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Equidependence in Qualitative And Duration Models with Application to Credit Risk

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Abstract

The aim of this paper is to introduce factor models for joint analysis of interdependent individual defaults. The default can be characterized either by its occurrence, or by the date of default. In the first case we introduce multivariate dependent models for dichotomous $0-1$ variables, whereas multivariate duration models are considered in the second framework. We study the stochastic process counting the number of defaults and discuss statistical inference.

Keywords : Equidependence, Exchangeability, Durations, Factor Model, Credit Risk, Default Correlation.
JEL number : C33, C41, G21.

Equidépendance dans les modèles qualitatifs et les modèles de durée.

Résumé

Nous introduisons des modèles à facteurs pour l'analyse jointe de défaillances individuelles. La défaillance peut être caractérisée soit par sa survenance, soit par la date à laquelle elle se produit. Dans le premier cas, nous considérons des modèles multivariés avec équidépendance pour variables qualitatives dichotomiques ; dans le second cas nous introduisons des spécifications multivariées pour variables de durée. Nous étudions les propriétés du processus comptant le nombre de défaillances et discutons l'inférence statistique.

Mots clés : équidépendance, échangeabilité, durée, modèle à facteur, risque de crédit, corrélation de défaut.

1 Introduction

Portfolio management for corporate bonds requires a careful analysis of default risk, including the possibility of interdependent default, called default correlation. The credit risk models depend on the definition of default [see e.g. Crouhy, Galai, Mark (2000), Gordy (2000)].

In a crude approach the time of default is often neglected and the analysis focuses on default occurrence. This approach implicitly assumes a portfolio of bonds with identical residual maturities, in order to avoid a bias due to heterogenous maturities. Then corporation default can be characterized by a $0-1$ variable Z_i, where $Z_i = 1$, if there is no default for corporation i, $Z_i = 0$, otherwise.

A more accurate specification takes into account the time of default D_i for any corporation i, $i = 1, ..., n$. Then the analysis can be performed for portfolios of bonds with various maturities.

Finally the analysis can be completed by also considering the recovery aspect. As before recovery can be treated without considering the timing and then is characterized by a recovery rate assigned at default time. Alternatively we can describe the sequence of recovery dates and rates for any given failure.

The aim of this paper is to introduce factor models for joint analysis of default. In section 2 we focus on default occurrence, characterized by $0-1$ variables $Z_i, i = 1, ..., n$. We consider a model with equidependence, in which the distribution of $Z_1, ..., Z_n$ is invariant by permutation. We describe a minimal set of parameters for the joint distribution and recall the factor interpretation of the equidependent (exchangeable) model due to de Finetti [de Finetti (1937)]. We also study the distribution of the count variable $N = \sum_{i=1}^{n} Z_i$ measuring the number of firms which are still alive at the end of the period. In section 3 the timing of default is taken into account. We introduce a factor model for the durations before default $D_i, i = 1, ..., n$, in which the factors have a direct influence on the individual survivor intensities. We explain how to derive the joint duration distribution when the future factor values have been integrated out. We also discuss the properties of the associated count process N_t, providing the number of firms in the portfolio which are still alive at time t. The general results are applied in section 4 to credit porfolios and statistical inference is discussed in section 5, both in a parametric and a nonparametric framework.

2 Dichotomous qualitative models

In this section we consider a set of n dichotomous qualitative variables $Z_1, ..., Z_n$, and assume that their distribution is invariant by permutation. Thus the distribution of $Z_{\sigma(1)}, ..., Z_{\sigma(n)}$ is the same as the distribution of $Z_1, ..., Z_n$ for any permutation σ. The variables are said equidependent, symmetric [Savage (1954) p. 50] or exchangeable in the terminology proposed by Frechet.

2.1 Characterization of the distribution

Equidependence implies restrictions on the joint distribution of $Z_1, ..., Z_n$, which can be described by n independent parameters [Khinchine (1932)].

> **Proposition 1 :** Let us consider equidependent dichotomous qualitative variables $Z_1, ..., Z_n$ and denote by $\mu(k) = E(Z_1...Z_k)$, $k = 1, ..., n$ the cross moments between the variables. Then the joint distribution of $Z_1, ..., Z_n$ is characterized by the set $\mu(1), ..., \mu(n)$.

Proof : First note that the cross moments depend on the number of dichotomous variables involved only, and not on individual indexes since :

$$E[Z_{\sigma(1)}...Z_{\sigma(k)}] = E[Z_1...Z_k],$$

because of equidependence.

Then we have just to explain how to reconstitute the joint probabilities from the cross moments. Let us consider a given probability :

$$P[Z_1 = 1, ..., Z_k = 1, Z_{k+1} = 0, ..., Z_n = 0]$$

$$= E[Z_1...Z_k(1 - Z_{k+1})...(1 - Z_n)],$$

(with appropriate convention for $k = 0$).

By expanding the latter expectation and by using the equidependence property, we immediately deduce the expression of the joint probability in terms of cross-moments $\mu(1), ..., \mu(n)$.

Q.E.D.

In fact we directly get the expression of the joint probability since :

$$
\begin{aligned}
E[Z_1 ... Z_k \quad & (1-Z_{k+1})...(1-Z_n)] \\
& = \sum_{h=0}^{n-k} C_{n-k}^h (-1)^h \mu(k+h) \\
& = \sum_{h=0}^{n-k} C_{n-k}^h (-1)^{n-k-h} \mu(n-h) \\
& = [\sum_{h=0}^{n-k} C_{n-k}^h (-1)^{n-k-h} L^h] \mu(n) \\
& = (-1)^{n-k} \Delta^{n-k} \mu(n),
\end{aligned}
$$

where $\Delta = I - L$ denotes the differencing operator, and $\mu(0) = 1$.

We deduce from the expression of the joint probabilities the constraints that have to be satisfied by the cross-moments to define a genuine probability distribution.

Corollary 1 : Under the equidependence assumption, the cross-moments are such that :

$$(-1)^k \Delta^k \mu(n) \geq 0, \qquad \forall k = 0, ..., n.$$

Therefore, when n is infinitely large, the sequence $\mu(n)$ is completely monotone [see Feller (1966) p. 225]. Then, by applying Hausdorff theorem [see Castelnuovo (1930), (1933), Shohat and Tamarkin (1943), Feller (1966) p. 224], we deduce that $\mu(n)$ is the moment of order n of a distribution G defined on the interval $[0, 1]$:

$$\mu(n) = \int_0^1 \omega^n dG(\omega), \forall n. \tag{1}$$

Thus we get :

$$
\begin{aligned}
P[Z_1 &= 1, ..., Z_k = 1, Z_{k+1} = 0, ..., Z_n = 0] \\
&= (-1)^{n-k} \Delta^{n-k} \mu(n) \\
&= \textstyle\sum_{h=0}^{n-k} C_{n-k}^h (-1)^h \mu(k+h) \\
&= \textstyle\sum_{h=0}^{n-k} C_{n-k}^h (-1)^h \int_0^1 w^{k+h} dG(w) \\
&= \textstyle\int_0^1 w^k (1-w)^{n-k} dG(w).
\end{aligned}
$$

This result due to de Finetti [de Finetti (1937), Savage (1954)] has a simple factor interpretation and explains why the model is also known as a conditionally independent credit risk model [Schoenbucher (2000)].

Corollary 2 : Let us assume a population of infinite size. The dichotomous qualitative variables $Z_1, ..., Z_n, ...$ are equidependent if and only if there exists a random variable W on $[0, 1]$ such that, conditionally to W, $Z_1, ..., Z_n, ...$ are independent identically distributed as Bernoulli distribution $B(1, W)$.

It is important to note that the joint distribution of default is characterized by the sequence $\mu(k), k = 1, 2, ...$ or equivalently (for n large) by the distribution G of the factor. Although it has become common to speak of default correlation, the term correlation is misleading. Indeed the linear correlation between the Z variables involve $\mu(1)$ and $\mu(2)$ only, and neglects the higher moments, that is the possibility of default clustering (for instance when defaults occur ten per ten, or twenty per twenty). Similarly the whole pattern of the factor distribution has to be considered, not only the mean and variance of W. Typically multimodes in distribution F can summarize different regimes for default.

The characterization of equidependence by a one-factor representation is valid for a population with infinite size n [see also Frey, McNeil (2001)]. It implies a positive dependence when $n = \infty$, since :

$$
\begin{aligned}
\text{cov}(Z_i, Z_j) &= E\text{cov}(Z_i, Z_j \mid W) + \text{cov}[E(Z_i \mid W), E(Z_j \mid W)] \\
&= VW > 0.
\end{aligned}
$$

When the size n is finite, negative equidependence can exist. For instance for $n = 2$, $Z_2 = 1 - Z_1$ and $Z_1 \sim B(1, 1/2)$, the equidependence is satisfied with a correlation equal to -1. However when n increases, the range of admissible values of the correlation diminishes and asymptotically does not intersect the set of negative real numbers[3]. Thus if a model with equidependence has to be specified for studying default in a pool of credits and if the size of the pool is not bounded a priori, the correlation coefficient has to be nonnegative.

However negative default correlation can be observed in small pools. For instance let us consider an industrial sector with two firms of similar size. The failure of a firm will likely increase the monopolistic power of the remaining firm and thus decreases its default probability.

2.2 Examples

2.2.1 Independence

The Corollary below is a direct consequence of Proposition 1.

Corollary 3 : Under the equidependence assumption, the variables $Z_1, ..., Z_n$ are independent if and only if :

$$\mu(k) = \mu(1)^k, k = 1, ..., n.$$

2.2.2 Probit model with equicorrelation

A standard model introduced for default correlation is based on a multivariate probit specification [see e.g. Gupton, Finger, Bhatia (1997), Belkin, Suchover, Forest (1998), Finger (1999), Schoenbucher(2000)]. Let us consider latent variables X_i such that :

$$X_i = m + F + u_i, i = 1, ..., n, \tag{2}$$

where m is a constant, F, $u_1, ..., u_n$ independent zero mean gaussian variables, such that $VF = \rho^2, Vu_i = 1 - \rho^2$. Thus the joint distribution of $(X_1, ..., X_n)$

[3] This property is similar to a property of gaussian model with equicorrelation. If $Y_1, ..., Y_n$ are homoscedastic gaussian variables with the same correlation ρ, the condition of positivity for the variance-covariance matrix requires $1+(n-1)\rho > 0$, or $\rho > -1/(n-1)$. Thus, for infinite n, the correlation is nonnegative.

is multivariate gaussian with mean $(m, ..., m)'$ and a variance-covariance matrix Σ which features equicorrelation : $\Sigma = \begin{pmatrix} 1\rho^2 & ... & \rho^2 \\ \rho^2 & & \rho^2 \\ \vdots & & \\ \rho^2 & ... & \rho^2 1 \end{pmatrix}$.

