
\end{aligned}
$$

Then the first order condition yields :

$$
\begin{aligned}
\frac{\partial \Pi_2\left(y, a_{r,2}^{\kappa}\right)}{\partial y} &= 0, \text{ or} \\
\bar{F}\left(v_1(y)\right)(1+\kappa / Q)+v_1^{\prime}(y) \times \bar{F}^{\prime}\left(v_1(y)\right) \times\left(y-a_{r,2}^{\kappa}\right) &= 0
\end{aligned}
$$

Now, at equilibrium, if a_2 is the optimal strategy, then $v_2(y)$ must verify the first order condition of dealer D_2 such that for each y :

$$\bar{F}(v_1(y))(1+\kappa/Q) + v_1'(y) \times \bar{F}'(v_1(y)) \times (y - v_2(y)) = 0 \tag{11}$$

At last, the equations (10) and (11) give the following system :

$$\begin{aligned} \frac{-\bar{F}_\kappa'(v_2(y))}{\bar{F}_\kappa(v_2(y))} \times v_2'(y) &= \frac{1}{y - v_1(y)}, \\ \frac{-\bar{F}'(v_1(y))}{\bar{F}(v_1(y))} \times v_1'(y) &= \frac{(1+\kappa/Q)}{y - v_2(y)}. \end{aligned}$$

STEP 2 : **Existence of an equilibrium**

Given that $\bar{F}(x) = \frac{a_{r,d}-x}{a_{r,d}-a_{r,u}}$ and $\bar{F}_\kappa(x) = \frac{a^\kappa_{r,d}-x}{a_{r,d}-a_{r,u}}$, the system writes also :

$$v_1'(y) = \frac{(a_{r,d} - v_1(y)) \times (1+\kappa/Q)}{y - v_2(y)}, \tag{12}$$

$$v_2'(y) = \frac{a^\kappa_{r,d} - v_2(y)}{y - v_1(y)}. \tag{13}$$

Following Theorem 3 of Griesmer et al. (1967), since $\frac{a_{r,d}+a^\kappa_{r,d}}{2} > a^\kappa_{r,u}$, we can prove that there exists a multiplicity of [37] equilibria parameterized by $a^{\sup}$. In such an equilibrium :

(i) $\max\left(a_{r,d}, a^\kappa_{r,u}\right) \leq a^{\sup} \leq \frac{a_{r,d}+a^\kappa_{r,d}}{2}$,

(ii) $v_2(a^{\sup}) = a^{\sup}, v_1(a^{\sup}) = a_{r,d}$

(iii) $a^{\inf}$ is such that $v_1\left(a^{\inf}\right) = a_{r,u}$ and $v_2\left(a^{\inf}\right) = a^\kappa_{r,u}$.

■

7.7 Proof of Proposition 1

We prove the Proposition 1 by a sequence of lemmata. First we prove that the inverse functions $v_1$and v_2 are uniformly decreasing with $a^{\sup}$. Then we use the result to compare dealers' profit according to the initial condition considered and we prove that a Pareto-dominant equilibrium is obtained when $a^{\sup} = \left(a^\kappa_{r,d} + a_{r,d}\right)/2$. Finally, we show that this Pareto-Dominant equilibrium is unique.

Lemma 6 *When $a^{\sup(1)} < a^{\sup(2)}$, then $a^{\inf(1)} < a^{\inf(2)}$, $v_1^{(1)}(y) > v_1^{(2)}(y)$ and $v_2^{(1)}(y) > v_2^{(2)}(y)$ for $y \in \left[a^{\inf(2)}, a^{\sup(1)}\right]$.*

Proof :

[37] Both dealers have a positive probability to accommodate the unpreferenced order flow $+Q$.

Let us define $v_j^{(i)}$ for v_j under the set of initial conditions (i) for $j = 1, 2$. Remind that at bounds the bidding functions must be equal to $v_2^{(i)}\left(a^{\sup(i)}\right) = a^{\sup(i)}$, $v_2^{(i)}\left(a^{\inf(i)}\right) = a_{r,u}^{\kappa}$, $v_1^{(i)}\left(a^{\sup(i)}\right) = a_{r,d}$ and $v_1^{(i)}\left(a^{\inf(i)}\right) = a_{r,u}$.

We are going to prove the different points of the lemma by contradiction.

1. Proof related to the following result : $a^{\inf(1)} < a^{\inf(2)}$

Suppose that $a^{\inf(1)} > a^{\inf(2)}$, then there exists y_0 such that

$$\begin{aligned} v_2^{(1)}(y_0) &= v_2^{(2)}(y_0), \\ \left(v_2^{(1)}\right)'(y_0) &> \left(v_2^{(2)}\right)'(y_0), \\ v_2^{(1)}(y) &< v_2^{(2)}(y) \text{ for each } y < y_0. \end{aligned}$$

Using equation (13), the latter expression writes also

$$\frac{a_{r,d}^{\kappa} - v_2^{(1)}(y_0)}{y_0 - v_1^{(1)}(y_0)} > \frac{a_{r,d}^{\kappa} - v_2^{(2)}(y_0)}{y_0 - v_1^{(2)}(y_0)}.$$

Consequently,

$$v_1^{(1)}(y_0) > v_1^{(2)}(y_0).$$

There necesseraly exists y_1 such that $y_1 < y_0$ and

$$\begin{aligned} v_1^{(1)}(y_1) &= v_1^{(2)}(y_1), \\ \left(v_1^{(1)}\right)'(y_1) &> \left(v_1^{(2)}\right)'(y_1). \end{aligned}$$

Using equation (12), the latter expression writes also

$$\frac{\left(a_{r,d} - v_1^{(1)}(y_1)\right) \times (1 + \kappa/Q)}{y_1 - v_2^{(1)}(y_1)} > \frac{\left(a_{r,d} - v_1^{(2)}(y_1)\right) \times (1 + \kappa/Q)}{y_1 - v_2^{(2)}(y_1)},$$

and

$$v_2^{(1)}(y_1) > v_2^{(2)}(y_1).$$

which contradicts the existence of y_0.

2. Proof related to the following result : $v_1^{(1)}(y) > v_1^{(2)}(y)$ and $v_2^{(1)}(y) > v_2^{(2)}(y)$ for $y \in \left[a^{\inf(2)}, a^{\sup(1)}\right]$

Suppose that there exists y_2 such that :

(i) $v_2^{(1)}(y_2) = v_2^{(2)}(y_2)$ and
(ii) $v_2^{(1)}(y) > v_2^{(2)}(y)$ for each $a^{\inf(2)} < y < y_2$

We deduce that :

$$\left(v_2^{(1)}\right)'(y_2) < \left(v_2^{(2)}\right)'(y_2).$$

Using equation (13), we obtain,

$$\frac{a_{r,d}^{\kappa} - v_2^{(1)}(y_2)}{y_2 - v_1^{(1)}(y_2)} < \frac{a_{r,d}^{\kappa} - v_2^{(2)}(y_2)}{y_2 - v_1^{(2)}(y_2)},$$

or, equivalently,

$$v_1^{(1)}(y_2) < v_1^{(2)}(y_2).$$

Then, there exists $y_3 < y_2$ such that

$$\begin{aligned} v_1^{(1)}(y_3) &= v_1^{(2)}(y_3), \\ \left(v_1^{(1)}\right)'(y_3) &< \left(v_1^{(2)}\right)'(y_3). \end{aligned}$$

Following the expression of equation (12), we get

$$\frac{\left(a_{r,d} - v_1^{(1)}(y_3)\right) \times (1+\kappa/Q)}{y_3 - v_2^{(1)}(y_3)} < \frac{\left(a_{r,d} - v_1^{(2)}(y_3)\right) \times (1+\kappa/Q)}{y_3 - v_2^{(2)}(y_3)},$$

$$v_2^{(1)}(y_3) < v_2^{(2)}(y_3).$$

This latter result contradicts the existence of y_2.