The dichotomous qualitative variables are defined by :

$$Z_i = 1, \text{if} \quad X_i > 0, \quad Z_i = 0, \text{otherwise.} \tag{3}$$

In the so-called firm's value model [see e.g. Black, Scholes (1973), Merton (1974), Vasicek (1997)], the latent variable is the difference between the asset value of the firm and the liability. Default occurs when the obligor's assets fall beneath the value of liabilities.

This model with equidependence involves two parameters only that are m and ρ.

Note that the dichotomous variables $Z_1, ..., Z_n$ are independent conditionally to F, with distribution $B(1, W)$, where :

$$\begin{aligned} W &= P[Z_1 > 0 \mid F] = P[u_i + m + F > 0 \mid F] \\ &= P[-u_i < m + F \mid F] = \Phi(\tfrac{m+F}{\sqrt{1-\rho^2}}). \end{aligned}$$

We deduce how the canonical factor W of Corollary 2 is related with the factor F introduced in the standard probit model.

It has been proposed to extend the model above by suppressing the normality assumption and assuming different distributions G and H for the factor F and the error term u, respectively. A direct consequence of Corollary 2 is that G and H cannot be identified nonparametrically.

2.2.3 Factor with beta distribution

When W follows a beta distribution with positive parameters α, β and density :

$$g(w) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} w^{\alpha-1}(1-w)^{\beta-1} I_{[0,1]}(w),$$

we get :

$P[Z_1 = 1, ..., Z_k = 1, Z_{k+1} = 0, ..., Z_n = 0]$

$= \int_0^1 w^k (1-w)^{n-k} f(w) dw$

$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 w^{k+\alpha-1}(1-w)^{n-k+\beta-1} dw$

$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(k+\alpha)\Gamma(n-k+\beta)}{\Gamma(n+\alpha+\beta)}.$

In particular :

$$\mu(k) = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} \frac{\Gamma(\alpha+\beta)}{\Gamma(k+\alpha+\beta)}.$$

2.3 Count variable

It is often interesting to summarize the values of the set of dichotomous variables $Z_1, ..., Z_n$ by means of the count variable :

$$N = \sum_{i=1}^{n} Z_i. \tag{4}$$

For instance N measures the number of firms, which are still alive at the end of the period.

Under the equidependence assumption the moments of the count variable are easily computed.

Proposition 2 : Under the equidependence assumption, we get :
$E[N^k] = \sum_{l=1}^{k} \alpha_{kl} A_n^l \mu(l), k = 1, 2, ..., n,$
where : $A_n^l = n(n-1)...(n-l+1),$
and the coefficients α_{kl} are uniquely defined by :
$n^k = \sum_{l=1}^{k} \alpha_{kl} A_n^l, \forall k \leq n.$

Proof : See Appendix 1.

Moreover due to the factor representation given in Corollary 2, we immediately deduce the distribution of the count variable for a portfolio of infinite size.

Corollary 4 : Let us consider a population of infinite size $n = \infty$. Under the equidependence assumption, we get :

$$P[N = k] = C_n^k \int_0^1 w^k (1 - w)^{n-k} dG(w),$$

where G is the distribution of factor W. In particular the factorial moments of the count variable are :
$E[N(N-1)...(N-k+1)] = n(n-1)...(n-k+1)\mu(k), \forall k.$

Note that the expression of the moments of the count variable given in Corollary 4 provides an interpretation of the result of Proposition 2 (valid for any finite population size n). Moreover moment conditions of Corollary 4 explain how to identify the factor distribution G from the distribution of the count variable. Indeed we deduce the power moments by :

$$\mu(k) = E[N(N-1)\dots(N-k+1)]/[n(n-1)\dots(n-k+1)], \forall k.$$

These moments can be used to construct the characteristic function of factor W by :

$$E[\exp itW] = \sum_{k=0}^{\infty} \frac{i^k t^k}{k!} \mu(k),$$

which exists since $\mu(k) \in [0, 1], \forall k$. Finally the pdf of the factor is derived by inversion of the Fourier transform.

3 Duration models

The results of section 2 can be extended to study a set of equidependent duration variables. In section 3.1 we introduce a factor model for duration, where the factor has a direct effect on the survivor intensity. Then we explain how to compute at date t the joint distribution of durations for individuals who are still alive. This conditional distribution depends on the available information by means of the current value of the factor and of the number of remaining individuals. The distribution of the count process is studied in section 3.2. Finally the results are particularized to the case of compound autoregressive factor processes.

3.1 Factor duration model

Let us consider at date 0 a set of N_0 individuals (firms) and denote $D_i, i = 1, ..., N_0$ the lifetime of individual i. By analogy with Corollary 2[4], we introduce the following factor duration model.

Definition 1 : The set of durations $D_i, i = 1, ..., N_0$ satisfy a factor duration model with equidependence, if there exist factors $F = \{F_t, t = 0, 1, ...\}$ such that :

i) The durations $D_i, i = 1, ..., N_0$ are independent conditionally to $(F) = (F_t, t$ varying) and N_0, with survivor intensity :

$P[D_i > h + 1 \mid D_i > h, (F), N_0] = \mu(F_{h+1})$, h varying,

where μ is a function into $[0, 1]$.

ii) The factor process is a Markov process, that is the conditional distribution of F_{t+1} given $(F_t, F_{t-1}, ..., F_0, N_0)$ depends on F_t only.

Thus the individual survivor intensities depend on a common factor which creates intensity correlation. Moreover this influence depends on time by means of the time varying factor. [5]

It can be noted that the discrete time duration model is defined by means of the survivor intensity whereas the continuous time duration models usually involve default intensity (or hazard function) [see e.g. Heath, Jarrow, Morton (1992), Lando (1998), Duffie, Singleton (1999) for specifications in continuous time applied to credit risk]. In the continuous time framework the survivor function is given by :

[4] Factor representations of exchangeable distributions of continuous variables have been proposed in the literature (see e.g. Hewitt, Savage (1955),Buhlmann (1960), Freedman (1962)]. In this section we directly specify a convenient factor representation.

[5] When the factor is time independent we get the so-called multivariate mixed proportional hazard (MMPH) model described in Van den Berg (1997), (2001).

$$P[D_i > t \mid (F), N_o] = \exp - \int_0^t \lambda(F_u)du,$$

where $\lambda(F_u)$ is the stochastic infinitesimal default intensity. Thus the discrete time survivor intensity is :

$$P[D_i > h+1 \mid D_i > h, (F), N_o] = \exp - \int_h^{h+1} \lambda(F_u)du.$$

A first order expansion provides :

$$P[D_i > h+1 \mid D_i > h, (F), N_o]$$

$$\sim \exp\left[-\lambda(F_{h+1})\right] \text{ (if } \lambda(F_u) \text{ is smooth).}$$

Thus the discrete time survivor intensity is approximately related to the continuous time default intensity by : $\mu = \exp(-\lambda)$. This relationship gives a link between discrete and continuous time approaches.

The advantage of discrete time factor duration specification is double. First it corresponds to the demand of the Basle Committee which requires the computation at discrete dates of the CreditVaR, that is the amount of reserve necessary to balance the risk included in a credit portfolio. Second the defaults are easy to simulate by a recursive scheme. Let us denote :

$$Z_{i,t} = 1, \text{if} \quad D_i > t, = 0, \text{otherwise},$$

the dichotomous $0-1$ variables saying if firm i is still alive at t. For given initial values F_o, N_o the simulation can be performed along the following steps.

step 1 : draw the future factor value F_1^s in the conditional distribution of F_1 given F_o, N_o ;

step 2 : for each firm i in the set I_o, draw $Z_{i,1}$ in the Bernoulli distribution $B(1, \mu(F_1^s))$.

Then iterate the procedure.

The durations are given by :

$$D_i = \inf\{t : Z_{i,t} = 0\}.$$

3.2 Conditional intensities

In the factor duration model the basic survivor intensity is defined conditionally to the past, current and future values of the factors. However the notion of intensity depends on the conditioning set and it is interesting to study how it evolves with time when only past and current values of the factors are known [6]. Let us consider the situation at date t. There is a set $I_t = \{i : D_i > t\}$ of $N_t = \text{Card } I_t$ individuals who are still alive. The information available at time t includes : this set I_t, the lifetimes of individuals already disappeared $D_j, j \notin I_t$, the current and lagged values $\underline{F_t}$ of the factor and the initial population size N_0. The conditional joint survivor function of the durations is :

$$P_t[D_i > t + h_i, i\epsilon I_t]$$

$$= E_t(P[D_i > t + h_i, i\epsilon I_t \mid I_t, D_j, j \notin I_t, (F), N_0])$$

$$= E_t[\Pi_{i\epsilon I_t} P[D_i > t + h_i \mid D_i > t, (F), N_0]]$$

$$= E_t[\Pi_{i\epsilon I_t} \Pi_{k=t}^{t+h_i-1} \mu(F_{k+1})]$$

$$= E_t[\mu(F_{t+1})^{N_t} \mu(F_{t+2})^{N_{t+1}(h)} ......\mu(F_{t+H})^{N_{t+H-1}(h)}],$$

where $h_i \geq 1$, i varying, $H = \max h_i$ and $N_{t+k}(h) = \text{Card } \{i : i\epsilon I_t$ and $h_i \geq k\}$.

The successive numbers $N_{t+k}(h)$ are deterministic functions of the population size at t, that is N_t. Then from assumptions of Definition 1, a sufficient

[6]It can be checked that in an affine framework underlying values of the factors can be recovered if we observed both the term structures of T-bonds and corporate bonds [see e.g. Gourieroux, Monfort, Polimenis (2002)].

information set includes N_t, F_t only.

Proposition 3 : When the durations satisfy a factor duration model with equidependence :
$P_t[D_i > t + h_i, i\epsilon I_t] = E[\mu(F_{t+1})^{N_t}\mu(F_{t+2})^{N_{t+1}(h)}....\mu(F_{t+H})^{N_{t+H-1}(h)} \mid F_t, N_t]$.

Thus by integrating out the future values of the factors, we have introduced equidependence between the durations and implicitely selected a specific copula. The specification of the copula by means of unobservable factors has the advantage of being easily understandable and implementable by simulation. Moreover the factor representation explains how the copula is modified with the population size n [see e.g. Van den Berg (1997) or Li (2000) for direct specifications of copulas on duration data].

A similar computation provides the conditional survivor function of a given firm $i\epsilon I_t$:

$$P_t[D_i > t + h] = E[\mu(F_{t+1})...\mu(F_{t+h}) \mid F_t, N_t].$$

We deduce the associated survivor intensity.