In the same way, it can be proved that $v_2^{(1)}(y) > v_2^{(2)}(y)$ for $y \in \left[a^{\inf(2)}, a^{\sup(1)}\right]$.

■

Corollary 2 *When $a^{\sup(1)} < a^{\sup(2)}$, then $a_1^{(1)}(z) < a_1^{(2)}(z)$ for each $z \in [a_{r,u}, a_{r,d}]$ and $a_2^{(1)}(z) < a_2^{(2)}(z)$ for each $z \in \left[a_{r,u}^{\kappa}, a^{\sup(1)}\right]$.*

Proof : Straightforward.■

We abuse the following notations and define

$$\begin{aligned} \Pi_1^{(i)}(z) &= \Pi_1^{(i)}\left(a_1^{(i)}(z), z\right) \text{ for } z \in [a_{r,u}, a_{r,d}], \\ \Pi_2^{(i)}(z) &= \Pi_2^{(i)}\left(a_2^{(i)}(z), z\right) \text{ for } z \in \left[a_{r,u}^{\kappa}, a_{r,d}^{\kappa}\right]. \end{aligned}$$

Lemma 7 *When $a^{\sup(1)} < a^{\sup(2)}$, then $\Pi_1^{(1)}(z) < \Pi_1^{(2)}(z)$ for each $z \in [a_{r,u}, a_{r,d}]$.*

Proof :

STEP 1 : **Dealer $\mathbf{D_1}$'s profit at bounds**

On the lower bound $a_{r,u}$, dealer D_1's profit is such that

$$\begin{aligned}\Pi_1^{(i)}\left(a_{r,u}\right) &= \bar{F}_\kappa\left(v_2^{(i)}\left(a_1^{(i)}(a_{r,u})\right)\right)\times\left(a_1^{(i)}(a_{r,u})-a_{r,u}\right)\times Q\\ &= \left(a^{\inf(i)}-a_{r,u}\right)\times Q\end{aligned}$$

On the upper bound $a_{r,d}$, the expected profit is such that :

$$\begin{aligned}\Pi_1^{(i)}\left(a_{r,d}\right) &= \bar{F}_\kappa\left(v_2^{(i)}\left(a_1^{(i)}(a_{r,d})\right)\right)\times\left(a_1^{(i)}(a_{r,d})-a_{r,d}\right)\times Q\\ &= \bar{F}_\kappa\left(a^{\sup(i)}\right)\times\left(a^{\sup(i)}-a_{r,d}\right)\times Q\end{aligned}$$

STEP 2 : **Comparison of dealer $\mathbf{D_1}$'s profit under different initial conditions**

$$\begin{aligned}\frac{d\Pi_1^{(i)}\left(a_{r,1}\right)}{da_{r,1}} &= \bar{F}_\kappa\left(v_2^{(i)}\left(a_1^{(i)}(a_{r,1})\right)\right)\times\left(\left(a_1^{(i)}\right)'(a_{r,1})-1\right)\times Q\\ &\quad+\left(a_1^{(i)}\right)'(a_{r,1})\times\left(v_2^{(i)}\right)'\left(a_1^{(i)}(a_{r,1})\right)\times\bar{F}'_\kappa\left(v_2^{(i)}\left(a_1^{(i)}(a_{r,1})\right)\right)\times\left(a_1^{(i)}(a_{r,1})-a_{r,1}\right)\times Q\\ &= \left(a_1^{(i)}\right)'(a_{r,1})\\ &\quad\times\left(\left(v_2^{(i)}\right)'\left(a_1^{(i)}(a_{r,1})\right)\times\bar{F}'_\kappa\left(v_2^{(i)}\left(a_1^{(i)}(a_{r,1})\right)\right)\times\left(a_1^{(i)}(a_{r,1})-a_{r,1}\right)+\bar{F}_\kappa\left(v_2^{(i)}\left(a_1^{(i)}(a_{r,1})\right)\right)\right)\times Q\\ &\quad-\bar{F}_\kappa\left(v_2^{(i)}\left(a_1^{(i)}(a_{r,1})\right)\right)\times Q.\end{aligned}$$

Using the equation (10), we obtain

$$\frac{d\Pi_1^{(i)}\left(a_{r,1}\right)}{da_{r,1}}=-\bar{F}_\kappa\left(v_2^{(i)}\left(a_1^{(i)}(a_{r,1})\right)\right)\times Q.$$

Since $a^{\sup(1)}<a^{\sup(2)}$, then, by using Lemma (6), we get

$$\Pi_1^{(1)}\left(a_{r,u}\right)<\Pi_1^{(2)}\left(a_{r,u}\right),$$

Given that the function $y\rightarrow\bar{F}_\kappa\left(y\right)\times\left(y-a_{r,d}\right)\times Q$ is increasing on $\left[a_{r,d},\frac{a_{r,d}+a_{r,d}^\kappa}{2}\right]$, we deduce

$$\Pi_1^{(1)}\left(a_{r,d}\right)<\Pi_1^{(2)}\left(a_{r,d}\right).$$

Now, suppose that there exists z_0 such that :

(i) $\Pi_1^{(1)}\left(z_0\right)=\Pi_1^{(2)}\left(z_0\right)$ and

(ii) $\Pi_1^{(1)}\left(z\right)<\Pi_1^{(2)}\left(z\right)$ for each $a_{r,u}<z<z_0$,

We deduce that

$$\frac{d\Pi_1^{(1)}}{dz}\left(z_0\right)>\frac{d\Pi_1^{(2)}}{dz}\left(z_0\right).$$

This expression writes also

$$\bar{F}_\kappa\left(v_2^{(1)}\left(a_1^{(1)}(z_0)\right)\right) < \bar{F}_\kappa\left(v_2^{(2)}\left(a_1^{(2)}(z_0)\right)\right)$$

Now, by assumption

$$\bar{F}_\kappa\left(v_2^{(1)}\left(a_1^{(1)}(z_0)\right)\right)\left(a_1^{(1)}(z_0)-z_0\right)Q = \bar{F}_\kappa\left(v_2^{(2)}\left(a_1^{(2)}(z_0)\right)\right)\left(a_1^{(2)}(z_0)-z_0\right)Q$$

which implies that

$$a_1^{(1)}(z_0) > a_1^{(2)}(z_0)$$

However this inequality contradicts Corollary 2.■

If $a^{\sup} = \frac{a_{r,d}+a^\kappa_{r,d}}{2}$, then, under this initial condition, the expected profit of dealer D_1 is uniformly larger than any other profits determined under other initial conditions.

Lemma 8 *When $a^{\sup(1)} < a^{\sup(2)}$, then $\Pi_2^{(1)}(z) < \Pi_2^{(2)}(z)$ for each $z \in \left[a^\kappa_{r,u}, a^\kappa_{r,d}\right]$.*

Proof :

STEP 1 : **Dealer D_2's profit at bounds**

On the lower bound $a^\kappa_{r,u}$, dealer D_2's profit of is such that

$$\begin{aligned}
\Pi_2^{(i)}\left(a^\kappa_{r,u}\right) &= \bar{F}\left(v_1^{(i)}\left(a_2^{(i)}(a^\kappa_{r,u})\right)\right)\times\left(a_2^{(i)}(a^\kappa_{r,u})-a^\kappa_{r,u}\right)\times(Q+\kappa)\\
&+\left(\int_{a_{r,u}}^{v_1^{(i)}\left(a_2^{(i)}(a^\kappa_{r,u})\right)} a_1^{(i)}(s)\,f(s)\,ds-\left(1-\bar{F}\left(v_1^{(i)}\left(\left(a_2^{(i)}(a^\kappa_{r,u})\right)\right)\right)\right)\times a^\kappa_{r,u}\right)\times\kappa\\
&+\frac{\rho\sigma_v^2}{2}\kappa\times(Q+\kappa)\\
&= \bar{F}\left(v_1^{(i)}\left(a^{\inf}\right)\right)\times\left(a^{\inf}-a^\kappa_{r,u}\right)\times(Q+\kappa)\\
&+\left(\int_{a_{r,u}}^{v_1^{(i)}\left(a^{\inf}\right)} a_1^{(i)}(s)\,f(s)\,ds-\left(1-\bar{F}\left(v_1^{(i)}\left(a^{\inf}\right)\right)\right)\times a^\kappa_{r,u}\right)\times\kappa\\
&+\frac{\rho\sigma_v^2}{2}\kappa\times(Q+\kappa)\\
&= \left(a^{\inf}-a^\kappa_{r,u}\right)\times(Q+\kappa)+\frac{\rho\sigma_v^2}{2}\kappa\times(Q+\kappa)
\end{aligned}$$

On the upper bound $a^\kappa_{r,d}$, expected profit is such that

$$
\begin{aligned}
\Pi_2^{(i)}\left(a_{r,d}^{\kappa}\right) &= \bar{F}\left(v_1^{(i)}\left(a_2(a_{r,d}^{\kappa})\right)\right)\times\left(a_2^{(i)}(a_{r,d}^{\kappa})-a_{r,d}^{\kappa}\right)\times(Q+\kappa)\\
&+\left(\int_{a_{r,u}}^{v_1^{(i)}\left(a_2^{(i)}(a_{r,d}^{\kappa})\right)} a_1^{(i)}(s)\,f(s)\,ds-\left(1-\bar{F}\left(v_1^{(i)}\left(\left(a_2^{(i)}(a_{r,d}^{\kappa})\right)\right)\right)\right)\times a_{r,d}^{\kappa}\right)\times\kappa\\
&+\frac{\rho\sigma_v^2}{2}\kappa\times(Q+\kappa)\\
&= \bar{F}\left(v_1^{(i)}\left(a^{\sup}\right)\right)\times\left(a^{\sup}-a_{r,d}^{\kappa}\right)\times(Q+\kappa)\\
&+\left(\int_{a_{r,u}}^{v_1^{(i)}(a^{\sup})} a_1^{(i)}(s)\,f(s)\,ds-\left(1-\bar{F}\left(v_1^{(i)}\left(a^{\sup}\right)\right)\right)\times a_{r,d}^{\kappa}\right)\times\kappa\\
&+\frac{\rho\sigma_v^2}{2}\kappa\times(Q+\kappa)\\
&= \left(\int_{a_{r,u}}^{a_{r,d}} a_1^{(i)}(s)\,f(s)\,ds-a_{r,d}^{\kappa}\right)\times\kappa+\frac{\rho\sigma_v^2}{2}\kappa\times(Q+\kappa)
\end{aligned}
$$

STEP 2 : **Comparison of the dealer $\mathbf{D_2}$'s profit under different initial conditions**

$$
\begin{aligned}
\frac{d\Pi_2^{(i)}\left(a_{r,2}^{\kappa}\right)}{da_{r,2}^{\kappa}} &= \bar{F}\left(v_1^{(i)}\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\right)\times\left(\left(a_2^{(i)}\right)'(a_{r,2}^{\kappa})-1\right)\times(Q+\kappa)\\
&+\left(a_2^{(i)}\right)'(a_{r,2}^{\kappa})\times\left(v_1^{(i)}\right)'\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\times\bar{F}'\left(v_1^{(i)}\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\right)\times\left(a_2^{(i)}\left(a_{r,2}^{\kappa}\right)-a_{r,2}^{\kappa}\right)\times(Q+\kappa)\\
&+\left(\begin{array}{c}\left(a_2^{(i)}\right)'(a_{r,2}^{\kappa})\times\left(v_1^{(i)}\right)'\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\times a_2^{(i)}(a_{r,2}^{\kappa})\times f\left(v_1^{(i)}\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\right)\\ -1+\bar{F}\left(v_1^{(i)}\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\right)+\left(a_2^{(i)}\right)'(a_{r,2}^{\kappa})\times\left(v_1^{(i)}\right)'\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\times\bar{F}'\left(v_1^{(i)}\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\right)\times a_{r,2}^{\kappa}\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\Pi_2^{(i)}\left(a_{r,2}^{\kappa}\right)}{da_{r,2}^{\kappa}} &= \bar{F}\left(v_1^{(i)}\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\right)\times\left(a_2^{(i)}\right)'\left(a_{r,2}^{\kappa}\right)\times(Q+\kappa)\\
&+\left(a_2^{(i)}\right)'(a_{r,2}^{\kappa})\times\left(\left(v_1^{(i)}\right)'\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\times\bar{F}'\left(v_1^{(i)}\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\right)\times\left(a_2^{(i)}\left(a_{r,2}^{\kappa}\right)-a_{r,2}^{\kappa}\right)\times Q\right)\\
&+\left(-1+\bar{F}\left(v_1^{(i)}\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\right)\right)\times\kappa-\bar{F}\left(v_1^{(i)}\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\right)\times(Q+\kappa)
\end{aligned}
$$

Using the expression (11), this expression is simplified :

$$
\frac{d\Pi_2^{(i)}\left(a_{r,2}^{\kappa}\right)}{da_{r,2}^{\kappa}}=-\kappa-\bar{F}\left(v_1^{(i)}\left(a_2^{(i)}(a_{r,2}^{\kappa})\right)\right)\times Q.
$$