Corollary 5 : The conditional survivor intensity is given by :

$$P_t[D_i > t + h + 1 \mid D_i > t + h] = \mu(h; F_t, N_t),$$

where :

$$\begin{aligned}\mu(h; F_t, N_t) = \quad & E(\mu(F_{t+1})...\mu(F_{t+h+1}) \mid F_t, N_t) \\ & [E(\mu(F_{t+1})...\mu(F_{t+h}) \mid F_t, N_t)]^{-1}.\end{aligned}$$

For structural interpretations it is important to distinguish the basic and conditional survivor intensities given in Definition 1 and Corollary 5, respectively. The basic intensity is a deterministic function of factor F, and is stochastic, if the future values of F are not observable. On the contrary, at date t, F_t and N_t are known and the conditional intensity $\mu(h, F_t, N_t)$ is a deterministic function of h. To illustrate the difference between both types of intensity, we can consider their limiting behavior when h tends to infinity. If the factor process is stationary, the basic intensity $\mu(F_h)$ is a stationary process and cannot converge in the long run. On the contrary we can expect that $\mu(h; F_t, N_t)$ tends to a limit when h tends to infinity. Thus a "mean reverting" effect can be observed on the conditional intensity whereas it does not exist for the underlying basic intensity [7].

Finally note that the conditional survivor intensity can be written as :

$$\mu(h; F_t, N_t) = \overset{Q_t}{E} [\mu(F_{t+h+1}) | F_t, N_t],$$

where Q_t is the probability distribution with pdf :

$$\mu(F_{t+1}) \dots \mu(F_{t+h}) / E[\mu(F_{t+1}) \dots \mu(F_{t+h}) | F_t, N_t].$$

Thus it is equal to the expected underlying intensity with respect to a modified measure.

3.3 Interpretation with failure indicators

We can relate the previous results to those of section 2. From Proposition 3 the distribution of the durations conditionally to the information available at date t is invariant by permutation, which is the equidependence (exchangeability) property. The equidependence is also satisfied by the indicator functions characterizing failure occurrence :

$$Z_{i,t+1} = 1, \text{if} \quad D_i > t+1, = 0, \text{otherwise}, \tag{5}$$

for any $i \epsilon I_t$.

Conditionally to the information at date t, the set $Z_{i,t+1}, i\epsilon I_t$, corresponds to equidependent $0-1$ variables. From Section 2 the conditional joint distribution of these $0-1$ variables is characterized by the cross-moments :

$$\mu_t(1) = E_t(Z_{1,t+1}), ..., \mu_t(k) = E_t(Z_{1,t+1}...Z_{k,t+1}), .. \tag{6}$$

for $k \leq N_t$.

[7] In our discussion the mean-reverting effect is a consequence of a different conditioning and can arise even in a stationary framework. Mean-reverting basic intensities can also be introduced in a nonstationary framework [see Duffie, Singleton (1999), section 6.1, for an example].

The interpretation of the cross-moment $\mu_t(k)$ as a moment of order k of a distribution defined on $[0,1]$ is immediate, since :

$$\mu_t(1) = E_t[\mu(F_{t+1})], ..., \mu_t(k) = E_t[\mu(F_{t+1})^k]. \tag{7}$$

Thus the canonical latent factor corresponding to this set of qualitative variables is simply : $W_{t+1} = \mu(F_{t+1}) \in [0,1]$.

3.4 The count process

i) Joint distribution of (N_t, F_t)

The distribution of the count process (N_t, t varying) can also be easily characterized from Proposition 3 and Corollary 5.

> **Proposition 4 :** i) The bivariate process (N_t, F_t) is a Markov process of order 1.
>
> ii) The conditional distribution of N_{t+1} given F_{t+1}, F_t, N_t is binomial $B[N_t, \mu(F_{t+1})]$.
>
> iii) The conditional distribution of F_{t+1} given F_t, N_t coincides with the conditional distribution of F_{t+1} given F_t only.
>
> Proof : see Appendix 2

In particular the count process satisfies a set of autoregressive models with stochastic coefficients [Andel (1976)]. Indeed we get :

$$E[N_{t+1} \mid N_t, (F)] = \mu(F_{t+1})N_t, \tag{8}$$

$$E[N_{t+1}(N_{t+1}-1) \mid N_t, (F)] = \mu(F_{t+1})^2 N_t(N_t - 1). \tag{9}$$

These autoregressive equations can be used to derive predictions at any horizon H, when the past, current and future values of the factors are known :

$$E[N_{t+H} \mid N_t, (F)] = \Pi_{k=1}^{H} \mu(F_{t+k}) N_t, \tag{10}$$

$$E[N_{t+H}(N_{t+H}-1) \mid N_t, (F)] = [\Pi_{k=1}^{H} \mu(F_{t+k})^2] N_t(N_t-1). \tag{11}$$

ii) Second order properties for factors observable up to time t

We can now deduce the second order properties of the count process, when the factors are observable up to time t. Since the count process is decreasing, thus nonstationary, the first and second order moments have to be computed for any date t and horizon H.

We get :

$$E_t(N_{t+H}) = E(N_{t+H} \mid N_t, F_t) = N_t E(\Pi_{k=1}^{H} \mu(F_{t+k}) \mid F_t), \tag{12}$$

$$E_t[N_{t+H}(N_{t+H}-1)] = E[N_{t+H}(N_{t+H}-1) \mid N_t, F_t] = N_t(N_t-1) E(\Pi_{k=1}^{H} \mu(F_{t+k})^2 \mid F_t). \tag{13}$$

The conditional autocovariance function is derived by iterated expectations. We get :

$$\begin{aligned}
&\gamma_t(H,K) \\
&= cov_t(N_{t+H}, N_{t+H+K}) \\
&= E_t(N_{t+H}N_{t+H+K}) - E_t(N_{t+H})E_t(N_{t+H+K}) \\
&= E_t[N_{t+H}^2 \mu(F_{t+H+1}) \ldots \mu(F_{t+H+K})] - E_t(N_{t+H})E_t(N_{t+H+K}) \\
&= N_t(N_t-1) E_t[\Pi_{k=1}^{H} \mu(F_{t+k})^2 \Pi_{k=H+1}^{H+K} \mu(F_{t+k})] + N_t E_t[\Pi_{k=1}^{H+K} \mu(F_{t+k})] \\
&- N_t^2 E_t[\Pi_{k=1}^{H} \mu(F_{t+k})] E_t[\Pi_{k=1}^{H+K} \mu(F_{t+k})], \text{for} H \geq 0, K \geq 0.
\end{aligned}$$

Therefore :

$$\gamma_t(H,K) = N_t^2 \text{cov}_t[\Pi_{k=1}^H \mu(F_{t+k}), \Pi_{k=1}^{H+K} \mu(F_{t+k})]$$

$$+N_t\{E_t[\Pi_{k=1}^{H+K} \mu(F_{t+k})] - E_t[\Pi_{k=1}^H \mu(F_{t+k})^2 \Pi_{k=H+1}^{H+K} \mu(F_{t+k})]\}. \qquad (14)$$

Both terms in the decomposition of the conditional autocovariance are nonnegative. In particular the conditional variance is given by :

$$\gamma_t(H,0) = V_t(N_{t+H})$$

$$= N_t^2 V_t[\Pi_{k=1}^H \mu(F_{t+k})] + N_t\{E_t(\Pi_{k=1}^H \mu(F_{t+k})] - E_t[\Pi_{k=1}^H \mu(F_{t+k})^2]\}. \qquad (15)$$

3.5 Exponential affine duration model

Closed form expressions of cross-moments and prediction of count variables can be derived for particular specifications of the survivor intensity and of the factor dynamics.

i) The specification

Definition 2 : An exponential affine duration model requires :

i) an exponential affine survivor intensity :

$$\mu(F) = \exp(\alpha' F + \beta),$$

where β is a scalar and α a vector with a dimension equal to the number m of factors ;

ii) a compound autoregressive [CAR] factor process, with a conditional real Laplace transform (moment generating function) given by :

$$E(\exp u' F_{t+1} \mid F_t) = \exp[a(u)' F_t + b(u)],$$

where b is a scalar function and a a m-dimensional function.

An exponential affine factor representation of heterogeneity is usually assumed in the duration models used in the applications to labour [see e.g. Flinn, Heckman (1982), or Heckman, Walker (1990)]. Some constraints on the parameters α, β and on the factor process have to be introduced to ensure survivor intensity between 0 and 1. This condition is satisfied if the components of the factor process are nonnegative and the coefficients α, β are nonpositive. Also note that the exponental affine specification of the survivor intensity in discrete time is the analogue of an affine specification of the infinitesimal default intensity in continuous time.

The CAR processes have been introduced in Darolles, Gouriéroux, Jasiak (2002) [DGJ thereafter] and their analogue in continuous time (called affine processes) in Duffie, Filipovich, Schachermayer (2001). Their importance is due to simple computations of predictions at any horizon H.

Some computations can be simplified if the factor process is stationary with a marginal distribution admitting the Laplace transform :

$$E(\exp u' F_t) = \exp c(u). \tag{16}$$

By applying the iterated expectation theorem, we get :

$$\begin{aligned} \exp\,[c(u)] &= E[E(\,\exp\, u' F_{t+1} \mid F_t)] \\ &= E\,\{\,\exp\,[a(u)' F_t + b(u)]\} \\ &= \exp\,\{c[a(u)] + b(u)\}, \end{aligned}$$

or equivalently :

$$b(u) = c(u) - c[a(u)]. \tag{17}$$

Thus in the stationary framework the dynamics of the factor process can be characterized either by a and b, or by a and c.

ii) Second order properties of the count process for an observable factor

Explicit expressions of the first and second order conditional moments of the count process are deduced from the expressions of the conditional Laplace transform of the factor at horizon H. For instance we have [see DGJ (2002)] :

$$E[\exp(\alpha' F_{t+1} + ... + \alpha' F_{t+H}) \mid F_t] = \exp[A(H)' F_t + B(H)], \qquad (18)$$

$$\text{where}: \quad A(H) = a_\alpha^{oH}(0),$$

$$B(H) = \sum_{h=0}^{H-1} b_\alpha[a_\alpha^{oh}(0)],$$

$$\text{where} \quad a_\alpha(u) = a(u+\alpha), b_\alpha(u) = b(u+\alpha),$$

and a^{oH} denotes function a compounded H times with itself.