Since $a^{\sup(1)}<a^{\sup(2)}$, then, by using the Lemma (6), we get

$$
\begin{aligned}
\Pi_2^{(1)}\left(a_{r,u}^{\kappa}\right) &< \Pi_2^{(2)}\left(a_{r,u}^{\kappa}\right),\\
\Pi_2^{(1)}\left(a_{r,d}^{\kappa}\right) &< \Pi_2^{(2)}\left(a_{r,d}^{\kappa}\right).
\end{aligned}
$$

Now, we are going to prove that if

$$\frac{d\Pi_2^{(1)}\left(a_{r,2}^{\kappa}\right)}{da_{r,2}^{\kappa}} > \frac{d\Pi_2^{(2)}\left(a_{r,2}^{\kappa}\right)}{da_{r,2}^{\kappa}} \text{ for each } a_{r,2}^{\kappa} \in \left[a_{r,u}^{\kappa}, a_{r,d}^{\kappa}\right],$$

then, given the initial and final conditions, we get $\Pi_2^{(1)}\left(a_{r,2}^{\kappa}\right) < \Pi_2^{(2)}\left(a_{r,2}^{\kappa}\right)$ for each $a_{r,2}^{\kappa} \in \left[a_{r,u}^{\kappa}, a_{r,d}^{\kappa}\right]$. In this purpose, we introduce the following function : $g_{1,2} = v_1 \circ a_2$

Using the system of equations (12) and (13), we get

$$\begin{aligned}
\left(y - v_2\left(y\right)\right) \times v_1'\left(y\right) &= \left(a_{r,d} - v_1\left(y\right)\right) \times (1+\alpha) \\
\left(y - v_1\left(y\right)\right) \times v_2'\left(y\right) &= \left(a_{r,d}^{\kappa} - v_2\left(y\right)\right)
\end{aligned}$$

where $\alpha = \kappa/Q$.

Now, let $y = a_2(a_{r,2}^{\kappa})$, the previous system writes also,

$$\begin{aligned}
\left(a_2\left(a_{r,2}^{\kappa}\right) - a_{r,2}^{\kappa}\right) \times v_1'\left(a_2(a_{r,2}^{\kappa})\right) &= \left(a_{r,d} - v_1\left(a_2(a_{r,2}^{\kappa})\right)\right) \times (1+\alpha) \\
\left(a_2(a_{r,2}^{\kappa}) - v_1\left(a_2(a_{r,2}^{\kappa})\right)\right) \times v_2'\left(a_2(a_{r,2}^{\kappa})\right) &= \left(a_{r,d}^{\kappa} - a_{r,2}^{\kappa}\right)
\end{aligned}$$

Given that

$$\begin{aligned}
v_2'\left(a_2(a_{r,2}^{\kappa})\right) &= \frac{1}{a_2'(a_{r,2}^{\kappa})} \\
g_{1,2}'\left(a_{r,2}^{\kappa}\right) &= v_1'\left(a_2(a_{r,2}^{\kappa})\right) \times a_2'(a_{r,2}^{\kappa})
\end{aligned}$$

then,

$$\begin{aligned}
\left(a_2\left(a_{r,2}^{\kappa}\right) - a_{r,2}^{\kappa}\right) \times g_{1,2}'\left(a_{r,2}^{\kappa}\right) &= \left(a_{r,d} - v_1\left(a_2(a_{r,2}^{\kappa})\right)\right) \times a_2'(a_{r,2}^{\kappa}) \times (1+\alpha) \\
\left(a_2(a_{r,2}^{\kappa}) - v_1\left(a_2(a_{r,2}^{\kappa})\right)\right) &= \left(a_{r,d}^{\kappa} - a_{r,2}^{\kappa}\right) a_2'(a_{r,2}^{\kappa})
\end{aligned}$$

Then, the new system of ordinary differential equations is :

$$\begin{aligned}
a_2'(z) &= \frac{\left(a_2(z) - g_{1,2}(z)\right)}{\left(a_{r,d}^{\kappa} - z\right)} \\
g_{1,2}'(z) &= \frac{a_2'(z) \times \left(a_{r,d} - g_{1,2}(z)\right) \times (1+\alpha)}{\left(a_2(z) - z\right)} \\
&= \frac{\left(a_2(z) - g_{1,2}(z)\right) \times \left(a_{r,d} - g_{1,2}(z)\right) \times (1+\alpha)}{\left(a_{r,d}^{\kappa} - z\right) \times \left(a_2(z) - z\right)}
\end{aligned}$$

with the following boundary conditions

$$\begin{aligned} a_2(a^{\kappa}_{r,u}) &= a^{\inf},\ g_{1,2}(a^{\kappa}_{r,u}) = a_{r,u}, \\ a_2\left(a^{\sup}\right) &= a^{\sup},\ g_{1,2}(a^{\sup}) = a_{r,d}. \end{aligned}$$

Note that

$$\begin{aligned} \left(g^{(1)}_{1,2}\right)'(a^{\kappa}_{r,u}) &= \frac{\left(a^{(1)}_2(a^{\kappa}_{r,u}) - g^{(1)}_{1,2}(a^{\kappa}_{r,u})\right) \times \left(a_{r,d} - g^{(1)}_{1,2}(a^{\kappa}_{r,u})\right) \times (1+\alpha)}{\left(a^{\kappa}_{r,d} - a^{\kappa}_{r,u}\right) \times \left(a^{(1)}_2(a^{\kappa}_{r,u}) - a^{\kappa}_{r,u}\right)} \\ &= \frac{\left(a^{\inf(1)} - a^{\kappa}_{r,u} + a^{\kappa}_{r,u} - a_{r,u})\right) \times (a_{r,d} - a_{r,u}) \times (1+\alpha)}{\left(a^{\kappa}_{r,d} - a^{\kappa}_{r,u}\right) \times \left(a^{\inf(1)} - a^{\kappa}_{r,u}\right)} \\ &= \frac{(a_{r,d} - a_{r,u}) \times (1+\alpha)}{\left(a^{\kappa}_{r,d} - a^{\kappa}_{r,u}\right)} + \frac{\left(a^{\kappa}_{r,u} - a_{r,u}\right) \times (a_{r,d} - a_{r,u}) \times (1+\alpha)}{\left(a^{\kappa}_{r,d} - a^{\kappa}_{r,u}\right) \times \left(a^{\inf(1)} - a^{\kappa}_{r,u}\right)} \\ &> \frac{(a_{r,d} - a_{r,u}) \times (1+\alpha)}{\left(a^{\kappa}_{r,d} - a^{\kappa}_{r,u}\right)} + \frac{\left(a^{\kappa}_{r,u} - a_{r,u}\right) \times (a_{r,d} - a_{r,u}) \times (1+\alpha)}{\left(a^{\kappa}_{r,d} - a^{\kappa}_{r,u}\right) \times \left(a^{\inf(2)} - a^{\kappa}_{r,u}\right)} = \left(g^{(2)}_{1,2}\right)'(a^{\kappa}_{r,u}) \end{aligned}$$

Since $\left(g^{(2)}_{1,2}\right)'(z) > 0$ for each z, we get : $g^{(1)}_{1,2}(a^{\sup(1)}) = a_{r,d} > g^{(2)}_{1,2}(a^{\sup(1)})$.

Now let us assume that there exists z_1 such that

(i) $g^{(1)}_{1,2}(z_1) = g^{(2)}_{1,2}(z_1)$

(ii) $g^{(1)}_{1,2}(z) > g^{(2)}_{1,2}(z)$ for each $z < z_1$,

we deduce that $\left(g^{(1)}_{1,2}\right)'(z_1) < \left(g^{(2)}_{1,2}\right)'(z_1)$.

$$\begin{aligned} \left(g^{(1)}_{1,2}\right)'(z_1) &= \frac{\left(a^{(1)}_2(z_1) - g^{(1)}_{1,2}(z_1)\right) \times \left(a_{r,d} - g^{(1)}_{1,2}(z_1)\right) \times (1+\alpha)}{\left(a^{\kappa}_{r,d} - z_1\right) \times \left(a^{(1)}_2(z_1) - z_1\right)} \\ &= \frac{\left(a^{(1)}_2(z_1) - z_1 + z_1 - g^{(1)}_{1,2}(z_1)\right) \times \left(a_{r,d} - g^{(1)}_{1,2}(z_1)\right) \times (1+\alpha)}{\left(a^{\kappa}_{r,d} - z_1\right) \times \left(a^{(1)}_2(z_1) - z_1\right)} \\ &= \frac{\left(a_{r,d} - g^{(1)}_{1,2}(z_1)\right) \times (1+\alpha)}{\left(a^{\kappa}_{r,d} - z_1\right)} + \frac{\left(z_1 - g^{(1)}_{1,2}(z_1)\right) \times \left(a_{r,d} - g^{(1)}_{1,2}(z_1)\right) \times (1+\alpha)}{\left(a^{\kappa}_{r,d} - z_1\right) \times \left(a^{(1)}_2(z_1) - z_1\right)} \\ &< \frac{\left(a_{r,d} - g^{(2)}_{1,2}(z_1)\right) \times (1+\alpha)}{\left(a^{\kappa}_{r,d} - z_1\right)} + \frac{\left(z_1 - g^{(2)}_{1,2}(z_1)\right) \times \left(a_{r,d} - g^{(2)}_{1,2}(z_1)\right) \times (1+\alpha)}{\left(a^{\kappa}_{r,d} - z_1\right) \times \left(a^{(2)}_2(z_1) - z_1\right)} = \left(g^{(2)}_{1,2}\right)'(z_1) \end{aligned}$$

Given the assumption (i) related to z_1, we obtain

$$a^{(2)}_2(z_1) < a^{(1)}_2(z_1)$$

which contracdicts Corollary (2).

Then,

$$\begin{aligned}
g_{1,2}^{(1)}(a_{r,2}^{\kappa}) &> g_{1,2}^{(2)}(a_{r,2}^{\kappa}) \text{ for each } a_{r,2}^{\kappa} \in \left[a_{r,u}^{\kappa}, a_{r,d}^{\kappa}\right] \\
\frac{d\Pi_2^{(1)}\left(a_{r,2}^{\kappa}\right)}{da_{r,2}^{\kappa}} &= -\kappa - \bar{F}\left(g_{1,2}^{(1)}(a_{r,2}^{\kappa})\right) \times Q > -\kappa - \bar{F}\left(g_{1,2}^{(2)}(a_{r,2}^{\kappa})\right) \times Q = \frac{d\Pi_2^{(2)}\left(a_{r,2}^{\kappa}\right)}{da_{r,2}^{\kappa}}
\end{aligned}$$

under initial conditions, $\Pi_2^{(1)}\left(a_{r,2}^{\kappa}\right) < \Pi_2^{(2)}\left(a_{r,2}^{\kappa}\right)$.■

By using Lemmata (7) and (8), we deduce that there exists a Pareto-Dominant equilibrium when $a^{\sup} = \left(a_{r,d} + a_{r,d}^{\kappa}\right)/2$.

Lemma 9 *The Pareto-dominant equilibrium is unique.*

We are going to show that the ask price quoted by each dealer is a global maximum on $\left[a^{\inf}, a^{\sup}\right]$ where $a^{\sup} = \left(a_{r,d} + a_{r,d}^{\kappa}\right)/2$.

Step 1 : **The unicity of a maximum sell price for dealer D_1**

Let us define the function g_1 by

$$g_1(y) = \left(a_{r,d}^{\kappa} - v_2(y)\right) - \left(y - a_{r,1}\right) v_2'(y) \text{ for } y \in \left[\max\left(a^{\inf}, a_{r,1}\right), \infty\right[$$

Notice that

$$\begin{aligned}
g_1\left(\max\left(a^{\inf}, a_{r,1}\right)\right) &= a_{r,d}^{\kappa} - v_2\left(\max\left(a^{\inf}, a_{r,1}\right)\right) > a_{r,d}^{\kappa} - v_2\left(a^{\sup}\right) = \frac{\rho\sigma_v^2\kappa}{2} > 0, \\
g_1\left(a^{\sup}\right) &= a_{r,d}^{\kappa} - a^{\sup} - \left(a^{\sup} - a_{r,1}\right) = \left(a_{r,1} - a_{r,d}\right) < 0.
\end{aligned}$$

We observe that $g_1'(y) = -2v_2'(y) - (y - a_{r,1})\, v_2''(y)$. Then, using the derivative of equation (13), we obtain

$$g_1'(y) = -v_2'(y)\left(2 - \left(y - a_{r,1}\right)\frac{\left(2 - v_1'(y)\right)}{\left(y - v_1(y)\right)}\right)$$

Then, assume that there exist two maxima y_1 and y_2 i.e $g_1(y_1) = g_1(y_0) = 0$ with $g_1'(y_1) > 0$ and $g_1'(y_0) < 0$ (which is equivalent to assume that conditions are only local). Since $v_2'(y_0) > 0$ (see equation (13), then

$$\begin{aligned}
2 - \frac{\left(y_0 - a_{r,1}\right)\left(2 - v_1'(y_0)\right)}{\left(y_0 - v_1(y_0)\right)} &< 0 \\
2 - \frac{\left(y_0 - a_{r,1}\right)\left(2 - v_1'(y_0)\right)}{\left(a_{r,d}^{\kappa} - v_2(y_0)\right)} v_2'(y_0) &< 0 \text{ (using equation(13)}
\end{aligned}$$

Given that $g_1(y_0) = 0$ i.e. $\left(a_{r,d}^{\kappa} - v_2(y_0)\right) - \left(y_0 - a_{r,1}\right) v_2'(y_0) = 0$,

$$2 - \frac{\left(y_0 - a_{r,1}\right)\left(2 - v_1'(y_0)\right)}{\left(a_{r,d}^{\kappa} - v_2(y_0)\right)} v_2'(y_0) = v_1'(y_0) < 0$$

which contradicts with equation (12).