Since the stochastic autoregressive coefficient at horizon H

$$\Pi_{k=1}^{H} \mu(F_{t+k}) = \exp[\alpha'(F_{t+1} + ... + F_{t+H}) + H\beta],$$

admits an exponential form, formula (18) can be directly used to derive the conditional expectation and the conditional variance of the count variable. We get :

$$E_t(N_{t+H}) = N_t \exp[a_\alpha^{oH}(0) F_t + \beta H + \sum_{k=0}^{H-1} b_\alpha[a_\alpha^{oh}(o)]], \qquad (19)$$

$$\begin{aligned} V_t(N_{t+H}) &= N_t(N_t - 1)\exp[a_{2\alpha}^{oH}(o) F_t + 2\beta H + \textstyle\sum_{k=0}^{H-1} b_{2\alpha}[a_{2\alpha}^{ok}(o)]] \\ &+ N_t \exp(a_\alpha^{oH}(o) F_t + \beta H + \textstyle\sum_{k=0}^{H-1} b_\alpha[a_\alpha^{ok}(o)]) \\ &- N_t^2 \exp(2a_\alpha^{oH}(o) F_t + 2\beta H + 2\textstyle\sum_{k=0}^{H-1} b_\alpha[a_\alpha^{ok}(o)]. \end{aligned}$$

ii) Autoregressive gamma factor

The autoregressive gamma process is the discrete time counterpart of the Cox-Ingersoll-Ross process [Cox, Ingersoll, Ross (1985)]. This is a one-dimensional process. Conditionally on F_t, the future value F_{t+1} is such that F_{t+1}/c follows a gamma distribution $\gamma(\nu + M_t)$, where M_t is drawn in the Poisson distribution $P[\rho F_t/c]$ (see e.g. Gouriéroux, Jasiak (2001)b]. The parameters are constrained by $c > 0, 1 \geq \rho > 0$. The conditional real Laplace transform (moment generating function) is given by :

$$E(\exp uF_{t+1} \mid F_t) = -\nu \log(1-uc) + \frac{\rho u}{1-uc} F_t. \tag{20}$$

Thus we get :

$$a(u) = \frac{\rho u}{1-uc}, b(u) = -\nu \log(1-uc). \tag{21}$$

We deduce that :

$$a_\alpha(u) = \frac{\rho(u+\alpha)}{1-(u+\alpha)c}.$$

Let us denote by $\lambda_1 < \lambda_2$ the solutions of the associated fixed point equation :

$$\lambda = \frac{\rho(\lambda+\alpha)}{1-(\lambda+\alpha)c}.$$

Then the shifted function a_α satisfies the rational formula :

$$\frac{a_\alpha(u)-\lambda_1}{a_\alpha(u)-\lambda_2} = k\frac{u-\lambda_1}{u-\lambda_2},$$

for some constant k. By recursion we deduce :

$$\frac{a_\alpha^{oh}(0)-\lambda_1}{a_\alpha^{oh}(0)-\lambda_2} = k^h \frac{\lambda_1}{\lambda_2}.$$

4 Credit portfolio

Equidependent models can be used to study the risk included in an homogenous portfolio of credits [8] and compute the required capital (CreditVaR)

[8]The joint analysis of several homogenous portfolios of credit is beyond the scope of the present paper. It can be done by introducing different types of factors, where some are general and others are portfolio specific [see e.g. Gourieroux, Monfort, Polimenis (2002)].

appropriate to hedge credit risk. In the first section we define the portfolio value with and without default and discuss its stochastic properties. Then we give in section 4.2 an illustration of the seasoning effect. Finally large portfolio approximation is discussed in section 4.3.

4.1 Description of the portfolio

Let us consider a portfolio of credits, including N_o contracts of the same type at date $t=0$. Each contract provides a deterministic payoff m_t at date $t=1,...,T$.

With zero default probability the value of the portfolio at date t is deterministic and is equal to :

$$W_t^* = N_o \sum_{h=1}^{T-t} B(t,t+h) m_{t+h}, \tag{22}$$

where $B(t,t+h)$ denotes the price at t of the zero-coupon bond with residual maturity h.

When default can occur, the portfolio value at date t becomes random and is equal to :

$$W_t = \sum_{h=1}^{T-t} B(t,t+h) N_{t+h} m_{t+h}, \tag{23}$$

where the recovery rate is equal to zero[9].

The conditional distribution of W_t at date t can be deduced from the conditional distribution of the future portfolio sizes $N_{t+1},...,N_T$. For instance the first and second order conditional moments of the portfolio value are given by :

[9] W_t is an actuarial value of the portfolio. Interpreted in the standard risk neutral framework, it is implicitely assumed that the shocks on the T-bond interest rates and the factors with effect on default can be priced independently. This assumption is usual in a first step analysis of default risk.

$$E_tW_t = \sum_{h=1}^{T-t} B(t,t+h)m_{t+h}E_t(N_{t+h}), \tag{24}$$

$$V_tW_t = \sum_{h=1}^{T-t} B(t,t+h)^2 m_{t+h}^2 V_t(N_{t+h})$$
$$+2\sum_{h<k}\sum B(t,t+h)B(t,t+k)m_{t+h}m_{t+k}\mathrm{cov}_t(N_{t+h},N_{t+k}).$$

They depend on the information available at date t to compute the conditional expectations and covariances, and on the dynamics of the count process (see section 3.4).

4.2 The seasoning effect

Let us provide some illustration of the seasoning effect, that is of the way the characteristics of the bond evolve with the age of the portfolio. To simplify, we assume credits with constant monthly payments $m_t = m$, $\forall t$, no payment in fine, an interest rate $r^\star$, a maturity T, and a flat term structure of interest rate $B(t,t+h) = (1+r)^{-h}$, with $r = 0.03$. In particular the monthly payment is given by :
$m = C_o \frac{r^\star(1+r^\star)^T}{(1+r^\star)^T-1}$, where C_o denotes the initial balance.

If the initial size of portfolio is N_o, the portfolio value with zero default probability $W_t^f = N_o m \sum_{h=1}^{T-t} \frac{1}{(1+r)^h}$ diminishes exponentially. When default can occur the portfolio value is smaller and given by :

$$W_t = m\sum_{h=1}^{T-t} \frac{1}{(1+r)^h} N_{t+h}.$$

This exercise is easily extended when default is insured, as on the market for mortgage backed securities (MBS). When a default occurs, the insurance company reimburses the remaining balance at the default date, transforming a default into a prepayment or equivalently into a default with a recovery rate equal to one. The value of the insured portfolio is :

$$W_t^s = \sum_{h=1}^{T-t} \frac{1}{(1+r)^h}\{N_{t+h}m + (N_{t+h-1} - N_{t+h})C_{t+h}\}, \tag{25}$$

where C_t denotes the remaining balance at date t :

$$C_t = m \sum_{j=1}^{T-t} \frac{1}{(1+r^\star)^j}.$$

In the insured case the difference between the expected values of portfolio with and without default is due to a timing problem, in which all payments due after default are replaced by a single payment at the random date of default.

To study the effect of default with zero or unitary recovery rate, we provide below some comparative studies on values W_t^f, W_t and W_t^s. We assume $N_o = 100$, a maturity $T = 20$ and a factor survivor intensity process corresponding to a gamma autoregressive scheme (see 3.5 ii) with parameters $\nu = 2$, $\rho = 0.9$, $c = 1$ and initial value $F_o = 1$. We provide in Figure 1 the dynamics of the expected portfolio size.

[Insert Figure 1 : Expected Portfolio Size]

The parameter values of survivor intensity correspond to a very risky pool of credits, that is a junk bond. Approximately 70 % of the credits default before maturity.

In the first simulation study the credit rate is equal to the riskfree rate $r^\star = r = 0.03$. There is no compensation for default. We provide in Figure 2 the expected portfolio values corresponding to the computation which neglects default (W_t^f) and to the correct computations for insured and non insured portfolio (W_t^s and W_t, respectively).

[Insert Figure 2 : Expected Portfolio Value]

We observe the different decreasing patterns of the expected portfolio values, and note that $W_o^f = W_o^s$ because of the standard actuarial equality for $r^\star = r$.

More detailed results are given in Figure 3 and Figure 4, where the distribution of the portfolio value is displayed for the different dates.

[Insert Figure 3 : Distribution of the Portfolio Values of the Noninsured Portfolio]

[Insert Figure 4 : Distribution of the Portfolio Value of the Insured Portfolio]

For the insured portfolio, the distribution is more concentrated, just after the starting date of the credit, where the value is close to the deterministic value W_o^f, and when we are close to maturity.

In Figure 5 we provide the quantiles at 5 % for the insured and noninsured portfolios evaluated at the previous dates. In this scheme the value N_t has been replaced by its expectation taken at date $t = 0$. Since the portfolio values are necessarily positive, the Value at Risk, which is equal to the opposite of the quantile, is negative. This corresponds to the admissible investment line which can be backed on the risky bond. The updating of the CreditVaR for the portfolio without insurance has to be read as follows. Let us denote by VaR(t) the values reported on Figure 5. At date 0, the CreditVaR is VaR(0). At date 1, the credit institution receives an amount of cash equal to mN_1. Thus the VaR at date 1 associated to both the cash and the residual credit portfolio becomes $mN_1 + VaR(1)$, which can be compared with the value $(1 + r)VaR(0)$, and so on.

[Insert Figure 5 : Value at Risk of the Credit Portfolio]

A second simulation study has been performed when a spread is introduced to compensate default. The credit corresponds to the same initial balance as in the previous example $C_o = 14.87$, a rate $r^\star = 0.05$, which implies a constant monthly payment $m = 1.19$. Figure 6 displays the expected portfolio values which can be directly compared with Figure 2, where the rate was smaller. Similarly Figure 7 displays the quantile and Figure 8 the distributions of the portfolio values of the insured portfolio.

[Insert Figure 6 : Expected Portfolio Value $r^\star = 0.05$]

[Insert Figure 7 : Value at Risk $r^\star = 0.05$]

[Insert Figure 8 : Distribution of the Portfolio Value (with insurance and

$r^\star = 0.05)]$

4.3 Large portfolio approximation

The Value at Risk for a credit portfolio is generally computed by simulation as in the examples of section 4.2. However analytical approximations can be derived for portfolios with large size N_o, when the default probabilities are small and the residual maturity $T-t$ is not too large. In this framework, the portfolio has a large size N_t at any date t, $t \leq T$, and normal approximation based on Central Limit Theorem (CLT) can be used. For expository purpose let us consider a residual maturity $T-t=2$. Conditionally to the past and future values of the factors the durations are independent and CLT applies[10]. More precisely the conditional distribution of N_{t+1} given $(\underline{N_t}), (F)$ can be approximated by :

$$N_{t+1} \mid N_t, (F) \sim N[N_t\mu(F_{t+1}), N_t\mu(F_{t+1})[1-\mu(F_{t+1})]].$$

Equivalently we can write :

$$N_{t+1} \sim N_t\mu(F_{t+1}) + \sqrt{N_t\mu(F_{t+1})[1-\mu(F_{t+1})]}u_{t+1}, \tag{26}$$

where the variables u_t are independent standard normal variables.