STEP 2 : **The unicity of a maximum sell price for dealer D_2**

Let us define the function g_2 by

$$g_2(z) = (a_{r,d} - v_1(z)) \times (1+\alpha) - v_1'(z) \times \left(z - a_{r,2}^{\kappa}\right) \text{ for each } z \in \left[\max\left(a^{\inf}, a_{r,2}^{\kappa}\right), \infty\right[$$

Suppose that there exist two local maxima z_0 and z_1, i.e. $g_2(z_0) = g_2(z_1) = 0$. Then, we must have $g_2'(z_0) < 0$ and $g_2'(z_1) > 0$.

Observe that

$$g_2'(z) = -v_1'(z) \times (2+\alpha) - v_1''(z) \times \left(z - a_{r,2}^{\kappa}\right)$$

Using the derivative of equation (12) ($v_1''(z) = v_1'(z) \times \frac{\left(v_2'(z)-(2+\alpha)\right)}{(z-v_2(z))}$), we obtain

$$g_2'(z) = -v_1'(z) \times \left((2+\alpha) - \frac{((2+\alpha) - v_2'(z))}{(z - v_2(z))} \times \left(z - a_{r,2}^{\kappa}\right)\right).$$

$$\begin{aligned} g_2'(z_1) &> 0 \\ (2+\alpha) - \frac{((2+\alpha) - v_2'(z_1))}{(z_1 - v_2(z_1))} \times \left(z_1 - a_{r,2}^{\kappa}\right) &< 0 \end{aligned}$$

Using equation (12) , then the latter expression writes also :

$$(2+\alpha) - \frac{((2+\alpha) - v_2'(z_1))}{(a_{r,d} - v_1(z_1)) \times (1+\alpha)} \times v_1'(z_1) \times \left(z_1 - a_{r,2}^{\kappa}\right) < 0$$

Given that $g_2(z_1) = 0$ i.e. $(a_{r,d} - v_1(z_1)) \times (1+\alpha) = \left(z_1 - a_{r,2}^{\kappa}\right) v_1'(z_1)$,

$$\begin{aligned} (2+\alpha) - \frac{((2+\alpha) - v_2'(z_1))}{\left(z_1 - a_{r,2}^{\kappa}\right) v_1'(z_1) \times (1+\alpha)} \times v_1'(z_1) \times \left(z_1 - a_{r,2}^{\kappa}\right) &< 0 \\ v_2'(z_1) &< 0 \end{aligned}$$

Consequently, there cannot exist two local maxima.

■

7.8 Proof of Proposition 2

The captive order flow κ is such that $a_{r,u}^{\kappa} > \left(a_{r,d} + a_{r,d}^{\kappa}\right)/2$ i.e. $\kappa \geq 2(I_u - I_d)$

Then,

$$a_{r,u} \leq a_{r,d} \leq \frac{a_{r,d} + a_{r,d}^{\kappa}}{2} \leq a_{r,u}^{\kappa} \leq a_{r,d}^{\kappa}$$

Now, we suppose that the preferred dealer D_2 quotes an ask price equal to her reservation price : $a_2\left(a_{r,2}^{\kappa}\right) = a_{r,2}^{\kappa}$ (we will prove ultimately that this reply is the best one).

When $a_1 \geq a^{\kappa}_{r,u}$, dealer D_1 chooses a selling quote that maximizes his profit,

$$\begin{aligned}
\Pi_1(a_{r,1}) &= \Pr(a_1 < a_2) \times (a_1 - a_{r,1}) \times Q \\
&= \Pr\left(a_1 < a^{\kappa}_{r,2}\right) \times (a_1 - a_{r,1}) \times Q \\
&= \bar{F}_{\kappa}(a_1) \times (a_1 - a_{r,1}) \times Q
\end{aligned}$$

The first order condition yields to

$$\begin{aligned}
\bar{F}_{\kappa}(a_1) - f_{\kappa}(a_1) \times (a_1 - a_{r,1}) &= 0 \\
\left(a^{\kappa}_{r,d} - a_1\right) - (a_1 - a_{r,1}) &= 0
\end{aligned}$$

Then, we deduce that

$$a_1 = \frac{a^{\kappa}_{r,d} + a_{r,1}}{2}$$

a_1 is increasing in $a_{r,1} \leq a_{r,d}$. Setting $a_1 = \frac{a^{\kappa}_{r,d}+a_{r,d}}{2} = a^{\sup} \leq a^{\kappa}_{r,u}$ gives dealer D_1 an equal probability to win the auction. However dealer D_1 maximizes his profit when he quotes $a_1 = a^{\kappa}_{r,u}$. Given the dealer D_1' s best reply, dealer D_2 has no chance to execute the unpreferenced order flow and quotes indeed $a_2\left(a^{\kappa}_{r,2}\right) = a^{\kappa}_{r,2}$ (given that dealer D_2 never quotes a price under her cutoff price).

■

7.9 Proofs related to the characterization of the way to quote

STEP 1 : **The symmetric case**

1. **Aggressiveness**

$$\begin{aligned}
\theta_i(a_{r,i}) &= \frac{a^{*}_{NP}(a_{r,i}) - a_{r,i}}{a_{ri}} \\
&= \frac{a_{r,d}}{2a_{r,i}} - \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
E(\theta_i) &= \int_{a_{r,u}}^{a_{r,d}} \left(\frac{a_{r,d}}{2x} - \frac{1}{2}\right) f(x)\,dx \\
&= \frac{1}{(a_{r,d} - a_{r,u})} \times \left[\frac{a_{r,d}}{2}\ln x - \frac{x}{2}\right]_{a_{r,u}}^{a_{r,d}} \\
&= \frac{a_{r,d}}{2(a_{r,d} - a_{r,u})} \ln\left(\frac{a_{r,d}}{a_{r,u}}\right) - \frac{1}{2}
\end{aligned}$$

2. **Probability to post the best price**

Given that $\Pr(D_i \text{ posts the best price} \mid a_{r,i}) = \bar{F}(v(a_i)) = \frac{a_{r,d}-v(a_i)}{a_{r,d}-a_{r,i}}$. At equilibrium, we must have $v(a_i) = a_{r,i}$, then

$$
\begin{aligned}
\Pr\left(D_i \text{ posts the best price}\right) &= \int_{a_{r,u}}^{a_{r,d}} \frac{a_{r,d}-x}{a_{r,d}-a_{r,u}} f(x)\, dx \\
&= \frac{\left[\left(a_{r,d}-x\right)^2\right]_{a_{r,u}}^{a_{r,d}}}{2\left(a_{r,d}-a_{r,u}\right)^2} \\
&= \frac{1}{2}
\end{aligned}
$$

STEP 2 : **Case 2 (when $\kappa > 2\left(I_u - I_d\right)$)**

1. **Aggressiveness**

$$\theta\left(a_{r,1}\right) = \frac{a^{\kappa}_{r,u} - a_{r,1}}{a_{r,1}}$$

Hence,

$$
\begin{aligned}
E\left(\theta_1\right) &= \int_{a_{r,u}}^{a_{r,d}} \left(\frac{a^{\kappa}_{r,u}}{x} - 1\right) f(x)\, dx \\
&= \frac{\left[\left(a^{\kappa}_{r,u} \ln x - x\right)\right]_{a_{r,u}}^{a_{r,d}}}{a_{r,d}-a_{r,u}} \\
&= \frac{a^{\kappa}_{r,u} \ln \frac{a_{r,d}}{a_{r,u}}}{a_{r,d}-a_{r,u}} - 1
\end{aligned}
$$

$$E\left(\theta_2\right) = 0$$

2. **Dealers' expected profits**

$$
\begin{aligned}
E\left(\Pi_2\right) &= \left(\int_{a^{\kappa}_{r,u}}^{a^{\kappa}_{r,d}} \left(a^{\kappa}_{r,u} + \frac{\rho\sigma_v^2}{2}\kappa + \frac{\rho\sigma_v^2}{2}Q - x\right) f_{\kappa}(x)\, dx\right) \times \kappa \\
&= \frac{\kappa}{2\left(a_{r,d}-a_{r,u}\right)} \left(-\left[\left(a^{\kappa}_{r,u} + \frac{\rho\sigma_v^2}{2}\kappa + \frac{\rho\sigma_v^2}{2}Q - x\right)^2\right]_{a^{\kappa}_{r,u}}^{a^{\kappa}_{r,d}}\right) \\
&= \frac{\kappa}{2\left(a_{r,d}-a_{r,u}\right)} \left(\left(a^{\kappa}_{r,u} + \frac{\rho\sigma_v^2}{2}\kappa + \frac{\rho\sigma_v^2}{2}Q - a^{\kappa}_{r,u}\right) - \left(a^{\kappa}_{r,u} + \frac{\rho\sigma_v^2}{2}\kappa + \frac{\rho\sigma_v^2}{2}Q - a^{\kappa}_{r,d}\right)\right) \\
&\quad \times \left(a^{\kappa}_{r,u} - a^{\kappa}_{r,d} + \rho\sigma_v^2\left(\kappa + Q\right)\right) \\
&= \frac{\kappa}{2\left(a_{r,d}-a_{r,u}\right)} \left(\frac{\rho\sigma_v^2}{2}\left(\kappa+Q\right) - \left(a^{\kappa}_{r,u} + \frac{\rho\sigma_v^2}{2}\left(\kappa+Q\right) - a^{\kappa}_{r,d}\right)\right) \times \left(a^{\kappa}_{r,u} - a^{\kappa}_{r,d} + \rho\sigma_v^2\left(\kappa+Q\right)\right) \\
&= \frac{\left(a_{r,u} - a_{r,d} + \rho\sigma_v^2\left(\kappa+Q\right)\right)}{2} \times \kappa
\end{aligned}
$$

$$
\begin{aligned}
E\left(\Pi_1\right) &= \left[\int_{a_{r,u}}^{a_{r,d}} \left(a_{r,u}^{\kappa} - x\right) f\left(x\right) dx\right] \times Q \\
&= \frac{\left[\frac{-1}{2}\left(a_{r,u}^{\kappa} - x\right)^2\right]_{a_{r,u}}^{a_{r,d}}}{a_{r,d} - a_{r,u}} \times Q \\
&= \frac{\left[\left(a_{r,u}^{\kappa} - a_{r,u}\right)^2 - \left(a_{r,u}^{\kappa} - a_{r,d}\right)^2\right]}{2\left(a_{r,d} - a_{r,u}\right)} \times Q \\
&= \frac{2\rho\sigma_v^2\kappa - \left(a_{r,d} - a_{r,u}\right)}{2} \times Q
\end{aligned}
$$

■

7.10 Comments on Effect 2

In Effect 2 , we analyze the asymmetry created solely by the payoff function coming from the execution of the preferenced trade. To do so, we analyze the ODE system in restricting the distributions' supports of $a_{r,1}$ and $a_{r,2}^S$ to be equal to the same interval $[a_{r,u}, a_{r,d}]$. Then, the system of ODE that results from the first order condition of the asymmetric Nash equilibrium is :

$$
\begin{aligned}
\frac{-\bar{F}'\left(v_2\left(y\right)\right)}{\bar{F}\left(v_2\left(y\right)\right)} \times v_2'\left(y\right) &= \frac{1}{y - v_1\left(y\right)} \\
\frac{-\bar{F}'\left(v_1\left(y\right)\right)}{\bar{F}\left(v_1\left(y\right)\right)} \times v_1'\left(y\right) &= \frac{1+\alpha}{y - v_2\left(y\right)}
\end{aligned}
$$

where $\alpha = \kappa/Q$. Now, let us suppose that the utility function of dealer D_2 is $U\left(y\right) = \left(y - a_{r,2}\right)^{1+\alpha}$, then it is direct to verify that we get an identical system of ODE. Note that since $U''\left(y\right) > 0$, it characterizes a risk-lover agent. We could have, equivalently, set $\bar{H} = \bar{F}^{\frac{1}{1+\alpha}}$ and get the same system of ODE. ■

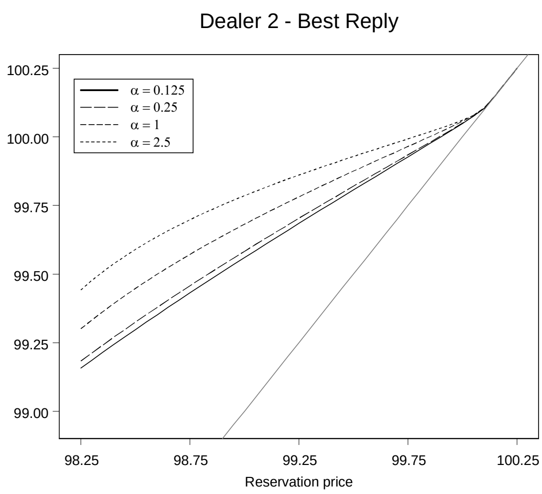

FIGURE A1 where $\alpha = \kappa/Q$.

7.11 Proofs included in the section 'Discussions'

EXTENSION 1 : **Payment for Order Flow**

As we mentioned in the framework the reservation price is obtained in a mean variance framework, as in Biais (1993). If now we include a payment for order flow denoted by τ, the reservation prices change to incorporate this incremental cost.