Similarly we get :

$$N_{t+2} \sim N_{t+1}\mu(F_{t+2}) + \sqrt{N_{t+1}\mu(F_{t+2})(1-\mu(F_{t+2}))}u_{t+2}.$$

After the replacement of N_{t+1} by its expression and the elimination of negligeable terms we get :

$$\begin{aligned} N_{t+2} \sim N_t\mu(F_{t+1})\mu(F_{t+2}) &+ \sqrt{N_t}\sqrt{\mu(F_{t+1})(1-\mu(F_{t+1})}\mu(F_{t+2})u_{t+1} \\ &+ \sqrt{N_t}\sqrt{\mu(F_{t+1})\mu(F_{t+2})[1-\mu(F_{t+2})]}u_{t+2}. \end{aligned} \tag{27}$$

[10] The portfolio is called infinitely fine-grained portfolio by the Basle Committee, when CLT applies (conditionally to the factor). This terminology can be misleading since we will see below that CLT and even the law of large numbers will not apply when the factor is not observed.

Thus given $N_t, (F)$, the joint distribution of $(N_{t+1}, N_{t+2})'$ is approximately gaussian with mean :

$$m_t = N_t[\mu(F_{t+1}), \mu(F_{t+1})\mu(F_{t+2})]',$$

and variance-covariance matrix :

$$\Sigma_t = N_t A_t A_t',$$

where :

$$A_t = \begin{bmatrix} \sqrt{\mu(F_{t+1})(1-\mu(F_{t+1}))} & 0 \\ \mu(F_{t+2})\sqrt{\mu(F_{t+1})(1-\mu(F_{t+1}))} & \sqrt{\mu(F_{t+1})\mu(F_{t+2})[1-\mu(F_{t+2})]} \end{bmatrix},$$

$$A_t A_t' = \mu(F_{t+1}) \begin{bmatrix} 1-\mu(F_{t+1}) & \mu(F_{t+2})[1-\mu(F_{t+1})] & \\ \mu(F_{t+2})[1-\mu(F_{t+1})] & \mu(F_{t+2})[1-\mu(F_{t+2})] & +\mu^2(F_{t+2})[1-\mu(F_{t+1})] \end{bmatrix}.$$

However the future values of the factors have to be reintegrated out, when we consider the observable information at date t. The conditional distribution of $(N_{t+1}, N_{t+2})'$ given N_t, F_t admits the density :

$$\begin{aligned} & g_t(N_{t+1}, N_{t+2}) \\ = \; & \int\int \tfrac{1}{2\Pi} \tfrac{1}{(\det \sum_t)^{1/2}} \exp\{-\tfrac{1}{2}[(N_{t+1}, N_{t+2}) - m_t'] \textstyle\sum_t^{-1} [(N_{t+1}, N_{t+2})' - m_t]\} \\ & f(F_{t+2} \mid F_{t+1}) f(F_{t+1} \mid F_t) dF_{t+2} dF_{t+1}. \end{aligned}$$

Note that the factor heterogeneity affects both the conditional mean, which can create long memory effects on the count process [see e.g. Granger (1980)] and the conditional variance-covariance matrix, which can create heavy tails [Clark (1973)]. Concerning VaR computation both effects have to be taken into account since long memory implies a serial smoothing of the required capital whereas the heavy tails imply large required capital.

5 Statistical inference

In this section we assume that we have observations on J independent pools of credits with the same initial size $N_o = n$. For such a panel model different asymptotic theories can be considered with either the size n, or the number of pools J tending to infinity. As usual for models with equidependence consistent estimators of the parameters of interest can only be derived if J tends to infinity with n fixed, or J and n tend jointly to infinity. We consider qualitative models in section 5.1 and duration models in section 5.2. Finally we discuss in the last section the links between estimation and prediction problems.

5.1 Qualitative models

Let us consider independent identically distributed variables $(Z_{1j}, ..., Z_{nj}), j = 1, ..., J$, measuring default occurrence for J pools of credits. We assume that the n dichotomous $0-1$ variables satisfy a model with equidependence. Thus they correspond to independent factors $W_j, j = 1, ..., J$ with identical distribution G. This distribution G can be parametrized or left unconstrained. In both cases the count variables $N_j = \sum_{i=1}^{n} Z_{ij}, j = 1, ..., J$ are sufficient statistics for the estimation of the factor distribution[11].

i) Parametric framework

Let us consider a parametric family G_θ for the factor distribution. The parameter θ can be estimated by maximum likelihood based on the sufficient statistics N_j, $j = 1, ..., J$. The log-likelihood function is given by :

$$\begin{aligned} L_J(\theta) \propto & \sum_{j=1}^{J} \log[\int_0^1 w^{n_j}(1-w)^{n-n_j} dG_\theta(w)] \\ & = \sum_{j=1}^{J} \log[\int_0^1 w^{n_j}(1-w)^{n-n_j} g_\theta(w) dw], \end{aligned}$$

where g_θ denotes the factor density. Under standard regularity conditions the ML estimator $\hat{\theta}_J$ is consistent, asymptotically normal, when J tends to

[11] This explains why consistent estimators cannot be derived when n tends to infinity with J fixed [see also Frey, McNeil(2001)]. Indeed the knowledge of N_j is equivalent to the knowledge of N_j/n, which is equal to W_j, when n tends to infinity. Thus we observe a finite number of i.i.d. variables $W_j, j = 1, ..., J$ with distribution G, which is not sufficient to estimate consistently G.

infinity. Its asymptotic variance-covariance matrix can be estimated by :

$$\hat{V}(\hat{\theta}_J) = [\frac{1}{J}\sum_{j=1}^{J} h(n_j)h(n_j)']^{-1}, \tag{28}$$

where :

$$h(n_j) = \frac{\int_0^1 w^{n_j}(1-w)^{n-n_j}\frac{\partial \log g_{\hat{\theta}}}{\partial \theta}(w)g_{\hat{\theta}}(w)dw}{\int_0^1 w^{n_j}(1-w)^{n-n_j}g_{\hat{\theta}}(w)dw}. \tag{29}$$

In a general framework the log-likelihood function and the asymptotic variance-covariance matrix involve integrals which have to be computed numerally or by Monte-Carlo methods [see e.g. Gouriéroux, Monfort (1993)]. In special cases, for instance for a beta distribution of the factor, the integrals can admit closed form expressions.

ii) Nonparametric framework

Two simple nonparametric estimation method of distribution G can also be proposed, when both n and J tend to infinity.

Firstly it is possible to estimate the moments of the heterogeneity distribution, which can be used to recover the distribution of the factor by means of the characteristic function. More precisely let us consider a moment order K_n, function of n. Then, from Corollary 4, the moment of order k, $k \leq K_n$, can be consistently estimated by :

$$\hat{\mu}_{J,n}(k) = \frac{1}{J}\frac{\sum_{j=1}^{J} N_j(N_j-1)...(N_j-k+1)}{n(n-1)...(n-k+1)}. \tag{30}$$

We deduce a consistent estimator of the characteristic function of factor W :

$$\hat{\phi}(u) = \sum_{k=0}^{K_n} \hat{\mu}_{J,n}(k)\frac{i^k u^k}{k!}, \tag{31}$$

when the number of pools J and the order K_n tend to infinity at appropriate rates with n. The associated probability density function of the factor is deduced by inverting the estimated characteristic function.

This approach can be used to introduce a diagnostic tool for testing the independence hypothesis. We can plot the quantities $\hat{\gamma}(k) = \frac{1}{k} \log \hat{\mu}_{J,n}(k)$ as function of k and check if they are approximately constant [see Corollary 3].

A second approach is based on the remark that $\hat{W}_j = N_j/n$ is a consistent estimator of W_j, when n tends to infinity. Then a consistent estimator of G, when n and J tend to infinity is the sample cdf of $\hat{W}_1, \ldots, \hat{W}_J$.

iii) The assumption of independent pools

The previous inference is based on the assumption of a large number J of independent pools of credits. It is useful to discuss this assumption[12] in practice. Three approaches can be distinguished to construct the pools.

a) Let us first consider all the credits granted at a given month, which can include several hundred thousands of credits in the consumer credit case. A natural idea is to partition this very large set of credits into a large number of pools, each of them being sufficiently large. For instance a set of 100 000 credits can be partitioned into 100 pools of 1 000 credits. However this approach has the following drawback. If within a pool the individual defaults are equidependent, they are also likely equidependent between the pools. The equidependence between the pools contradicts the independence assumption, and the estimators introduced in the previous section are no longer consistent when J, n both tend to infinity.

b) A way to circumvent this difficulty consists in stratifying the population of credits according to an observed characteristic. For instance the firms can be classified according to their industrial sector (or to their rating). Then it is more natural to assume the independence of default between sectors (or classes of rating grade) whereas equidependence can exist within sectors. However it is difficult to assume the same magnitude of default correlation in all the sectors, that is identical distributions for factors $W_j, j = 1, ..., J$.

[12]This assumption underlies the estimation approaches proposed in the literature on credit risk [see e.g. Gordy, Heitfield (2002)].

c) Alternatively the pools can correspond to sets of credits granted at different months t, with periods of repayment, which do not intersect. As an illustration, let us consider for date t credits with maturity 1. Then the pool index is $j = t$, $t = 1, ..., J$ and the factor is $W_j = W_t$. The factor values are likely serially dependent, which contradicts the independence assumption between the pools. However, if the factor process (W_t) is strongly stationary, the estimators above are still consistent, even if their asymptotic accuracy is modified to account for serial dependence.

More precisely let us assume that the factor W_t satisfies a Markov process, with conditional distribution $g_\theta(w_t \mid w_{t-1})$. The log-likelihood becomes :

$$L_T(\theta) = \log \int ... \int \Pi_{t=1}^{T} \{w_t^{n_t}(1 - w_t)^{n-n_t} g_\theta(w_t \mid w_{t-1}) dw_t\},$$

(with a given initial condition for w_o), and can be optimized numerically with respect to θ. In this time series framework, the number of dates T can be small leading to inaccurate estimations of parameter θ, that is of default correlation [see Gordy, Heitfield (2002) for a discussion of finite sample properties of maximum likelihood estimators, when the factor values are independent].

5.2 Duration models

i) The observations

When the time of default is taken into account the observations concern J independent sets of duration variables $(D_{1j}, ..., D_{nj})$, $j = 1, ..., J$. From these observations we can compute the counts $N_{h,j}$ giving the number of credits in pool j, which are still alive at age h. Similarly, for these credits, we can define the default indicator variables for the period $(h, h+1)$, denoted by $Z_{h,i,j}$.

ii) Parametric framework

In a parametric framework two types of parameters are generally involved. For illustration let us consider the exponential affine model of section 3.5. The parameters α, β explain how the survivor intensity depend on the factor.

Other parameters can be introduced to specify the factor dynamics, defined by functions a and b.

When the factors are observable, the two types of parameters can be estimated separately. The parameters characterizing the factor dynamics can be deduced from the observations of the factors, whereas α and β parameters are deduced from the durations conditional to the factors.