In case of dealer D_2 should accommodate her preferenced order flow and the nonpreferenced trade, her reservation price $a_r^\tau(I_2, (Q+\kappa))$ is defined such that :

$$E\left(-\exp\left(-\rho \times \tilde{\pi}_2\left(a_r^\tau\left(I_2, (Q+\kappa)\right)\right)\right) \mid I_2\right) = E\left(-\exp\left(-\rho \times \tilde{\pi}_2(0)\right) \mid I_2\right)$$

where $\tilde{\pi}_2\left(a_r^\tau\left(I_2, (Q+\kappa)\right)\right) = a_r^\tau\left(I_2, (Q+\kappa)\right) \times (Q+\kappa) - \tau \times \kappa + \left(I_2 - (Q+\kappa)\right) \times \tilde{v}$ and $\tilde{\pi}_2(0) = I_2 \times \tilde{v}$. Given that

$$\begin{aligned} & E\left(-\exp\left(-\rho \times \tilde{\pi}_2(0)\right) \mid I_2\right) \\ = \; & -\exp\left(-\rho \times \left(E\left(\tilde{\pi}_2(0) \mid I_2\right) - \frac{\rho}{2}\operatorname{Var}\left(\tilde{\pi}_2(0) \mid I_2\right)\right)\right) \\ = \; & -\exp\left(-\rho \times \left(\mu \times I_2 - \frac{\rho\sigma_v^2}{2} I_2^2\right)\right) \end{aligned}$$

$$\begin{aligned} & E\left(-\exp\left(-\rho \times \tilde{\pi}_2\left(a_r^\tau\left(I_2, (Q+\kappa)\right)\right)\right) \mid I_2\right) \\ = \; & -\exp\left(-\rho \times \left(a_r^\tau\left(I_2, (Q+\kappa)\right) \times (Q+\kappa) - \tau \times \kappa + \mu \times \left(I_2 - (Q+\kappa)\right) - \frac{\rho\sigma_v^2}{2}\left(I_2 - (Q+\kappa)\right)^2\right)\right) \\ = \; & -\exp\left(-\rho \times \left(\left(\mu \times I_2 - \frac{\rho\sigma_v^2}{2}\left(I_2\right)^2\right)\right)\right) \\ & \times \exp\left(-\rho \times \left(a_r^\tau\left(I_2, (Q+\kappa)\right) \times (Q+\kappa) - \tau \times \kappa - \mu \times (Q+\kappa) - \frac{\rho\sigma_v^2}{2}\left((Q+\kappa)^2 - 2I_2(Q+\kappa)\right)\right)\right) \end{aligned}$$

It is straightforward to show that :

$$a_r^\tau\left(I_2, (Q+\kappa)\right) \times (Q+\kappa) = \mu + \tau \times \frac{\kappa}{(Q+\kappa)} + \frac{\rho\sigma_v^2}{2}\left((Q+\kappa) - 2I_2\right)$$

Using $\mu^\tau = \mu + \tau \times \frac{\kappa}{(Q+\kappa)}$, the reservation price is similar to the one obtained in the previous section, given that $a_r^\tau\left(I_2, (Q+\kappa)\right)$ writes also

$$a_r^\tau\left(I_2, (Q+\kappa)\right) = \mu^\tau + \frac{\rho\sigma_v^2}{2}\left((Q+\kappa) - 2I_2\right)$$

In case of dealer D_2 should accommodate only her preferenced order flow, her reservation price $a_r^\tau(I_2, \kappa)$ is defined such that :

$$E\left(-\exp\left(-\rho \times \tilde{\pi}_2\left(a_r^\tau\left(I_2, \kappa\right)\right)\right) \mid I_2\right) = E\left(-\exp\left(-\rho \times \tilde{\pi}_2(0)\right) \mid I_2\right)$$

where $\tilde{\pi}_2 \left(a_r^{\tau}(I_2,\kappa)\right) = a_r^{\tau}(I_2,\kappa) \times \kappa - \tau \times \kappa + (I_2 - \kappa) \times \tilde{v}$

$$\begin{aligned}
& E\left(-\exp\left(-\rho \times \tilde{\pi}_2\left(a_r^{\tau}(I_2,\kappa)\right)\right) \mid I_2\right) \\
= \; & -\exp\left(-\rho \times \left(a_r^{\tau}(I_2,\kappa) \times \kappa - \tau \times \kappa + \mu \times (I_2 - \kappa) - \frac{\rho\sigma_v^2}{2}(I_2-\kappa)^2\right)\right) \\
= \; & -\exp\left(-\rho \times \left(\mu \times I_2 - \frac{\rho\sigma_v^2}{2} I_2^2 + a_r^{\tau}(I_2,\kappa) \times \kappa - \tau \times \kappa - \mu \times \kappa - \frac{\rho\sigma_v^2}{2}\left(-2I_2 \times \kappa + \kappa^2\right)\right)\right)
\end{aligned}$$

$$a_r^{\tau}(I_2,\kappa) = \mu + \tau + \frac{\rho\sigma_v^2}{2}(\kappa - 2I_2)$$

Then, the cutoff price $a_{r,2}^{\kappa,\tau}$ is defined such that

$$\left(a_{r,2}^{\kappa,\tau} - a_r^{\tau}(I_2,\kappa)\right) \times \kappa = \left(a_{r,2}^{\kappa,\tau} - a_r^{\tau}(I_2,(Q+\kappa))\right) \times (Q+\kappa)$$

and we get

$$a_{r,2}^{\kappa,\tau} = \mu + \frac{\rho\sigma_v^2}{2}\left(Q - 2(I_2 - \kappa)\right) = a_{r,2}^{\kappa}$$

Then, including a payment for order flow does not change our equilibrium bidding strategies (whether the ranking of reservation prices is observed or not), since they depend on the cutoff price of the preferred dealer which is unchanged. The payment for order flow has however an impact on the profit earned from the trade by dealer D_2.■

EXTENSION 2 : **Another trading process**

Now, according to the new trading process where dealer D_2 receives a preferenced order flow with probability α, the new payoff function is :

$$A_2(a_2,a_1,I_2) = \begin{cases} \alpha \times (\, a_1 - a_r(I_2,\kappa)) \times \kappa & \text{if } a_2 > a_1 \\ (a_2 - a_r(I_2,Q)) \times Q + \alpha \times (\, a_2 - a_r((I_2-Q),\kappa)) \times \kappa & \text{if } a_2 < a_1 \end{cases}$$

where the expression $(a_2 - a_r(I_2,Q)) \times Q + \alpha \times (\, a_2 - a_r((I_2-Q),\kappa)) \times \kappa$ also writes $(1-\alpha)(a_2 - a_r(I_2,Q)) \times Q + \alpha(a_2 - a_r(I_2,Q+\kappa)) \times (Q+\kappa)$ .Then the cutoff $a_{r,2}^{\kappa,\alpha}$ price is defined such that :

$$\alpha \times (a_{r,2}^{\kappa,\alpha} - a_r(I_2,\kappa)) \times \kappa = \left(a_{r,2}^{\kappa,\alpha} - a_r(I_2,Q)\right) \times Q + \alpha \times (a_{r,2}^{\kappa,\alpha} - a_r((I_2-Q),\kappa)) \times \kappa$$

or, equivalently

$$\begin{aligned}
a_{r,2}^{\kappa,\alpha} &= \mu + \frac{\rho\sigma_v^2}{2}(Q - 2I_2) + \rho\sigma_v^2 \times \alpha \times \kappa \\
&= \mu + \frac{\rho\sigma_v^2}{2}(Q - 2I_2) + \rho\sigma_v^2 \times \kappa_\alpha
\end{aligned}$$

where $\kappa_\alpha = \alpha \times \kappa$. Then, including an uncertainty on the reception of a preferenced order flow does not alter the equilibrium bidding strategies which are now computed as some function of $a_{r,2}^{\kappa,\alpha}$.■