When the factors are unobservable, the likelihood function involves multiple integrals of large dimension. The parameters can be estimated by a simulation based estimation method for dynamic factor models [see e.g. Gouriéroux, Jasiak (2001)a].

iii) Nonparametric framework

When the factors are unobservable, the joint duration distribution is characterized by the sequence of distributions of the indicator variables ($Z_{h,i,j}, i$ varying) given the past. Moreover, it is known that the counts $N_{o,j} = n, N_{1,j}, ...N_{h,j}$ define a sufficient information set. Thus for age h we can reconstitute the conditional heterogeneity distribution of the basic factor $W_{h,j}$, or equivalently the associated moments at order k : $\mu_h(k; n, n_1..., n_h)$. Consistent estimators of these moments are :

$$\begin{aligned}
& \hat{\mu}_{h,J,n}(k; n, n_1, ..., n_h) \\
= \; & [\textstyle\sum_{j=1}^J N_{h+1,j}(N_{h+1,j} - 1)...(N_{h+1,j} - k + 1)\delta_j(n, n_1, ..., n_h)] \\
& [\textstyle\sum_{j=1}^J (N_{h,j} - 1)...(N_{h,j} - k + 1)\delta_j(n, n_1, ..., n_h)]^{-1},
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad & \delta_j(n, n_1, ..., n_h) = 1, \text{ if } N_{1,j} = n_1, ..., N_{h,j} = n_h, \\
= \; & 0, \text{ otherwise.}
\end{aligned}$$

However these moments are very numerous since they are conditional to the path of the counting process, and this approach will encounter the curse of dimensionality problem.

5.3 Prediction versus estimation

In the sections above we have discussed the estimation of the parameters, but actually our interest is in predicting the risk included in a portfolio of credits, for which default has not yet been observed. Thus in the model we have to specify the links between the pools $j = 1, ..., J$, which are used for estimation purpose, but also the links between these pools and the pool $j = J + 1$ for which risk has to be predicted.

For convenience let us consider qualitative models with two extreme correlation schemes.

Scheme 1 : The factor W_j is identical for all the pools $j = 1, ..., J + 1$. There is the same level of default correlation within and between pools.

Scheme 2 : The factors W_j $j = 1, ..., J + 1$ are i.i.d. There is equidependence within the pools and independence between the pools.

The prediction formulas of the number of default in pool $J + 1$ are different for the two schemes. Let us assume for sake of simplicity that the size n of the pool is large.

In scheme 1 the conditional distribution of N_{J+1} given $N_1, ..., N_J$ coincides with the binomial distribution $B(n, W_j)$. Indeed the common value W_j can be reconstituted from the count data $W_j \sim \frac{N_1 + ... N_J}{nJ} = \bar{N}_J$. It is remarkable that the prediction problem does not involve the factor distribution. Therefore, the prediction can be performed even if the parameters of this distribution are not identifiable.

In scheme 2 the conditional distribution of N_{J+1} given $N_1, ..., N_J$ coincides with a mixture of binomial distributions $B(n, W)$, where W follows the factor distribution. In this scheme this distribution has to be identified in order to get the risk prediction.

Let us finally remark that in scheme 1 the estimated conditional distribution will be $B(n, \bar{N}_J)$, whereas in scheme 2 it will be a mixture of binomial distributions with an estimated heterogeneity distribution with mean $\bar{N}_J$. Thus a model assuming the independence between pools, even if it is misspecified, will imply larger required capital [i.e. CreditVaR] than a model assuming correlation between pools. This explains why the assumption of independence between pools could be suggested by the regulators, since it overestimates the risk.

6 Concluding remarks

In this paper we have introduced model with equidependence for qualitative and duration data, and discussed their factorial interpretations. These models are useful for the analysis of default correlation and the determination of the required capital necessary to hedge the risk of a credit portfolio. For such an application the models are estimated from data on obligors' defaults.

Similar models can be used for pricing credit derivatives. For such an application they concern a risk neutral probability (or pricing kernel) and will be estimated from data on credit prices. In the risk neutral world the default correlation has also to be taken into account, if we consider a credit derivative with a payoff depending jointly of the defaults of the credits included in the portfolio, for instance a "first to default basket" paying 1 euro at date T if all credits of the pool are still alive at this date [see e.g. Gouriéroux, Monfort, Polimenis (2002)].

Appendix 1
Moments of the count variable

We have :

$$
\begin{aligned}
& E(N^k) \\
= \quad & E\{(\textstyle\sum_{i=1}^n Z_i)^k\} \\
= \quad & \textstyle\sum_{i_1=1}^n \ldots \sum_{i_k=1}^n E(Z_{i_1} \ldots Z_{i_k}).
\end{aligned}
$$

Since $Z^p = Z$ for any integer p, the generic term in the summation is equal to $\mu(\ell)$, where ℓ is the number of different indices among $i_1, \ldots, i_k$. Moreover the number of choices of ℓ different indices among $1, \ldots, n$ is A_n^ℓ. Thus we deduce :

$$E[N^k] = \sum_{\ell=1}^{k} \alpha_{k\ell} A_n^\ell \mu(\ell), \qquad k = 1, 2, \ldots, n,$$

where $\alpha_{k\ell}$ is a constant independent of the population size n.

Since the computation is valid for any distribution of Z_i, we get for $Z_i = 1$, $i = 1, \ldots, n$:

$$n^k = \sum_{\ell=1}^{k} \alpha_{k\ell} A_n^\ell \qquad k = 1, 2, \ldots, n.$$

The system uniquely defines the coefficients $\alpha_{k\ell}$.

Appendix 2
Proof of Proposition 4

• Let us denote by $p(n_1, ..., n_{t+1}, f_0, ..., f_{t+1})$ the p.d.f. (with respect to the appropriate measure) of $N_1, ..., N_{t+1}, F_0, ..., F_{t+1}$, conditionally to $N_0 = n_0$. We have :

$$p(n_1, ..., n_{t+1}, f_0, ..., f_{t+1}) = p(n_1, ..., n_{t+1}|f_0, ..., f_{t+1}) \times \Pi_{s=1}^{t+1} p(f_s|f_{s+1})p(f_0).$$

• Moreover the event :

$$\{N_1 = n_1, ..., N_{t+1} = n_{t+1}\}$$

is identical to the event :

$$\{D_i = 1, n_0 - n_1 \text{ times}; D_i = 2, n_1 - n_2 \text{ times}; ...; D_i = t+1, n_t - n_{t+1} \text{ times}\}$$

(and, therefore $D_i > t+1, n_{t+1}$ times).

So this event has the following multinomial conditional probability given $f_0, ..., f_{t+1}$:

$$\frac{n_0!}{(n_0 - n_1)!..(n_t - n_{t+1})!n_{t+1}}![1 - \mu(f_1)]^{n_o - n_1}![\mu(f_1)[1 - \mu(f_2)]]^{n_1 - n_2}...$$

$$[\mu(f_1)...\mu(f_t)[1 - \mu(f_{t+1})]]^{n_t - n_{t+1}}[\mu(f_1)...\mu(f_{t+1})]^{n_{t+1}}$$

• So the conditional p.d.f. $p(n_{t+1}, f_{t+1}|\underline{n}_t, \underline{f}_t)$

is equal to :

$$\frac{p(n_1,...,n_{t+1},f_0,...,f_{t+1})}{p(n_1,...,n_t,f_0,...,f_t)} = \frac{n_t!}{(n_t - n_{t+1})!n_{t+1}!}[1 - \mu(f_{t+1})]^{n_t - n_{t+1}} \times [\mu(f_{t+1})]^{n_{t+1}} p(f_{t+1}|f_t)$$

- Summing $p(n_{t+1}, f_{t+1} | \underline{n_t}, \underline{f_t})$ over the values of n_{t+1} we get :

$$p(f_{t+1} | \underline{n_t}, \underline{f_t}) = p(f_{t+1}/f_t)$$

and, therefore :

$$p(n_{t+1} | \underline{n_t}, \underline{f_{t+1}}) = \frac{n_t!}{(n_t - n_{t+1})! n_{t+1}!} [1 - \mu(f_{t+1})]^{n_t - n_{t+1}} [\mu(f_{t+1})]^{n_{t+1}}$$

which are the point masses of the binomial distribution $B(n_t, \mu(f_{t+1}))$.

Figure 1. Expected Portfolio Size

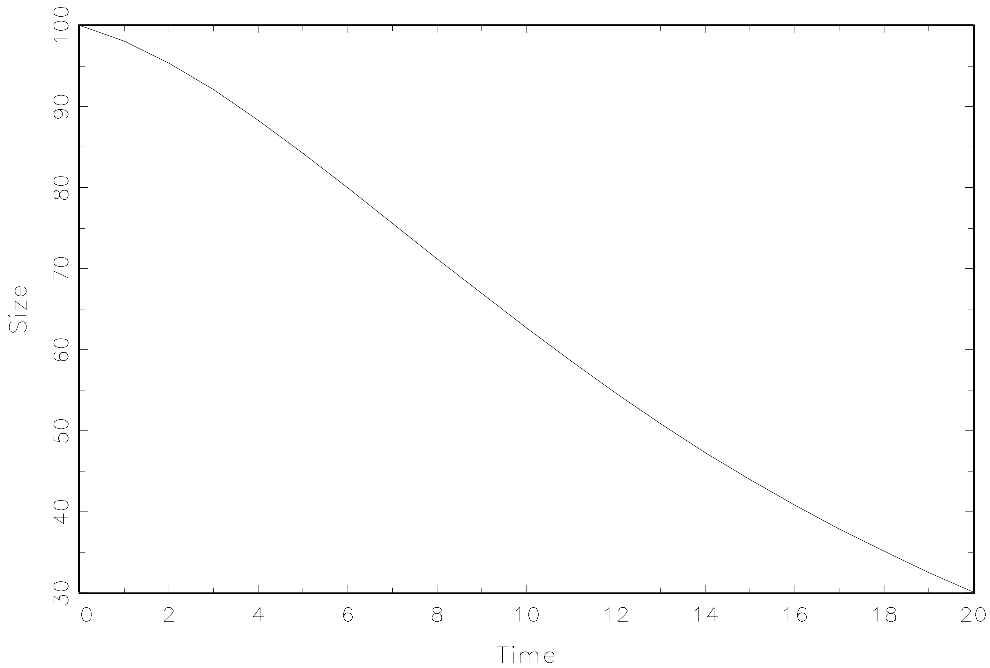


Figure 2. Expected Portfolio Value(riskfree rate):without default(solid), with default and insurance(dashed),with default and without insurance (short dashes)

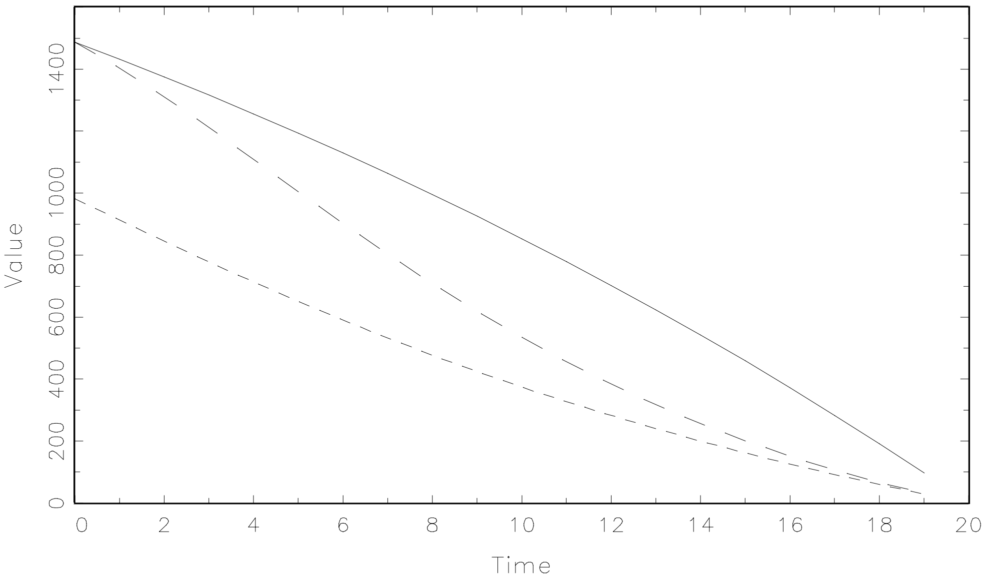


Figure 3. P.D.F. of the Portfolio Value,riskfree rate
(without insurance),T=20,for t=0(right) to 19(left)

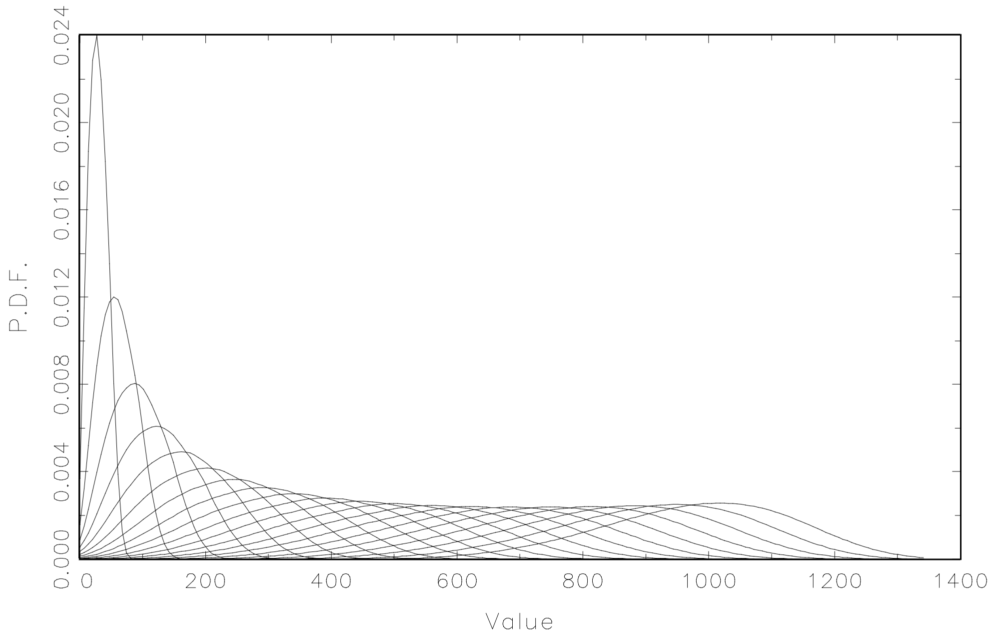


Figure 4. P.D.F. of the Portfolio Value,riskfree rate (with insurance),T=20,for t=1(right) to 19(left)

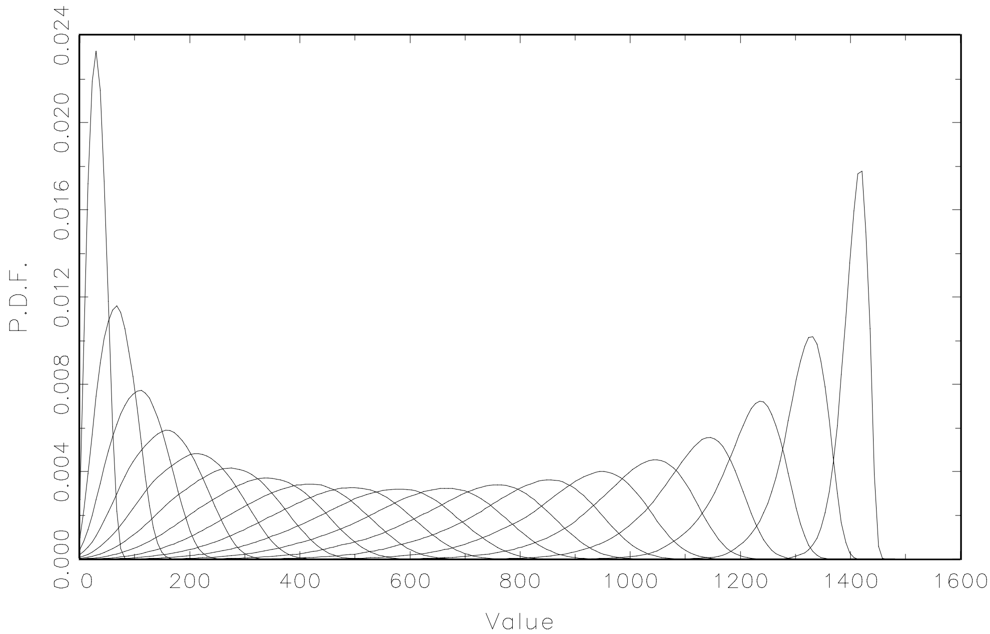


Figure 5. 5% quantile(riskfree rate):with insurance(solid) and without insurance(dashed)

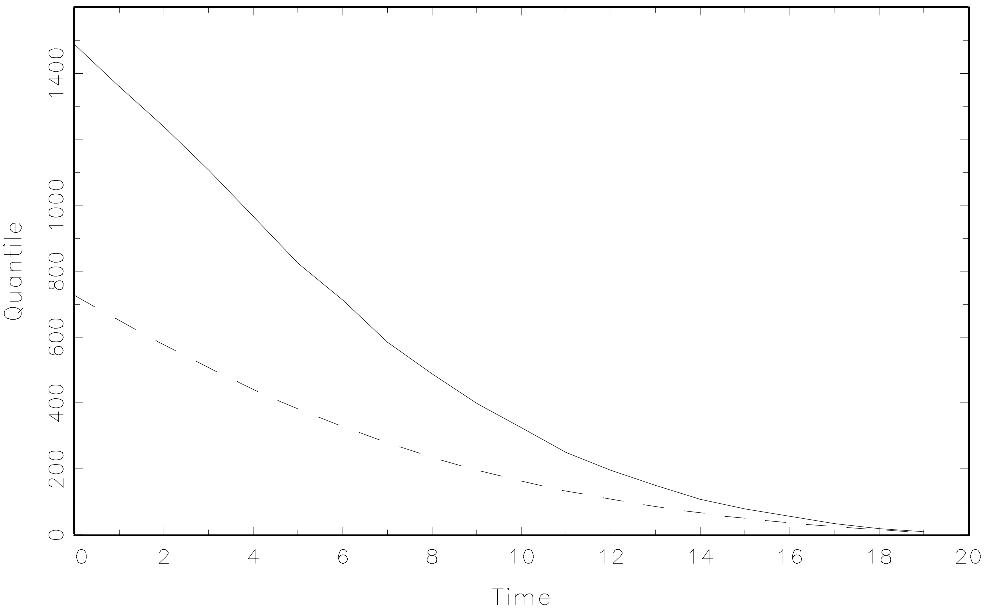
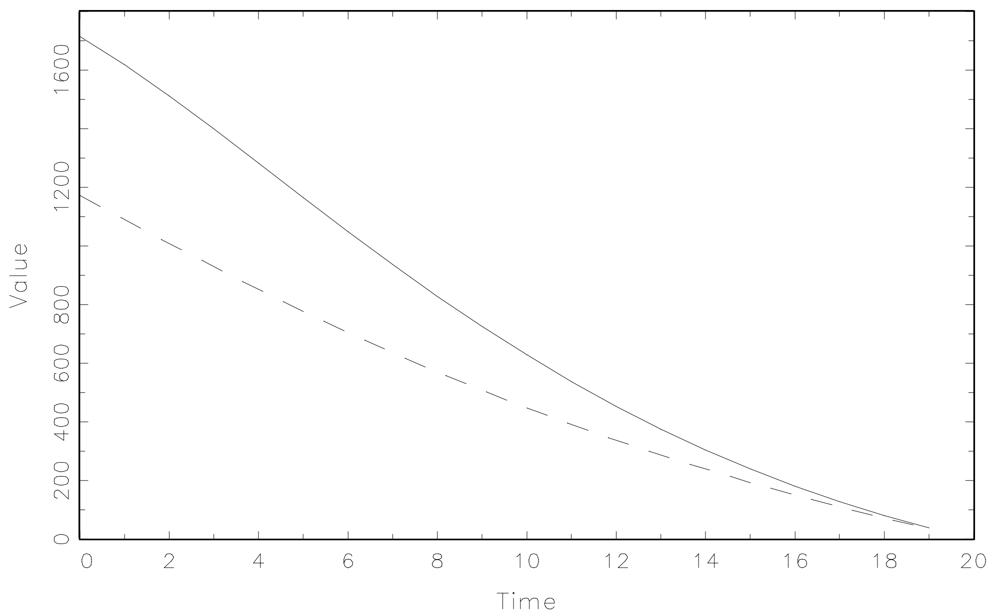


Figure 6. Expected Portfolio Value(with spread): with default and insurance(solid),with default and without insurance(dashed)



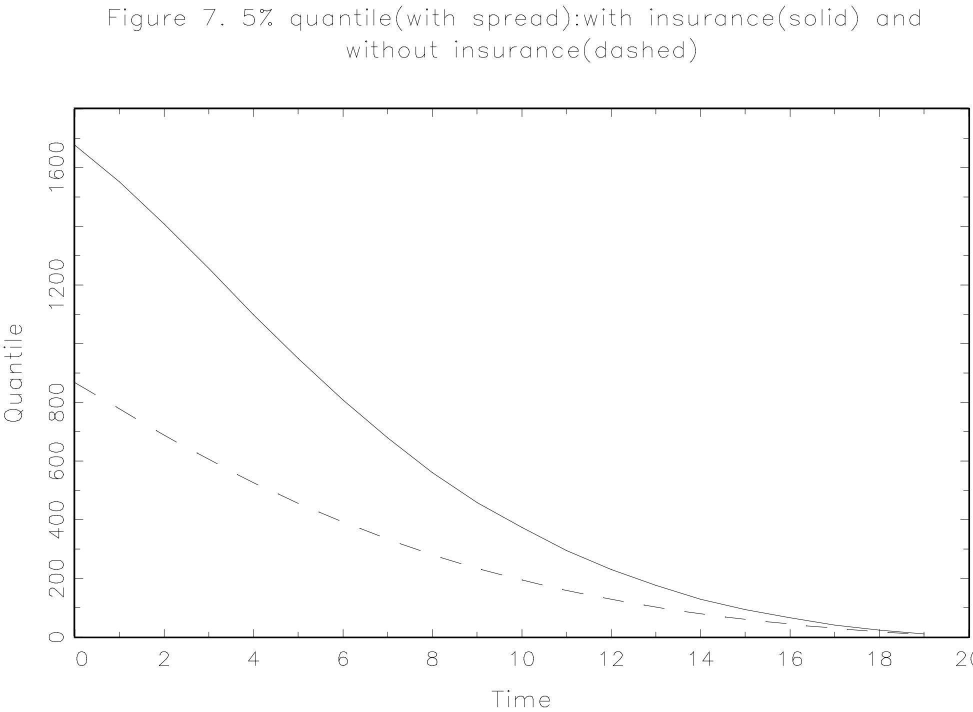
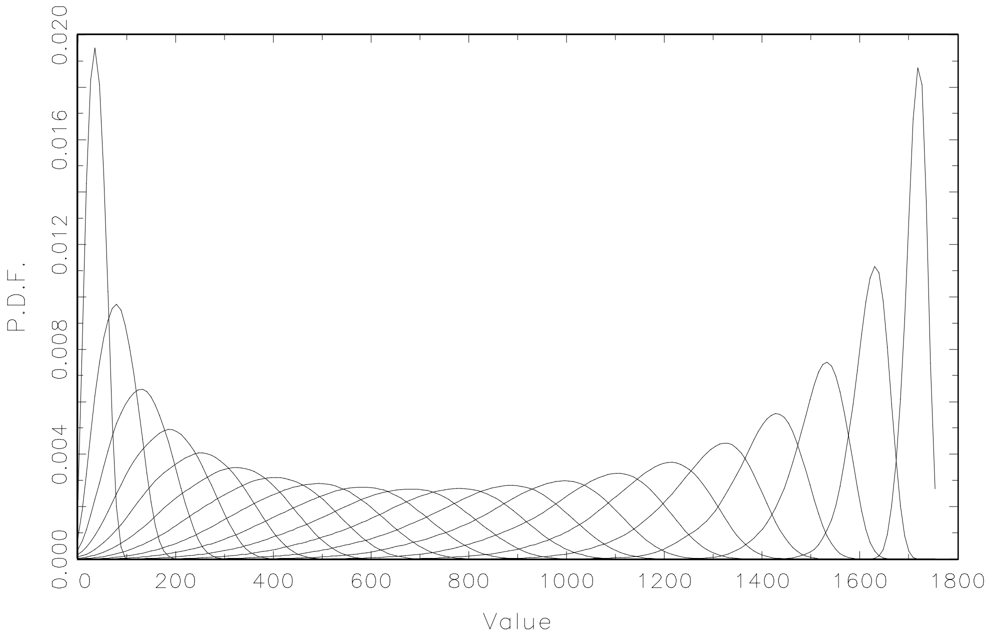


Figure 7. 5% quantile(with spread):with insurance(solid) and without insurance(dashed)
Quantile
0
200
400
600
800
1200
1600
0
2
4
6
8
10
12
14
16
18
20
Time

Figure 8. P.D.F. of the Portfolio Value(with insurance and spread),T=20,
for t=0(right) to 19(left)



References

Andel, J. (1976) : "Autoregressive Series with Random Parameters", Math. Operationsforsch. U. Statistics, 7, 735-471.

Belkin, B., Suchover, S. and L. Forest (1998) : "A One-Parameter Representation of Credit Risk and Transition Matrices", Credit Metrics Monitor, 1, 46-56.

Black, F., and M., Scholes (1973) : "The Pricing of Options and Corporate Liabilities", Journal of Political Economy, 81, 637-654.

Buhlmann, H. (1960) : "Austauschbare Stockastische Variablen und ihre Grenzwertsatze", Univ. Calif. Publ. Stat., 3, 1-35.

Castelnuovo, G. (1930) : "Sul problema dei momenti", Gior. Ist. Ital. Attuari, Vol. 1-2.

Castelnuovo, G. (1933) : "Sur quelques problèmes se rattachant au calcul des probabilités", Ann. Inst. H. Poincaré, Vol. III, fasc. 4.

Clark, P. (1973) : "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices", Econometrica, 41, 135-155.

Cox, J., Ingersoll, J. and S., Ross (1985) : "A Theory of the Term Structure of Interest Rates", Econometrica, 53, 385-408.

Crouhy, M., Galai, D. and R., Mark (2000) : "A Comparative Analysis of Current Credit Risk Models", Journal of Banking and Finance, 24, 59-117.

Darolles, S., Gouriéroux, C. and J., Jasiak (2002) : "Structural Laplace Transform and Compound Autoregressive Model", CREST DP.

Davis, M. (1999) : "Contagion Modeling of Collaterized Bond Obligations", Working Paper, Bank of Tokyo, Mitsubishi.

De Finetti, B. (1937) : "Foresight : Its Logical Laws, Its Subjective Sources", Annales de l'Institut Henri Poincaré, Vol. 7, 1-68, reproduced in Studies in Subjective Probability, ed. Kyburg, H., and H., Smoker (1964), Wiley, p. 93-158.

Delianedis, G., and R., Geske (1998) : "Credit Risk and Risk Neutral Default Probabilities : Information About Rating Migrations and Defaults", Working Paper 19.98, Anderson Graduate School of Business, University of California, Los Angeles.

Duffie, D., Filipovich, D. and W., Schachermayer (2001) : "Affine Processes and Applications in Finance", DP. Stanford Univ.

Duffie, D., and K., Singleton (1999) : "Simulating Correlated Default", DP Standford Univ.

Feller, W. (1966) : "An Introduction to Probability Theory and its Applications", vol. II, Wiley.

Finger, C. (1999) : "Conditional Approach for Credit Metrics Portfolio Distribution", Credit Metrics Monitor, 2, 14-33.

Flinn, C., and J., Heckman (1982) : "Models for the Analysis of Labor Force Dynamics", in R. Basmann and G., Rhodes, eds, Advances in Econometrics, JAI Press, Greenwich.

Freedman, D. (1962) : "Invariants under Mixing which Generalize de Finetti's Theorem", Ann. Math. Stat., 33, 916-923.

Frey, R., and A., McNeil (2001) : "Modelling Dependent Defaults", Discussion paper, ETH Zentrum, Zurich.

Gordy, M. (2000) : "A Comparative Anatomy of Credit Risk Models", Journal of Banking and Finance, 24, 119-149.

Gordy, M., and E., Heitfield (2002) : "Estimating Default Correlation from Short Panels of Credit Rating Performance Data", Federal Reserve Board, Washington.

Gouriéroux, C. and J., Jasiak (2001)a : "Dynamic Factor Models", Econometric Reviews, 20, 385-424.

Gouriéroux, C. and J., Jasiak (2001)b : "Autoregressive Gamma Processes", CREST DP.

Gouriéroux, C., and A., Monfort (1993) : "Simulation Based Inference : a Survey with Special References to Panel Data Models", Journal of Econometrics, 59, 5-33.

Gouriéroux, C., Monfort, A., and V., Polimenis (2002) : "Affine Model for Credit Risk Analysis", CREST DP.

Granger, C. (1980) : "Long Memory Relationship and the Aggregation of Dynamic Models", Journal of Econometrics, 14, 227-238.

Granger, C. and N., Swanson (1997) : "An Introduction to Stochastic Unit Root Processes", Journal of Econometrics, 80, 35-62.

Gupton, G., Finger, G., and M., Bhatia (1997) : "Creditmetrics Technical Document", The Riskmetrics Group, www.riskmetrics.com.

Heath, D., Jarrow, R., and A., Morton (1992) : "Bond Pricing and the Term Structure of Interest Rates", Econometrica, 60, 77-106.

Heckman, J., and R., Walker (1990) : "The Relationship Between Wages and Income and the Timing and Spacing of Births", Econometrica, 58,1411-1441.

Hewitt, E., and L., Savage (1955) : "Symmetric Measures on Cartesian Products", Trans. Amer. Math. Soc., 80, 470-501.

Kealhofer, S. (1995) : "Managing Default Risk in Portfolio of Derivatives", in E. Van Hertsen and P. Fields, editors, Derivative Credit Risk, Advances in Measurement and Management. London, Risk Publications.

Khinchine, A. (1932) : "Sur les classes d'évènements équivalentes", Mathematiceskii Sbornik, Recueil. Math., Moscow, 39-3.

Lando, D. (1998) : "On Cox Processes and Credit Risky Securities", Review of Derivatives Research, 2, 99-120.

Li, D. (2000) : "On Default Correlation : A Copula Function Approach", Working Paper 99.07, The Riskmetrics Group.

Lucas, A., Klaassen, P., and S., Straetmans (1999) : "An Analytic Approach to Credit Risk of Large Corporate Bond and Loan Portfolios", Research Memorandum 1999-18, Vrije Universiteit, Amsterdam.

Lucas, A., Klaassen, P., and S., Straetmans (2001) : "Tail Behavior of Credit Loss Distribution for General Latent Factor Model", Vrije Universiteit, Amsterdam.

Merton, R. (1974) : "On the Pricing of Corporate Debt : The Risk Structure of Interest Rates", Journal of Finance, 29, 449-470.

Savage, L. (1954) : "The Foundations of Statistics", New-York, Wiley. Revised and enlarged edition, New-York, Dover Publication, 1972.

Schoenbucher, P. (2000) : "Factor Models for Portfolio Credit Risk", Bonn Univ., DP.

Schoenbucher, P. and D., Schubert (2001) : "Copula Dependent Default Risk in Intensity Models", Dept. of Statistics, Bonn Univ.

Shohat, J., and J., Tamarkin (1943) : "The Problem of Moments", Mathematical Surveys, 1, American Mathematical Society, New-York.

Sun, Y. (2001) : "Asymptotic Theory for Panel Structure Models", Yale Univ., DP.

Van den Berg, G. (1997) : "Association Measures for Durations in Bivariate Hazard Rate Models", Journal of Econometrics, 79, 221-245.

Van den Berg, G. (2001) : "Duration Models : Specification, Identification and Multiple Durations", in Heckman, J., and E., Leamer, eds., Handbook of Econometrics, Vol 5, Amsterdam, North-Holland.

Vasicek, O. (1987) : "Probability of Loss on Loan Portfolio", Working Paper, KMV Corporation.

Vasicek, O. (1997) : "The Loan Loss Distribution", Working Paper, KMV Corporation.